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Excluding subdivisions of bounded degree graphs ☆,☆☆

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ABSTRACT

Let H be a fixed graph. What can be said about graphs G that have no subgraph isomorphic to a subdivision of H? Grohe and Marx proved that such graphs G satisfy a certain structure theorem that is not satisfied by graphs that contain a subdivision of a (larger) graph H_1. Dvořák found a clever strengthening—his structure is not satisfied by graphs that contain a subdivision of a graph H_2, where H_2 has "similar embedding properties" as H. Building upon Dvořák's theorem, we prove that said graphs G satisfy a similar structure theorem. Our structure is not satisfied by graphs that contain a subdivision of a graph H_3 that has similar embedding properties as H and has the same maximum degree as H. This will be important in a forthcoming application to well-quasi-ordering.

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1. Introduction

In this paper *graphs* are finite and are permitted to have loops and parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. The cornerstone of the Graph Minors project of Robertson and Seymour is the following excluded minor theorem. (The missing definitions are as in [15] and are given at the end of this section.)

Theorem 1.1 *([15, Theorem (1.3)]). Let L be a graph. Then there exist integers $\kappa, \rho, \xi > 0$ such that every graph G with no L-minor can be constructed by clique-sums, starting from graphs that are an $\leq \xi$-extension of an outgrowth by $\leq \kappa$ ρ-rings of a graph that can be drawn in a surface in which L cannot be drawn.*

In this paper we are concerned with excluding topological minors. The first such theorem was obtained by Grohe and Marx.

Theorem 1.2 *([3, Corollary 4.4]). For every graph H there exist integers ξ, κ, ρ, g, D such that every graph G with no H-subdivision can be constructed by clique-sums, starting from graphs that are an $\leq \xi$-extension of either*

(a) a graph of maximum degree at most D, or
(b) an outgrowth by $\leq \kappa$ ρ-rings of a graph that can be drawn in a surface of genus at most g.

Thus the second outcome includes graphs drawn on surfaces in which H *can* be drawn. Dvořák [2, Theorem 3] strengthened the result by restricting the graphs in (b) to those that can be drawn in a surface Σ in which H can possibly be drawn, but only "in a way in which H cannot be drawn in Σ". We omit the precise statement of Dvořák's theorem, because it requires a large amount of definitions that we otherwise do not need. Instead, let us remark that the meaning of "the way in which H cannot be drawn in Σ" has to do with the function mf, defined as follows.

Let H be a graph and Σ a surface in which H can be embedded. We define $\mathrm{mf}(H, \Sigma)$ as the minimum of $|S|$, over all embeddings of H in Σ and all sets S of regions of the embedded graph such that every vertex of H of degree at least four is incident with a region in S. When H cannot be embedded in Σ, we define $\mathrm{mf}(H, \Sigma)$ to be infinity.

Our objective is to strengthen the theorems of Grohe and Marx, and Dvořák by reducing the value of the constant D to the maximum degree of H, which is best possible. (We will prove that this value D cannot be replaced by any number smaller than the maximum degree of H in Section 7.) However, we are not able to extend the theorems verbatim; our theorem gives a structure relative to a tangle, as follows. (Tangles, vortices and segregations are defined in Section 2.)

Theorem 1.3. *Let $d \geq 4$ and $h > 0$ be integers. Then there exist integers $\theta, \kappa, \rho, \xi, g \geq 0$ such that the following holds. If H is a graph of maximum degree d on h vertices, and a graph G does not admit an H-subdivision, then for every tangle $\mathcal{T}$ in G of order at least θ there exists a set $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either*

1. *for every vertex $v \in V(G) - Z$ there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d - 1$ such that $v \in V(A) - V(B)$, or*
2. *there exists a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$ such that $\mathcal{S}$ has a proper arrangement in some surface Σ of genus at most g, every society (S_1, Ω_1) in $\mathcal{S}_1$ satisfies $|\overline{\Omega}_1| \leq 3$, every society (S_2, Ω_2) in $\mathcal{S}_2$ is a ρ-vortex, and either*

   (a) *H cannot be drawn in Σ, or*
   (b) *H can be drawn in Σ and $\mathrm{mf}(H, \Sigma) \geq 2$, and there exists $\mathcal{S}'_2 \subseteq \mathcal{S}_2$ with $|\mathcal{S}'_2| \leq \mathrm{mf}(H, \Sigma) - 1$ such that for every vertex $v \in V(G) - Z$ either $v \in V(S) - \bar{\Omega}$ for some $(S, \Omega) \in \mathcal{S}'_2$ or there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d - 1$ such that $v \in V(A) - V(B)$.*

In fact, we will prove a stronger statement (Theorem 6.8) that provides the structure information for graphs with no H-subdivision with branch vertices prescribed and immediately implies Theorem 1.3. In addition, Theorem 1.3 has the following immediate corollary.

Corollary 1.4. *Let $d \geq 4$ and $h > 0$ be integers. Then there exist θ and ξ such that for every graph H of order h and of maximum degree d that can be drawn in the plane such that every vertex of degree at least four is incident with the infinite region, and for every graph G, either G admits an H-subdivision, or for every tangle $\mathcal{T}$ of order at least θ in G, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that for every vertex $v \in V(G) - Z$ there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d - 1$ such that $v \in V(A) - V(B)$.*

Proof. Let $d \geq 4$ and h be given, let θ and ξ be as in Theorem 1.3, and let H be as in the statement of the corollary. Then $\mathrm{mf}(H, \Sigma) = 1$ for every surface Σ, and hence the second outcome of Theorem 1.3 cannot hold. Thus the first outcome holds, as desired. □

Corollary 1.4 will be used in a forthcoming series of papers to prove the following theorem, conjectured by Robertson. In the application it will be important that the order of the separation in Corollary 1.4 is at most $d - 1$.

Theorem 1.5 *([6]). Let $k \geq 1$ be an integer, let R denote the graph obtained from a path of length k by replacing each edge by a pair of parallel edges, and let $G_1, G_2, \ldots$ be an infinite sequence of graphs such that none of them has an R-subdivision. Then there exist integers i, j such that $1 \leq i < j$ and G_j has a G_i-subdivision.*

Let us now introduce the missing definitions. Given a subset X of the vertex-set $V(G)$ of a graph G, the subgraph of G induced by X is denoted by $G[X]$. We say that a graph G is the *clique-sum* of graphs G_1, G_2 if there exist $V_1 = \{v_{1,1}, ..., v_{1,|V_1|}\} \subseteq V(G_1), V_2 = \{v_{2,1}, v_{2,2}, ..., v_{2,|V_2|}\} \subseteq V(G_2)$ with $|V_1| = |V_2|$ such that $G_1[V_1]$ and $G_2[V_2]$ are complete graphs, and G can be obtained from $G_1 \cup G_2$ by identifying $v_{1,i}$ and $v_{2,i}$ for each i and deleting a subset of edges with both ends in $V_1 \cup V_2$. A graph G' is a $\leq r$*-extension* of a graph G if G can be obtained from G' by deleting at most r vertices of G. A graph G is an r*-ring with perimeter* $t_1, ..., t_n$ if $t_1, ..., t_n \in V(G)$ are distinct and there is a sequence $X_1, ..., X_n$ of subsets of $V(G)$ such that

- $X_1 \cup ... \cup X_n = V(G)$, and every edge of G has both ends in some X_i,
- $t_i \in X_i$ for $1 \leq i \leq n$,
- $X_i \cap X_k \subseteq X_j$ for $1 \leq i \leq j \leq k \leq n$,
- $|X_i| \leq r$ for $1 \leq i \leq n$.

Let G_0 be a graph drawn in a surface Σ, and let $\Delta_1, ..., \Delta_d \subseteq \Sigma$ be pairwise disjoint closed disks, each meeting the drawing only in vertices of G_0, and each containing no vertices of G_0 in its interior. For $1 \leq i \leq d$, let the vertices of G_0 in the boundary of Δ_i be $t_1, ..., t_n$ say, in order, and choose an r-ring G_i with perimeter $t_1, ..., t_n$ meeting G_0 just in $t_1, ..., t_n$ and disjoint from every other G_j; and let G be the union of $G_0, G_1, ..., G_d$. We call such a graph G an *outgrowth by* d r*-rings of* G_0.

The paper is organized as follows. In Section 2 we review the notions of tangles and graph minors. In Section 3 we prove an Erdős–Pósa-type result for "spiders", trees with one vertex of degree d and all other vertices of degree one or two. In Section 4 we prove a lemma that will allow us to find a large well-behaved family of spiders, given a huge number of spiders. In Section 5 we review some theorems related to graphs embedded on a surface, and prove some other lemmas. In Section 6 we prove a structure theorem for excluding subdivisions of a fixed graph with branch vertices prescribed (Theorem 6.8), which immediately implies Theorem 1.3. In Section 7 we prove that the order of the separations mentioned in the first conclusion of Theorem 1.3 cannot be decreased, and the constants D mentioned in the first conclusion of Theorem 1.2 and the first conclusion of [2, Theorem 3] cannot be improved to be a number less than the constant $d-1$ in the first conclusion of Theorem 1.3.

2. Tangles and minors

In this section, we review some theorems about tangles and graph minors.

A *separation* of a graph G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and the *order* of (A, B) is $|V(A) \cap V(B)|$. A *tangle* $\mathcal{T}$ in G of *order* θ is a set of separations of G, each of order less than θ such that

(T1) for every separation (A, B) of G of order less than θ, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$;
(T2) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$;
(T3) if $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

The notion of tangle was first defined by Roberson and Seymour in [10]. (T1), (T2) and (T3) are called the first, second and third tangle axiom, respectively. In addition, we say that G *contains* $\mathcal{T}$ in this case. Furthermore, for $Z \subseteq V(G)$ with $|Z| < \theta$, we define $\mathcal{T} - Z$ to be the set of all separations (A', B') of $G - Z$ of order less than $\theta - |Z|$ such that there exists $(A, B) \in \mathcal{T}$ with $Z \subseteq V(A \cap B)$, $A' = A - Z$ and $B' = B - Z$. It is proved in [10, Theorem 8.5] that $\mathcal{T} - Z$ is a tangle in $G - Z$ of order $\theta - Z$.

Given a graph H, an *H-minor* of a graph G is a map α with domain $V(H) \cup E(H)$ such that the following hold.

- $\alpha(h)$ is a nonempty connected subgraph of G, for every $h \in V(H)$.
- If h_1 and h_2 are different vertices of H, then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint.
- For each edge e of H,
  - if e is not a loop, $\alpha(e)$ is an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$, where h_1, h_2 are the ends of e;
  - if e is a loop, then $\alpha(e) \in E(G) - E(\alpha(h_1))$ and every end of $\alpha(e)$ is in $\alpha(h_1)$, where h_1 is the end of e.
- If e_1, e_2 are two different edges of H, then $\alpha(e_1) \neq \alpha(e_2)$.

We say that G *contains an H-minor* if such a function α exists. For every $h \in V(H)$, $\alpha(h)$ is called a *branch set* of α. A tangle $\mathcal{T}$ in G *controls* an H-minor α if α is an H-minor such that there does not exist $(A, B) \in \mathcal{T}$ of order less than $|V(H)|$ and $h \in V(H)$ such that $V(\alpha(h)) \subseteq V(A)$.

The following theorem offers a way to obtain a tangle in a graph from a minor.

Theorem 2.1 *([10, Theorem (6.1)]). Let G and H be graphs. Let $\mathcal{T}'$ be a tangle in H of order $\theta \geq 2$. If G admits an H-minor α, and $\mathcal{T}$ is the set of separations (A, B) of G of order less than θ such that there exists $(A', B') \in \mathcal{T}'$ with $\alpha(E(A')) = E(A) \cap \alpha(E(H))$, then $\mathcal{T}$ is a tangle in G of order θ.*

The tangle $\mathcal{T}$ in Theorem 2.1 is called the *tangle induced by* $\mathcal{T}'$. We say that $\mathcal{T}'$ is *conformal* with a tangle $\mathcal{T}''$ in G if $\mathcal{T} \subseteq \mathcal{T}''$.

A *society* is a pair (S, Ω), where S is a graph and Ω is a cyclic permutation of a subset $\bar{\Omega}$ of $V(S)$. Let ρ be a nonnegative integer. A society (S, Ω) is a *ρ-vortex* if for all distinct $u, v \in \bar{\Omega}$, there do not exist $\rho + 1$ mutually disjoint paths of S between $I \cup \{u\}$ and $J \cup \{v\}$, where I is the set of vertices in $\bar{\Omega}$ after u and before v in Ω, and J is the set of vertices in $\bar{\Omega}$ after v and before u in Ω.

A *segregation* of a graph G is a set $\mathcal{S}$ of societies such that the following hold.

- S is a subgraph of G for every $(S,\Omega) \in \mathcal{S}$, and $\bigcup\{S : (S,\Omega) \in \mathcal{S}\} = G$.
- For every distinct (S,Ω) and $(S',\Omega') \in \mathcal{S}$, $V(S \cap S') \subseteq \bar{\Omega} \cap \overline{\Omega}'$ and $E(S \cap S') = \emptyset$.

We write $V(\mathcal{S}) = \bigcup\{\bar{\Omega} : (S,\Omega) \in \mathcal{S}\}$. If $\mathcal{T}$ is a tangle in G, a segregation $\mathcal{S}$ of G is *$\mathcal{T}$-central* if for every $(S,\Omega) \in \mathcal{S}$, there is no $(A,B) \in \mathcal{T}$ of order at most half of the order of $\mathcal{T}$ with $B \subseteq S$. Note that our definition of a $\mathcal{T}$-central segregation is different from the one in [15], but every segregation that is $\mathcal{T}$-central in the sense of [15] is $\mathcal{T}$-central in the sense of this paper.

A *surface* is a nonnull compact connected 2-manifold without boundary. Let Σ be a surface and $\mathcal{S} = \{(S_1,\Omega_1),...,(S_k,\Omega_k)\}$ a segregation of G. For every subset Δ of Σ, we denote the closure of Δ by $\bar{\Delta}$, and the boundary of Δ by $\partial\Delta$. An *arrangement* of $\mathcal{S}$ in Σ is a function α with domain $\mathcal{S} \cup V(\mathcal{S})$, such that the following hold.

- For $1 \leq i \leq k$, $\alpha(S_i,\Omega_i)$ is a closed disk $\Delta_i \subseteq \Sigma$, and $\alpha(x) \in \partial\Delta_i$ for each $x \in \overline{\Omega_i}$.
- For $1 \leq i \leq k$, if $x \in \Delta_i \cap \Delta_j$, then $x = \alpha(v)$ for some $v \in \overline{\Omega_i} \cap \overline{\Omega_j}$.
- For all distinct $x, y \in V(\mathcal{S})$, $\alpha(x) \neq \alpha(y)$.
- For $1 \leq i \leq k$, Ω_i is mapped by α to a natural order of $\alpha(\overline{\Omega_i})$ determined by $\partial\Delta_i$.

An arrangement is *proper* if $\Delta_i \cap \Delta_j = \emptyset$ for all $1 \leq i < j \leq k$ such that $|\overline{\Omega_i}|, |\overline{\Omega_j}| > 3$.

Given a graph H, an *H-subdivision* is a pair of functions (π_V, π_E) such that the following hold.

- $\pi_V : V(H) \to V(G)$ is an injective function.
- π_E maps loops of H to cycles in G and maps other edges of H to paths in G such that $\pi_E(e)$ contains $\pi_V(v)$, and $\pi_E(e')$ has ends $\pi_V(x)$ and $\pi_V(y)$ for every loop e with end v and every edge $e' \in E(H)$ with distinct ends x and y.
- If f_1, f_2 are two different edges in H, then $\pi_E(f_1) \cap \pi_E(f_2) \subseteq \pi_V(X)$, where X is the set of common ends of f_1 and f_2.

We say that *G admits an H-subdivision* if such a pair of functions (π_V, π_E) exists. The vertices in the image of π_V are called the *branch vertices* of (π_V, π_E).

3. Finding disjoint spiders

First, we introduce a lemma proved by Robertson and Seymour [13].

Lemma 3.1 *([13, Theorem (5.4)]). Let G be a graph, and let Z be a subset of $V(G)$ with $|Z| = \xi$. Let $k \geq \lfloor \frac{3}{2}\xi \rfloor$, and let α be a K_k-minor in G. If there is no separation (A,B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(h) = \emptyset$ for some $h \in V(K_k)$, then for every partition $(Z_1,...,Z_n)$ of Z into non-empty subsets, there are n connected graphs $T_1,...,T_n$ of G, mutually disjoint and $V(T_i) \cap Z = Z_i$ for $1 \leq i \leq n$.*

A *d-spider with head* v is a tree such that every vertex other than v in the tree has degree at most 2, and the degree of v is d. A *leaf* is a vertex of degree one. Let G be a graph, and let S, Y be subsets of $V(G)$. A *d-spider from* S *to* Y is a d-spider with head $v \in S$ whose leaves are in Y.

Let G be a graph and $\mathcal{T}$ a tangle in G. We say that a subset X of $V(G)$ is *free* if there exists no $(A, B) \in \mathcal{T}$ of order less than $|X|$ such that $X \subseteq V(A)$.

Lemma 3.2. *Let G be a graph and H be a graph on h vertices of maximum degree at most d. Let $t \geq \lfloor \frac{3hd}{2} \rfloor$. Let $\mathcal{T}$ be a tangle of order at least hd in G that controls a K_t-minor. If there exist pairwise disjoint sets $X_1, X_2, ..., X_h$ such that for $1 \leq i \leq h$ the set X_i consists of a vertex v_i of G and $d-1$ of its neighbors and $\bigcup_{i=1}^{h} X_i$ is free with respect to $\mathcal{T}$, then G has an H-subdivision with branch vertices $v_1, v_2, ..., v_h$.*

Proof. Let $Z = \bigcup_{i=1}^{h} X_i$, and let α be a K_t-minor controlled by $\mathcal{T}$. Suppose that there exists a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(v) = \emptyset$ for some $v \in V(K_t)$. By the first tangle axiom, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$. Since Z is free, $(B, A) \in \mathcal{T}$. But it is a contradiction since $t \geq hd$ and $\mathcal{T}$ controls α. Therefore, there does not exist a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(v) = \emptyset$ for some $v \in V(K_t)$.

