VOLUME DIFFERENCE INEQUALITIES

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ABSTRACT. We prove several inequalities estimating the distance between volumes of two bodies in terms of the maximal or minimal difference between areas of sections or projections of these bodies. We also provide extensions in which volume is replaced by an arbitrary measure.

1. INTRODUCTION

Volume difference inequalities are designed to estimate the error in computations of volume of a body out of the areas of its sections and projections. We start with the case of sections. For $1 \leq k < n$, let $\gamma_{n,k}$ be the smallest constant $\gamma > 0$ satisfying the inequality

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \leq \gamma^k \max_{F \in \mathrm{Gr}_{n-k}} \big(|K \cap F| - |L \cap F|\big) \tag{1.1}$$

for all origin-symmetric convex bodies K and L in $\mathbb{R}^n$ such that $L \subset K$. Here Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $|K|$ stands for volume of appropriate dimension.

Question 1.1. *Does there exist an absolute constant C so that $\sup_{n,k} \gamma_{n,k} \leq C$?*

Question 1.1 is stronger than the slicing problem, a major open problem in convex geometry [6, 7, 2, 35]. In fact, putting $L = \beta B_2^n$ in (1.1), where B_2^n is the unit Euclidean ball in $\mathbb{R}^n$, and then sending β to zero, one gets the slicing problem: does there exist an absolute constant C so that for any $1 \leq k < n$, and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-k}{n}} \leq C^k \max_{H \in \mathrm{Gr}_{n-k}} |K \cap H| \ ? \tag{1.2}$$

The best-to-date general estimate $C \leq O(n^{1/4})$ follows from the inequality

$$|K|^{\frac{n-k}{n}} \leq (cL_K)^k \max_{H \in \mathrm{Gr}_{n-k}} |K \cap H|,$$

where L_K is the isotropic constant of K (see e.g. [10, Proposition 5.1]), and the estimate $L_K = O(n^{1/4})$ of Klartag [19] who improved an earlier estimate $L_K = O(n^{1/4} \log n)$ of Bourgain [8]. For several special classes of bodies the isotropic constant is uniformly bounded, and hence the answer to the slicing problem is known to be affirmative; see [9].

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In the case where K is a generalized k-intersection body in $\mathbb{R}^n$ (we write $K \in \mathcal{BP}_k^n$; see definition in Section 2) and L is any origin-symmetric star body in $\mathbb{R}^n$, inequality (1.1) was proved in [23] for $k=1$, and in [25] for $1<k<n$:

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \le c_{n,k}^k \max_{F\in \mathrm{Gr}_{n-k}} \big(|K\cap F| - |L\cap F|\big), \tag{1.3}$$

where $c_{n,k}^k = \omega_n^{\frac{n-k}{n}}/\omega_{n-k}$, and ω_n is the volume of the unit Euclidean ball in $\mathbb{R}^n$. One can check that $c_{n,k} \in (\frac{1}{\sqrt{e}}, 1)$ for all n, k.

Note that in Question 1.1 we added an extra assumption that $L \subset K$, compared to (1.3). Without extra assumptions on K and L, inequality (1.1) cannot hold with any $\gamma > 0$, as follows from counterexamples to the Busemann-Petty problem. The Busemann-Petty problem asks whether, for any origin-symmetric convex bodies K and L, inequalities $|K\cap F| \le |L \cap F|$ for all $F \in \mathrm{Gr}_{n-k}$ necessarily imply $|K| \le |L|$. The answer is negative in general; see [22, Chapter 5] for details. Every counterexample provides a pair of bodies K and L that contradict inequality (1.1). However, if K is a generalized k-intersection body, the answer to the question of Busemann and Petty is affirmative, as proved by Lutwak [33] for $k=1$, and by Zhang [40] for $k>1$. Inequality (1.3) is a quantified version of this fact.

Our first result extends (1.3) to arbitrary origin-symmetric star bodies. For a star body K in $\mathbb{R}^n$ and $1 \le k < n$, denote by

$$d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) = \inf\left\{ \left(\frac{|D|}{|K|}\right)^{1/n} :\ K\subset D,\ D\in \mathcal{BP}_k^n \right\} \tag{1.4}$$

the outer volume ratio distance from K to the class of generalized k-intersection bodies.

Theorem 1.2. *Let $1 \le k < n$, and let K and L be origin-symmetric star bodies in $\mathbb{R}^n$ such that $L \subset K$. Then*

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \le c_{n,k}^k d_{\mathrm{ovr}}^k(K, \mathcal{BP}_k^n) \max_{F\in \mathrm{Gr}_{n-k}} \big(|K\cap F| - |L\cap F|\big). \tag{1.5}$$

By John's theorem [18] and the fact that ellipsoids are intersection bodies, if K is origin-symmetric and convex, then $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \le \sqrt{n}$. In fact the same is true for any convex body by K. Ball's volume ratio estimate in [4]. The outer volume ratio distance was also estimated in [31]. If K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \le c\sqrt{n/k}\,[\log(en/k)]^{\frac{3}{2}}, \tag{1.6}$$

where $c>0$ is an absolute constant. The proof of (1.6) in [31] employs the existence of an α-regular position for any symmetric convex body, Pisier's extension of Milman's M-position. In conjunction with Theorem 1.2, the estimate (1.6) provides an affirmative answer to Question 1.1 for sections of proportional dimensions.

Corollary 1.3. *Let $1 \le k < n$, let K be an origin-symmetric convex body in $\mathbb{R}^n$, and let L be an origin-symmetric star body in $\mathbb{R}^n$ such that $L \subset K$. Then*

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \le C^k \left(\sqrt{n/k}\,[\log(en/k)]^{\frac{3}{2}}\right)^k \max_{F\in \mathrm{Gr}_{n-k}} \big(|K\cap F| - |L\cap F|\big), \tag{1.7}$$

where C is an absolute constant.

It is also known that for several classes of origin-symmetric convex bodies the distance $d_{\rm ovr}(K,\mathcal{BP}_k^n)$ is bounded by an absolute constant. These classes include unconditional convex bodies, duals of bodies with bounded volume ratio (see [27]) and the unit balls of normed spaces that embed in $L_p,\ -n<p<\infty$ (see [28, 34, 30]).

The inequality of Theorem 1.2 can be extended to arbitrary measures in place of volume, as follows. Let f be a bounded non-negative measurable function on $\mathbb{R}^n$. Let μ be the measure with density f so that $\mu(B)=\int_B f$ for every Borel set B in $\mathbb{R}^n$. Also, for every $F\in {\rm Gr}_{n-k}$ we write $\mu(B\cap F)=\int_{B\cap F} f$, where we integrate the restriction of f to F against Lebesgue measure on F.

It was proved in [27] that for any $1\le k<n$, any origin-symmmetric star body K in $\mathbb{R}^n$ and any measure μ with even non-negative continuous density f in $\mathbb{R}^n$,

$$\mu(K)\le \frac{n}{n-k}c_{n,k}^k\ |K|^{\frac{k}{n}}\ d_{\rm ovr}^k(K,\mathcal{BP}_k^n)\max_{F\in{\rm Gr}_{n-k}}\mu(K\cap F). \tag{1.8}$$

Considering measures with densities supported in $K\setminus L$ in inequality (1.8), we get the following measure difference inequality.

Theorem 1.4. *Let $1\le k<n$, let K and L be origin-symmetric star bodies in $\mathbb{R}^n$ such that $L\subset K$, and let μ be a measure with even non-negative continuous density. Then*

$$\mu(K)-\mu(L)\le \frac{n}{n-k}c_{n,k}^k\ |K|^{\frac{k}{n}}\ d_{\rm ovr}^k(K,\mathcal{BP}_k^n)\max_{F\in{\rm Gr}_{n-k}}\big(\mu(K\cap F)-\mu(L\cap F)\big). \tag{1.9}$$

In Section 2 we provide an alternative proof of this result.

Moreover, using an approach recently developed in [10], we prove a different version of Theorem 1.4, where the symmetry and continuity assumptions are dropped, but the body K is required to be convex.

Theorem 1.5. *Let $1\le k<n$, let K be a convex body with $0\in K$ and let $L\subseteq K$ be a Borel set in $\mathbb{R}^n$. For any measure μ with a bounded measurable non-negative density, we have*

$$\mu(K)^{n-k}-\mu(L)^{n-k}\le\Big(c_0\sqrt{n-k}\Big)^{k(n-k)}|K|^{\frac{k(n-k)}{n}}\max_{F\in G_{n,n-k}}\big(\mu(K\cap F)^{n-k}-\mu(L\cap F)^{n-k}\big) \tag{1.10}$$

where $c_0>0$ is an absolute constant.

