ON UNIQUENESS SETS OF ADDITIVE EIGENVALUE PROBLEMS AND APPLICATIONS

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Abstract. In this paper, we provide a simple way to find uniqueness sets for additive eigenvalue problems of first and second order Hamilton–Jacobi equations by using a PDE approach. An application in finding the limiting profiles for large time behaviors of first order Hamilton–Jacobi equations is also obtained.

1. Introduction

Let $\mathbb{T}^n$ be the usual n-dimensional torus. Let the Hamiltonian $H = H(x,p) \in C^2(\mathbb{T}^n \times \mathbb{R}^n)$ be such that

(H1) for every $x \in \mathbb{T}^n$, $p \mapsto H(x,p)$ is convex,

(H2) uniformly for $x \in \mathbb{T}^n$,

$$\lim_{|p|\to\infty} \frac{H(x,p)}{|p|} = +\infty \quad \text{and} \quad \lim_{|p|\to\infty} \left(\frac{1}{2}H(x,p)^2 + D_x H(x,p)\cdot p\right) = +\infty.$$

The first order additive eigenvalue (ergodic) problem corresponding to H is

$$\text{(E)} \qquad H(x,Dw) = c \qquad \text{in } \mathbb{T}^n.$$

Here, $(w,c) \in C(\mathbb{T}^n) \times \mathbb{R}$ is a pair of unknowns. It was shown in [14] that there exists a unique constant $c \in \mathbb{R}$, which is called the *ergodic constant* of (E), such that (E) has a viscosity solution $w \in C(\mathbb{T}^n)$. Without loss of generality, we normalize the ergodic constant c to be zero henceforth.

We emphasize here that since (E) is not monotone in w, viscosity solutions of (E) are not unique even up to additive constants in general (see examples in [14], [13, Chapter 5.5], [12, Chapter 6]). It is therefore fundamental to understand why this *nonuniqueness phenomenon* appears, and in particular, to find a *uniqueness set* for (E). Here, a uniqueness set for (E) denote a set $A \subset \mathbb{T}^n$ satisfying that for any viscosity solutions $v, w \in C(\mathbb{T}^n)$ of (E), if $v = w$ on A, then $v = w$ on $\mathbb{T}^n$. It turns out that this has deep relations to Hamiltonian dynamics and weak KAM theory. In fact, uniquenesss set for (E) has already been studied in [7, 8] in the context of weak KAM theory.

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In this short paper, we provide a new and simple way to look at this phenomenon for (E) by using PDE techniques. Our approach is quite general and robust, which is indeed applicable in studying the nonuniqueness phenomenon for second order (degenerate viscous) Hamilton–Jacobi equations which appears in stochastic optimal control of the form

$$\text{(VE)} \qquad H(x, Dw) = \operatorname{tr}\big(A(x)D^2 w\big) + c \quad \text{ in } \mathbb{T}^n,$$

as well. Here, H is the Hamiltonian as above, and $A : \mathbb{T}^n \to \mathbb{M}^{n\times n}_{\rm sym}$ is the diffusion matrix, where $\mathbb{M}^{n\times n}_{\rm sym}$ is the set of all $n \times n$ real symmetric matrices, and $(w, c) \in C(\mathbb{T}^n) \times \mathbb{R}$ is a pair of unknowns.

1.1. **Settings and main results.** We first recall the definition of Mather measures. Consider the following minimization problem

$$\min_{\mu\in\mathcal{F}} \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x, v)\, d\mu(x, v), \tag{1.1}$$

where L is the Legendre transform of H, that is,

$$L(x, v) = \sup_{p\in\mathbb{R}^n} (p \cdot v - H(x, p)) \quad \text{for } (x, v) \in \mathbb{T}^n \times \mathbb{R}^n,$$

and

$$\mathcal{F} = \left\{ \mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n) \,:\, \iint_{\mathbb{T}^n\times\mathbb{R}^n} v \cdot D\phi(x)\, d\mu(x, v) = 0 \ \text{ for all } \phi \in C^1(\mathbb{T}^n) \right\}.$$

Here, $\mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ is the set of all Radon probability measures on $\mathbb{T}^n \times \mathbb{R}^n$. Measures which belong to $\mathcal{F}$ are called *holonomic measures* associated with (E).

Definition 1 (Mather measures). *Let $\widetilde{\mathcal{M}} \subset \mathcal{F}$ be the set of all minimizers of* (1.1). *Each measure in $\widetilde{\mathcal{M}}$ is called a Mather measure.*

As we normalize $c = 0$, we actually have that (see [16, 15, 7, 8] for instance)

$$\min_{\mu\in\mathcal{F}} \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x, v)\, d\mu(x, v) = -c = 0. \tag{1.2}$$

See [17], [12, Lemma 6.12] for a proof of a more general version this fact. Here is our first main result.

Theorem 1.1. *Assume* (H1)–(H2). *Let w_1, w_2 be any viscosity solutions of ergodic problem* (E). *Assume further that*

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} w_1(x)\, d\mu(x, v) \le \iint_{\mathbb{T}^n\times\mathbb{R}^n} w_2(x)\, d\mu(x, v) \quad \textit{for all } \mu \in \widetilde{\mathcal{M}}.$$

Then $w_1 \le w_2$ in $\mathbb{T}^n$.

