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Quantitative K-theory and the Künneth formula for operator algebras

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ABSTRACT

In this paper, we apply quantitative operator K-theory to develop an algorithm for computing K-theory for the class of filtered C^*-algebras with asymptotic finite nuclear decomposition. As a consequence, we prove the Künneth formula for C^*-algebras in this class. Our main technical tool is a quantitative Mayer–Vietoris sequence for K-theory of filtered C^*-algebras.

Contents

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0. Introduction

The classical Künneth formula computes the (co)homology groups of the product of two topological spaces X and Y in terms of the (co)homology groups of X and Y. By Gelfand's theorem, the category of locally compact spaces being naturally equivalent to the category of commutative C^*-algebra by considering continuous functions vanishing at infinity, it is therefore natural to consider C^*-algebras as functions algebras on noncommutative spaces. In this setting, cohomology is substituted by K-theory for C^*-algebras. K-theory for C^*-algebras has found important applications in topology, geometry and mathematical physics. The Cartesian product for topological spaces corresponds to the (minimal) tensor product in the category of C^*-algebras. The Künneth formula in K-theory computes the K-theory of the tensor product $A \otimes B$ of two C^*-algebras A and B in terms of the K-theory of A and B. In an important article [12], C. Schochet proved the Künneth formula in K-theory when one of the C^*-algebra is in the so called Bootstrap class. Other examples of C^*-algebras for which the Künneth formula holds arise from crossed-product by groups that satisfy the Baum–Connes conjecture with coefficients [1]. If a given C^*-algebra A satisfies the Künneth formula in K-theory together with any C^*-algebra B, then A is exact in K-theory, i.e. every exten-

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sion of C^*-algebra gives rise by tensorization to a six-term exact sequence in K-theory [1, Remark 4.3]. In [11], inspired by [6], N. Ozawa provided a counterexample for exactness in K-theory and hence to the Künneth formula.

The aim of this paper is to develop techniques of quantitative operator K-theory to compute K-theory of C^*-algebras. In particular, we apply these techniques to provide new examples of C^*-algebras that satisfy the Künneth formula in K-theory. The concept of quantitative operator K-theory was first introduced in [14] and was set-up in full generality for filtered C^*-algebras in [9]. A standard way to compute (co)homology groups is by cutting and pasting using Mayer–Vietoris long exact sequence in (co)homology. In the category of C^*-algebras, the usual Mayer–Vietoris six terms exact sequence in K-theory requires the existence of non trivial ideals. But for simple C^*-algebras, non trivial ideal does not exist. The full power of quantitative K-theory is that a controlled version of the Mayer–Vietoris six terms exact sequence for a C^*-algebra A filtered by $(A_r)_{r>0}$ can be stated, only involving neighborhood algebras of a suitable decomposition of A_r into closed linear subspaces Δ_1 and Δ_2. This neighborhood algebras can be viewed as the "ideal generated up to a certain order" by Δ_1 and Δ_2.

We introduce a concept of finite asymptotic nuclear decomposition for filtered C^*-algebras. This C^*-algebraic concept can be viewed as the noncommutative analogue of metric spaces with finite asymptotic dimension. We establish an algorithm for computing K-theory of C^*-algebras with finite asymptotic nuclear decomposition. As a consequence, we prove the Künneth formula for C^*-algebras in this class. Essentially, the assumption of finite asymptotic nuclear dimension allows for some integer n to decompose at order r a C^*-algebra a A in n steps under controlled Mayer–Vietoris into C^*-algebras which are locally in the Bootstrap class. We can then compute inductively quantitative K-theory for C^*-algebras in this class using the controlled Mayer–Vietoris exact sequence. The K-theory is computed by taking limit of quantitative K-theory when the order goes to infinity.

The paper is organized as follows. In Section 1, we give from [9,10] an overview of quantitative K-theory. In Section 2 we introduce the concept of a controlled Mayer–Vietoris pair. This is the key ingredient to define later on the class of C^*-algebras with finite asymptotic nuclear decomposition. Typical examples of these objects arise from Roe algebras and more generally from C^*-algebras of étale groupoïds. In Section 3 is stated for a controlled Mayer–Vietoris pair the controlled six term exact sequence. We apply this sequence to K-contractibility of C^*-algebra. Section 4 is devoted to the quantitative Künneth formula, which implies the Künneth formula in K-theory. We show that examples of filtered C^*-algebras for which the quantitative Künneth formula holds are provided by crossed product of C^*-algebras by finitely generated groups satisfying the Baum–Connes conjecture with coefficients. The main result of these section in that the quantitative Künneth formula is stable under decomposition by controlled Mayer–Vietoris pair. In Section 5 we introduce the class of C^*-algebras with finite asymptotic nuclear decomposition and we show that these C^*-algebras satisfy the quantitative Künneth formula. As a consequence, we show that the uniform Roe algebra of a discrete metric space with bounded geometry and with finite asymptotic dimension satisfies the Künneth formula in K-theory.

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1. Overview of quantitative K-theory

In this section, we recall the basic concepts of quantitative K-theory for filtered C^*-algebras and collect the main results of [9] concerning quantitative K-theory that we shall use throughout this paper. Roughly speaking, quantitative K-theory is the abelian groups of K-theory elements with a prescribed propagation and K-theory can be obtained as an inductive limit of quantitative K-groups (see Corollary 1.11). The key point is that quantitative K-theory is in numerous geometric situations more computable that usual K-theory. The structure of filtered C^*-algebras allows us to talk about the scale of elements in the C^*-algebras.

Definition 1.1. A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive numbers such that:

- $A_r \subset A_{r'}$ if $r \leqslant r'$;
- A_r is stable by involution;
- $A_r \cdot A_{r'} \subset A_{r+r'}$;
- the subalgebra $\bigcup\limits_{r>0} A_r$ is dense in A.

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r. The elements of A_r are said to have **propagation** r.

Many examples of filtered C^*-algebras arise from geometry. Typical examples are provided by Roe algebras, group and crossed-product algebra [9], groupoid algebras (see Section 2.4) and finitely generated C^*-algebras. Indeed all filtered C^*-algebras are associated with a **length function**: let A be a C^*-algebra and assume that there exists a function $\ell : A \to \mathbb{R}^+ \cup \{\infty\}$ such that

- $\ell(0) = 0$;
- $\ell(x + y) \leqslant \max\{\ell(x), \ell(y)\}$ for all x and y in A;
- $\ell(x^*) = \ell(x)$ for all x in A;
- $\ell(\lambda x) = \ell(x)$ for all x in A and λ in $\mathbb{C} \setminus \{0\}$;
- $\ell(xy) \leqslant \ell(x) + \ell(y)$ for all x and y in A;
- the set $\{x \in A$ such that $\ell(x) \leqslant r\}$ is closed in A for all positive numbers r;
- the disjoint union $\bigcup_{r>0}\{x \in A$ such that $\ell(x) \leqslant r\}$ is dense in A.

If we set $A_r = \{x \in A$ such that $\ell(x) \leqslant r\}$, then A is filtered by $(A_r)_{r>0}$. It is straightforward to show that the category of filtered C^*-algebras is equivalent to the category of

C^*-algebras equipped with a length function. For this reason, a C^*-algebra with a lengh function (or a filtration) is called a geometric C^*-algebra. In essence, we study geometric C^*-algebras just as group theorists study geometric group theory.

Let A and A' be respectively C^*-algebras filtered by $(A_r)_{r>0}$ and $(A'_r)_{r>0}$. A homomorphism of C^* -algebras $\phi : A \longrightarrow A'$ is a **filtered homomorphism** (or a **homomorphism of filtered C^*-algebras**) if $\phi(A_r) \subset A'_r$ for any positive number r. If A is not unital, let us denote by $\tilde{A}$ its unitarization, i.e.,

$$\tilde{A} = \{(x, \lambda);\, x \in A\,, \lambda \in \mathbb{C}\}$$

with the product

$$(x, \lambda)(x', \lambda') = (xx' + \lambda x' + \lambda' x, \lambda\lambda')$$

for all (x, λ) and (x', λ') in $\tilde{A}$. Then $\tilde{A}$ is filtered with

$$\tilde{A}_r = \{(x, \lambda);\, x \in A_r\,, \lambda \in \mathbb{C}\}.$$

We also define $\rho_A : \tilde{A} \to \mathbb{C};\, (x, \lambda) \mapsto \lambda$.

1.1. Definition of quantitative K-theory

Let A be a unital filtered C^*-algebra. For any positive numbers r and ε with $\varepsilon < 1/4$, we call

- an element u in A an ε-r-unitary if u belongs to A_r, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. The set of ε-r-unitaries on A will be denoted by $\mathrm{U}^{\varepsilon,r}(A)$;
- an element p in A an ε-r-projection if p belongs to A_r, $p = p^*$ and $\|p^2 - p\| < \varepsilon$. The set of ε-r-projections on A will be denoted by $\mathrm{P}^{\varepsilon,r}(A)$.

Then ε is the called the control and r is called the propagation of the ε-r-projection or of the ε-r-unitary. Notice that an ε-r-unitary is invertible, and that if p is an ε-r-projection in A, then it has a spectral gap around 1/2 and then gives rise by functional calculus to a projection $\kappa_0(p)$ in A such that $\|p - \kappa_0(p)\| < 2\varepsilon$.

Recall the following from [9, Lemma 1.7] the following result that will be used quite extensively throughout the paper.

Lemma 1.2. *Let A be a C^*-algebra filtered by $(A_r)_{r>0}$.*

(i) *If p is an ε-r-projection in A and q is a self-adjoint element of A_r such that $\|p - q\| < \frac{\varepsilon - \|p^2 - p\|}{4}$, then q is an ε-r-projection. In particular, if p is an ε-r-projection in A and if q is a self-adjoint element in A_r such that $\|p - q\| < \varepsilon$, then q is a 5ε-r-projection in A and p and q are connected by a homotopy of 5ε-r-projections.*

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(ii) *If A is unital and if u is an ε-r-unitary and v is an element of A_r such that $\|u-v\| < \frac{\varepsilon - \|u^*u-1\|}{3}$, then v is an ε-r-unitary. In particular, if u is an ε-r-unitary and v is an element of A_r such that $\|u-v\| < \varepsilon$, then v is an 4ε-r-unitary in A and u and v are connected by a homotopy of 4ε-r-unitaries.*

(iii) *If p is a projection in A and q is a self-adjoint element of A_r such that $\|p-q\| < \frac{\varepsilon}{4}$, then q is an ε-r-projection.*

(iv) *If A is unital and if u is a unitary in A and v is an element of A_r such that $\|u-v\| < \frac{\varepsilon}{3}$, then v is an ε-r-unitary.*

Let us also mention the following result concerning homotopy up to stabilization of products of ε-r-unitaries [9, Corollary 1.8].

Lemma 1.3. *Let ε and r be positive numbers with $\varepsilon < 1/12$ and let A be a unital filtered C^*-algebra.*

(i) *Let u and v be ε-r-unitaries in A, then* $\operatorname{diag}(u, v)$ *and* $\operatorname{diag}(uv, 1)$ *are homotopic as 3ε-$2r$-unitaries in $M_2(A)$;*

(ii) *Let u be an ε-r-unitary in A, then* $\operatorname{diag}(u, u^*)$ *and I_2 are homotopic as 3ε-$2r$-unitaries in $M_2(A)$.*

For purpose of rescaling the control and the propagation of an ε-r-projection or of an ε-r-unitary, we introduce the following concept of control pair.

Definition 1.4. A control pair is a pair (λ, h), where

- λ is a positive number with $\lambda > 1$;
- $h : (0, \frac{1}{4\lambda}) \to (1, +\infty)$; $\varepsilon \mapsto h_\varepsilon$ is a map such that there exists a non-increasing map $g : (0, \frac{1}{4\lambda}) \to (1, +\infty)$, with $h \leqslant g$.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leqslant (\lambda', h')$ if $\lambda \leqslant \lambda'$ and $h_\varepsilon \leqslant h'_\varepsilon$ for all ε in $(0, \frac{1}{4\lambda'})$.

Recall the following from [9, Corollary 1.31].

Proposition 1.5. *There exists a control pair (α, k) such that the following holds:*

For any unital filtered C^*-algebra A, any positive numbers ε and r with $\varepsilon < \frac{1}{4\alpha}$ and any homotopic ε-r-projections q_0 and q_1 in $\mathrm{P}_n^{\varepsilon,r}(A)$, then there is for some integers k and l an $\alpha\varepsilon$-$k_\varepsilon r$-unitary W in $M_{n+k+l}(A)$ such that

$$\|\operatorname{diag}(q_0, I_k, 0_l) - W \operatorname{diag}(q_1, I_k, 0_l) W^*\| < \alpha\varepsilon.$$

For any n integer, we set $\mathrm{U}_n^{\varepsilon,r}(A) = \mathrm{U}^{\varepsilon,r}(M_n(A))$ and $\mathrm{P}_n^{\varepsilon,r}(A) = \mathrm{P}^{\varepsilon,r}(M_n(A))$. For any unital filtered C^*-algebra A, any positive numbers ε and r and any positive integer n, we consider inclusions

$$\mathrm{P}_n^{\varepsilon,r}(A) \hookrightarrow \mathrm{P}_{n+1}^{\varepsilon,r}(A);\, p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathrm{U}_n^{\varepsilon,r}(A) \hookrightarrow \mathrm{U}_{n+1}^{\varepsilon,r}(A);\, u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

This allows us to define

$$\mathrm{U}_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}_n^{\varepsilon,r}(A)$$

and

$$\mathrm{P}_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}_n^{\varepsilon,r}(A).$$

For a unital filtered C^*-algebra A, we define the following equivalence relations on $\mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and on $\mathrm{U}_\infty^{\varepsilon,r}(A)$:

- if p and q are elements of $\mathrm{P}_\infty^{\varepsilon,r}(A)$, l and l' are positive integers, $(p,l) \sim (q,l')$ if there exists a positive integer k and an element h of $\mathrm{P}_\infty^{\varepsilon,r}(A[0,1])$ such that $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$;
- if u and v are elements of $\mathrm{U}_\infty^{\varepsilon,r}(A)$, $u \sim v$ if there exists an element h of $\mathrm{U}_\infty^{3\varepsilon,2r}(A[0,1])$ such that $h(0) = u$ and $h(1) = v$.

If p is an element of $\mathrm{P}_\infty^{\varepsilon,r}(A)$ and l is an integer, we denote by $[p,l]_{\varepsilon,r}$ the equivalence class of (p,l) modulo $\sim$ and if u is an element of $\mathrm{U}_\infty^{\varepsilon,r}(A)$ we denote by $[u]_{\varepsilon,r}$ its equivalence class modulo $\sim$.

Definition 1.6. Let r and ε be positive numbers with $\varepsilon < 1/4$. We define:

(i) $K_0^{\varepsilon,r}(A) = \mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}/\sim$ for A unital and

$$K_0^{\varepsilon,r}(A) = \{[p,l]_{\varepsilon,r} \in \mathrm{P}^{\varepsilon,r}(\tilde{A}) \times \mathbb{N}/\sim \ \text{such that}\ \operatorname{rank} \kappa_0(\rho_A(p)) = l\}$$

for A non unital ($\kappa_0(\rho_A(p))$ being the spectral projection associated to $\rho_A(p)$);

(ii) $K_1^{\varepsilon,r}(A) = \mathrm{U}_\infty^{\varepsilon,r}(\tilde{A})/\sim$, with $\tilde{A} = A$ if A is already unital.

Then $K_0^{\varepsilon,r}(A)$ turns to be an abelian group [9, Lemma 1.15], where

$$[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,r}$$

for any $[p,l]_{\varepsilon,r}$ and $[p',l']_{\varepsilon,r}$ in $K_0^{\varepsilon,r}(A)$. According to Corollary 1.3, $K_1^{\varepsilon,r}(A)$ is equipped with a structure of abelian group such that

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$$[u]_{\varepsilon,r} + [u']_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r},$$

for any $[u]_{\varepsilon,r}$ and $[u']_{\varepsilon,r}$ in $K_1^{\varepsilon,r}(A)$.

Recall from [9, Corollaries 1.19 and 1.21] that for any positive numbers r and ε with $\varepsilon < 1/4$, then the map

$$K_0^{\varepsilon,r}(\mathbb{C}) \to \mathbb{Z};\ [p,l]_{\varepsilon,r} \mapsto \mathrm{rank}\,\kappa_0(p) - l$$

is an isomorphism and $K_1^{\varepsilon,r}(\mathbb{C}) = \{0\}$.

We have for any filtered C^*-algebra A and any positive numbers r, r', ε and ε' with $\varepsilon \leqslant \varepsilon' < 1/4$ and $r \leqslant r'$ natural group homomorphisms called the structure maps:

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A);\ [p,l]_{\varepsilon,r} \mapsto [\kappa_0(p)] - [I_l]$ (where $\kappa_0(p)$ is the spectral projection associated to p);
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A);\ [u]_{\varepsilon,r} \mapsto [u]$;
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$;
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A);\ [p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'}$;
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon',r'}(A);\ [u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}$;
- $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$.

If some of the indices r, r' or $\varepsilon, \varepsilon'$ are equal, we shall not repeat them in $\iota_*^{\varepsilon,\varepsilon',r,r'}$. The structure maps satisfy the obvious compatibilitity rules with respect to compositions. We have in the formalism of quantitative K-theory the analogue of the standard form for a K-theory class.

Lemma 1.7. *Let A be a non unital filtered C^*-algebra. Let ε and s be positive numbers with $\varepsilon < \frac{1}{36}$. Then for any x in $K_0^{\varepsilon,s}(A)$, there exist*

- *two integers k and n with $k \leqslant n$;*
- *a 9ε-s-projection q in $M_n(\widetilde{A})$*

such that $\rho_A(q) = \mathrm{diag}(I_k, 0)$ and $x = [q,k]_{9\varepsilon,s}$ in $K_0^{9\varepsilon,s}(A)$.

Proof. Let x be an element in $K_0^{\varepsilon,s}(A)$, let p be an ε-s-projection in some $M_n(\tilde{A})$ and let k be an integer with $\mathrm{rank}\,\kappa_0(\rho_A(q)) = k$ and such that $x = [p,k]_{\varepsilon,s}$. We can assume without loss of generality that $k \leq n$. Let U be a unitary in $M_n(\mathbb{C})$ such that $U \cdot \kappa_0(\rho_A(q)) \cdot U^* = \mathrm{diag}(I_k, 0)$. Since U is homotopic to I_n as a unitary of $M_n(\mathbb{C})$, we see that $U \cdot p \cdot U^*$ and p are homotopic as ε-s-projections in $M_n(\tilde{A})$. Set then

$$q' = U \cdot q \cdot U^* + \mathrm{diag}(I_k, 0) - U \cdot \rho_A(q) \cdot U^*.$$

Since

$$\begin{aligned}\|q' - U\cdot q\cdot U^*\| &= \|U\cdot(\kappa_0(\rho_A(p)) - \rho_A(p))\cdot U^*\|\\ &= \|\kappa_0(\rho_A(p)) - \rho_A(p)\|\\ &< 2\varepsilon,\end{aligned}$$

we get according to Lemma 1.2 that q and q' are homotopic 9ε-s-projections in $M_n(\tilde{A})$. Since $\rho_A(q') = \mathrm{diag}(I_k, 0)$, we get the result. □

We have a similar result in the odd case.

Lemma 1.8. *Let A be a non unital C^*-algebra filtered by $(A_s)_{s>0}$. Let ε and s be positive numbers with $\varepsilon < \frac{1}{84}$.*

(i) *for any x in $K_1^{\varepsilon,s}(A)$, there exists an 21ε-s-unitary u in $M_n(\tilde{A})$, such that $\rho_A(u) = I_n$ and $\iota_1^{\varepsilon,21\varepsilon,s}(x) = [u]_{21\varepsilon,s}$ in $K_1^{21\varepsilon,s}(A)$;*

(ii) *if u and v are two ε-r-unitaries in $M_n(\tilde{A})$ such that $\rho_A(u) = \rho_A(v) = I_n$ and $[u]_{\varepsilon,s} = [v]_{\varepsilon,s}$ in $K_1^{\varepsilon,r}(A)$, then there exists an integer k and a homotopy $(w_t)_{t\in[0,1]}$ of 21ε-s-unitaries of $M_{n+k}(\tilde{A})$ between $\mathrm{diag}(u, I_k)$ and $\mathrm{diag}(v, I_k)$ such that $\rho_A(w_t) = I_{n+k}$ for every t in $[0,1]$.*

Proof. Let v be an ε-r-unitary in some $M_n(\tilde{A})$ such that $x = [v]_{\varepsilon,r}$. According to [9, Remark 1.4], we have that $\|\rho_A(v)^{-1} - \rho_A(v^*)\| < 2\varepsilon$. In particular, $\rho_A(v)^{-1}$ is a 7ε-$r$unitary and $\rho_A(v)^{-1}$ is homotopic to I_n as a 7ε-s-unitary of $M_n(\mathbb{C})$, where $\mathbb{C}$ is provided with the trivial filtration [9, Lemma 1.20]. Hence, if we set $u = \rho_A(v)^{-1}v$, then u is a 21ε-s unitary of $M_n(\tilde{A})$ such that $\rho_A(u) = I_n$ and u and v are homotopic as 21ε-s unitaries of $M_n(\tilde{A})$. Hence we have the equality

$$\iota_1^{\varepsilon,21\varepsilon,s}(x) = [v]_{21\varepsilon,s} = [u]_{21\varepsilon,s}. \quad \square$$

Let $\phi : A \to B$ be a homomorphism of filtered C^*-algebras. Then ϕ preserves ε-r-projections and ε-r-unitaries and hence ϕ induces for any positive number r and any $\varepsilon \in (0, 1/4)$ a group homomorphism

$$\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r}(B).$$

Moreover quantitative K-theory is homotopy invariant with respect to homotopies that preserves propagation [9, Lemma 1.26]. There is also a quantitative version of Morita equivalence [9, Proposition 1.28].

Proposition 1.9. *If A is a filtered algebra and $\mathscr{H}$ is a separable Hilbert space, then the homomorphism*

$$A \to \mathscr{K}(\mathscr{H}) \otimes A;\, a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces a ($\mathbb{Z}_2$-graded) group isomorphism (the Morita equivalence)

$$\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(\mathscr{K}(\mathscr{H}) \otimes A)$$

for any positive number r and any $\varepsilon \in (0, 1/4)$.

The following observation establishes a connection between quantitative K-theory and classical K-theory (see [9, Remark 1.17]).

Proposition 1.10.

(i) *Let A be a filtered C^*-algebra. For any positive number ε with $\varepsilon < \frac{1}{4}$ and any element y of $K_*(A)$, there exists a positive number r and an element x of $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$;*

(ii) *There exists a positive number λ_0 such that for any C^*-algebra A, any positive numbers ε and r with $\varepsilon < \frac{1}{4\lambda_0}$ and any element x of $K_*^{\varepsilon,r}(A)$ for which $\iota_*\varepsilon, r(x) = 0$ in $K_*(A)$, then there exists a positive number r' with $r' \geqslant r$ such that $\iota_*^{\varepsilon,\lambda_0\varepsilon,r,r'}(x) = 0$ in $K_*^{\lambda\varepsilon,r'}(A)$.*

As a consequence, we get the following approximation property.

Corollary 1.11. *Let λ_0 be as in Proposition 1.10. Then for any positive number ε with $\varepsilon < \frac{1}{4\lambda_0}$ and for any filtered C^*-algebra A, then*

$$K_*(A) = \lim_r \iota_*^{\varepsilon,\lambda_0\varepsilon,r}(K_*^{\varepsilon,r}(A)).$$

1.2. Quantitative objects

In order to study the functorial properties of quantitative K-theory, we introduced in [10] the concept of quantitative object.

Definition 1.12. A quantitative object is a family $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<1/4, r>0}$ of abelian groups, together with a family of group homomorphisms

$$\iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} : O^{\varepsilon,r} \to O^{\varepsilon',r'}$$

for $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leq r'$ called the structure maps such that

- $\iota_{\mathcal{O}}^{\varepsilon,\varepsilon,r,r} = Id_{O^{\varepsilon,r}}$ for any $0 < \varepsilon < 1/4$ and $r > 0$;
- $\iota_{\mathcal{O}}^{\varepsilon',\varepsilon'',r',r''} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} = \iota_{\mathcal{O}}^{\varepsilon,\varepsilon'',r,r''}$ for any $0 < \varepsilon \leqslant \varepsilon' \leqslant \varepsilon'' < 1/4$ and $0 < r \leqslant r' \leqslant r''$.

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Example 1.13. Our prominent example will be of course quantitative K-theory $\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0<\varepsilon<1/4,r>0}$ of a filtered C^*-algebra A with structure maps $\iota_*^{\varepsilon,\varepsilon',r,r'}$: $K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon',r'}(A)$ for $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$.

1.3. Controlled morphisms

In this subsection, we recall from [9, Section 2] the relevant notion of morphisms in the framework of quantitative objects.

Definition 1.14. Let (λ, h) be a control pair and let $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<1/4,r>0}$ and $\mathcal{O}' = (O'^{\,\varepsilon,r})_{0<\varepsilon<1/4,r>0}$ be quantitative objects. A (λ, h)-controlled morphism

$$\mathcal{F} : \mathcal{O} \to \mathcal{O}'$$

is a family $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\lambda},r>0}$ of groups homomorphisms

$$F^{\varepsilon,r} : \mathcal{O}^{\varepsilon,r} \to \mathcal{O}'^{\,\lambda\varepsilon,h_\varepsilon r}$$

such that for any positive numbers ε, ε', r and r' with $0 < \varepsilon \leqslant \varepsilon' < \frac{1}{4\lambda}$, $r \leqslant r'$ and $h_\varepsilon r \leqslant h_{\varepsilon'} r'$, we have

$$F^{\varepsilon',r'} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} = \iota_{\mathcal{O}'}^{\lambda\varepsilon,\lambda\varepsilon',h_\varepsilon r,h_{\varepsilon'} r'} \circ F^{\varepsilon,r}.$$

When it is not necessary to specify the control pair, we will just say that $\mathcal{F}$ is a controlled morphism. In order to avoid overloading subscripts, from now on we shall not specify the range of ε and r in the quantitative objects and quantitative morphisms. Indeed, for a quantitative morphism, the range of ε is completely determined by the control pair. In the same way, to avoid overloading upperscript in the structure maps, we shall write $\iota_{\mathcal{O}}^{-,\varepsilon',r'}$ for $\iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'}$ when ε and r in the source are implicit and $\iota_{\mathcal{O}}^{\varepsilon,r,-}$ for $\iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'}$ when ε' and r' in the range are implicit. If both source and range are implicit we shall write $\iota_{\mathcal{O}}^{-,-}$

If $\mathcal{O} = (O^{\varepsilon,r})$ is a quantitative object, let us define the identity $(1, 1)$-controlled morphism

$$\mathcal{I}d_{\mathcal{O}} = (Id_{O^{\varepsilon,r}}) : \mathcal{O} \to \mathcal{O}.$$

Recall that if A and B are filtered C^*-algebras and if $\mathcal{F} : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ is a (λ, h)-controlled morphism, then $\mathcal{F}$ induces a morphism $F : K_*(A) \to K_*(B)$ uniquely defined by $\iota_*^{\varepsilon,r} \circ F^{\varepsilon,r} = F \circ \iota_*^{\varepsilon,r}$.

In some situation, as for instance control boundary maps of controlled Mayer–Vietoris pair (see Section 3.2), we deal with family $F^{\varepsilon,r} : \mathcal{O}^{\varepsilon,r} \to \mathcal{O}'^{\,\lambda\varepsilon,h_\varepsilon r}$ of group morphism defined indeed only up to a certain order.

Definition 1.15. Let (λ, h) be a control pair, let $\mathcal{O} = (O^{\varepsilon,s})$ and $\mathcal{O}' = (O'^{\,\varepsilon,s})$ be quantitative objects and let r be a positive number. A (λ, h)-controlled morphism of order r

$$\mathcal{F} : \mathcal{O} \to \mathcal{O}'$$

is a family $\mathcal{F} = (F^{\varepsilon,s})_{0<\varepsilon<\frac{1}{4\lambda},0<s<\frac{r}{h_\varepsilon}}$ of groups homomorphisms

$$F^{\varepsilon,s} : \mathcal{O}^{\varepsilon,s} \to \mathcal{O}'^{\,\lambda\varepsilon,h_\varepsilon s}$$

such that for any positive numbers ε, ε', s and s' with $0 < \varepsilon \leqslant \varepsilon' < \frac{1}{4\lambda}$, $s \leqslant s' < r$ and $h_\varepsilon s \leqslant h_{\varepsilon'} s'$, we have

$$F^{\varepsilon',s'} \circ \iota_{\mathcal{O}}^{\varepsilon,s,-} = \iota_{\mathcal{O}'}^{-,\lambda\varepsilon',h_{\varepsilon'}s'} \circ F^{\varepsilon,s}.$$

As for general controlled morphism, we shall not specify if not necessary the range of ε and s as there are uniquely determined by the underlying control pair and order.

If (λ, h) and (λ', h') are two control pairs, define

$$h * h' : (0, \frac{1}{4\lambda\lambda'}) \to (1, +\infty);\ \varepsilon \mapsto h_{\lambda'\varepsilon} h'_\varepsilon.$$

Then $(\lambda\lambda', h * h')$ is again a control pair. Let $\mathcal{O} = (O^{\varepsilon,r})$, $\mathcal{O}' = (O'^{\,\varepsilon,r})$ and $\mathcal{O}'' = (O''^{\,\varepsilon,r})$ be quantitative objects, let

$$\mathcal{F} = (F^{\varepsilon,r}) : \mathcal{O} \to \mathcal{O}'$$

be a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism, and let

$$\mathcal{G} = (G^{\varepsilon,r}) : \mathcal{O}' \to \mathcal{O}''$$

be a $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism. Then $\mathcal{G} \circ \mathcal{F} : \mathcal{O} \to \mathcal{O}''$ is for $(\alpha, k) = (\alpha_\mathcal{G}\alpha_F, k_\mathcal{G} * k_\mathcal{F})$ the (α, k)-controlled morphism defined by the family

$$(G^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}r} \circ F^{\varepsilon,r} : O^{\varepsilon,r} \to O''^{\alpha\varepsilon,kr}). \tag{1}$$

Notice that if let $\mathcal{F} : \mathcal{O} \to \mathcal{O}'$ and $\mathcal{G} : \mathcal{O}' \to \mathcal{O}''$ are respectively a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism and $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism of order r, then equation (1) defines a $(\alpha_\mathcal{G}\alpha_F, k_\mathcal{G} * k_\mathcal{F})$-controlled morphism $\mathcal{G} \circ \mathcal{F} : \mathcal{O} \to \mathcal{O}''$ of order r.

Notation 1.16. Let (λ, h) be a control pair and let $\mathcal{O} = (O^{\varepsilon,r})$ and $\mathcal{O}' = (O'^{\,\varepsilon,r})$ be quantitative objects and let $\mathcal{F} = (F^{\varepsilon,r}) : \mathcal{O} \to \mathcal{O}'$ (resp. $\mathcal{G} = (G^{\varepsilon,r}) : \mathcal{O} \to \mathcal{O}'$) be a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism (resp. a $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism). Then we write $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ if

- $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leqslant (\lambda, h)$ and $(\alpha_{\mathcal{G}}, k_{\mathcal{G}}) \leqslant (\lambda, h)$;
- for every ε in $(0, \frac{1}{4\lambda})$ and $r > 0$, then

$$\iota_{\mathcal{O}'}^{-,\lambda\varepsilon,h_\varepsilon r} \circ F^{\varepsilon,r} = \iota_{\mathcal{O}'}^{-,\lambda\varepsilon,h_\varepsilon r} \circ G^{\varepsilon,r}.$$

Definition 1.17. Let $\mathcal{F} : \mathcal{O}_1 \to \mathcal{O}'_1$, $\mathcal{F} : \mathcal{O}_2 \to \mathcal{O}'_2$, $\mathcal{G} : \mathcal{O}_1 \to \mathcal{O}_2$ and $\mathcal{G}' : \mathcal{O}'_1 \to \mathcal{O}'_2$ be controlled morphisms and let (λ, h) be a control pair. Then the diagram (or the square)

$$\begin{array}{ccc} \mathcal{O}_1 & \xrightarrow{\mathcal{G}} & \mathcal{O}_2 \\ {\scriptstyle\mathcal{F}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{F}'} \\ \mathcal{O}'_1 & \xrightarrow{\mathcal{G}'} & \mathcal{O}'_2 \end{array}$$

is called $(\lambda, \mathbf{h})$**-commutative** (or $(\lambda, \mathbf{h})$**-commutes**) if $\mathcal{G}' \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{F}' \circ \mathcal{G}$. The definition of (λ, h)-commutative diagram can be obviously extended to the setting of controlled morphism of order r.

Recall from [10] the definition of controlled isomorphisms.

Definition 1.18. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{O} \to \mathcal{O}'$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism with $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leqslant (\lambda, h)$. $\mathcal{F}$ is called (λ, h)-invertible or a (λ, h)-isomorphism if there exists a controlled morphism $\mathcal{G} : \mathcal{O}' \to \mathcal{O}$ such that $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{O}}$ and $\mathcal{F} \circ \mathcal{G} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{O}'}$. The controlled morphism $\mathcal{G}$ is called a (λ, h)-inverse for $\mathcal{G}$.

In particular, if A and B are filtered C^*-algebras and if $\mathcal{G} : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ is a (λ, h)-isomorphism, then the induced morphism $G : K_*(A) \to K_*(B)$ is an isomorphism and its inverse is induced by a controlled morphism (indeed induced by any (λ, h)-inverse for $\mathcal{F}$).

In order to state in Section 4 the quantitative Künneth formula, we will need the more general notion of quantitative isomorphism. Let $\mathcal{O}_1 = (O_1^{\varepsilon,s})$ and $\mathcal{O}_2 = (O_2^{\varepsilon,s})$ be quantitative objects. For a (α, h)-controlled morphism

$$\mathcal{F} = (F^{\varepsilon,s}) : \mathcal{O}_1 \to \mathcal{O}_2,$$

consider the following statements:

$QI_{\mathcal{F}}(\varepsilon, \varepsilon', s, s')$ we assume that $0 < \varepsilon \leqslant \varepsilon' < \frac{1}{4\alpha}$ and $0 < s \leqslant s'$. If x is an element in $O_1^{\varepsilon,s}$ such that $F^{\varepsilon,s}(x) = 0$ in $O_2^{\alpha\varepsilon,h_\varepsilon s}$, then $\iota_{\mathcal{O}_1}^{-,\varepsilon',s'}(x) = 0$ in $O_1^{\varepsilon',s'}$;

$QS_{\mathcal{F}}(\varepsilon, \varepsilon', s, s')$ we assume that $0 < \varepsilon \leqslant \varepsilon' < \frac{1}{4\alpha}$ and $0 < s \leqslant h_{\varepsilon'} s'$. If y is an element in $O_2^{\varepsilon,s}$, there exists an element x in $O_1^{\varepsilon',s'}$ such that $F^{\varepsilon',s'}(x) = \iota_{\mathcal{O}_2}^{-,\alpha\varepsilon',h_{\varepsilon'}s'}(y)$ in $O_2^{\alpha\varepsilon',h_{\varepsilon'}s'}$.

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Definition 1.19.

- Let $\mathcal{O}_1 = (O_1^{\varepsilon,s})$ and $\mathcal{O}_2 = (O_2^{\varepsilon,s})$ be quantitative objects. Then a quantitative isomorphism

$$\mathcal{F} : \mathcal{O}_1 \to \mathcal{O}_2$$

is an (α, h)-controlled morphism $\mathcal{F} = (F^{\varepsilon,r})$ for some control pair (α, h) that satisfies the following: there exists a positive number λ_0, with $\lambda_0 \geqslant 1$ such that
  - for any positive numbers ε and s with $\varepsilon < \frac{1}{4\alpha\lambda_0}$ there exists a positive number s' with $s \leqslant s'$ such that $QI_{\mathcal{F}}(\varepsilon, \lambda_0\varepsilon, s, s')$ is satisfied;
  - for any positive numbers ε and s with $\varepsilon < \frac{1}{4\alpha}$, there exists a positive number s' with $s \leqslant s'h_{\lambda_0\varepsilon}$ such that $QS_{\mathcal{F}}(\varepsilon, \lambda_0\varepsilon, s, s')$ is satisfied. The positive number λ_0 is called the **rescaling** of the quantitative isomorphism $\mathcal{F}$.
- A uniform family of quantitative isomorphisms if a family $(\mathcal{F}_i)_{i\in I}$ where, $\mathcal{F}_i : \mathcal{O}_i \to \mathcal{O}'_i$ is for any i in I an (α, h)-controlled morphism for a given control pair (α, h) such that there exists a positive number λ_0, with $\lambda_0 \geqslant 1$ for which the following holds
  - for any positive numbers ε and s with $\varepsilon < \frac{1}{4\alpha\lambda_0}$ there exists a positive number s' with $s \leqslant s'$ such that $QI_{\mathcal{F}_i}(\varepsilon, \lambda_0\varepsilon, s, s')$ is satisfied for any i in I;
  - for any positive numbers ε and s with $\varepsilon < \frac{1}{4\alpha}$, there exists a positive number s' with $s \leqslant s'h_{\alpha_0\varepsilon}$ such that $QS_{\mathcal{F}_i}(\varepsilon, \lambda_0\varepsilon, s, s')$ is satisfied for any i in I.

In particular, if A and B are filtered C^*-algebras and if $\mathcal{G} : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ is a quantitative isomorphism, then the induced morphism $G : K_*(A) \to K_*(B)$ is an isomorphism (but its inverse is no more given by a quantitative isomorphism).

1.4. Controlled exact sequences

In this subsection, we recall the controlled exact sequence for quantitative objects. This controlled exact sequence is an important tool in computing quantitative K-theory for filtered C^*-algebras.

Definition 1.20. Let (λ, h) be a control pair,

- Let $\mathcal{O} = (O^{\varepsilon,s})$, $\mathcal{O}' = (O'\varepsilon, s)$ and $\mathcal{O}'' = (O''_{\varepsilon,s})$ be quantitative objects and let

$$\mathcal{F} = (F^{\varepsilon,s}) : \mathcal{O} \to \mathcal{O}'$$

be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism and let

$$\mathcal{G} = (G^{\varepsilon,s}) : \mathcal{O}' \to \mathcal{O}''$$

be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism. Then the composition

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$$\mathcal{O} \stackrel{\mathcal{F}}{\to} \mathcal{O}' \stackrel{\mathcal{G}}{\to} \mathcal{O}''$$

is said to be (λ, h)-exact at $\mathcal{O}'$ if $\mathcal{G} \circ \mathcal{F} = 0$ and if for any $0 < \varepsilon < \frac{1}{4\max\{\lambda\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}\}}$, any $s > 0$ and any y in $O'^{\varepsilon,s}$ such that $G^{\varepsilon,s}(y) = 0$ in $O''\varepsilon, r$, there exists an element x in $O^{\lambda\varepsilon,h_\varepsilon s}$ such that

$$F^{\lambda\varepsilon,h_{\lambda\varepsilon}s}(x) = \iota_{\mathcal{O}'}^{-,\alpha_{\mathcal{F}}\lambda\varepsilon,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon s}(y)$$

in $O'^{\,\alpha_{\mathcal{F}}\lambda\varepsilon,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon s}$.