A different kind of volume difference inequality was proved in [14]. If K is any origin-symmetric star body in $\mathbb{R}^n$, L is an intersection body, and $\min_{\xi\in S^{n-1}}\big(|K\cap\xi^\perp|-|L\cap\xi^\perp|\big)>0$, where $\xi^\perp$ is the subspace of $\mathbb{R}^n$ perpendicular to ξ, then

$$|K|^{\frac{n-1}{n}}-|L|^{\frac{n-1}{n}}\ge c\frac{1}{\sqrt{n}M(\overline{L})}\min_{\xi\in S^{n-1}}\big(|K\cap\xi^\perp|-|L\cap\xi^\perp|\big), \tag{1.11}$$

where $c>0$ is an absolute constant, $\overline{L}=L/|L|^{\frac{1}{n}}$, $M(L)=\int_{S^{n-1}}\|\theta\|_L d\sigma(\theta)$, and σ is the normalized Lebesgue measure on the sphere.

As shown in [15], there exist constants $c_1,c_2>0$ such that for any $n\in\mathbb{N}$ and any origin-symmetric convex body K in $\mathbb{R}^n$ in the isotropic position,

$$\frac{1}{M(K)}\ge\ c_1\frac{n^{1/10}L_K}{\log^{2/5}(e+n)}\ge c_2\frac{n^{1/10}}{\log^{2/5}(e+n)}. \tag{1.12}$$

Also, if K is convex, has volume 1 and is in the minimal mean width position, then we have

$$\frac{1}{M(K)} \geq c_3 \frac{\sqrt{n}}{\log(e+n)}. \tag{1.13}$$

Inserting these estimates into (1.11) we obtain estimates independent from the bodies.

For a star body K in $\mathbb{R}^n$ and $1 \leq k < n$, we define

$$d_k(K, \mathcal{BP}_k^n) = \inf \left\{ \left(\frac{\int_{S^{n-1}} \|\theta\|_K^{-k} d\sigma(\theta)}{\int_{S^{n-1}} \|\theta\|_D^{-k} d\sigma(\theta)} \right)^{\frac{1}{k}} : \ D \subset K, \ D \in \mathcal{BP}_k^n \right\}.$$

By John's theorem, if K is origin-symmetric and convex, then $d_k(K, \mathcal{BP}_k^n) \leq \sqrt{n}$.

We prove the following generalization of (1.11).

Theorem 1.6. *Let $1 \leq k < n$, and let K and L be origin-symmetric star bodies in $\mathbb{R}^n$ such that $L \subset K$. Then*

$$d_k^k(L, \mathcal{BP}_k^n) \left(|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \right) \geq c^k \frac{1}{(\sqrt{n} M(\overline{L}))^k} \min_{F \in \mathrm{Gr}_{n-k}} \big(|K \cap F| - |L \cap F| \big), \tag{1.14}$$

where $c > 0$ is an absolute constant.

We introduce another method that gives a different generalization of (1.11).

Theorem 1.7. *Let $1 \leq k < n$, and let K and L be bounded Borel sets in $\mathbb{R}^n$ with $L \subset K$. Then*

$$\big(|K| - |L| \big)^{\frac{n-k}{n}} \geq c_{n,k}^k \min_{F \in \mathrm{Gr}_{n-k}} \big(|K \cap F| - |L \cap F| \big), \tag{1.15}$$

where $c_{n,k}^k = \omega_n^{\frac{n-k}{n}} / \omega_{n-k}$.

Note that Theorem 1.7 holds true for an arbitrary pair of bounded Borel sets $L \subseteq K$ and it no longer involves the distance d_k and $M(\overline{L})$. Actually, the constant $c_{n,k}$ is sharp as one can check from the example of the ball $K = B_2^n$ and $L = \beta B_2^n$ where $\beta \to 0$. Nevertheless, it is formally not stronger than Theorem 1.6 because $|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}}$ is smaller than $(|K| - |L|)^{\frac{n-k}{n}}$.

We deduce Theorem 1.7 from a more general statement for arbitrary measures.

Theorem 1.8. *Let $1 \leq k < n$, and let K and L be two bounded Borel sets in $\mathbb{R}^n$ such that $L \subset K$. Let μ a measure in $\mathbb{R}^n$ with bounded density g. Then,*

$$\big(\mu(K) - \mu(L) \big)^{\frac{n-k}{n}} \geq c_{n,k}^k \frac{1}{\|g\|_\infty^{\frac{k}{n}}} \left(\int_{\mathrm{Gr}_{n-k}} \big(\mu(K \cap F) - \mu(L \cap F) \big)^{\frac{n}{n-k}} d\nu_{n,n-k}(F) \right)^{\frac{n-k}{n}}, \tag{1.16}$$

where $\nu_{n,n-k}$ is the Haar probability measure on Gr_{n-k}. In particular,

$$\big(\mu(K) - \mu(L) \big)^{\frac{n-k}{n}} \geq c_{n,k}^k \frac{1}{\|g\|_\infty^{\frac{k}{n}}} \min_{F \in \mathrm{Gr}_{n-k}} \big(\mu(K \cap F) - \mu(L \cap F) \big). \tag{1.17}$$

An inequality going in the direction opposite to (1.14) was proved in [27]. Suppose that K is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, with strictly positive curvature, that is not an intersection body. Then there exists an origin-symmetric convex body L in $\mathbb{R}^n$ such that $L \subset K$ and

$$|K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} < c_{n,1} \min_{\xi \in S^{n-1}} \big(|K \cap \xi^\perp| - |L \cap \xi^\perp|\big). \tag{1.18}$$

Here we prove a similar inequality going in the direction opposite to (1.5).

Theorem 1.9. *Let L be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ with strictly positive curvature that is not an intersection body. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $L \subset K$ and*

$$|K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} > c \frac{1}{\sqrt{n} M(\overline{L})} \max_{\xi \in S^{n-1}} \big(|K \cap \xi^\perp| - |L \cap \xi^\perp|\big), \tag{1.19}$$

where $c > 0$ is an absolute constant.

Let us pass to projections. For $\xi \in S^{n-1}$ and a convex body L, we denote by $L|\xi^\perp$ the orthogonal projection of L to $\xi^\perp$. Let β_n be the smallest constant $\beta > 0$ satisfying

$$\beta\big(|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}\big) \geq \min_{\xi \in S^{n-1}} \big(|L|\xi^\perp| - |K|\xi^\perp|\big) \tag{1.20}$$

for all origin-symmetric convex bodies K, L in $\mathbb{R}^n$ whose curvature functions f_K and f_L exist and satisfy $f_K(\xi) \leq f_L(\xi)$ for all $\xi \in S^{n-1}$. We prove

Theorem 1.10. $\beta_n \simeq \sqrt{n}$, *i.e. there exist absolute constants $a, b > 0$ such that for all $n \in \mathbb{N}$*

$$a\sqrt{n} \leq \beta_n \leq b\sqrt{n}.$$

It was proved in [23, 26] that if L is a projection body (see definition in Section 3) and K is an origin-symmetric convex body, then

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \geq c_{n,1} \min_{\xi \in S^{n-1}} \big(|L|\xi^\perp| - |K|\xi^\perp|\big). \tag{1.21}$$

Note that we formulate (1.20) with the condition $f_K \leq f_L$, which is not needed for (1.21). The reason is that without an extra condition inequality (1.20) simply cannot hold in general with any $\beta > 0$. This follows from counterexamples to the Shephard problem asking whether, for any origin-symmetric convex bodies K and L, inequalities $|K|\xi^\perp| \leq |L|\xi^\perp|$ for all $\xi \in S^{n-1}$ necessarily imply $|K| \leq |L|$. The answer is negative in general; see [36, 38] or [22, Chapter 8] for details. However, if L is a projection body, the answer to the question of Shephard is affirmative, as proved by Petty [36] and Schneider [38]. Inequality (1.21) is a quantified version of this fact.

For a convex body L in $\mathbb{R}^n$ denote by

$$d_{\mathrm{vr}}(L, \Pi) = \inf \left\{ \left(\frac{|L|}{|D|} \right)^{1/n} : \ D \subset L, \ D \in \Pi \right\}$$

the volume ratio distance from L to the class of projection bodies. We extend (1.21) to arbitrary origin-symmetric convex bodies, as follows.