Let $\mathcal{M}$ be the projected Mather set on $\mathbb{T}^n$, that is,

$$\mathcal{M} = \overline{\bigcup_{\mu\in\widetilde{\mathcal{M}}} \operatorname{supp}(\operatorname{proj}_{\mathbb{T}^n}\mu)}.$$

Theorem 1.1 gives the following straightforward result.

Corollary 1.2. *Assume* (H1)–(H2). *Let w_1, w_2 be any viscosity solutions of ergodic problem* (E). *Assume further that $w_1 = w_2$ on $\mathcal{M}$. Then $w_1 = w_2$ in $\mathbb{T}^n$.*

Corollary 1.2 was derived in [7, Theorem 4.12.6], [8, Theorem 10.4] earlier. We provide a simple proof for Theorem 1.1 in Section 2, which is a new application of the nonlinear adjoint method introduced in [5] (see also [18]). A generalization of Theorem 1.1 to the second order (degenerate viscous) setting, Theorem 4.1, is given in Section 4. It is worth mentioning that the result of Theorem 4.1 is new in the literature.

1.2. **Application.** We provide here an application in large time behavior. In this context, we need to strengthen the convexity of H in (H1).

(H1') There exists $\gamma > 0$ such that

$$D^2_{pp}H(x,p) \geq \gamma I_n \quad \text{for all } (x,p) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Here, I_n is the identity matrix of size n.

Under assumptions (H1'), (H2) and that the ergodic constant $c = 0$, for given $u_0 \in \mathrm{Lip}\,(\mathbb{T}^n)$, the viscosity solution $u \in C(\mathbb{T}^n \times [0,\infty))$ of the Cauchy problem

$$\text{(C)} \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{T}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{T}^n \end{cases}$$

has the following large time behavior

$$\lim_{t\to\infty} \|u(\cdot,t) - v\|_{L^\infty(\mathbb{T}^n)} = 0, \tag{1.3}$$

where $v \in \mathrm{Lip}\,(\mathbb{T}^n)$ is a viscosity solution of (E). This result was first proved in [6]. Notice that there are various different ways to prove it (see [2, 3, 12] and the references therein). We say that v is the *asymptotic profile* of u, and denote it by u^∞, or $u^\infty[u_0]$ to display the clear dependence on the initial data u_0.

We now give a representation formula for $u^\infty[u_0]$.

Theorem 1.3 (Asymptotic profiles)**.** *Assume that* (H1') *and* (H2) *hold, and the ergodic constant $c = 0$. For given $u_0 \in \mathrm{Lip}\,(\mathbb{T}^n)$, let $u^\infty[u_0]$ be the corresponding asymptotic profile. Then, we have*

(i) $u^\infty[u_0](y) = u_0^-(y)$ *for all* $y \in \mathcal{M}$,
(ii) $u^\infty[u_0](x) = \min\left\{d(x,y) + u_0^-(y) \,:\, y \in \mathcal{M}\right\}$ *for all* $x \in \mathbb{T}^n$.

Here,

$$u_0^-(x) = \sup\{v(x) \,:\, v \leq u_0 \text{ on } \mathbb{T}^n, \text{ and } v \text{ is a subsolution to (E)}\},$$
$$d(x,y) = \sup\{v(x) - v(y) \,:\, v \text{ is a subsolution to (E)}\}.$$

Theorem 1.3 was first proved in [4, Theorem 3.1], and we give an elementary proof of this in Section 3, which is simpler.

2. Uniqueness set of the ergodic problem

We present in this section the proof of Theorem 1.1.

Proof of Theorem 1.1. We use ideas introduced in [3].

For each $i=1,2$ and each $\varepsilon>0$, let u_i^ε be the viscosity solution to the Cauchy problem

$$\begin{cases} \varepsilon(u_i^\varepsilon)_t + H(x,Du_i^\varepsilon) = \varepsilon^4 \Delta u_i^\varepsilon & \text{in } \mathbb{T}^n\times(0,1),\\ u_i^\varepsilon(x,0)=w_i(x) & \text{on } \mathbb{T}^n. \end{cases} \tag{2.1}$$

Without the viscosity term, (2.1) becomes

$$\begin{cases} \varepsilon(u_i)_t + H(x,Du_i) = 0 & \text{in } \mathbb{T}^n\times(0,1),\\ u_i(x,0)=w_i(x) & \text{on } \mathbb{T}^n. \end{cases} \tag{2.2}$$

It is clear that the unique viscosity solution to (2.2) is $u_i(x,t)=w_i(x)$ for all $(x,t)\in\mathbb{T}^n\times[0,1)$ because w_i is a viscosity solution to (E). Thanks to (H2), by a standard argument, there exists $C>0$ independent of ε such that

$$\|Du_i^\varepsilon\|_{L^\infty(\mathbb{T}^n\times(0,1))} \le C \tag{2.3}$$

and

$$\|u_i^\varepsilon - w_i\|_{L^\infty(\mathbb{T}^n\times(0,1))} \le C\varepsilon. \tag{2.4}$$

See [12, Propositions 4.15 and 5.5] for the proofs of similar versions of (2.3) and (2.4) for instance. Our plan is to use $u_1^\varepsilon, u_2^\varepsilon$ to deduce the conclusion as $\varepsilon\to0$.