Denote $V(H)$ by $\{u_1, u_2, ..., u_h\}$ and $E(H)$ by $\{e_1, e_2, ..., e_{|E(H)|}\}$. Since the maximum degree of H is at most d, there exist $Z_0 \subseteq Z$ and a partition $(Z_1, Z_2, ..., Z_{|E(H)|})$ of $Z - Z_0$ such that for every $1 \leq \ell \leq |E(H)|$, Z_ℓ consists of two distinct vertices where one is in X_i and one is in X_j, where the ends of e_ℓ are u_i and u_j. By Lemma 3.1, there exist $|E(H)|$ pairwise disjoint paths in $G - Z_0$ connecting the two vertices of each part of $(Z_1, Z_2, ..., Z_{|E(H)|})$. This creates a subdivision of H with branch vertices $v_1, v_2, ..., v_h$. □

Theorem 3.3 *([8, Theorem 3.3]). Let G be a graph and $\mathcal{T}$ a tangle in G of order θ. Let $\{X_j \subseteq V(G) : j \in J\}$ be a family of subsets of $V(G)$ indexed by J. Let d, k be integers with $\theta \geq (k+d)^{d+1} + d$. If $|X_j| = d$ for every $j \in J$, then there exists a set $J' \subseteq J$ satisfying the following.*

1. *For distinct elements j, j' of J', X_j and $X_{j'}$ are disjoint.*
2. *$\bigcup_{j \in J'} X_j$ is free.*
3. *If $|\bigcup_{j \in J'} X_j| < k$, then there exists a set Z with $\bigcup_{j \in J'} X_j \subseteq Z$ and $|Z| \leq (k+d)^{d+1}$ satisfying that for all $j \in J$, either $X_j \cap Z \neq \emptyset$, or X_j is not free in $\mathcal{T} - Z$.*

Theorem 3.4. *Let h and d be positive integers. Let G be a graph, and let S be a subset of vertices of degree at least $d-1$ in G. Let $\mathcal{T}$ be a tangle in G of order θ. If $\theta \geq (hd+1)^{d+1} + d$, then either*

1. *there exist h vertices $v_1, v_2, ..., v_h \in S$ and h pairwise disjoint subsets $X_1, X_2, ..., X_h$ of $V(G)$, where X_i consists of v_i and $d-1$ neighbors of v_i for each $1 \leq i \leq h$, such that $\bigcup_{i=1}^{h} X_i$ is free in $\mathcal{T}$, or*
2. *there exists a set $C \subseteq V(G)$ with $|C| \leq (hd+1)^{d+1}$ such that for every $v \in S - C$, there exists $(A, B) \in \mathcal{T} - C$ of order less than d such that $v \in V(A) - V(B)$.*

Proof. Let $\{X_j : j \in J\}$ be the collection of the d-element subsets consisting of one vertex v_j in S and $d-1$ of its neighbors. Applying Theorem 3.3 by further taking $k = (h-1)d+1$, we know there exists $J' \subseteq J$ such that $X_j \cap X_{j'} = \emptyset$ for every distinct j, j' in J', and $\bigcup_{j \in J'} X_j$ is free. Furthermore, if $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists a set C with $\bigcup_{j \in J'} X_j \subseteq C \subseteq V(G)$ and $|C| \leq (hd+1)^{d+1}$ satisfying that for all $j \in J$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$.

Observe that if $|\bigcup_{j \in J'} X_j| > (h-1)d$, then $|J'| \geq h$ and the first statement holds. So we assume that $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, and we shall prove that the second statement of this theorem holds. Let $v \in S - C$. Suppose that there does not exist $(A, B) \in \mathcal{T} - C$ of order less than d such that $v \in V(A) - V(B)$. In particular, v has at least d neighbors in $G - C$. Let U be the collection of those X_j that are disjoint from C and consist of v and $d-1$ neighbors of v. For every member X_j of U, we define the *rank* of X_j to be the minimum order of a separation $(A, B) \in \mathcal{T} - C$ such that $X_j \subseteq V(A)$. As no member of U is free, the rank of each member of U is at most $d-1$. Let r be the maximum rank of any member of U, and let X be a member of U of rank r. Let $(A, B) \in \mathcal{T} - C$ of order r such that $X \subseteq V(A)$, and subject to that, $|V(B) - V(A)|$ is as small as possible. By the assumption, $v \in V(A) \cap V(B)$ and $r \leq d-1$. On the other hand, there exist r disjoint paths in $G - C$ from $X - \{v\}$ to $V(B)$, as v is adjacent to all vertices in $X - \{v\}$. We denote these r disjoint paths by $P_1, P_2, ..., P_r$, and denote the end of P_i in $X - \{v\}$ by u_i for $1 \leq i \leq r$. As $v \in V(A) \cap V(B)$ and $|V(A) \cap V(B)| = r$, $v \in V(P_i)$ for some $1 \leq i \leq r$. Without loss of generality, we may assume that $v \in V(P_r)$. In addition, v is adjacent to a vertex u in $V(B) - V(A)$, otherwise, the rank of X is smaller than r. As $(X - \{u_r\}) \cup \{u\}$ is a member of U, its rank is at most r. Let $(A', B') \in \mathcal{T} - C$ be a separation of order at most r such that $(X - \{u_r\}) \cup \{u\} \subseteq V(A')$. $X \subseteq V(A \cup A')$ and $u \in (V(B) - V(A)) - (V(B \cap B') - V(A \cup A'))$, so the order of $(A \cup A', B \cap B')$ is at least $r+1$ by the choice of (A, B). It implies that the order of $(A \cap A', B \cup B')$ is at most $r-1$. Notice that $v \in V(A') \cap V(B')$ by the assumption, so $((A \cap A') - \{v\}, (B \cup B') - \{v\})$ is a separation of $G - (C \cup \{v\})$ of order less than $r-1$. But $P_1, P_2, ..., P_{r-1}$ are $r-1$ disjoint paths from $V(A \cap A') - \{v\}$ to $V(B \cup B') - \{v\}$ in $G - (C \cup \{v\})$, a contradiction. This proves the second statement. □

We need the following variation of Theorem 3.4. A version for edge-disjoint spiders was proved in [7] and [8, Theorem 3.1].

Theorem 3.5. *Let G be a graph, and let X, Y be disjoint subsets of $V(G)$. Let h, d be nonnegative integers. Then either there exist h disjoint d-spiders from X to Y, or there*

exists $C \subseteq V(G)$ *with* $|C| \leq \frac{3}{2}(hd+1)^{d+1} + \frac{d}{2} + 1$ *such that every* d*-spider from* X *to* Y *intersects* C*.*

Proof. Note that for every subset C of Y such that $|Y - C| \leq d-1$, every d-spider from X to Y intersects C. So we may assume that $|Y| \geq \frac{3}{2}((hd+1)^{d+1} + d)$, otherwise we are done. Let G' be the graph obtained from G by adding edges such that Y is a clique in $G'[Y]$. As every complete graph on k vertices contains a tangle of order $\lfloor 2k/3 \rfloor$, $G'[Y]$ contains a tangle of order $(hd+1)^{d+1} + d$. And $G'[Y]$ is a minor of G', so G' contains a tangle $\mathcal{T}$ of order $(hd+1)^{d+1} + d$ induced by a tangle of the same order in $G'[Y]$ by Theorem 2.1 such that $Y \subseteq V(B)$ for every $(A, B) \in \mathcal{T}$. Let $\{X_j : j \in J\}$ be the collection of d-element subsets of $V(G)$ such that every X_j consists of one vertex x in X and $d-1$ neighbors of x. By Theorem 3.3, there exists $J' \subseteq J$ such that $X_j \cap X_{j'} = \emptyset$ for every distinct j, j' in J', and $\bigcup_{j \in J'} X_j$ is free. Furthermore, if $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd+1)^{d+1}$ satisfying that for all $j \in J$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$.

First, assume that $|\bigcup_{j \in J'} X_j| > (h-1)d$, so $|J'| \geq h$. Let $\{1, 2, ..., h\} \subseteq J'$, and for $1 \leq j \leq h$, let x_j be a vertex in $X_j \cap X$ adjacent to all other vertices in X_j. Suppose that there do not exist dh disjoint paths from $\bigcup_{j=1}^{h} X_j$ to Y in G'. Then there exists a separation (A, B) of G' of order less than dh such that $\bigcup_{j=1}^{h} X_j \subseteq V(A)$ and $Y \subseteq V(B)$. Since $Y \subseteq V(B)$, we know that $(A, B) \in \mathcal{T}$. But it implies that $\bigcup_{j=1}^{h} X_j$ is not free, a contradiction. Hence, there exist dh disjoint paths from $\bigcup_{j=1}^{h} X_j$ to Y in G'. That is, there exist h disjoint d-spiders from $\{x_j : 1 \leq j \leq h\}$ to Y in G'. We are done in this case since every d-spider from X to Y in G' contains a d-spider from X to Y in G as a subgraph.

So we may assume that $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $\bigcup_{j \in J'} X_j \subseteq C$ and $|C| \leq (hd+1)^{d+1}$ satisfying that for all $j \in J$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$. Let $v \in X - C$, and let D be a d-spider from v to Y in G. Note that D is also a d-spider from v to Y in G'. Suppose that D is disjoint from C. So $V(D)$ contains some X_j such that $v \in X_j$ and $X_j \cap C = \emptyset$. Since X_j is not free in $\mathcal{T} - C$, there exists $(A, B) \in \mathcal{T} - C$ of order less than d such that $X_j \subseteq V(A)$ and $Y - C \subseteq V(B)$. It is a contradiction since there exist d disjoint paths in D from $V(A)$ to $V(B)$. This proves that D intersects C. □

4. Taming spiders

A *surface* is a compact 2-manifold. An *O-arc* is a subset homeomorphic to a circle, and a *line* is a subset homeomorphic to $[0, 1]$. Let Σ be a surface. A *drawing* Γ in Σ is a pair (U, V), where $V \subseteq U \subseteq \Sigma$, U is closed, V is finite, $U - V$ has only finitely many arc-wise connected components, called *edges*, and for every edge e, either $\bar{e}$ is a line whose set of ends are $\bar{e} \cap V$, or $\bar{e}$ is an O-arc and $|\bar{e} \cap V| = 1$. The components of $\Sigma - U$ are called *regions*. The members of V are called *vertices*. For a drawing $\Gamma = (U, V)$, we write $U = U(\Gamma), V = V(\Gamma)$, and $E(\Gamma), R(\Gamma)$ are defined to be the set of edges and the set of

regions, respectively. If v is a vertex of a drawing Γ and e is an edge or a region of Γ, we say that e is *incident with* v if v is contained in the closure of e. Note that the incidence relation between $V(\Gamma)$ and $E(\Gamma)$ defines a graph, and we say that Γ is a *drawing of* G in Σ if G is defined by this incidence relation. In this case, we say that G is *embeddable* in Σ, or G can be *drawn* in Σ.

We say that (S, Ω, Ω_0) is a *neighborhood* if S is a graph and Ω, Ω_0 are cyclic permutations with $\bar{\Omega}, \overline{\Omega_0} \subseteq V(S)$. A neighborhood (S, Ω, Ω_0) is *rural* if S has a drawing Γ in the plane and there are disks $\Delta_0 \subseteq \Delta$ such that

- Γ uses no point outside Δ and none in the interior of Δ_0, and
- $\bar{\Omega}$ are the vertices in $\Gamma \cap \partial\Delta$, and $\overline{\Omega_0}$ are the vertices in $\Gamma \cap \Delta_0$, and
- the cyclic permutations of $\bar{\Omega}$ and $\overline{\Omega_0}$ coincide with the natural cyclic order on Δ and Δ_0.

In this case, we say that $(\Gamma, \Delta, \Delta_0)$ is a *presentation* of (S, Ω, Ω_0). For a fixed presentation $(\Gamma, \Delta, \Delta_0)$ of a neighborhood (S, Ω, Ω_0) and an integer $s \geq 0$, an *s-nest* for $(\Gamma, \Delta, \Delta_0)$ is a sequence $(C_1, C_2, ..., C_s)$ of pairwise disjoint cycles of S such that $\Delta_0 \subseteq \Delta_1 \subseteq ... \subseteq \Delta_s \subseteq \Delta$, where Δ_i is the closed disk in the plane bounded by C_i in the drawing Γ.

If (S, Ω, Ω_0) is a neighborhood and (S_0, Ω_0) is a society, then $(S \cup S_0, \Omega)$ is a society and we call this society the *composition* of the society (S_0, Ω_0) with the neighborhood (S, Ω, Ω_0). A society (S, Ω) is *s-nested* if it is the composition of a society with a rural neighborhood that has an s-nest for some presentation of it.

A subgraph $F \subseteq S$ for a rural neighborhood (S, Ω, Ω_0) with presentation $(\Gamma, \Delta, \Delta_0)$ is *perpendicular* to an s-nest $(C_1, C_2, ..., C_s)$ for $(\Gamma, \Delta, \Delta_0)$ if for every component P of F

- P is a path with one end in $\bar{\Omega}$ and the other in $\overline{\Omega_0}$, and
- $P \cap C_i$ is a path for all $i = 1, 2, ..., s$.

We shall use the following theorem, which was proved in [4], to prove the main theorem of this section. We present a simplified restatement of it.

Theorem 4.1 *([4, Theorem 10.5]). For every three positive integers $s, k \geq 3, c$, there exists an integer $s'(s, k, c)$ such that for every s'-nested society (S, Ω) that is a composition of a society (S_0, Ω_0) with a rural neighborhood with an s'-nest, and for every union of c pairwise disjoint k-spiders F_0 from $V(S_0) - \overline{\Omega_0}$ to $\bar{\Omega}$, where every vertex in $V(F_0) \cap \bar{\Omega}$ is a leaf in F_0, there exists a union of c pairwise disjoint k-spiders F in (S, Ω) from the set of the heads of F_0 to the set of leaves of F_0 such that (S, Ω) can be expressed as a composition of some society with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $S' \cap F$ is perpendicular to $(C_1, C_2, ..., C_s)$.*

Given a cyclic ordering Ω on a set $\bar{\Omega}$, an *interval* of Ω is a subset I of $\bar{\Omega}$ such that the elements of I are consecutive elements in Ω.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{X})$, where T is a tree and $\mathcal{X}$ is a collection $\{X_t : t \in V(G)\}$ of subsets of $V(G)$, such that the following hold.

- $\bigcup_{t \in V(T)} X_t = V(G)$.
- For each edge e of G, there exists $t \in V(T)$ such that X_t contains the ends of e.
- For each vertex v of G, the subgraph of T induced by $\{t : v \in X_t\}$ is connected.

The *width* of $(T, \mathcal{X})$ is $\max\{|X_t|-1 : t \in V(T)\}$. The *adhesion* of $(T, \mathcal{X})$ is $\max\{|X_t \cap X_{t'}| : tt' \in E(T)\}$. For each $t \in V(T)$, the *torso* at t is the graph L obtained from the subgraph of G induced by X_t by adding edges such that for each neighbor t' of t in T, $X_t \cap X_{t'}$ is a clique in L. When the tree T is a path, we say $(T, \mathcal{X})$ is a *path-decomposition*. The *tree-width* of G is the minimum width of a tree-decomposition of G.

Now, we are ready to prove the main theorem of this section.

Theorem 4.2. *For every positive integers $d \geq 3, \rho, k$ and s, there exist integers $s' = s'(k, d, s, \rho)$ and $k' = k'(k, \rho)$ such that for every s'-nested society (S, Ω) that is a composition of a ρ-vortex (S_0, Ω_0) with a rural neighborhood that has an s'-nest, and for every k' pairwise disjoint d-spiders $D_1, D_2, ..., D_{k'}$ from $V(S_0) - \overline{\Omega_0}$ to $\bar{\Omega}$ such that every vertex of $D_i \cap \bar{\Omega}$ is a leaf of D_i, there exist k pairwise disjoint d-spiders $D'_1, D'_2, ..., D'_k$ from $V(S_0)$ to $\bar{\Omega}$ such that the following hold.*

1. *(S, Ω) can be expressed as a composition of a society (S'_0, Ω') with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $D'_i \cap S'$ is perpendicular to $(C_1, C_2, ..., C_s)$ for every $1 \leq i \leq k$.*
2. *For every $1 \leq i \leq k$, the head of D'_i is the head of $D_{i'}$ for some $1 \leq i' \leq k'$.*
3. *For every $1 \leq i \leq k$, every leaf of D'_i is a leaf of $D_1 \cup D_2 \cup ... \cup D_{k'}$.*
4. *For every $1 \leq i \leq k$, there exists an interval I_i of Ω containing all leaves of D'_i such that I_i is disjoint from I_j for every $j \neq i$.*

Proof. Let $s'(k, d, s, \rho) = s'_{4.1}(s, d, (12\rho+7)k)$ and $k'(k, \rho) = (12\rho+7)k$, where $s'_{4.1}$ is the function s' mentioned in Theorem 4.1. By Theorem 4.1, there exist $(12\rho + 7)k$ pairwise disjoint d-spiders $D'_1, D'_2, ..., D'_{k'}$ from the set of the heads of $D_1, D_2, ..., D_{k'}$ to the union of the set of leaves of $D_1, D_2, ..., D_{k'}$ such that (S, Ω) can be expressed as a composition of some society with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $D'_i \cap S'$ is perpendicular to $(C_1, C_2, ..., C_s)$ for every $1 \leq i \leq k'$.

Let $\overline{\Omega_0} = \{v_1, v_2, ..., v_{|\overline{\Omega_0}|}\}$ in order. Since the head of each D'_i is contained in $V(S_0) - \overline{\Omega_0}$, each component of $D'_i - (V(S_0) - \overline{\Omega_0})$ is a path. For each $1 \leq i \leq k'$, let W_i be the subset of $[|\overline{\Omega_0}|]$ such that for each $j \in W_i$, some component of $D'_i - (V(S_0) - \overline{\Omega_0})$ is a path from a leaf of D'_i to v_j, and let $a_i = \min W_i$ and $b_i = \max W_i$. For each $1 \leq i \leq k'$, define ℓ_i and r_i to be the leaves of D'_i such that there exists a path in $S - (V(S_0) - \overline{\Omega_0}) \cap D'_i$ from ℓ_i to v_{a_i} and from r_i to v_{b_i}, respectively; define I_i to be the interval of Ω with

ends ℓ_i and r_i containing all leaves of D'_i. Then it is sufficient to prove that there exist $1 \leq i_1 < i_2 < ... < i_k \leq k'$ such that $I_{i_1}, I_{i_2}, ..., I_{i_k}$ are pairwise disjoint.

Since (S_0, Ω_0) is a ρ-vortex, by Theorem 8.1 in [9], there exists a path-decomposition $(t_1 t_2 ... t_{|\overline{\Omega_0}|}, \mathcal{X})$ of S_0, where $\mathcal{X} = (X_{t_i} : 1 \leq i \leq |\overline{\Omega_0}|)$, such that $|X_{t_i} \cap X_{t_j}| \leq \rho$ for every $1 \leq i < j \leq |\overline{\Omega_0}|$ and $v_i \in X_{t_i}$ for every $1 \leq i \leq |\overline{\Omega_0}|$. And we define $X_0 = \emptyset$ and $X_{|\overline{\Omega_0}|+1} = \emptyset$. Furthermore, for each $1 \leq i \leq k'$, let Q_i be the path in D'_i from r_i to ℓ_i, L_i be the path in D'_i from ℓ_i to v_{a_i}, and R_i be the path in D'_i from r_i to v_{b_i}. We say that I_i is *naughty* if $Q_i - ((S_0 - \overline{\Omega_0}) \cup V(L_i) \cup V(R_i))$ contains a path from v_c to $v_{c'}$, where $1 \leq c < a_i < b_i < c' \leq |\overline{\Omega_0}|$ such that Q_i passes through $\ell_i, v_c, v_{c'}, r_i$ in the order listed; otherwise we say I_i is *nice*.