- A sequence of controlled morphisms

$$\cdots \mathcal{O}_{k-1} \stackrel{\mathcal{F}_{k-1}}{\longrightarrow} \mathcal{O}_k \stackrel{\mathcal{F}_k}{\longrightarrow} \mathcal{O}_{k+1} \stackrel{\mathcal{F}_{k+1}}{\longrightarrow} \mathcal{O}_{k+2} \cdots$$

is called (λ, h)-exact if for every k, the composition

$$\mathcal{O}_{k-1} \stackrel{\mathcal{F}_{k-1}}{\longrightarrow} \mathcal{O}_k \stackrel{\mathcal{F}_k}{\longrightarrow} \mathcal{O}_{k+1}$$

is (λ, h)-exact at $\mathcal{O}_k$.

- the notion of (λ, h)-exactness of a composition and of a sequence can obviously be extended to the setting of controlled morphism of order r.

Notice that the constraint on the range of ε is the definition of (λ, h)-exatness is fixed in such a way that $F^{\lambda\varepsilon,\bullet}$ and $G^{\varepsilon,\bullet}$ make sense.

1.5. Six terms controlled exact sequence in quantitative K-theory

Examples of controlled exact sequences in quantitative K-theory are provided by controlled six term exact sequences associated to a completely filtered extensions of C^*-algebras [9, Section 3].

Definition 1.21. Let A be a C^*-algebra filtered by $(A_r)_{r>0}$, let J be an ideal of A and set $J_r = J \cap A_r$. The extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

is called a completely filtered extension of C^*-algebras if the bijective continuous linear map

$$A_r/J_r \longrightarrow (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry i.e. for any integer n, any positive number r and any x in $M_n(A_r)$, then

$$\inf_{y\in M_n(J_r)} \|x+y\| = \inf_{y\in M_n(J)} \|x+y\|.$$

Notice that in this case, the ideal J is filtered by $(A_r \cap J)_{r>0}$ and A/J is filtered by $(A_r + J)_{r>0}$. A particular case of completely filtered extension of C^*-algebra is the case of filtered and semi-split extension of C^*-algebras [9, Lemma 3.3] (or a semi-split extension of filtered algebras) i.e. extension

$$0 \to J \to A \to A/J \to 0,$$

where

- A is filtered by $(A_r)_{r>0}$;
- there exists a completely positive (complete) norm decreasing cross-section

  $$s : A/J \to A$$

  such that

  $$s(A_r + J) \subseteq A_r$$

  for any number $r > 0$.

For any extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0,$$

we denote by $\partial_{J,A} : K_*(A/J) \to K_*(J)$ the associated (odd degree) boundary map in K-theory.

Proposition 1.22. *There exists a control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ such that for any completely filtered extension of C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \stackrel{q}{\longrightarrow} A/J \longrightarrow 0,$$

there exists a $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$-controlled morphism of odd degree

$$\mathcal{D}_{J,A} = (\partial^{\varepsilon,r}_{J,A}) : \mathcal{K}_{*+1}(A/J) \to \mathcal{K}_*(J)$$

which induces in K-theory $\partial_{J,A} : K_*(A/J) \to K_{*+1}(J)$.

Moreover the controlled boundary map enjoys the usual naturally properties with respect to extensions (see [9, Remark 3.8]).

Theorem 1.23. *There exists a control pair (λ, h) such that for any completely filtered extension of C^*-algebras*

$$0 \longrightarrow J \xrightarrow{\jmath} A \xrightarrow{q} A/J \longrightarrow 0,$$

then the following six-term sequence is (λ, h)-exact

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{\jmath_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\uparrow & & & & \downarrow{\scriptstyle \mathcal{D}_{J,A}} \\ \mathcal{K}_1(A/J) & \xleftarrow{q_*} & \mathcal{K}_1(A) & \xleftarrow{\jmath_*} & \mathcal{K}_1(J) \end{array}$$

1.6. KK-theory and controlled morphisms

In this subsection, we discuss compatibility of Kasparov's KK-theory with quantitative K-theory of filtered C^*-algebras.

Let A be a C^*-algebra and let B be a filtered C^*-algebra filtered by $(B_r)_{r>0}$. Let us define $A \otimes B_r$ as the closure in the spatial tensor product $A \otimes B$ of the algebraic tensor product of A and B_r. Then the C^*-algebra $A \otimes B$ is filtered by $(A \otimes B_r)_{r>0}$. If $f : A_1 \to A_2$ is a homomorphism of C^*-algebras, let us set

$$f_B : A_1 \otimes B \to A_2 \otimes B; \; a \otimes b \mapsto f(a) \otimes b.$$

Recall from [4] that for C^*-algebras A_1, A_2 and B, Kasparov defined a tensorization map

$$\tau_B : KK_*(A_1, A_2) \to KK_*(A_1 \otimes B, A_2 \otimes B).$$

If B is a filtered C^*-algebra, then for any z in $KK_*(A_1, A_2)$ the morphism

$$K_*(A_1 \otimes B) \longrightarrow K_*(A_2 \otimes B); \; x \mapsto x \otimes_{A_1 \otimes B} \tau_B(z)$$

is induced by a controlled morphism which enjoys compatibility properties with Kasparov product [9, Theorem 4.4].

Theorem 1.24. *There exists a control pair $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ such that*

- *for any filtered C^*-algebra B;*
- *for any C^*-algebras A_1 and A_2;*
- *for any element z in $KK_*(A_1, A_2)$,*

there exists a $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$-controlled morphism $\mathcal{T}_B(z) : \mathcal{K}_*(A_1 \otimes B) \to \mathcal{K}_*(A_2 \otimes B)$ with $\mathcal{T}_B(z) = (\tau_B^{\varepsilon,r})$ of same degree as z that induces in K-theory the right multiplication by $\tau_B(z)$.

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Moreover $\mathcal{T}_B(\bullet)$ enjoys the following properties:

Proposition 1.25. *For any filtered C^*-algebra B and any C^*-algebras A_1 and A_2,*

(i) *for any elements z and z' in $KK_*(A_1, A_2)$, we have*

$$\mathcal{T}_B(z + z') = \mathcal{T}_B(z) + \mathcal{T}_B(z').$$

(ii) *Let A_1' be a C^*-algebras and let $f : A_1 \to A_1'$ be a homomorphism of C^*-algebras, then $\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_{B,*}$ for all z in $KK_*(A_1', A_2)$.*

(iii) *Let A_2' be a C^*-algebra and let $g : A_2' \to A_2$ be a homomorphism of C^*-algebras then $\mathcal{T}_B(g_*(z)) = g_{B,*} \circ \mathcal{T}_B(z)$ for any z in $KK_*(A_1, A_2')$.*

(iv) $\mathcal{T}_B([Id_{A_1}]) \stackrel{(\alpha_\mathcal{T}, k_\mathcal{T})}{\sim} \mathcal{I}d_{\mathcal{K}_*(A_1 \otimes B)}$.

(v) *For any C^*-algebra D and any element z in $KK_*(A_1, A_2)$, we have $\mathcal{T}_B(\tau_D(z)) = \mathcal{T}_{B \otimes D}(z)$.*

For any element in KK_1 corresponding to a semi-split extension, up to a rescaling, the $\mathcal{T}_B$ is given by the controlled boundary map associated to the tensorized extension:

Proposition 1.26. *For any filtered C^*-algebra B and any semi-split extension of C^*-algebras $0 \to J \to A \to A/J \to 0$ with corresponding element $[\partial_{J,A}]$ of $KK_1(A/J, J)$ that implements the boundary map, we have*

$$\mathcal{T}_B([\partial_{J,A}]) \stackrel{(\alpha_\mathcal{T}, k_\mathcal{T})}{\sim} \mathcal{D}_{J \otimes B, A \otimes B}.$$

The controlled tensorization morphism $\mathcal{T}_B$ is compatible with Kasparov products.

Theorem 1.27. *There exists a control pair (λ, h) such that the following holds:*

let A_1, A_2 and A_3 be separable C^*-algebras and let B be a filtered C^*-algebra. Then for any z in $KK_*(A_1, A_2)$ and any z' in $KK_*(A_2, A_3)$, we have

$$\mathcal{T}_B(z \otimes_{A_2} z') \stackrel{(\lambda, h)}{\sim} \mathcal{T}_B(z') \circ \mathcal{T}_B(z).$$

We also have in the case of finitely generated group a controlled version of the Kasparov transformation. Let Γ be a finitely generated group. Recall that a length on Γ is a map $\ell : \Gamma \to \mathbb{R}^+$ such that

- $\ell(\gamma) = 0$ if and only if γ is the identity element e of Γ;
- $\ell(\gamma\gamma') \leqslant \ell(\gamma) + \ell(\gamma')$ for all element γ and γ' of Γ.
- $\ell(\gamma) = \ell(\gamma^{-1})$.

In what follows, we will assume that ℓ is a word length arising from a finite generating symmetric set S, i.e., $\ell(\gamma) = \inf\{d \text{ such that } \gamma = \gamma_1 \cdots \gamma_d \text{ with } \gamma_1, \ldots, \gamma_d \text{ in } S\}$. Let

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us denote by $B(e,r)$ the ball centered at the neutral element of Γ with radius r, i.e., $B(e,r)=\{\gamma\in\Gamma \text{ such that } \ell(\gamma)\leqslant r\}$. Let A be a separable Γ-C^*-algebra, i.e., a separable C^*-algebra provided with an action of Γ by automorphisms. For any positive number r, we set

$$(A\rtimes_{red}\Gamma)_r \overset{\text{def}}{=\!=} \{f\in C_c(\Gamma,A) \text{ with support in } B(e,r)\}.$$

Then the C^*-algebra $A\rtimes_{red}\Gamma$ is filtered by $((A\rtimes_{red}\Gamma)_r)_{r>0}$. Moreover if $f:A\to B$ is a Γ-equivariant morphism of C^*-algebras, then the induced homomorphism $f_\Gamma : A\rtimes_{red}\Gamma\to B\rtimes_{red}\Gamma$ is a filtered homomorphism. Recall from [4] that for any Γ-C^*-algebras A and B, there exists a natural transformation

$$J_\Gamma : KK_*^\Gamma(A,B)\to KK_*(A\rtimes_{red}\Gamma, B\rtimes_{red}\Gamma)$$

called the Kasparov transformation that preserves Kasparov products. The Kasparov transformation admits a quantitative version [9, Section 5].

Theorem 1.28. *There exists a control pair $(\alpha_{\mathcal{J}},k_{\mathcal{J}})$ such that*

- *for any separable Γ-C^*-algebras A and B;*
- *For any z in $KK_*^\Gamma(A,B)$,*

there exists a $(\alpha_{\mathcal{J}},k_{\mathcal{J}})$-controlled morphism

$$\mathcal{J}_\Gamma^{red}(z):\mathcal{K}_*(A\rtimes_{red}\Gamma)\to\mathcal{K}_*(B\rtimes_{red}\Gamma)$$

with $\mathcal{J}_\Gamma^{red}(z)=(J_\Gamma^{red,\varepsilon,r}(z))$ of same degree as z that induces in K-theory right multiplication by $J_\Gamma^{red}(z)$.

Moreover, $\mathcal{J}_\Gamma^{red}(\bullet)$ satisfies the following properties.

Proposition 1.29. *For any separable Γ-C^*-algebras A and B,*

(i) *for any z and z' in $KK_*^\Gamma(A,B)$, then*

$$\mathcal{J}_\Gamma^{red}(z+z')=\mathcal{J}_\Gamma^{red}(z)+\mathcal{J}_\Gamma^{red}(z').$$

(ii) *For any Γ-C^*-algebra A', any homomorphism $f:A\to A'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A',B)$, then $\mathcal{J}_\Gamma^{red}(f^*(z))=\mathcal{J}_\Gamma^{red}(z)\circ f_{\Gamma,*}$.*

(iii) *For any Γ-C^*-algebra B', any homomorphism $g:B\to B'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A,B)$, then $\mathcal{J}_\Gamma^{red}(g_*(z))=g_{\Gamma,*}\circ\mathcal{J}_\Gamma^{red}(z)$.*

(iv) $\mathcal{J}_\Gamma^{red}(Id_A)\overset{(\alpha_{\mathcal{J}},k_{\mathcal{J}})}{\sim}\mathcal{I}d_{\mathcal{K}_*(A\rtimes_{red}\Gamma)}$.

For any element in KK_1^Γ corresponding to a Γ-equivariant semi-split extension, up to a rescaling, then $\mathcal{J}_\Gamma^{red}$ is given by the controlled boundary map associated to the reduced crossed-product extension:

Proposition 1.30. *Let*

$$0 \to J \to A \to A/J \to 0$$

be a semi-split exact sequence of Γ-C^**-algebras and let* $[\partial_{J,A}]$ *be the element of* $KK_1^\Gamma(A/J, J)$ *that implements the boundary map* $\partial_{J,A}$*. Then we have*

$$\mathcal{J}_\Gamma^{red}([\partial_{J,A}]) \overset{(\alpha_{\mathcal{J}},k_{\mathcal{J}})}{\sim} \mathcal{D}_{J\rtimes_{red}\Gamma, A\rtimes_{red}\Gamma}.$$

Eventually, the controlled Kasparov transformation is compatible with Kasparov products.

Theorem 1.31. *There exists a control pair* (λ, h) *such that the following holds: for every separable* Γ-C^**-algebras* A, B *and* D*, any elements* z *in* $KK_*^\Gamma(A, B)$ *and* z' *in* $KK_*^\Gamma(B, D)$*, then*

$$\mathcal{J}_\Gamma^{red}(z \otimes_B z') \overset{(\lambda,h)}{\sim} \mathcal{J}_\Gamma^{red}(z') \circ \mathcal{J}_\Gamma^{red}(z).$$

Remark 1.32. We can choose $(\alpha_{\mathcal{J}}, k_{\mathcal{J}})$ such that $(\alpha_{\mathcal{J}}, k_{\mathcal{J}}) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$. In this case, for any Γ-C^*-algebra A, any C^*-algebras D_1 and D_2 equipped with the trivial action of Γ and any z in $KK_*(D_1, D_2)$, then

$$\mathcal{T}_{A\rtimes_{red}\Gamma,*}(z) = \mathcal{J}_\Gamma^{red}(\tau_{A,*}(z)).$$

We have a similar result for maximal crossed products.

1.7. Quantitative assembly maps

In this subsection, we discuss a quantitative version of the Baum–Connes assembly map.

Let Γ be a finitely generated group and let B be a Γ-C^*-algebra. We equip Γ with any word metric. Recall that if d is a positive number, then the Rips complex of degree d is the set $P_d(\Gamma)$ of probability measures with support of diameter less than d. Then $P_d(\Gamma)$ is a locally finite simplicial complex and provided with the simplicial topology, $P_d(\Gamma)$ is endowed with a proper and cocompact action of Γ by left translation. In [9], for any Γ-C^*-algebra B, we construct quantitative assembly maps

$$\mu_{\Gamma,B,*}^{\varepsilon,r,d} : KK_*^\Gamma(C_0(P_d(\Gamma)), B) \to K_*^{\varepsilon,r}(B\rtimes_{red}\Gamma),$$

with $d > 0$, $\varepsilon \in (0, 1/4)$ and $r \geqslant r_{d,\varepsilon}$, where

$$[0, +\infty) \times (0, 1/4) \longrightarrow (0, +\infty) : (d, \varepsilon) \mapsto r_{d,\varepsilon}$$

is a function independent on B, non decreasing in d and non increasing in ε. Moreover, the maps $\mu^{\varepsilon,r,d}_{\Gamma,B,*}$ induce the usual assembly maps

$$\mu^d_{\Gamma,B,*} : KK^\Gamma_*(C_0(P_s(\Gamma)), B) \to K_*(B \rtimes_{red} \Gamma),$$

i.e., $\mu^d_{\Gamma,B,*} = \iota^{\varepsilon,r}_* \circ \mu^{\varepsilon,r,d}_{\Gamma,B,*}$. Let us recall now the definition of the quantitative assembly maps. Observe first that any x in $P_d(\Gamma)$ can be written down in a unique way as a finite convex combination

$$x = \sum_{\gamma \in \Gamma} \lambda_\gamma(x) \delta_\gamma,$$

where δ_γ is the Dirac probability measure at γ in Γ. The functions

$$\lambda_\gamma : P_d(\Gamma) \to [0, 1]$$

are continuous and $\gamma(\lambda_{\gamma'}) = \lambda_{\gamma\gamma'}$ for all γ and γ' in Γ. The function

$$p_{\Gamma,d} : \Gamma \to C_0(P_d(\Gamma)); \, \gamma \mapsto \sum_{\gamma \in \Gamma} \lambda_e^{1/2} \lambda_\gamma^{1/2}$$

is a projection of $C_0(P_d(\Gamma)) \rtimes_{red} \Gamma$ with propagation less than d. Let us set then $r_{d,\varepsilon} = k_{\mathcal{J},\varepsilon/\alpha_\mathcal{J}} d$, where the control pair $(\alpha_\mathcal{J}, k_\mathcal{J})$ is as in Theorem 1.28. Recall that $k_\mathcal{J}$ can be chosen non increasing and in this case, $r_{d,\varepsilon}$ is non decreasing in d and non increasing in ε.

Definition 1.33. For any Γ-C^*-algebra A and any positive numbers ε, r and d with $\varepsilon < 1/4$ and $r \geqslant r_{d,\varepsilon}$, we define the quantitative assembly map

$$\begin{aligned} \mu^{\varepsilon,r,d}_{\Gamma,A,*} : KK^\Gamma_*(C_0(P_d(\Gamma)), A) &\to K^{\varepsilon,r}_*(A \rtimes_{red} \Gamma) \\ z &\mapsto \left(J^{red,\varepsilon',r'}_\Gamma(z)\right)\left([p_{\Gamma,d}, 0]_{\varepsilon',r'}\right), \end{aligned}$$

with $\varepsilon' = \frac{\varepsilon}{\alpha_\mathcal{J}}$ and $r' = \frac{r}{k_{\mathcal{J},\varepsilon/\alpha_\mathcal{J}}}$ and where the notation $[p_{\Gamma,d}, 0]_{\varepsilon',r'}$ is as in Definition 1.6.

Then according to point (ii) of Proposition 1.29, the map $\mu^{\varepsilon,r,d}_{\Gamma,A}$ is a group homomorphism. For any positive numbers d and d' such that $d \leqslant d'$, we denote by $q_{d,d'} : C_0(P_{d'}(\Gamma)) \to C_0(P_d(\Gamma))$ the homomorphism induced by the restriction from $P_{d'}(\Gamma)$ to $P_d(\Gamma)$. It is straightforward to check that if d, d' and r are positive numbers such that $d \leqslant d'$ and $r \geqslant r_{d',\varepsilon}$, then $\mu^{\varepsilon,r,d}_{\Gamma,A} = \mu^{\varepsilon,r,d'}_{\Gamma,A} \circ q_{d,d',*}$. Moreover, for every positive numbers ε, ε', d, r and r' such that $\varepsilon \leqslant \varepsilon' \leqslant 1/4$, $r_{d,\varepsilon} \leqslant r$, $r_{d,\varepsilon'} \leqslant r'$, and $r < r'$, we get by definition of a controlled morphism that

$$\iota_*^{-,\varepsilon',r'} \circ \mu_{\Gamma,A,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,*}^{\varepsilon',r',d}.$$

In [9] we introduced quantitative statements for the quantitative assembly maps. For a Γ-C^*-algebra A and positive numbers $d, d', r, r', \varepsilon$ and ε' with $d \leqslant d'$, $\varepsilon \leqslant \varepsilon' < 1/4$, $r_{d,\varepsilon} \leqslant r'$ and $r \leqslant r'$ we set:

$QI_{\Gamma,A,*}(d,d',r,\varepsilon)$ for any element x in $KK_*^\Gamma(C_0(P_d(\Gamma)),A)$, if $\mu_{\Gamma,A,*}^{\varepsilon,r,d}(x) = 0$ in $K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$, then $q_{d,d'}^*(x) = 0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)),A)$.

$QS_{\Gamma,A,*}(d,r,r',\varepsilon,\varepsilon')$ for every y in $K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$, there exists an element x in $KK_*^\Gamma(C_0(P_d(\Gamma)),A)$ such that $\mu_{\Gamma,A,*}^{\varepsilon',r',d}(x) = \iota_*^{-,\varepsilon',r'}(y)$.

The following results were then proved [9, Theorem 6.6].

Theorem 1.34. *Let Γ be a discrete group.*

(i) *Assume that for any Γ-C^*-algebra A, the assembly map $\mu_{\Gamma,A,*}$ is one-to-one. Then for any positive numbers d, ε and $r \geqslant r_{d,\varepsilon}$ with $\varepsilon < 1/4$ and $r \geqslant r_d$, there exists a positive number d' with $d' \geqslant d$ such that $QI_{\Gamma,A}(d,d',r,\varepsilon)$ is satisfied for every Γ-C^*-algebra A;*

(ii) *Assume that for any Γ-C^*-algebra A, the assembly map $\mu_{\Gamma,A,*}$ is onto. Then for some positive number α_0 which not depends on Γ or on A and such that with $\alpha_0 > 1$ and for any positive numbers ε and r with $\varepsilon < \frac{1}{4\alpha_0}$, there exist positive numbers d and r' with $r_{d,\varepsilon} \leqslant r'$ and $r \leqslant r'$ such that $QS_{\Gamma,A}(d,r,r',\varepsilon,\alpha_0\varepsilon)$ is satisfied for every Γ-C^*-algebra A.*

In particular, if Γ satisfies the Baum–Connes conjecture with coefficients, then Γ satisfies points (i) *and* (ii) *above.*

In [10] we developed a geometric version of the controlled assembly maps and of the quantitative statements in the following setting. Let Σ be a proper discrete metric space and let A be a C^*-algebra. Then the distance d on Σ induces a filtration on $A \otimes \mathcal{K}(\ell^2(\Sigma))$ in the following way: let r be a positive number and $T = (T_{\sigma,\sigma'})_{(\sigma,\sigma') \in \Sigma^2}$ be an element in $A \otimes \mathcal{K}(\ell^2(\Sigma))$, with $T_{\sigma,\sigma'}$ in A for any σ and σ' in Σ^2. Then T has propagation less that r if $T_{\sigma,\sigma'} = 0$ for σ and σ' in Σ such that $d(\sigma,\sigma') > r$. As for finitely generated group, we define the Rips complex of degree d of Σ as the set $P_d(\Sigma)$ of probability measure with support of diameter less than d. Then $P_d(\Sigma)$ is a locally finite simplicial complex and is locally compact when endowed with the simplicial topology. Let us define then

$$K_*(P_d(\Sigma),A) \stackrel{\text{def}}{=\!=} \lim_{Z \subseteq P_d(\Sigma)} KK_*(C(Z),A),$$

where Z runs through compact subsets of $P_d(\Sigma)$. In turn we constructed in [10] local quantitative coarse assembly maps

$$\nu_{\Sigma,A,*}^{\varepsilon,r,d} : K_*(P_d(\Sigma)), A) \longrightarrow K_*^{\varepsilon,r}(A \otimes \mathcal{K}(\ell^2(\Sigma))),$$

with $d > 0$, $\varepsilon \in (0, 1/4)$ and $r \geqslant r_{d,\varepsilon}$. The map $\nu_{\Sigma,\bullet,*}^{\varepsilon,r,d}$ is natural in the C^*-algebra and induces in K-theory the index map, i.e. the maps $\iota_*^{\varepsilon,r} \circ \nu_{\Sigma,A,*}^{\varepsilon,r,d}$ is up to Morita equivalence given for any compact subset Z of $P_d(\Sigma)$ by the morphism in the inductive limit $KK_*(C(Z), A) \to K_*(A)$ induced by the map $Z \to \{pt\}$. Moreover, the maps $\nu_{\Sigma,A,*}^{\bullet,\bullet,\bullet}$ are compatible with structure morphisms and with inclusion of Rips complexes:

- $\iota_*^{-,\varepsilon',r'} \circ \nu_{\Sigma,A,*}^{\varepsilon,r,d} = \nu_{\Sigma,A,*}^{\varepsilon',r',d}$ for any positive numbers $\varepsilon, \varepsilon', r, r'$ and s such that $\varepsilon \leqslant \varepsilon' < 1/4$, $r_{d,\varepsilon} \leqslant r$, $r_{d,\varepsilon'} \leqslant r'$ and $r \leqslant r'$;
- $\nu_{\Sigma,A,*}^{\varepsilon,r,d'} \circ q_{d,d'}^* = \nu_{\Sigma,A,*}^{\varepsilon,r,d}$ for any positive numbers ε, r, d and d' such that $\varepsilon < 1/4$, $d \leqslant d'$ and $r_{d',\varepsilon} \leqslant r$, where

$$q_{d,d'}^* : KK_*(C_0(P_{d'}(\Sigma)), A) \to KK_*(C_0(P_d(\Sigma)), A)$$

  is the homomorphism induced by the restriction from $P_{d'}(\Sigma)$ to $P_d(\Sigma)$.

For $d, d', r, r', \varepsilon$ and ε' positive numbers with $d \leqslant d'$, $\varepsilon' \leqslant \varepsilon < 1/4$, $r_{d,\varepsilon} \leqslant r$ and $r' \leqslant r$, we consider the following statements:

$QI_{\Sigma,A,*}(d, d', r, \varepsilon)$ for any element x in $K_*(P_d(\Sigma), A)$, then $\nu_{\Sigma,A,*}^{\varepsilon,r,d}(x) = 0$ in $K_*^{\varepsilon,r}(A \otimes \mathcal{K}(\ell^2(\Sigma)))$ implies that $q_{d,d'}^*(x) = 0$ in $K_*(P_{d'}(\Sigma), A)$.

$QS_{\Sigma,A,*}(d, r, r', \varepsilon, \varepsilon')$ for every y in $K_*^{\varepsilon',r'}(A \otimes \mathcal{K}(\ell^2(\Sigma)))$, there exists an element x in $K_*(P_d(\Sigma), A)$ such that

$$\nu_{\Sigma,A,*}^{\varepsilon,r,d}(x) = \iota_*^{-,\varepsilon,r}(y).$$

Recall that a proper discrete metric space Σ with bounded geometry coarsely embeds in a Hilbert space if there exist

- a map $\phi : \Sigma \to \mathscr{H}$ where $\mathscr{H}$ is a Hilbert space;
- two maps $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{+\infty} \rho_\pm = +\infty$,

such that

$$\rho_-(d(x, y)) \leqslant \|\phi(x) - \phi(y)\| \leqslant \rho_+(d(x, y))$$

for any x and y in Σ. Proper discrete metric spaces with bounded geometry that coarsely embed into a Hilbert space provide numerous examples that satisfy the following statement called the Quantitative Assembly Map estimates [10, Theorems 4.9 and 4.10].

Theorem 1.35. *Let Σ be a discrete metric space with bounded geometry that coarsely embeds into a Hilbert space.*

(i) *For any positive numbers d, ε and r with $\varepsilon < 1/4$ and $r \geqslant r_{d,\varepsilon}$, there exists a positive number d' with $d' \geqslant d$ for which $QI_{\Sigma,A,*}(d, d', r, \varepsilon)$ is satisfied for any C^*-algebra A.*

(ii) *There exists a positive number $\lambda > 1$ such that for any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{d,\varepsilon} \leqslant r$ and $r' \leqslant r$ for which $QS_{\Sigma,A,*}(d, r, r', \lambda\varepsilon, \varepsilon)$ is satisfied for any C^*-algebra A.*

2. Controlled Mayer–Vietoris pairs

In the construction of the boundary map of the Mayer–Vietoris six terms exact sequence in K-theory and for establishing exactness, the following result is a key point: let A be a unital C^*-algebra which is the sum of two closed ideals J_1 and J_2. Then any unitary u in A connected to the identity can be written as a product $u = v_1v_2$ of two unitaries v_1 and v_2 lying respectively in the unitarization of J_1 and J_2 and as such connected to the identity. In this section and in order to state a controlled version of the K-theory Mayer–Vietoris six terms exact sequence, we investigate an analogue of this result for a so-called coercive decomposition at a given order r into closed linear subspaces Δ_1 and Δ_2. Every ε-r-unitary connected to the identity is then up to rescaling by a (universal) control pair and to stabilization, closed to a product of ε-r-unitaries lying respectively in the unitarization of some neighborhood C^*-algebras of Δ_1 and Δ_2 and as such connected to the identity. These neighborhood C^*-algebras can be viewed as the ideals generated up to certain order respectively by Δ_1 and Δ_2. The strategy to prove this result is first to obtain an approximation by a product of ε-r-N-invertibles and then to use in the setting of ε-r-N-invertibles an analogue of the polar decomposition. We then give the definition of a controlled Mayer–Vietoris pair which allows to define C^*-algebras with finite asymptotic nuclear decomposition in Section 5 and which is the framework to state in Section 3 the Mayer–Vietoris controlled exact for quantitative K-theory. We also discuss a few technical lemmas useful for establishing the latter.

2.1. ε-r-N-invertible elements of a filtered C^*-algebra

In [9, Section 7] is introduced the notion of ε-r-N-invertible element of a unital Banach algebra. In this subsection, we study ε-r-N-invertible elements for C^*-algebras. In particular, we state an analogue of the polar decomposition in the setting of ε-r-unitaries.

Definition 2.1. Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$ and let ε, r and N be positive numbers with $\varepsilon < 1$. An element x in A_r is called ε-r-N-invertible if $\|x\| \leqslant N$ and there exists y in A_r such that $\|y\| \leqslant N$, $\|xy - 1\| < \varepsilon$ and $\|yx - 1\| < \varepsilon$. Such an element y is called an ε-r-N-inverse for x.

Remark 2.2. If x is ε-r-N-invertible, then x is invertible and for any ε-r-N-inverse y for x, we have $\|x^{-1} - y\| \leq \frac{\varepsilon N}{1-\varepsilon}$.

Definition 2.3. Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$ and let ε, r and N be positive numbers with $\varepsilon < 1$. Two ε-r-N-invertibles in A are called homotopic if there exists $Z : [0,1] \to A$ an ε-r-N-invertible in $A[0,1]$ such that $Z(0) = x$ and $Z(1) = y$.

In the setting of ε-r-N-invertibles and of ε-r-unitaries, there is the analogue of the polar decomposition.

Lemma 2.4. *For any positive number N there exists a control pair (α, l) and a positive number N' with $N' \geqslant N$ such that the following holds.*

For any filtered unital C^*-algebra A filtered by $(A_s)_{s>0}$, any positive numbers ε and r with $\varepsilon < \frac{1}{4\alpha}$ and every ε-r-N-invertible element x of A, there exist h a positive $\alpha\varepsilon$-$l_\varepsilon r$-N'-invertible in A and u an $\alpha\varepsilon$-$l_\varepsilon r$-unitary in A such that $\||x| - h\| < \alpha\varepsilon$ and $\|x - uh\| < \alpha\varepsilon$. Moreover we can choose u and h such that

- *there exists a real polynomial function Q with $Q(1) = 1$ such that $u = xQ(x^*x)$ and $h = x^*xQ(x^*x)$;*
- *h admits a positive $\alpha\varepsilon$-$l_\varepsilon r$-N'-inverse;*
- *If x is homotopic to 1 as an ε-r-N-invertible, then u is homotopic to 1 as an $\alpha\varepsilon$-$l_\varepsilon r$-unitary.*

Proof. According to Remark 2.2 and since $\varepsilon < 1/4$, if x is an ε-r-N-invertible, then x is invertible and $\|x^{-1}\| < 2N$ and hence $\|(x^*x)^{-1}\| < 4N^2$. This implies that the spectrum of x^*x is included in $[\frac{1}{4N^2}, N^2]$. Let t_0 and t_1 be positive numbers such that $t_0 < \min(\frac{1}{4N^2}, 1)$ and $\max(N^2, 1) < t_1$. Let us consider the power series $\sum a_n t^n$ of $t \mapsto \frac{1}{\sqrt{1+t}}$ for t in $[0,1]$ and let l_ε be the smallest integer such that

$$\sum_{k=l_\varepsilon+1}^{+\infty} |a_k| \left(\frac{1-t_1}{t_1}\right)^k < \frac{\min(\sqrt{t_1}, 1)\varepsilon}{2}$$

and

$$\sum_{k=0}^{l_\varepsilon} a_k(-t)^k \geqslant 1/2$$

for all t in $[0, 1 - \frac{t_0}{t_1}]$. Since $\sum a_n \left(\frac{x^*x - t_1}{t_1}\right)^n$ converges to $\sqrt{t_1}(x^*x)^{-1/2} = \sqrt{t_1}|x|^{-1}$, if we set

$$Q(t) = \frac{1}{\sqrt{t_1}} \sum_{k=0}^{l_\varepsilon} a_k \left(\frac{t-t_1}{t_1}\right)^k + \frac{1}{\sqrt{t_1}} \sum_{k=l_\varepsilon+1}^{+\infty} a_k \left(\frac{1-t_1}{t_1}\right)^k$$

then Q is a polynomial of degree l_ε such that $Q(1)=1$, $Q(t)\geqslant 0$ for every t in $[t_0,t_1]$ and $\|Q(x^*x)-(x^*x)^{-1/2}\|<\varepsilon$. If we set $u=xQ(x^*x)$, then u is a α-$2l_\varepsilon+1$-unitary for some $\alpha>1$ depending only on N. Set now $h=u^*x=x^*xQ(x^*x)$, then up to taking a large α, there exists a control pair (α,k) and a positive number N' depending only on N, with $N'\geqslant N$ and such that u and h satisfy the required properties and $Q(x^*x)$ is a positive $\alpha\varepsilon$-$k_\varepsilon r$-N'-inverse for h. Moreover, if $(x_t)_{t\in[0,1]}$ is a homotopy of ε-r-invertibles between 1 and x, then $(x_tQ(x_t^*x_t))_{t\in[0,1]}$ is a homotopy of $\alpha\varepsilon$-$k_\varepsilon r$-unitaries between 1 and u. □

The first step in order to obtain the main result of this section is to approximate element of the form $\left(\begin{smallmatrix}u&0\\0&u^*\end{smallmatrix}\right)$ for u an ε-r unitary that decomposes into $u=x_1+x_2$ by a product elementary matrices with entries involving x_1 and x_2. Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$. For x and y in A, set

$$X(x)=\begin{pmatrix}1&x\\0&1\end{pmatrix}$$

and

$$Y(y)=\begin{pmatrix}1&0\\y&1\end{pmatrix}$$

and consider the commutators

$$Z(x,y)=X(x)Y(y)X(x)^{-1}Y(y)^{-1}=\begin{pmatrix}1+xy+xyxy&-xyx\\yxy&1-yx\end{pmatrix}$$

and

$$Z'(x,y)=Y(y)^{-1}X(x)^{-1}Y(y)X(x)=\begin{pmatrix}1-xy&-xyx\\yxy&1+yx+yxyx\end{pmatrix}.$$

Lemma 2.5. *Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$ and let ε and r be positive numbers with $\varepsilon<1/4$. Let x_1 and x_2 in A_r such that x_1+x_2 is an ε-r-unitary. Then we have the inequality*

$$\left\|X(x_1)Z(x_2,-x_1^*)Y(-x_1^*)X(x_1)X(x_2)Y(-x_2^*)\,Z'(x_1,-x_2^*)X(x_2)\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right.$$
$$\left.-\begin{pmatrix}x_1+x_2&0\\0&x_1^*+x_2^*\end{pmatrix}\right\|<3\varepsilon.$$

Proof. Let us set $u=x_1+x_2$. Consider the matrix

$$W(u)=X(u)Y(-u^*)X(u)\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}2u-uu^*u&uu^*-1\\1-u^*u&u^*\end{pmatrix}.$$

Since u is an ε-r-unitary, then

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$$\left\| \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} - W(u) \right\| < 3\varepsilon.$$

We have

$$W(u) = X(x_1)X(x_2)Y(-x_1^*)Y(-x_2^*)X(x_1)X(x_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This, together with the definition of Z and Z', implies that

$$W(u) = X(x_1)Z(x_2, -x_1^*)Y(-x_1^*)X(x_1)X(x_2)Y(-x_2^*)Z'(x_1, -x_2^*) \\ \times X(x_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad \square$$

2.2. Coercive decomposition pair and R-neighborhood C^*-algebras

We introduce in this subsection the basic ingredient to define controlled Mayer–Vietoris pairs.

If Δ and Δ' are two closed linear subspaces of a C^*-algebra A such that $\Delta \subseteq \Delta'$, we equip $M_n(\Delta/\Delta') \cong M_n(\Delta)/M_n(\Delta')$ with the quotient C^*-algebra norm, i.e. if x is a element of $M_n(\Delta)$, then $\|x + M_n(\Delta')\| = \inf\{\|x + y\|;\ y \in M_n(\Delta')\}$. Then this family of norms is a matrix norm on Δ/Δ'.

Definition 2.6. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$ and let r be a positive number.

- a coercive decomposition pair of degree r for A (or a coercive decomposition r-pair) is a pair (Δ_1, Δ_2) of closed linear subspaces of A_r such that there exists a positive number C satisfying the following: for any positive number s with $s \leqslant r$ the inclusion $\Delta_1 \cap A_s \hookrightarrow A_s$ induces an isomorphism

$$\frac{\Delta_1 \cap A_s}{\Delta_1 \cap \Delta_2 \cap A_s} \xrightarrow{\cong} \frac{A_s}{\Delta_2 \cap A_s}$$

  whose inverse is bounded in norm by C.
- a completely coercive decomposition pair of degree r for A (or a completely coercive decomposition r-pair) is a pair (Δ_1, Δ_2) of closed linear subspaces of A_r such that there exists a positive number C satisfying the following: for any positive number s with $s \leqslant r$ the inclusion $\Delta_1 \cap A_s \hookrightarrow A_s$ induces a complete isomorphism

$$\frac{\Delta_1 \cap A_s}{\Delta_1 \cap \Delta_2 \cap A_s} \xrightarrow{\cong} \frac{A_s}{\Delta_2 \cap A_s}$$

  whose inverse has complete norm bounded by C.

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Remark 2.7. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$, let r be a positive number and let (Δ_1, Δ_2) be a pair of closed linear subspaces of A_r. Then (Δ_1, Δ_2) is a coercive decomposition pair of degree r for A if and only if there exists a positive number c such that for every positive number s with $s \leqslant r$ and any x in A_s, there exists x_1 in $\Delta_1 \cap A_s$ and x_2 in $\Delta_2 \cap A_s$, both with norm at most $c\|x\|$ and such that $x = x_1 + x_2$. In the same way, (Δ_1, Δ_2) is a completely coercive decomposition pair of degree r for A if and only if there exists a positive number c such that for every positive number s with $s \leqslant r$, any integer n and any x in $M_n(A_s)$, there exists x_1 in $M_n(\Delta_1 \cap A_s)$ and x_2 in $M_n(\Delta_2 \cap A_s)$, both with norm at most $c\|x\|$ and such that $x = x_1 + x_2$. The (completely) coercive decomposition r-pair (Δ_1, Δ_2) is said to have **coercitivity c**.