Theorem 1.11. *Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and their curvature functions exist and satisfy $f_K(\xi) \le f_L(\xi)$ for all $\xi \in S^{n-1}$. Then*

$$d_{\rm vr}(L,\Pi)\left(|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}\right) \ge c_{n,1} \min_{\xi\in S^{n-1}} \left(|L|\xi^\perp| - |K|\xi^\perp|\right). \tag{1.22}$$

Again by K. Ball's volume ratio estimate, for any convex body K in $\mathbb{R}^n$, $d_{\rm vr}(K,\Pi) \le \sqrt{n}$. In Section 3 we show that this distance can be of the order $\sqrt{n}$, up to an absolute constant. The same argument is used to deduce Theorem 1.10 from Theorem 1.11.

Denote by h_K the support function, and by

$$w(K) = \int_{S^{n-1}} h_K(\xi) d\sigma(\xi)$$

the mean width of the body K. Denote by

$$d_w(K,\Pi) = \inf\left\{\frac{w(D)}{w(K)} : K\subset D,\ D\in\Pi\right\}$$

the mean width distance from K to the class of projection bodies.

Theorem 1.12. *Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and their curvature functions exist and satisfy $f_K(\xi) \le f_L(\xi)$ for all $\xi \in S^{n-1}$. Then*

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \le c\ d_{\rm w}(K,\Pi)\frac{w(\overline{K})}{\sqrt{n}} \max_{\xi\in S^{n-1}} \left(|L|\xi^\perp| - |K|\xi^\perp|\right), \tag{1.23}$$

where c is an absolute constant.

In Section 3 we show that the distance d_w can be of the order $\sqrt{n}$, up to a logarithmic term. Note that if K is a symmetric convex body of volume 1 in $\mathbb{R}^n$ and is in the minimal mean width position, then $w(K) \le c\sqrt{n}(\log n)$.

Theorems 1.11 and 1.12 are complemented by the following results, going in the opposite directions, that were proved in [29]. The constant in Theorem 1.14 is written in a more general form than in [29].

Theorem 1.13. *Suppose that L is an origin-symmetric convex body in $\mathbb{R}^n$, with strictly positive curvature, that is not a projection body. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ so that $f_L(\xi) \ge f_K(\xi)$ for all $\xi \in S^{n-1}$ and*

$$\max_{\xi\in S^{n-1}} \left(|L|\xi^\perp| - |K|\xi^\perp|\right) \le \frac{1}{c_{n,1}}\left(|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}\right).$$

Theorem 1.14. *Suppose that K is an origin-symmetric convex body in $\mathbb{R}^n$ that is not a projection body. Then there exists an origin-symmetric convex body L in $\mathbb{R}^n$ so that $f_L(\xi) \ge f_K(\xi)$ for all $\xi \in S^{n-1}$ and*

$$\min_{\xi\in S^{n-1}} \left(|L|\xi^\perp| - |K|\xi^\perp|\right) \ge \frac{c\sqrt{n}}{w(\overline{K})}\left(|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}\right),$$

where c is an absolute constant.

In Section 2 we provide the proofs of the volume difference inequalities for sections, and in Section 3 we give the proofs of the volume difference inequalities for projections. As we proceed, we introduce notation and the necessary background information. We refer to the books [12] and [39] for basic facts from the

Brunn-Minkowski theory and to the book [1] for basic facts from asymptotic convex geometry.

2. Volume difference inequalities for sections

We need several definitions from convex geometry. A closed bounded set K in $\mathbb{R}^n$ is called a star body if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\} \tag{2.1}$$

is a continuous function on $\mathbb{R}^n$.

The radial function of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n, \ x \neq 0. \tag{2.2}$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x.

We use the polar formula for the volume of a star body:

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta, \tag{2.3}$$

where $d\theta$ stands for the uniform measure on the sphere with density 1.

The class $\mathcal{BP}_k^n$ of *generalized k-intersection bodies* was introduced by Lutwak [33] for $k = 1$, and by Zhang [40] for $k > 1$. For $1 \leq k \leq n-1$, the $(n-k)$-*dimensional spherical Radon transform* $R_{n-k} : C(S^{n-1}) \to C(\mathrm{Gr}_{n-k})$ is a linear operator defined by

$$R_{n-k}g(E) = \int_{S^{n-1}\cap E} g(\theta) \ d\theta, \qquad E \in \mathrm{Gr}_{n-k} \tag{2.4}$$

for every function $g \in C(S^{n-1})$. We say that an origin-symmetric star body D in $\mathbb{R}^n$ is a *generalized k-intersection body*, and write $D \in \mathcal{BP}_k^n$, if there exists a finite non-negative Borel measure μ_D on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \rho_D^k(\theta) g(\theta) \ d\theta = \int_{\mathrm{Gr}_{n-k}} R_{n-k}g(H) \ d\mu_D(H). \tag{2.5}$$

The class $\mathcal{BP}_1^n$ is the original class of intersection bodies introduced by Lutwak.

Proof of Theorem 1.2. For every $H \in \mathrm{Gr}_{n-k}$ we have

$$|K \cap H| - |L \cap H| \leq \max_{F \in \mathrm{Gr}_{n-k}} \left(|K \cap F| - |L \cap F|\right).$$

Writing volume in terms of the Radon transform, we get

$$\frac{1}{n-k} \left(R_{n-k}(\|\cdot\|_K^{-n+k})(H) - R_{n-k}(\|\cdot\|_L^{-n+k})(H)\right) \leq \max_{F \in \mathrm{Gr}_{n-k}} \left(|K \cap F| - |L \cap F|\right).$$

Let $D \in \mathcal{BP}_k^n, \ K \subset D$. Integrating both sides by $H \in \mathrm{Gr}_{n-k}$ with the measure μ_D corresponding to D by (2.5), we get

$$\frac{1}{n-k} \int_{S^{n-1}} \|\theta\|_D^{-k} \left(\|\theta\|_K^{-n+k} - \|\theta\|_L^{-n+k}\right) d\theta \leq \max_{F \in \mathrm{Gr}_{n-k}} \left(|K \cap F| - |L \cap F|\right) \mu_D(\mathrm{Gr}_{n-k}). \tag{2.6}$$

We have $\|\theta\|_D^{-1} \geq \|\theta\|_K^{-1} \geq \|\theta\|_L^{-1}$, because $L \subset K \subset D$. Using this, Hölder's inequality and the polar formula for volume, we estimate the left-hand side of (2.6) by

$$\frac{1}{n-k}\int_{S^{n-1}} \|\theta\|_K^{-k}\left(\|\theta\|_K^{-n+k} - \|\theta\|_L^{-n+k}\right) d\theta \geq \frac{n}{n-k}\left(|K| - |K|^{\frac{k}{n}}|L|^{\frac{n-k}{n}}\right).$$

To estimate $\mu_D(\mathrm{Gr}_{n-k})$ from above, we combine the fact that $1 = R_{n-k}\mathbf{1}(E)/|S^{n-k-1}|$ for every $E \in \mathrm{Gr}_{n-k}$ with (2.5) and Hölder's inequality to write

$$\begin{aligned}(2.7)\qquad \mu_D(\mathrm{Gr}_{n-k}) &= \frac{1}{|S^{n-k-1}|}\int_{\mathrm{Gr}_{n-k}} R_{n-k}\mathbf{1}(E)d\mu_D(E)\\ &= \frac{1}{|S^{n-k-1}|}\int_{S^{n-1}} \|\theta\|_D^{-k}\, d\theta\\ &\leq \frac{1}{|S^{n-k-1}|}\left|S^{n-1}\right|^{\frac{n-k}{n}}\left(\int_{S^{n-1}} \|\theta\|_D^{-n}\, d\theta\right)^{\frac{k}{n}}\\ &= \frac{1}{|S^{n-k-1}|}\left|S^{n-1}\right|^{\frac{n-k}{n}} n^{\frac{k}{n}}|D|^{\frac{k}{n}}.\end{aligned}$$

These estimates show that

$$\begin{aligned}(2.8)\qquad \frac{n}{n-k}\left(|K| - |K|^{\frac{k}{n}}|L|^{\frac{n-k}{n}}\right) &\leq \frac{1}{|S^{n-k-1}|}\left|S^{n-1}\right|^{\frac{n-k}{n}} n^{\frac{k}{n}}|D|^{\frac{k}{n}} \max_{F\in\mathrm{Gr}_{n-k}}\left(|K\cap F| - |L\cap F|\right)\\ &= \frac{n}{n-k}\, c_{n,k}^k|D|^{\frac{k}{n}} \max_{F\in\mathrm{Gr}_{n-k}}\left(|K\cap F| - |L\cap F|\right).\end{aligned}$$