Claim 1. *If $I_i \cap I_j \neq \emptyset$ for some $i \neq j$, and either both I_i, I_j are naughty or both I_i, I_j are nice, then there exists $a \in [|\overline{\Omega}|]$ such that $X_a \cap X_{a+1} \cap V(D'_i) \neq \emptyset \neq X_a \cap X_{a+1} \cap V(D'_j)$.*

Proof of Claim 1. Since $V(D'_i) \cap V(D'_j) = \emptyset$, we may assume that $b_i < b_j$ by symmetry. Since $I_i \cap I_j \neq \emptyset$, $a_j < b_i < b_j$. Since (S, Ω) is a composition of (S_0, Ω_0) with a rural neighborhood, and $V(D'_i) \cap V(D'_j) = \emptyset$, the vertices ℓ_j, r_i, r_j appear in Ω in the order listed. Note that Q_i passes through v_{b_i}, a vertex in $V(S_0) - \overline{\Omega_0}$, and v_{a_i} in the order listed. So $V(Q_i) \cap X_{t_{b_i}} \cap (X_{t_{b_i-1}} \cup X_{t_{b_i+1}}) \neq \emptyset$. Similarly, $V(Q_i) \cap X_{t_{a_i}} \cap (X_{t_{a_i-1}} \cup X_{t_{a_i+1}})$, $V(Q_j) \cap X_{t_{a_j}} \cap (X_{t_{a_j-1}} \cup X_{t_{a_j+1}})$ and $V(Q_j) \cap X_{t_{b_j}} \cap (X_{t_{b_j-1}} \cup X_{t_{b_j+1}})$ are non-empty sets. If $V(Q_i) \cap X_{t_{b_j-1}} \cap X_{t_{b_j}} \neq \emptyset \neq V(Q_i) \cap X_{t_{b_j}} \cap X_{t_{b_j+1}}$, then we are done by choosing $a = b_j - 1$ or $a = b_j$. So we may assume that $V(Q_i) \cap X_{t_{b_j-1}} \cap X_{t_{b_j}} = \emptyset$ or $V(Q_i) \cap X_{t_{b_j}} \cap X_{t_{b_j+1}} = \emptyset$.

Let Y_r be one of $X_{t_{b_j-1}} \cap X_{t_{b_j}}$ and $X_{t_{b_j}} \cap X_{t_{b_j+1}}$ such that $V(Q_i) \cap Y_r$ is empty. Similarly, there exists $Y_\ell \in \{X_{t_{a_j-1}} \cap X_{t_{a_j}}, X_{t_{a_j}} \cap X_{t_{a_j+1}}\}$ such that $V(Q_i) \cap Y_\ell = \emptyset$; otherwise we are done by taking $a = a_j - 1$ or $a = a_j$. Hence $V(L_j) \cup V(R_j) \cup Y_\ell \cup Y_r$ is disjoint from $V(D'_i)$. So $a_j < a_i < b_i < b_j$ and I_i is nice.

If $V(D'_i) \cap X_{t_{b_i-1}} \cap X_{t_{b_i}} = \emptyset$, then v_{a_i} and the neighbor of v_{b_i} in $Q_i - V(R_i)$ belong to different components of $S - (V(R_i) \cup (X_{t_{b_i-1}} \cap X_{t_{b_i}}) \cup V(R_j) \cup Y_r)$, but $Q_i - (V(R_i) \cup (X_{t_{b_i-1}} \cap X_{t_{b_i}}) \cup V(R_j) \cup Y_r)$ contains a path passing through these two vertices, a contradiction. So $V(D'_i) \cap X_{t_{b_i-1}} \cap X_{t_{b_i}} \neq \emptyset$. Since I_i is nice, I_j is nice. If $V(D'_j) \cap X_{t_{b_i-1}} \cap X_{t_{b_i}} = \emptyset$, then since I_j is nice, ℓ_j and r_j belong to different components of $D'_j - (X_{t_{b_i-1}} \cup X_{t_{b_i}})$, but $Q_j - (X_{t_{b_i-1}} \cup X_{t_{b_i}})$ contains a path passing through these two vertices, a contradiction. Therefore, $V(D'_j) \cap X_{t_{b_i-1}} \cap X_{t_{b_i}} \neq \emptyset$. This proves the claim by taking $a = b_i - 1$. □

Suppose that there do not exist such k pairwise disjoint intervals among $I_1, I_2, ..., I_{k'}$. Let H be the intersection graph of $I_1, I_2, ..., I_{k'}$. Then H does not contain an independent set of size k. We claim that H contains a clique with size at least $12\rho + 7$. Let u_1, u_2 be consecutive vertices in Ω. If there are at least $12\rho + 7$ members of $\{I_1, I_2, ..., I_{k'}\}$ containing both u_1, u_2, then these $12\rho + 7$ members form a clique in H. So we may assume that there exist at most $12\rho + 6$ members of $\{I_1, I_2, ..., I_{k'}\}$ containing both

u_1, u_2. Let H' be the intersection graph of the members of $\{I_1, I_2, ..., I_{k'}\}$ not containing both u_1, u_2. Then H' is an interval graph on at least $k' - 12\rho - 6$ vertices, and H' is an induced subgraph of H. Since H has no independent set of size k, H' does not have an independent set of size k. Since interval graphs are prefect, H' (and hence H) has a clique with size at least $(k' - 12\rho - 6)/(k-1) \geq 12\rho + 7$.

Therefore, there exist $12\rho+7$ pairwise intersecting members of $\{I_1, I_2, ..., I_{k'}\}$. So there exist $6\rho + 4$ pairwise intersecting members of $\{I_1, I_2, ..., I_{k'}\}$ such that either all of them are naughty or all of them are nice. Let G be the graph obtained from S by adding edges such that vertices in $X_i \cap X_{i+1}$ are pairwise adjacent in G, for every $1 \leq i \leq |\overline{\Omega_0}| - 1$. Since there exist $6\rho+4$ pairwise intersecting members of $\{I_1, I_2, ..., I_{k'}\}$ where either all of them are naughty or all of them are nice, Claim 1 implies that G contains a $K_{6\rho+4}$-minor, where each branch set is D'_i for some i. Without loss of generality, we may assume that the branch sets of the $K_{6\rho+4}$-minor are $D'_1, D'_2, ..., D'_{6\rho+4}$.

Let G' be the graph obtained from G by the following operations.

- Deleting vertices not in $D'_1 \cup D'_2 \cup ... \cup D'_{6\rho+4}$.
- Contracting each path in $D'_i - E(S_0)$ from $\overline{\Omega_0}$ to $\overline{\Omega}$ internally disjoint from $V(S_0)$ into a vertex for every $1 \leq i \leq 6\rho + 4$.
- For every $1 \leq i \leq 6\rho + 4$ and for each path in $D'_i - E(S_0)$ from $\overline{\Omega_0}$ to $\overline{\Omega_0}$ internally disjoint from $V(S_0)$, contracting all edges incident with at least one internal vertex of this path into an edge.
- Repeatedly contracting edges of $D'_i[X_{t_j}]$ having an end not in $\{v_j\} \cup ((X_{t_{j-1}} \cup X_{t_{j+1}}) \cap X_{t_j})$ until each remaining edge has ends in $\{v_j\} \cup ((X_{t_{j-1}} \cup X_{t_{j+1}}) \cap X_{t_j})$ for every $1 \leq i \leq 6\rho + 4$ and $1 \leq j \leq |\overline{\Omega_0}|$.

Note that G' contains a $K_{6\rho+4}$-minor, so the tree-width of G' is at least $6\rho + 3$. Observe that $V(G') \subseteq V(S_0)$. Furthermore, G' can be written as $G_0 \cup G_1$ such that $V(G_0 \cap G_1) = V(G_0) \subseteq \overline{\Omega_0}$, and G_0 is an outerplanar graph that can be drawn in the plane such that the vertices of $V(G_0 \cap G_1)$ are in the boundary of a region in order, and G_1 has a path-decomposition of width at most 2ρ such that each bag is a subset of $\{v_j\} \cup (X_{t_j} \cap (X_{t_{j-1}} \cup X_{t_{j+1}}))$ (for some j) and contains a vertex in $V(G_0 \cap G_1)$ in order. By Lemma 8.1 in [1], G' has tree-width less than $6\rho + 3$, a contradiction. This proves the theorem. □

5. Theorems on surfaces

In this section, we recall some results about graphs embedded in surfaces.

Let Σ be a surface and let Γ be a drawing in Σ. The sets $\{v\}$, for $v \in V(\Gamma)$, the edges and regions of Γ are called the *atoms* of Γ. A *subdrawing* Γ' of Γ is a drawing satisfying $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$; we write $\Gamma' \subseteq \Gamma$ if Γ' is a subdrawing of Γ. A drawing is 2-*cell* if every region is an open disk.

Let Γ be a 2-cell drawing in a surface Σ. We say that a drawing K in Σ is a *radial drawing* of Γ if it satisfies the following conditions.

- $U(\Gamma) \cap U(K) = V(\Gamma) \subseteq V(K)$.
- Each region r of Γ contains a unique vertex of K.
- K is a drawing of a bipartite graph, and $(V(\Gamma), V(K) - V(\Gamma))$ is a bipartition of it.
- For every $v \in V(\Gamma)$, the edges of $K \cup \Gamma$ incident with v belong alternately to Γ and to K (in their cyclic order around v).

Let Σ be a surface, and let Γ be a drawing in Σ. A subset Z of Σ is Γ-*normal* if $Z \cap U(\Gamma) \subseteq V(\Gamma)$. If Σ is not a sphere, we say that Γ is θ-*representative* if $|F \cap V(\Gamma)| \geq \theta$ for every non-null-homotopic Γ-normal O-arc F in Σ. If $\Delta \subseteq \Sigma$ is a closed set such that either $\bar{e} \subseteq \Delta$ or $e \cap \Delta = \emptyset$ for each $e \in E(\Gamma)$, then we define $\Gamma \cap \Delta$ to be the drawing $(U(\Gamma) \cap \Delta, V(\Gamma) \cap \Delta)$.

Let Σ be a surface, and let Γ be a drawing of a graph G in Σ. A *tangle* in Γ and a *separation* of Γ are a tangle in G and a separation of G, respectively. A tangle $\mathcal{T}$ in Γ of order θ is said to be *respectful (towards* Σ*)* if Σ is connected and for every Γ-normal O-arc F in Σ with $|F \cap V(\Gamma)| < \theta$, there is a closed disk $\Delta \subseteq \Sigma$ with $\partial\Delta = F$ such that $(\Gamma \cap \Delta, \Gamma \cap \overline{\Sigma - \Delta}) \in \mathcal{T}$. It is clear that Δ has to be unique, and we write $\Delta = \mathrm{ins}(F)$; the function *ins* is called the *inside function* of $\mathcal{T}$. Assume that Γ is 2-cell, and let K be a radial drawing of Γ. If W is a closed walk of K, we define $K|W$ to be the subdrawing of K formed by the vertices and the edges in W. If the length of W is less than 2θ, then we define $\mathrm{ins}(W)$ to be the union of $U(K|W)$ and $\mathrm{ins}(C)$, taken over all cycles C of $K|W$. For every two atoms a, b of K, define a function $m_{\mathcal{T}}(a, b)$ as follows:

- if $a = b$, then $m_{\mathcal{T}}(a, b) = 0$;
- if $a \neq b$ and $a, b \subseteq \mathrm{ins}(W)$ for some closed walk W of K of length less than 2θ, then $m_{\mathcal{T}}(a, b) = \min \frac{1}{2}|E(W)|$, taking over all such closed walks W;
- otherwise, $m_{\mathcal{T}}(a, b) = \theta$.

Note that K is bipartite, so $m_{\mathcal{T}}$ is integral. In addition, for every atom c of Γ, we define $a(c)$ to be an atom of K such that

- $a(c) = c$ if $c \subseteq V(\Gamma)$;
- $a(c)$ is the region of K including c if c is an edge of Γ;
- $a(c) = \{v\}$, where v is the vertex of K in c, if c is a region of Γ.

For every atoms b, c of Γ, we define $m_{\mathcal{T}}(b, c) = m_{\mathcal{T}}(a(b), a(c))$. If X, Y are sets of atoms of G, then we define $m_{\mathcal{T}}(X, Y) = \min\{m_{\mathcal{T}}(x, y) : x \in X, y \in Y\}$. When one of X and Y, say Y, has size one, then we denote $m_{\mathcal{T}}(X, Y)$ by $m_{\mathcal{T}}(X, y)$, where y is the unique element of Y.

The following is a consequence of Theorem 9.1 of [11].

Theorem 5.1. *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ. If $\mathcal{T}$ is a respectful tangle in Γ, then $m_{\mathcal{T}}$ is a metric on the atoms of Γ.*

The following theorem is useful.

Theorem 5.2 *([12, Theorem (1.1)]). Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ with $E(\Gamma) \neq \emptyset$. Let $\mathcal{T}$ be a respectful tangle of order θ in Γ, and let K be a radial drawing of Γ. Let (A, B) be a separation of Γ of order less than θ. Then $(A, B) \in \mathcal{T}$ if and only if for every edge e of A, there exists a cycle C of K with $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$ and with $e \subseteq \text{ins}(C)$.*

Theorem 5.3. *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ with $E(\Gamma) \neq \emptyset$. Let $\mathcal{T}$ be a respectful tangle of order θ in Γ. Then the following hold.*

1. *If $x \in V(\Gamma)$ and $(A, B) \in \mathcal{T}$ is a separation of Γ such that $x \in V(A) - V(B)$ and there exists a path P in A from x to a vertex $y \in V(A)$ internally disjoint from $V(B)$, then $m_{\mathcal{T}}(x, y) \leq |V(A) \cap V(B)|$.*
2. *If $X \subseteq V(\Gamma)$ and $(A, B) \in \mathcal{T}$ is a separation of Γ of order less than $|X|$ such that $X \subseteq V(A)$, subject to this, A is minimal, then $m_{\mathcal{T}}(X, y) \leq |V(A) \cap V(B)|$ for every $y \in V(A)$.*

Proof. We first prove Statement 1. Let x and (A, B) be a vertex and a separation mentioned in the statement of Conclusion 1. Let $y \in V(A)$ be a vertex different from x. Let P be a path in A from x to y internally disjoint from $V(B)$. Let e be the edge in P incident with x. By Theorem 5.2, there exists a cycle C of the radial drawing K of Γ with $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$ and with $e \subseteq \text{ins}(C)$. So $x \in \text{ins}(C)$. If $y \notin \text{ins}(C)$, then C intersects P at an internal vertex of P. However, $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$. This implies that some internal vertex of P is in $V(A) \cap V(B)$, a contradiction. Hence, $y \in \text{ins}(C)$. Therefore, $m_{\mathcal{T}}(x, y) \leq |V(A) \cap V(B)|$.

Now we prove Statement 2. Let X and (A, B) be a set and a separation mentioned in the statement of Conclusion 2. Since $|V(A \cap B)| < |X|$ and $X \subseteq V(A)$, $X - V(B) \neq \emptyset$. Since A is minimal, every component of $A - V(A \cap B)$ intersects $X - V(B)$, and each vertex in $V(A \cap B)$ is either in X or adjacent to some vertex in $V(A) - V(B)$. So for every $y \in V(A)$, either $y \in X$, or there exists a path in A from a vertex $x_y \in X - V(B)$ to y internally disjoint from $V(B)$. If $y \in X$, then $m_{\mathcal{T}}(X, y) = 0$; otherwise, $m_{\mathcal{T}}(x_y, y) \leq |V(A \cap B)|$ by Statement 1. Hence $m_{\mathcal{T}}(X, y) \leq |V(A \cap B)|$ for every $y \in V(A)$. □

Theorem 5.4 *([11, Theorem (8.12)], [12, Theorem (1.2)]). Let $\mathcal{T}$ be a respectful tangle of order θ, where $\theta \geq 2$, in a 2-cell drawing Γ in a connected surface Σ. If c is an atom in Γ, then there exists an edge e of Γ such that $m_{\mathcal{T}}(c, e) = \theta$.*

Let Γ be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle of order θ in Γ. Let x be an atom of Γ. A *λ-zone around x* is an open disk Δ in Σ with $x \subseteq \Delta$, such that $\partial\Delta$ is an O-arc, $\partial\Delta \subseteq U(\Gamma)$, $m_{\mathcal{T}}(x, y) \leq \lambda$ for every atom y of Γ with $y \subseteq \bar{\Delta}$, and if $x \in E(\Gamma)$, then $\lambda \geq 2$. A *λ-zone* is a λ-zone around some atom.

Let Δ be a λ-zone. Note that $U(\Gamma) \cap \partial\Delta$ is a cycle, and the drawing $\Gamma' = \Gamma \cap (\Sigma - \Delta)$ is 2-cell in Σ. We say that Γ' is the *drawing obtained from* Γ *by clearing* Δ. We say that $\mathcal{T}'$ is a *tangle of order* $\theta - 4\lambda - 2$ *obtained by clearing* Δ if $\mathcal{T}'$ is a tangle in Γ' of order $\theta - 4\lambda - 2$, and

- $\mathcal{T}'$ is respectful with a metric $m_{\mathcal{T}'}$, and
- $\mathcal{T}'$ is conformal with $\mathcal{T}$, and
- if x, y are atoms of Γ and x', y' are atoms of Γ' with $x \subseteq x'$ and $y \subseteq y'$, then $m_{\mathcal{T}}(x, y) \geq m_{\mathcal{T}'}(x', y') \geq m_{\mathcal{T}}(x, y) - 4\lambda - 2$.

Theorem 5.5 *([12, Theorem (7.10)]). Let Δ be a λ-zone. If $\theta \geq 4\lambda + 3$, then there exists a unique respectful tangle of order $\theta - 4\lambda - 2$ obtained by clearing Δ.*

Theorem 5.6 *([14, Theorem (9.2)]). Let Γ be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle in Γ of order θ. Let x be an atom of Γ, and λ an integer with $2 \leq \lambda \leq \theta - 4$. Then there exists a $(\lambda + 3)$-zone Δ around x such that $x' \subseteq \Delta$ for every atom x' of Γ with $m_{\mathcal{T}}(x, x') \leq \lambda$.*

Lemma 5.7. *Let Γ be a 2-cell drawing in a surface, z an atom, and $\mathcal{T}$ a respectful tangle in Γ of order θ. Let λ be a nonnegative integer, and let C be the cycle of the boundary of a λ-zone around z. If $\theta \geq \lambda + 8$, then there exists a $(\lambda + 7)$-zone Λ around z such that the cycle bounding Λ is disjoint from C, and Λ contains the λ-zone bounded by C.*

Proof. For every atom x of Γ, let Λ_x be a 4-zone around x containing all atoms y with $m_{\mathcal{T}}(x, y) \leq 1$, and let Δ_x be the closure of Λ_x, and let C_x be the boundary cycle of Δ_x. For every $v \in V(C)$, since every region incident with v has distance 1 from v, v is an interior point of Δ_v. Let $\Delta = \Delta' \cup \bigcup_{v \in V(C)} \Delta_v$, where Δ' is the open disk with the boundary C. So $V(C)$ are interior points of Δ. By the triangle-inequality, for every $v \in V(C)$ and for every vertex u in Δ_v, $m_{\mathcal{T}}(z, u) \leq \lambda + 4$. Therefore, there exists a $(\lambda + 7)$-zone Λ around z that contains Δ by Theorem 5.6. Since any vertex in C is an interior point of Δ, it is an interior point of Λ, so C is disjoint from the cycle that bounds Λ. □

Let Γ be a 2-cell drawing in a surface having a respectful tangle $\mathcal{T}$. Let Λ be a λ-zone (with respect to $m_{\mathcal{T}}$) around some atom of Γ for some nonnegative integer λ. For every $v \in V(\Gamma) \cap \partial\Lambda$, a *loose component with respect to* $(\Lambda, v, \mathcal{T})$ is a component L of $\Gamma - v$ such that some vertex of L is adjacent to v, and there exists no separation $(A, B) \in \mathcal{T}$ with $V(A \cap B) = \{v\}$ and $V(B) = V(L) \cup \{v\}$; we call v the *attachment* of L. *A loose component with respect to* $(\Lambda, \mathcal{T})$ is a loose component with respect to $(\Lambda, v, \mathcal{T})$ for some $v \in V(\Gamma) \cap \partial\Lambda$.