The aim of this subsection is to show that for any coercive decomposition r-pair (Δ_1, Δ_2), there exists a control pair (α, h) depending indeed only on the coercitivity, such that up to stabilization, any ε-s-unitary of A with $0 < \varepsilon \leqslant \frac{1}{4\alpha}$ and $0 < s \leqslant \frac{r}{h_\varepsilon}$ can be approximated by a product of two $\alpha\varepsilon$-$h_\varepsilon s$-unitaries lying respectively in the unitarization of some suitable neighborhood algebras of Δ_1 and Δ_2. We first show using Lemma 2.5 that this approximation exists in term of ε-r-N-invertibles. Then we use the analogue of the polar decomposition of Lemma 2.4 to conclude.

Definition 2.8. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$. Let r and R be positive numbers and let Δ be a closed linear subspace of A_r. We define $C^*N_\Delta^{(r,R)}$, the R-neighborhood C^*-algebra of Δ, as the C^*-subalgebra of A generated by its R-neighborhood $N_\Delta^{(r,R)} = \Delta + A_R \cdot \Delta + \Delta \cdot A_R + A_R \cdot \Delta \cdot A_R$.

Notice that $C^*N_\Delta^{(r,R)}$ inherits from A a structure of filtered C^*-algebra with $C^*N_{\Delta,s}^{(r,R)} = C^*N_\Delta^{(r,R)} \cap A_s$ for every positive number s. For a positive number s satisfying $s \leqslant r$, we also denote by $C^*N_\Delta^{(s,R)}$ for the R-neighborhood C^*-algebra of $\Delta \cap A_s$.

Lemma 2.9. *For any positive number c, there exist positive numbers λ, C and N, with $\lambda > 1$ and $C > 1$ such that the following holds.*

Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$, let r and ε be positive numbers such that $\varepsilon < \frac{1}{4\lambda}$ and let (Δ_1, Δ_2) be a coercive decomposition pair for A of degree r with coercitivity c. Then for any ε-r-unitary u in A homotopic to 1, there exist an integer k and P_1 and P_2 in $M_k(A_{Cr})$ such that

- *P_1 and P_2 are invertible;*
- *$P_i - I_k$ and $P_i^{-1} - I_k$ are elements in the matrix algebra $M_n(C^*N_{\Delta_i,Cr}^{(r,4r)})$ for $i = 1, 2$;*
- *$\|P_i\| < N$ and $\|P_i^{-1}\| < N$ for $i = 1, 2$;*
- *for $i = 1, 2$, there exists a homotopy $(P_{i,t})_{t\in[0,1]}$ of invertible elements in $M_k(A_{Cr})$ between 1 and P_i such that $\|P_{i,t}\| < N$, $\|P_{i,t}^{-1}\| < N$ and $P_{i,t} - I_k$ and $P_{i,t}^{-1} - I_k$ are elements in the matrix algebra $M_n(C^*N_{\Delta_i,Cr}^{(r,4r)})$ for every t in $[0, 1]$.*
- *$\|\operatorname{diag}(u, I_{k-1}) - P_1P_2\| < \lambda\varepsilon$.*

Proof. Let $(u_t)_{t\in[0,1]}$ be a homotopy of ε-r-unitaries of A between $u=u_0$ and $1=u_1$ and let $t_0=0<t_1<\cdots<t_k=1$ be a partition of $[0,1]$ such that $\|u_{t_i}-u_{t_{i-1}}\|<\varepsilon$ for $i=1,\cdots,k$. Set

$$V=\operatorname{diag}(u_{t_0},\dots,u_{t_k},u^*_{t_0},\dots,u^*_{t_k})$$

and

$$W=\operatorname{diag}(1,u^*_{t_0},\dots,u^*_{t_{k-1}},u_{t_0},\dots,u_{t_{k-1}},1).$$

Then we have

$$\begin{aligned}
&\|\operatorname{diag}(u,I_{2k+1})-VW\|\leqslant\|\operatorname{diag}(u,I_{2k+1})-\operatorname{diag}(u_0,u_{t_1}u^*_{t_1},\dots,u_{t_k}u^*_{t_k},I_{k+1})\|\\
&+\|\operatorname{diag}(u_{t_0},u_{t_1}u^*_{t_1},\dots,u_{t_k}u^*_{t_k},I_{k+1})-\operatorname{diag}(u_{t_0},u_{t_1}u^*_{t_0},\dots,u_{t_k}u^*_{t_{k-1}},I_{k+1})\|\\
&+\|\operatorname{diag}(u_{t_0},u_{t_1}u^*_{t_0},\dots,u_{t_k}u^*_{t_{k-1}},I_{k+1})\\
&-\operatorname{diag}(u_{t_0},u_{t_1}u^*_{t_0},\dots,u_{t_k}u^*_{t_{k-1}},u_{t_0}u^*_{t_0},u_{t_1}u^*_{t_1},\dots,u_{t_k}u^*_{t_k})\|\\
&<4\varepsilon.
\end{aligned}$$

For any matrix X in $M_{2k}(A)$, let us set $\widetilde{X}=\operatorname{diag}(1,X,1)$ in $M_{2k+2}(A)$. For every integer $i=-1,0,\dots,k$, pick v_i in Δ_1 and w_i in Δ_2 such that $u_{t_i}=v_i+w_i$ with $\|v_i\|\leqslant c\|u_{t_i}\|$ and $\|w_i\|\leqslant c\|u_{t_i}\|$. Set $x_1=\operatorname{diag}(v_0,\dots,v_k)\,,x_2=\operatorname{diag}(w_0,\dots,w_k)\,,y_1=\operatorname{diag}(v^*_0,\dots,v^*_{k-1})$ and $y_2=\operatorname{diag}(w^*_0,\dots,w^*_{k-1})$. Since we have

$$V=\operatorname{diag}(x_1+x_2,x^*_1+x^*_2)$$

and

$$W=\widetilde{\operatorname{diag}}(y_1+y_2,y^*_1+y^*_2),$$

then if we set

$$T(x,y)=X(x)Z(y,-x^*)Y(-x^*)X(x),$$

we deduce from Lemma 2.5 that

$$\left\|VW-T(x_1,x_2)T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{T}(y_1,y_2)\widetilde{T}^{-1}(-y_2,-y_1)\widetilde{U_k}\right\|<9\varepsilon$$

with $U_k=\begin{pmatrix}0&-I_k\\I_k&0\end{pmatrix}$ in $M_{2k}(\mathbb{C})$ and hence

$$\left\|\operatorname{diag}(u,I_{2k+1})-T(x_1,x_2)T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{T}(y_1,y_2)\widetilde{T}^{-1}(-y_2,-y_1)\widetilde{U_k}\right\|<13\varepsilon\quad(2)$$

Let us show that

$$S(x_1,x_2,y_1,y_2) \stackrel{\text{def}}{=\!=} T(x_1,x_2)T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{T}(y_1,y_2)\widetilde{T}^{-1}(-y_2,-y_1)\widetilde{U_k} \tag{3}$$

can be decomposed as a product P_1P_2 with P_1 and P_2 satisfying the required properties. Notice that as a product of elementary matrices, the matrix $T(x_1,x_2)$ is invertible. By the definition of the neighborhood C^*-algebra, $T(x_1,x_2)-I_{2k+2}$ and $T^{-1}(x_1,x_2)-I_{2k+2}$ are elements in the matrix algebra $M_{2k+2}(C^*N^{(r,r)}_{\Delta_1,7r})$. The same holds for $\widetilde{T}(y_1,y_2)$, and we have similar properties for $T(-x_2,-x_1)$ and $\widetilde{T}(-y_2,-y_1)$ with respect to $C^*N^{(r,r)}_{\Delta_2,7r}$. In order to bring out some commutators, let us study the term

$$\widetilde{T}(y_1,y_2)^{-1}U^*_{k+1}T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{T}(y_1,y_2)$$

which is the product of the four first terms in equation (3). Let us make the following observations:

- $\widetilde{X}(-y_1)U^*_{k+1}T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{X}(y_1)-I_{2k+2}$ is an element in the matrix algebra $M_{2k+2}(C^*N^{(r,2r)}_{\Delta_2,9r})$;
- $\widetilde{Z}^{-1}(y_1,y_2)\widetilde{X}(-y_1)U^*_{k+1}T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{X}(y_1)\widetilde{Z}(y_1,y_2)-I_{2k+2}$ is an element in the matrix algebra $M_{2k+2}(C^*N^{(r,2r)}_{\Delta_2,17r})$ (because $Z^{-1}(y_1,y_2)-I_{2k}=Z'(y_1,y_2)-I_{2k}$ and $Z(y_1,y_2)-I_{2k}$ are elements of the matrix algebra $M_{2k}(C^*N^{(r,r)}_{\Delta_2,4r})$);
- $\widetilde{Y}(y_1^*)\widetilde{Z}^{-1}(y_1,y_2)\widetilde{X}(-y_1)U^*_{k+1}T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{X}(y_1)\widetilde{Z}(y_1,y_2)\widetilde{Y}(y_1)-I_{2k+2}$ is an element in the matrix algebra $M_{2k+2}(C^*N^{(3r,r)}_{\Delta_2,19r})$;
- $\widetilde{X}(-x_1)\widetilde{Y}(y_1^*)\widetilde{Z}^{-1}(y_1,y_2)\widetilde{X}(-y_1)U^*_{k+1}T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{X}(y_1)\widetilde{Z}(y_1,y_2)\widetilde{Y}(y_1)X(x_1)$ $-I_{2k+2}$ is an element in the matrix algebra $M_{2k+2}(C^*N^{(r,4r)}_{\Delta_2,21r})$.

Hence $\widetilde{T}(y_1,y_2)^{-1}U^*_{k+1}T^{-1}(-x_2,-x_1)U_{k+1}\widetilde{T}(y_1,y_2)-I_{2k+1}$ is an element in the matrix algebra $M_{2k+2}(C^*N^{(r,r)}_{\Delta_2,21r})$. Since $1=v_k+w_k$, then we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = T(v_k,w_k)T^{-1}(-w_k,-v_k).$$

Therefore there exists for $i=1,2$ an invertible matrix $Q_i(v_k,w_k)$ in $M_{2k+2}(A)$ such that $Q_i(v_k,w_k)-I_{2k+2}$ and $Q_i^{-1}(v_k,w_k)-I_{2k+2}$ lie in $M_{2k+2}(C^*N^{(r,r)}_{\Delta_i,7r})$ and

$$Q_1(v_k,w_k)Q_2(v_k,w_k)=U_{k+1}\widetilde{U_k}.$$

Therefore if we write $S(x_1,x_2,y_1,y_2)=P_1P_2$ with

$$P_1=T(x_1,x_2)U_{k+1}\widetilde{T}(y_1,y_2)U^*_{k+1}Q_1(v_k,w_k)$$

and

$$P_2 = Q_2(v_k, w_k)\widetilde{U_k^*}\widetilde{T}(y_1, y_2)^{-1}U_{k+1}^*T^{-1}(-x_2, -x_1)U_{k+1}\widetilde{T}(y_1, y_2)\widetilde{T}^{-1}(-y_2, -y_1)\widetilde{U_k}$$

are invertible matrices of $M_{2k+2}(A)$ such that

- $P_1 - I_{2k+2}$ and $P_1^{-1} - I_{2k+2}$ are elements in the matrix algebra $M_{2k+2}(C^*N^{(r,r)}_{\Delta_1,21r})$.
- $P_2 - I_{2k+2}$ and $P_2^{-1} - I_{2k+2}$ are elements in the matrix algebra $M_{2k+2}(C^*N^{(r,4r)}_{\Delta_2,35r})$.

Since P_1 and P_2 can be written as a product of a fixed number, say p, of matrices $X(x)$ or $Y(x)$ with $\|x\| < 2c$, we see that P_1 and P_2 have norm less than $(2c+1)^p$. According to equation (2), we have

$$\|\mathrm{diag}(u, I_{2k+1}) - P_1P_2\| < 13\varepsilon.$$

The required homotopies are then

$$(T(tx_1, tx_2)U_{k+1}\widetilde{T}(ty_1, ty_2)U_{k+1}^*Q_1(tv_k, tw_k))_{t\in[0,1]}$$

and

$$(Q_2(tv_k, tw_k)\widetilde{U_k^*}\widetilde{T}(ty_1, ty_2)^{-1}U_{k+1}T^{-1}(-tx_2, -tx_1)U_{k+1}^*\widetilde{T}(ty_1, ty_2)$$
$$\widetilde{T}^{-1}(-ty_2, -ty_1)\widetilde{U_k})_{t\in[0,1]}. \quad \square$$

Let us briefly explain how we deal with the non unital case. Let A be a non unital filtered C^*-algebra and let u be an ε-r unitary in $\tilde{A}$ such that $u - 1$ is in A. Assume that $u = x_1 + x_2$ with $1 - x_1$ and x_2 lie in A. Proceeding as in the proof of Lemma 2.5, we see that $\mathrm{diag}(u, u^*)$ is 3ε-close to the product of

$$P_1 = X(x_2)X(x_1)Y(-x_1^*)X(x_1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}X(-x_2)$$

and

$$P_2 = X(x_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}X(-x_1)Y(-x_2^*)X(x_1)X(x_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now if (Δ_1, Δ_2) be a coercive decomposition pair for A of degree r with coercitivity c and assume that in the above decomposition of $u = x_1 + x_2$ we have $1 - x_1$ in Δ_1 and x_2 in Δ_2, we get then by a straightforward computation that $P_1 - I_2$ has coefficient in $C^*N^{(r,2r)}_{\Delta_1,5r}$ and $P_2 - I_2$ has coefficient in $C^*N^{(r,2r)}_{\Delta_2,5r}$. Notice that in view of the proof of Lemma 1.8, if under above assumption, u is connected to 1 as a ε-r-unitary of $\tilde{A}$, then u is connected to 1 by a homotopy of 21ε-r-unitaries $(u_t)_{t\in[0,1]}$ of $\tilde{A}$ such that $u_t - 1$ lies in A for all t in $[0, 1]$. Hence proceeding as in the proof of Lemma 2.9 we get:

Lemma 2.10. *For any positive number c, there exist positive numbers λ, C and N, with $\lambda > 1$ and $C > 1$ such that the following holds.*

Let A be a non unital C^*-algebra filtered by $(A_s)_{s>0}$, let r and ε be positive numbers such that $\varepsilon < \frac{1}{4\lambda}$ and let (Δ_1, Δ_2) be a coercive decomposition pair for A of degree r with coercitivity c. Then for any ε-r-unitary u in $\tilde{A}$ homotopic to 1 and such that $u - 1$ lies in A, there exist an integer k and P_1 and P_2 in $M_k(\tilde{A}_{Cr})$ such that

- *P_1 and P_2 are invertible;*
- *$P_i - I_k$ and $P_i^{-1} - I_k$ are elements in the matrix algebra $M_n(C^* N^{(r,5r)}_{\Delta_i,Cr})$ for $i = 1, 2$;*
- *$\|P_i\| < N$ and $\|P_i^{-1}\| < N$ for $i = 1, 2$;*
- *for $i = 1, 2$, there exists a homotopy $(P_{i,t})_{t\in[0,1]}$ of invertible elements in $M_k(\tilde{A}_{Cr})$ between 1 and P_i such that $\|P_{i,t}\| < N$, $\|P_{i,t}^{-1}\| < N$ and $P_{i,t} - I_k$ and $P_{i,t}^{-1} - I_k$ are elements in the matrix algebra $M_n(C^* N^{(r,5r)}_{\Delta_i,Cr})$ for every t in $[0, 1]$.*
- *$\|\operatorname{diag}(u, I_{k-1}) - P_1 P_2\| < \lambda\varepsilon$.*

Using the analogue of the polar decomposition stated in Lemma 2.4, we are now in position to prove the approximation result in terms of ε-r-unitaries.

Proposition 2.11. *For every positive number c, there exists a control pair (α, l) such that the following holds.*

Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$, let r and ε be positive numbers such that $\varepsilon < \frac{1}{4\alpha}$ and let (Δ_1, Δ_2) be a coercive decomposition pair for A of degree r with coercitivity c. Then for any ε-r-unitary u in A homotopic to 1, there exist a positive integer k and w_1 and w_2 two $\alpha\varepsilon$-$l_\varepsilon r$-unitaries in $M_k(A)$ such that

- *$w_i - I_k$ is an element in the matrix algebra $M_k(C^* N^{(r,4r)}_{\Delta_i,l_\varepsilon r})$ for $i = 1, 2$;*
- *for $i = 1, 2$, there exists a homotopy $(w_{i,t})_{t\in[0,1]}$ of $\alpha\varepsilon$-$l_\varepsilon r$-unitaries between 1 and w_i such that $w_{i,t} - I_k \in M_k(C^* N^{(r,4r)}_{\Delta_i,l_\varepsilon r})$ for all t in $[0, 1]$.*
- *$\|\operatorname{diag}(u, I_{k-1}) - w_1 w_2\| < \alpha\varepsilon$.*

Proof. As in Lemma 2.9, let λ, C and N be positive numbers, k be an integer and P_1 and P_2 be matrices of $M_k(A_{Cr})$ such that $\|\operatorname{diag}(u, I_{k-1}) - P_1 P_2\| < \lambda\varepsilon$. Since P_1 and P_2 are ε-Cr-N-invertible for every ε in $(0, \frac{1}{4\lambda})$, then according to Lemma 2.4, there exists

- a control pair (α, l);
- w_1 an ε-$l_{\varepsilon/\alpha} r$-unitary and h_1 an ε-$l_{\varepsilon/\alpha} r$-N-invertible both in

$$M_{2k+2}(C^* N^{(r,4r)}_{\Delta_1} + \mathbb{C}),$$

  with h_1 positive and admitting a positive ε-$l_{\varepsilon/\alpha} r$-N-inverse;

- w_2 an ε-$l_{\varepsilon/\alpha}r$-unitary and h_2 an ε-$k_{\varepsilon/\alpha}r$-N-invertible both in

$$M_{2k+2}(C^*N_{\Delta_2}^{(r,4r)}+\mathbb{C}),$$

with h_2 positive and admitting a positive ε-$l_{\varepsilon/\alpha}r$-N-inverse,

such that $\|P_1 - w_1h_1\| < \varepsilon$, $\|P_2^* - w_2h_2\| < \varepsilon$, $\||P_1| - h_1\| < \varepsilon$ and $\||P_2^*| - h_2\| < \varepsilon$. Then

$$\|w_1h_1h_2w_2^* - \mathrm{diag}(u, I_{2k+1})\| < (2N + \lambda + 1)\varepsilon \tag{4}$$

and hence, up to replacing λ by $4(2N + \lambda + 1)$, we know according to Lemma 1.2 that $w = w_1h_1h_2w_2^*$ is a $\lambda\varepsilon$-$4l_{\varepsilon/\alpha}r$-unitary. Let us prove that h_1h_2 is close to I_{2k+2}.

Let h_1' be a positive ε-$l_{\varepsilon/\alpha}r$-N-inverse for h_1. Then we have $\|w_1^*w - h_1h_2w_2^*\| < 2\lambda\varepsilon$ and then $\|h_1'w_1^*w - h_2w_2^*\| < 4\lambda N\varepsilon$. This implies that

$$\|h_1'w_1^*w(h_1'w_1^*w)^* - h_2w_2^*(h_2w_2^*)^*\| < 16\lambda N\varepsilon.$$

Since $\|h_1'w_1^*w(h_1'w_1^*w)^* - {h_1'}^2\| < 3\lambda N^2\varepsilon$ and $\|h_2w_2^*(h_2w_2^*)^* - h_2^2\| < 3\lambda N^2\varepsilon$, we deduce that there exists $\lambda' \geqslant \lambda$ depending only on λ and N such that $\|{h_1'}^2 - h_2^2\| < \lambda'\varepsilon$. But, since h_1' and h_2 are ε-$l_{\varepsilon/\alpha}r$-N-invertible with $\varepsilon < 1/2$, their spectrum is bounded below by $\frac{1}{2N}$. The square root is Lipschitz on the set of positive elements of A with spectrum bounded below by $\frac{1}{2N}$ (this can be checked easily by holomorphic functional calculus), thus there exists a positive number M, depending only on N such that $\|h_1' - h_2\| < M\lambda'\varepsilon$. Since h_1' is an ε-$l_{\varepsilon/\alpha}r$-N-inverse for h_1, we finally obtain that $\|h_1h_2 - I_{2k+2}\| < (1 + \lambda'MN)\varepsilon$.

Combining this inequality with equation (4), we know that there exist a positive number $\lambda'' > 1$, depending only on N and λ' such that

$$\|w_1w_2^* - \mathrm{diag}(u, I_{2k+1})\| < \lambda''\varepsilon.$$

According to Lemma 2.4, $w_1 = P_1Q(P_1^*P_1)$ where Q is polynomial and such that $Q(1) = 1$. Since $P_1 - I_{2k+2}$ lies in $C^*N_{\Delta_1}^{(r,4r)}$, then the same holds for $w_1 - I_{2k+2}$ and similarly, $w_2 - I_{2k+2}$ lies in $C^*N_{\Delta_2}^{(r,4r)}$. □

Proceeding similarly, Lemma 2.10 allows to deal with the non unital case.

Proposition 2.12. *For every positive number c, there exists a control pair (α, l) such that the following holds.*

Let A be a non unital C^*-algebra filtered by $(A_r)_{r>0}$, let r and ε be positive numbers such that $\varepsilon < \frac{1}{4\alpha}$ and let (Δ_1, Δ_2) be a coercive decomposition pair for A of degree r with coercitivity c. Then for any ε-r-unitary u in $\tilde{A}$ homotopic to 1 and such that $u - 1$ lies in A, there exist a positive integer k and w_1 and w_2 two $\alpha\varepsilon$-$l_\varepsilon r$-unitaries in $M_k(\tilde{A})$ such that

- $w_i - I_k$ *is an element in the matrix algebra* $M_k(C^*N^{(r,5r)}_{\Delta_i,l_\varepsilon r})$ *for* $i = 1, 2$;
- *for* $i = 1, 2$, *there exists a homotopy* $(w_{i,t})_{t\in[0,1]}$ *of* $\alpha\varepsilon$-$l_\varepsilon r$*-unitaries between* 1 *and* w_i *such that* $w_{i,t} - I_k \in M_k(C^*N^{(r,5r)}_{\Delta_i,l_\varepsilon r})$ *for all* t *in* $[0, 1]$.
- $\|\operatorname{diag}(u, I_{k-1}) - w_1w_2\| < \alpha\varepsilon$.

2.3. Controlled Mayer–Vietoris pair

In this subsection, we introduce the controlled Mayer–Vietoris pair that allows to decompose at a given order r a filtered C^*-algebra A into a completely coercive decomposition (Δ_1, Δ_2). This controlled Mayer–Vietoris pair gives rise to a controlled six-terms exact sequence that compute the quantitative K-theory at order r of A in terms of the controlled K-theory of some attached neighborhood C^*-algebras of Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$. These neighborhood C^*-algebras are roughly speaking C^*-algebras that contains a quantitative ideal associated to the underlying linear subspaces. Our prominent examples of controlled Mayer–Vietoris pair will be given by Roe algebras.

Definition 2.13. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$, let r be a positive number and let Δ be a closed linear subspace of A_r. Then a sub-C^*-algebra B of A is called an r-controlled Δ-neighborhood-C^*-algebra if

- B is filtered by $(B \cap A_r)_{r>0}$;
- $C^*N^{(r,5r)}_\Delta \subseteq B$.

Remark 2.14. In view of Propositions 2.11 and 2.12, the second assumption in the above definition guarantees that the controlled boundary in the controlled Mayer–Vietoris exact sequence in well defined.

Example 2.15. Let Σ be a discrete metric space with bounded geometry and consider $C^*(\Sigma)$ the Roe Algebra of Σ. Recall that $C^*(\Sigma)$ is the closure of the algebra of locally compact and finite propagation operators on $\ell^2(\Sigma)\otimes\mathscr{H}$, where $\mathscr{H}$ is a fixed separable Hilbert space. Then $C^*(\Sigma)$ is filtered by the propagation. For r a positive number, let $(X_i)_{i\in\mathbb{N}}$ be a family of finite subsets of Σ with uniformly bounded diameter which is R-disjoint (i.e., $d(X_i, X_j) \geqslant R$ if $i \neq j$) for some positive number $R \geqslant 12r$. Let us consider the set $\Delta \subseteq C^*(\Sigma)_r$ of locally compact operators on $\ell^2(\Sigma)\otimes\mathscr{H}$ with support in

$$\Big\{(x, y) \in \Sigma \times \Sigma;\ x \in \bigcup_{i\in\mathbb{N}} X_i,\ d(x, y) \leqslant r\Big\}.$$

For a positive number s, let us set $X_{i,s} = \{x \in X_i \text{ such that } d(x, X_i) < s\}$. If $s < R/2$, then $(X_{i,s})_{i\in\mathbb{N}}$ is a family of $(R - 2s)$-disjoint subsets of Σ with uniformly bounded diameter. Consider then the subalgebra A_Δ of $C^*(\Sigma)$ of operators with support in $\bigsqcup_{i\in\mathbb{N}} X_{i,s} \times X_{i,s}$. Then

$$A_\Delta \cong \prod_{i\in\mathbb{N}} \mathscr{K}(\ell^2(X_{i,s})\otimes\mathscr{H}) \cong \left(\prod_{i\in\mathbb{N}} \mathscr{K}(\ell^2(X_{i,s})\otimes\mathscr{K}(\mathscr{H})\right)$$

and A_Δ is for every s with $5r < s < R/2$ an r-controlled Δ-neighborhood-C^*-algebra.

Definition 2.16. Let S_1 and S_2 be two subsets of a C^*-algebra A. The pair (S_1, S_2) is said to have *complete intersection approximation* property (CIA) if there exists $c > 0$ such that for any positive number ε, any $x \in M_n(S_1)$ and $y \in M_n(S_2)$ for some n and $||x - y|| < \varepsilon$, then there exists $z \in M_n(S_1 \cap S_2)$ satisfying

$$||z - x|| < c\varepsilon, \quad ||z - y|| < c\varepsilon.$$

The positive number c is called the **coercitivity** of the pair (S_1, S_2).

In the above definition, we note that the inequalities $||x - y|| < \varepsilon$ and $||z - x|| < c\varepsilon$ implies $||z - y|| < (c + 1)\epsilon$. Hence we can remove the condition $||z - y|| < c\varepsilon$ up to replacing the constant c by $c + 1$.

Definition 2.17. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$ and let r be a positive number. An r-controlled weak Mayer–Vietoris pair for A is a quadruple $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ such that for some positive number c.

(i) (Δ_1, Δ_2) is a completely coercive decomposition pair for A of order r with coercitivity c.
(ii) A_{Δ_i} is an r-controlled Δ_i-neighborhood-C^*-algebra for $i = 1, 2$;
(iii) the pair $(A_{\Delta_1,s}, A_{\Delta_2,s})$ has the CIA property with coercitivity c as defined above for any positive number s with $s \leqslant r$.

The positive number c is called the **coercitivity** of the r-controlled weak Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

Remark 2.18. In the above definition,

(i) $(\Delta_1\cap A_s, \Delta_2\cap A_s, A_{\Delta_1}, A_{\Delta_2})$ is an s-controlled Mayer–Vietoris pair for any $0 < s \leqslant r$ with same coercitivity as $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.
(ii) $A_{\Delta_1} \cap A_{\Delta_2}$ is filtered by $(A_{\Delta_1,r} \cap A_{\Delta_2,r})_{r>0}$.

In order to ensure some persistence properties for the controlled Mayer–Vietoris exact sequence (see Corollary 3.6), we need to strenghen condition (iii) of Definition 2.17.

Definition 2.19. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$ and let r be a positive number. An r-controlled Mayer–Vietoris pair for A is a quadruple $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ such that for some positive number c.

(i) (Δ_1, Δ_2) is a completely coercive decomposition pair for A of order r with coercitivity c.
(ii) A_{Δ_i} is an r-controlled Δ_i-neighborhood-C^*-algebra for $i = 1, 2$;
(iii) the pair $(A_{\Delta_1,s}, A_{\Delta_2,s})$ has the CIA property for any positive number s with coercitivity c as defined above.

The positive number c is called the **coercitivity** of the r-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

If A is a unital C^*-algebra filtered by $(A_s)_{s>0}$ and if $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ is an r-controlled Mayer–Vietoris pair, we will view $\widetilde{A_{\Delta_1}}$ the unitarization of A_{Δ_1} as $A_{\Delta_1} + \mathbb{C} \cdot 1 \subseteq A$ and similarly for A_{Δ_2} and $A_{\Delta_1} \cap A_{\Delta_2}$.

Example 2.20. Let (Σ, d) be a proper metric discrete space, let $X^{(1)}$ and $X^{(2)}$ be subsets in Σ such that $\Sigma = X^{(1)} \cup X^{(2)}$ and let r be a positive number. Assume that $X^{(1)} = \cup_{i\in\mathbb{N}} X_i^{(1)}$ and $X^{(2)} = \cup_{i\in\mathbb{N}} X_i^{(2)}$, where $(X_i^{(1)})_{i\in\mathbb{N}}$ and $(X_i^{(2)})_{i\in\mathbb{N}}$ are families of R-disjoint subsets of Σ with uniformly bounded diameter for some positive number $R \geqslant 10r$. Let us consider as in Example 2.15 for $j = 1, 2$ the sets $\Delta_j \subseteq C^*(\Sigma)_r$ of locally compact operators on $\ell^2(\Sigma)\otimes\mathscr{H}$ with support in

$$\left\{(x, y) \in \Sigma \times \Sigma;\, x \in X^{(j)},\, d(x, y) \leqslant r\right\}$$

and let us consider then the subalgebra A_{Δ_j} of $C^*(\Sigma)$ of operators with support in $\bigcup_{i\in\mathbb{N}} X_{i,s}^{(j)} \times X_{i,s}^{(j)}$ for some fixed positive number s with $5r < s < R/2$. Let $\chi_{X_{i,5r}^{(2)}}$ for i integer be the characteristic function of

$$\{x \in \Sigma \text{ such that } d(x, X_i^{(2)}) \leqslant 5r\}.$$

Set

$$\Psi : C^*(\Sigma) \longrightarrow C^*(\Sigma);\; x \mapsto \sum_{i\in\mathbb{N}} \chi_{X_{i,5r}^{(2)}}\, x\, \chi_{X_{i,5r}^{(2)}}.$$

Then Ψ is norm decreasing. Since $\Psi(x_2) = x_2$ for every x_2 in A_{Δ_2}, we obtain

$$\|\Psi(x_1) - x_2\| \leqslant \|x_1 - x_2\|$$

for every x_1 in $M_n(A_{\Delta_1})$ and x_2 in $M_n(A_{\Delta_2})$. Since $\Psi(x_1)$ lies in $M_n(A_{\Delta_1} \cap A_{\Delta_2})$, we see that $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ is an r-controlled Mayer–Vietoris pair with coercitivity 1.

In next lemma, we show that in the context of controlled Mayer–Vietoris pairs, the pairs as in Proposition 2.11 arising respectively from an ε-s-unitary and from its adjoint are up to stabilization homotopically adjoint. This result will be needed for the proof of Theorem 4.12 in Section 4.3.

Proposition 2.21. *For every positive number c, there exists a control pair (α, l) such that the following holds.*

Let A be any unital filtered C^*-algebra, let r be any positive number, let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be any r-controlled Mayer–Vietoris pair for A at order r with coercitivity c, and let ε and r' be positive numbers with $\varepsilon \in (0, \frac{1}{4\alpha})$ and $r \leqslant r'$. Assume that for some ε-r-unitary u in some $M_n(A)$, there exist two ε-r'-unitaries v_1 and v_1' in $M_n(\widetilde{A_{\Delta_1}})$ and two ε-r'-unitaries v_2 and v_2' in $M_n(\widetilde{A_{\Delta_2}})$ such that $\|u - v_1v_2\| < \varepsilon$ and $\|u^* - v_1'v_2'\| < \varepsilon$. Then there exists an integer k and v_1'' and v_2'' respectively $\alpha\varepsilon$-$l_\varepsilon r'$-unitaries in $M_{n+k}(\widetilde{A_{\Delta_1}})$ and $M_{n+k}(\widetilde{A_{\Delta_2}})$ such that

- $\|\operatorname{diag}(u^*, I_k) - v_1''v_2''\| < \alpha\varepsilon$;
- *v_i'' is homotopic to $\operatorname{diag}(v_i^*, I_k)$ as an $\alpha\varepsilon$-$l_\varepsilon r'$-unitary in $M_{n+k}(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$.*

Moreover, if $v_i - I_n$ and $v_i' - I_n$ lie in $M_n(A_{\Delta_i})$ for $i = 1, 2$ then v_1'' and v_2'' can be chosen such that $v_i'' - I_{n+k}$ lies in $M_{n+k}(A_{\Delta_i})$ for $i = 1, 2$.

Proof. Let (α, l) be a control pair as in Proposition 2.11. Since $\rho^{-1}_{A_{\Delta_j}}(v_j)$ and $\rho^{-1}_{A_{\Delta_j}}(v_j')$ are for $j = 1, 2$ homotopic to I_n as 8ε-s-unitaries of $M_n(\mathbb{C})$ for every positive number s [9, Lemma 1.20], then up to replacing α by 90α, there exists an integer k and w_1 and w_2 be two $\alpha\varepsilon$-$2l_\varepsilon r$-unitaries respectively in $M_{2n+k}(A_{\Delta_1})$ and $M_{2n+k}(A_{\Delta_2})$ such that if we set $W_j = \operatorname{diag}(\rho^{-1}_{A_{\Delta_j}}(v_j), \rho^{-1}_{A_{\Delta_1}}(v_j'), I_k)$ for $j = 1, 2$, then

- $\|W_1 \operatorname{diag}(u, u^*, I_k) W_2 - w_1 w_2\| < \alpha\varepsilon$;
- $w_j - I_{2n+k}$ is in $M_{2n+k}(A_{\Delta_j})$ for $j = 1, 2$.
- w_j is homotopic to I_{2n+k} as an $\alpha\varepsilon$-$2l_\varepsilon r$-unitaries in $M_{2n+k}(A_{\Delta_j})$ for $j = 1, 2$.

Then

$$\|W_1 diag(v_1v_2, v_1'v_2', I_k) W_2 - w_1w_2\| < (\alpha + 2)\varepsilon$$

and hence we have

$$\|\operatorname{diag}(v_1^*, {v'}_1^*, I_k) W_1^* w_1 - \operatorname{diag}(v_2, v_2', I_k) W_2 w_2^*\| < 5(\alpha + 1)\varepsilon.$$

Since $\rho_{\mathcal{A}_{\Delta_j}}(\operatorname{diag}(v_j, v_j', I_k) W_j) = I_{2n+k}$ for $j = 1, 2$ and in view of CIA property, there exist v in $M_{2n+k}(A)$ with propagation less than $(2l_\varepsilon + 1)r'$ such that

- $v - I_{2n+k}$ is in $M_{2n+k}(A_{\Delta_1} \cap A_{\Delta_2})$;
- $\|\operatorname{diag}(v_2, v_2', I_k) W_2 w_2^* - v\| < 5c(\alpha + 1)\varepsilon$;
- $\|\operatorname{diag}(v_1^*, {v'}_1^*, I_k) W_1^* w_1 - v\| < 5c(\alpha + 1)\varepsilon$.

In particular some control pair (α', l') depending only on c.

- v is an α'-l'_ε-r'-unitary in $M_{2n+k}(\widetilde{A_{\Delta_1}\cap A_{\Delta_2}})$;
- v is homotopic to $\mathrm{diag}(v_1^*, {v'_1}^*, I_k)$ as an α'-l'_ε-r'-unitary in $M_{2n+k}(\widetilde{A_{\Delta_1}})$;
- v is homotopic to $\mathrm{diag}(v_2, v'_2, I_k)$ as an α'-l'_ε-r'-unitary in $M_{2n+k}(\widetilde{A_{\Delta_2}})$.

Let us set $v''_1 = \mathrm{diag}(v'_1, I_{n+k}) \cdot v$ and $v''_2 = v^* \cdot \mathrm{diag}(v'_2, I_{n+k})$. Then v''_1 and v''_2 satisfy the required properties for some suitable control pair depending only on c. □

2.4. Controlled Mayer–Vietoris pair associated to groupoïds

In this section, we discuss the example of a controlled Mayer–Vietoris pair associated to groupoïds. Our method in this paper provides a different approach to the controlled Mayer–Vietoris sequence in the context of crossed product C^*-algebras in [5]. Recall first the definition of a proper symmetric length on an étale groupoïd.

Definition 2.22. Let $\mathcal{G}$ be an étale groupoïd, with compact base space X. A proper symmetric length on $\mathcal{G}$ is a continuous proper map $\ell : \mathcal{G} \to \mathbb{R}^+$ such that

- $\ell(\gamma) = 0$ if and only if γ is a unit of $\mathcal{G}$;
- $\ell(\gamma) = \ell(\gamma^{-1})$ for any γ in $\mathcal{G}$;
- $\ell(\gamma \cdot \gamma') \leqslant \ell(\gamma) + \ell(\gamma')$ for any γ and γ' in $\mathcal{G}$ composable.

Let $\mathcal{G}$ be an étale groupoïd with compact base space X and source map and range map $\mathbf{r}, \mathbf{s} : \mathcal{G} \to X$ equipped with a symmetric and proper length ℓ. Then, if we set

$$\mathcal{G}_r = \{\gamma \in \mathcal{G}; \text{ such that } \ell(\gamma) \leqslant r\},$$

then the reduced C^*-algebra $C^*_r(\mathcal{G})$ of $\mathcal{G}$ is filtered by $(C^*_r(\mathcal{G})_r)_{r>0}$ with

$$C^*_r(\mathcal{G})_r = \{f \in C_c(\mathcal{G}) \text{ with support in } \mathcal{G}_r\}$$

for all positive number r.

Remark 2.23. In [2] is developed a more general notion of filtered C^*-algebras and the definition of quantitative K-theory is extended to this setting. A filtered structure in this sense can be defined on reduced C^*-algebras of étale without involving a length.