Finally, we choose D so that $|D|^{1/n} \leq (1+\delta)d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)|K|^{1/n}$, and then send δ to zero. □

Next, we extend Theorem 1.2 to arbitrary measures in place of volume. Let f be a bounded non-negative measurable function on $\mathbb{R}^n$ and let μ be the measure with density f. Writing integrals in polar coordinates, we get

$$(2.9)\qquad \mu(K) = \int_K f(x)dx = \int_{S^{n-1}}\left(\int_0^{\rho_K(\theta)} r^{n-1}f(r\theta)dr\right)d\theta,$$

and for $H \in \mathrm{Gr}_{n-k}$

$$\begin{aligned}(2.10)\qquad \mu(K\cap H) &= \int_{K\cap H} f(x)dx = \int_{S^{n-1}\cap H}\left(\int_0^{\rho_K(\theta)} r^{n-k-1}f(r\theta)dr\right)d\theta\\ &= R_{n-k}\left(\int_0^{\rho_K(\cdot)} r^{n-k-1}f(r\cdot)dr\right)(H).\end{aligned}$$

Proof of Theorem 1.4. Let f be the density of the measure μ. For every $H \in \mathrm{Gr}_{n-k}$ we have

$$\mu(K\cap H) - \mu(L\cap H) \leq \max_{F\in\mathrm{Gr}_{n-k}}\left(\mu(K\cap F) - \mu(L\cap F)\right).$$

Using (2.10), we get

$$R_{n-k}\left(\int_{\rho_L(\cdot)}^{\rho_K(\cdot)} r^{n-k-1}f(r\cdot)dr\right)(H) \leq \max_{F\in\mathrm{Gr}_{n-k}}\left(\mu(K\cap F) - \mu(L\cap F)\right).$$

Let $D \in \mathcal{BP}_k^n,\ K \subset D$. Integrating both sides by $H \in \mathrm{Gr}_{n-k}$ with the measure μ_D corresponding to D by (2.5), we get

(2.11)

$$\int_{S^{n-1}} \rho_D^k(\theta) \left(\int_{\rho_L(\theta)}^{\rho_K(\theta)} r^{n-k-1} f(r\theta) dr \right) d\theta \le \max_{F \in \mathrm{Gr}_{n-k}} \left(\mu(K \cap F) - \mu(L \cap F)\right) \mu_D(\mathrm{Gr}_{n-k}).$$

We have $\rho_D \ge \rho_K \ge \rho_L$, because $L \subset K \subset D$. Using this and (2.9), we estimate the left-hand side of (2.11) from below

$$\int_{S^{n-1}} \rho_D^k(\theta) \left(\int_{\rho_L(\theta)}^{\rho_K(\theta)} r^{n-k-1} f(r\theta) dr \right) d\theta \ge \int_{S^{n-1}} \rho_K^k(\theta) \left(\int_{\rho_L(\theta)}^{\rho_K(\theta)} r^{n-k-1} f(r\theta) dr \right) d\theta$$

$$\ge \int_{S^{n-1}} \left(\int_{\rho_L(\theta)}^{\rho_K(\theta)} r^{n-1} f(r\theta) dr \right) d\theta = \mu(K) - \mu(L).$$

Now estimate $\mu_D(G_{n-k})$ and then choose D in the same way as in the proof of Theorem 1.2. □

Remark 2.1. Note that in the case of volume ($f \equiv 1$), Theorem 1.4 implies that if K is an origin-symmetric convex body in $\mathbb{R}^n$, and L is an origin-symmetric star body in $\mathbb{R}^n$ such that $L \subset K$ then

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \le \frac{|K| - |L|}{|K|^{\frac{k}{n}}} \le \frac{n}{n-k} c_{n,k}^k \, d_{\mathrm{ovr}}^k(K, \mathcal{BP}_k^n) \max_{F \in \mathrm{Gr}_{n-k}} \left(|K \cap F| - |L \cap F|\right).$$

This estimate differs from the one of Theorem 1.2 by a factor $\frac{n}{n-k}$; however, note that also $(|K| - |L|)/|K|^{\frac{k}{n}}$ is greater than $|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}}$.

To prove Theorem 1.5 we use a technique that was introduced in [10]. It is based on the following generalized Blaschke-Petkantschin formula (see [13]).

Lemma 2.2. *Let $1 \le q \le s \le n$. There exists a constant $p(n,s,q) > 0$ such that, for every non-negative bounded Borel measurable function $f : (\mathbb{R}^n)^q \to \mathbb{R}$,*

(2.12)

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(x_1, \dots, x_q) dx_1 \cdots dx_q$$
$$= p(n,s,q) \int_{G_{n,s}} \int_F \cdots \int_F f(x_1, \dots, x_q) \left|\mathrm{conv}(0, x_1, \dots, x_q)\right|^{n-s} dx_1 \dots dx_q \, d\nu_{n,s}(F),$$

where $\nu_{n,s}$ is the Haar probability measure on Gr_s. The exact value of the constant $p(n,s,q)$ is

$$p(n,s,q) = (q!)^{n-s} \frac{(n\omega_n) \cdots ((n-q+1)\omega_{n-q+1})}{(s\omega_s) \cdots ((s-q+1)\omega_{s-q+1})}. \tag{2.13}$$

We will also use Grinberg's inequality: If D is a bounded Borel set of positive Lebesgue measure in $\mathbb{R}^n$ then, for any $1 \le k \le n-1$,

(2.14)

$$\tilde{R}_k(D) := \frac{1}{|D|^{n-k}} \int_{G_{n,n-k}} |D \cap F|^n \, d\nu_{n,n-k}(F) \le \frac{1}{|B_2^n|^{n-k}} \int_{G_{n,n-k}} |B_2^n \cap F|^n \, d\nu_{n,n-k}(F).$$

This fact was proved by Grinberg in [16]. It is stated for convex bodies D but the proof applies to bounded Borel sets (see also [13]). For the Euclidean ball we have

$$\tilde{R}_k(B_2^n) := \frac{1}{|B_2^n|^{n-k}} \int_{G_{n,n-k}} |B_2^n \cap F|^n \, d\nu_{n,n-k}(F) = \frac{\omega_{n-k}^n}{\omega_n^{n-k}} = c_{n,k}^{-kn}, \tag{2.15}$$

where as before

$$c_{n,k}^k := \omega_n^{\frac{n-k}{n}} / \omega_{n-k}. \tag{2.16}$$

For any $1 \leq k \leq n-1$ we define

$$p(n,s) := p(n,s,s).$$

It was proved in [10] that for every $1 \leq k \leq n-1$ we have

$$[c_{n,k}^{-n} \, p(n,n-k)]^{\frac{1}{k(n-k)}} \simeq \sqrt{n-k}. \tag{2.17}$$

Proof of Theorem 1.5. Let g be the density of the measure μ. Applying Lemma 2.2 with $q = s = n-k$ for the functions $f(x_1, \ldots, x_{n-k}) = \prod_{i=1}^{n-k} g(x_i) \mathbf{1}_K(x_i)$ and $h(x_1, \ldots, x_{n-k}) = \prod_{i=1}^{n-k} g(x_i) \mathbf{1}_L(x_i)$ we get

(2.18)

$$\begin{aligned}
\mu(K)^{n-k} - \mu(L)^{n-k} &= \prod_{i=1}^{n-k} \int_K g(x_i) dx - \prod_{i=1}^{n-k} \int_L g(x_i) dx \\
&= p(n,n-k) \int_{G_{n,n-k}} \Big[\int_{K\cap F} \cdots \int_{K \cap F} g(x_1) \cdots g(x_{n-k}) \, |\mathrm{conv}(0,x_1,\ldots,x_{n-k})|^k dx_1 \ldots dx_{n-k} \\
&\quad - \int_{L\cap F} \cdots \int_{L\cap F} g(x_1) \cdots g(x_{n-k}) \, |\mathrm{conv}(0,x_1,\ldots,x_{n-k})|^k dx_1 \ldots dx_{n-k} \Big] \, d\nu_{n,n-k}(F) \\
&= p(n,n-k) \int_{G_{n,n-k}} \int_{P_{n-k}(K,L;F)} g(x_1) \cdots g(x_{n-k}) \, |\mathrm{conv}(0,x_1,\ldots,x_{n-k})|^k dx_1 \ldots dx_{n-k} \, d\nu_{n,n-k}(F),
\end{aligned}$$

where

$$P_{n-k}(K,L;F) = (K\cap F)^{n-k} \setminus (L \cap F)^{n-k}.$$

Note that

$$|\mathrm{conv}(0,x_1,\ldots,x_{n-k})|^k \leq |K \cap F|^k$$

for all $(x_1, \ldots, x_{n-k}) \in P_{n-k}(K,L;F)$ by the convexity of $K \cap F$ and the assumption that $0 \in K$. Therefore,