Lemma 5.8. *Let Γ be a 2-cell drawing in a surface, z an atom, and $\mathcal{T}$ a respectful tangle in Γ of order θ. Let λ be a nonnegative integer, and let Λ be a λ-zone around z. If*

$\theta \geq 4\lambda + 31$, then there exists a $(\lambda + 7)$-zone Λ' around z containing Λ such that the following hold.

1. *$\partial\Lambda'$ is obtained from $\partial\Lambda$ by adding pairwise disjoint paths in Γ, where each of them has both ends in $\partial\Lambda$ and for each of its internal vertices v, every edge incident in Γ with v is either contained in $\overline{\Lambda'}$ or incident with a vertex in a loose component with respect to $(\Lambda', v, \mathcal{T})$.*
2. *The drawing Γ' obtained from Γ by clearing Λ', deleting $E(\Gamma) \cap \partial\Lambda'$ and all loose components with respect to $(\Lambda', \mathcal{T})$ and deleting all resulting isolated vertices is 2-cell and has a respectful tangle $\mathcal{T}'$ of order at least $\theta - (4\lambda + 30)$ conformal with $\mathcal{T}$ such that $m_{\mathcal{T}}(x', y') \geq m_{\mathcal{T}'}(x, y) \geq m_{\mathcal{T}}(x', y') - (4\lambda + 30)$ for every two atoms x, y of Γ', where x', y' are atoms of Γ with $x' \subseteq x$ and $y' \subseteq y$.*

Proof. By Lemma 5.7, there exists a $(\lambda+7)$-zone Λ_0 such that $\Lambda_0 \supseteq \Lambda$ and $\partial\Lambda_0 \cap \partial\Lambda = \emptyset$. Let Γ_0 be the drawing obtained from Γ by clearing Λ_0. Then Γ_0 has a respectful tangle $\mathcal{T}_0$ of order at least $\theta - (4\lambda + 30)$ obtained from $\mathcal{T}$ by clearing Λ_0 conformal with $\mathcal{T}$, and Γ_0 is 2-cell.

Let Γ_1 be the drawing obtained from Γ by clearing Λ and deleting $E(\Gamma) \cap \partial\Lambda$ and isolated vertices. Since Γ_0 is a subgraph of Γ_1, Γ_1 contains a respectful tangle $\mathcal{T}_1$ of order at least $\theta - (4\lambda + 30)$ conformal with $\mathcal{T}$. We may assume that Γ_1 is not 2-cell; otherwise we are done by choosing $\Lambda' = \Lambda$.

Since $\theta - (4\lambda + 30) \geq 1$, there exists a component of Γ_1, say Q_1, such that $(\Gamma_1 - V(Q_1), Q_1) \in \mathcal{T}_1$. Since Γ_0 is 2-cell, Γ_0 is connected. So $V(\Gamma_0) \subseteq V(Q_1)$. Hence, every component of $\Gamma_1 - V(Q_1)$ is contained in Λ_0.

Let Q be a component of $\Gamma_1 - V(Q_1)$ with $|V(Q) \cap \partial\Lambda| \geq 2$. Hence there exists a path P_Q in Q on at least two vertices with ends contained in $\partial\Lambda$. We choose P_Q such that the closed set, denoted by Δ_Q, bounded by $\partial\Lambda \cup P_Q$ with interior disjoint from $V(Q_1)$ is maximal.

Define Λ' to be the open disk whose closure is $\overline{\Lambda} \cup \bigcup_{Q'} \Delta_{Q'}$, where the union is over all components Q' of $\Gamma_1 - V(Q_1)$ with $|V(Q') \cap \partial\Lambda| \geq 2$. Hence $\partial\Lambda'$ is obtained from $\partial\Lambda$ by adding $\bigcup_{Q'} P_{Q'}$, where the union is over some components Q' of $\Gamma_1 - V(Q_1)$. Clearly, Λ' is bounded by a cycle C' in Γ. Note that those $P_{Q'}$ are pairwise disjoint, and for every internal vertex v of some $P_{Q'}$, every edge incident in Γ with v is either contained in $\overline{\Lambda'}$ or incident with a vertex in a loose component with respect to $(\Lambda', v, \mathcal{T})$. Furthermore, the drawing Γ' obtained from Γ by clearing Λ' and deleting $E(C')$, all loose components with respect to $(\Lambda', \mathcal{T})$ and all isolated vertices is Q_1. Since $\partial\Lambda \cap \partial\Lambda_0 = \emptyset$, every region of Q_1 is either a region of Γ or a subset of Λ_0. Since Q_1 is connected, Q_1 and hence Γ' is 2-cell. Since Γ_0 is a subgraph of Γ', there exists a respectful tangle $\mathcal{T}'$ in Γ' of order at least $\theta - (4\lambda + 30)$ conformal with $\mathcal{T}$, and $m_{\mathcal{T}}(x', y') \geq m_{\mathcal{T}'}(x, y) \geq m_{\mathcal{T}_0}(x'', y'') \geq m_{\mathcal{T}}(x', y') - (4\lambda + 30)$ for all atoms x, y of Γ', where x', y' are atoms of Γ and x'', y'' are atoms of Γ_0 with $x' \subseteq x \subseteq x''$ and $y' \subseteq y \subseteq y''$. Since $\Lambda' \subseteq \Lambda_0$, Λ' is a $(\lambda + 7)$-zone. This proves the lemma. □

Let Σ be a connected surface, and let $\Delta_1, ..., \Delta_t$ be pairwise disjoint closed disks in Σ. Let Γ be a drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for $1 \leq i \leq t$. Let $Z = \bigcup_{i=1}^{t} V(\Gamma) \cap \partial\Delta_i$. We say that a partition $(Z_1, Z_2, ..., Z_p)$ of Z satisfies the *topological feasibility condition* if there exist pairwise disjoint disks $D_1, D_2, ..., D_p$ in Σ such that $D_j \cap (\bigcup_{i=1}^{t} \Delta_i) = Z_j$ for $1 \leq j \leq p$.

Theorem 5.9 *([12, Theorem (3.2)]). For every connected surface Σ and all integers $t \geq 0$ and $z \geq 0$, there exists a positive integer $\theta \geq 1$ such that the following is true. Let $\Delta_1, ..., \Delta_t$ be pairwise disjoint closed disks in Σ, and let Γ be a 2-cell drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for $1 \leq i \leq t$. Let $|Z| \leq z$, where $Z = \bigcup_{i=1}^{t} (V(\Gamma) \cap \partial\Delta_i)$, and let $(Z_1, Z_2, ..., Z_p)$ be a partition of Z satisfying the topological feasibility condition. Let $\mathcal{T}$ be a respectful tangle of order at least θ in Γ with metric $m_{\mathcal{T}}$ such that $m_{\mathcal{T}}(r_i, r_j) \geq \theta$ for $1 \leq i < j \leq t$, where r_i is the region of Γ meeting Δ_i for $1 \leq i \leq t$, and $V(\Gamma) \cap \partial\Delta_i$ is free for $1 \leq i \leq t$. Then there are mutually disjoint connected subdrawings $\Gamma_1, \Gamma_2, ..., \Gamma_p$ of Γ with $V(\Gamma_j) \cap Z = Z_j$ for $1 \leq j \leq p$.*

6. Excluding a subdivision of a fixed graph

Let G be a graph and $\mathcal{T}$ a tangle in G. Given an integer k, a vertex v of G is said to be *k-free* (with respect to $\mathcal{T}$) if there is no $(A, B) \in \mathcal{T}$ of order less than k such that $v \in V(A) - V(B)$.

Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ be a segregation of $\mathcal{S}$ with $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ such that $|\overline{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1$. The *skeleton* of a proper arrangement α of $\mathcal{S}$ in Σ (with respect to $(\mathcal{S}_1, \mathcal{S}_2)$) is the drawing $\Gamma = (U, V)$ in Σ with $V(\Gamma) = \{\alpha(v) : v \in V(\mathcal{S})\}$ such that $U(\Gamma)$ consists of the boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}_1$ with $|\bar{\Omega}| = 3$, and a line in the boundary $\alpha(S', \Omega')$ with ends $\overline{\Omega'}$ for each $(S', \Omega') \in \mathcal{S}_1$ with $|\overline{\Omega'}| = 2$. Note that we do not add any edges into the skeleton for (S, Ω) with $|\bar{\Omega}| \leq 1$ or $(S, \Omega) \in \mathcal{S}_2$. Furthermore, the skeleton of α is unique up to the choice of the line in $\alpha(S', \Omega')$ for each $(S', \Omega') \in \mathcal{S}$ with $|\overline{\Omega'}| = 2$, and the choices of those lines do not affect whether the skeleton is a 2-cell drawing or not.

Lemma 6.1. *Let t, ρ, θ be nonnegative integers. Let G be a graph. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ be a segregation of G such that $|\overline{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1$. Let α be a proper arrangement of $\mathcal{S}$ with respect to $(\mathcal{S}_1, \mathcal{S}_2)$ of G in a surface Σ. Let $(S, \Omega) \in \mathcal{S}$ be a ρ-vortex. Let G' be the skeleton of α. Let $\mathcal{T}'$ be a respectful tangle in G' of order θ. If G' is 2-cell and $\theta \geq 4t + 59$, then there exists a cycle C such that the following hold.*

1. *C bounds a $(t + 14)$-zone Λ in G' around some vertex in $\bar{\Omega}$.*
2. *Λ contains every atom x of G' with $m_{\mathcal{T}'}(x, y) \leq t$ for some $y \in \bar{\Omega}$.*
3. *The closure of Λ contains $\alpha(S, \Omega)$.*
4. *Let S' be the union of S'' over all societies $(S'', \Omega'') \in \mathcal{S}$ with*
   - *either $\alpha(S'', \Omega'') \subseteq \overline{\Lambda}$, or*

- $|\overline{\Omega''}| = 2$ *and* $\alpha(S'', \Omega'') \cap E(C) \neq \emptyset$, *or*
- $\overline{\Omega''}$ *contained in the union of some loose component with respect to* $(\Lambda, \mathcal{T}')$ *and its attachment, or*
- $|\overline{\Omega''}| = 1$ *and* $\overline{\Omega''} \subseteq V(C)$.

Let $\overline{\Omega'} = V(C) - \{x \in V(C)$: *every edge of* G' *incident with* x *is either contained in* $\overline{\Lambda}$ *or incident in* G' *with a vertex in a loose component with respect to* $(\Lambda, \mathcal{T}')\}$, *and let* Ω' *be a cyclic ordering consistent with the cyclic ordering of* C. *If every* $(S'', \Omega'') \in \mathcal{S}_2$ *with* $\alpha(S'', \Omega'') \subseteq \overline{\Lambda}$ *is a* $\rho_{S''}$*-vortex for some nonnegative integer* $\rho_{S''}$, *then* (S', Ω') *is a* $(\rho + t + 8 + \sum_{S''} \rho_{S''})$*-vortex, where the sum is over all societies* $(S'', \Omega'') \in \mathcal{S}_2 - \{(S, \Omega)\}$ *with* $\alpha(S'', \Omega'') \subseteq \overline{\Lambda}$.

5. *Let* $\mathcal{S}_1^* = \mathcal{S}_1 - \{(S'', \Omega'') \in \mathcal{S}_1 : S'' \subseteq S'\}$ *and* $\mathcal{S}_2^* = (\mathcal{S}_2 - \{(S'', \Omega'') \in \mathcal{S}_2 : S'' \subseteq S'\}) \cup \{(S', \Omega')\}$. *If* $m_{\mathcal{T}'}(x, y) \geq 3$ *for every atom* $x \subseteq \partial\Lambda$ *and* $y \in V(\overline{\Omega''})$ *with* $(S'', \Omega'') \in \mathcal{S}_2^* - \{(S', \Omega')\}$, *then* $\mathcal{S}^*$ *is a segregation, and there exists a proper arrangement* α^* *of* $\mathcal{S}_1^* \cup \mathcal{S}_2^*$ *with respect to* $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ *such that the skeleton* G^* *of* α^*
   - *is obtained from* G' *by clearing* Λ *and deleting some edges in* $E(C)$ *and all loose components with respect to* $(\Lambda, \mathcal{T}')$ *and deleting all resulting isolated vertices,*
   - *is 2-cell, and*
   - *has a respectful tangle* $\mathcal{T}^*$ *conformal with* $\mathcal{T}'$ *of order at least* $\theta - 4t - 58$ *such that* $m_{\mathcal{T}'}(x', y') \geq m_{\mathcal{T}^*}(x, y) \geq m_{\mathcal{T}'}(x', y') - 4t - 58$ *for all atoms* x, y *of* G^*, *where* x', y' *are atoms of* G' *with* $x' \subseteq x$ *and* $y' \subseteq y$.

Proof. Let y be a vertex in $\bar{\Omega}$. By Theorem 5.6, there exists a $(t+5)$-zone Λ' around y in G' such that $x \in \Lambda'$ for every atom x of G' with $m_{\mathcal{T}'}(x, y) \leq t + 2$. Since $m_{\mathcal{T}'}(y', y'') \leq 2$ for every two vertices y', y'' in $\bar{\Omega}$, $x \in \Lambda'$ for every atom x of G' with $m_{\mathcal{T}'}(x, z) \leq t$ for some $z \in \bar{\Omega}$. Let H be the drawing obtained from G' by deleting every atom $x \in V(G')$ with $m_{\mathcal{T}'}(x, y) \leq t + 2$. It follows from [11, Theorem (8.10)] that H has a region f homeomorphic to an open disk that contains $\alpha(S, \Omega)$ and all deleted vertices.

In the rest of the proof, we fix a radial drawing of G'.

Claim 1. *For every vertex* v *of* H *incident with* f, *there exists a closed walk* ℓ_v *of length at most* $2t + 8$ *in the radial drawing of* G' *with* $\{v, y\} \subseteq \text{ins}(\ell_v) \subseteq \bar{f} \subseteq \overline{\Lambda'}$ *such that* v *is adjacent to only one vertex in* ℓ_v *and* $V(\ell_v) \cap V(H) = \{v\}$.

Proof of Claim 1. Since v is incident with f, there exists a path $P \subseteq \bar{f}$ of length two in the radial drawing of G' containing v and a vertex v' of $G' - V(H)$ internally disjoint from $V(H)$. As $m_{\mathcal{T}'}(v', y) \leq t + 2$, there exists a closed walk $W_{v'}$ of length at most $2t + 4$ in the radial drawing of G' such that $\{v', y\} \subseteq \text{ins}(W_{v'})$. Note that $v \in V(H)$, so $m_{\mathcal{T}'}(v, y) > t + 2$ and $\{v\} \nsubseteq \text{ins}(W_{v'})$. Hence, there exists a closed walk ℓ_v of length at most $2t+8$ in $W_{v'} \cup P$ with $\{v, y\} \subseteq \text{ins}(\ell_v)$ and such that v is adjacent to only one vertex in ℓ_v. Note that $m_{\mathcal{T}'}(u, y) \leq t + 2$ for every $u \in V(W_{v'}) \cap V(G)$, so $\text{ins}(W_{v'}) \subseteq \bar{f}$. Hence $\text{ins}(\ell_v) \subseteq \bar{f}$. Since Λ' contains all atoms u of G' with $m_{\mathcal{T}'}(u, y) \leq t + 2$, $\bar{f} \subseteq \overline{\Lambda'}$. □

By Claim 1, $m_{\mathcal{T}'}(y,v) \leq t+4$ for every vertex v incident with f. So there exists a $(t+7)$-zone Λ'' containing all vertices incident with f. Hence $f \subseteq \Lambda''$. By Lemma 5.4, there exists an edge y' of G' such that $m_{\mathcal{T}'}(y,y') = \theta > t+8$. So $y' \notin \Lambda''$. Each cycle Y contained in the boundary of f bounds an open disk Δ_Y disjoint from y'. Let C_0 be the cycle contained in the boundary of f such that Δ_{C_0} is maximal among all cycles Y contained in the boundary of f. Then $f \subseteq \Delta_{C_0}$ and hence Δ_{C_0} contains $\alpha(S,\Omega)$ and all atoms x with $m_{\mathcal{T}'}(x,y) \leq t+2$. Since $\Delta_{C_0} \subseteq \Lambda''$, C_0 bounds a $(t+7)$-zone.

Let S'_0 be the union of S'' over all societies $(S'',\Omega'') \in \mathcal{S}$ with $\alpha(S'',\Omega'')$ contained in the closure of the disk bounded by C_0 disjoint from y'. We call such a society (S'',Ω'') an *inner vortex*. Let Ω'_0 be the cyclic ordering of C_0. Since (S,Ω) is a ρ-vortex, for every two intervals I,J that partition $\bar{\Omega}$, there exists $X_{I,J} \subseteq V(S)$ with $|X_{I,J}| \leq \rho$ such that there exists no path in $S - X_{I,J}$ from $I - X_{I,J}$ to $J - X_{I,J}$. Similarly, since (S'',Ω'') is a $\rho_{S''}$-vortex for every $(S'',\Omega'') \in \mathcal{S}_2 - \{(S,\Omega)\}$, for every two intervals I,J that partition $\overline{\Omega''}$, there exists $X^{S''}_{I,J} \subseteq V(S)$ with $|X^{S''}_{I,J}| \leq \rho_{S''}$ such that there exists no path in $S'' - X^{S''}_{I,J}$ from $I - X^{S''}_{I,J}$ to $J - X^{S''}_{I,J}$.

Claim 2. *(S'_0,Ω'_0) is a $(\rho+t+6+\sum \rho_{S''})$-vortex, where the sum is over all inner vortices.*

Proof of Claim 2. Let I',J' be two intervals that partition $\overline{\Omega'}$, let u,v be the first vertex in I',J', respectively, under the ordering Ω', and let ℓ^*_u and ℓ^*_v be a closed walk ℓ_u and a closed walk ℓ_v mentioned in Claim 1, respectively.

Note that for each path Q in $\ell^*_u \cup \ell^*_v$ and for each vertex $w \in V(Q)$ corresponding to the region containing $\alpha(S'',\Omega'')$ for some inner vortex (S'',Ω'') or vortex $(S'',\Omega'') = (S,\Omega)$, the edges in Q incident with w define a partition of $\overline{\Omega''}$ into two intervals I'',J'', and we denote $X^{S''}_{I'',J''}$ by $X^{S''}_{w,Q}$.