For every open subset V of X and every positive number r, set

$$V_r = \{\mathbf{s}(\gamma)\ , \gamma \in \mathring{\mathcal{G}}_r \text{ and } \mathbf{r}(\gamma) \in V\} = \{\mathbf{r}(\gamma)\ , \gamma \in \mathring{\mathcal{G}}_r \text{ and } \mathbf{s}(\gamma) \in V\}.$$

Then V_r is an open subset of X and $V \subseteq V_r$. If Y and Z are subsets of X, then we set $\mathcal{G}_Y = \{\gamma \in \mathcal{G}; \mathbf{s}(\gamma) \in Y\}, \mathcal{G}^Z = \{\gamma \in \mathcal{G}; \mathbf{r}(\gamma) \in Z\}$ and $\mathcal{G}_X^Y = \mathcal{G}_Y \cap \mathcal{G}^Z$. For every open subset V of X and every positive number r, let $\mathcal{G}_V^{V,(r)}$ be the subgroupoïd of $\mathcal{G}_V^V$ generated by $\mathcal{G}_{V,r}^V = \mathcal{G}_V^V \cap \mathring{\mathcal{G}}_r$. Then $\mathcal{G}_V^{V,(r)}$ is an open subgroupoïd of $\mathcal{G}$. Let us set

$$\Delta_V = \{f \in C_0(\mathcal{G}_V) \text{ with support in } \mathcal{G}_r\}.$$

Then Δ_V is a closed linear subspace of $C_r^*(\mathcal{G})_r$ and for every positive number R and R' with $r \leq R < R'$, then we have $C^*N_{\Delta_V}^{r,R} \subseteq C_r^*\left(\mathcal{G}_{V'_R}^{V'_R,(R')}\right)$. In particular, if $R > 5r$, then $C_r^*\left(\mathcal{G}_{V_R}^{V_R,(R)}\right)$ is a r-controlled Δ_V-neighborhood-C^*-algebra.

Let $V^{(1)}$ and $V^{(2)}$ be two open subsets of X such that $X = V^{(1)} \cup V^{(2)}$. Fix $R > 5r$. Set $\Delta_1 = \Delta_{V^{(1)}}$ and $\Delta_2 = \Delta_{V^{(2)}}$. Using partition of unity relatively to $V^{(1)}$ and $V^{(2)}$, we see that (Δ_1, Δ_2) is a completely coercive decomposition pair of order r for $C_{red}^*(\mathcal{G})$ with coercitivity 1. Let us set also $A_{\Delta_1} = C_{red}^*\left(\mathcal{G}_{V_R^{(1)}}^{V_R^{(1)},(R)}\right)$ and $A_{\Delta_2} = C_{red}^*\left(\mathcal{G}_{V_R^{(2)}}^{V_R^{(2)},(R)}\right)$. Let s be a positive number with $s \leqslant r$, let ε be a positive number, let x_1 be an element of $M_n\left(A_{\Delta_1,s}\right)$ and let x_2 be an element of $M_n\left(A_{\Delta_2,s}\right)$ such that $\|x_1 - x_2\| < \varepsilon$. Let x'_1 and x'_2 be respectively elements in $M_n\left(C_c\left(\mathcal{G}_{V_R^{(1)}}^{V_R^{(1)},(R)}\right)\right)$ and $M_n\left(C_c\left(\mathcal{G}_{V_R^{(2)}}^{V_R^{(2)},(R)}\right)\right)$ such that $\|x'_1 - x_1\| < \varepsilon$ and $\|x'_2 - x_2\| < \varepsilon$. Let K be a compact subset of $V_R^{(2)}$ such that all coefficients of x'_2 have support in $\mathcal{G}_K^K$ and let $h : X \to [0,1]$ be a continuous function with support in $V_R^{(2)}$ and such that $h(z) = 1$ for all z in K. The Schur multiplication (i.e. the pointwise multiplication) by $h \circ \mathbf{s} \cdot h \circ \mathbf{r}$

$$C_c(\mathcal{G}) \to C_c(\mathcal{G});\ f \mapsto f \cdot h \circ \mathbf{s} \cdot h \circ \mathbf{r}$$

extends to a completely positive map

$$\phi : C_{red}^*(\mathcal{G}) \to C_{red}^*(\mathcal{G})$$

of complete norm less than 1 and such that $\phi(x'_2) = x'_2$ and $\phi(x'_1)$ belongs to $C_{red}^*\left(\mathcal{G}_{V_R^{(1)}}^{V_R^{(1)},(R)}\right)_s \bigcap C_{red}^*\left(\mathcal{G}_{V_R^{(2)}}^{V_R^{(2)},(R)}\right)_s$. Moreover

$$\begin{aligned}
\|\phi(x'_1) - x_2\| &\leqslant \|\phi(x'_1) - x'_2\| + \|x'_2 - x_2\| \\
&\leqslant \|\phi(x'_1) - \phi(x'_2)\| + \varepsilon \\
&\leqslant \|x'_1 - x'_2\| + \varepsilon \\
&\leqslant 4\varepsilon.
\end{aligned}$$

Hence, $\left(\Delta_1, \Delta_2, C_{red}^*\left(\mathcal{G}_{V_R^{(1)}}^{V_R^{(1)},(R)}\right), C_{red}^*\left(\mathcal{G}_{V_R^{(2)}}^{V_R^{(2)},(R)}\right)\right)$ is for every $R > 5r$ a r-controlled weak Mayer–Vietoris pair for $C_{red}^*(\mathcal{G})$) with coercitivity 5. Assume that there exists a positive number C such that for every compact subset K of $V^{(2)}$, there exists a continuous function $h : X \to [0,1]$ with support in $V^{(2)}$ that satisfies the following:

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- $h(z) = 1$ for all z in K;
- the Schur multiplication by $h \circ \mathbf{s} \cdot h \circ \mathbf{r}$ extends to a completely bounded map $\Phi : C^*_{red}(\mathcal{G}) \to C^*_{red}(\mathcal{G})$ with complete norm bounded by C and such that for any x in $C_c\left(\mathcal{G}^{V_R^{(2)},(R)}_{V_R^{(2)}}\right)$ with support in $\mathcal{G}^K_K$, then $\Phi(x) = x$,

then $\left(\Delta_1, \Delta_2, C^*_{red}\left(\mathcal{G}^{V_R^{(1)},(R)}_{V_R^{(1)}}\right), C^*_{red}\left(\mathcal{G}^{V_R^{(2)},(R)}_{V_R^{(2)}}\right)\right)$ is for every $R > 5r$ a r-controlled Mayer–Vietoris pair for $C^*_{red}(\mathcal{G})$) with coercitivity c depending only on C.

3. Controlled Mayer–Vietoris six terms exact sequence in quantitative K-theory

In this section, we establish for a control Mayer–Vietoris pair associated to a filtered C^*-algebra A a controlled exact sequence that allows to compute quantitative K-theory of A up to a certain order. We follow the route of the proof in the K-theory case. We first check controlled exactness in the middle. Using Propositions 2.11 and 2.12 and the CIA property, we then define the quantitative boundary map mimicking the construction of the Mayer–Vietoris boundary map in usual K-theory. The thickness of the neighborhood algebras guarantees that this quantitative boundary map is well defined. We prove eventually the control exactness at the source and at the range of the quantitative boundary map to complete the statement of the controlled six terms exact sequence. Notice that exactness at the source and at the range is indeed persistent at any order (see Corollary 3.6 Lemma 3.8). This strenghening of controlled exactness is crucial to compute quantitative K-theory out of the controlled Mayer–Vietoris exact sequence (see the proof of Theorems 3.14 and 4.12). We end the section with an application to computation of K-theory of obstruction C^*-algebras.

Notation 3.1. Let A be a unital C^*-algebra filtered by $(A_r)_{r>0}$, let r be a positive number and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a r-controlled Mayer–Vietoris pair for A. We denote by $\jmath_{\Delta_1} : A_{\Delta_1} \to A$, $\jmath_{\Delta_2} : A_{\Delta_2} \to A$, $\jmath_{\Delta_1,\Delta_2} : A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_1}$ and $\jmath_{\Delta_2,\Delta_1} : A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_2}$ the obvious inclusion maps.

3.1. Controlled half-exactness in the middle

Proposition 3.2. *For every positive number c, there exists a control pair (α, l) such that for any filtered C^*-algebra A, any positive number r and any r-controlled weak Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ for A with coercitivity c, then the composition*

$$\mathcal{K}_*(A_{\Delta_1} \cap A_{\Delta_2}) \xrightarrow{(\jmath_{\Delta_1,\Delta_2,*}, \jmath_{\Delta_2,\Delta_1,*})} \mathcal{K}_*(A_{\Delta_1}) \oplus \mathcal{K}_*(A_{\Delta_2}) \xrightarrow{(\jmath_{\Delta_1,*} - \jmath_{\Delta_2,*})} \mathcal{K}_*(A)$$

is (α, l)-exact at order r.

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Proof. Let us first assume that A is unital. In the even case let y_1 and y_2 be respectively element in $K_0^{\varepsilon,s}(A_{\Delta_1})$ and $K_0^{\varepsilon,s}(A_{\Delta_2})$ such that $\jmath^{\varepsilon,s}_{\Delta_1,*}(y_1) = \jmath^{\varepsilon,s}_{\Delta_2,*}(y_2)$ in $K_0^{\varepsilon,s}(A)$. In view of Lemma 1.7, we can assume up to rescaling ε that there exist integer m and n with $m \leqslant n$ and two ε-r-projections q_1 and q_2 in $M_n(A)$ such that

- $q_1 - \mathrm{diag}(I_m, 0)$ is an element in the matrix algebra $M_n(A_{\Delta_1})$;
- $q_2 - \mathrm{diag}(I_m, 0)$ is an element in the matrix algebra $M_n(A_{\Delta_2})$;
- $y_1 = [q_1, m]_{\varepsilon,s}$;
- $y_2 = [q_2, m]_{\varepsilon,s}$.

Up to stabilization, we can also assume that q_1 and q_2 are homotopic as ε-s-projections in $M_n(A)$. Let (α, k) be the control pair of Proposition 1.5. Up to stabilization there exists u a $\alpha\varepsilon$-$k_\varepsilon s$-unitary in $M_n(A)$ such that $\|u^* q_1 u - q_2\| < \alpha\varepsilon$. Up to replacing u by $\mathrm{diag}(u, u^*)$, q_1 by $\mathrm{diag}(q_1, 0)$ and q_2 by $\mathrm{diag}(q_2, 0)$, we can assume in view of Lemma 1.3 that u is homotopic to I_n as a $3\alpha\varepsilon$-$2k_\varepsilon s$-unitary in $M_n(A)$. According to Proposition 2.11, then for some control pair (λ, l) depending only on (α, k) and c with $(\alpha, k) \leqslant (\lambda, l)$ and up to stabilization, there exist w_1 and w_2 some $\lambda\varepsilon$-$k_\varepsilon s$ unitaries in $M_n(A)$ such that

- $w_i - I_k$ is an element in the matrix algebra $M_n(C^* N^{(r,4r)}_{\Delta_i, l_\varepsilon s})$ for $i = 1, 2$;
- $\|w_1^* q_1 w_1 - w_2 q_2 w_2^*\| < \lambda\varepsilon$.

Notice that $w_1^* q_1 w_1 - \mathrm{diag}(I_m, 0)$ is an element in the matrix algebra $M_n(A_{\Delta_1,(2l_\varepsilon+1)s})$ and $w_2 q_2 w_2^* - \mathrm{diag}(I_m, 0)$ is an element in the matrix algebra $M_n(A_{\Delta_2,(2l_\varepsilon+1)s})$. By Definition 2.16 of the CIA property, there exists y in

$$M_n(A_{\Delta_1,(2l_\varepsilon+1)s} \cap A_{\Delta_2,(2l_\varepsilon+1)s})$$

such that

$$\|y - (w_1^* q_1 w_1^* - \mathrm{diag}(I_n, 0))\| < \lambda c\varepsilon$$

and

$$\|y - (w_2 q_2 w_2^* - \mathrm{diag}(I_n, 0))\| < \lambda c\varepsilon.$$

Let us set then

$$p = y + \mathrm{diag}(I_m, 0).$$

Since $\|p - w_1^* q_1 w_1\| < \lambda c\varepsilon$ and $\|p - w_2 q_2 w_2^*\| < \lambda c\varepsilon$, up to stabilization, in view of the proof of [9, Lemma 1.9] and according to [9, Lemma 1.7], we know that for some control pair (α', l') depending only on (α, k) and c and such that $((c+1)\lambda, 2l+1) \leqslant (\alpha', l')$, then for $j = 1, 2$

- $w_1^* q_1 w_1$ is an α'-$l'_\varepsilon s$-projection in $M_n(\widetilde{A}_{\Delta_1})$;
- $w_1^* q_1 w_1$ is homotopic to q_1 as an α'-$l'_\varepsilon s$-projections in $M_n(\widetilde{A}_{\Delta_1})$;
- p is connected to $w_1^* q_1 w_1$ as an α'-$l'_\varepsilon s$-projections in $M_n(\widetilde{A}_{\Delta_1})$;

and

- $w_2 q_2 w_2^*$ is an α'-$l'_\varepsilon s$-projection in $M_n(\widetilde{A}_{\Delta_2})$;
- $w_2 q_2 w_2$ is homotopic to q_2 as an α'-$l'_\varepsilon s$-projections in $M_n(\widetilde{A}_{\Delta_2})$;
- p is connected to $w_2^* q_2 w_2$ as an α'-$l'_\varepsilon s$-projections in $M_n(\widetilde{A}_{\Delta_2})$.

Now if we set $x = [p, m]_{\alpha'\varepsilon, l'_\varepsilon s}$ in $K_0^{\alpha'\varepsilon, l'_\varepsilon s}(A_{\Delta_1} \cap A_{\Delta_2})$, we have that

$$\jmath_{\Delta_1,\Delta_2}^{\alpha'\varepsilon, l'_\varepsilon s}(x) = \iota^{-,\alpha'\varepsilon, l'_\varepsilon s}(y_1)$$

in $K_*^{\alpha'\varepsilon, l'_\varepsilon s}(A_{\Delta_1})$ and

$$\jmath_{\Delta_2,\Delta_1}^{\alpha'\varepsilon, l'_\varepsilon s}(x) = \iota^{-,\alpha'\varepsilon, l'_\varepsilon s}(y_2)$$

in $K_*^{\alpha'\varepsilon, l'_\varepsilon s}(A_{\Delta_2})$.

A similar proof can be carried out in the odd case but we can also use the controlled Bott periodicity [10, Lemma 4.6]. The non unital case can be proved in a similar way using Lemma 2.12, noticing that in view of the proof of Proposition 1.5 and following the proof of the unital case above, we can assume that u which is now a $\alpha_h\varepsilon$-$k_{h,\varepsilon}s$-unitary in $M_n(\tilde{A})$ is such that $u - I_n$ has coefficient in A (see the proofs of [9, Lemma 1.11 and Corollary 1.31]). □

3.2. Quantitative boundary maps for controlled Mayer–Vietoris pair

In this subsection, we introduce the quantitative boundary map that fits into the controlled Mayer–Vietoris sequence for quantitative K-theory of filtered C^*-algebras.

Lemma 3.3. *For every positive number c, there exists a control pair (λ, k) such that the following holds:*

Let A be a unital C^*-algebra filtered by $(A_s)_{s>0}$, let r be a positive number and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a r-controlled weak Mayer–Vietoris pair for A with coercitivity c. Let ε and s be positive numbers with $\varepsilon < \frac{1}{4\lambda}$ and $s \leqslant r/2$, let m and n be integers and let u in $U_n^{\varepsilon,s}(A)$, v in $U_m^{\varepsilon,s}(A)$ and w_1, w_2 be ε-s-unitaries in $M_{n+m}^{\varepsilon,s}(A)$ such that

- *$w_i - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_i})$ for $i = 1, 2$;*
- *$\|\operatorname{diag}(u, v) - w_1 w_2\| < \varepsilon$.*

Then,

(i) *there exists a $\lambda\varepsilon$-$k_\varepsilon s$-projection q in $M_{n+m}(A)$ such that*
- $q - \operatorname{diag}(I_n, 0)$ *is an element in the matrix algebra* $M_{n+m}(A_{\Delta_1} \cap A_{\Delta_2})$;
- $\|q - w_1^* \operatorname{diag}(I_n, 0) w_1\| < \lambda\varepsilon$;
- $\|q - w_2 \operatorname{diag}(I_n, 0) w_2^*\| < \lambda\varepsilon$;

(ii) *if q and q' are two $\lambda\varepsilon$-$k_\varepsilon s$-projections in $M_{n+m}(A)$ that satisfy the first point, then* $[q, n]_{\lambda\varepsilon, k_\varepsilon s} = [q', n]_{\lambda\varepsilon, k_\varepsilon s}$ *in* $K_0(A_{\Delta_1} \cap A_{\Delta_2})$.

(iii) *Let (w_1, w_2) and (w_1', w_2') be two pairs of ε-s-unitaries in $M_{n+m}^{\varepsilon,s}(A)$ satisfying the assumption of the lemma and let q and q' be $\lambda\varepsilon$-$k_\varepsilon s$-projections in $M_{n+m}(A)$ that satisfy the first point relatively to respectively (w_1, w_2) and (w_1', w_2'), then* $[q, n]_{\lambda\varepsilon, k_\varepsilon s} = [q', n]_{\lambda\varepsilon, k_\varepsilon s}$ *in* $K_0(A_{\Delta_1} \cap A_{\Delta_2})$.

Proof. Since $\operatorname{diag}(u, v)$ is an ε-s-unitary, we have that

$$\begin{aligned}\|w_1^* \operatorname{diag}(I_n, 0) w_1 - w_1^* \operatorname{diag}(u, v) \operatorname{diag}(I_n, 0) \operatorname{diag}(u^*, v^*) w_1\| \\ = \|w_1^* \operatorname{diag}(I_n - u^* u, 0) w_1\| \\ < 2\varepsilon.\end{aligned}$$

Since $\|w_1^* \operatorname{diag}(u, v) - w_2\| < 4\varepsilon$, we deduce that

$$\|w_1^* \operatorname{diag}(I_n, 0) w_1 - w_2 \operatorname{diag}(I_n, 0) w_2^*\| < 8\varepsilon.$$

With notations as in Definition 2.19, let y be an element in $M_{n+m}(A_{\Delta_{1,2s}} \cap A_{\Delta_{2,2s}})$ such that

$$\|w_1^* \operatorname{diag}(I_n, 0) w_1 - \operatorname{diag}(I_n, 0) - y\| < 8c\varepsilon$$

and

$$\|y - w_2 \operatorname{diag}(I_n, 0) w_2^* - \operatorname{diag}(I_n, 0)\| < 8c\varepsilon$$

and set

$$q = y + \operatorname{diag}(I_n, 0).$$

Then q is close to a 2ε-$2s$-projection and thus we obtain in view of Lemma 1.2 that there exists a control pair (λ, k), depending only on c such that the conclusion of the first point is satisfied. With notations as in Lemma 3.3 and in view of Lemma 1.2, if q and q' are $\lambda\varepsilon$-$k_\varepsilon s$-projections of $M_{n+m}(A)$ that satisfies the first point, then

$$[q, n]_{10\lambda\varepsilon, k_\varepsilon s} = [q', n]_{10\lambda\varepsilon, k_\varepsilon s}.$$

If (w_1, w_2) and (w_1', w_2') are two pairs of ε-s-unitaries in $M_{n+m}^{\varepsilon,s}(A)$ that satisfy the assumption of the lemma and let q and q' be $\lambda\varepsilon$-$k_\varepsilon s$-projections in $M_{n+m}(A)$ that satisfy the first point relatively to (w_1, w_2) and (w'_1, w'_2). Then $\|w_1w_2 - w'_1w'_2\| < 2\varepsilon$ and hence $\|{w'_2}^*w_2 - w'_1{w_1}^*\| < 10\varepsilon$. Hence using the CIA condition, we see that there exists v in $M_{n+m}(A_{2s})$ such that $v - I_{n+m}$ is in $M_{n+n}(A_{\Delta_1} \cap A_{\Delta_2})$, $\|v - w'_1{w_1}^*\| < 10c\varepsilon$ and $\|{w'_2}^*w_2 - v\| < 10c\varepsilon$. Since we have then $\|w_1 - v^*w'_1\| < 30c\varepsilon$ and $\|w_2 - w'_2v\| < 30c\varepsilon$ the last point is consequence of [9, Lemma 1.9] and of the second point applied to $45c\varepsilon$, (w_1, w_2), q and $v^*q'v$. □

Remark 3.4. We have a similar statement in the non-unital case with u in $U_n^{\varepsilon,s}(\tilde{A})$ and v in $U_m^{\varepsilon,s}(\tilde{A})$ such that $u - I_n$ and $v - I_m$ have coefficients in A

We are now in position to define the boundary map associated to a controlled Mayer–Vietoris pair. Let A be a filtered C^*-algebra and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a r-controlled weak Mayer–Vietoris pair for A with coercitivity c. Assume first that A is unital.

Let (α, l) be a control pair as is Proposition 2.11. For any positive numbers ε and s with $\varepsilon < \frac{1}{4\alpha}$ and $s \leqslant r/2$ and any ε-s-unitary u in $M_n(A)$, let v be an ε-s-unitary in some $M_m(A)$ such that $\operatorname{diag}(u, v)$ is homotopic to I_{n+m} as a 3ε-$2s$-unitary in $M_{n+m}(A)$, we can take for instance $v = u^*$ (see Lemma 1.3). Since $C^*N_{\Delta_i}^{(2s,8s)} \subset A_{\Delta_i}$ as a filtered subalgebra for $i = 1, 2$, then according to Proposition 2.11 and up to replacing v by $\operatorname{diag}(v, I_k)$ for some integer k, there exists w_1 and w_2 two $3\alpha\varepsilon$-$2l_{3\varepsilon}r$-unitaries in $M_{n+m}(A)$ such that

- $w_i - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_i, 2l_{3\varepsilon}s})$ for $i = 1, 2$;
- for $i = 1, 2$, there exists a homotopy $(w_{i,t})_{t\in[0,1]}$ of $3\alpha\varepsilon$-$2l_{3\varepsilon}s$-unitaries between 1 and w_i such that $w_{i,t} - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_i, l_{3\varepsilon}s})$ for all t in $[0, 1]$.
- $\|\operatorname{diag}(u, v) - w_1w_2\| < 3\alpha\varepsilon$.

Let (λ, k) be the control pair of Lemma 3.3 (recall that (λ, k) depends only on c). Then if ε is in $(0, \frac{1}{12\alpha\lambda})$, there exists a $3\alpha\lambda\varepsilon$-$2l_{3\varepsilon}k_{3\alpha\varepsilon}s$-projection q in $M_{n+m}(A)$ such that

- $q - \operatorname{diag}(I_n, 0)$ is an element in the matrix algebra
$$M_{n+m}(A_{\Delta_1, 2l_{3\varepsilon}k_{3\alpha\varepsilon}s} \cap A_{\Delta_2, 2l_\varepsilon k_{3\alpha\varepsilon}s});$$
- $\|q - w_1^* \operatorname{diag}(I_n, 0)w_1\| < 3\alpha\lambda\varepsilon$;
- $\|q - w_2 \operatorname{diag}(I_n, 0)w_2^*\| < 3\alpha\lambda\varepsilon$.

In view of second point of Lemma 3.3, the class $[q, n]_{3\alpha\lambda\varepsilon, 2l_{3\varepsilon}k_{3\alpha\varepsilon}s}$ in
$$K_0^{3\alpha\lambda\varepsilon, 2l_{3\varepsilon}k_{3\alpha\varepsilon}s}(A_{\Delta_1} \cap A_{\Delta_2})$$
does not depend on the choice of q. Set then $\alpha_c = 3\alpha\lambda$ and
$$k_c : \left(0, \frac{1}{4\alpha_c}\right) \longrightarrow (1, +\infty),\ \varepsilon \mapsto 2l_{3\varepsilon}k_{3\alpha\varepsilon}$$

and define $\partial^{\varepsilon,s,1}_{\Delta_1,\Delta_2,*}([u]_{\varepsilon,s}) = [q,n]_{\alpha_c\varepsilon,k_cs}$ and let us prove that we define in this way a morphism

$$\partial^{\varepsilon,s,1}_{\Delta_1,\Delta_2,*} : K_1^{\varepsilon,s}(A) \to K_0^{\alpha_c\varepsilon,k_cs}(A_{\Delta_1} \cap A_{\Delta_2}).$$

It is straightforward to check that (compare with [13, Chapter 8]).

- two different choices of elements satisfying the conclusion of Lemma 3.3 relative to $\mathrm{diag}(u,v)$ give rise to homotopic elements in $\mathrm{P}_{n+j}^{\alpha_{\mathcal{D}}\varepsilon,k_{\mathcal{D}}s}(A_{\Delta_1} \cap A_{\Delta_2})$ (this is a consequence of Lemma 3.3).
- Replacing u by $\mathrm{diag}(u,I_m)$ and v by $\mathrm{diag}(v,I_k)$ gives also rise to the same element of $K_0^{\alpha_c\varepsilon,k_cs}(A_{\Delta_1} \cap A_{\Delta_2})$.

Applying now Proposition 2.11 to the r-controlled Mayer–Vietoris pair

$$(C([0,1],\Delta_1), C([0,1],\Delta_2), C([0,1],A_{\Delta_1}), C([0,1],A_{\Delta_2}))$$

for the C^*-algebra $C([0,1],A)$ filtered by $(C([0,1],A_s))_{s>0}$, we see that $\partial^{\varepsilon,s,1}_{\Delta_1,\Delta_2,*}([u]_{\varepsilon,s})$

- only depends on the class of u in $K_1^{\varepsilon,s}(A)$;
- does not depend on the choice of v such that $\mathrm{diag}(u,v)$ is connected to I_{n+j} in $U_{n+j}^{3\varepsilon,2s}(A)$.

In the non unital case $\partial^{\varepsilon,s,1}_{\Delta_1,\Delta_2,*}$ is defined similarly by using point (ii) of Remark 3.4, noticing that in view of Lemma 1.8 and up to replacing ε by 3ε, every element x in $K_1(A)$ is the class of a ε-r-unitary u in $M_n(\tilde{A})$ such that $u - I_n$ has coefficients in A. It is straightforward to check that $\partial^{\bullet,\bullet,1}_{\Delta_1,\Delta_2,*}$ is compatible with the structure morphisms. Let us consider $\mathcal{D}^1_{\Delta_1,\Delta_2,*} = (\partial^{\varepsilon,s,1}_{\Delta_1,\Delta_2,*})$ where ε runs through $(0,\frac{1}{4\alpha_c})$ and s runs through $(0,\frac{r}{k_{c,\varepsilon}})$. Then

$$\mathcal{D}^1_{\Delta_1,\Delta_2,*} : \mathcal{K}_1(A) \to \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_1})$$

is a odd degree (α_c,k_c)-controlled morphism of order r.

Let us now define the boundary map in the even case using controlled Bott periodicity. For Δ a closed subspace in an C^*-algebra, let us define its suspension as $S\Delta = C_0((0,1),\Delta)$. Let $[\partial]$ be the element of $KK_1(\mathbb{C}, C_0(0,1))$ that implements the extension

$$0 \to C_0(0,1) \to C_0[0,1) \stackrel{ev_0}{\to} \mathbb{C} \to 0,$$

where $ev_0 : C_0[0,1) \to \mathbb{C}$ is the evaluation at 0. Then $[\partial]$ is an invertible element of $KK_1(\mathbb{C}, C_0(0,1))$ and in view of Proposition 1.26 and according to [9, Lemma 4.6],

$$\mathcal{T}_B([\partial]) : \mathcal{K}_*(B) \to \mathcal{K}_*(SB)$$

is a $(\alpha_\mathcal{T}, k_\mathcal{T})$-controlled isomorphism of degree one with controlled inverse

$$\mathcal{T}_B([\partial]^{-1}) : \mathcal{K}_*(SB) \to \mathcal{K}_*(B).$$

Let A be a C^*-algebra filtered by $(A_s)_{s>0}$, let r be a positive number and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a r-controlled weak Mayer–Vietoris pair for A with coercitivity c. Then $(S\Delta_1, S\Delta_2, SA_{\Delta_1}, SA_{\Delta_2})$ is a r-controlled weak Mayer–Vietoris pair for SA (filtered by $(SA_r)_{r>0}$) with coercitivity c. Set then $\lambda = \alpha_\mathcal{T}^2 \alpha_c$ and $h_\varepsilon = k_{\mathcal{T},\alpha_\mathcal{T}\alpha_c\varepsilon} k_{c,\alpha_c\varepsilon} k_{c,\varepsilon}$. Let us define in the even case the quantitative boundary map for the r-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ as the (λ, h)-controlled morphism of order r

$$\mathcal{D}^0_{\Delta_1,\Delta_2,*} \stackrel{\text{def}}{=\!=} \mathcal{T}_{A_{\Delta_1} \cap A_{\Delta_2},*}([\partial]^{-1}) \circ \mathcal{D}^1_{S\Delta_1,S\Delta_2,*} \circ \mathcal{T}_A([\partial]) : \mathcal{K}_0(A) \longrightarrow \mathcal{K}_1(A_{\Delta_1} \cap A_{\Delta_2}).$$

For sake of simplicity, we will rescale (α_c, k_c) to (λ, h) and use the same control pair in the odd and in the even case for the definition of

$$\mathcal{D}_{\Delta_1,\Delta_2,*} \stackrel{\text{def}}{=\!=} \mathcal{D}^0_{\Delta_1,\Delta_2,*} \oplus \mathcal{D}^1_{\Delta_1,\Delta_2,*} : \mathcal{K}_*(A) \longrightarrow \mathcal{K}_{*+1}(A_{\Delta_1} \cap A_{\Delta_2})$$

as a odd degree (α_c, k_c)-controlled morphism of order r. Notice that the quantitative boundary map of a r-controlled weak Mayer–Vietoris pair is natural in the following sense: let A and B be filtered C^*-algebras, let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ and $(\Delta'_1, \Delta'_2, B_{\Delta'_1}, B_{\Delta'_2})$ be respectively r-controlled weak Mayer–Vietoris pairs for A and B with coercitivity c and let $f : A \to B$ be a filtered morphism such that $f(\Delta_1) \subseteq \Delta'_1$, $f(\Delta_2) \subseteq \Delta'_2$, $f(A_{\Delta_1}) \subseteq B_{\Delta'_1}$ and$f(A_{\Delta_2}) \subseteq B_{\Delta'_2}$. Then we have

$$f_{/A_{\Delta_1} \cap A_{\Delta_2},*} \circ \mathcal{D}_{\Delta_1,\Delta_2,*} = \mathcal{D}_{\Delta'_1,\Delta'_2,*} \circ f_*, \tag{5}$$

where $f_{/A_{\Delta_1} \cap A_{\Delta_2}} : A_{\Delta_1} \cap A_{\Delta_2} \to B_{\Delta'_1} \cap B_{\Delta'_2}$ is the restriction of f to $A_{\Delta_1} \cap A_{\Delta_2}$.

3.3. The controlled six-term exact sequence

In this subsection, we prove the controlled exactness at order r at the source and at the range of $\mathcal{D}_{\Delta_1,\Delta_2,*}$, stating as a consequence the Mayer–Vietoris controlled six term exact sequence associated to a r-controlled Mayer–Vietoris. Let us start with controlled exactness at the source.

Lemma 3.5. *There exists a control pair (λ, l) such that*

- *for any unital filtered C^*-algebra A filtered by $(A_s)_{s>0}$ and any subalgebras A_1 and A_2 of A such that A_1, A_2 and $A_1 \cap A_2$ are respectively filtered by $(A_1 \cap A_r)_{r>0}$, $(A_2 \cap A_r)_{r>0}$ and $(A_1 \cap A_2 \cap A_r)_{r>0}$;*

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- *for any positive number ε with $\varepsilon < \frac{1}{4\lambda}$ any integers n and m and any ε-r-unitaries u_1 in $M_n(A)$ and u_2 in $M_m(A)$;*
- *for any ε-r-unitaries v_1 and v_2 respectively in $M_{n+m}(\widetilde{A_1})$ and $M_{n+m}(\widetilde{A_2})$ such that*
  - $\|\operatorname{diag}(u_1, u_2) - v_1v_2\| < \varepsilon$;
  - *there exists an ε-r-projection q in $M_{n+m}(A)$ such that $q - \operatorname{diag}(I_n, 0)$ is in $M_{n+m}(A_1 \cap A_2)$, $\|q - v_1^* \operatorname{diag}(I_n, 0)v_1\| < \varepsilon$ and $[q, n]_{\varepsilon,r} = 0$ in $K_0^{\varepsilon,r}(A_1 \cap A_2)$.*

Then there exists an integer k and $\lambda\varepsilon$-$l_\varepsilon r$-unitaries w_1 and w_2 respectively in $M_{n+k}(\widetilde{A_1})$ and $M_{n+k}(\widetilde{A_2})$ such that $\|\operatorname{diag}(u_1, I_k) - \operatorname{diag}(w_1w_2)\| < \lambda\varepsilon$. Moreover, if $v_i - I_{n+k}$ lies in $M_{n+k}(A_i)$ for $i = 1, 2$ then w_1 and w_2 can be chosen such that $w_i - I_{n+k}$ lies in $M_{n+m}(A_i)$ for $i = 1, 2$

Proof. Up to replacing u_2, v_1 and v_2 respectively by $\operatorname{diag}(u_2, I_k)$, $\operatorname{diag}(v_1, I_k)$ and $\operatorname{diag}(v_2, I_k)$ for some integer k, we can assume that q is homotopic to $\operatorname{diag}(I_n, 0)$ as an ε-r-projection in $M_{n+m}(\widetilde{A_1 \cap A_2})$. According to Lemma 1.5, there exist

- a control pair (α, h);
- up to stabilization an $\alpha\varepsilon$-$h_\varepsilon r$-unitary v in $M_{n+m}(\widetilde{A_{\Delta_1} \cap A_{\Delta_2}})$ with $v - I_{n+m}$ in $M_{n+m}(A_{\Delta_1} \cap A_{\Delta_2})$

such that

$$\|q - v\operatorname{diag}(I_n, 0)v^*\| < \alpha\varepsilon.$$

Up to take a larger control pair (α, h), we can assume that

$$\|v_1^* \operatorname{diag}(I_n, 0)v_1 - v\operatorname{diag}(I_n, 0)v^*\| < \alpha\varepsilon$$

and

$$\|v_2 \operatorname{diag}(I_n, 0)v_2^* - v\operatorname{diag}(I_n, 0)v^*\| < \alpha\varepsilon$$

and hence even indeed that

$$\|v^* v_1^* \operatorname{diag}(I_n, 0)v_1 v - \operatorname{diag}(I_n, 0)\| < \alpha\varepsilon$$

and

$$\|v^* v_2 \operatorname{diag}(I_n, 0)v_2^* v - \operatorname{diag}(I_n, 0)\| < \alpha\varepsilon.$$

Hence, for some control pair (α', h') depending only on (α, h), there exist $\alpha'\varepsilon$-$h'_\varepsilon s$-unitaries v_1' in $M_n(\widetilde{A_{\Delta_1}})$, v_1'' in $M_m(\widetilde{A_{\Delta_1}})$, v_2' in $M_n(\widetilde{A_{\Delta_2}})$, v_2'' in $M_m(\widetilde{A_{\Delta_2}})$ such that $\|v_1 v -$

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$\operatorname{diag}(v_1', v_1'')\| < \alpha'\varepsilon$ and $\|v^* v_2 - \operatorname{diag}(v_2', v_2'')\| < \alpha'\varepsilon$. Thus, for a control pair (α'', h'') depending only on (α', h') we have,

$$\| \operatorname{diag}(u_1, u_2) - \operatorname{diag}(v_1' v_2', v_1'' v_2'')\| < \alpha''\varepsilon.$$

Hence we deduce that $\|u_1 - v_1' v_2'\| < \alpha''\varepsilon$. □

As a consequence, we get the following controlled exactness result at the source of $\mathcal{D}_{\Delta_1,\Delta_2,*}$ that persists at any order.

Corollary 3.6. *For any positive number c, there exists a control pair (λ, l) such that*

- *for any filtered C^*-algebra A;*
- *for any positive number r and any r-controlled weak Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ for A with coercitivity c;*
- *for any positive numbers ε, ε' and r' with $0 < \alpha_c\varepsilon \leqslant \varepsilon' < \frac{1}{4\lambda}$ and $r' \geqslant k_{c,\varepsilon} r$*

then for any y in $K_1^{\varepsilon,r}(A)$ such that

$$\iota_*^{-,\varepsilon',r'} \circ \partial^{\varepsilon,r}_{\Delta_1,\Delta_2,*}(y) = 0$$

in $K_1^{\varepsilon',r'}(A_{\Delta_1} \cap A_{\Delta_2})$, there exist x_1 in $K_1^{\lambda\varepsilon',l_{\varepsilon'}r'}(A_{\Delta_1})$ and x_2 in $K_1^{\lambda\varepsilon',l_{\varepsilon'}r'}(A_{\Delta_2})$ such that

$$\iota_*^{-,\lambda\varepsilon',l_{\varepsilon'}r'}(y) = \jmath^{\lambda\varepsilon',l_{\varepsilon'}r'}_{\Delta_1,*}(x_1) - \jmath^{\lambda\varepsilon',l_{\varepsilon'}r'}_{\Delta_2,*}(x_2).$$

Proof. Let us assume for sake of simplicity that A is unital, the non unital being similar (just extra notation are added). Let y be an element in $K_1^{\varepsilon,r}(A)$ such that $\iota_*^{-,\varepsilon',r'} \circ \partial^{\varepsilon,r}_{\Delta_1,\Delta_2,*}(y) = 0$ in $K_1^{\varepsilon',r'}(A_{\Delta_1} \cap A_{\Delta_2})$. Let (λ, l) be the controlled pair of Lemma 3.5 and let u be an ε-r-unitary in some $M_n(A)$ such that $y = [u]_{\varepsilon,r}$. Then according to the definition of $\partial^{\varepsilon,r}_{\Delta_1,\Delta_2,*}(y)$, we see by using Lemma 3.5 that up to replacing u by $\operatorname{diag}(u, I_m)$ for some integer m, there exists two $\lambda\varepsilon'$-$l_{\varepsilon'}r'$-unitaries w_1 and w_2 respectively in $M_n(\widetilde{A_{\Delta_1}})$ and $M_n(\widetilde{A_{\Delta_2}})$ such that $\|u - w_1 w_2\| < \lambda\varepsilon'$. Then u is homotopic to $w_1 w_2$ as a $4\lambda\varepsilon'$-$l_{\varepsilon'}r'$-unitary in $M_{2n}(A)$. From this we deduce that

$$\begin{aligned}
\iota_*^{-,4\lambda\varepsilon',2l_{\varepsilon'}r'}(y) &= [w_1 w_2]_{4\lambda\varepsilon',2l_{\varepsilon'}r'} \\
&= [w_1]_{4\lambda\varepsilon',2l_{\varepsilon'}r'} + [w_2]_{4\lambda\varepsilon',2l_{\varepsilon'}r'} \\
&= \jmath^{4\lambda\varepsilon',2l_{\varepsilon'}r'}_{\Delta_1,*}(x_1) + \jmath^{4\lambda\varepsilon',2l_{\varepsilon'}r'}_{\Delta_2,*}(x_2)
\end{aligned}$$

with $x_1 = [w_1]_{4\lambda\varepsilon',2l_{\varepsilon'}r'}$ in $K_1^{4\lambda\varepsilon,2l_\varepsilon r'}(A_{\Delta_1})$ and $x_2 = [w_2]_{4\lambda\varepsilon,2l_{\varepsilon'}r'}$ in $K_1^{4\lambda\varepsilon,2l_{\varepsilon'}r'}(A_{\Delta_2})$. □

In particular, at order r, we obtain the following controlled exactness result.