(2.19)

$$\begin{aligned}
&\mu(K)^{n-k} - \mu(L)^{n-k} \\
&\quad \leq p(n,n-k) \int_{G_{n,n-k}} |K\cap F|^k \int_{P_{n-k}(K,L;F)} g(x_1) \cdots g(x_{n-k}) \, dx_1 \ldots dx_{n-k} \, d\nu_{n,n-k}(F) \\
&\quad = p(n,n-k) \int_{G_{n,n-k}} |K \cap F|^k [\mu(K\cap F)^{n-k} - \mu(L \cap F)^{n-k}] \, d\nu_{n,n-k}(F) \\
&\quad \leq \max_{F \in G_{n,n-k}} [\mu(K \cap F)^{n-k} - \mu(L\cap F)^{n-k}] \cdot p(n,n-k) \int_{G_{n,n-k}} |K \cap F|^k \, d\nu_{n,n-k}(F).
\end{aligned}$$

From Grinberg's inequality (2.14) we have

$$\int_{G_{n,n-k}} |K \cap F|^k \, d\nu_{n,n-k}(F) \leq c_{n,k}^{-kn} \, |K|^{\frac{k(n-k)}{n}}. \tag{2.20}$$

Using also (2.17) we see that
(2.21)

$$\mu(K)^{n-k}-\mu(L)^{n-k} \le \left(c_0\sqrt{n-k}\right)^{k(n-k)} |K|^{\frac{k(n-k)}{n}} \max_{F\in G_{n,n-k}} [\mu(K\cap F)^{n-k}-\mu(L\cap F)^{n-k}],$$

as claimed. □

Remark 2.3. Theorem 1.5 implies [10, Theorem 1.1]:

$$\mu(K) \le \left(c_0\sqrt{n-k}\right)^{k} |K|^{\frac{k}{n}} \max_{F\in G_{n,n-k}} \mu(K\cap F) \tag{2.22}$$

for every convex body K with $0 \in K$ and any measure μ. Considering measures with densities supported in $K \setminus L$ in (2.22), we get the following measure difference inequality:

$$\mu(K)-\mu(L) \le \left(c_0\sqrt{n-k}\right)^{k} |K|^{\frac{k}{n}} \max_{F\in G_{n,n-k}} \big(\mu(K\cap F)-\mu(L\cap F)\big) \tag{2.23}$$

under the assumptions of Theorem 1.5.

The next inequalities estimate the distance between volumes of two bodies in $\mathbb{R}^n$ in terms of the minimal difference between areas of their $(n-k)$-dimensional sections.

Proof of Theorem 1.6. For every $H \in \mathrm{Gr}_{n-k}$ we have

$$|K\cap H| - |L\cap H| \ge \min_{F\in \mathrm{Gr}_{n-k}} \left(|K\cap F| - |L\cap F|\right).$$

Writing volume in terms of the Radon transform, we get

$$\frac{1}{n-k}\left(R_{n-k}(\|\cdot\|_K^{-n+k})(H) - R_{n-k}(\|\cdot\|_L^{-n+k})(H)\right) \ge \min_{F\in \mathrm{Gr}_{n-k}} \left(|K\cap F| - |L\cap F|\right).$$

Let $D \in \mathcal{BP}_k^n,\ D \subset L$. Integrating both sides by $H \in \mathrm{Gr}_{n-k}$ with the measure μ_D corresponding to D by (2.5), we get
(2.24)

$$\frac{1}{n-k}\int_{S^{n-1}} \|\theta\|_D^{-k}\left(\|\theta\|_K^{-n+k} - \|\theta\|_L^{-n+k}\right)d\theta \ge \min_{F\in \mathrm{Gr}_{n-k}} \left(|K\cap F| - |L\cap F|\right)\mu_D(\mathrm{Gr}_{n-k}).$$

We have $\|\theta\|_D^{-1} \le \|\theta\|_L^{-1} \le \|\theta\|_K^{-1}$, because $D \subset L \subset K$. Using this, Hölder's inequality and the polar formula for volume, we estimate the left-hand side of (2.24) from above by

$$\frac{1}{n-k}\int_{S^{n-1}} \|\theta\|_L^{-k}\left(\|\theta\|_K^{-n+k} - \|\theta\|_L^{-n+k}\right)d\theta \le \frac{n}{n-k}\left(|L|^{\frac{k}{n}}|K|^{\frac{n-k}{n}} - |L|\right).$$

To estimate $\mu_D(\mathrm{Gr}_{n-k})$ from below, we combine the fact that $1 = R_{n-k}\mathbf{1}(E)/|S^{n-k-1}|$ for every $E \in \mathrm{Gr}_{n-k}$ with (2.5) to write
(2.25)

$$\mu_D(\mathrm{Gr}_{n-k}) = \frac{1}{|S^{n-k-1}|}\int_{\mathrm{Gr}_{n-k}} R_{n-k}\mathbf{1}(E)d\mu_D(E) = \frac{|S^{n-1}|}{|S^{n-k-1}|}\int_{S^{n-1}} \|\theta\|_D^{-k}\ d\sigma(\theta).$$

These estimates show that

$$\frac{n}{n-k}\left(|L|^{\frac{k}{n}}|K|^{\frac{n-k}{n}} - |L|\right) \ge \frac{|S^{n-1}|}{|S^{n-k-1}|}\int_{S^{n-1}} \|\theta\|_D^{-k}d\sigma(\theta) \min_{F\in \mathrm{Gr}_{n-k}} \left(|K\cap F| - |L\cap F|\right).$$

Finally, for $\delta > 0$, we choose D so that

$$\int_{S^{n-1}} \|\theta\|_D^{-k} d\sigma(\theta) \geq \frac{1}{(1+\delta) d_k^k(L, \mathcal{BP}_k^n)} \int_{S^{n-1}} \|\theta\|_L^{-k} d\sigma(\theta),$$

and send δ to zero. Then use Jensen's inequality and homogeneity to get

$$\left(\int_{S^{n-1}} \|\theta\|_L^{-k}\, d\sigma(\theta)\right)^{\frac{1}{k}} \geq \left(\int_{S^{n-1}} \|\theta\|_L d\sigma(\theta)\right)^{-1} = \frac{1}{M(\overline{L})}|L|^{\frac{1}{n}}, \tag{2.26}$$

and apply standard estimates for the Γ-function. □

Next we prove Theorem 1.8, which directly implies Theorem 1.7. For the proof we will use some basic facts about Sylvester-type functionals. Let C be a bounded Borel set of positive measure in $\mathbb{R}^m$. For every $p > 0$ we consider the normalized p-th moment of the expected volume of the random simplex $\mathrm{conv}(0, x_1, \ldots, x_m)$, the convex hull of the origin and m points from C, defined by

$$S_p(C) = \left(\frac{1}{|C|^{m+p}} \int_C \cdots \int_C |\mathrm{conv}(0, x_1, \ldots, x_m)|^p dx_1 \cdots dx_m\right)^{1/p}. \tag{2.27}$$

It was proved by Pfiefer [37] (see also [13]) that

$$S_p(C) \geq S_p(B_2^m).$$

More generally, for any Borel probability measure ν on $\mathbb{R}^m$, for any $1 \leq q \leq m$ and every $p > 0$, we define

$$S_{p,q}(\nu) = \left(\int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} |\mathrm{conv}(0, x_1, \ldots, x_q)|^p d\nu(x_1) \cdots d\nu(x_q)\right)^{1/p}. \tag{2.28}$$

A generalization of Pfiefer's result appears in [11]. Let ν be a measure in $\mathbb{R}^n$ with a bounded non-negative measurable density g. Then

$$S_{p,q}^p(\nu) \geq \frac{\|g\|_1^{q+\frac{pq}{m}}}{\omega_m^{q+\frac{pq}{m}} \|g\|_\infty^{\frac{pq}{m}}} S_{p,q}^p(\mathbf{1}_{B_2^m}). \tag{2.29}$$