Assume that there exists a path Q in $\ell^*_u \cup \ell^*_v$ from u to v on at most $2t+11$ vertices. Let $X'' = (V(Q) \cap V(G')) \cup \bigcup_{w \in V(Q)-V(G')} \bigcup_{S''} X^{S''}_{w,Q}$, where the last union is over all (S'',Ω'') such that either $(S'',\Omega'') = (S,\Omega)$ or (S'',Ω'') is an inner vortex with $\alpha(S'',\Omega'')$ corresponding to w. Then there exists no path in $S'_0 - X''$ from $I' - X''$ to $J' - X''$. Note that $|X''| \leq \lceil \frac{2t+11}{2} \rceil + \rho + \sum_{S''} \rho_{S''}$, where the sum is over all inner vortices (S'',Ω''). Therefore, (S'_0,Ω'_0) is a $(\rho+t+6+\sum_{S''} \rho_{S''})$-vortex, where the sum is over all inner vortices (S'',Ω'').

So we may assume that there does not exist a path Q in $\ell^*_u \cup \ell^*_v$ from u to v on at most $2t+11$ vertices. In particular, ℓ^*_u is disjoint from ℓ^*_v. So one of ℓ^*_u and ℓ^*_v does not contain y. By symmetry, we may assume that ℓ^*_u does not contain y. Since $\{y\} \subseteq \text{ins}(\ell^*_u)$, ℓ^*_u contains a cycle C_u such that C_u bounds an open disk $\Delta_u \subseteq \text{ins}(\ell^*_u)$ containing $\{y\}$. Since $\overline{\Delta_u} \subseteq \bar{f} \subseteq \overline{\Lambda'}$, $\{v\}$ is not contained in the closure of Δ_u, for if it did, then, since v belongs to the boundary of f, it would belong to the boundary of Δ_u, contrary to the fact that ℓ^*_u and ℓ^*_v are disjoint. Since ℓ^*_u is disjoint from ℓ^*_v, ℓ^*_v is disjoint from the closure of Δ_u. In particular, $y \notin V(\ell^*_v)$. Hence ℓ^*_v contains a cycle C_v such that C_v bounds an open disk $\Delta_v \subseteq \text{ins}(\ell^*_v)$ containing $\{y\} \cup \Delta_u$. Since $\overline{\Delta_v} \subseteq \bar{f} \subseteq \overline{\Lambda'}$, $\{u\}$ is not contained in the closure of Δ_v, for if it did, then, since u belongs to the boundary of f, it would

belong to the boundary of Δ_v, contrary to the fact that ℓ_u^* and ℓ_v^* are disjoint. Since ℓ_u^* intersects the closure of Δ_v and the complement of the closure of Δ_v, ℓ_u^* intersects $C_v \subseteq \ell_v^*$, a contradiction. This proves the claim. □

By Lemma 5.8, there exists a $(t+14)$-zone Λ' containing the $(t+7)$-zone bounded by C_0 satisfying the conclusion of Lemma 5.8. Define C to be the cycle bounding Λ'. So Conclusions 1–3 hold.

Let (S', Ω') be the society mentioned in Conclusion 4. To prove Conclusion 4, it suffices to prove the following claim.

Claim 3. *If (S'_0, Ω'_0) is a p-vortex for some nonnegative integer p, then (S', Ω') is a $(p+2)$-vortex.*

Proof of Claim 3. Suppose that (S', Ω') is not a $(p+2)$-vortex. So there exist a partition I, J of Ω' into cyclic intervals and $p+3$ disjoint paths $P_1, ..., P_{p+3}$ in S' from I to J. By the definition of Ω', we know $\overline{\Omega'} \subseteq \overline{\Omega'_0}$. Let I' be the minimal cyclic interval in Ω'_0 containing I, and let J' be the cyclic interval such that $I' \cup J' = \Omega'_0$. Note that the ends of I and I' are the same. By changing the indices, we may assume that $P_1, P_2, ..., P_{p+1}$ do not intersect the ends of I'. So for each i with $1 \le i \le p+1$, some subpath Q_i of P_i in S'_0 is from I' to J'. This contradicts that (S'_0, Ω'_0) is a p-vortex and proves the claim. □

Then Conclusion 4 follows from Claims 2 and 3.

Define $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ as mentioned in Conclusion 5. Now we assume that $m_{\mathcal{T}'}(x, y) \ge 3$ for every $x \subseteq \partial\Lambda$ and $y \in \overline{\Omega''}$ with $(S'', \Omega'') \in \mathcal{S}_2^* - \{(S', \Omega')\}$. Hence every loose component with respect to $(\Lambda, \mathcal{T}')$ does not intersect $\overline{\Omega''}$ for every $(S'', \Omega'') \in \mathcal{S}_2^* - \{(S', \Omega')\}$. So $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$ is a segregation of G. Then it is clear that there exists a proper arrangement α^* of $\mathcal{S}^*$ with respect to $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ such that the skeleton G^* of α^* can be obtained from G' by deleting some edges in $E(C)$ and all loose components with respect to $(\Lambda, \mathcal{T}')$, and deleting all resulting isolated vertices. Conclusion 2 of Lemma 5.8 implies that G^* is 2-cell. Furthermore, Conclusion 2 of Lemma 5.8 implies that there exists a respectful tangle $\mathcal{T}^*$ in G^* conformal with $\mathcal{T}'$ of order at least $\theta - (4(t+7)+30) = \theta - 4t - 58$ such that $m_{\mathcal{T}'}(x', y') \ge m_{\mathcal{T}^*}(x, y) \ge m_{\mathcal{T}'}(x', y') - (4t+58)$ for all atoms x, y of G^*, where x', y' are atoms of G' with $x' \subseteq x$ and $y' \subseteq y$. This proves Conclusion 5. □

Lemma 6.2. *Let $d \ge 3$, and let $\kappa, h, h_1, h_2, ..., h_\kappa, \rho, \theta''$ be nonnegative integers. Then there exist integers $\theta_0(d, h, \rho, \kappa, \theta'')$, $\beta(d, h, \rho)$ and $f(d, h, \rho, \kappa)$ such that the following holds. Suppose that*

1. *G is a graph and $\mathcal{T}$ is a tangle in G, and*
2. *τ is a proper arrangement of a $\mathcal{T}$-central segregation $\mathcal{S}$ of G with respect to $(\mathcal{S}_1, \mathcal{S}_2)$ in a surface Σ, and*
3. *G' is the skeleton of τ, G' is 2-cell and is a minor of G, and $\mathcal{T}'$ is a respectful tangle in G' of order θ, for some $\theta \ge \theta_0$, such that $\mathcal{T}'$ is conformal with $\mathcal{T}$, and*

4. *let* $X \subseteq V(G)$ *and let* $(S_1, \Omega_1), ..., (S_\kappa, \Omega_\kappa)$ *be societies in* $\mathcal{S}$*, where each* (S_i, Ω_i) *is a* ρ*-vortex such that* $V(S_i) \cap X$ *contains at least one* d*-free vertex with respect to* $\mathcal{T}$*, such that for every* $1 \leq i < j \leq \kappa$*, and for every* $x \in \overline{\Omega_i}$ *and* $y \in \overline{\Omega_j}$*,* $m_{\mathcal{T}'}(x, y) \geq 2f + 1$*, and*
5. $m_{\mathcal{T}'}(x, y) \geq f + 1$*, for every* $x \in \overline{\Omega_i}$ *with* $1 \leq i \leq \kappa$*, and for every* $y \in \bar{\Omega}$ *with* $(S, \Omega) \in \mathcal{S} - \{(S_i, \Omega_i)\}$ *and* $|\bar{\Omega}| > 3$*, and*
6. $h_i \leq h$ *for* $1 \leq i \leq \kappa$*.*

Then there exist $Z_1, Z_2, ..., Z_\kappa \subseteq V(G)$*, a subdrawing* G'' *of* G' *and a tangle* $\mathcal{T}''$ *in* G'' *obtained from* G' *and* $\mathcal{T}'$ *by clearing at most* κ f*-zones* $\Lambda_1, ..., \Lambda_\kappa$ *such that* $\mathcal{T}'' - \bigcup_{i=1}^{\kappa} Z_i$ *has order at least* θ'' *and is conformal with* $\mathcal{T}'$*, and for every* $i \in \{1, 2, ..., \kappa\}$*, either*

1. $h_i \geq 2$*,* $\Lambda_i = \emptyset$ *and* $|Z_i| \leq \beta$ *such that every vertex in* $(V(S_i) - Z_i) \cap X$ *is not* d*-free with respect to* $\mathcal{T} - \bigcup_{i=1}^{\kappa} Z_i$*, or*
2. $Z_i = \emptyset$*, and* Λ_i *is an* f*-zone in* G' *around a vertex in* $\overline{\Omega_i}$ *with the boundary cycle* Y_i*, and there exist* h_i *subsets* $A_{i,1}, A_{i,2}, ..., A_{i,h_i}$ *of* Y_i *such that the following hold.*

   (a) $V(S_i) \subseteq \Lambda_i$*.*
   (b) *Each* $A_{i,j}$ *has size* d *and* $\bigcup_{j=1}^{h_i} A_{i,j}$ *is free in* G'' *with respect to* $\mathcal{T}'' - \bigcup_{i=1}^{\kappa} Z_i$*.*
   (c) *For each* $1 \leq j \leq h_i$*, there exists a minimum interval* I_j *of* Y_i *containing* $A_{i,j}$ *such that* $I_{j'} \cap I_{j''} = \emptyset$ *for every* $1 \leq j' < j'' \leq h_i$*.*
   (d) *There exist* $v_{i,1}, v_{i,2}, ..., v_{i,h_i} \in \Lambda_i \cap X$ *such that there are* h_i *disjoint* d*-spiders in* G *contained in* $\Lambda_i \cup \bigcup_{j=1}^{h_i} A_{i,j}$*, where the* j*-th spider is from* $v_{i,j}$ *to* $A_{i,j}$*.*

Proof. Define k' to be the value $k'(h, \rho)$ mentioned in Theorem 4.2, and let $\beta(d, h, \rho) = 2(k'd + 1)^{d+1}$. Define $s' = s'_{4.2}(h, d, 4hd + 2\kappa\beta + 3, \rho) + 2hd + \kappa\beta$, where $s'_{4.2}$ is the value s' mentioned in Theorem 4.2. Let $f(d, h, \rho, \kappa) = 19 + 10s'$ and $\theta_0(d, h, \rho, \kappa, \theta'') = \theta'' + \kappa(4f + \beta + 2)$. Let $i \in \{1, 2, ..., \kappa\}$ be fixed. For simplicity, we denote (S_i, Ω_i) by (S, Ω), and let v_S be a vertex in $\bar{\Omega}$.

We may assume that $\kappa \geq 1$ and $h \geq 1$, for otherwise this lemma is obvious. In particular, $\theta_0 \geq 6$. By Theorem 5.6, there exists a 5-zone $\Lambda'_{S,0}$ in G' around v_S such that $\Lambda'_{S,0}$ contains all atoms y of G' with $m_{\mathcal{T}'}(v_S, y) \leq 2$. Note that every vertex in $\bar{\Omega}$ has distance at most 2 from v_S with respect to the metric $m_{\mathcal{T}'}$, so $\Lambda'_{S,0}$ contains $\tau(S, \Omega)$. Let $\Lambda_{S,0}$ be a 19-zone in G' around v_S such that $\Lambda_{S,0}$ satisfies Lemma 6.1 and contains $\Lambda'_{S,0}$. Let $(G_{S,0}, \Omega_{S,0})$ be the society (S', Ω') mentioned in Lemma 6.1 by taking Λ to be $\Lambda_{S,0}$. Note that Lemma 6.1 ensures that $(G_{S,0}, \Omega_{S,0})$ is a $(\rho + 13)$-vortex.

For $1 \leq j \leq s'$, let $\Lambda_{S,j}$ be a $(19 + 10j)$-zone around v_S such that $\Lambda_{S,j}$ contains every vertex x of G'' with $m_{\mathcal{T}'}(x, v_S) \leq 19 + 10(j - 1)$ and $\partial\Lambda_{S,j} \cap \partial\Lambda_{S,j-1} = \emptyset$. Note that the existence of $\Lambda_{S,j}$ follows from Lemmas 5.6 and 5.7. Let $C_{S,j}$ be the boundary cycle of $\Lambda_{S,j}$ for $1 \leq j \leq s'$. Let $\Lambda_S = \Lambda_{S,s'}$. Let G_S be the union of S' over all societies (S', Ω') with $\tau(S', \Omega')$ contained in the closure of Λ_S, and let Ω_S be the cyclic ordering on the

boundary cycle of Λ_S. So (G_S, Ω_S) is a composition of a $(\rho+13)$-vortex $(G_{S,0}, \Omega_{S,0})$ with a rural neighborhood which has a presentation with an s'-nest $(C_{S,1}, C_{S,2}, ..., C_{S,s'})$.

Let $h'_i = k'$ if $h_i \neq 1$, and $h'_i = 1$ if $h_i = 1$. Let X_S be the set of d-free vertices in $V(S) \cap X$ with respect to $\mathcal{T}$. Note that $X_S \neq \emptyset$ by assumption. By Theorem 3.5, either there exist h'_i disjoint d-spiders from X_S to $\overline{\Omega_S}$, or there exists $W_S \subseteq V(G_S)$ with $|W_S| \leq 2(h'_i d+1)^{d+1} \leq \beta$ such that every d-spider from X_S to $\overline{\Omega_S}$ intersects W_S.

We first assume that $h_i > 1$ and the set W_S mentioned above exists. Then for every vertex $v \in X_S - W_S$, there exists a separation (A, B) of $G_S - W_S$ of order less than d such that $v \in V(A) - V(B)$ and $\overline{\Omega_S} - W_S \subseteq V(B)$. So there exists a separation (A', B') of $G - W_S$ with $V(A \cap B) = V(A' \cap B')$ such that $V(A') = V(A)$ and $V(B') = V(B) \cup (V(G) - V(G_S))$. Note that $(A', B') \in \mathcal{T} - W_S$; otherwise since $\mathcal{T}'$ is conformal with $\mathcal{T}$, there exists a separation $(B'', A'') \in \mathcal{T}' - W_S$, where $V(A'') = V(A') \cap V(G')$ and $V(B'') = V(B') \cap V(G')$, but A'' is contained in a $(19+10s')$-zone around v_S in G', a contradiction. So every vertex in $X_S - W_S$ is not d-free in $\mathcal{T} - W_S$. Hence every vertex in $(V(S) - W_S) \cap X$ is not d-free in $\mathcal{T} - W_S$. In other words, the first statement of this theorem holds by taking $\Lambda_i = \emptyset$ and $Z_i = W_S$.

When $h_i = 1$, the former case holds by Menger's Theorem and the fact that $V(S) \cap X$ contains a d-free vertex. Therefore, we may assume that there exist h'_i disjoint d-spiders from X_S to $\overline{\Omega_S}$.

Define Z_i to be the empty set. Let $D_{i,1}, D_{i,2}, ..., D_{i,h'_i}$ be disjoint d-spiders from X_{S_i} to $\overline{\Omega_{S_i}}$. Apply Theorem 4.2 by taking $(S, \Omega) = (G_{S_i}, \Omega_{S_i})$, $(S_0, \Omega_0) = (S_i, \Omega_i)$ and $D_j = D_{i,j}$ for $1 \leq j \leq h'_i$. There exist pairwise disjoint d-spiders $D'_{i,1}, D'_{i,2}, ..., D'_{i,h_i}$ from X_{S_i} to $V(C_{S_i,s'})$, a $(4hd+2\kappa\beta+3)$-nest $(N_{S_i,1}, ..., N_{S_i,4hd+2\kappa\beta+3})$ and intervals $I_{i,1}, I_{i,2}, ..., I_{i,h_i}$ of $C_{S_i,s'}$ satisfying the conclusions of Theorem 4.2. For every $1 \leq j \leq h_i$, since each $D'_{i,j}$ is perpendicular to $(N_{S_i,1}, ..., N_{S_i,4hd+2\kappa\beta+3})$, there exists a set $A_{i,j}$ of $h_i d$ vertices in $D'_{i,j} \cap V(N_{S_i,1})$ such that there exist $h_i d$ disjoint paths from $A_{i,j}$ to $V(C_{S_i,s'})$, but there exists no path from the head of $D'_{i,j}$ to $V(N_{S_i,1})$ in $D'_{i,j} - A_{i,j}$. Note that $N_{S_i,1}$ is contained in the disk bounded by $C_{S_i,s'}$ which bounds an f-zone, so $N_{S_i,1}$ is the boundary of an f-zone. Define Λ_i to be the f-zone bounded by $N_{S_i,1}$. Define G'' to be the drawing and $\mathcal{T}''$ to be the tangle obtained from G' and $\mathcal{T}'$, respectively, by clearing $\bigcup_{i=1}^{\kappa} \Lambda_i$. Note that $\mathcal{T}'' - \bigcup_{i=1}^{\kappa} Z_i$ has order $\theta - \kappa\beta - \kappa(4f+2) \geq \theta''$ and is conformal with $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$. On the other hand, by planarity, for every $1 \leq j \leq h_i$, there exists an interval $J_{i,j}$ of $N_{S_i,1}$ containing $A_{i,j}$, such that $J_{i,j} \cap J_{i,j'} = \emptyset$ for every $j' \neq j$.

To prove this lemma, it is sufficient to show that $\bigcup_{j=1}^{h_i} A_{i,j}$ is free with respect to $\mathcal{T}'' - \bigcup_{j=1}^{\kappa} Z_j$. Suppose that $\bigcup_{j=1}^{h_i} A_{i,j}$ is not free with respect to $\mathcal{T}'' - \bigcup_{j=1}^{\kappa} Z_j$ for some i, then there exists $(A', B') \in \mathcal{T}'' - \bigcup_{j=1}^{\kappa} Z_j$ with order less than dh_i such that $\bigcup_{j=1}^{h_i} A_{i,j} \subseteq V(A')$. Let $(A, B) \in \mathcal{T}''$ with $V(A) = V(A') \cup \bigcup_{j=1}^{\kappa} Z_j$ and $V(B) = V(B') \cup \bigcup_{j=1}^{\kappa} Z_j$. We assume that A is as small as possible, so $m_{\mathcal{T}''}(\bigcup_{j=1}^{h_i} A_{i,j}, u) < dh + \kappa\beta$ for every $u \in V(A)$ by Theorem 5.3.

We claim that $\overline{\Omega_{S_i}} \subseteq V(B) - V(A)$. Suppose to the contrary that there exists $u \in \overline{\Omega_{S_i}} \cap V(A)$. So there exists a closed walk W of a radial drawing of G'' with length less

than $2dh_i+2\kappa\beta$ and $\{u,v\}\subseteq \mathrm{ins}(W)$ for some vertex $v\in\bigcup_{j=1}^{h_i} A_{i,j}$. Since $\{v\}\subseteq \mathrm{ins}(W)$, if W contains the vertex of a radial drawing of G'' corresponding to the region bounded by $N_{S_\ell,1}$ for any $\ell\neq i$, then $W\cap V(N_{s_\ell,j})\neq\emptyset$ for every $2\leq j\leq 4hd+2\kappa\beta+2$, so $|W|\geq 4hd+2\kappa\beta+1$, a contradiction. So W does not contain the vertex of a radial drawing of G'' corresponding to the region bounded by $N_{S_\ell,1}$ for any $\ell\neq i$. Similarly, since $\{u\}\subseteq \mathrm{ins}(W)$, if W contains the vertex of a radial drawing of G'' corresponding to the region bounded by $N_{S_i,1}$, then $W\cap V(N_{s_i,j})\neq\emptyset$ for every $2\leq j\leq 4hd+2\kappa\beta+2$, a contradiction. So W does not contain the vertex of a radial drawing of G'' corresponding to the region bounded by $N_{S_i,1}$. Therefore, W is a closed walk in a radial drawing of G'. This implies that $m_{\mathcal{T}'}(u,v)<dh_i+\kappa\beta$. So u is a vertex in $C_{S_i,s'}$ contained in the disk bounded by $C_{S_i,dh+\kappa\beta}$ containing $C_{S_i,1}$, a contradiction.