Proposition 3.7. *For any positive number c, there exists a control pair* (α, l) *such that for any* C^**-algebra* A *filtered by* $(A_s)_{s>0}$*, any positive number* r *and any* r*-controlled weak Mayer–Vietoris pair* $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ *for* A *with coercitivity* c *then the composition*

$$\mathcal{K}_1(A_{\Delta_1}) \oplus \mathcal{K}_1(A_{\Delta_2}) \xrightarrow{\jmath_{\Delta_1,*}-\jmath_{\Delta_2,*}} \mathcal{K}_1(A) \xrightarrow{\mathcal{D}_{\Delta_1,\Delta_2,*}} \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2})$$

is (α, l)*-exact at order* r.

Let us prove now the controlled exactness at the range of $\mathcal{D}_{\Delta_1,\Delta_2,*}$.

Lemma 3.8. *There exists a control pair* (λ, h) *such that the following holds:*

- *Let* A *be a unital* C^**-algebra filtered by* $(A_r)_{r>0}$ *and let* A_1 *and* A_2 *be subalgebras of* A *such that* A_1*,* A_2 *and* $A_1 \cap A_2$ *are respectively filtered by* $(A_1 \cap A_r)_{r>0}$*,* $(A_2 \cap A_r)_{r>0}$ *and* $(A_1 \cap A_2 \cap A_r)_{r>0}$*;*
- *let* ε *and* s *be positive numbers with* $\varepsilon < \frac{1}{4\lambda}$*;*
- *let* n *and* N *be positive integers with* $n \leqslant N$ *and let* p *an* ε*-*s *projection in* $M_N(\widetilde{A_1 \cap A_2})$ *such that* $\rho_{A_1 \cap A_2}(p) = \operatorname{diag}(I_n, 0)$*.*

Assume that

- p *is homotopic to* $\operatorname{diag}(I_n, 0)$ *as an* ε*-*s*-projection in* $M_N(\widetilde{A_1})$*;*
- p *is homotopic to* $\operatorname{diag}(I_n, 0)$ *as an* ε*-*s*-projection in* $M_N(\widetilde{A_2})$*.*

Then there exist an integer N' *with* $N' \geqslant N$*, and four* $\lambda\varepsilon$*-*$h_\varepsilon s$*-unitaries* w_1 *and* w_2 *in* $M_{N'}(A)$*,* u *in* $M_n(A)$ *and* v *in* $M_{N'-n}(A)$ *such that*

- $w_i - I_{N'}$ *is an element in* $M_{N'}(A_i)$ *for* $i = 1, 2$*;*
- $$\|w_1^* \operatorname{diag}(I_n, 0) w_1 - diag(p, 0)\| < \lambda\varepsilon$$

  and

  $$\|w_2 \operatorname{diag}(I_n, 0) w_2^* - \operatorname{diag}(p, 0)\| < \lambda\varepsilon.$$
- *for* $i = 1, 2$*, then* w_i *is connected to* $I_{N'}$ *by a homotopy of* $\lambda\varepsilon$*-*$h_\varepsilon s$*-unitaries* $(w_{i,t})_{t\in[0,1]}$ *in* $M_{N'}(A)$ *such that* $w_{i,t} - I_{N'}$ *is in* $M_{N'}(A_i)$ *for all* t *in* $[0, 1]$*.*
- $\|\operatorname{diag}(u, v) - w_1 w_2\| < \lambda\varepsilon$*.*

Proof. Let (α, k) be the control pair of Proposition 1.5, then there exist up to stabilization

- w_1 an $\alpha\varepsilon$-$k_\varepsilon s$-unitary in $M_N(\widetilde{A_1})$;
- w_2 an $\alpha\varepsilon$-$k_\varepsilon s$-unitary in $M_N(\widetilde{A_2})$,

such that

$$\|w_1^* \operatorname{diag}(I_n,0)w_1 - p\| < \alpha_h\varepsilon$$

and

$$\|w_2 \operatorname{diag}(I_n,0)w_2^* - p\| < \alpha_h\varepsilon.$$

Up to replacing $w_1$and w_2 respectively by $\rho_{A_1}(w_1^{-1})w_1$ and $w_2\rho_{A_2}(w_2^{-1})$ and up to replacing α by 4α, we can assume that $w_1 - I_N$ is an element in the matrix algebra $M_N(A_1)$ and $w_2 - I_N$ is an element in the matrix algebra $M_N(A_2)$. Hence there exists a control pair (α', k'), depending only on (α, k) and that we can choose larger such that

$$\|w_1w_2 \operatorname{diag}(I_n,0)w_2^*w_1^* - \operatorname{diag}(I_n,0)\| < \alpha'\varepsilon \tag{6}$$

and

$$\|w_2^*w_1^* \operatorname{diag}(I_n,0)w_2w_1 - \operatorname{diag}(I_n,0)\| < \alpha'\varepsilon. \tag{7}$$

Up to replacing w_1 , w_2, p and (α', k') respectively by $\operatorname{diag}(w_1, w_1^*)$, $\operatorname{diag}(w_2, w_2^*)$, $\operatorname{diag}(p, 0)$ and $(3\alpha, 2k)$, we can assume that w_i for $i = 1, 2$ is connected to I_N by a homotopy of $\alpha\varepsilon$-$k_\varepsilon s$-unitaries $(w_{i,t})_{t\in[0,1]}$ in $M_N(A)$ such that $w_{i,t} - I_N$ is in $M_N(A_i)$ for all t in $[0, 1]$. Equations (6) and (7) imply that for a control pair (α'', k''), depending only on (α', k'), there exist u and v some $\alpha''\varepsilon$-$k''_\varepsilon s$-unitaries respectively in $M_n(A)$ and $M_{N-n}(A)$ such that

$$\|\operatorname{diag}(u, v) - w_1w_2\| < \alpha''\varepsilon. \quad \square$$

Proposition 3.9. *For every positive number c, there exists a control pair (α, l) such that for any filtered C^*-algebra A, any positive number r and any r-controlled weak Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ for A of order r with coercitivity c, then the composition*

$$\mathcal{K}_1(A) \xrightarrow{\mathcal{D}_{\Delta_1,\Delta_2,*}} \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2}) \xrightarrow{(\jmath_{\Delta_1,\Delta_2,*},\jmath_{\Delta_2,\Delta_1,*})} \mathcal{K}_0(A_{\Delta_1}) \oplus \mathcal{K}_0(A_{\Delta_2})$$

is (α, l)-exact at order r.

Proof. As in the previous proposition, let us assume that A is unital. Let y be an element in $K_0^{\varepsilon,s}(A_{\Delta_1} \cap A_{\Delta_2})$ such that $\jmath^{\varepsilon,s}_{\Delta_1,\Delta_2,*}(y) = 0$ in $K_0^{\varepsilon,s}(A_{\Delta_1})$ and $\jmath^{\varepsilon,s}_{\Delta_2,\Delta_1,*}(y) = 0$ in $K_0^{\varepsilon,s}(A_{\Delta_2})$. Let p be an ε-r-projection in some $M_N(\widetilde{A_{\Delta_1} \cap A_{\Delta_2}})$ and n be an integer

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such that $y = [p, n]_{\varepsilon,s}$. In view of Lemma 1.7 and up to replacing ε by 5ε, we can assume without loss of generality that $N \geqslant n$ and that

$$\rho_{A_{\Delta_1} \cap A_{\Delta_2}}(p) = \operatorname{diag}(I_n, 0).$$

Up to stabilization, we can also assume that

- p is homotopic $\operatorname{diag}(I_n, 0)$ as an ε-s-projection in $M_N(\widetilde{A_{\Delta_1}})$;
- p is homotopic $\operatorname{diag}(I_n, 0)$ as an ε-s-projection in $M_N(\widetilde{A_{\Delta_2}})$.

Let (α, k) be a control pair, N' be an integer with $N' \geqslant N$, let w_1 and w_2 be in $U^{\alpha\varepsilon,k_\varepsilon s}_{N'}(A)$, let u be in $U^{\alpha\varepsilon,k_\varepsilon s}_{n}(A)$ and let v be in $U^{\alpha\varepsilon,k_\varepsilon s}_{N'-n}(A)$ that satisfy all together the conclusion of Lemma 3.8. Since $\| \operatorname{diag}(u, v) - w_1 w_2\| < \alpha\varepsilon$, we can up to replacing (α, k) by $(4\alpha, 2k)$ and in view of Lemma 1.2 moreover assume that $\operatorname{diag}(u, v)$ is homotopic to I'_N as an $\alpha\varepsilon$-$h_\varepsilon r$-unitary of $M_{N'}(A)$. Since

$$\|w_1^* \operatorname{diag}(I_n, 0) w_1 - \operatorname{diag}(p, 0)\| < \alpha\varepsilon$$

and

$$\|w_2 \operatorname{diag}(I_n, 0) w_2^* - \operatorname{diag}(p, 0)\| < \alpha\varepsilon$$

and in view of the definition of the quantitative boundary map of a control Mayer–Vietoris pair, there exists a control pair (α', k') depending only on (α, k) and c such that

$$\partial^{\alpha'\varepsilon,k'_\varepsilon s}_{\Delta_1,\Delta_2,*}([u]_{\alpha''\varepsilon,k'_\varepsilon s}) = [p, l]_{\varepsilon,\alpha'\alpha_c\varepsilon,s,k'_\varepsilon k_{c,\lambda_\varepsilon s}}.$$

Hence, if we set $x = [u]_{\alpha'\varepsilon,k'_\varepsilon s}$, we get

$$\partial^{\alpha'\varepsilon,k'_\varepsilon s}_{\Delta_1,\Delta_2,*}(x) = \iota_*^{\varepsilon,\alpha'\alpha_c\varepsilon,s,k'_\varepsilon k_{c,\lambda_\varepsilon s}}(y). \quad \square$$

Collecting together Propositions 3.2, 3.7 and 3.9 and using naturality of quantitative Bott isomorphism, we obtain the controlled six terms exact sequence (at order r) associated to a weak r-controlled Mayer–Vietoris sequence.

Theorem 3.10. *For every positive number c, there exists a control pair (λ, h) such that for any C^*-algebra A filtered by $(A_s)_{s>0}$, any positive number r and any r-controlled weak Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ for A, we have a (λ, h)-exact six term exact sequence at order r:*

$$\begin{array}{ccccc}
\mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2}) & \xrightarrow{(\jmath_{\Delta_1,\Delta_2,*},\, \jmath_{\Delta_2,\Delta_1,*})} & \mathcal{K}_0(A_{\Delta_1}) \oplus \mathcal{K}_0(A_{\Delta_2}) & \xrightarrow{\jmath_{\Delta_1,*} - \jmath_{\Delta_2,*}} & \mathcal{K}_0(A) \\
\uparrow {\scriptstyle \mathcal{D}_{\Delta_1,\Delta_2,*}} & & & & {\scriptstyle \mathcal{D}_{\Delta_1,\Delta_2,*}} \downarrow \\
\mathcal{K}_1(A) & \xleftarrow{\jmath_{\Delta_1,*} - \jmath_{\Delta_2,*}} & \mathcal{K}_1(A_{\Delta_1}) \oplus \mathcal{K}_1(A_{\Delta_2}) & \xleftarrow{(\jmath_{\Delta_1,\Delta_2,*},\, \jmath_{\Delta_2,\Delta_1,*})} & \mathcal{K}_1(A_{\Delta_1} \cap A_{\Delta_2})
\end{array}.$$

3.4. Quantitatively K-contractible C^*-algebra

In numerous cases, the proof of the Baum–Connes conjecture and of its generalization amounts to proving that the K-theory of some obstruction algebra vanishes. In [14], the second author proved the Novikov conjecture for finitely generated groups with finite asymptotic dimension by showing that the obstruction C^*-algebra corresponding to the localization C^*-algebra is quantitatively K-contractible (which implies that its K-theory vanishes). In this subsection, we apply the controlled Mayer–Vietoris six-term exact sequence to quantitative K-contractibility.

Definition 3.11. Let A be a filtered C^*-algebra. A is called **quantitatively K-contractible** if there exists a positive number $\lambda_0 \geqslant 1$ that satisfies the following:

for any positive numbers ε and r with $\varepsilon < \frac{1}{4\lambda_0}$, there exists a positive number r' with $r' \geqslant r$ such that $\iota_*^{\varepsilon,\lambda_0\varepsilon,r,r'} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\lambda_0\varepsilon,r'}(A)$ is the zero map (we say that A is K-contractible with **rescaling** λ_0).

Example 3.12.

- (i) Recall that a separable C^*-algebra B is K-contractible if the class of the identity map $Id_B : B \to B$ vanishes in $KK_*(B,B)$. According to point (iv) of Proposition 1.25, if B is K-contractible, then $A \otimes B$ is quantitatively K-contractible for any filtered C^*-algebra A. Moreover, the rescaling does not depend on A or on B;
- (ii) Let Γ be a finitely generated group and let A be a C^*-algebra provided with an action of Γ by automorphisms. Assume that
  - the group Γ satisfies the Baum–Connes conjecture with coefficients;
  - for any finite subgroup F of Γ the $K_*(A \rtimes F) = 0$.

Then $A \rtimes_{red} \Gamma$ is quantitatively K-contractible and the rescaling does not depend on Γ or on A. Indeed, under these assumptions, the left hand side of the (quantitative) Baum–Connes assembly map is vanishing [1]. The quantitative K-contractibility for $A \rtimes_{red} \Gamma$ is then a consequence of the Quantitative Assembly Map estimates of Theorem 1.34.

Remark 3.13. It can be proved that there exists a universal rescaling for quantitative K-contractibility, i.e. there exists a positive number λ_0 with $\lambda_0 \geqslant 1$ such that every quantitatively K-contractible C^*-algebra is indeed quantitatively K-contractible with rescaling λ_0.

Theorem 3.14. *Let A be a filtered C^*-algebra. Assume that there exist positive numbers λ_0 and c, with $\lambda_0 \geqslant 1$ such that for every positive number r there exists a weak r-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ with coercitivity c and with A_{Δ_1}, A_{Δ_2} and $A_{\Delta_1} \cap A_{\Delta_2}$ quantitatively K-contractible with rescaling λ_0. Then there exists a positive number*

λ_1 depending only on λ_0 and on c such that A is quantitatively K-contractible with rescaling λ_1.

Proof. Let (λ, l) be the controlled pair of Corollary 3.6 and set $\lambda_1 = \lambda\lambda_0^2\alpha_c$, let ε and r be positive numbers with $\varepsilon < \frac{1}{4\lambda_1}$ and let y be an element in $K_*^{\varepsilon,r}(A)$. Let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be a weak r-controlled Mayer–Vietoris pair for A with coercitivity c and A_{Δ_1}, A_{Δ_2} and $A_{\Delta_1} \cap A_{\Delta_2}$ quantitatively K-contractible with rescaling λ_0. Let r' be a positive number with $r' \geqslant k_{c,\varepsilon} r$ such that

$$\iota^{-,\lambda_0\alpha_c\varepsilon,r'}(z) = 0$$

in $K_*^{\lambda_0\alpha_c\varepsilon,r'}(A_{\Delta_1} \cap A_{\Delta_2})$ for all z in $K_*^{\alpha_c\varepsilon,k_{c,\varepsilon}r}(A_{\Delta_1} \cap A_{\Delta_2})$. Since

$$\iota^{-,\lambda_0\alpha_c\varepsilon,r'} \circ \partial^{\varepsilon,r}_{\Delta_1,\Delta_2,*}(y) = 0$$

in $K_*^{\alpha_c\varepsilon,k_{c,\varepsilon}r}(A_{\Delta_1} \cap A_{\Delta_2})$ and according to Corollary 3.6, then if we set $\lambda' = \alpha_c\lambda_0$, there exist an element x_1 in $K_1^{\lambda\lambda'\varepsilon,l_{\lambda'\varepsilon}r'}(A_{\Delta_1})$ and an element x_2 in $K_1^{\lambda\lambda'\varepsilon,l_{\lambda'\varepsilon}r'}(A_{\Delta_2})$ such that

$$\iota_*^{-,\lambda\lambda'\varepsilon,l_{\lambda'\varepsilon}r'}(y) = \jmath_{\Delta_1,*}^{\lambda\lambda'\varepsilon,l_{\lambda'\varepsilon}r'}(x_1) - \jmath_{\Delta_2,*}^{\lambda\lambda',l_{\lambda'\varepsilon}r'}(x_2).$$

Let r'' be a positive number with $r'' \geqslant l_{\lambda'\varepsilon}r'$ such that for $i = 1, 2$,

$$\iota^{-,\lambda_0\lambda\lambda'\varepsilon,r''}(z) = 0$$

in $K_1^{\lambda_0\lambda\lambda'\varepsilon,r''}(A_{\Delta_i})$ for all z in $K_1^{\lambda\lambda'\varepsilon,l_{\lambda'\varepsilon}r'}(A_{\Delta_i})$. Then we eventually obtain that

$$\begin{aligned}\iota_*^{-,\lambda_1\varepsilon,r''}(y) &= \jmath_{\Delta_1,*}^{\lambda_1\varepsilon,r''} \circ \iota_*^{-,\lambda_1\varepsilon,r''}(x_1) - \jmath_{\Delta_2,*}^{\lambda_1\varepsilon,r''} \circ \iota_*^{-,\lambda_1\varepsilon,r''}(x_2) \\ &= 0.\end{aligned}$$

Hence A is quantitatively K-contractible with rescaling λ_1. □

4. Quantitative Künneth formula

The Künneth formula computes the K-theory of the minimal tensor product $A \otimes B$ of two C^*-algebras A and B in terms of the K-theory of A and B. More precisely, $K_*(A \otimes B)$ fits into a natural extension

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow \mathrm{Tor}(K_*(A), K_*(B)) \longrightarrow 0, \tag{8}$$

where the inclusion map is given by the external product in K-theory. We say that a C^*-algebra A satisfies the Künneth formula in K-theory if the formula of equation (8) holds for any C^*-algebra B. In [12], C. Schochet proved using a geometric resolution that a C^*-algebra A satisfies the Künneth formula in K-theory if and only if for every

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C^*-algebra B such that $K_*(B)$ is a free abelian group, then the external product in K-theory

$$K_*(A)\otimes K_*(B)\longrightarrow K_*(A\otimes B)$$

is an isomorphism. The quantitative Künneth formula in K-theory was then proved for any C^*-algebra A in the Bootstrap class (see Definition 5.1). Using the above characterization, we formulate in this section a quantitative Künneth formula for filtered C^*-algebras which implies the classical one. We show that this quantitative version of the Künneth formula is asymptotically hereditary with respect to decomposition under controlled Mayer–Vietoris (nuclear) pairs. We also show that finitely generated groups for which the Baum–Connes conjecture with coefficient holds provides numerous examples of filtered C^*-algebras that satisfy the quantitative Künneth formula in K-theory.

4.1. Statement of the formula

Recall that if A and B are C^*-algebras, then there is a morphism

$$\omega_{A,B,*}: K_*(A)\otimes K_*(B)\longrightarrow K_*(A\otimes B)$$

given by the external Kasparov product i.e., $\omega_{A,B,*}(x\otimes y)=x\otimes\tau_A(y)$ for all x in $K_*(A)$ and y in $K_*(B)$. Indeed, in the case of unital C^*-algebras, if p and q are respectively projections in $M_n(A)$ and $M_k(B)$ and if u and v are respectively unitary elements in $M_n(A)$ and $M_k(B)$, then

$$\begin{aligned}
\omega_{A,B,*}([p]\otimes[q]) &= [p\otimes q];\\
\omega_{A,B,*}([u]\otimes[q]) &= [u\otimes q+I_n\otimes(I_k-q)];\\
\omega_{A,B,*}([p]\otimes[v]) &= [p\otimes v+(I_n-p)\otimes I_k].
\end{aligned}$$

Let A be a C^*-algebra filtered by $(A_r)_{r>0}$ and let B be a C^*-algebra (with a trivial filtration). Recall that $A\otimes B$ is then filtered by $(A_r\otimes B)_{r>0}$. Let us consider then the quantitative object $\mathcal{K}_*(A)\otimes K_*(B)=(K_*^{\varepsilon,r}(A)\otimes K_*(B))$. With notations of Theorem 1.24, define the $(\alpha_{\mathcal{T}},k_{\mathcal{T}})$-control morphism

$$\Omega_{A,B,*}=(\omega^{\varepsilon,r}_{A,B}):\mathcal{K}_*(A)\otimes K_*(B)\to\mathcal{K}_*(A\otimes B),$$

by

$$\omega^{\varepsilon,r}_{A,B,*}:K_*^{\varepsilon,r}(A)\otimes K_*(B)\to\mathcal{K}_*^{\varepsilon,r}(A\otimes B);\ x\otimes y\mapsto\tau_A^{\alpha_{\mathcal{T}}\varepsilon,h_{\mathcal{T},\varepsilon}r}(y)(x).$$

Then the controlled morphism $\Omega_{A,B,*}$ induces the map $\omega_{A,B,*}$ in K-theory, i.e.,

$$\iota_*^{\varepsilon,r}\circ\omega^{\varepsilon,r}_{A,B,*}=\omega_{A,B,*}\circ\left(\iota_*^{\varepsilon,r}\otimes\mathcal{I}d_{K_*(B)}\right)\tag{9}$$

for every positive numbers r and ε with $0<\varepsilon<\frac{1}{4\alpha_{\mathcal{T}}}$.

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Remark 4.1. Let A be a unital filtered C^*-algebra and let B be a unital C^*algebra. Let ε and r be positive numbers with $\varepsilon < \frac{1}{4\alpha_{\mathcal{T}}}$,

(i) for any ε-r-projection p in some $M_n(A)$, any integer l and any projection q in some $M_k(B)$ then $\omega^{\varepsilon,r}_{A,B,*}([p,l]_{\varepsilon,r}\otimes[q]) = [p\otimes q + I_l\otimes(I_k - q), lk]_{\alpha_{\mathcal{T}}\varepsilon,h_{\mathcal{T},\varepsilon}r}$ in $K_0^{\alpha_{\mathcal{T}}\varepsilon,h_{\mathcal{T},\varepsilon}r}(A\otimes B)$;
(ii) for any ε-r-unitary u in some $M_n(A)$ and any projection q in some $M_k(B)$ then $\omega^{\varepsilon,r}_{A,B,*}([u]_{\varepsilon,r}\otimes[q]) = [u\otimes q + I_n\otimes(I_k - q)]_{\alpha_{\mathcal{T}}\varepsilon,h_{\mathcal{T},\varepsilon}r}$ in $K_1^{\alpha_{\mathcal{T}}\varepsilon,h_{\mathcal{T},\varepsilon}r}(A\otimes B)$.

The quantitative morphism $\Omega_{\bullet,\bullet,*}$ is compatible with the Kasparov tensorization (controlled) morphism.

Lemma 4.2. *There exists a control pair (α, k) such that the following assertion holds:*
For any filtered C^*-algebra A, for any separable C^*-algebras B_1, B_2, D_1 and D_2, any z in $KK_*(B_1, B_2)$ and any z' in $KK_*(D_1, D_2)$, the following diagram is (α, k)-commutative.

$$\begin{array}{ccc} \mathcal{K}_*(A\otimes B_1)\otimes K_*(D_1) & \xrightarrow{\omega_{A\otimes B_1,D_1,*}} & \mathcal{K}_*(A\otimes B_1\otimes D_1) \\ {\scriptstyle \mathcal{T}_A(z)\otimes(\bullet\otimes z')}\downarrow & & \downarrow{\scriptstyle \mathcal{T}_A(\tau_{D_1}(z)\otimes\tau_{B_2}(z'))} \\ \mathcal{K}_*(A\otimes B_2)\otimes K_*(D_2) & \xrightarrow{\omega_{A\otimes B_2,D_2,*}} & \mathcal{K}_*(A\otimes B_2\otimes D_2) \end{array},$$

where $\bullet\otimes z' : K_*(D_1) \to K_*(D_2)$ is right multiplication by z'.

Proof. Let y be an element of $K_*(D_1)$. According to point (v) of Proposition 1.25 and to Theorem 1.27, there exists a control pair (λ, h) such that

$$\mathcal{T}_{A\otimes B_2}(y\otimes z') \circ \mathcal{T}_A(z) \overset{(\lambda,h)}{\sim} \mathcal{T}_A(z\otimes\tau_{B_2}(y)\otimes\tau_{B_2}(z')).$$

Since the external Kasparov is commutative, we have

$$z\otimes\tau_{B_2}(y) = \tau_{B_1}(y)\otimes\tau_{D_1}(z)$$

Using once again Theorem 1.27 and up to rescaling the control pair (λ, h), we get that

$$\mathcal{T}_{A\otimes B_2}(y\otimes z') \circ \mathcal{T}_A(z) \overset{(\lambda,h)}{\sim} \mathcal{T}_A(\tau_{D_1}(z)\otimes\tau_{B_2}(z')) \circ \mathcal{T}_{A\otimes B_1}(y)$$

and hence the diagram is commutative. □

Definition 4.3. Let A be a filtered C^*-algebra and let λ_0 be a positive number with $\lambda_0 \geqslant 1$. We say that A satisfies the quantitative Künneth formula with rescaling λ_0 if

$$\Omega_{A,B,*} : \mathcal{K}_*(A)\otimes K_*(B) \to \mathcal{K}_*(A\otimes B)$$

is a quantitative isomorphism with rescaling λ_0 for every C^*-algebra B such that $K_*(B)$ is free abelian group.

Remark 4.4. If a filtered C^*-algebra A satisfies the quantitative Künneth formula, then according to equation (9)

$$\omega_{A,B,*} : K_*(A)\otimes K_*(B) \longrightarrow K_*(A\otimes B)$$

is an isomorphism for every C^*-algebra B such that $K_*(B)$ is free and hence the C^*-algebra A satisfies the Künneth formula in K-theory

The next theorem provides many examples of filtered C^*-algebras that satisfy the quantitative Künneth formula and will be proved in Section 4.4.

Theorem 4.5. *Let Γ be a finitely generated group, let A be a Γ-C^*-algebra. Assume that*

- *Γ satisfies the Baum–Connes conjecture with coefficients.*
- *For each subgroup K of Γ, the crossed product algebra $A \rtimes K$ satisfies the Künneth formula.*

Then $A \rtimes_r \Gamma$ satisfies the quantitative Künneth formula, i.e., for any C^*-algebra B such that $K_*(B)$ is a free abelian group,

$$\Omega_{A\rtimes_r\Gamma,B,*} : \mathcal{K}_*(A \rtimes_r \Gamma)\otimes K_*(B) \to \mathcal{K}_*((A \rtimes_r \Gamma)\otimes B)$$

is a quantitative isomorphism with rescaling that does not depend on Γ or on A.

Moreover, under the above assumption, when the C^*-algebra A runs through family of Γ-C^*-algebras and B runs through C^*-algebras such that $K_*(B)$ is a free abelian group, the family of quantitative isomorphisms $(\Omega_{A\rtimes_r\Gamma,B})_{A,B}$ is uniform.

The proof of these two results relies indeed on the quantitative statements of Theorem 1.34 which hold for groups that satisfy the Baum–Connes conjecture with coefficients. Similarly, using the geometric quantitative statements of Section 1.7 and Theorem 1.35, we can prove the following result:

Theorem 4.6. *Let Σ be a discrete proper metric space with bounded geometry that coarsely embeds into a Hilbert space. If A satisfies the Künneth formula, then $A\otimes\mathscr{K}(\ell^2(\Sigma))$ satisfies the quantitative Künneth formula with rescaling that does not depend on Σ or on A.*

Remark 4.7. As for quantitative K-contractibility (see Remark 3.13), it can be proved that there exists a universal rescaling for the quantitative Künneth formula.

4.2. Quantitative Künneth formula and controlled Mayer–Vietoris pairs

In this subsection, we state a permanence result (that we shall prove in next subsection) for the quantitative Künneth formula with respect to controlled Mayer–Vietoris pairs which satisfy a nuclear type condition.

Definition 4.8. Let A be a C^*-algebra filtered by $(A_s)_{s>0}$ and let r be a positive number. An r-controlled nuclear Mayer–Vietoris pair is a quadruple $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$, where Δ_1 and Δ_2 are closed linear subspaces of A_r stable under involution and A_{Δ_1} and A_{Δ_2} are respectively r-controlled Δ_1 and Δ_2-neighborhood-C^*-algebras such that for some positive number c and for any C^*-algebra B

(i) $(\Delta_1 \otimes B, \Delta_2 \otimes B)$ is a coercive decomposition pair for $A \otimes B$ of order r with coercitivity c;
(ii) the pair $(A_{\Delta_1,s} \otimes B, A_{\Delta_2,s} \otimes B)$ has the CIA property with coercitivity c for any positive number s.

The positive number c is called the **coercitivity** of the r-controlled nuclear Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

Remark 4.9.

(i) Notice that $A_{\Delta_1} \otimes B$ and $A_{\Delta_2} \otimes B$ are respectively r-controlled $\Delta_1 \otimes B$ and $\Delta_2 \otimes B$-neighborhood-C^*-algebras.
(ii) Condition (ii) amounts to the following: for any positive numbers ε and s, any x in $A_{\Delta_1,s} \otimes B$ and any y in $A_{\Delta_2,s} \otimes B$ such that $||x - y|| < \varepsilon$, then there exists z in $(A_{\Delta_1,s} \cap A_{\Delta_2,s}) \otimes B$ satisfying

$$||z - x|| < c\varepsilon, \qquad ||z - y|| < c\varepsilon.$$

Example 4.10. Replacing $M_n(\mathbb{C})$ with B, we see that Example 2.20 and examples of Section 2.4 are indeed r-controlled nuclear Mayer–Vietoris pairs (with the same coercitivity).

Next lemma shows that the controlled boundary maps of a r-controlled nuclear Mayer–Vietoris pair are indeed compatible with Kasparov external product.

Lemma 4.11. *For any positive number c, there exists a control pair (α, h) such that the following is satisfied: let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be an r-controlled nuclear Mayer–Vietoris pair with coercitivity c, let B and B' be two C^*-algebras and z be an element in $KK_*(B, B')$, then*

$$\mathcal{T}_{(A_{\Delta_1} \cap A_{\Delta_2}),*}(z) \circ \mathcal{D}_{\Delta_1 \otimes B, \Delta_2 \otimes B} \overset{(\alpha,h)}{\sim} \mathcal{D}_{\Delta_1 \otimes B', \Delta_2 \otimes B'} \circ \mathcal{T}_{A,*}(z).$$

Proof. We first deal with the case z even. According to [7, Lemma 1.6.9], there exists a C^*-algebra D and homomorphisms $\theta : D \to B$ and $\eta : D \to B'$ such that

- the element $[\theta]$ of $KK_*(D, B)$ induced by θ is invertible.
- $z = \eta_*([\theta]^{-1})$.

Let $\theta_A : A\otimes B' \to A\otimes D$ and $\theta_{A_{\Delta_1}\cap A_{\Delta_2}} : (A_{\Delta_1} \cap A_{\Delta_2})\otimes B' \to (A_{\Delta_1} \cap A_{\Delta_2})\otimes D$ be the homorphisms induced by θ on tensor products and define similarly $\eta_A : A\otimes B' \to A\otimes D$ and $\eta_{A_{\Delta_1}\cap A_{\Delta_2}} : (A_{\Delta_1} \cap A_{\Delta_2})\otimes B' \to (A_{\Delta_1} \cap A_{\Delta_2})\otimes D$ the homomorphisms induced by η.

By naturality of quantitative boundary morphism of r-controled Mayer–Vietoris pairs (see equation (5) of Section 3.2), we get that

$$\theta_{A_{\Delta_1}\cap A_{\Delta_2},*} \circ \mathcal{D}_{\Delta_1\otimes B',\Delta_2\otimes B'} = \mathcal{D}_{\Delta_1\otimes D,\Delta_2\otimes D} \circ \theta_{A,*}.$$

According to Proposition 1.25, the control morphisms $\theta_{A,*}$ and $\theta_{A_{\Delta_1}\cap A_{\Delta_2},*}$ are $(\alpha_\mathcal{T}, k_\mathcal{T})$-invertible with $(\alpha_\mathcal{T}, k_\mathcal{T})$-inverses respectively given by $\mathcal{T}_{A,*}([\theta]^{-1})$ and $\mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}([\theta]^{-1})$ and hence, there exists a control pair (α, k) depending only on c and $(\alpha_\mathcal{T}, k_\mathcal{T})$ such that

$$\mathcal{D}_{\Delta_1\otimes B',\Delta_2\otimes B'} \circ \mathcal{T}_{A,*}([\theta]^{-1}) \overset{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}([\theta]^{-1}) \circ \mathcal{D}_{\Delta_1\otimes D,\Delta_2\otimes D}.$$

Then, using the bifunctoriality of $\mathcal{T}_{A,*}$ (see Proposition 1.25), we obtain

$$\begin{aligned}
\mathcal{D}_{\Delta_1\otimes B',\Delta_2\otimes B'} \circ \mathcal{T}_{A,*}(z) &= \mathcal{D}_{\Delta_1\otimes B',\Delta_2\otimes B'} \circ \mathcal{T}_{A,*}([\theta]^{-1}) \circ \eta_{A,*} \\
&\overset{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}([\theta]^{-1}) \circ \mathcal{D}_{\Delta_1\otimes D,\Delta_2\otimes D} \circ \eta_{A,*} \\
&\overset{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}([\theta]^{-1}) \circ \eta_{A_{\Delta_1}\cap A_{\Delta_2},*} \circ \mathcal{D}_{\Delta_1\otimes B,\Delta_2\otimes B} \\
&\overset{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}(z) \circ \mathcal{D}_{\Delta_1\otimes B,\Delta_2\otimes B},
\end{aligned}$$

where the third line is once again the consequence of naturality of quantitative boundary morphism of r-controled Mayer–Vietoris pairs (see equation (5) of Section 3.2). If z is an odd element, recall that $[\partial]$ is the invertible element of $KK_1(\mathbb{C}, C(0, 1))$ implementing the boundary morphism of the extension

$$0 \longrightarrow C(0, 1) \longrightarrow C(0, 1] \overset{ev_0}{\longrightarrow} \mathbb{C} \longrightarrow 0.$$

Then there exists a control pair (α, k) depending only on c and on the control pairs of Theorems 1.24 and 1.27 such that

$$\begin{aligned}
&\mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}(z) \circ \mathcal{D}^0_{\Delta_1\otimes B,\Delta_2\otimes B} \\
&\overset{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}(z) \circ \mathcal{T}_{(A_{\Delta_1}\cap A_{\Delta_2})\otimes B,*}([\partial]^{-1}) \circ \mathcal{D}^1_{S\Delta_1\otimes B,S\Delta_2\otimes B,*} \circ \mathcal{T}_{A\otimes B,*}([\partial])
\end{aligned}$$

$$\stackrel{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}(z)\circ \mathcal{T}_{(A_{\Delta_1}\cap A_{\Delta_2}),*}(\tau_B([\partial]^{-1}))\circ \mathcal{D}^1_{S\Delta_1\otimes B,S\Delta_2\otimes B,*}\circ \mathcal{T}_{A\otimes B,*}([\partial])$$
$$\stackrel{(\alpha,k)}{\sim} \mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}(\tau_B([\partial]^{-1})\otimes_B z)\circ \mathcal{D}^1_{S\Delta_1\otimes B,S\Delta_2\otimes B,*}\circ \mathcal{T}_{A\otimes B,*}([\partial])$$
$$\stackrel{(\alpha,k)}{\sim} \mathcal{D}^1_{S\Delta_1\otimes B',S\Delta_2\otimes B',*}\circ \mathcal{T}_{A,*}(\tau_B([\partial]^{-1})\otimes_B z)\circ \mathcal{T}_{A,*}(\tau_{B,*}([\partial]))$$
$$\stackrel{(\alpha,k)}{\sim} \mathcal{D}^1_{S\Delta_1\otimes B',S\Delta_2\otimes B',*}\circ \mathcal{T}_{A,*}(z),$$

where

- the first $\stackrel{(\alpha,k)}{\sim}$ holds by definition of $\mathcal{D}^0_{\Delta_1\otimes B,\Delta_2\otimes B}$;
- the second $\stackrel{(\alpha,k)}{\sim}$ is a consequence of point (v) of Proposition 1.25;
- the third and fifth $\stackrel{(\alpha,k)}{\sim}$ are consequences of Theorem 1.27;
- the fourth $\stackrel{(\alpha,k)}{\sim}$ is a consequence of the even case.

Similarly, we can prove that there exists a control pair (α,k) depending only on c and on the control pair of Theorems 1.24 and 1.27 such that

$$\mathcal{T}_{A_{\Delta_1}\cap A_{\Delta_2},*}(z)\circ \mathcal{D}^1_{\Delta_1\otimes B,\Delta_2\otimes B} \stackrel{(\alpha,k)}{\sim} \mathcal{D}^0_{S\Delta_1\otimes B,S\Delta_2\otimes B,*}\circ \mathcal{T}_{A,*}(z)$$

and hence we obtain the result in the odd case. □

We are known in position to state the main result of this section. If A a filtered C^*-algebra admits at every order r a decomposition into an r-controlled nuclear Mayer–Vietoris pair such that for every pieces involved in the decomposition, the quantitative Künneth formula holds, then the C^*-algebra A satisfies the quantitative Künneth formula.

Theorem 4.12. *Let A be a filtered C^*-algebra. Assume there exist positive numbers c and λ_0 with $\lambda_0 \geqslant 1$ that satisfies the following: for any positive number r, there exists an r-controlled nuclear Mayer–Vietoris pair $(\Delta_1,\Delta_2,A_{\Delta_1},A_{\Delta_2})$ with coercitivity c such that A_{Δ_1}, A_{Δ_2} and $A_{\Delta_1}\cap A_{\Delta_2}$ satisfies the quantitative Künneth formula with rescaling λ_0.*

Then A satisfies the quantitative Künneth formula with rescaling λ_1 for a positive number λ_1 with $\lambda_1 \geqslant 1$ depending only on c and λ_0.

The proof of this theorem will be given in next subsection.

4.3. Proof of Theorem 4.12

The idea of the proof of Theorem 4.12 is to use the controlled exactness persistence properties stated in Corollary 3.6 and Lemma 3.8 to carry out a controlled five lemma argument. This requires some preliminary work in order to formulate the quantitative

Künneth formula in terms of an (even degree) controlled morphism between quantitative K-theory groups.

4.3.1. Preliminaries

We denote by $SB = C_0((0,1), B)$ the suspension algebra of a C^*-algebra B. Let A be a filtered C^*-algebra, let B be a C^*-algebra and let $[\partial]$ be the invertible element in $KK_1(\mathbb{C}, C(0,1))$ that implements the boundary of the Bott extension, $0 \to C_0(0,1) \to C_0[0,1) \stackrel{ev_0}{\to} \mathbb{C} \to 0$. Applying Lemma 4.2 to $z = Id_{\mathbb{C}}$ and $z' = \tau_B([\partial])$, we see that we only have to consider the odd case

$$\Omega_{A,B,*} : \mathcal{K}_1(A)\otimes K_0(B) \oplus \mathcal{K}_0(A)\otimes K_1(B) \longrightarrow \mathcal{K}_1(A\otimes B).$$

Let $\mathbb{T}_2 = \{(z_1, z_2) \text{ such that } |z_1| = |z_2| = 1\}$ be the two torus. Let us view $(0,1)^2$ as an open subset of $\mathbb{T}^2$ via the inclusion map $(0,1)^2 \hookrightarrow \mathbb{T}^2$; $(s,t) \mapsto (e^{2\imath\pi s}, e^{2\imath\pi t})$.