Proof of Theorem 1.8. Let $u(x) = g(x)\mathbf{1}_K(x)$ and $v(x) = g(x)\mathbf{1}_L(x)$. Using Lemma 2.2 with $s = n-k$ and $q = 1$, we start by writing

(2.30)
$$\begin{aligned} \mu(K) - \mu(L) &= \int_{\mathbb{R}^n} u(x)dx - \int_{\mathbb{R}^n} v(x)dx \\ &= p(n, n-k, 1) \int_{G_{n,n-k}} \left[\int_{K\cap F} g(x)\, \|x\|_2^k dx - \int_{L\cap F} g(x)\, \|x\|_2^k dx\right] d\nu_{n,n-k}(F) \\ &= p(n, n-k, 1) \int_{G_{n,n-k}} \int_{(K\cap F)\setminus(L\cap F)} g(x)\, \|x\|_2^k dx\, d\nu_{n,n-k}(F). \end{aligned}$$

(Note that $|\mathrm{conv}(0, x)| = \|x\|_2$, the Euclidean norm of x). For every F set $C_F = (K \cap F) \setminus (L \cap F)$ and consider the measure ν_F with density g on C_F. Applying

(2.29) with $p = k$, $q = 1$ and $m = n - k$ we have

$$\begin{aligned}
\mu(K) - \mu(L) &\geq p(n, n-k, 1) \int_{\mathrm{Gr}_{n-k}} S^k_{k,1}(\nu_F)\, d\nu_{n,n-k}(F) \\
&\geq p(n, n-k, 1) \int_{\mathrm{Gr}_{n-k}} \frac{\|g\,|_{C_F}\|_1^{1+\frac{k}{n-k}}}{\omega_{n-k}^{1+\frac{k}{n-k}} \|g\,|_{C_F}\|_\infty^{\frac{k}{n-k}}} S^k_k(\mathbf{1}_{B_2^{n-k}})\, d\nu_{n,n-k}(F) \\
&= \frac{p(n, n-k, 1)}{\omega_{n-k}^{\frac{n}{n-k}}} S^k_2(\mathbf{1}_{B_2^{n-k}}) \int_{\mathrm{Gr}_{n-k}} \frac{\|g\,|_{C_F}\|_1^{\frac{n}{n-k}}}{\|g\,|_{C_F}\|_\infty^{\frac{k}{n-k}}}\, d\nu_{n,n-k}(F).
\end{aligned} \tag{2.31}$$

Note that

$$p(n, n-k, 1) = \frac{n\omega_n}{(n-k)\omega_{n-k}}$$

and

$$S^k_{k,1}(\mathbf{1}_{B_2^{n-k}}) = \int_{B_2^{n-k}} \|x\|_2^k dx = \frac{n-k}{n}\omega_{n-k}.$$

Therefore,

$$\frac{p(n, n-k, 1)}{\omega_{n-k}^{\frac{n}{n-k}}} S^k_2(\mathbf{1}_{B_2^{n-k}}) = \frac{\omega_n}{\omega_{n-k}^{\frac{n}{n-k}}} = c_{n,k}^{\frac{kn}{n-k}}.$$

On the other hand, for any $F \in \mathrm{Gr}_{n-k}$ we have

$$\|g\,|_{C_F}\|_1 = \mu(K \cap F) - \mu(L \cap F)$$

and

$$\|g\,|_{C_F}\|_\infty \leq \|g\|_\infty.$$

Combining the above we get

$$\mu(K) - \mu(L) \geq c_{n,k}^{\frac{kn}{n-k}} \frac{1}{\|g\|_\infty^{\frac{k}{n-k}}} \int_{\mathrm{Gr}_{n-k}} (\mu(K \cap F) - \mu(L \cap F)^{\frac{n}{n-k}}\, d\nu_{n,n-k}(F),$$

and the result follows. □

Remark 2.4. Theorem 1.7 is an immediate consequence of Theorem 1.8. It corresponds to the case $g \equiv \mathbf{1}$, for which we clearly have $\|g\|_\infty = 1$.

We pass to Theorem 1.9. We consider Schwartz distributions, i.e. continuous functionals on the space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$. For any even distribution f, we have $(\hat{f})^\wedge = (2\pi)^n f$.

If K is an origin-symmetric convex body and $0 < p < n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents a distribution acting by integration. Suppose that K is infinitely smooth, i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [22, Lemma 3.16], the Fourier transform of $\|\cdot\|_K^{-p}$ is an extension of some function $g \in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n + p$ on $\mathbb{R}^n$. When we write $\left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)$, we mean $g(\xi),\ \xi \in S^{n-1}$.

For $f \in C^\infty(S^{n-1})$ and $0 < p < n$, we denote by

$$(f \cdot r^{-p})(x) = f(x/\|x\|_2)\|x\|_2^{-p}$$

the extension of f to a homogeneous function of degree $-p$ on $\mathbb{R}^n$. Again by [22, Lemma 3.16], there exists $g \in C^\infty(S^{n-1})$ such that

$$(f \cdot r^{-p})^\wedge = g \cdot r^{-n+p}.$$

If K, L are infinitely smooth origin-symmetric convex bodies, the following spherical version of Parseval's formula can be found in [22, Lemma 3.22]: for any $p \in (-n, 0)$

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-p}\right)^\wedge(\xi) \left(\|\cdot\|_L^{-n+p}\right)^\wedge(\xi) = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}\, dx. \tag{2.32}$$

It was proved in [20, Theorem 1] that an origin-symmetric convex body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution. In the case where K is infinitely smooth, this means that the function $(\|\cdot\|_K^{-1})^\wedge$ is non-negative on the sphere.

We also need a result from [21] (see also [22, Theorem 3.8]) expressing volume of central hyperplane sections in terms of the Fourier transform. For any origin-symmetric star body K in $\mathbb{R}^n$, the distribution $(\|\cdot\|_K^{-n+1})^\wedge$ is a continuous function on the sphere extended to a homogeneous function of degree -1 on the whole of $\mathbb{R}^n$, and for every $\xi \in S^{n-1}$,

$$|K \cap \xi^\perp| = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi). \tag{2.33}$$

In particular, if $K = B_2^n$ then for every $\xi \in S^{n-1}$

$$(\|\cdot\|_2^{-n+1})^\wedge(\xi) = \pi(n-1)|B_2^{n-1}|. \tag{2.34}$$

Note that every non-intersection body can be approximated in the radial metric by infinitely smooth non-intersection bodies with strictly positive curvature; see [22, Lemma 4.10]. Different examples of convex bodies that are not intersection bodies (in dimensions five and higher, as in dimensions up to four such examples do not exist) can be found in [22, Chapter 4]. In particular, the unit balls of the spaces $\ell_q^n,\ q > 2,\ n \geq 5$ are not intersection bodies.

Proof of Theorem 1.9. Since L is infinitely smooth, the Fourier transform of $\|\cdot\|_L^{-1}$ is a continuous function on the sphere S^{n-1}. Also, L is not an intersection body, so $\left(\|\cdot\|_L^{-1}\right)^\wedge < 0$ on an open set $\Omega \subset S^{n-1}$. Let $\phi \in C^\infty(S^{n-1})$ be an even non-negative, not identically zero, infinitely smooth function on S^{n-1} with support in $\Omega \cup -\Omega$. Extend ϕ to an even homogeneous of degree -1 function $\phi \cdot r^{-1}$ on $\mathbb{R}^n \setminus \{0\}$. The Fourier transform of this function in the sense of distributions is $\psi \cdot r^{-n+1}$ where ψ is an infinitely smooth function on the sphere.

Let ε be a number such that $|B_2^{n-1}| \cdot \|\theta\|_L^{-n+1} > \varepsilon > 0$ for every $\theta \in S^{n-1}$. Define a star body K by

$$\|\theta\|_K^{-n+1} = \|\theta\|_L^{-n+1} - \delta\psi(\theta) + \frac{\varepsilon}{|B_2^{n-1}|}, \qquad \theta \in S^{n-1}, \tag{2.35}$$

where $\delta > 0$ is small enough so that for every θ

$$|\delta\psi(\theta)| < \min\left\{\|\theta\|_L^{-n+1} - \frac{\varepsilon}{|B_2^{n-1}|},\ \frac{\varepsilon}{|B_2^{n-1}|}\right\}.$$

The latter condition implies that $L \subset K$. Since L has strictly positive curvature, by an argument from [22, p. 96], we can make ε, δ smaller (if necessary) to ensure that the body K is convex.