Therefore, $\overline{\Omega_{S_i}}\subseteq V(B')-V(A')$. However, there exist dh_i disjoint paths in G'' from $\bigcup_{j=1}^{h_i} A_{i,j}$ to $\overline{\Omega_{S_i}}$, a contradiction. So $\bigcup_{j=1}^{h_i} A_{i,j}$ is free with respect to $\mathcal{T}''-\bigcup_{j=1}^{\kappa} Z_j$ for every i. This proves the lemma. □

Lemma 6.3. *Let $d\geq 3, h$ be positive integers, and let Σ be a surface. Then there exist integers $\theta(d,h,\Sigma),\phi(d,h,\Sigma)$ such that if G is a 2-cell drawing in Σ, $\mathcal{T}$ is a respectful tangle in G of order at least θ, and G contains h d-free vertices $v_1,v_2,...,v_h$ with $m_{\mathcal{T}}(v_i,v_j)>\phi$ for $1\leq i<j\leq h$, then G admits an H-subdivision with branch vertices $v_1,v_2,...,v_h$, for every graph H of order h and of maximum degree at most d embeddable in Σ.*

Proof. Let H be a graph of order h and of maximum degree at most d embeddable in Σ. Let $\theta_{5.9}$ be the positive integer θ mentioned in Theorem 5.9 by taking $t=h$ and $z=dh$. Note that $(\{v_i\},\{v_i\})$ is a 0-vortex for every i. For $1\leq i\leq h$, let Λ_i be the 12-zone around v_i of G mentioned in Lemma 6.1 such that Λ_i contains v_i and all its neighbors, and let S_i be the subgraph of G contained in the closure of Λ_i, and let $\overline{\Omega_i}=\partial\Lambda_i\cap V(G)$ with the cyclic order defined by the boundary cycle of Λ_i. So (S_i,Ω_i) is a 24-vortex by Lemma 6.1. Let $\theta'=\theta_{6.2}(d,1,24,h,\theta_{5.9})$, $\beta=\beta_{6.2}(d,1,24)$ and $f=f_{6.2}(d,1,24,\kappa)$, where $\theta_{6.2}$, $\beta_{6.2}$ and $f_{6.2}$ are the numbers θ_0,β,f mentioned in Lemma 6.2. Define $\theta=\theta'+h(4f+2)+2f+1$ and $\phi=\theta_{5.9}+h(4f+2)+2f+1$.

Applying Lemma 6.2 by taking $\kappa=h$, $h_i=1$ for $1\leq i\leq h$, $\rho=24$, $\theta''=\theta_{5.9}$, and $\mathcal{S}$ the segregation consisting of $(S_1,\Omega_1),(S_2,\Omega_2),...,(S_h,\Omega_h)$ and the societies in which each of them consists of exactly one edge that is not in $\bigcup_{i=1}^{h} S_i$, we obtain the desired subgraph G'' with a respectful tangle $\mathcal{T}''$, and $A_{i,1}$ for $1\leq i\leq h$, such that every $A_{i,1}$ is free with respect to $\mathcal{T}''$, as mentioned in the conclusion of Lemma 6.2. Then for every $x\in A_{i,1}$ and $y\in A_{j,1}$ for some $i\neq j$, we have that $m_{\mathcal{T}''}(x,y)\geq\theta_{5.9}$ by Theorem 5.5.

For $1\leq i\leq h$, let Δ_i be a closed disk in Σ contained in the closure of Λ_i such that $\Delta_i\cap G''=A_{i,1}$. Since H can be embedded in Σ, we can partition $\bigcup_{i=1}^{h} A_{i,1}$ and apply Theorem 5.9 to obtain a linear forest so that an H-subdivision in G can be obtained by concatenating these linear forests and h disjoint d-spiders $D_1,D_2,...,D_h$, where each D_i is from v_i to $A_{i,1}$ contained in S_i. We obtain an H-subdivision in G with branch vertices $v_1,v_2,...,v_h$. □

Lemma 6.4. *Let ρ be an integer, G a graph, $\mathcal{T}$ a tangle in G of order at least $2\rho+2$, and $\mathcal{S}$ a segregation of G. If $(S,\Omega)\in\mathcal{S}$ is a ρ-vortex and there exists no $(A,B)\in\mathcal{T}$ of order at most $2\rho+1$ such that $B\subseteq S$, then there exists no $(A',B')\in\mathcal{T}$ of order at most the half of the order of $\mathcal{T}$ such that $B'\subseteq S$.*

Proof. Suppose that there exists $(A,B)\in\mathcal{T}$ of order at most the half of the order of $\mathcal{T}$ such that $B\subseteq S$. Let $\bar{\Omega}=v_1,v_2,...,v_n$ in order, where $n=|\bar{\Omega}|$. We may assume that every v_i is adjacent to a vertex in $G-V(S)$, for otherwise we may remove it from $\bar{\Omega}$. As (S,Ω) is a ρ-vortex, by Theorem 8.1 in [9], there exists a path-decomposition $(P,\mathcal{X})$ of S of adhesion at most ρ such that the i-th bag X_i of $(P,\mathcal{X})$ contains v_i for every $1\leq i\leq n$. For every subgraph H of S, we define (A_H,B_H) to be the separation of G with minimum order such that $A_H=H$. In particular, for $1\leq i\leq n$, $(A_{S[X_i]},B_{S[X_i]})$ has order at most $2\rho+1$, so $(A_{S[X_i]},B_{S[X_i]})\in\mathcal{T}$. For $1\leq i\leq n$, define $(A_i,B_i)=(A\cup A_{S[\bigcup_{j=1}^{i}X_j]},B\cap B_{S[\bigcup_{j=1}^{i}X_j]})$. Note that if $v_j\in V(B)$ for some j, then $v_j\in V(A)$ since $B\subseteq S$ and v_j is adjacent to a vertex in $G-V(S)$. So the order of (A_i,B_i) is at most $|V(A)\cap V(B)|+|V(A_{S[\bigcup_{j=1}^{i}X_j]})\cap V(B_{S[\bigcup_{j=1}^{i}X_j]})\cap(V(B)-V(A))|\leq|V(A)\cap V(B)|+|(\bigcup_{j=1}^{i}X_j)\cap(\bigcup_{j=i+1}^{n}X_j)|\leq|V(A)\cap V(B)|+\rho$. Since the order of (A,B) is at most the half of the order of $\mathcal{T}$, and the order of $\mathcal{T}$ is greater than 2ρ, either $(A_i,B_i)\in\mathcal{T}$ or $(B_i,A_i)\in\mathcal{T}$ by the first tangle axiom. Let $(A_0,B_0)=(A,B)$. We shall prove that $(A_i,B_i)\in\mathcal{T}$ for $0\leq i\leq n$ by induction on i.

When $i=0$, $(A_0,B_0)=(A,B)\in\mathcal{T}$. Assume that $(A_i,B_i)\in\mathcal{T}$ for some i. Suppose that $(B_{i+1},A_{i+1})\in\mathcal{T}$. But $(A_i,B_i),(A_{S[X_{i+1}]},B_{S[X_{i+1}]})\in\mathcal{T}$, and $B_{i+1}\cup A_i\cup S[X_{i+1}]=G$, a contradiction. This proves that $(A_i,B_i)\in\mathcal{T}$ for every $0\leq i\leq n$.

Furthermore, $(A_n,B_n)=(A\cup S,B\cap B_S)$. Recall that $V(B\cap B_S)\subseteq V(B)\cap\bar{\Omega}\subseteq V(A)\cap V(B)$, so $|V(B_n)|\leq|V(A)\cap V(B)|$. Since $(B_n,G-E(B_n))$ has order less than the order of $\mathcal{T}$, $(B_n,G-E(B_n))\in\mathcal{T}$ by the first and third tangle axioms. However, $A_n\cup B_n=G$, contradicting the second tangle axiom. This completes the proof. □

Lemma 6.5. *For a positive nondecreasing function ϕ and integers $\rho,\lambda,\kappa,k,\theta^*,d$ with $d\geq 4$, there exist integers θ,ρ^* such that the following is true. Assume that G is a graph, X is a subset of $V(G)$, $\mathcal{T}$ is a tangle in G, and $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$ is a $\mathcal{T}$-central segregation that has a proper arrangement τ in a surface Σ such that the following hold.*

1. *For every $(S,\Omega)\in\mathcal{S}_1$, $|\bar{\Omega}|\leq 3$, and for every $x\in\bar{\Omega}$, there exist $|\bar{\Omega}|-1$ paths in S from x to $\bar{\Omega}-\{x\}$ intersecting only in $\{x\}$.*
2. *$|\mathcal{S}_2|\leq\kappa$.*
3. *(S,Ω) is a ρ-vortex for every $(S,\Omega)\in\mathcal{S}_2$.*
4. *The skeleton G' of $\mathcal{S}$ is a minor of G, is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}'$ of order at least θ conformal with $\mathcal{T}$.*
5. *There exist k λ-zones $\Lambda_1,\Lambda_2,...,\Lambda_k$ in G' around some vertices of G' with respect to the metric $m_{\mathcal{T}'}$ such that every d-free vertex of G' with respect to $\mathcal{T}'$ contained in X is contained in $\bigcup_{i=1}^{k}\Lambda_i$.*

Then there exist a $\mathcal{T}$-central segregation $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$ of G and a proper arrangement τ^* of $\mathcal{S}^*$ with respect to $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ in Σ such that the following hold.

1. *$\mathcal{S}_1^* \subseteq \mathcal{S}_1$; in particular, $|\bar{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1^*$.*
2. *$|\mathcal{S}_2^*| \leq \kappa + k$ and $\bigcup_{(S,\Omega)\in\mathcal{S}_2} S \cup \bigcup_{(S',\Omega')\in\mathcal{S},\alpha(S',\Omega')\subseteq\bigcup_{i=1}^k \overline{\Lambda_i}} S' \subseteq \bigcup_{(S,\Omega)\in\mathcal{S}_2^*} S$.*
3. *There exists an integer ρ' with $\rho' \leq \rho^*$ such that (S, Ω) is a ρ'-vortex for every $(S, \Omega) \in \mathcal{S}_2^*$.*
4. *The skeleton G^* of τ^* is a minor of G, is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}^*$ of order at least $\theta^* + \phi(\rho^*) + 2\rho^*$ conformal with $\mathcal{T}$.*
5. *There is no d-free vertex of G^* with respect to $m_{\mathcal{T}^*}$ contained in X.*
6. *$m_{\mathcal{T}^*}(x, y) \geq \phi(\rho')$ for every atoms x, y of G^* with $x \in S_x, y \in S_y$ for different members $(S_x, \Omega_x), (S_y, \Omega_y) \in \mathcal{S}_2^*$.*

Proof. Note that each society that consists of a single vertex is a 0-vortex. For each i, applying Lemma 6.1 by taking $(S, \Omega) = (\{v_i\}, \{v_i\})$, where v_i is a vertex of G' such that Λ_i is a λ-zone around v_i (we will choose θ to be greater than $4\lambda + 59$ so that Lemma 6.1 is applicable), we can find a $(\lambda + 14)$-zone Λ_i' containing Λ_i such that the society (S', Ω') mentioned in Lemma 6.1 by taking $\Lambda = \Lambda_i$ is a $(\lambda + 8 + \kappa\rho)$-vortex. Therefore, we can replace Λ_i by Λ_i' so that we may assume that every Λ_i is a λ'-zone and the subgraph of G inside the disk Λ_i is a λ'-vortex (S, Ω), where $\lambda' = \lambda + 14 + \kappa\rho$. Similarly, for each $(S, \Omega) \in \mathcal{S}_2$, there exists a 16-zone Λ_S containing the disk $\tau(S, \Omega)$, and the society (S', Ω') mentioned in Lemma 6.1 by taking $\Lambda = \Lambda_S$ is a $(\kappa\rho + 10)$-vortex.

Let $\mathcal{C} = \{\Lambda_i, \Lambda_S : 1 \leq i \leq k, (S, \Omega) \in \mathcal{S}_2\}$, and let $\lambda_{\mathcal{C}}$ be the minimum t such that every member of $\mathcal{C}$ is a t-zone. For each member Λ of $\mathcal{C}$, let $(S_\Lambda, \Omega_\Lambda)$ be the (S', Ω') mentioned in Lemma 6.1 by taking $\Lambda = \Lambda$. Let $M_{\mathcal{C}}$ be the minimum such that $(S_\Lambda, \Omega_\Lambda)$ are $M_{\mathcal{C}}$-vortices for all members Λ of $\mathcal{C}$. Let $\rho_0 = M_{\mathcal{C}}$ and $\lambda_0 = \lambda_{\mathcal{C}}$. Note that $|\mathcal{C}| \leq k + \kappa$, $\rho_0 \leq \max\{\lambda', \kappa\rho + 10\}$, and $\lambda_0 \leq \max\{\lambda', 16\}$. For $i \geq 1$, let $t_i = \phi(\rho_{i-1}) + (4\lambda_{i-1} + 58)(k+\kappa) + 2$, $\lambda_i = t_i + 14$ and $\rho_i = (k+\kappa-i+1)\rho_{i-1} + t_i + 8$. Then we consecutively test whether there exist two atoms of G' in different members of $\mathcal{C}$ with distance less than $\phi(M_{\mathcal{C}}) + (4\lambda_{\mathcal{C}} + 58)|\mathcal{C}|$ under the metric $m_{\mathcal{T}'}$, and if such two nearby vortices exist, then we do the following. Find the $(t_{k+\kappa+1-|\mathcal{C}|} + 14)$-zone Λ mentioned in the conclusion of Lemma 6.1 containing these two nearby members of $\mathcal{C}$, remove these two members from $\mathcal{C}$ and add Λ into $\mathcal{C}$, and update $M_{\mathcal{C}}$ and $\lambda_{\mathcal{C}}$. Since $|\mathcal{C}|$ decreases in each step, this process will terminate within $\kappa + k$ steps. Furthermore, during the process, $\lambda_{\mathcal{C}} \leq \lambda_{k+\kappa-|\mathcal{C}|}$ and $M_{\mathcal{C}} \leq \rho_{k+\kappa-|\mathcal{C}|}$. Therefore, when the process terminates, each member Λ of $\mathcal{C}$ is a $\lambda_{\mathcal{C}}$-zone in G' with $\lambda_{\mathcal{C}} \leq \lambda_{k+\kappa}$ and defines an $M_{\mathcal{C}}$-vortex $(S_\Lambda, \Omega_\Lambda)$ with $M_{\mathcal{C}} \leq \rho_{k+\kappa}$, and the distance between any two members of $\mathcal{C}$ is at least $\phi(M_{\mathcal{C}}) + (4\lambda_{\mathcal{C}} + 58)|\mathcal{C}|$ under the metric $m_{\mathcal{T}'}$. Let $\rho^* = \rho_{k+\kappa}$ and $\lambda^* = \lambda_{k+\kappa}$. We define $\theta = \theta^* + \phi(\rho^*) + 2\rho^* + 1 + (4\lambda^* + 58)(\kappa + k)$.

Define $\mathcal{S}_1^* = \mathcal{S}_1 - \{(S'', \Omega'') \in \mathcal{S}_1 : S'' \subseteq \bigcup_{\Lambda\in\mathcal{C}} S_\Lambda\}$ and $\mathcal{S}_2^* = \{(S_\Lambda, \Omega_\Lambda) : \Lambda \in \mathcal{C}\}$. Then Conclusions 1–3 hold. Since atoms belonging to different members of $\mathcal{S}_2^*$ have distance at least $3 + (|\mathcal{C}| - 1)(4\lambda_C + 58)$, there exists a proper arrangement τ^* of $\mathcal{S}^*$ such that the skeleton of τ^* with respect to $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ is 2-cell and has a respectful tangle $\mathcal{T}^*$ of

order at least $\theta^* + \phi(\rho^*) + 2\rho^*$ by repeatedly applying Lemma 6.1 $|\mathcal{C}|$ times. Note that G^* is a subgraph of G' and hence is a minor of G. So Conclusion 4 holds. Since G^* is a subgraph of G', Conclusion 5 holds. For any $x \in S_x, y \in S_y$ for different members $(S_x, \Omega_x), (S_y, \Omega_y) \in \mathcal{S}_2^*$, let Λ_x^*, Λ_y^* be the zones Λ^* containing S_x and S_y, respectively, then $m_{\mathcal{T}^*}(x, y) \geq m_{\mathcal{T}'}(x, y) - |\mathcal{C}|(4\lambda_{\mathcal{C}} + 58) \geq \phi(\rho^*)$. This proves Conclusion 6.

It remains to prove that $\mathcal{S}^*$ is a $\mathcal{T}$-central segregation of G. Since $\mathcal{T}'$ has order at least θ and is conformal with $\mathcal{T}$, the order of $\mathcal{T}$ is at least θ. Since $\mathcal{S}_1^* \subseteq \mathcal{S}_1$ and $\mathcal{S}$ is $\mathcal{T}$-central, by Lemma 6.4, it is sufficient to show that there is no $(A, B) \in \mathcal{T}$ of order at most $2\rho^* + 1$ such that $B \subseteq S$ for some $(S, \Omega) \in \mathcal{S}_2^*$. Suppose that such (A, B) exists. Since $B \subseteq S$, $V(B \cap G^*)$ consists of a subset of $V(A) \cap V(B)$ and some vertices of degree at most two in G^*. Let β be the G^*-minor of G witnessing that $\mathcal{T}^*$ is conformal with $\mathcal{T}$. Since $\mathcal{T}^*$ has order at least $\rho^* + 1$, there exists a separation $(A', B') \in \mathcal{T}^*$ such that $\beta(E(A')) = E(A) \cap \beta(E(G^*))$, and subject to this, $|V(B')|$ is minimum. Note that every vertex in $V(B') - V(A')$ has degree at most two in G^*. So if $V(B') - V(A') \neq \emptyset$, then one can move a path in B' with at least one internal vertex in $V(B') - V(A')$ into A', contradicting the minimality of $|V(B')|$. So $V(B') \subseteq V(A) \cap V(B)$ contains at most $2\rho^* + 1$ vertices. This implies that $(G^* - E(B'), B') \in \mathcal{T}^*$ by the second tangle axiom, contradicting the third tangle axiom. Hence $\mathcal{S}^*$ is $\mathcal{T}$-central. □

A segregation $\mathcal{S}$ of G is *maximal* if there exists no segregation $\mathcal{S}'$ such that $\{(S, \Omega) \in \mathcal{S} : |\bar{\Omega}| > 3\} = \{(S', \Omega') \in \mathcal{S}' : |\overline{\Omega'}| > 3\}$ and for every $(S, \Omega) \in \mathcal{S}$ with $2 \leq |\bar{\Omega}| \leq 3$, there exists $(S', \Omega') \in \mathcal{S}'$ with $|\overline{\Omega'}| \leq 3$ such that $S' \subseteq S$, and the containment is strict for at least one society. Note that if a segregation $\mathcal{S}$ of G is maximal, then G contains the skeleton of $\mathcal{S}$ as a minor, and for every $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \leq 3$ and $x \in \bar{\Omega}$, there exist $|\bar{\Omega}| - 1$ paths in S from x to $\bar{\Omega} - \{x\}$ intersecting only in $\{x\}$. Consequently, if H is a triangle-free graph and the skeleton of a maximal segregation $\mathcal{S}$ of G admits an H-subdivision, then G admits an H-subdivision.