Lemma 4.13. *Possibly rescaling the control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$, then for any filtered C^*-algebra A, the filtered and semi-split extension of filtered C^*-algebras*

$$0 \longrightarrow S^2A \stackrel{j_A}{\longrightarrow} C(\mathbb{T}_2, A) \stackrel{q_A}{\longrightarrow} C(\mathbb{T}_2 \setminus (0,1)^2, A) \longrightarrow 0 \tag{10}$$

has a vanishing controlled boundary map

$$\mathcal{D}_{S^2A, C(\mathbb{T}_2,A)} : \mathcal{K}_*(C(\mathbb{T}_2 \setminus (0,1)^2, A)) \longrightarrow \mathcal{K}_{*+1}(S^2A).$$

Proof. By using controlled Bott periodicity [9, Lemma 4.6] and in view of [9, Proposition 3.19], we only need to check that the result holds for

$$\mathcal{D}_{S^2A, C(\mathbb{T}_2,A)} : \mathcal{K}_1(C(\mathbb{T}_2 \setminus (0,1)^2, A)) \longrightarrow \mathcal{K}_0(S^2A).$$

But

$$C(\mathbb{T}_2 \setminus (0,1)^2, A) \cong \{(f_1, f_2) \in C(\mathbb{T}, A) \oplus \mathbb{C}(\mathbb{T}, A) \text{ such that } f_1(1) = f_2(1)\}. \tag{11}$$

Let (f_1, f_2) be an ε-r-unitary in $C(\mathbb{T}_2 \setminus (0,1)^2, A)$ and define then

$$u_{f_1,f_2} : \mathbb{T}^2 \longrightarrow A;\ (z_1, z_2) \mapsto f_1(z_1) f_1^*(1) f_2(z_2)$$

Then u_{f_1,f_2} is a 9ε-$3r$-unitary in $C(\mathbb{T}_2, A)$. Moreover, under the identification of equation (11), we have

$$q_A(u_{f_1,f_2}) = (f_1 f_1^*(1) f_1(1), f_1(1) f_1^*(1) f_2)$$

and hence

$$\|q_A(u_{f_1,f_2}) - (f_1, f_2)\| < 3\varepsilon.$$

We deduce that u_{f_1,f_2} is an almost lift for (f_1, f_2) in the extension of Equation (10). Hence the result holds by construction of the controlled boundary map in the odd case. □

In consequence, by using the controlled six term exact sequence (Theorem 1.23) associated to the semi-split and filtered extension of equation (10), then if we set

$$\ker q_{A,*} \stackrel{\text{def}}{=\!=} (\ker q^{\varepsilon,r}_{A,*} : K^{\varepsilon,r}_*(C(\mathbb{T}_2, A)) \longrightarrow K^{\varepsilon,r}_*(C(\mathbb{T}_2 \setminus (0,1)^2, A))),$$

we obtain the following corollary.

Corollary 4.14. *There exists a controlled pair (λ, h) such that for any filtered C^*-algebra A, then*

$$j_{A,*} : \mathcal{K}_*(S^2A) \longrightarrow \ker q_{A,*}$$

is a (λ, h)-controlled isomorphism.

Notice that, by construction of the controlled boundary map associated to controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$, we have that

$$q_{A_{\Delta_1} \cap A_{\Delta_2} \otimes B,*} \circ \mathcal{D}_{C(\mathbb{T}^2, A_{\Delta_1}), C(\mathbb{T}^2, A_{\Delta_2}),*} = \mathcal{D}_{C_0(\mathbb{T}^2 \setminus]0,1[^2, A_{\Delta_1}), C_0(\mathbb{T}^2 \setminus]0,1[^2, A_{\Delta_2}),*} \circ q_{A \otimes B,*}. \tag{12}$$

Let us define with notation of Corollary 4.14 the quantitative object $\mathcal{O}'_*(A, B) = (O'^{\varepsilon,r}_*(A, B))$ as

$$\mathcal{O}'_*(A, B) = \mathcal{O}'_0(A, B) \oplus \mathcal{O}'_1(A, B) \stackrel{\text{def}}{=\!=} \ker q_{A \otimes B,*}.$$

Set $z = [\partial] \otimes \tau_{C_0(0,1)}([\partial])$ and define then the $(\alpha^2_{\mathcal{T}}, h_{\mathcal{T}} * h_{\mathcal{T}})$-controlled morphism

$$j_{A \otimes B,*} \circ \mathcal{T}_{A \otimes B}(z) \circ \Omega_{A,B,*} : \mathcal{K}_*(A) \otimes K_0(B) \oplus \mathcal{K}_{*+1}(A) \otimes K_1(B) \longrightarrow \mathcal{O}'_*(A, B).$$

Since z is an invertible element in $KK_0(\mathbb{C}, \mathbb{C}_0((0,1)^2))$ and hence, according to Theorem 1.27 and to Corollary 4.14,

- if $\Omega_{A,B,*}$ is a quantitative isomorphism with rescaling λ_0 then there exists a positive number λ_1 depending only on λ_0 with $\lambda_1 \geqslant 1$ such that $j_{A \otimes B,*} \circ \mathcal{T}_{A \otimes B}(z) \circ \Omega_{A,B,*}$ is a quantitative isomorphism with rescaling λ_1;
- if $j_{A \otimes B,*} \circ \mathcal{T}_{A \otimes B}(z) \circ \Omega_{A,B,*}$ is a quantitative isomorphism with rescaling λ_1, then there exists a positive number λ_0 depending only on λ_1 with $\lambda_0 \geqslant 1$ such that $\Omega_{A,B,*}$ is a quantitative isomorphism with rescaling λ_0.

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According to Lemma 4.2, we get that

$$\mathcal{T}_A(z)\circ\Omega_{A,B,*}\overset{(\lambda,h)}{\sim}\Omega_{SA,SB,*}\circ((\mathcal{T}_A([\partial])\otimes\tau_B([\partial])) \tag{13}$$

for some control pair (λ,k) that does not depend on A or on B. Let us define the quantitative object $\mathcal{O}_*(A,B)=(O_*^{\varepsilon,r}(A,B))$ as

$$\mathcal{O}_*(A,B)\overset{\text{def}}{=\!=}\mathcal{K}_*(A)\otimes K_0(B)\oplus\mathcal{K}_*(SA)\otimes K_0(SB)$$

and denote for positive numbers ε, ε',r and r' with $\varepsilon\leqslant\varepsilon'<1/4$ and $r\leqslant r'$ respectively by $\iota_{\mathcal{O},*}^{\varepsilon,\varepsilon',r,r'}:O_*^{\varepsilon,r}(A,B)\longrightarrow O_*^{\varepsilon',r'}(A,B)$ and $\iota_{\mathcal{O}',*}^{\varepsilon,\varepsilon',r,r'}:O_*'^{\,\varepsilon,r}(A,B)\longrightarrow O_*'^{\,\varepsilon',r'}(A,B)$ the structure morphisms corresponding to $\mathcal{O}_*(A,B)$ and $\mathcal{O}'_*(A,B)$. For $(\alpha_{\mathcal{F}},k_{\mathcal{F}})=(\lambda,h)$, let us define the $(\alpha_{\mathcal{F}},k_{\mathcal{F}})$-controlled morphism

$$\mathcal{F}_{A,B,*}=(F_{A,B,*}^{\varepsilon,r}):\mathcal{O}_*(A,B)\longrightarrow\mathcal{O}'_*(A,B)$$

by

$$F_{A,B,*}^{\varepsilon,r}(x\oplus x')=j_{A\otimes B,*}^{\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\varepsilon}r}\circ\iota_{\mathcal{O}',*}^{-,-}\circ\tau_A^{\alpha_{\mathcal{T}}\varepsilon,k_{\mathcal{T},\varepsilon}r}(z)\circ\omega_{A,B,*}^{\varepsilon,r}(x)$$
$$+\iota_{\mathcal{O}',*}^{-,-}\circ j_{A\otimes B,*}^{\alpha_{\mathcal{T}}\varepsilon,k_{\mathcal{T},\varepsilon}r}\circ\omega_{A,B,*}^{\varepsilon,r}(x').$$

Notice that $\mathcal{F}_{\bullet,B,*}$ is obviously natural with respect to filtered morphisms. Since $[\partial]$ is invertible in $KK_1(\mathbb{C},C(0,1))$ and according to equation (13) and Theorem 1.27, Theorem 4.12 is equivalent to the following statement:

Let A be a filtered C^*-algebra. Assume there exist positive numbers c and λ_0 with $\lambda_0\geqslant 1$ that satisfies the following: for any positive number r, there exists an r-controlled nuclear Mayer–Vietoris pair $(\Delta_1,\Delta_2,A_{\Delta_1},A_{\Delta_2})$ with coercitivity c such that for any C^*-algebra B with $K_*(B)$ free abelian, then $\mathcal{F}_{A_{\Delta_1},B,*}$, $\mathcal{F}_{A_{\Delta_2},B,*}$ and $\mathcal{F}_{A_{\Delta_1}\cap A_{\Delta_2},B,*}$ are quantitative isomorphisms with rescaling λ_0. Then for any C^*-algebra B with $K_*(B)$ free abelian, the controlled morphism $\mathcal{F}_{A,B,*}$ is a quantitative isomorphism with coercitivity λ_1 for a positive number λ_1 with $\lambda_1\geqslant 1$ depending only on λ_0 and c.

Notice that, by controlled Bott periodicity, we only need to consider the odd case.

4.3.2. Notations

Let us introduce some notations that we will use throughout the proof. Let v be a unitary in a unital C^*-algebra B. Define the unitary R_v in $M_2(B)$

$$R_v:[0,1]\longrightarrow U_2(B);\ t\mapsto\begin{pmatrix}v&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}\cos\frac{t\pi}{2}&\sin\frac{t\pi}{2}\\-\sin\frac{t\pi}{2}&\cos\frac{t\pi}{2}\end{pmatrix}\cdot\begin{pmatrix}v^*&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}\cos\frac{t\pi}{2}&-\sin\frac{t\pi}{2}\\\sin\frac{t\pi}{2}&\cos\frac{t\pi}{2}\end{pmatrix},$$

and the projection of $M_2(B)$

$$P_v:[0,1]\longrightarrow M_2(B);\ t\mapsto R_v(t)\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}R_v^*(t).$$

Define also

$$P_{\mathrm{Bott}}:[0,1]\times[0,1]\to M_2(\mathbb{C});\ (s,t)\mapsto R_{e^{2\imath\pi s}}(t)\cdot\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)R^*_{e^{2\imath\pi s}}(t).$$

Then, if we view the 2-sphere $\mathbb{S}^2$ as the one point compactification of $(0,1)^2$, then P_{Bott} is a rank one projection in $M_2(C(\mathbb{S}^2))$ and $[P_{\mathrm{Bott}}]-[1]$ is a generator for $K_0(C_0((0,1)^2))\cong\mathbb{Z}$.

For an r-controlled nuclear Mayer–Vietoris pair $(\Delta_1,\Delta_2,A_{\Delta_1},A_{\Delta_2})$ with respect to a filtered C^*-algebra A, let

$$\begin{aligned}
\jmath_{\Delta_1,\mathcal{O},*}&=(\jmath^{\varepsilon,r}_{\Delta_1,\mathcal{O},*}):\mathcal{O}_*(A_{\Delta_1},B)\longrightarrow\mathcal{O}_*(A,B),\\
\jmath_{\Delta_2,\mathcal{O},*}&=(\jmath^{\varepsilon,r}_{\Delta_2,\mathcal{O},*}):\mathcal{O}_*(A_{\Delta_2},B)\longrightarrow\mathcal{O}_*(A,B),\\
\jmath_{\Delta_1,\Delta_2,\mathcal{O},*}&=(\jmath^{\varepsilon,r}_{\Delta_1,\Delta_2,\mathcal{O},*}):\mathcal{O}_*(A_{\Delta_1}\cap A_{\Delta_2},B)\longrightarrow\mathcal{O}_*(A_{\Delta_1},B)
\end{aligned}$$

and

$$\jmath_{\Delta_2,\Delta_1,\mathcal{O},*}=(\jmath^{\varepsilon,r}_{\Delta_2,\Delta_1,\mathcal{O},*}):\mathcal{O}_*(A_{\Delta_1}\cap A_{\Delta_2},B)\longrightarrow\mathcal{O}_*(A_{\Delta_2},B)$$

be the morphisms respectively induced by the inclusions of C^*-algebras $\jmath_{\Delta_1}:A_{\Delta_1}\hookrightarrow A$, $\jmath_{\Delta_2}:A_{\Delta_2}\hookrightarrow A$, $\jmath_{\Delta_1,\Delta_2}:A_{\Delta_1}\cap A_{\Delta_2}\hookrightarrow A_{\Delta_1}$ and $\jmath_{\Delta_2,\Delta_1}:A_{\Delta_1}\cap A_{\Delta_2}\hookrightarrow A_{\Delta_2}$. In the same way, we define

$$\begin{aligned}
\jmath_{\Delta_1,\mathcal{O}',*}&=(\jmath^{\varepsilon,r}_{\Delta_1,\mathcal{O}',*}):\mathcal{O}'_*(A_{\Delta_1},B)\longrightarrow\mathcal{O}'_*(A,B),\\
\jmath_{\Delta_2,\mathcal{O}',*}&=(\jmath^{\varepsilon,r}_{\Delta_2,\mathcal{O}',*}):\mathcal{O}'_*(A_{\Delta_2},B)\longrightarrow\mathcal{O}'_*(A,B),\\
\jmath_{\Delta_1,\Delta_2,\mathcal{O}',*}&=(\jmath^{\varepsilon,r}_{\Delta_1,\Delta_2,\mathcal{O}',*}):\mathcal{O}'_*(A_{\Delta_1}\cap A_{\Delta_2},B)\longrightarrow\mathcal{O}'_*(A_{\Delta_1},B)
\end{aligned}$$

and

$$\jmath_{\Delta_2,\Delta_1,\mathcal{O}',*}=(\jmath^{\varepsilon,r}_{\Delta_2,\Delta_1,\mathcal{O}',*}):\mathcal{O}'_*(A_{\Delta_1}\cap A_{\Delta_2},B)\longrightarrow\mathcal{O}'_*(A_{\Delta_2},B)$$

4.3.3. Computation of $F^{\varepsilon,r}_{A,B,*}$

Let us now compute $F^{\varepsilon,r}_{A,B}(x)$ for A a unital filtered C^*-algebra, B a unital C^*-algebra and x an element in $O^{\varepsilon,r}_*(A,B)$.

Let us consider first the case $x=[u]_{\varepsilon,r}\otimes[p]$ where u is an ε-r-unitary in some $M_n(A)$ and p is a projection in some $M_m(B)$. Let us set $v_{u,p}=u\otimes p+I_n\otimes(I_m-p)$. Then $v_{u,p}$ is an ε-r-unitary in $M_{nm}(A\otimes B)$. According to Remark 4.1, then

$$F^{\varepsilon,r}_{A,B,*}([u]_{\varepsilon,r}\otimes[p])=\jmath^{\alpha_{\mathcal{T}}\varepsilon,h_{k_{\mathcal{T}},\varepsilon}r}_{A,*}\circ\tau^{\varepsilon,r}_A(z)(v_{u,p}).$$

It is well known that $z=[P_{\mathrm{Bott}}]-[P_1]$ (with $P_1=\mathrm{diag}(1,0)$). Let us define for u and u' some ε-r-unitaries in $M_n(A)$ and p a projection in $M_m(B)$

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$$\begin{aligned} W_{u,u',p} : \mathbb{T}_2 &\longrightarrow M_n(A)\otimes M_m(B)\otimes M_4(\mathbb{C}) \\ (e^{2\imath\pi s}, e^{2\imath\pi t}) &\mapsto \begin{pmatrix} v_{u,p\otimes P_{\text{Bott}}(s,t)} & 0 \\ 0 & v_{u',p\otimes P_1} \end{pmatrix}. \end{aligned}$$

Then $W_{u,u',p}$ is an ε-r-unitary in $C(\mathbb{T}_2, M_n(A)\otimes M_m(B)\otimes M_4(\mathbb{C}))$. Moreover, if $u' = u^*$ then we see that $q_{A,B,*}[W_{u,u^*,p}]_{\varepsilon,r} = 0$ in $K_1^{\varepsilon,r}(C(\mathbb{T}_2 \setminus (0,1)^2, A\otimes B))$ and

$$F^{\varepsilon,r}_{A,B,*}([u]_{\varepsilon,r}\otimes[p]) = [W_{u,u^*,p}]_{\alpha_{\mathcal{F}}\varepsilon,h_{\mathcal{F},\varepsilon}r}. \tag{14}$$

Consider now the case $x = [u]_{\varepsilon,r}\otimes([p] - [p(0)])$, where u is an ε-r-unitary in some $M_n(\widetilde{SA})$ and p is a projection in some $M_m(\widetilde{SB})$.

For ε-r-unitaries u and u' in $M_n(\widetilde{SA})$ and a projection p in $M_n(\widetilde{SA})$, set

$$\begin{aligned} W'_{u,u',p} : \mathbb{T}_2 &\longrightarrow M_n(A)\otimes M_m(B)\otimes M_2(\mathbb{C}) \\ (e^{2\imath\pi s}, e^{2\imath\pi t}) &\mapsto \begin{pmatrix} v_{u(s),p(t)} & 0 \\ 0 & v_{u'(s),p(0)} \end{pmatrix}. \end{aligned}$$

Then $W'_{u,u',p}$ is an ε-r-unitary in $C(\mathbb{T}_2, M_n(A)\otimes M_m(B)\otimes M_2(\mathbb{C}))$ and we can easily check that $[W'_{u,u^*,p}]_{\varepsilon,r}$ is in

$$\mathcal{O}'^{\,\varepsilon,r}_1(A,B) = \ker q^{\varepsilon,r}_{A,*} : K_1^{\varepsilon,r}(C(\mathbb{T}_2, A)) \longrightarrow K_1^{\varepsilon,r}(C(\mathbb{T}_2 \setminus (0,1)^2, A))$$

for all positive numbers ε and r with $\varepsilon \in (0, 1/4)$. Moreover, according to Remark 4.1, we have

$$F^{\varepsilon,r}_{A,B,*}([u]_{\varepsilon,r}\otimes([p] - [I_k])) = [W'_{u,u^*,p}]_{\alpha_{\mathcal{F}}\varepsilon,h_{\mathcal{F},\varepsilon}r}. \tag{15}$$

Consider now the case $x = [q, m]_{\varepsilon,r}\otimes[p]$, where q is an ε-r-projection in some $M_n(A)$, m is an integer with $m \leqslant n$ and p is a projection in some $M_k(B)$. Let us set when q is an ε-r-projection in some $M_n(A)$, m is a positive number and p_1 and p_2 are projections in some $M_k(B)$

$$E_{q,m,p_1,p_2} = \begin{pmatrix} q\otimes p_1 + P_m\otimes(I_k - p_1) & 0 \\ 0 & P_{n-m}\otimes(I_k - p_2) + (I_n - q)\otimes p_2 \end{pmatrix}$$

with $P_m = \operatorname{diag}(I_m, 0)$ in $M_n(\mathbb{C}) \subseteq M_n(A)$. Then E_{q,m,p_1,p_2} is an ε-r-projection in $M_{2kn}(A\otimes B)$. According to Lemma 4.2, we have that

$$F_{A,B,*}(x) = \Omega_{A,B,*}([q, m]_{\varepsilon,r}\otimes([p]\otimes z)).$$

Since $[p]\otimes z = [p\otimes P_{Bott}] - [p\otimes P_1]$ in $K_0(S^2 B)$, we see according to Remark 4.1 that

$$F^{\varepsilon,r}_{A,B,*}([q, m]_{\varepsilon,r}\otimes[p]) = [E_{q,m,p\otimes P_{Bott},p\otimes P_1}, 2nk]_{\varepsilon,r}. \tag{16}$$

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Consider finally the case of $x = [q', m']_{\varepsilon,r} \otimes ([p'_1] - [p'_2])$, where q' is an ε-r-projection in some $M_n(\widetilde{SA})$, m' is an integer and p'_1 and p'_2 are projections in some $M_k(\widetilde{SB})$, with $p'_1(0) = p'_2(0)$. Then

$$F^{\varepsilon,r}_{A,B,*}([q', m']_{\varepsilon,r} \otimes ([p'_1] - [p'_2])) = [E_{q',m',p'_1,p'_2}, nk]_{\varepsilon,r}. \tag{17}$$

4.3.4. QI-condition

Let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be an r-controlled nuclear Mayer–Vietoris pair with coercivity c such that for any C^*-algebra B with $K_*(B)$ free abelian, then $\mathcal{F}_{A_{\Delta_1},B,*}$, $\mathcal{F}_{A_{\Delta_2},B,*}$ and $\mathcal{F}_{A_{\Delta_1}\cap A_{\Delta_2},B,*}$ are quantitative isomorphisms with rescaling λ_0. Let us check that there exists a positive number λ_1 depending only on λ_0 and c, with $\lambda_1 \geqslant 1$ such that for any positive numbers ε and s with ε in $(0, \frac{1}{4\lambda_1})$ and $s \leqslant \frac{r}{\alpha_{\mathcal{F},\varepsilon}}$, then $F^{\varepsilon,s}_{A,B,*}$ satisfies the QI-condition of Definition 1.19. We can assume without loss of generality that A and B are unital. Moreover, up to replacing B by $B \oplus \mathbb{C}$, we can assume that there exists a system of generators of $K_0(B)$ given by classes of projections. Let us fix such a system of generators for $K_0(B)$ and let us fix also a system of generators for $K_1(B)$. As discussed in Section 4.3.1, we only need to consider the odd case, i.e. show that the control morphism $\mathcal{F}_{A,B,*} : \mathcal{O}_1(A, B) \longrightarrow \mathcal{O}'_1(A, B)$ satisfies the required conditions.

According to Lemma 4.11, there exists a control pair (α, h) depending only on c with $(\alpha_c, k_c) \leqslant (\alpha, h)$ such that

$$\begin{aligned} &\mathcal{F}_{A_{\Delta_1}\cap A_{\Delta_2},B,*} \circ \left(\mathcal{D}_{\Delta_1,\Delta_2,*} \otimes \mathcal{I}d_{K_0(B)} \oplus \mathcal{D}_{S\Delta_1,S\Delta_2,*} \otimes \mathcal{I}d_{K_0(SB)}\right) \\ &\overset{(\alpha,h)}{\sim} \mathcal{D}_{\Delta_1\otimes B,\Delta_2\otimes B,*} \circ \mathcal{F}_{A,B,*} \end{aligned} \tag{18}$$

at order r. For ε in $(0, \frac{1}{4\alpha\lambda_0})$, let $r^{\mathcal{F}}_\varepsilon$ be a positive numbers with $r^{\mathcal{F}}_\varepsilon \geqslant h_\varepsilon r$ such that

$$\iota^{-,\lambda_0\alpha\varepsilon,r^{\mathcal{F}}_\varepsilon}_{\mathcal{O},*}(x) = 0$$

for all x in $K^{\alpha\varepsilon,h_\varepsilon r}_*(A_{\Delta_1} \cap A_{\Delta_2}) \otimes K_*(B)$ such that $F^{\alpha\varepsilon,h_\varepsilon r}_{A_{\Delta_1}\cap A_{\Delta_2},B,*}(x) = 0$.

The following proposition is the analogue of the first steps of the injectivity part of the classical five lemma.

Proposition 4.15. *There exists a control pair (λ, h) depending only on λ_0 and c such that for any positive numbers ε, s and r' with $\varepsilon < \frac{1}{4\lambda}$, $s \leqslant \frac{r}{k_{\mathcal{F},\varepsilon}}$ and $r' \geqslant r^{\mathcal{F}}_\varepsilon$, for any x in $O^{\varepsilon,s}_1(A, B)$ such that $F^{\varepsilon,s}_{A,B}(x) = 0$ in $O'^{\,\alpha_{\mathcal{F}},k_{\mathcal{F},\varepsilon}s}_1(A, B)$, there exist*

- *an element $x^{(1)}$ in $O^{\lambda\varepsilon,h_\varepsilon r'}_1(A_{\Delta_1}, B)$ and an element $x^{(2)}$ in $O^{\lambda\varepsilon,h_\varepsilon r'}_1(A_{\Delta_2}, B)$;*
- *an integer n and an ε-s-unitary W_x in $M_n(C(\mathbb{T}_2, A \otimes B))$;*
- *for $i = 1, 2$, a $\lambda\varepsilon$-$h_\varepsilon r'$-unitary $W^{(i)}_x$ in $M_n(C(\mathbb{T}_2, A \otimes B))$ with $W^{(i)}_x - I_n$ in $M_n(C(\mathbb{T}^2, A_{\Delta_i} \otimes B))$;*

such that

(i) $\iota_{\mathcal{O},1}^{-,\lambda\varepsilon,h_\varepsilon r'}(x) = \jmath_{\Delta_1,\mathcal{O},*}^{\lambda\varepsilon,h_\varepsilon r'}(x^{(1)}) + \jmath_{\Delta_2,\mathcal{O},*}^{\lambda\varepsilon,h_\varepsilon r'}(x^{(2)})$;

(ii) $[W_x]_{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}r} = 0$ *in* ${O'_1}^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}r}(A,B)$;

(iii) $[W_x^{(i)}]_{\lambda\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\lambda_\varepsilon}h_\varepsilon r'}$ *is in* ${O'_1}^{\lambda\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\lambda_\varepsilon}h_\varepsilon r'}(A_{\Delta_i},B)$ *and* $F_{A_{\Delta_i},B,*}^{\lambda\varepsilon,h_\varepsilon r'}(x^{(i)}) = [W_x^{(i)}]_{\lambda\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\lambda_\varepsilon}h_\varepsilon r'}$ *for* $i = 1,2$;

(iv) $\|W_x - W_x^{(1)}W_x^{(2)}\| < \lambda\varepsilon$.

Proof. The proof of the proposition is quite long so we split it in several steps.

Step 1: Let x be an element in ${O'_1}^{\varepsilon,s}(A,B)$. Then there exist integers l and l' and

- for $i = 1,\dots l$
  - an ε-s-unitary u_i in some $M_{n_i}(A)$;
  - a projection p_i in some $M_{m_i}(B)$ with $[p_i]$ in the prescribed system of generators for $K_0(B)$;
- for $i = 1,\dots l'$
  - an ε-s-unitary u'_i in some $M_{n'_i}(\widetilde{SA})$;
  - a projection p'_i in some $M_{m'_i}(\widetilde{SB})$ with $[p'_i] - [p'_i(0)]$ in the prescribed system of generators for $K_1(SB)$;

such that

$$x = \sum_{i=1}^{l}[u_i]_{\varepsilon,s}\otimes[p_i] + \sum_{i=1}^{l'}[u'_i]_{\varepsilon,s}\otimes([p'_i] - [p'_i(0)]).$$

Assume that $F_{A,B,*}^{\varepsilon,s}(x) = 0$ in ${O'_1}^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}s}(A,B)$. Using Morita equivalence, and up to replacing A by $M_n(A)$ and B by $M_m(B)$ with $n = n_1 + \dots n_l + n'_1 + \dots n'_{l'}$ and $m = m_1 + \dots m_l + m'_1 + \dots m'_{l'}$, we can assume that $n_i = m_i = 1$ for $i = 1,\dots,l$ and $n'_i = m'_i = 1$ for $i = 1,\dots,l'$. Using Lemma 1.8, we can moreover assume without loss of generality that $u'_i(0) = u'_i(1) = I_{k_i}$ for $i = 1,\dots,l'$. Let us set

$$W_x = \begin{pmatrix} W_{u_1,u_1^*,p_1} & & & & & \\ & \ddots & & & & \\ & & W_{u_l,u_l^*,p_l} & & & \\ & & & W'_{u'_1,{u'_1}^*,p'_1} & & \\ & & & & \ddots & \\ & & & & & W'_{u'_{l'},{u'_{l'}}^*,p'_{l'}} \end{pmatrix}. \tag{19}$$

Then W_x is an ε-s-unitary in $M_{2(2l+l')}(C(\mathbb{T}_2, A\otimes B))$ and $[W_x]_{\varepsilon,s}$ is in ${O'_1}^{\varepsilon,s}(A,B)$. In view of equations (14) and (15), we see that $[W_x]_{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}s} = F_{A,B,*}^{\varepsilon,s}(x) = 0$ in ${O'_1}^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}s}(A,B) \subseteq K_1^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}s}(C(\mathbb{T}_2, A\otimes B))$.

Step 2: Let (α, h) be the control pair of equation (18), then we have

$$F^{\alpha\varepsilon,h_\varepsilon s}_{A_{\Delta_1}\cap A_{\Delta_2},B,*} \circ \iota^{-,-}_{\mathcal{O},*} \circ \left(\partial^{\varepsilon,s}_{\Delta_1,\Delta_2,*}\otimes \mathcal{I}d_{K_0(B)}\right) \oplus \partial^{\varepsilon,s}_{S\Delta_1,S\Delta_2,*}\otimes \mathcal{I}d_{K_0(SB)}\Big)(x) = 0. \qquad (20)$$

Since $[p_1], \dots, [p_l]$ belong to the prescribed system of generator for $K_0(B)$ and $[p'_1] - [p'_1(0)], \dots, [p'_{l'}] - [p'_{l'}(0)]$ belong to the prescribed system of generator for $K_1(B)$, we deduce from equation (20) and from the choice of r' that

$$\iota^{-,\lambda_0\alpha\varepsilon,r'}_* \circ \partial^{\varepsilon,s}_{\Delta_1,\Delta_2,*}[u_i]_{\varepsilon,s} = 0$$

for $i = 1, \dots, l$ and

$$\iota^{-,\lambda_0\alpha\varepsilon,r'}_* \circ \partial^{\varepsilon,s}_{S\Delta_1,S\Delta_2,*}[u'_i]_{\varepsilon,s} = 0$$

for $i = 1, \dots, l'$. According to Lemma 3.5 and in view of the definition of $\partial^{\varepsilon,s}_{\Delta_1,\Delta_2,*}$, then for a control pair (λ, k) depending only on c, there exists for $i = 1, \dots, n$ and up to replacing u_i by some $\mathrm{diag}(u_i, I_{n_i-1})$ for some integer n_i two $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries $v^{(1)}_i$ and $v^{(2)}_i$ respectively in $M_{n_i}(\widetilde{A_{\Delta_1}})$ and $M_{n_i}(\widetilde{A_{\Delta_2}})$, with $v^{(1)}_i - I_{n_i}$ and $v^{(2)}_i - I_{n_i}$ respectively in $M_{n_i}(A_{\Delta_1})$ and $M_{n_i}(A_{\Delta_2})$ and such that

$$\|u_i - v^{(1)}_i v^{(2)}_i\| < \lambda\lambda_0\alpha_c\varepsilon.$$

Since we also have

$$\iota^{-,\lambda_0\alpha\varepsilon,r'}_* \circ \partial^{\varepsilon,r}_{\Delta_1,\Delta_2}[u^*_i]_{\varepsilon,s} = 0,$$

according to Proposition 2.21 and up to rescaling (λ, k), we can assume that there exists two $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries $v'^{(1)}_i$ and $v'^{(2)}_i$ respectively in $M_{n_i}(\widetilde{A_{\Delta_1}})$ and $M_{n_i}(\widetilde{A_{\Delta_2}})$ with $v'^{(1)}_i - I_{n_i}$ and $v'^{(2)}_i - I_{n_i}$ respectively in $M_{n_i}(A_{\Delta_1})$ and $M_{n_i}(A_{\Delta_2})$ and such that

$$\|u^*_i - v'^{(1)}_i v'^{(2)}_i\| < \lambda\lambda_0\alpha\varepsilon$$

and $v'^{(j)}_i$ is homotopic to $v^{(j)*}_i$ as an $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitary in $M_{n_i}(\widetilde{A_{\Delta_i}})$ for $j = 1, 2$.

In the same way, up to replacing u'_i by some $\mathrm{diag}(u'_i, I_{n'_i-1})$ for some integer n'_i, there exists two $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries $w_i{}^{(1)}$ and $w'^{(1)}_i$ in $M_{n'_i}(\widetilde{SA_{\Delta_1}})$ and two $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries $w_i{}^{(2)}$ and $w'^{(2)}_i$ in $M_{n'_i}(\widetilde{SA_{\Delta_2}})$ such that

$$\|u'_i - w_i{}^{(1)} w_i{}^{(2)}\| < \lambda\lambda_0\alpha\varepsilon,$$
$$\|u'^*_i - w'^{(1)}_i w'^{(2)}_i\| < \lambda\lambda_0\alpha\varepsilon,$$

and $w'^{(j)}_i$ is homotopic to $w^{(j)*}_i$ as an $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitary in $M_{n'_i}(\widetilde{SA_{\Delta_i}})$ for $j = 1, 2$ and $w_i{}^{(1)}(0) = w_i{}^{(2)}(0) = w'^{(1)}_i(0) = w'^{(2)}_i(0) = I_{n'_i}$ for $i = 1, \dots, l'$. In particular, $W_{v_i^{(j)},v_i^{(j)*}}$ and $W_{v_i^{(j)},v_i'^{(j)}}$ for $j=1,2$ and $i=1,\dots,l$ are homotopic $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries in $M_{4n_i}(\widetilde{A_{\Delta_j}})$ and $W'_{w_i^{(j)},w_i^{(j)*}}$ and $W_{w_i^{(j)},w_i'^{(j)}}$ for $j=1,2$ and $i=1,\dots,l'$ are homotopic $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries in $M_{2n_i'}(\widetilde{A_{\Delta_j}})$.

Step 3: According to Lemmas 1.2 and 1.3 and up to replacing λ by 12λ, we have that

$$\begin{aligned}\iota_{\mathcal{O},*}^{-,\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}(x) =& \jmath_{\Delta_1,*}^{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes Id_{K_0(B)}(\sum_{i=1}^{l}[v_i^{(1)}]_{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes[p_i])\\ &+\jmath_{\Delta_2,*}^{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes Id_{K_0(B)}(\sum_{i=1}^{l}[v_i^{(2)}]_{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes[p_i])\\ &+\jmath_{S\Delta_1,*}^{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes Id_{K_0(SB)}(\sum_{i=1}^{l'}[w_i^{(1)}]_{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes([p_i']-[p_i'(0)]))\\ &+\jmath_{S\Delta_2,*}^{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes Id_{K_0(SB)}(\sum_{i=1}^{l'}[w_i^{(2)}]_{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes([p_i']-[p_i'(0)]))\end{aligned} \tag{21}$$

Let us set for $j=1,2$

$$\begin{aligned}x^{(j)} =& \jmath_{\Delta_1,*}^{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes Id_{K_0(B)}\left(\sum_{i=1}^{l}[v_i^{(j)}]_{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes[p_i]\right)\\ &+\jmath_{S\Delta_1,*}^{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes Id_{K_0(SB)}\left(\sum_{i=1}^{l'}[w_i^{(j)}]_{\lambda\lambda_0\alpha\varepsilon,2k_{\lambda_0\alpha\varepsilon}r'}\otimes([p_i']-[p_i'(0)]\right)\end{aligned}$$

and

$$W_x^{(j)} = \begin{pmatrix} W_{v_1^{(j)},v_1'^{(j)},p_1} &&&&& \\ & \ddots &&&& \\ && W_{v_l^{(j)},v_l'^{(j)},p_l} &&& \\ &&& W'_{w_1^{(j)},w_1'^{(j)},p_1'} && \\ &&&& \ddots & \\ &&&&& W'_{w_{l'}^{(j)},w_{l'}'^{(j)},p_{l'}'} \end{pmatrix}.$$

Since $W_x^{(j)}$ and

$$\begin{pmatrix} W_{v_1^{(j)},v_1{}^{(j)*},p_1} &&&&& \\ & \ddots &&&& \\ && W_{v_l^{(j)},v_l{}^{(j)*},p_l} &&& \\ &&& W'_{w_1^{(j)},w_1{}^{(j)*},p'_1} && \\ &&&& \ddots & \\ &&&&& W'_{w_{l'}^{(j)},w_{l'}{}^{(j)*},p'_{l'}} \end{pmatrix},$$

are homotopic as $\lambda\lambda_0\alpha\varepsilon$-$k_{\lambda_0\alpha\varepsilon}r'$-unitaries in $M_N(\widetilde{A_{\Delta_j}})$, with $N = n_1 + \dots n_l + 2(n'_1 + \dots + n'_{l'})$, we deduce that

$$q_{A_{\Delta_j}}^{\lambda\lambda_0\alpha\varepsilon,k_{\lambda_0\alpha\varepsilon}r'}[W_x^{(j)}]_{\lambda\lambda_0\alpha\varepsilon,k_{\lambda_0\alpha\varepsilon}r'} = 0.$$

Hence $x^{(1)}$, $x^{(2)}$, W_x, $W_x^{(1)}$ and $W_x^{(2)}$ satisfy the required condition for some suitable control pair. □

End of the proof of the QI-statement. To prove the QI-statement, we follow the steps of the proof of the injectivity part of the five lemma.

Step 1: Let x be an element in $O_1^{\varepsilon,s}(A,B)$ such that $F_{A,B,*}^{\varepsilon,s}(x) = 0$. Let r' be a positive number such that $r' \geqslant r_\varepsilon^{\mathcal{F}}$. With notations of Proposition 4.15, applying Proposition 2.11 to W_x, up to rescaling (λ, h) and to replacing $W_x^{(1)}$ and $W_x^{(2)}$ respectively by $\operatorname{diag}(W_x^{(1)}, I_j)$ and $\operatorname{diag}(W_x^{(2)}, I_j)$ for some integer j, there exist for any positive number ε in $(0, \frac{1}{4\lambda})$ two $\lambda\varepsilon$-$h_\varepsilon r'$-unitaries W'_1 and W'_2 in some $M_n(C(\mathbb{T}_2, A\otimes B))$ such that

- $W'_i - I_n$ is an element in the matrix algebra $M_n(C(\mathbb{T}_2, A_{\Delta_i}\otimes B))$ for $i = 1, 2$;
- for $i = 1, 2$, there exists a homotopy $(W'_{i,t})_{t\in[0,1]}$ of $\lambda\varepsilon$-$h_\varepsilon r'$-unitaries between I_n and W'_i such that $W'_{i,t} - I_k \in M_n(C(\mathbb{T}_2, A_{\Delta_i}\otimes B))$ for all t in $[0, 1]$.
- $\|W_x^{(1)}W_x^{(2)} - W'_1W'_2\| < \lambda\varepsilon$.