Now we extend the functions in (2.35) from the sphere to $\mathbb{R}^n\setminus\{0\}$ as homogeneous functions of degree $-n+1$ and apply the Fourier transform. We get that for every $\xi \in S^{n-1}$

$$\left(\|\cdot\|_K^{-n+1}\right)^\wedge(\xi) = \left(\|\cdot\|_L^{-n+1}\right)^\wedge(\xi) - (2\pi)^n \delta\phi(\xi) + \pi(n-1)\varepsilon. \tag{2.36}$$

Here, we used (2.34) to compute the last term. By (2.36), (2.33) and the fact that the function ϕ is non-negative and is equal to zero at some points, we have

$$\varepsilon = \max_{\xi\in S^{n-1}} (|K\cap\xi^\perp| - |L\cap\xi^\perp|). \tag{2.37}$$

Multiplying both sides of (2.36) by $\left(\|\cdot\|_L^{-1}\right)^\wedge(\xi)$, integrating over S^{n-1} and using Parseval's formula on the sphere, we get

$$(2\pi)^n \int_{S^{n-1}} \|\theta\|_L^{-1}\ \|\theta\|_K^{-n+1} d\theta = (2\pi)^n n|L| - (2\pi)^n\delta \int_{S^{n-1}} \phi(\theta)\left(\|\cdot\|_L^{-1}\right)^\wedge(\theta) d\theta$$
$$+ \pi(n-1)\varepsilon \int_{S^{n-1}} \left(\|\cdot\|_L^{-1}\right)^\wedge(\theta) d\theta.$$

Since ϕ is a non-negative function supported in Ω, where $\left(\|\cdot\|_L^{-1}\right)^\wedge$ is negative, the latter equality implies

$$\begin{aligned}(2\pi)^n n|L| + \pi(n-1)\varepsilon \int_{S^{n-1}} \left(\|\cdot\|_L^{-1}\right)^\wedge(\theta) d\theta &< (2\pi)^n \int_{S^{n-1}} \|\theta\|_L^{-1}\ \|\theta\|_K^{-n+1} d\theta \\ &\le (2\pi)^n \left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{\frac{n-1}{n}} \left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{\frac{1}{n}} \\ &= (2\pi)^n n|L|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}.\end{aligned}$$

Finally, by (2.34), Parseval's formula and Jensen's inequality,

$$\begin{aligned}\pi(n-1)\int_{S^{n-1}} \left(\|\cdot\|_L^{-1}\right)^\wedge(\theta) d\theta &= \frac{1}{|B_2^{n-1}|}\int_{S^{n-1}} \left(\|\cdot\|_L^{-1}\right)^\wedge(\theta)\left(\|\cdot\|_2^{-n+1}\right)^\wedge(\theta) d\theta \\ &= \frac{(2\pi)^n |S^{n-1}|}{|B_2^{n-1}|}\int_{S^{n-1}} \|\theta\|_L^{-1} d\sigma(\theta) \\ &\ge \frac{(2\pi)^n |S^{n-1}|}{|B_2^{n-1}|}\frac{1}{M(\overline{L})}|L|^{\frac{1}{n}} \\ &\ge c\frac{(2\pi)^n\sqrt{n}|L|^{\frac{1}{n}}}{M(\overline{L})}.\end{aligned}$$

Combining these estimates we get

$$(2\pi)^n n|L| + c\varepsilon\frac{(2\pi)^n\sqrt{n}|L|^{\frac{1}{n}}}{M(\overline{L})} \le (2\pi)^n n|L|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}.$$

The result follows after we recall (2.37). □

3. Volume difference inequalities for projections

The *support function* of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}, \qquad x \in \mathbb{R}^n.$$

If K is origin-symmetric, then h_K is a norm on $\mathbb{R}^n$.

The *surface area measure* $S(K,\cdot)$ of a convex body K in $\mathbb{R}^n$ is defined as follows. For every Borel set $E \subset S^{n-1}$, $S(K,E)$ is equal to Lebesgue measure of the part of the boundary of K where normal vectors belong to E. We usually consider bodies with absolutely continuous surface area measures. A convex body K is said to have the *curvature function*

$$f_K : S^{n-1} \to \mathbb{R},$$

if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to Lebesgue measure σ_{n-1} on S^{n-1}, and

$$\frac{dS(K,\cdot)}{d\sigma_{n-1}} = f_K \in L_1(S^{n-1}),$$

so f_K is the density of $S(K,\cdot)$.

By the approximation argument of [39, Theorem 3.3.1], we may assume in the formulation of Shephard's problem that the bodies K and L are such that their support functions h_K, h_L are infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$. Using [22, Lemma 3.16] we get in this case that the Fourier transforms $\widehat{h_K}$, $\widehat{h_L}$ are the extensions of infinitely differentiable functions on the sphere to homogeneous distributions on $\mathbb{R}^n$ of degree $-n-1$. Moreover, by a similar approximation argument (see e.g. [17, Section 5]), we may assume that our bodies have absolutely continuous surface area measures. Therefore, in the rest of this section, K and L are convex symmetric bodies with infinitely smooth support functions and absolutely continuous surface area measures.

The following version of Parseval's formula was proved in [32] (see also [22, Lemma 8.8]):

$$\int_{S^{n-1}} \widehat{h_K}(\xi)\widehat{f_L}(\xi)\, d\xi = (2\pi)^n \int_{S^{n-1}} h_K(x) f_L(x)\, dx. \tag{3.1}$$

The volume of a body can be expressed in terms of its support function and curvature function:

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K(x) f_K(x)\, dx. \tag{3.2}$$

If K and L are two convex bodies in $\mathbb{R}^n$ the *mixed volume* $V_1(K,L)$ is equal to

$$V_1(K,L) = \frac{1}{n}\lim_{\varepsilon\to+0}\frac{|K+\varepsilon L| - |K|}{\varepsilon}.$$

We use the following first Minkowski inequality (see [39] or [22, p.23]): for any convex bodies K, L in $\mathbb{R}^n$,

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}. \tag{3.3}$$

The mixed volume $V_1(K,L)$ can also be expressed in terms of the support and curvature functions:

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(x) f_K(x)\, dx. \tag{3.4}$$

Let K be an origin-symmetric convex body in $\mathbb{R}^n$. The *projection body* ΠK of K is defined as an origin-symmetric convex body in $\mathbb{R}^n$ whose support function in every direction is equal to the volume of the hyperplane projection of K to this direction: for every $\xi \in S^{n-1}$,

$$h_{\Pi K}(\xi) = |K|\xi^\perp|. \tag{3.5}$$

If L is the projection body of some convex body, we simply say that L is a projection body. The Minkowski (vector) sum of projection bodies is also a projection body. Every projection body is the limit in the Hausdorff metric of Minkowski sums of symmetric intervals. An origin-symmetric convex body in $\mathbb{R}^n$ is a projection body if and only if its polar body is the unit ball of an n-dimensional subspace of L_1; see [39, 12, 22] for proofs and more properties of projection bodies.

Proof of Theorem 1.11. By approximation (see [39, Theorem 3.3.1]), we can assume that K, L are infinitely smooth. We have

$$|L|\xi^\perp| - |K|\xi^\perp| \geq \min_{\eta\in S^{n-1}}(|L|\eta^\perp| - |K|\eta^\perp|). \tag{3.6}$$

It was proved in [32] that

$$|K|\xi^\perp| = -\frac{1}{\pi}\widehat{f_K}(\xi), \qquad \xi \in S^{n-1}, \tag{3.7}$$

where f_K is extended from the sphere to a homogeneous function of degree $-n-1$ on the whole $\mathbb{R}^n$. Therefore, (3.6) can be written as

$$-\frac{1}{\pi}\widehat{f_L}(\xi) + \frac{1}{\pi}\widehat{f_K}(\xi) \geq \min_{\eta\in S^{n-1}}(|L|\eta^\perp| - |K|\eta^\perp|), \qquad \xi \in S^{n-1}. \tag{3.8}$$

Let D be a projection body such that $D \subset L$, then $h_D \leq h_L$ in every direction. It was proved in [32] that an infinitely smooth origin-symmetric convex body D in $\mathbb{R}^n$ is a projection body if and only if $\widehat{h_D} \leq 0$ on the sphere S^{n-1}. Integrating (3.8) with respect to this negative density, we get

$$-\int_{S^{n-1}} \widehat{h_D}(\xi)\widehat{f_L}(\xi)\,d\xi + \int_{S^{n-1}} \widehat{h_D}(\xi)\widehat{f_K}(\xi)\,d\xi \leq \pi\int_{S^{n-1}} \widehat{h_D}(\xi)d\xi \min_{\eta\in S^{n-1}}(|L|\eta^\perp| - |K|\eta^\perp|).$$