The following theorem is proved in [2] and is a stronger form of the structure theorem for excluding minors in [15]. (We remark that our definition of maximal segregations is slightly different from the one in [2]. But our alternative is required, for otherwise the skeleton mentioned in Theorem 6.6 might not be 2-cell and the metric $m_{\mathcal{T}'}$ might not be defined. And the proof of Theorem 6.6 in [2] works under this setting.)

Theorem 6.6 *([2, Theorem 7]). For every graph L, there exists an integer κ such that for any nondecreasing positive function ϕ, there exist integers θ, ξ, ρ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G controlling no L-minor of G. Then there exist $Z \subseteq V(G)$ with size at most ξ and a maximal $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ properly arranged by an arrangement α in a surface Σ in which L cannot be drawn, where every $(S, \Omega) \in \mathcal{S}_1$ has the property that $|\bar{\Omega}| \leq 3$, and $|\mathcal{S}_2| \leq \kappa$ and there exists $p \leq \rho$ such that every member in $\mathcal{S}_2$ is a p-vortex. Furthermore, the skeleton G' of α with respect to $(\mathcal{S}_1, \mathcal{S}_2)$ is 2-cell embedded in Σ with a respectful tangle*

$\mathcal{T}'$ of order at least $\phi(p)$ conformal with $\mathcal{T} - Z$, and if x and y are two vertices in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x, y) \geq \phi(p)$.

Let us recall that the function mf was defined prior to Theorem 1.3. A graph H has a *nice* embedding in Σ if H can be 2-cell embedded in Σ and it has a set F of regions such that every vertex of H of degree at least 4 is incident with exactly one region in F, and $|F| = \mathrm{mf}(H, \Sigma)$.

Lemma 6.7 *([2, Lemma 12]). Let H be a graph of maximum degree d that can be embedded in a surface Σ. Then there exists a triangle-free graph H' of maximum degree d admitting an H-subdivision such that $\mathrm{mf}(H', \Sigma) = \mathrm{mf}(H, \Sigma)$ and H' has a nice embedding in Σ.*

Recall that a vertex v in a graph G is d-free with respect to a tangle $\mathcal{T}$ in G if there does not exist a separation $(A, B) \in \mathcal{T}$ of order less than d such that $v \in V(A) - V(B)$. Now, we are ready to prove Theorem 6.8, which is the main theorem of this paper. Note that Theorem 1.3 immediately follows if we take $X = V(G)$ in Theorem 6.8.

Theorem 6.8. *Let d, h be positive integers. Then there exist $\theta, \kappa, \rho, \xi, g \geq 0$ satisfying the following properties. If $d \geq 4$, H is a graph of maximum degree at most d on h vertices, G is a graph, and X is a subset of $V(G)$ such that G does not admit an H-subdivision whose branch vertices corresponding to vertices of degree at least four in H are contained in X, then for every tangle $\mathcal{T}$ in G of order at least θ, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either*

1. *no vertex in $V(G - Z) \cap X$ is d-free with respect to $\mathcal{T} - Z$, or*
2. *there exist a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$, having a proper arrangement in some surface Σ of genus at most g such that every society (S_1, Ω_1) in $\mathcal{S}_1$ satisfies that $|\overline{\Omega_1}| \leq 3$, and every society (S_2, Ω_2) in $\mathcal{S}_2$ is a ρ-vortex, and satisfies the following property: either*

   (a) *H cannot be drawn in Σ, or*
   (b) *H can be drawn in Σ and $\mathrm{mf}(H, \Sigma) \geq 2$, and there exists $\mathcal{S}'_2 \subseteq \mathcal{S}_2$ with $|\mathcal{S}'_2| \leq \mathrm{mf}(H, \Sigma) - 1$ such that every d-free vertex in $V(G - Z) \cap X$ with respect to $\mathcal{T} - Z$ is in $S - \bar{\Omega}$ for some $(S, \Omega) \in \mathcal{S}'_2$.*

Furthermore, if $\mathcal{T}$ controls a $K_{\lfloor \frac{3}{2} dh \rfloor}$-minor and G does not admit an H-subdivision with branch vertices contained in X, then the first conclusion always holds even when $d \leq 3$.

Proof. Note that there are only finitely many graphs of maximum degree at most d on h vertices, and there are only finitely many surfaces in which H can be drawn but $K_{\lfloor \frac{3}{2} dh \rfloor}$ cannot. So there exists h^* such that for every graph H on h vertices of maximum degree at most d and surface in which H can be drawn but $K_{\lfloor \frac{3}{2} dh \rfloor}$ cannot, the graph H' mentioned in Lemma 6.7 can be chosen such that it has at most h^* vertices.

We define the following.

- Let $\kappa_{6.6}$ be the number κ mentioned in Theorem 6.6 by taking $L = K_{\lfloor \frac{3}{2}dh \rfloor}$.
- Let $\theta_{6.2}, \beta_{6.2}, f_{6.2}$ be the functions θ_0, β, f mentioned in Lemma 6.2, respectively.
- Let ϕ' be the maximum of $\phi_{6.3}(d, h^*, \Sigma)$ among all surfaces Σ in which $K_{\lfloor \frac{3}{2}dh \rfloor}$ cannot be drawn, where $\phi_{6.3}$ is the number ϕ mentioned in Lemma 6.3.
- Let $\theta_{5.9}$ be the maximum of θ mentioned in Theorem 5.9 by taking all surfaces in which $K_{\lfloor \frac{3}{2}dh \rfloor}$ cannot be drawn, $t = (d+2)h^*$ and $z = dh^*$.
- Let $\phi^*(x) = \theta_{6.2}(d, h^*, x, 2\kappa_{6.6}+h^*, (dh^*+h^*+1)(\theta_{5.9}+1))+2f_{6.2}(d, h^*, x, 2\kappa_{6.6}+h^*)+(2\kappa_{6.6}+h^*)(6\beta_{6.2}(d, h^*, x)+4f_{6.2}(d, h^*, x, 2\kappa_{6.6}+h^*)+2)+2(dh^*+h^*+1)(\theta_{5.9}+6)$.
- Let $\theta'_{6.5}(x)$ be the function θ obtained by applying Lemma 6.5 by taking $\phi = \phi^*$, $\rho = x$, $\lambda = d+\phi'+11$, $\kappa = \kappa_{6.6}$, $k = h^*+\kappa_{6.6}$, $\theta^* = \theta_{6.2}(d, h^*, x, 2\kappa_{6.6}+h^*, (dh^*+h^*+1)(\theta_{5.9}+1))$ and $d = d$.
- Let $\theta_{6.3}$ be the maximum of $\theta(d, h^*, \Sigma)$ mentioned in Lemma 6.3 among all surfaces Σ in which $K_{\lfloor \frac{3}{2}dh \rfloor}$ cannot be drawn.
- Let $\theta_{6.6}$, $\xi_{6.6}$, $\rho_{6.6}$ be the numbers θ, ξ, ρ mentioned in Theorem 6.6, respectively, by taking $L = K_{\lfloor \frac{3}{2}dh \rfloor}$ and further taking $\phi(x) = \theta'_{6.5}(x) + \theta_{6.3}$.
- Let $\theta_{6.5}$ and $\rho_{6.5}$ be the numbers θ and ρ^* obtained by applying Lemma 6.5 by taking $\phi = \phi^*$, $\rho = \rho_{6.6}$, $\lambda = d+\phi'+11$, $\kappa = \kappa_{6.6}$, $k = h^*+\kappa_{6.6}$, $\theta^* = \theta_{6.2}(d, h^*, \rho_{6.6}, 2\kappa_{6.6}+h^*, (dh^*+h^*+1)(\theta_{5.9}+1))$ and $d = d$.
- Let $\theta_{3.4} = (hd)^{d+1} + d$.

Now we are ready to define the numbers for the conclusion of this theorem.

- Let $\xi = \max\{\xi_{6.6} + (2\kappa_{6.6}+h^*)\beta_{6.2}(d, h^*, \rho_{6.5}), (hd+1)^{d+1}\}$.
- Let $\theta = \theta_{6.5} + \theta_{6.3} + \theta_{6.6} + \xi + d + \rho_{6.6} + \rho_{6.5}$.
- Let $\kappa = \kappa_{6.6} + h^*$.
- Let $\rho = \rho_{6.6} + \rho_{6.5}$.
- Let g be the maximum genus of a surface in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn.

Let $\mathcal{T}$ be a tangle of order at least θ in G, and assume that G has no H-subdivision with branch vertices contained in X. We may assume that X contains at least h vertices of degree at least d in G, otherwise the first statement holds by letting Z be the set of vertices in X of degree at least d. We first assume that $\mathcal{T}$ controls a $K_{\lfloor \frac{3}{2}dh \rfloor}$-minor. By Lemma 3.2 and Theorem 3.4, since G does not admit an H-subdivision with branch vertices contained in X, there exists a set of vertices Z of G with $|Z| \leq \xi$ such that for every vertex $v \in V(G-Z) \cap X$ of degree at least d in G, there exists a separation $(A_v, B_v) \in \mathcal{T} - Z$ of $G-Z$ of order at most $d-1$ such that $v \in V(A_v) - V(B_v)$. Therefore, the first statement holds.

So we may assume that $\mathcal{T}$ does not control a $K_{\lfloor \frac{3}{2}dh \rfloor}$-minor. To prove this theorem, we may assume that G does not contain an H-subdivision whose branch vertices corresponding to vertices of degree at least four in H are contained in X.

By Theorem 6.6, there exist a surface Σ in which $K_{\lfloor \frac{3}{2}dh \rfloor}$ cannot be drawn, $Z \subseteq V(G)$ with $|Z| \leq \xi_{6.6}$, a number p with $p \leq \rho_{6.6}$, and a maximal $(\mathcal{T}-Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa_{6.6}$, having a proper arrangement τ in Σ such that every society (S, Ω) in $\mathcal{S}_1$ satisfies that $|\bar{\Omega}| \leq 3$, and every society in $\mathcal{S}_2$ is a p-vortex, and the skeleton G' of $\mathcal{S}$ is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}'$ of order at least $\theta'_{6.5}(p) + \theta_{6.3}$ conformal with $\mathcal{T} - Z$, and if x, y are two vertices in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x, y) \geq \theta'_{6.5}(p)$. If H cannot be drawn in Σ, then Statement 2(a) holds, so we may assume that H can be drawn in Σ.

In addition, we may assume that $V(G-Z) \cap X$ contains d-free vertices with respect to $\mathcal{T}-Z$, for otherwise Statement 1 holds. Note that every vertex in $\bigcup_{(S,\Omega)\in\mathcal{S}_1} V(S) - V(G')$ is not d-free with respect to $\mathcal{T} - Z$ since $d \geq 4$. If v is in $V((G - Z) \cap G')$ but is not d-free with respect to $\mathcal{T}'$, then there exists a separation $(A', B') \in \mathcal{T}'$ of order less than d such that $v \in V(A') - V(B')$. We choose (A', B') such that A' is as small as possible. Note $m_{\mathcal{T}'}(v, x) < d$ for every $x \in V(A')$ by Theorem 5.3. Suppose that there is no vertex $x \in \overline{\Omega}$ with $(S, \Omega) \in \mathcal{S}_2$ and $m_{\mathcal{T}'}(v, x) < d$. Then there exists $(A, B) \in \mathcal{T} - Z$ of order less than d such that $V(A) = \bigcup_{(S,\Omega)\in\mathcal{S},\overline{\Omega}\subseteq V(A')} V(S)$ and $V(A) \cap V(B) = V(A') \cap V(B')$. So v is not d-free with respect to $\mathcal{T} - Z$. Therefore, if v is a vertex in $(G - Z) \cap G'$ that is d-free with respect to $\mathcal{T} - Z$ but not d-free with respect to $\mathcal{T}'$, then $m_{\mathcal{T}'}(v, x) < d$ for some $x \in V(S)$ with $(S, \Omega) \in \mathcal{S}_2$. By Theorem 5.6 and Lemma 5.7, for every $(S, \Omega) \in \mathcal{S}_2$, there exists a $(d+11)$-zone Λ_S with respect to $\mathcal{T}'$ around a vertex in $\bar{\Omega}$ containing every atom y with $m_{\mathcal{T}'}(x, y) \leq d + 1$ for one such x. Thus every vertex of $(G - Z) \cap G'$ that is d-free with respect to $\mathcal{T} - Z$ but not d-free with respect to $\mathcal{T}'$ is in $\bigcup_{(S,\Omega)\in\mathcal{S}_2} \Lambda_S$.

Let H' be a graph that has a nice embedding mentioned in Lemma 6.7 such that $|V(H')| \leq h^*$. By Lemma 6.3, $V(G') \cap X$ does not contain $|V(H')|$ d-free vertices with respect to $\mathcal{T}'$ such that every pair of them has distance at least ϕ' under the metric $m_{\mathcal{T}'}$, for otherwise G contains an H-subdivision with branch vertices contained in X. So by Theorem 5.6 and Lemma 5.7, there exist integer k with $0 \leq k \leq h^*$, d-free vertices $v_1, v_2, ..., v_k$ of G' with respect to $\mathcal{T}'$, and $(\phi' + 10)$-zones $\Lambda_1, \Lambda_2, ..., \Lambda_k$ around $v_1, v_2, ..., v_k$, respectively, such that every d-free vertex in $V(G') \cap X$ with respect to $\mathcal{T}'$ is in $\bigcup_{i=1}^{k} \Lambda_i$. Then every d-free vertex in $V(G-Z) \cap X$ with respect to $\mathcal{T} - Z$ is a vertex of $G' \cup \bigcup_{(S,\Omega)\in\mathcal{S}_2} S$, and it is in $\bigcup_{i=1}^{k} \Lambda_i \cup \bigcup_{(S,\Omega)\in\mathcal{S}_2} \Lambda_S$.

Then let $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$, ρ', $\mathcal{T}^*$ and G^* be the $\mathcal{S}^*$, ρ', $\mathcal{T}^*$ and G^*, respectively, mentioned in the conclusion of Lemma 6.5 by taking $\phi = \phi^*$, $\rho = p$, $\lambda = d + \phi' + 11$, $\kappa = \kappa_{6.6}$, $k = h^* + \kappa_{6.6}$, $\theta^* = \theta_{6.2}(d, h^*, \rho_{6.5}, 2\kappa_{6.6} + h^*, (dh^* + h^* + 1)(\theta_{5.9} + 1))$ and $d = d$, and further taking $G = G - Z$, $\mathcal{T} = \mathcal{T} - Z$, $\mathcal{S} = \mathcal{S}$, $\tau = \tau$, $\Sigma = \Sigma$, and G' to be the skeleton of $\mathcal{S}$. Note that $|\mathcal{S}_2^*| \leq 2\kappa_{6.6} + h^*$, and the order of $\mathcal{T}^*$ is at least $\theta_{6.2}(d, h^*, \rho_{6.5}, 2\kappa_{6.6} + h^*, (dh^* + h^* + 1)(\theta_{5.9} + 1)) + \phi^*(\rho_{6.5}) + 2\rho_{6.5} \geq \theta_{6.2}(d, h^*, \rho_{6.5}, 2\kappa_{6.6} + h^*, (dh^* + h^* + 1)(\theta_{5.9} + 1))$.

Let κ' be the number of members of $\mathcal{S}_2^*$ containing d-free vertices with respect to $\mathcal{T} - Z$ belonging to X. Note that $\kappa' \leq |\mathcal{S}_2^*| \leq 2\kappa_{6.6} + h^*$. Let $Z_1, Z_2, ..., Z_{\kappa'}, \Lambda_1, \Lambda_2, ..., \Lambda_{\kappa'}$ be the sets obtained by applying Lemma 6.2 by taking $\kappa = \kappa'$, $h_i = h^*$ for every i, $\rho = \rho'$, $\theta'' = (dh^* + h^* + 1)(\theta_{5.9} + 1)$, $G = G - Z$, $G' = G^*$ and $(S_1, \Omega_1), (S_2, \Omega_2), ..., (S_{\kappa'}, \Omega_{\kappa'})$ as the vortices in $\mathcal{S}_2^*$ containing d-free vertices with respect to $\mathcal{T} - Z$ belonging to X. Define

$\mathcal{S}_2^{*\prime} \subseteq \mathcal{S}_2^*$ to consist of the members in which $\Lambda_i \neq \emptyset$. We define Z' to be $Z \cup \bigcup_{1\leq i\leq \kappa'} Z_i$. Note that $|Z'| \leq \xi$.

We may assume that there exist d-free vertices of $V(G-Z') \cap X$ with respect to $\mathcal{T}-Z'$; otherwise Statement 1 holds. So $|\mathcal{S}_2^{*\prime}| \geq 1$. In addition, we may assume that $\mathrm{mf}(H,\Sigma) \geq 1$ since otherwise H contains no vertex of degree at least four, and hence G has no H minor and Statement 2(a) holds by Theorem 6.6. If $\mathrm{mf}(H,\Sigma) \geq 2$ and $|\mathcal{S}_2^{*\prime}| \leq \mathrm{mf}(H,\Sigma) - 1$, then Statement 2(b) holds. So we may assume that $|\mathcal{S}_2^{*\prime}| \geq \mathrm{mf}(H,\Sigma) \geq 1$ and that $\Lambda_i \neq \emptyset$ for $i = 1, 2, ..., |\mathcal{S}_2^{*\prime}|$.

Let G'' be the drawing and $\mathcal{T}''$ the tangle in G'' conformal with $\mathcal{T}^*$ mentioned in the conclusion of Lemma 6.2. For $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$ and $1 \leq j \leq h^*$, let Y_i and $A_{i,j}$ be the cycles and sets mentioned in Conclusion 2 of Lemma 6.2, respectively. So for every $1 \leq i < i' \leq |\mathcal{S}_2^{*\prime}|$, $j, j' \in \{1, 2, ..., h^*\}$, $x \in A_{i,j}, y \in A_{i',j'}$, we have that $m_{\mathcal{T}''}(x,y) \geq m_{\mathcal{T}^*}(x,y) - (2\kappa_{6.6} + h^*)(4f_{6.2}(d, h^*, \rho', 2\kappa_{6.6} + h^*) + 2)$.