Up to replacing λ by 5λ, we can assume indeed that $\|W_1'^*W_x^{(1)} - W'_2W_x^{(2)*}\| < \lambda\varepsilon$. If we apply the CIA property to $W_1'^*W_x^{(1)} - I_n$ and $W'_2W_x^{(2)*} - I_n$, we get that there exists V' in $M_n(C(\mathbb{T}_2, A\otimes B)_{h_\varepsilon r'})$ such that

- $\|W_1'^*W_x^{(1)} - V'\| < c\lambda\varepsilon$;
- $\|W'_2W_x^{(2)*} - V'\| < c\lambda\varepsilon$;
- $V' - I_n$ lies in $M_n(C(\mathbb{T}_2, (A_{\Delta_1} \cap A_{\Delta_2})\otimes B))$.

In particular, in view of Lemma 1.2, V' is a $4(c+3)\lambda\varepsilon$-$2h_\varepsilon r'$-unitary in $M_n(\widetilde{C}(\mathbb{T}_2, (A_{\Delta_1} \cap A_{\Delta_2})\otimes B)))$ homotopic to $W_1'^*W_x^{(1)}$ (resp. to $W'_2W_x^{(2)*}$) as a $4(c+3)\lambda\varepsilon$-$2h_\varepsilon r'$-unitary in $M_n(\widetilde{C}(\mathbb{T}_2, A_{\Delta_1}\otimes B))$ (resp. in $M_n(\widetilde{C}(\mathbb{T}_2, A_{\Delta_2}\otimes B))$).

Step 2: We construct now out of V' an almost unitary whose class is (up to rescaling) in $O_1'^{4(c+3)\lambda\varepsilon,4h_\varepsilon r'}(A_{\Delta_1} \cap A_{\Delta_2}, B)$ and which have the same image as $[V']_{4(c+3)\lambda\varepsilon,2h_\varepsilon r'}$

in $K_1^{4(c+3)\lambda\varepsilon,4h_\varepsilon r'}(C(\mathbb{T}_2,A_{\Delta_1}\otimes B))$ (resp. $K_1^{4(c+3)\lambda\varepsilon,4h_\varepsilon r'}(C(\mathbb{T}_2,A_{\Delta_2}\otimes B))$ under the map $\jmath_{C(\mathbb{T}_2,\Delta_1\otimes B),C(\mathbb{T}_2,\Delta_2\otimes B),*}^{4(c+3)\lambda\varepsilon,2h_\varepsilon r'}$ (resp. $\jmath_{C(\mathbb{T}_2,\Delta_2\otimes B),C(\mathbb{T}_2,\Delta_1\otimes B),*}^{4(c+3)\lambda\varepsilon,2h_\varepsilon r'}$). Let us define

$$\begin{aligned} V_x:\mathbb{T}_2:&\longrightarrow M_{3n}(A\otimes B)\\ (z_1,z_2)\ &\mapsto \begin{pmatrix} V'^*(1,1)V'(z_1,z_2) & & \\ & V'^*(z_1,1)V'(1,1) & \\ & & V'^*(1,z_2)V'(1,1)\end{pmatrix}. \end{aligned}$$

If we set $\lambda'=12(c+3)\lambda$, then

- V_x is a $\lambda'\varepsilon$-$4h_\varepsilon r'$-unitary in $M_n(C(\mathbb{T}_2,A\otimes B))$;
- V_x-I_{3n} is in $M_{3n}(C(\mathbb{T}_2,(A_{\Delta_1}\cap A_{\Delta_2})\otimes B))$;
- $[V_x]_{\lambda'\varepsilon,4h_\varepsilon r'}$ lies in $O_1'^{\lambda'\varepsilon,4h_\varepsilon r'}(A_{\Delta_1}\cap A_{\Delta_2},B)$.

Then we have

$$\jmath_{\Delta_1,\Delta_2,\mathcal{O}',*}^{\lambda'\varepsilon,4h_\varepsilon r'}([V_x]_{\lambda'\varepsilon,4h_\varepsilon r'})=[W_1'^*W_x^{(1)}]_{\lambda'\varepsilon,4h_\varepsilon r'}=[W_x^{(1)}]_{\lambda'\varepsilon,4h_\varepsilon r'},\tag{22}$$

where the second equality holds because W_1' is connected to I_n as a $\lambda'\varepsilon$-$4h_\varepsilon r'$-unitary of $M_n(\widetilde{SA_{\Delta_1}})$. In the same way, we have

$$\jmath_{\Delta_2,\Delta_1,\mathcal{O}',*}^{\lambda'\varepsilon,4h_\varepsilon r}([V_x]_{\lambda'\varepsilon,4h_\varepsilon r'})=-[W_x^{(2)}]_{\lambda'\varepsilon,4h_\varepsilon r'}.\tag{23}$$

Step 3: Let r'' be a positive integer with $k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''\geqslant 4k_\varepsilon r'$ such that for any z in $O_*'^{\lambda'\varepsilon,4k_\varepsilon r'}(A_{\Delta_1}\cap A_{\Delta_2},B)$, there exists y in $O_*^{\lambda_0\lambda'\varepsilon,r''}(A_{\Delta_1}\cap A_{\Delta_2},B)$ such that

$$\iota_{\mathcal{O}',*}^{-,\lambda_0\lambda'\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}(z)=F_{A_{\Delta_1}\cap A_{\Delta_2},B,*}^{\lambda_0\lambda'\varepsilon,r''}(y).$$

Then there exists x' in $O_1^{\lambda_0\lambda',r''}(A_{\Delta_1}\cap A_{\Delta_2},B)$ such that

$$[V_x]_{\lambda_0\lambda'\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}=F_{A_{\Delta_1}\cap A_{\Delta_2},B,*}^{\lambda_0\lambda'\varepsilon,r''}(x').$$

Hence we have

$$\begin{aligned} F_{A_{\Delta_1},B,*}^{\lambda_0\lambda'\varepsilon,r''}\circ\jmath_{\Delta_1,\Delta_2,\mathcal{O},*}^{\lambda_0\lambda'\varepsilon,r''}(x')&=\jmath_{\Delta_1,\Delta_2,\mathcal{O}',*}^{\lambda_0\lambda'\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}([V_x]_{\lambda_0\lambda'\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''})\\ &=[W_x^{(1)}]_{\lambda_0\lambda'\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}\\ &=F_{A_{\Delta_1},B,*}^{\lambda_0\lambda'\varepsilon,r''}\circ\iota_{\mathcal{O},*}^{-,\lambda_0\lambda'\varepsilon,r''}(x^{(1)}) \end{aligned}\tag{24}$$

where the first equality holds by naturality of $\mathcal{F}_{\bullet,B,*}$, the second equality holds in view of equation (22) and the last equality is a consequence of Proposition 4.15. In the same way, using equation (23), we get that

$$F^{\lambda_0\lambda'\varepsilon,r''}_{A_{\Delta_2},B,*}\circ \jmath^{\lambda_0\lambda'\varepsilon,r''}_{\Delta_2,\Delta_1,\mathcal{O},*}(x') = -F^{\lambda_0\lambda'\varepsilon,r''}_{A_{\Delta_2},B,*}\circ \iota^{-,\lambda_0\lambda'\varepsilon,r''}_{\mathcal{O},*}(x^{(2)}). \tag{25}$$

Step 4: Let R be a positive number, with $R \geqslant r''$ such that

$$\iota^{-,\lambda_0^2\lambda''\varepsilon,R}_{\mathcal{O},*}(y) = 0$$

for all y in $O_1^{\lambda_0\lambda'\varepsilon,r''}(A_{\Delta_j},B)$ such that $F^{\lambda_0\lambda'\varepsilon,r''}_{A_{\Delta_j},B,*}(y) = 0$ and $j = 1,2$. In particular, from equations (24) and (25), we deduce

$$\iota^{-,\lambda_0^2\lambda''\varepsilon,R}_{\mathcal{O},*}\circ \jmath^{\lambda_0\lambda''\varepsilon,r''}_{\Delta_1,\Delta_2,\mathcal{O},*}(x') = \iota^{-,\lambda_0^2\lambda'\varepsilon,R}_{\mathcal{O},*}(x^{(1)})$$

and

$$\iota^{-,\lambda_0^2\lambda''\varepsilon,R}_{\mathcal{O},*}\circ \jmath^{\lambda_0\lambda''\varepsilon,r''}_{\Delta_2,\Delta_1,\mathcal{O},*}(x') = -\iota^{-,\lambda_0^2\lambda'\varepsilon,R}_{\mathcal{O},*}(x^{(2)}).$$

Since $\iota^{-,\lambda\varepsilon,h_\varepsilon r'}_{\mathcal{O},*}(x) = x^{(1)} + x^{(2)}$, this establishes the QI-statement.

4.3.5. QS-condition

Let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be an r-controlled nuclear Mayer–Vietoris pair with coercitivity c for A such that for any C^*-algebra B with $K_*(B)$ free abelian, then $\mathcal{F}_{A_{\Delta_1},B,*}$, $\mathcal{F}_{A_{\Delta_2},B,*}$ and $\mathcal{F}_{A_{\Delta_1}\cap A_{\Delta_2},B,*}$ are quantitative isomorphisms with rescaling λ_0. Let us check that $\mathcal{F}_{A,B,*}$ satisfies the QS-condition of Definition 1.19. As for the QI-condition, we can assume without loss of generality that A and B are unital and that there exists a system of generators of $K_0(B)$ given by classes of projections. Let us fix such a system of generator for $K_0(B)$ and fix also a system of generator for $K_0(SB)$. We also only need to consider the odd case, i.e. show that the control morphism $\mathcal{F}_{A,B,*} : \mathcal{O}_1(A,B)\longrightarrow\mathcal{O}'_1(A,B)$ satisfies the required conditions.

Let ε be a positive number with ε in $(0, \frac{1}{4\lambda_0\alpha_c})$ and let $r^{\mathcal{F},(1)}_\varepsilon$ be a positive number with $k_{c,\varepsilon}r \leqslant k_{\mathcal{F},\lambda_0\alpha_c\varepsilon}r^{\mathcal{F},(1)}_\varepsilon$ such that for any y in ${O'}_*^{\alpha_c\varepsilon,k_{c,\varepsilon}r}(A_{\Delta_1}\cap A_{\Delta_2},B)$, there exists an element x in $O_*^{\lambda_0\alpha_c\varepsilon,r^{\mathcal{F}_\varepsilon,(1)}}(A_{\Delta_1}\cap A_{\Delta_2},B)$ such that

$$\iota^{-,\lambda_0\alpha_c\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\alpha_c}r^{\mathcal{F}_\varepsilon,(1)}}_{\mathcal{O}',*}(y) = F^{\lambda_0\alpha_c\varepsilon,r^{\mathcal{F}_\varepsilon,(1)}}_{A_{\Delta_1}\cap A_{\Delta_2},B,*}(x).$$

Let $r^{\mathcal{F},(2)}_\varepsilon$ be a positive number with $r^{\mathcal{F},(1)}_\varepsilon \leqslant r^{\mathcal{F},(2)}_\varepsilon$ such that

$$\iota^{-,\lambda_0^2\alpha_c\varepsilon,r^{\mathcal{F},(2)}_\varepsilon}_{\mathcal{O},*}(x) = 0$$

for all x in $K_*^{\lambda_0\alpha_c\varepsilon,r^{\mathcal{F},(1)}_\varepsilon}(A_{\Delta_1}\cap A_{\Delta_2})\otimes K_*(B)$ such that $F^{\lambda_0\alpha_c\varepsilon,r^{\mathcal{F},(1)}_\varepsilon}_{A_{\Delta_1}\cap A_{\Delta_2},B,*}(x) = 0$. Next proposition can be viewed as the first steps of the surjectivity part of the five lemma.

Proposition 4.16. *There exists a control pair (λ, h) depending only on c and λ_0 with $(\alpha_c, k_c) \leqslant (\lambda, h)$ such that for any positive numbers ε and r' with ε in $(0, \frac{1}{4\lambda})$ and $r_\varepsilon^{\mathcal{F},(2)} \leqslant r'$ the following is satisfied:*

For any element y in ${O'_1}^{\varepsilon,r}(A,B) \subseteq K_1^{\varepsilon,r}(C(\mathbb{T}^2, A \otimes B))$, there exists

- *an element z_y in $\mathcal{O}_1^{\lambda\varepsilon, h_\varepsilon r'}(A,B)$;*
- *a positive integer n_y and two $\lambda'\varepsilon$-$h'_\varepsilon r'$-unitaries W_y and W'_y in $M_{n_y}(C(\mathbb{T}_2, A \otimes B))$;*
- *for $j = 1, 2$ a $\lambda\varepsilon$-$h_\varepsilon r'$-unitary $W_y^{(j)}$ in $M_{2n_y}(C(\mathbb{T}_2, A \otimes B))$ with $W_y^{(j)} - I_{2n_y}$ in $M_{2n_y}(C(\mathbb{T}_2, A_{\Delta_j} \otimes B))$;*
- *a $\lambda\varepsilon$-$h_\varepsilon r'$-projection q_y in $M_{2n_y}(C(\mathbb{T}_2, (A_{\Delta_1} \cap A_{\Delta_2}) \otimes B))$*

such that

- *$[W_y]_{\lambda\alpha_\mathcal{F}\varepsilon, k_{\mathcal{F},\lambda\varepsilon} h_\varepsilon r'}$ is in ${O'_1}^{\lambda\alpha_\mathcal{F}\varepsilon, k_{\mathcal{F},\lambda\varepsilon} r'}(A,B)$ and $F_{A,B,*}^{\lambda\varepsilon, h_\varepsilon r'}(z_y) = [W_y]_{\lambda\alpha_\mathcal{F}\varepsilon, k_{\mathcal{F},\lambda\varepsilon} h_\varepsilon r'}$;*
- *$[q_y, n_y]_{\lambda\varepsilon, h_\varepsilon r'} = \iota_{\mathcal{O}',*}^{\alpha_c\varepsilon, \lambda\varepsilon, k_{c,\varepsilon} r, h'_\varepsilon r'} \circ \partial^{\varepsilon,r}_{C(\mathbb{T}_2, \Delta_1 \otimes B), C(\mathbb{T}_2, \Delta_2 \otimes B)}(y)$.*
- *$\| \operatorname{diag}(W_y, W'_y) - W_y^{(1)} W_y^{(2)} \| < \lambda\varepsilon$;*
- *$\| W_y^{(1)*} \operatorname{diag}(I_{n_y}, 0) W_y^{(1)} - q_y \| < \lambda\varepsilon$.*

Proof. We follow the steps of the beginning of the proof of the surjectivity part of the five lemma.

Step 1: According to equation (12), we see that $y' = \partial^{\varepsilon,r}_{C(\mathbb{T}_2, A_{\Delta_1}), C(\mathbb{T}_2, A_{\Delta_2}),*}(y)$ belongs to ${O'_0}^{\alpha_c\varepsilon, k_{c,\varepsilon} r}(A_{\Delta_1} \cap A_{\Delta_2}, B)$. Set $R = r_\varepsilon^{\mathcal{F},(1)}$. Then there exists an element x in ${O_0}^{\lambda_0\alpha_c\varepsilon, R}(A_{\Delta_1} \cap A_{\Delta_2}, B)$ such that

$$\iota_{\mathcal{O}',*}^{-,\lambda_0\alpha_c\alpha_\mathcal{F}\varepsilon, k_{\mathcal{F},\lambda_0\alpha_c\varepsilon} R}(y') = F^{\lambda_0\alpha_c\varepsilon, R}_{A_{\Delta_1} \cap A_{\Delta_2}, B,*}(x).$$

There exist two integers l and l' and

- for $i = 1, \dots l$
  - an $\lambda_0\alpha_c\varepsilon$-$R$-projection q_i in some $M_{n_i}(\widetilde{A_{\Delta_1} \cap A_{\Delta_2}})$ and an integer k_i;
  - a projection p_i in some $M_{m_i}(B)$ with $[p_i]$ in the prescribed system of generators for $K_0(B)$;
- for $i = 1, \dots l'$
  - an $\lambda_0\alpha_c\varepsilon$-$R$-projection q'_i in some $M_{n'_i}(S\widetilde{A_{\Delta_1} \cap A_{\Delta_2}})$ and an integer k'_i;
  - a projection p'_i in some $M_{m'_i}(\widetilde{SB})$ such that $[p'_i] - [p'_i(0)]$ is in the prescribed system of generators for $K_1(SB)$;

such that

$$x = \sum_{i=1}^{l} [q_i, k_i]_{\lambda_0\alpha_c\varepsilon, R} \otimes [p_i] + \sum_{i=1}^{l'} [q'_i, k'_i]_{\lambda_0\alpha_c\varepsilon, R} \otimes ([p'_i] - [p'_i(0)]).$$

By Morita equivalence, up to replacing B by $M_m(B)$ with $m = m_1 + \dots m_l + m'_1 + \dots m'_{l'}$, we can assume that $m_i = 1$ for $i = 1, \dots, l$ and $m'_i = 1$ for $i = 1, \dots, l'$. Using Lemma 1.7, we can moreover assume without loss of generality that $q_i - \mathrm{diag}(I_{k_i}, 0)$ is in $M_{n_i}(A_{\Delta_1} \cap A_{\Delta_2})$ for $i = 1, \dots, l$ and that $q'_i(0) = q'_i(1) = \mathrm{diag}(I_{k'_i}, 0)$ for $i = 1, \dots, l'$. Set

$$q'_y = \begin{pmatrix} E_{q_1,n_1,p_1\otimes P_{Bott},p_1\otimes P_1} & & & & & \\ & \ddots & & & & \\ & & E_{q_l,n_l,p_l\otimes P_{Bott},p_l\otimes P_1} & & & \\ & & & E_{q'_1,n'_1,p'_1,p'_1(0)} & & \\ & & & & \ddots & \\ & & & & & E_{q'_l,n'_l,p'_l,p'_l(0)} \end{pmatrix}$$

In view of equations (16) and (17),

$$\iota^{-,\lambda_0\alpha_c\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\alpha_c}R}_{\mathcal{O}',*}(y') = [q'_y, n_y]_{\lambda_0\alpha_c\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\alpha_c}R},$$

with $n_y = 2(n_1 + \dots + n_l) + n'_1 + \dots + n'_l$.

Step 2: By naturality of $\mathcal{F}_{\bullet,B,*}$, we obtain

$$F^{\lambda_0\alpha_c\varepsilon,R}_{A_{\Delta_1},B,*} \circ \jmath^{\lambda_0\alpha_c\varepsilon,R}_{\Delta_1,\Delta_2,\mathcal{O},*}(x) = 0$$

in $O'^{\lambda_0\alpha_c\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\alpha_c}R}_0(A_{\Delta_1}, B)$ and

$$F^{\lambda_0\alpha_c\varepsilon,r'}_{A_{\Delta_2},B,*} \circ \jmath^{\lambda_0\alpha_c\varepsilon,R}_{\Delta_2,\Delta_1,\mathcal{O},*}(x) = 0$$

in $O'^{\lambda_0\alpha_c\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\lambda_0\alpha_c}R}_0(A_{\Delta_2}, B)$.

Let r' be a positive number with $r^{\mathcal{F},(2)}_\varepsilon \leqslant r'$. Since $[p_i]$ for $i = 1, \dots, l$ and $[p'_i] - [p'_i(0)]$ for $i = 1, \dots, l'$ are respectively in the prescribed system of generator of $K_0(B)$ and $K_0(SB)$ and by definition of $r^{\mathcal{F},(2)}_\varepsilon$, we deduce that

- $\jmath^{\lambda_0^2\alpha_c\varepsilon,r'}_{\Delta_1,\Delta_2,*}([q_i, k_i]_{\lambda_0^2\alpha_c\varepsilon,r'}) = 0$ in $K_0^{\lambda_0^2\alpha_c\varepsilon,r'}(A_{\Delta_1})$ for $i = 1, \dots, l$;
- $\jmath^{\lambda_0^2\alpha_c\varepsilon,r'}_{S\Delta_1,S\Delta_2,*}([q'_i, k'_i]_{\lambda_0^2\alpha_c\varepsilon,r'}) = 0$ in $K_0^{\lambda_0^2\alpha_c\varepsilon,r'}(SA_{\Delta_1})$ for $i = 1, \dots, l'$;
- $\jmath^{\lambda_0^2\alpha_c\varepsilon,r'}_{\Delta_2,\Delta_1,*}([q_i, k_i]_{\lambda_0^2\alpha_c\varepsilon,r'}) = 0$ in $K_0^{\lambda_0^2\alpha_c\varepsilon,r'}(A_{\Delta_2})$ for $i = 1, \dots, l$;
- $\jmath^{\lambda_0^2\alpha_c\varepsilon,r'}_{S\Delta_2,S\Delta_1,*}([q'_i, k'_i]_{\lambda_0^2\alpha_c\varepsilon,r'}) = 0$ in $K_0^{\lambda_0^2\alpha_c\varepsilon,r'}(SA_{\Delta_2})$ for $i = 1, \dots, l'$.

Let (λ, h) be the control pair of Lemma 3.8. Then for $i = 1, \dots l$ and up to stabilization, we can assume that $n_i = 2k_i$ and that there exists $v_i^{(1)}$ and $v_i^{(2)}$ two $\lambda\lambda_0^2\alpha_c\varepsilon$-$h_{\lambda_0^2\alpha_c\varepsilon}r'$ unitaries in $M_{2n_i}(A)$ and u_i and u'_i two $\lambda\lambda_0^2\alpha_c\varepsilon$-$h_{\lambda_0^2\alpha_c\varepsilon}r'$ unitaries in $M_{n_i}(A)$ such that

- $v_i^{(j)} - I_{n_i}$ is an element in $M_{n_i}(A_{\Delta_j})$ for $j = 1, 2$;
-

$$\|v_i^{(1)*}\,\mathrm{diag}(I_{k_i}, 0)v_i^{(1)} - q_i\| < \lambda\lambda_0^2\alpha_c\varepsilon$$

and

$$\|v_i^{(2)}\operatorname{diag}(I_{k_i},0)v_i^{(2)*}-q_i\|<\lambda\lambda_0^2\alpha_c\varepsilon;$$

- for $j=1,2$, then $v_i^{(j)}$ is connected to I_{n_i} by a homotopy of $\lambda\lambda_0^2\alpha_c\varepsilon$-$h_{\lambda_0^2\alpha_c\varepsilon}r''$-unitaries $(v_{i,t}^{(j)})_{t\in[0,1]}$ in $M_{n_i}(A)$ such that $v_{i,t}^{(j)}-I_{2n_i}$ is in $M_{n_i}(A_{\Delta_j})$ for all t in $[0,1]$;
- $\|\operatorname{diag}(u_i,u_i')-v_i^{(1)}v_i^{(2)}\|<\lambda\lambda_0^2\alpha_c\varepsilon$.

In the same way, for $i=1,\dots l'$ and up to stabilization, we can assume that $n_i'=2k_i'$ and that there exists $w_i^{(1)}$ and $w_i^{(2)}$ two $\lambda\lambda_0^2\alpha_c\varepsilon$-$h_{\lambda_0^2\alpha_c\varepsilon}r'$ unitaries in $M_{n_i'}(\widetilde{SA})$ and u_i'' and u_i''' two $\lambda\lambda_0^2\alpha_c\varepsilon$-$h_{\lambda_0\alpha_c\varepsilon}r'$ unitaries in $M_{n_i'}(\widetilde{SA})$ such that

- $w_i^{(j)}-I_{n_i'}$ is an element in $M_{n_i'}(SA_{\Delta_j})$ for $j=1,2$;
- $$\|w_i^{(1)*}\operatorname{diag}(I_{k_i},0)w_i^{(1)}-q_i'\|<\lambda\lambda_0^2\alpha_c\varepsilon$$

  and

  $$\|w_i^{(2)}\operatorname{diag}(I_{k_i'},0)w_i^{(2)*}-q_i'\|<\lambda\lambda_0^2\alpha_c\varepsilon;$$
- for $j=1,2$, then $w_i^{(j)}$ is connected to $I_{n_i'}$ by a homotopy of $\lambda\lambda_0^2\alpha_c\varepsilon$-$h_{\lambda_0^2\alpha_c\varepsilon}r'$-unitaries $(w_{i,t}^{(j)})_{t\in[0,1]}$ in $M_{n_i'}(\widetilde{SA})$ such that $w_{i,t}^{(j)}-I_{n_i'}$ is in $M_{n_i'}(SA_{\Delta_j})$ for all t in $[0,1]$;
- $u_i''(0)=u_i''(1)=u_i'''(0)=u_i'''(1)=I_{2n_i'}$;
- $\|\operatorname{diag}(u_i',u_i'')-w_i^{(1)}w_i^{(2)}\|<\lambda\lambda_0^2\alpha_c\varepsilon$.

Step 3: Let us set $\lambda'=\lambda\lambda_0^2\alpha_c$ and for any ε in $(0,\frac{1}{4\lambda'})$ set $h'_\varepsilon=h_{\lambda\lambda_0^2\alpha_c\varepsilon}$. Consider the element of $O_1^{\lambda'\varepsilon,h'_\varepsilon r'}(A,B)$

$$z_y=[u_1]_{\lambda'\varepsilon,h'_\varepsilon r'}\otimes[p_1]+\ldots+[u_l]_{\lambda'\varepsilon,h'_\varepsilon r'}\otimes[p_l]+[u_1'']_{\lambda'\varepsilon,h'_\varepsilon r'}\otimes([p_1']-[p_1'(0)])+\ldots \\ +[u_{l'}'']_{\lambda'\varepsilon,h'_\varepsilon r'}\otimes([p_{l'}']-[p_{l'}'(0)]).$$

If we set

$$W_y=\begin{pmatrix} W_{u_1,u_1',p_1} &&&&& \\ & \ddots &&&& \\ && W_{u_l,u_l',p_l} &&& \\ &&& W'_{u_1'',u_1''',p_1'} && \\ &&&& \ddots & \\ &&&&& W'_{u_{l'}'',u_{l'}''',p_{l'}'} \end{pmatrix},$$

then W_y is a $\lambda'\varepsilon$-$h'_\varepsilon r'$-unitary in $M_{2n_y}(C(\mathbb{T}_2, A\otimes B))$. Using Lemmas 1.2 and 1.3 and up to replacing (λ', h') by $(12\lambda', 2h')$, then for $i = 1, \dots l$ (resp. for $i = 1, \dots, l'$) $\mathrm{diag}(u'_i, I_{k_i})$ is homotopic to $\mathrm{diag}(u_i^*, I_{k_i})$ (resp. $\mathrm{diag}(u'''_i, I_{k'_i})$ is homotopic to $\mathrm{diag}(u''^*_i, I_{k'_i}))$ as a $\lambda'\varepsilon$-$h'_\varepsilon r'$-unitary in $M_{n_i}(A)$ (resp. in $M_{n'_i}(\widetilde{SA})$). Hence we deduce that $[W_y]_{\alpha_\mathcal{F}\lambda'\varepsilon, k_{\mathcal{F},\lambda'\varepsilon}h'_\varepsilon r'}$ belongs to ${O'_1}^{\alpha_\mathcal{F}\lambda'\varepsilon, k_{\mathcal{F},\lambda'\varepsilon}h'_\varepsilon r'}(A, B)$ and in view of equations (14) and (15), we see that

$$[W_y]_{\alpha_\mathcal{F}\lambda'\varepsilon, k_{\mathcal{F},\lambda'\varepsilon}h'_\varepsilon r'} = F_{A,B,*}^{\lambda'\varepsilon, h'_\varepsilon r'}(z_y)$$

in ${O'_1}^{\alpha_\mathcal{F}\lambda'\varepsilon, k_{\mathcal{F},\lambda'\varepsilon}h'_\varepsilon r'}(A, B)$. In the same way, if we set

$$W'_y = \begin{pmatrix} W_{u'_1,u_1,p_1} & & & & & \\ & \ddots & & & & \\ & & W_{u'_l,u_l,p_l} & & & \\ & & & W'_{u'''_1,u''_1,p'_1} & & \\ & & & & \ddots & \\ & & & & & W'_{u'''_{l'},u''_{l'},p'_{l'}} \end{pmatrix},$$

then W'_y is also an $\lambda'\varepsilon$-$h'_\varepsilon r'$-unitary in $M_{2n_y}(C(\mathbb{T}_2, A\otimes B))$.

Step 4: For $i = 1, \dots, l$ and $j = 1, 2$, let ${v'_i}^{(j)}$ be the matrix in $M_{n_i}(A)$ obtained from ${v_i}^{(j)}$ by flipping the k_i first and the k_i-last coordinates, and define similarly ${w'_i}^{(j)}$ in $M_{n'_i}(\widetilde{A})$ for $i = 1, \dots, l'$ and $j = 1, 2$. Up to replacing λ' by $2\lambda'$, we have that

- ${v'_i}^{(1)}$ and ${v'_i}^{(2)}$ are $\lambda'\varepsilon$-$h'_\varepsilon r'$ unitaries in $M_{n_i}(A)$;
- $$\|{v'_i}^{(1)*}\,\mathrm{diag}(I_{k_i}, 0){v'_i}^{(1)} - (I_{n_i} - \tilde{q}_i)\| < \lambda'\varepsilon$$

  and

  $$\|{v'_i}^{(2)}\,\mathrm{diag}(I_{n_i}, 0){v'_i}^{(2)*} - (I_{n_i} - \tilde{q}_i)\| < \lambda'\varepsilon.$$
- $\|\,\mathrm{diag}(u'_i, u_i) - {v'_i}^{(1)}{v'_i}^{(2)}\| < \lambda'\varepsilon$,

where $\tilde{q}_i$ is obtained from q_i by flipping the k_i first and the k_i last coordinates. Similarly, for $i = 1, \dots l'$, we have

- ${w'_i}^{(1)}$ and ${w'_i}^{(2)}$ are two $\lambda'\varepsilon$-$h'_\varepsilon r'$ unitaries in $M_{n'_i}(\widetilde{SA})$;
- ${w'_i}^{(j)} - I_{n'_i}$ is an element in $M_{2n'_i}(SA_{\Delta_j})$ for $j = 1, 2$;
- $$\|{w'_i}^{(1)*}\,\mathrm{diag}(I_{k'_i}, 0){w'_i}^{(1)} - (I_{n'_i} - \tilde{q'_i})\| < 2\lambda'\varepsilon$$

and

$$\|{w'_i}^{(2)}\operatorname{diag}(I_{n'_i},0){w'_i}^{(2)*}-(I_{n'_i}-\tilde{q'_i})\|<\lambda'\varepsilon.$$

- $\|\operatorname{diag}(u''_i,u'_i)-{w'_i}^{(1)}{w'_i}^{(2)}\|<\lambda'\varepsilon,$

where $\tilde{q'_i}$ is obtained from q'_i by flipping the k'_i first and the k'_i last coordinates. Then we have

$$\|\operatorname{diag}(W_{u_i,u'_i,p_l},W_{u'_i,u_i,p_i})-W_{{v_i}^{(1)},{v'_i}^{(1)},p_i}W_{{v_i}^{(2)},{v'_i}^{(2)},p_i}\|<\lambda'\varepsilon$$

and

$$\|W^*_{{v_i}^{(1)},{v'_i}^{(1)},p_i}\cdot\operatorname{diag}(P_{k_i},P_{k_i})\cdot W_{{v_i}^{(1)},{v'_i}^{(1)},p_i}-E'_{q_i,k_i,p_i\otimes P_{Bott},p_i\otimes P_1}\|<\lambda'\varepsilon,$$

with

$$E'_{q_i,k_i,p_i\otimes P_{Bott},p_i\otimes P_1}=\begin{pmatrix}q_i\otimes p_i\otimes P_{Bott}+P_{k_i}\otimes(I_2-p_i\otimes P_{Bott}) & 0\\ 0 & P_{k_i}\otimes(I_2-p_i\otimes P_1)+(I_{n_i}-\tilde{q}_i)\otimes p_i\otimes P_1\end{pmatrix}$$

for $i=1\dots l$. Similarly, we have

$$\|\operatorname{diag}(W'_{u''_i,u'''_i,p'_i},W_{u'''_i,u''_i,p'_i})-W_{{w_i}^{(1)},{w'_i}^{(1)},p'_i},W_{{w_i}^{(2)}{w'_i}^{(2)},p'_i})\|<\lambda'\varepsilon,$$

and

$$\|{W'}^*_{{w_i}^{(1)},{w'_i}^{(1)},p'_i}\cdot\operatorname{diag}(P_{k'_i},P_{k'_i})\cdot W'_{{w_i}^{(1)},{w'_i}^{(1)},p'_i}-E'_{q'_i,k'_i,p'_i,p'_i(0)}\|<\lambda'\varepsilon,$$

for $i=1\dots l'$ with

$$E'_{q'_i,k'_i,p'_i,p'_i(0)\otimes P_1}=\begin{pmatrix}q'_i\otimes p'_i+P'_{k'_i}\otimes(I_2-p'_i) & 0\\ 0 & P_{k'_i}\otimes(I_2-p'_i(0))+(I_{n'_i}-\tilde{q'_i})\otimes p'_i(0))\end{pmatrix}.$$

From this we deduce that there exist $W_y^{(1)}$ and $W_y^{(2)}$ two λ'-$h'_\varepsilon r'$-unitaries in $M_{2n_y}(C(\mathbb{T}_2,A\otimes B))$ such that

- $W_y^{(i)}-I_{2n_y}$ is in $M_{2n_x}(C(\mathbb{T}_2,A_{\Delta_i}\otimes B))$ for $i=1,\dots n$;
- $\|\operatorname{diag}(W_y,W'_y)-W_y^{(1)}W_y^{(2)}\|<\lambda'\varepsilon$;
- $\|W_y^{(1)*}\operatorname{diag}(I_{n_x},0)W_y^{(1)}-q_y\|<\lambda'\varepsilon,$

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where

$$q_y = \begin{pmatrix} E'_{q_1,k_1,p_1\otimes P_{Bott},p_1\otimes P_1} & & & & & \\ & \ddots & & & & \\ & & E'_{q_l,k_l,p_l\otimes P_{Bott},p_l\otimes P_1} & & & \\ & & & E'_{q'_1,k'_1,p'_1,p'_1(0)} & & \\ & & & & \ddots & \\ & & & & & E_{q'_{l'},k'_l,p'_{l'},p'_{l'}(0)} \end{pmatrix}.$$

Clearly q_y is $\lambda'\varepsilon$-$h'_\varepsilon r'$-projection in $M_{2n_y}(C(\mathbb{T}_2, A\otimes B))$ such that

$$[q_y, n_y]_{\lambda'\varepsilon,h'_\varepsilon r'} = [q'_y, n_y]_{\lambda'\varepsilon,h'_\varepsilon r'} = \iota_{\mathcal{O}',*}^{-,\lambda'\varepsilon,h'_\varepsilon r'} \circ \partial^{\varepsilon,r}_{C(\mathbb{T}_2,\Delta_1\otimes B),C(\mathbb{T}_2,\Delta_2\otimes B)}(y)$$

and hence z_y, q_y, n_y, W_y, W'_y, $W_y^{(1)}$ and $W_y^{(2)}$ satisfy the required conditions for some suitable control pair. □

End of the proof of the QS-statement. Let (λ, h) be a control pair as in Proposition 4.16, let ε be a positive number in $(0, \frac{1}{4\lambda})$, let y be an element in ${O'_1}^{\varepsilon,r}(A, B)$ and let r', z_y, q_y, n_y, W_y, W'_y, $W_y^{(1)}$ and $W_y^{(2)}$ as in the proposition. Let u be an ε-r unitary in some $M_n(C(\mathbb{T}_2, A\otimes B))$, let u_1 and u_2 be $\alpha_c\varepsilon$-$k_{c,\varepsilon}r$-unitaries in some $M_{2n}(C(\mathbb{T}_2, A\otimes B))$ and let q be an $\alpha_c\varepsilon$-$k_{c,\varepsilon}r$-projection in $M_{2n}(C(\mathbb{T}_2, A\otimes B))$ such that

- $u_i - I_{2n}$ is in $M_{2n}(C(\mathbb{T}_2, A\otimes B)$ for $i = 1, 2$;
- $\|\operatorname{diag}(u, u^*) - u_1u_2\| < \alpha_c\varepsilon$;
- $q - \operatorname{diag}(I_n, 0)$ is in $M_{2n}(C(\mathbb{T}_2, (A_{\Delta_1} \cap A_{\Delta_2})\otimes B))$;
- $\|q - v_1^* \operatorname{diag}(I_n, 0)v_1\| < \alpha_c\varepsilon$;
- $\|q - v_2 \operatorname{diag}(I_n, 0)v_2^*\| < \alpha_c\varepsilon$;
- $-y = [u]_{\varepsilon,r}$;
- $\partial^{\varepsilon,r}_{C(\mathbb{T}_2,\Delta_1\otimes B),C(\mathbb{T}_2,\Delta_2\otimes B)}(-y) = [q, n]_{\alpha_c\varepsilon,k_{c,\varepsilon}r}$.

Then applying Lemma 3.5 to $\operatorname{diag}(u, W_y)$, $\operatorname{diag}(u^*, W'_y)$ and to the matrices respectively obtained from $\operatorname{diag}(u_1, W_y^{(1)})$, $\operatorname{diag}(u_2, W_y^{(2)})$ and $\operatorname{diag}(q, q_y)$ by swapping the order of coordinates $n + 1, \ldots, 2n$ and $2n + 1, \ldots, 2n + n_x$, we see that for a controlled pair (λ', h') depending only on λ_0 and c, and if ε is in $(0, \frac{1}{4\lambda'})$, there exist U_1 and U_2 some $\lambda'\varepsilon$-$h'_\varepsilon r'$-unitary in some $M_{n'}(C(\mathbb{T}_2, A\otimes B))$ with $U_1 - I_{n'}$ in $M_{n'}(C(\mathbb{T}_2, A_{\Delta_1}\otimes B))$ and $U_2 - I_{n'}$ in $M_{n'}(C(\mathbb{T}_2, A_{\Delta_2}\otimes B))$ such that

$$[U_1]_{\lambda'\varepsilon,h'_\varepsilon r'} + [U_2]_{\lambda'\varepsilon,h'_\varepsilon r'} = [W_y]_{\lambda'\varepsilon,h'_\varepsilon r'} - \iota_{\mathcal{O}',*}^{-,\lambda'\varepsilon,h'_\varepsilon r'}(y)$$

in $K_1^{\lambda',h'_\varepsilon r'}(C(\mathbb{T}_2, A\otimes B))$. Up to replacing U_j for $j = 1, 2$ by

$$\mathbb{T}_2 \longrightarrow M_{3n'}; (z_1, z_2) \mapsto \operatorname{diag}(U_j^*(1, 1)U_j(z_1, z_2), U_j^*(z_1, 1)U_j(1, 1), U_j^*(1, z_2)U_j(1, 1)),$$

and (λ', h') by $(3\lambda', 2h')$, we can assume that $[U_j]_{\lambda'\varepsilon,h'_\varepsilon r'}$ belongs to ${O'_1}^{\lambda'\varepsilon,h'_\varepsilon r'}(A_{\Delta_j}, B)$.