Using Parseval's formula (3.1), we get

$$(2\pi)^n\int_{S^{n-1}} h_D(\xi)(f_L(\xi) - f_K(\xi))d\xi \geq -\pi\int_{S^{n-1}} \widehat{h_D}(\xi)d\xi \min_{\eta\in S^{n-1}}(|L|\eta^\perp| - |K|\eta^\perp|). \tag{3.9}$$

We estimate the left-hand side of (3.9) from above using (3.2) and (3.4) (recall that $f_K \leq f_L$):

$$\begin{aligned}(2\pi)^n\int_{S^{n-1}} h_D(\xi)(f_L(\xi) - f_K(\xi))d\xi &\leq (2\pi)^n\int_{S^{n-1}} h_L(\xi)(f_L(\xi) - f_K(\xi))d\xi \\ &\leq (2\pi)^n n(|L| - |K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}).\end{aligned} \tag{3.10}$$

To estimate the right-hand side of (3.10) from below, note that, by (3.7), the Fourier transform of the curvature function f_2 of the unit Euclidean ball is equal to

$$\widehat{f_2}(\xi) = -\pi|B_2^{n-1}|, \qquad \xi \in S^{n-1}.$$

Therefore, by (3.1) and (3.4) (recall that $f_2 \equiv 1$) ,

$$\begin{aligned}-\pi\int_{S^{n-1}} \widehat{h_D}(\xi)\,d\xi &= \frac{1}{|B_2^{n-1}|}\int_{S^{n-1}} \widehat{h_D}(\xi)\widehat{f_2}(\xi)\,d\xi = \frac{(2\pi)^n}{|B_2^{n-1}|}\int_{S^{n-1}} h_D(x)f_2(x)\,dx \\ &= \frac{(2\pi)^n}{|B_2^{n-1}|}nV_1(B_2^n, D) \geq \frac{(2\pi)^n n}{|B_2^{n-1}|}|D|^{\frac{1}{n}}|B_2^n|^{\frac{n-1}{n}} \\ &= (2\pi)^n n\; c_{n,1}|D|^{\frac{1}{n}}.\end{aligned}$$

Now for $\delta > 0$ choose D so that $(1+\delta)\ d_{\rm vr}(L,\Pi)\ |D|^{\frac{1}{n}} \geq |L|^{\frac{1}{n}}$. Combine the resulting inequality with (3.9) and (3.10) and send δ to zero. □

Proof of Theorem 1.10. Putting $K = \delta B_2^n$ in (1.20) and sending δ to zero, we get

$$\beta |L|^{\frac{n-1}{n}} \geq \min_{\xi\in S^{n-1}} |L|\xi^\perp|.$$

By a result of K. Ball [3], there exists an absolute constant c_1 so that for each $n\in\mathbb{N}$ there is an origin-symmetric convex body L_n in $\mathbb{R}^n$ satisfying

$$\min_{\xi\in S^{n-1}} |L_n|\xi^\perp| \geq c_1\sqrt{n}|L_n|^{\frac{n-1}{n}}.$$

This shows that $\beta_n \geq c_1\sqrt{n}$. On the other hand, since ellipsoids are projection bodies, we have $d_{\rm vr}(L,\Pi)\leq \sqrt{n}$ for every origin-symmetric convex body L in $\mathbb{R}^n$. By approximation (see [17]), one can assume that each of the bodies L_n has a curvature function, so we can apply Theorem 1.11 to the bodies L_n and $K=\delta B_2^n,\ \delta\to 0$, to see that $\beta_n \leq (1/c_{n,1})\sqrt{n} < \sqrt{en}$. □

Remark 3.1. From Theorem 1.11 we see that the bodies L_n defined in the proof of Theorem 1.10 satisfy

$$d_{\rm vr}(L_n,\Pi)|L_n|^{\frac{n-1}{n}} \geq c_{n,1} \min_{\xi\in S^{n-1}} |L|\xi^\perp| \geq c_{n,1}\, c_1\sqrt{n}|L_n|^{\frac{n-1}{n}}.$$

This shows that $d_{\rm vr}(L_n,\Pi)\geq c_1\sqrt{n/e}$, and hence

$$\sup_L d_{\rm vr}(L,\Pi_n) \simeq \sqrt{n},$$

where the supremum is over all origin-symmetric convex bodies L in $\mathbb{R}^n$.

Proof of Theorem 1.12. Again, by approximation, we can assume that K, L are infinitely smooth. Let D be a projection body such that $K\subset D$, then $h_K\leq h_D$ in every direction. Similarly to the proof of Theorem 1.11,

(3.11)

$$(2\pi)^n \int_{S^{n-1}} h_D(\xi)(f_L(\xi)-f_K(\xi))d\xi \leq -\pi \int_{S^{n-1}} \widehat{h_D}(\xi)d\xi \max_{\eta\in S^{n-1}}(|L|\eta^\perp| - |K|\eta^\perp|).$$

We estimate the left-hand side of (3.11) from below using (3.2) and (3.4) (recall that $f_K\leq f_L$ and $h_K\leq h_D$):

(3.12)

$$\begin{aligned}(2\pi)^n \int_{S^{n-1}} h_D(\xi)(f_L(\xi)-f_K(\xi))d\xi &\geq (2\pi)^n \int_{S^{n-1}} h_K(\xi)(f_L(\xi)-f_K(\xi))d\xi \\ &\geq (2\pi)^n n(|L|^{\frac{n-1}{n}}|K|^{\frac{1}{n}} - |K|).\end{aligned}$$

Now for $\delta>0$ choose D so that

$$w(D)\leq (1+\delta)d_{\rm w}(K,\Pi)w(\overline{K})|K|^{\frac{1}{n}}.$$

As in the proof of Theorem 1.11,

$$\begin{aligned}-\pi\int_{S^{n-1}} \widehat{h_D}(\xi)\ d\xi = \frac{(2\pi)^n}{|B_2^{n-1}|}\int_{S^{n-1}} h_D(x)\ dx = \frac{(2\pi)^n|S^{n-1}|}{|B_2^{n-1}|}w(D) \\ \leq (1+\delta)(2\pi)^n c\ d_{\rm w}(K,\Pi)\sqrt{n}\ w(\overline{K})|K|^{\frac{1}{n}}.\end{aligned}$$

We get the result combining the latter with (3.11) and (3.12) and sending δ to zero. □

Finally, we show that the distance d_w can be of the order $\sqrt{n}$, up to a logarithmic term. We will use the fact that projection bodies have positions with "small diameter". More precisely, we have the following statement: For every $D \in \Pi$ there exists $T \in GL(n)$ such that

$$R(T(D)) \leq \frac{\sqrt{n}}{2}|T(D)|^{1/n}. \tag{3.13}$$

In particular, this holds true if T is chosen so that $T(D)$ in Lewis or Löwner or minimal mean width position (see e.g. [9, Chapter 4]). Let $K = B_1^n$ be the cross-polytope, and consider a projection body D such that $B_1^n \subseteq D$. We may find T so that (3.13) is satisfied. We will use the next well-known result of Bárány and Füredi from [5]: if $x_1, \dots, x_N \in RB_2^n$ then

$$|\mathrm{conv}\{x_1, \dots, x_N\}|^{1/n} \leq \frac{c_3 R\sqrt{\log(1+N/n)}}{n}.$$

Since

$$T(B_1^n) = \mathrm{conv}\{\pm Te_1, \dots, \pm Te_n\} \subseteq R(T(D))B_2^n,$$

we get

$$|T(B_1^n)|^{1/n} \leq \frac{c_4}{\sqrt{n}}|T(D)|^{1/n}.$$

It follows that

$$|B_1^n|^{1/n} \leq \frac{c_4}{\sqrt{n}}|D|^{1/n}.$$

From Urysohn's inequality (see [1]) we know that $w(D) \geq c_5\sqrt{n}\,|D|^{1/n}$, and a direct computation shows that $w(B_1^n) \leq c_6\sqrt{n\log n}|B_1^n|^{1/n}$. This shows that

$$w(D) \geq c_7\sqrt{n/\log n}\,w(B_1^n).$$

Since $D \supset B_1^n$ was arbitrary, we conclude that

$$d_w(B_1^n) \geq c\sqrt{n/\log n}, \tag{3.14}$$

where $c > 0$ is an absolute constant.

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