Note that for such x, y, we know $m_{\mathcal{T}^*}(x,y) \geq \phi^*(\rho') - 2f_{6.2}(d, h^*, \rho', 2\kappa_{6.6} + h^*)$, since each x, y is within distance at most $f_{6.2}(d, h^*, \rho', 2\kappa_{6.6} + h^*)$ (with respect to $m_{\mathcal{T}^*}$) away from a member of $\mathcal{S}_2^{*\prime}$, and the distance (with respect to $m_{\mathcal{T}^*}$) between those two members is at least $\phi^*(\rho')$. Therefore, for every $1 \leq i < i' \leq |\mathcal{S}_2^{*\prime}|$, $j, j' \in \{1, 2, ..., h^*\}$, $x \in A_{i,j}, y \in A_{i',j'}$, we have that $m_{\mathcal{T}''}(x,y) \geq \phi^*(\rho') - 2f_{6.2}(d, h^*, \rho', 2\kappa_{6.6} + h^*) - (2\kappa_{6.6} + h^*)(4f_{6.2}(d, h^*, \rho', 2\kappa_{6.6} + h^*) + 2) \geq 2(dh^* + h^* + 1)(\theta_{5.9} + 6)$.

Let $x \in A_{1,1}$. By Lemma 5.4, there exists an edge e^* of G'' with $m_{\mathcal{T}''}(e^*, x) \geq (dh^* + h^* + 1)(\theta_{5.9} + 6)$. As in the proof of Theorem 4.3 in [12], there exist non-loop edges $e_1, e_2, ..., e_{dh^*+h^*}$ of G'' such that $(\theta_{5.9} + 6)i \leq m_{\mathcal{T}''}(x, e_i) \leq (\theta_{5.9} + 6)i + 3$ for $1 \leq i \leq dh^* + h^*$, and the set of the ends of each e_i is free for each $1 \leq i \leq dh^* + h^*$. Therefore, $m_{\mathcal{T}''}(e_i, e_j) \geq \theta_{5.9} + 3$ for every $1 \leq i < j \leq dh^* + h^*$. Note that $m_{\mathcal{T}''}(x,y) \leq 2$ for $y \in \bigcup_{j=1}^{h^*} A_{1,j}$ and $m_{\mathcal{T}''}(x,y) \geq 2(dh^* + h^* + 1)(\theta_{5.9} + 6)$ for $y \in \bigcup_{i=2}^{|\mathcal{S}_2^{*\prime}|} \bigcup_{j=1}^{h^*} A_{i,j}$. Hence, $m_{\mathcal{T}''}(y, e_\ell) \geq \theta_{5.9} + 1$ for every $y \in \bigcup_{i=1}^{|\mathcal{S}_2^{*\prime}|} \bigcup_{j=1}^{h^*} A_{i,j}$ and $1 \leq \ell \leq dh^* + h^*$. For $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$, define Δ_i to be a disk in Σ contained in $\overline{\Lambda_i}$ such that $\Delta_i \cap G'' = \bigcup_{j=1}^{h^*} A_{i,j}$. For $1 \leq i \leq dh^* + h^*$, define $\Delta_{|\mathcal{S}_2^{*\prime}|+i}$ to be a disk in Σ such that $\Delta_i \cap G''$ is the set of the ends of e_i. Let $W = \bigcup_{i=1}^{\mathrm{mf}(H',\Sigma)} \bigcup_{j=1}^{h^*} A_{i,j} \cup \bigcup_{i=1}^{|E(H')|+|V(H')|} \{a_i, b_i\}$, where a_i, b_i are the ends of e_i. By Theorem 5.9, for any positive integer p and any partition $(W_1, ..., W_p)$ of W satisfying the topological feasibility condition, there exist pairwise disjoint connected subdrawings $\Gamma_1, ..., \Gamma_p$ of G'' with $V(\Gamma_j) \cap W = W_j$ for every $1 \leq j \leq p$.

Since H' has a nice embedding in Σ, we can embed H' into Σ such that the vertices of degree at least four of H' are incident with $\mathrm{mf}(H',\Sigma)$ regions. Then there exists a partition $(W_z : z \in V(H') \cup E(H'))$ (with possible empty part) of a subset of W such that the following hold.

- Fix an nice embedding of H' in Σ, a set of $\mathrm{mf}(H',\Sigma)$ faces incident with all vertices of degree at least four in H', an injection ι_1 that maps each of those $\mathrm{mf}(H',\Sigma)$ faces to a member of $\mathcal{S}_2^{*\prime}$, a set of h^* disjoint d-spiders mentioned in Conclusion 2(d) in Lemma 6.2 for each member of $\mathcal{S}_2^{*\prime}$, and an injection ι_2 that maps each vertex u

of H' of degree at least four in H' to a d-spider in the set for $\iota_1(f)$, where f is the face among those $\mathrm{mf}(H', \Sigma)$ faces incident with u.

- There exist an injection ι_3 from $E(H')$ to $\{e_t : 1 \leq t \leq dh^*\}$, and an injection ι_4 that maps each vertex-edge incidence pair (v, e) of H' to an end of $\iota_3(e)$, where every loop contributes two vertex-edge incidence pairs.
- For each vertex v of H' of degree at most three, $W_v = \{\iota_4((v, e)) : e \text{ is an edge of } H' \text{ incident with } v\}$.
- For each edge e of H' incident with both ends, say u, v, of degree at least four, where u, v are not necessarily distinct, W_e is a set of size two consisting of one leaf of $\iota_2(u)$ and one leaf of $\iota_2(v)$.
- For each edge e of H' incident with one vertex, say u, of degree at most three and one vertex, say v, of degree at least four, W_e consists of $\iota_3((u, e))$ and one leaf of $\iota_2(v)$.
- $W_z = \emptyset$ if z is a vertex of H' of degree at least four or z is an edge of H' not incident with any vertex of degree at least four.
- $(W_z : z \in V(H') \cup E(H'))$ satisfies the topological feasibility condition.

Since the partition $(W_z : z \in V(H') \cup E(H'))$ satisfies the topological feasibility condition, by Theorem 5.9, there exist pairwise disjoint connected subdrawings $(\Gamma_z : z \in V(H') \cup E(H'))$ of G'' with $V(\Gamma_z) \cap W = W_z$ for every $z \in V(H') \cup E(H')$. Observe that the union of the image of ι_2 and $\bigcup_{z \in V(H') \cup E(H')} \Gamma_z \cup \bigcup_{i=1}^{dh^*} e_i$ contains an H'-subdivision (π_V, π_E) such that its every branch vertex corresponding to a vertex of degree at least four in H' is contained in X.

Finally, we shall obtain a contradiction by showing that G admits an H-subdivision whose branch vertices corresponding to vertices of degree at least four in H are contained in X. Recall that $\mathcal{S}^*$ is maximal, so for every $(S, \Omega) \in \mathcal{S}_1^*$ and for every $a \in \bar{\Omega}$, there exist $|\bar{\Omega}| - 1$ paths in S from a to $\bar{\Omega} - \{a\}$ intersecting in a and otherwise disjoint. Since H' is triangle-free, one edge of the triangle in G'' formed by $\bar{\Omega}$ is not contained in the image of π_E for any $(S, \Omega) \in \mathcal{S}_1^*$ with $|\bar{\Omega}| = 3$. Therefore, G admits an H-subdivision whose branch vertices corresponding to vertices of degree at least four in H are contained in X. This completes the proof. □

7. Remarks about optimality

In this section we shall prove that the order of the separations mentioned in the first conclusion of Theorem 1.3 cannot be improved, and the value D mentioned in Theorem 1.2 and [2, Theorem 3] cannot be replaced by any number smaller than the number $d - 1$ mentioned in the first conclusion of Theorem 1.3.

Let θ be a positive integer. A $2\theta \times 2\theta$-*wall* is the graph W with $V(W) = \{(i, j) : 1 \leq i \leq 2\theta, 1 \leq j \leq 2\theta\}$ and $E(W) = \{(i, j)(i, j + 1) : 1 \leq i \leq 2\theta, 1 \leq j \leq 2\theta - 1\} \cup \{(2k - 1, j)(2k, j) : 1 \leq k \leq \theta, 1 \leq j \leq 2\theta,\ j \text{ is odd}\} \cup \{(2k, j)(2k + 1, j) : 1 \leq j \leq \theta - 1, 1 \leq j \leq 2\theta,\ j \text{ is even}\}$. For each $1 \leq i \leq 2\theta$, the *i-th row* is the subgraph induced by the vertices $\{(i, j) : 1 \leq j \leq 2\theta\}$.

We fix d to be a positive integer with $d \geq 4$ in the rest of this section. Let r be a positive integer. Let Y be the collection of all $(d-1)$-element subsets of the edge-set of the $2r \times 2r$-wall. Define G_r to be a graph obtained from the $2r \times 2r$-wall by the following procedure.

- Subdividing each edge $|Y|$ times.
- For each member y of Y, add a vertex v_y. Let $V_Y = \{v_y : y \in Y\}$.
- For each $y \in Y$ and $e \in y$, add an edge between v_y and a vertex obtained by subdividing e in a way that every vertex not in V_Y has degree at most three in G_r.

Note that G_r has maximum degree $d-1$, so G_r does not contain any graph with maximum degree at least d as a subdivision.

Theorem 7.1. *For all positive integers $d, \theta, \kappa, \rho, \xi, g$, there exist a positive integer r and a tangle $\mathcal{T}$ in G_r of order at least θ such that the following hold.*

1. *For every $Z \subseteq V(G_r)$ with $|Z| \leq \xi$, there exists a vertex $v \in V_Y - Z$ such that there exists no separation $(A, B) \in \mathcal{T} - Z$ of order at most $d-2$ with $v \in V(A) - V(B)$.*
2. *For every $Z \subseteq V(G_r)$ with $|Z| \leq \xi$, there exists no $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G_r - Z$ with $|\mathcal{S}_2| \leq \kappa$ such that $\mathcal{S}$ can be properly arranged in some surface Σ with genus at most g, every $(S_1, \Omega_1) \in \mathcal{S}_1$ satisfies $|\overline{\Omega_1}| \leq 3$, and every $(S_2, \Omega_2) \in \mathcal{S}_2$ is a ρ-vortex.*
3. *G_r cannot be constructed by clique-sums, starting with graphs that are an $\leq \xi$-extension of either a graph of maximum degree at most $d-2$, or an out-growth by $\leq \kappa$ ρ-rings of a graph that can be drawn in a surface of genus at most g.*

Proof. We may assume $d \geq 2$; otherwise the first conclusion holds. Let r be a fixed positive integer. We shall prove that every sufficiently large r satisfies the conclusion of this theorem.

Define $\mathcal{T}$ to be the collection of all separations (A, B) of G_r of order less than $\lfloor \frac{2r}{3} \rfloor$ such that B contains all vertices in some row of the $2r \times 2r$-wall. It is straightforward to verify that $\mathcal{T}$ is a tangle of order $\lfloor \frac{2r}{3} \rfloor$. We choose r to be a number with $\lfloor \frac{2r}{3} \rfloor > \theta$, then $\mathcal{T}$ is a tangle in G_r of order at least θ.

There exists a positive integer k such that if $\mathcal{T}$ controls a K_k-minor, then there exist no $Z \subseteq V(G_r)$ with $|Z| \leq \xi$ and a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G_r - Z$ with $|\mathcal{S}_2| \leq \kappa$ such that $\mathcal{S}$ can be properly arranged in a surface of genus at most g, every $(S_1, \Omega_1) \in \mathcal{S}_1$ satisfies $|\Omega_1| \leq 3$, and every $(S_2, \Omega_2) \in \mathcal{S}_2$ is a ρ-vortex. It is not hard to show that there exists an integer r_k such that if $r \geq r_k$, then G_r contains a K_k-minor such that each branch set intersects every row of the $2r \times 2r$-wall in G_r. That is, if $r \geq r_k$, then $\mathcal{T}$ controls a K_k-minor. We choose $r \geq r_k$, so the second conclusion of this theorem holds.

Choose $r \geq d^{2\xi+1}2^{2d^2}$. Now we prove the first conclusion of this theorem. Let $Z \subseteq V(G)$ with $|Z| \leq \xi$. Then $G_r - Z$ contains a subgraph R which is a subdivision of $G_{2^{2d^2}d}$ and only the edges contained in the subdivided wall are subdivided. Let v be a vertex of $V(R) \cap V_Y$ such that there exists a row of the subdivided wall in $G_{2^{2d^2}d}$ such that all neighbors of v are vertices obtained by subdividing edges of that row. Then for any row of the subdivided wall in $G_{2^{2d^2}d}$, there exist $d-1$ disjoint paths in the subdivided wall from the neighbors of v to that row. So if $(A, B) \in \mathcal{T} - Z$ with $v \in V(A) - V(B)$, then the order of (A, B) is at least $d-1$. This proves the first conclusion of this theorem.

Finally, we prove the third conclusion of this theorem. Suppose that G_r can be constructed by clique-sums starting from graphs mentioned in the third conclusion. Then G_r has a tree-decomposition $(T, \mathcal{X})$ such that each torso is a graph involved in the clique-sum. For each node t of the tree, let X_t be the bag at t. Since the clique size of each graph involved in the clique-sum is bounded, the adhesion of $(T, \mathcal{X})$ is bounded. We choose r to be larger than the adhesion of $(T, \mathcal{X})$. For each edge tt' of the tree T, let T_t and $T_{t'}$ be the components of $T - tt'$ containing t and t', respectively. Then there exists a separation $(A_{tt'}, B_{tt'})$ of G_r such that $V(A_{tt'}) = \bigcup_{t'' \in V(T_t)} X_{t''}$ and $V(B_{tt'}) = \bigcup_{t'' \in V(T_{t'})} X_{t''}$. Since the order of $\mathcal{T}$ is larger than the adhesion of $(T, \mathcal{X})$, either $(A_{tt'}, B_{tt'}) \in \mathcal{T}$ or $(B_{tt'}, A_{tt'}) \in \mathcal{T}$. We orient the edge tt' from t to t' if $(A_{tt'}, B_{tt'}) \in \mathcal{T}$. Then there exists a node t^* of T with out-degree zero. Let L be the torso at t^*. Since $\mathcal{T}$ controls a K_k-minor, L is a $\leq \xi$-extension of a graph of maximum degree at most $d-2$. Let $Z \subseteq V(L)$ with $|Z| \leq \xi$ such that $L - Z$ has maximum degree at most $d-2$.

Note that $G_r - Z$ contains a subgraph R which is a subdivision of $G_{2^{2d^2}d}$ and only the edges contained in the subdivided wall are subdivided. And there exists a vertex v of $V(R) \cap V_Y$ such that there exists a row of the subdivided wall in $G_{2^{2d^2}d}$ such that all neighbors of v are vertices obtained by subdividing edges of that row.

Since $L - Z$ has maximum degree at most $d-2$, $|X_{t^*} \cap X_t - Z| \leq d-2$ for every neighbor t of t^*. Since there exists no separation $(A, B) \in \mathcal{T} - Z$ of order at most $d-2$ with $v \in V(A) - V(B)$, $v \in X_{t^*}$. But since v has degree $d-1$ in $G_r - Z$, v belongs to $X_{t^*} \cap X_c$ for some neighbor c of t^*. Let $c_1, c_2, ..., c_\ell$ be the neighbors of t^* such that v is adjacent in G_r to some vertex in $(\bigcup_{i=1}^{\ell} \bigcup_{t \in T_{c_i}} X_t) - X_{t^*}$, where T_{c_i} is the component of $T - t^*c_i$ containing c_i. Since v has degree $d-1$ in G_r, $\ell \leq d-1$. Let (A', B') be a separation of G_r such that $V(A') = Z \cup \bigcup_{i=1}^{\ell} \bigcup_{t \in V(T_{c_i})} X_t$ and $V(B') = Z \cup \bigcup_{t \in V(T) - \bigcup_{i=1}^{\ell} V(T_{c_i})} X_t$. Note that the order of (A', B') is at most $\xi + d\ell \leq \xi + d^2 < \lfloor \frac{2r}{3} \rfloor$. So $(A', B') \in \mathcal{T}$. Let $(A^*, B^*) \in \mathcal{T} - Z$ be a separation of $G_r - Z$ with $V(A^*) = V(A') - Z$ and $V(B^*) = V(B') - Z$. Since the order of (A^*, B^*) is at most d^2, $R \cap B^*$ contains a subgraph L' isomorphic to a subdivision of G_d, where only the edges contained in the subdivided wall are subdivided. So there exists a row of the subdivided wall in L' contained in B^*. Note that there are $d-1$ disjoint paths $P_1, P_2, ..., P_{d-1}$ in $G_r - Z$ from the neighbors of v to that row, where each neighbor of v in $G_r - Z$ is contained in some P_i. Let p be the number of neighbors of v in $G_r - Z$ contained in $V(A^*) - V(B^*)$, and we may assume that $P_1, P_2, ..., P_p$ contain such neighbors. So for each i with $1 \leq i \leq p$, P_i intersects $V(A^* \cap B^*)$. Hence $p \leq |V(A^* \cap B^*) - \{v\}|$. By the definition of a torso, v is adjacent

in L to every vertex in $V(A^* \cap B^*) - \{v\}$. Therefore, the degree of v in L is at least the degree of v in $G_r - Z$, which is $d - 1$, a contradiction. This proves the theorem. □

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References

[1] R. Diestel, R. Thomas, Excluding a countable clique, J. Combin. Theory Ser. B 76 (1999) 41–67.
[2] Z. Dvořák, A stronger structure theorem for excluded topological minors, arXiv:1209.0129v1.
[3] M. Grohe, D. Marx, Structure theorem and isomorphism test for graphs with excluded topological subgraphs, SIAM J. Comput. 44 (2015) 114–159.
[4] K. Kawarabayasi, S. Norine, R. Thomas, P. Wollan, K_6 minors in large 6-connected graphs, http://people.math.gatech.edu/~thomas/PAP/k6large.pdf, 2014.
[5] C.-H. Liu, Graph Structures and Well-Quasi-Ordering, PhD Dissertation, Georgia Institute of Technology, 2014.
[6] C.-H. Liu, R. Thomas, Robertson's conjecture I–IV. Well-quasi-ordering graphs by the topological minor relation, in preparation.
[7] D. Marx, Important separators and parameterized algorithms, http://www.cs.bme.hu/~dmarx/papers/marx-mds-separators-slides.pdf, 2011.
[8] D. Marx, P. Wollan, Immersions in highly edge connected graphs, SIAM J. Discrete Math. 28 (2014) 503–520.
[9] N. Robertson, P.D. Seymour, Graph minors. IX. Disjoint crossed paths, J. Combin. Theory Ser. B 49 (1990) 40–77.
[10] N. Robertson, P.D. Seymour, Graph minors. X. Obstructions to tree-decomposition, J. Combin. Theory Ser. B 52 (1991) 153–190.
[11] N. Robertson, P.D. Seymour, Graph minors. XI. Circuits on a surface, J. Combin. Theory Ser. B 60 (1994) 72–106.
[12] N. Robertson, P.D. Seymour, Graph minors. XII. Distance on a surface, J. Combin. Theory Ser. B 64 (1995) 240–272.
[13] N. Robertson, P.D. Seymour, Graph minors. XIII. The disjoint paths problem, J. Combin. Theory Ser. B 63 (1995) 65–110.
[14] N. Robertson, P.D. Seymour, Graph minors. XIV. Extending an embedding, J. Combin. Theory Ser. B 65 (1995) 23–50.
[15] N. Robertson, P.D. Seymour, Graph minors. XVI. Excluding a non-planar graph, J. Combin. Theory Ser. B 89 (2003) 43–76.