Let r'' be a positive number with $k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r'' \geqslant h'_\varepsilon r'$ such that for $j=1,2$, any positive number ε in $(0, \frac{1}{4\lambda_0\lambda'})$ and any z in ${O'_*}^{\lambda'\varepsilon,h'_\varepsilon r'}(A_{\Delta_j},B)$, there exists an element x in ${O_*}^{\lambda_0\lambda'\varepsilon,r''}(A_{\Delta_j},B)$ such that

$$\iota_{\mathcal{O}',*}^{-,\lambda_0\alpha_{\mathcal{F}}\lambda'\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}(z) = F^{\lambda_0\lambda'\varepsilon,r''}_{A_{\Delta_j},B,*}(x).$$

Let then $z_y^{(j)}$ be for $j=1,2$ an element in ${O_1}^{\lambda_0\lambda'\varepsilon,r''}(A_{\Delta_j},B)$ such that

$$\iota_{\mathcal{O}',*}^{-,\alpha_{\mathcal{F}}\lambda_0\lambda'\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}([U_i]_{\lambda'\varepsilon,h'_\varepsilon r'}) = F^{\lambda_0\lambda'\varepsilon,r''}_{A_{\Delta_j},B,*}(z_y^{(j)}).$$

Let us set

$$\tilde{z}_y = \iota_{\mathcal{O},*}^{-,\lambda_0\lambda'\varepsilon,h'_\varepsilon R}(z_y) - \jmath^{\lambda_0\lambda'\varepsilon,r''}_{\Delta_1,\mathcal{O},*}(z_y^{(1)}) - \jmath^{\lambda_0\lambda'\varepsilon,r''}_{\Delta_2,\mathcal{O},*}(z_y^{(2)})$$

in $\mathcal{O}_1^{\lambda_0\lambda'\varepsilon,h'_\varepsilon r''}(A,B)$. By naturality of $\mathcal{F}_{\bullet,B,*}$, we see then that

$$F^{\lambda_0\lambda'\varepsilon,r''}_{A,B,*}(\tilde{z}_y) = \iota_{\mathcal{O},*}^{-,\alpha_{\mathcal{F}}\lambda_0\lambda'\varepsilon,k_{\mathcal{F},\lambda_0\lambda'\varepsilon}r''}(y)$$

and hence the QS-condition is satisfied.

4.4. Quantitative Künneth formula for crossed-product C^*-algebras

We shall next discuss the connection between the Baum–Connes conjecture and the quantitative Künneth formula. The connection between the usual Baum–Connes conjecture and the Künneth formula was studied in [1].

Before proving Theorem 4.5, recall that article [1] introduced an equivariant analogue of the map $\omega_{\bullet,\bullet,*}$ for the topological K-theory of a locally compact group G (i.e., the left-hand side of the Baum–Connes assembly map). Let A be a G-C^*-algebra and let B be a C^*-algebra. The C^*-algebra B can be viewed as a G-C^*-algebra with the trivial action of G and we equip $A\otimes B$ with the diagonal action. Then the elements in $K_*(B)$ can be viewed as element of $K_*^G(B)$. If X is a G-proper space, the map

$$\omega^{G,X}_{A,B,*} : KK_*^G(C_0(X),A)\otimes K_*(B) \to KK_*^G(C_0(X),A\otimes B);\ x\otimes y \mapsto x\otimes\tau_A(y),$$

is compatible with inclusion of G-proper cocompact spaces and hence gives rise to a morphism

$$\omega^{G,top}_{A,B,*} : K_*^{top}(G,A)\otimes\mathcal{K}_*(B) \to K_*^{top}(G,A\otimes B).$$

Theorem 4.17. *Let Γ be a discrete group and let A be a Γ-C^*-algebra. Assume that for every finite subgroup F of Γ, the C^*-algebra $A\rtimes F$ satisfies the Künneth formula, then for any C^*-algebra B such that $K_*(B)$ is a free abelian group and any positive number d*

$$\omega^{\Gamma,P_d(\Gamma)}_{A,B,*}: KK^{\Gamma}_*(C_0(P_d(\Gamma)),A)\otimes K_*(B)\to KK^{\Gamma}_*(C_0(P_d(\Gamma)),A\otimes B)$$

is an isomorphism.

Proof. (Compare with the proof of [1, Lemma 1.7]) The action of Γ on $P_r(\Gamma)$ is simplicial and up take a barycentric subdivision of $P_d(\Gamma)$, we can assume that $P_d(\Gamma)$ is a locally finite and finite dimension typed simplicial complex, equipped with a simplicial and type preserving action of Γ. Let $Z_0,\cdots,Z_n$ be the skeleton decomposition of $P_d(\Gamma)$. Then Z_j is a simplicial complex of dimension j, locally finite and equipped with a proper, cocompact and type preserving simplicial action of Γ. Let us prove by induction on j that

$$\omega^{\Gamma,Z_j}_{A,B,*}: KK^{G}_*(C_0(Z_j),A)\otimes K_*(B)\to KK^{G}_*(C_0(Z_j),A\otimes B)$$

is an isomorphism. The 0-skeleton Z_0 is a finite union of orbits and thus, for $j=0$, it is enough to prove that

$$\omega^{\Gamma,\Gamma/F}_{A,B,*}: KK^{\Gamma}_*(C_0(\Gamma/F),A)\otimes K_*(B)\to KK^{G}_*(C_0(Z_j),A\otimes B)$$

is an isomorphism when F is a finite subgroup of Γ. Let us recall from [8] that for every C^*-algebra B equipped with an action of Γ, there is a natural restriction isomorphism

$$\mathrm{Res}^{B}_{F,\Gamma,*}: KK^{\Gamma}_*(\Gamma/F,B)\longrightarrow KK^{F}_*(\mathbb{C},B)\cong K_*(B\rtimes F).$$

By naturality, this isomorphism respects also Kasparov products (using the same argument as in the proof of Lemma 4.11). Therefore, we have the following commutative diagram

$$\begin{array}{ccc} KK^{\Gamma}_*(C_0(\Gamma/F),A)\otimes K_*(B) & \xrightarrow{\omega^{\Gamma/F,\Gamma}_{A,B,*}} & KK^{\Gamma}_*(C_0(\Gamma/F),A\otimes B)\\ \Big\downarrow{\scriptstyle \mathrm{Res}^{A}_{F,\Gamma,*}} & & \Big\downarrow{\scriptstyle \mathrm{Res}^{A\otimes B}_{F,\Gamma,*}}\\ K_*(A\rtimes F)\otimes K_*(B) & \xrightarrow{\omega_{A\rtimes F,B,*}} & K_*(A\rtimes F\otimes B)\end{array}.$$

The bottom row being by assumption an isomorphism, the top row is then also an isomorphism. Let us assume that we have proved that $\omega^{Z_{j-1},\Gamma}_{A,B,*}$ is an isomorphism. Then the short exact sequence

$$0\longrightarrow C_0(Z_j\setminus Z_{j-1})\longrightarrow C_0(Z_j)\longrightarrow C_0(Z_{j-1})\longrightarrow 0$$

gives rise to an natural long exact sequence

$$\begin{aligned}&\longrightarrow KK^{\Gamma}_*(C_0(Z_{j-1}),\bullet)\longrightarrow KK^{\Gamma}_*(C_0(Z_j),\bullet)\longrightarrow KK^{\Gamma}_*(C_0(Z_j\setminus Z_{j-1}),\bullet)\\ &\longrightarrow KK^{\Gamma}_{*+1}(C_0(Z_{j-1}),\bullet)\end{aligned}$$

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and thus by naturality and since $K_*(B)$ is a free abelian group, we get a commutative diagram

$$\begin{array}{ccccccc} KK_*^\Gamma(C_0(Z_{j-1}),A)\otimes\cdots & \longrightarrow & KK_*^\Gamma(C_0(Z_j),A)\otimes\cdots & \longrightarrow & KK_*^\Gamma(C_0(Z_j\setminus Z_{j-1}),A)\cdots & \longrightarrow & KK_{*+1}^\Gamma(C_0(Z_{j-1}),A)\otimes\cdots \\ \omega_{A,B,*}^{Z_{j-1},\Gamma}\downarrow & & \omega_{A,B,*}^{Z_j,\Gamma}\downarrow & & \omega_{A,B,*}^{Z_j\setminus Z_{j-1},\Gamma}\downarrow & & \omega_{A,B,*}^{Z_{j-1},\Gamma}\downarrow \\ KK_*^\Gamma(C_0(Z_{j-1}),\cdots & \longrightarrow & KK_*^\Gamma(C_0(Z_j),\cdots & \longrightarrow & KK_*^\Gamma(C_0(Z_j\setminus Z_{j-1}),\cdots & \longrightarrow & KK_{*+1}^\Gamma(C_0(Z_{j-1})\cdots \end{array},$$

Let $\mathring{\sigma}_j$ be the interior of the standard j-simplex. Since the action of Γ is type preserving, then $Z_j \setminus Z_{j-1}$ is equivariantly homeomorphic to $\mathring{\sigma}_j \times \Sigma_j$, where Σ_j is the set of center of j-simplices of Z_j, and where Γ acts trivially on $\mathring{\sigma}_j$. This identification, together with Bott periodicity, provides a commutative diagram

$$\begin{array}{ccc} KK_*^\Gamma(C_0(Z_j\setminus Z_{j-1}),A)\otimes K_*(B) & \longrightarrow & KK_{*+1}^\Gamma(C_0(\Sigma_j),A)\otimes K_*(B) \\ \omega_{A,B,*}^{Z_j\setminus Z_{j-1},\Gamma}\downarrow & & \omega_{A,B,*}^{\Sigma_j,\Gamma,*}\downarrow \\ KK_*^\Gamma(C_0(Z_j\setminus Z_{j-1}),A\otimes B) & \longrightarrow & KK_{*+1}^\Gamma(C_0(\Sigma_j),A\otimes B) \end{array}.$$

By the first step of induction, $\omega_{A,B,*}^{\Sigma_j,\Gamma}$ is an isomorphism, and hence $\omega_{A,B,*}^{Z_j\setminus Z_{j-1},\Gamma}$ is an isomorphism. Using the induction hypothesis and the five lemma, we conclude that $\omega_{A,B,*}^{Z_j,\Gamma}$ is an isomorphism. □

Lemma 4.18. *There exists a positive number λ_0 and a function*

$$(0,+\infty)\times\left(0,\frac{1}{4\lambda_0}\right):\ (d,\varepsilon)\mapsto r'_{d,\varepsilon}$$

non decreasing in d and non increasing in ε with $r_{d,\varepsilon}\leqslant r'_{d,\varepsilon}$ for all ε in $(0,\frac{1}{4\lambda_0})$ and $d>1$ such that the following holds:

for any finitely generated group Γ, any Γ-C^*-algebra A, any C^*-algebra B and any positive numbers ε, r and d with $\varepsilon<\frac{1}{4\lambda_0}$ and $r\geqslant r'_{d,\varepsilon}$, then we have

$$\omega_{A\rtimes\Gamma,B,*}^{\varepsilon,r}\circ(\mu_{\Gamma,A,*}^{\varepsilon,r,d}\otimes\mathcal{I}d_{K_*(B)})=\mu_{\Gamma,A\otimes B,*}^{\alpha_{\mathcal{T}}\varepsilon,k_{\mathcal{T},\varepsilon}r,d}\circ\omega_{A,B,*}^{\Gamma,P_d(\Gamma)}.$$

Proof. Let z be an element in $KK_*^\Gamma(C_0(P_d(\Gamma)),A)$ and let y be an element in $K_*(B)$. Then

$$\omega_{A\rtimes\Gamma,B,*}^{\varepsilon,r}\left(J_\Gamma^{red,\varepsilon',r'}(z)\left([p_{\Gamma,d},0]_{\varepsilon',r'}\right)\otimes y\right)$$
$$=\tau_{A\rtimes_{red},*}^{\varepsilon,r}(y)\circ J_\Gamma^{red,\varepsilon',r'}(z)\left([p_{\Gamma,d},0]_{\varepsilon',r'}\right)$$

for $\varepsilon'=\frac{\varepsilon}{\alpha_J}$ and $r'=\frac{r}{k_{J,\varepsilon/\alpha_J}}$. The result is then a consequence of Remark 1.32. □

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Proof of Theorem 4.5. Let λ_0 and $(0,+\infty)\times\left(0,\frac{1}{4\lambda_0}\right); (d,\varepsilon)\mapsto r'_{d,\varepsilon}$ as in Lemma 4.18. Let α_0 be a positive number in as in Theorem 1.34, let ε and r be positive numbers with $\varepsilon<\frac{1}{4\lambda_0\alpha_0\alpha_{\mathcal{T}}}$. Let d and R be positive numbers with $R\geqslant r'_{d,\varepsilon}$ such that $QS_{\Gamma,A}(d,\varepsilon,\alpha_0\varepsilon,r,R)$ is satisfied for every Γ-C^*-algebra A. Let d' be a positive number such that $QI_{\Gamma,A}(d,d',\alpha_{\mathcal{T}}\alpha_0\varepsilon,k_{\mathcal{T},\alpha_0\varepsilon}R)$ is satisfied for every Γ-C^*-algebra A. Let us prove the QS-statement Definition 1.19.

Let y be an element in $K_*^{\varepsilon,r}((A\rtimes_{red}\Gamma)\otimes B)$. Since $(A\rtimes_{red}\Gamma)\otimes B\cong(A\otimes B)\rtimes_{red}\Gamma$ and $QS_{\Gamma,A\otimes B}(d,\varepsilon,\alpha_0\varepsilon,r,R)$ is satisfied, there exists an element z_1 in $KK_*^\Gamma(C_0(P_d(\Gamma)),A\otimes B)$ such that

$$\iota_*^{-,\alpha_0\varepsilon,R}(y)=\mu^{\alpha_0\varepsilon,R,d}_{\Gamma,A\otimes B,*}(z_1).$$

According to Theorem 4.17 there exists z_0 in $KK_*^\Gamma(C_0(P_d(\Gamma)),A)\otimes K_*(B)$ such that $z_1=\omega^{\Gamma,P_d(\Gamma)}_{A,B,*}(z_0)$. Now if we set $\varepsilon'=\alpha_0\lambda_0\varepsilon$ and $r'=k_{\mathcal{T},\alpha_0\varepsilon}R$, then $x=\mu^{\varepsilon',r',d}_{\Gamma,A,*}\otimes\mathcal{I}d_{K_*(B)}(z_0)$ is in $K_*^{\varepsilon',r'}(A\rtimes_{red}\Gamma)\otimes K_*(B)$ and from Lemma 4.18 we deduce that

$$\iota_*^{-,\alpha_{\mathcal{T}}\varepsilon',k_{\mathcal{T},\varepsilon'}r'}(y)=\omega^{\varepsilon',r'}_{A\rtimes\Gamma,B,*}(x).$$

Let us prove the QI-statement. Let x be an element in $K_*^{\varepsilon,r}(A\rtimes_{red}\Gamma)\otimes K_*(B)$ such that $\omega^{\varepsilon,r}_{A\rtimes\Gamma,B,*}(x)=0$ in $K_*^{\alpha_{\mathcal{T}}\varepsilon,k_{\mathcal{T},\varepsilon}r}((A\rtimes_{red}\Gamma)\otimes B)$, let z_0 be an element in $KK_*^\Gamma(C_0(P_d(\Gamma)),A)\otimes K_*(B)$ such that

$$\iota_*^{-,\alpha_0\varepsilon,R}\otimes\mathcal{I}d_{K_*(B)}(x)=\mu^{\alpha_0\varepsilon,R,d}_{\Gamma,A,*}\otimes\mathcal{I}d_{K_*(B)}(z_0)$$

and let us set

$$z_1=\omega^{\Gamma,P_d(\Gamma)}_{A,B,*}(z_0).$$

According to Lemma 4.18, we have that

$$\mu^{\alpha_{\mathcal{T}}\alpha_0\varepsilon,k_{\mathcal{T},\alpha_0\varepsilon}R,d}_{\Gamma,A\otimes B,*}(z_1)=0$$

in $K_*^{\alpha_{\mathcal{T}}\alpha_0\varepsilon,k_{\mathcal{T},\alpha_0\varepsilon}R}((A\rtimes_{red}\Gamma)\otimes B)$ and hence since $QI_{\Gamma,A\otimes B}(d,d',\alpha_{\mathcal{T}}\alpha_0\varepsilon,k_{\mathcal{T},\alpha_0\varepsilon}R)$ is satisfied, we have $q_{d,d',*}(z_1)=0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)),A\otimes B)$. According to Theorem 4.17 and since $\omega^{\Gamma,P_d(\Gamma)}_{A,B,*}$ is compatible with inclusion

$$P_d(\Gamma)\hookrightarrow P_{d'}(\Gamma),$$

we deduce that $q_{d,d',*}(z_0)=0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)),A)\otimes K_*(B)$. Set $\varepsilon'=\alpha_0\varepsilon$ and pick any positive number r' such that $r'\geqslant R$ and $r'\geqslant r_{d',\alpha_0\varepsilon}$. Then we have

$$\begin{aligned}\iota_*^{-,\varepsilon',r'}\otimes\mathcal{I}d_{K_*(B)}(x)&=(\iota_*^{-,\alpha_0\varepsilon,r'}\otimes\mathcal{I}d_{K_*(B)})\circ\mu^{\alpha_0\varepsilon,R,d'}_{\Gamma,A,*}\otimes\mathcal{I}d_{K_*(B)}(z_0)\\&=0.\quad\square\end{aligned}$$

5. C^*-algebras with finite asymptotic nuclear decomposition and quantitative Künneth formula

In this section, we introduce the concept of finite asymptotic nuclear decomposition for filtered C^*-algebras. For a C^*-algebra A in this class, there exist an integer n such that for any positive number r, we can decompose A_r in n steps using controlled Mayer–Vietoris pairs into locally Bootstrap C^*-algebras (see Definition 5.2). We prove the quantitative Künneth formula for C^*-algebras with finite asymptotic nuclear decomposition. We deduce from this that uniform Roe algebras of discrete metric spaces with bounded geometry and finite asymptotic dimension satisfy the Künneth formula.

5.1. Locally bootstrap C^*-algebras

Let us first recall the definition of the bootstrap category.

Definition 5.1. The bootstrap category $\mathcal{N}$ is the smallest class of nuclear separable C^*-algebras such that

(i) $\mathcal{N}$ contains $\mathbb{C}$;
(ii) $\mathcal{N}$ is closed under countable inductive limits;
(iii) $\mathcal{N}$ is stable under extension, i.e. for any extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0,$$

if any two of the C^*-algebras are in $\mathcal{N}$ then so is the third;
(iv) $\mathcal{N}$ is closed under KK-equivalence.

Next we introduce the concept of locally bootstrap C^*-algebras.

Definition 5.2. A filtered C^*-algebra A with filtration $(A_r)_{r>0}$ is called locally bootstrap if for all positive number s there exists a positive number r with $r \geqslant s$ and a sub-C^*-algebra $A^{(s)}$ of A such that

- $A^{(s)}$ belongs to the bootstrap class;
- $A_s \subseteq A^{(s)} \subseteq A_r$.

Proposition 5.3. *There exists a positive number λ_0 with $\lambda_0 \geqslant 1$ such that any locally bootstrap C^*-algebra satisfies the quantitative Künneth formula with rescaling λ_0.*

Proof. Let λ_0 be as in the second part of Proposition 1.10 and let B be a separable C^*-algebra with $K_*(B)$ free abelian. Let us prove first the QI-statement of Definition 1.19.

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Let ε and s be positive numbers with $\varepsilon < \frac{4}{\lambda_0\alpha_\tau}$. Let then r be a positive number with $r \geqslant k_{\tau,\varepsilon}s$ and let $A^{(s)}$ be a C^*-algebra such that $A^{(s)}$ belongs to the bootstrap class and $A_{k_{\tau,\varepsilon}s} \subseteq A^{(s)} \subseteq A_r$. Then $A^{(s)}$ is filtered by $(A^{(s)}\cap A_{s'})_{s'>0}$ and the filtration is indeed finite, i.e. $A^{(s)}\cap A_{s'} = A^{(s)}$ for any positive number s' with $s' \geqslant r$. Let us consider the commutative diagram

$$\begin{array}{ccccc}
K_*^{\varepsilon,s}(A)\otimes K_*(B) & \longrightarrow & K_*^{\varepsilon,s}(A^{(s)})\otimes K_*(B) & \xrightarrow{\iota_*^{-\varepsilon,r}\otimes Id_{K_*(B)}} & K_*^{\varepsilon,r}(A^{(s)})\otimes K_*(B)\\
\omega^{\varepsilon,s}_{A,B,*}\big\downarrow & & \omega^{\varepsilon,s}_{A^{(s)},B,*}\big\downarrow & & \big\downarrow\omega^{\varepsilon,r}_{A^{(s)},B,*}\\
K_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}s}(A\otimes B) & \longrightarrow & K_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}s}(A^{(s)}\otimes B) & \xrightarrow{\iota_*^{-,\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}} & K_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}(A^{(s)}\otimes B),
\end{array}$$

where the left bottom and left top maps are induced by the inclusion $A_{s'} \subseteq A^{(s)}_{s'}$ for any $s' \leqslant k_{\tau,\varepsilon}s$. Let x be an element in $K_*^{\varepsilon,s}(A)\otimes K_*(B)$ such that $\omega^{\varepsilon,s}_{A,B,*}(x) = 0$ and let then y in $K_*^{\varepsilon,r}(A^{(s)})\otimes K_*(B)$ be the image of x under the compositions of the top row. Then $\omega^{\varepsilon,r}_{A^{(s)},B,*}(y) = 0$ and hence

$$\omega_{A^{(s)},B,*}\circ(\iota_*^{\varepsilon,r}\otimes Id_{K_*(B)})(y) = \iota_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}\circ\omega^{\varepsilon,r}_{A^{(s)},B,*}(y) = 0.$$

Since $A^{(s)}$ is in the bootstrap class, then

$$\omega_{A^{(s)},B,*} : K_*(A^{(s)})\otimes K_*(B) \to K_*(A\otimes B)$$

is an isomorphism and hence $(\iota_*^{\varepsilon,r}\otimes Id_{K_*(B)})(y) = 0$ in $K_*(A^{(s)})\otimes K_*(B)$. Since $K_*(B)$ is free abelian and according to Proposition 1.10, there exists a positive number r', with $r' \geq r$ such that

$$(\iota_*^{-,\lambda_0\varepsilon,r'}\otimes Id_{K_*(B)})(y) = 0$$

in $K_*^{\lambda_0\varepsilon,r'}(A^{(s)})\otimes K_*(B)$. But since $A^{(s)}$ has propagation less than r, then $(\iota_*^{-,\lambda_0\varepsilon,r}\otimes Id_{K_*(B)})(y) = 0$ in $K_*^{\lambda_0\varepsilon,r}(A^{(s)})\otimes K_*(B)$. Hence composing with the map

$$K_*^{\lambda_0\varepsilon,r}(A^{(s)})\otimes K_*(B)\longrightarrow K_*^{\lambda_0\varepsilon,r}(A)\otimes K_*(B)$$

induced by the inclusion $A^{(s)} \hookrightarrow A$, we get then that

$$(\iota_*^{-,\lambda_0\varepsilon,r}\otimes Id_{K_*(B)})(x) = 0.$$

Let us prove now the QS-statement of Definition 1.19. Let s and ε be positive numbers with $\varepsilon < \frac{1}{4\lambda_0\alpha_\tau}$, let r be a positive number and let $A^{(s)}$ be a C^*-algebra such that $A^{(s)}$ belongs to the bootstrap class and $A_s \subseteq A^{(s)} \subseteq A_r$. Let z be an element in some $K_*^{\varepsilon,s}(A\otimes B)$ and let z' in $K_*^{\varepsilon,s}(A^{(s)}\otimes B)$ be the image of z under the map

$$K_*^{\varepsilon,s}(A\otimes B)\to K_*^{\varepsilon,s}(A^{(s)}\otimes B)$$

induced by the inclusion $A_s\subseteq A^{(s)}$. Since $A^{(s)}$ is in the bootstrap class, there exists y in $K_*(A^{(s)})\otimes K_*(B)$ such that $\iota_*^{\varepsilon,s}(z')=\omega_{A^{(s)},B,*}(y)$ in $K_*(A^{(s)}\otimes B)$. Since any element of $A^{(s)}$ has propagation less than r, there exists an element x in $K_*^{\varepsilon,r}(A^{(s)}\otimes B)$ such that $(\iota_*^{\varepsilon,r}\otimes Id_{K_*(B)})(x)=y$ in $K_*(A^{(s)})\otimes K_*(B)$. Since

$$\iota_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}\circ\omega^{\varepsilon,r}_{A^{(s)},B,*}=\omega_{A^{(s)},B,*}\circ(\iota_*^{\varepsilon,r}\otimes Id_{K_*(B)}),$$

we get that $\omega^{\varepsilon,r}_{A^{(s)},B,*}(x)$ and $\iota_*^{-,\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}(z')$ have same image under the map

$$\iota_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}:K_*^{\alpha_\tau\varepsilon,k_{\tau,\varepsilon}r}(A^{(s)}\otimes B)\longrightarrow K_*(A^{(s)}\otimes B).$$

Hence, according to Proposition 1.10, there exists a positive number r', with $r'\geq k_{\tau,\varepsilon}r$, such that

$$\iota_*^{-,\lambda_0\alpha_\tau\varepsilon,r'}\omega^{\varepsilon,r}_{A^{(s)},B,*}(x)=\iota_*^{-,\lambda_0\alpha_\tau\varepsilon,r'}(z').$$

But since $A_r^{(s)}=A_{r''}^{(s)}$ for all $r''\geqslant r$ we get that

$$\iota_*^{-,\lambda_0\alpha_\tau\varepsilon,k_{\tau,\lambda_0\varepsilon}r}\omega^{\varepsilon,r}_{A^{(s)},B,*}(x)=\iota_*^{-,\lambda_0\alpha_\tau\varepsilon,k_{\tau,\lambda_0\varepsilon}r}(z').$$

Composing with the map

$$K_*^{\alpha_\tau\lambda_0\varepsilon,k_{\tau,\varepsilon}r}(A^{(s)}\otimes B)\longrightarrow K_*^{\alpha_\tau\lambda_0\varepsilon,k_{\tau,\varepsilon}r}(A\otimes B)$$

induced by the inclusion $A^{(s)}\hookrightarrow A$, we get then that

$$\omega^{\lambda_0\varepsilon,r}_{A,B,*}(x')=\iota_*^{-,\lambda_0\alpha_\tau\varepsilon,k_{\tau,\lambda_0\varepsilon}r}(z),$$

where x' is the image of $\iota_*^{-,\lambda_0\varepsilon,r}\otimes Id_{K_*(B)}(x)$ under the composition

$$K_*^{\lambda_0\varepsilon,r}(A^{(s)}\otimes B)\longrightarrow K_*^{\lambda_0\varepsilon,r}(A\otimes B)$$

induced by the inclusion $A^{(s)}\hookrightarrow A$. □

We will need a uniform version of Proposition 5.3.

Definition 5.4. A family of filtered C^*-algebras $(A_i)_{i\in\mathbb{N}}$ is uniformly locally bootstrap if for all integer i and for all positive number s, there exist a positive number r with $r\geqslant s$ and a sub-C^*-algebra $A_i^{(s)}$ of A_i such that for all integer i,

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- $A_i^{(s)}$ belongs to the bootstrap class;
- $A_{i,s} \subseteq A_i^{(s)} \subseteq A_{i,r}$

(A_i being filtered by $(A_{i,r})_{r>0}$).

Proposition 5.3 can be extended to uniformly locally bootstrap families of C^*-algebras.

Proposition 5.5. *There exists a positive number λ_0 with $\lambda_0 \geqslant 1$ such that any uniformly locally bootstrap family $(A_i)_{i\in\mathbb{N}}$ of filtered C^*-algebras and any C^*-algebra B with $K_*(B)$-free abelian then*

$$(\Omega_{A_i,B,*} : \mathcal{K}_*(A_i)\otimes K_*(B)\longrightarrow\mathcal{K}_*(A_i\otimes B))_{i\in\mathbb{N}}$$

is a uniform family of quantitative isomorphisms with rescaling λ_0

5.2. Finite asymptotic nuclear decomposition

Let us define $\mathcal{C}_{fand}^{(0)}$ as the class of uniformly locally bootstrap families of C^*-algebras. Then we define by induction $\mathcal{C}_{fand}^{(n)}$ as the class of family $\mathcal{A}^{(1)}$ for which there exists a positive number c such that for every positive number r, the following is satisfied:

there exists a family $\mathcal{A}^{(2)}$ in $\mathcal{C}_{fand}^{(n-1)}$ and for any C^*-algebra A in $\mathcal{A}^{(1)}$ an r-controlled nuclear Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ with coercitivity c for A with A_{Δ_1}, A_{Δ_2} and $A_{\Delta_1} \cap A_{\Delta_2}$ in $\mathcal{A}^{(2)}$.

Define then $\mathcal{C}_{fand}$ as the class of families $\mathcal{A}$ such that $\mathcal{A}$ is in $\mathcal{C}_{fand}^{(n)}$ for some integer n. Theorem 4.12 obviously admits a uniform version for families and hence, together with Proposition 5.3, we obtain the following result.

Proposition 5.6. *Let $\mathcal{A}$ be a family in $\mathcal{C}_{fand}$. Then there exists a positive number $\lambda_{\mathcal{A}}$ with $\lambda_{\mathcal{A}} \geqslant 1$ such that for any C^*-algebra B with $K_*(B)$ free abelian, then*

$$(\Omega_{A,B,*} : \mathcal{K}_*(A)\otimes K_*(B)\longrightarrow\mathcal{K}_*(A\otimes B))_{A\in\mathcal{A}}$$

is a uniform family of quantitative isomorphisms with rescaling $\lambda_{\mathcal{A}}$ (indeed $\mathcal{A}$ only depends on n such that $\mathcal{A}$ lies in $\mathcal{C}_{fand}^{(n)}$.

Definition 5.7. A filtered C^*-algebra A has finite asymptotic nuclear decomposition if the single family $\{A\}$ is in $\mathcal{C}_{fand}$.

As a consequence of Proposition 5.6, we obtain

Theorem 5.8. *If A is a filtered C^*-algebra with finite asymptotic nuclear decomposition, then the quantitative Künneth formula holds for A.*

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Corollary 5.9. *If A is a filtered C^*-algebra with finite asymptotic nuclear decomposition, then A satisfies the Künneth formula in K-theory, i.e. there exists a natural short exact sequence*

$$0 \to K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to Tor(K_*(A), K_*(B)) \to 0$$

for any other C^*-algebra B.

Typical examples of family of filtered C^*-algebra in $\mathcal{C}_{fand}$ are provided by spaces with asymptotic dimension. Recall that for a metric space X and a positive number r, a cover $(U_i)_{i\in\mathbb{N}}$ has r-multiplicity n if any ball of radius r in X intersects at most n elements in $(U_i)_{i\in\mathbb{N}}$.

Definition 5.10. Let Σ be a proper discrete metric space. Then Σ has **finite asymptotic dimension** if there exists an integer m such that for any positive number r, there exists a uniformly bounded cover $(U_i)_{i\in\mathbb{N}}$ with finite r-multiplicity $m+1$. The smallest integer that satisfies this condition is called the **asymptotic dimension** of Σ.

Recall the following characterization of finite asymptotic dimension.

Proposition 5.11. *Let Σ be a proper discrete metric space and let m be an integer. Then the following assertions are equivalent:*

(i) *Σ has asymptotic dimension m;*
(ii) *For every positive number r there exist $m+1$ subsets $X^{(1)}, \dots, X^{(m+1)}$ of Σ such that*
  - *$\Sigma = X^{(1)} \cup \dots \cup X^{(m+1)}$;*
  - *for $i = 1, \dots m+1$, then $X^{(i)}$ is the r-disjoint union of a family $(X_k^{(i)})_{k\in\mathbb{N}}$ of subsets of $X^{(i)}$ with uniformly bounded diameter, i.e. $X^{(i)} = \cup_{k\in\mathbb{N}} X_k^{(i)}$, $d_i(X_k^{(i)}, X_l^{(i)}) \geqslant r$ if $k \neq l$ and there exists a positive number C such diam $X_k^{(i)} \leqslant C$ for all integer k.*

Example 5.12. If T is a tree, then T has asymptotic dimension equal to 1.

Let Σ be a proper metric space with asymptotic dimension m, then there exists a sequence of positive numbers $(R_k)_{k\in\mathbb{N}}$ and for any integer k a cover $(U_i^{(k)})_{i\in\mathbb{N}}$ of Σ such that

- $R_{k+1} > 4R_k$ for every integer k;
- $U_i^{(k)}$ has diameter less than R_k for every integer i and k;
- for any integer k, the R_k-multiplicity of $(U_i^{(k+1)})_{i\in\mathbb{N}}$ is $m+1$.

The sequence $(R_k)_{k\in\mathbb{N}}$ is called **the m-growth** of Σ.

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Lemma 5.13. *Let m be an integer, let $(R_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $R_{k+1} > 4R_k$ for every integer k. Let $(\Sigma_i)_{i\in\mathbb{N}}$ be a family of proper metric spaces with asymptotic dimension m and m-growth $(R_k)_{k\in\mathbb{N}}$. Then for any family $(A_i)_{i\in I}$ in the bootstrap category, the family $(A_i\otimes\mathscr{K}(\ell^2(\Sigma_i)))_{i\in\mathbb{N}}$ belongs to $\mathcal{C}_{fand}$.*

Proof. Let us equip $\Sigma = \coprod_{i\in\mathbb{N}}\Sigma_i$ with a distance d_Σ such that the inclusion $\Sigma_i \hookrightarrow \Sigma$ are isometric for all integer i and $d_\Sigma(\Sigma_i,\Sigma_j) \leqslant i+j$ for all integers i and j with $i \neq j$. Then Σ has asymptotic dimension m and hence according to [3], the metric space Σ embeds uniformly in a product of trees $\prod_{j=1}^n T_j$. Let d be the metric on $X = \prod_{j=1}^n T_j$ and d_i the distance on Σ_i when i runs through integers. Then there exist two non-decreasing functions $\rho_\pm : [0,+\infty) \to [0,+\infty)$ and for every integer i a map $f_i : \Sigma_i \to \prod_{j=1}^n T_j$ such that

- $\lim_{r\mapsto+\infty}\rho_\pm(r) = +\infty$;
- $\rho_-(d_i(x,y)) \leqslant d(f_i(x),f_i(y)) \leqslant \rho_+(d_i(x,y))$ for all integer i and all x and y in Σ_i.

If $n = 1$, then X is a tree and then the result holds in view of Examples 2.15, 2.20 and 5.12. A straightforward induction shows that if Σ embeds uniformly in a product of n trees, then $(A_i\otimes\mathscr{K}(\ell^2(\Sigma_i)))_{i\in\mathbb{N}}$ is in $\mathcal{C}_{fand}$. □

In order to study the structure of Roe algebras we need to add some infinite product decompositions in the quantitative decomposition process.

Definition 5.14. Let $(A^{(i)})_{i\in I}$ be a family of filtered C^*-algebras. Then the uniform products of $(A^{(i)})_{i\in I}$, denoted by $\prod_{i\in I}^u A^{(i)}$ is the closure of

$$\{(x_i)_{i\in I} \in \prod_{i\in I} A_r^{(i)},\ r > 0\}$$

in $\prod_{i\in I} A^{(i)}$ equipped with the supremium norm. The uniform product $\prod_{i\in I}^u A^{(i)}$ is then obviously a filtered C^*-algebra.

It is proved in [10, Lemma 1.11] that the quantitative K-theory of a uniform product of a stable filtered C^*-algebra is computable in term of the quantitative K-theory of the algebras of the family.

Definition 5.15. A C^*-algebra is said to be of finite asymptotic nuclear π-decomposition if there exists a positive number c and an integer n such that for any positive number r, there exists a r-controlled Mayer–Vietoris pair $(\Delta_1,\Delta_2,A_{\Delta_1},A_{\Delta_2})$ that satisfies the following.

There exist three families of filtered C^*-algebras $(B_k^{(1)})_{k\in\mathbb{N}}$, $(B_k^{(2)})_{k\in\mathbb{N}}$ and $(B_k^{(1,2)})_{k\in\mathbb{N}}$ in $\mathcal{C}_{fand}$ such that A_{Δ_1}, A_{Δ_2} and $A_{\Delta_1}\cap A_{\Delta_2}$ are respectively isomorphic as filtered C^*-algebras to $\prod_{k\in\mathbb{N}}^u B_k^{(1)}$, $\prod_{k\in\mathbb{N}}^u B_k^{(2)}$ and $\prod_{k\in\mathbb{N}}^u B_k^{(1,2)}$.

If the C^*-algebras in the families $(B_k^{(1)})_{k\in\mathbb{N}}$, $(B_k^{(2)})_{k\in\mathbb{N}}$ and $(B_k^{(1,2)})_{k\in\mathbb{N}}$ are stable, then A is said to be of stably asymptotic finite nuclear π-decomposition.

Proposition 5.16. *If Σ is a proper metric set of bounded geometry and with finite asymptotic dimension. Then the uniform Roe algebra $C^u_*(\Sigma)$ has asymptotic finite nuclear π-decomposition and the Roe algebra $C^*(\Sigma)$ has stably asymptotic finite nuclear π-decomposition.*

Proof. Let us prove de result for $C^u_*(\Sigma)$, the proof for $C^*(\Sigma)$ being similar. Let us fix x_0 in Σ and let r be a positive number. Let us fix s and R two positive numbers such that $10r < 2s < R$. Set for k integer

$$X_k^{(1)} = \{x \in \Sigma \text{ such that } 2kR \leqslant d(x, x_0) \leqslant (2k+1)R\}$$

and

$$X_k^{(2)} = \{x \in \Sigma \text{ such that } (2k+1)R \leqslant d(x, x_0) \leqslant (2k+2)R\}.$$

Then $\Sigma = X^{(1)} \cup X^{(2)}$ and $X^{(i)}$ is for $i = 1, 2$ the R-disjoint union of the family $(X_k^{(i)})_{k\in\mathbb{N}}$. Let Δ_i be for $i = 1, 2$ the set of element in $C^u_*(\Sigma)$ with support in

$$\{(x, y) \in \Sigma \times \Sigma \text{ such that } d(x, y) < r \text{ and } x \in X^{(i)}\}.$$

Since Σ has bounded geometry, then with notations of Example 2.15, there exists a controlled Mayer–Vietoris $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ of order r and coercitivity 1 such that

$$A_{\Delta_i} \cong \prod_{k\in\mathbb{N}}^{u} \mathscr{K}(\ell^2(X_{k,s}^{(i)}))$$

for $i = 1, 2$ and

$$A_{\Delta_1} \cap A_{\Delta_2} \cong \prod_{(k,l)\in\mathbb{N}^2}^{u} \mathscr{K}(\ell^2(X_{k,s}^{(i)} \cap X_{l,s}^{(i)})).$$

The result is now a consequence of Lemma 5.13. □

Proceeding similarly we can prove the quantitative Künneth formula for uniform Roe algebras of spaces with finite asymptotic dimension.

Theorem 5.17. *If Σ is a discrete proper metric set of bounded geometry and with finite asymptotic dimension. Then the uniform Roe algebra $C^u_*(\Sigma)$ satisfies the quantitative Künneth formula for some rescaling λ.*

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