For any $x_0\in\mathbb{T}^n$, let σ^ε be the solution to

$$\begin{cases} -\varepsilon\sigma_t^\varepsilon - \operatorname{div}(D_pH(x,Du_2^\varepsilon)\sigma^\varepsilon) = \varepsilon^4\Delta\sigma^\varepsilon & \text{in } \mathbb{T}^n\times(0,1),\\ \sigma^\varepsilon(x,1)=\delta_{x_0} & \text{on } \mathbb{T}^n. \end{cases}$$

Here δ_{x_0} is the Dirac delta mass at x_0.

By convexity of H in (H1), we have

$$\varepsilon(u_1^\varepsilon-u_2^\varepsilon)_t + D_pH(x,Du_2^\varepsilon)\cdot D(u_1^\varepsilon-u_2^\varepsilon) \le \varepsilon^4\Delta(u_1^\varepsilon-u_2^\varepsilon).$$

Multiply this by σ^ε, integrate on $\mathbb{T}^n$, and note that

$$\begin{aligned} &\int_{\mathbb{T}^n}\left(-D_pH(x,Du_2^\varepsilon)\cdot D(u_1^\varepsilon-u_2^\varepsilon) + \varepsilon^4\Delta(u_1^\varepsilon-u_2^\varepsilon)\right)\sigma^\varepsilon\,dx\\ &= \int_{\mathbb{T}^n}\left(\operatorname{div}(D_pH(x,Du_2^\varepsilon)\sigma^\varepsilon)+\varepsilon^4\Delta\sigma^\varepsilon\right)(u_1^\varepsilon-u_2^\varepsilon)\,dx = -\int_{\mathbb{T}^n}\varepsilon\sigma_t^\varepsilon(u_1^\varepsilon-u_2^\varepsilon)\,dx. \end{aligned}$$

Thus,

$$\frac{d}{dt}\int_{\mathbb{T}^n}(u_1^\varepsilon-u_2^\varepsilon)\sigma^\varepsilon\,dx \le 0,$$

which yields, for each $t\in[0,1]$,

$$(u_1^\varepsilon-u_2^\varepsilon)(x_0,1) \le \int_{\mathbb{T}^n}(u_1^\varepsilon-u_2^\varepsilon)(x,t)\sigma^\varepsilon(x,t)\,dx,$$

and hence,

$$(u_1^\varepsilon-u_2^\varepsilon)(x_0,1) \le \int_0^1\int_{\mathbb{T}^n}(u_1^\varepsilon-u_2^\varepsilon)\sigma^\varepsilon\,dxdt. \tag{2.5}$$

In light of the Riesz theorem, there exists $\nu^\varepsilon \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ such that

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} \varphi(x,p)\,d\nu^\varepsilon(x,p) = \int_0^1 \int_{\mathbb{T}^n} \varphi(x, Du_2^\varepsilon)\sigma^\varepsilon\,dxdt \quad \text{for all } \varphi \in C_c(\mathbb{T}^n \times \mathbb{R}^n). \tag{2.6}$$

Then, (2.5) becomes

$$(u_1^\varepsilon - u_2^\varepsilon)(x_0, 1) \le \iint_{\mathbb{T}^n\times\mathbb{R}^n} (u_1^\varepsilon - u_2^\varepsilon)\,d\nu^\varepsilon(x,p). \tag{2.7}$$

Thanks to (2.3), we have that $\operatorname{supp}(\nu^\varepsilon) \subset \mathbb{T}^n \times \overline{B}(0,C)$. There exists $\{\varepsilon_j\} \to 0$ such that $\nu^{\varepsilon_j} \rightharpoonup \nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ as $j \to \infty$ weakly in the sense of measures. We set $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ be such that

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} \varphi(x,p)\,d\nu(x,p) = \iint_{\mathbb{T}^n\times\mathbb{R}^n} \varphi(x, D_vL(x,v))\,d\mu(x,v). \tag{2.8}$$

We provide a proof that μ is a Mather measure in Lemma 2.1 below for completeness (see also [17, Proposition 2.3], [12, Proposition 6.11]).

Sending $j \to \infty$ in (2.7) and using (2.4) to yield

$$w_1(x_0) - w_2(x_0) \le \iint_{\mathbb{T}^n\times\mathbb{R}^n} (w_1 - w_2)\,d\mu(x,v) \le 0. \qquad \square$$

Lemma 2.1. *For each $\varepsilon > 0$, let ν^ε be the measure defined in* (2.6)*. Assume that there exists a sequence $\{\varepsilon_j\} \to 0$ such that $\nu^{\varepsilon_j} \rightharpoonup \nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ as $j \to \infty$ weakly in the sense of measures. Let μ be a measure defined through ν by* (2.8)*. Then μ is a Mather measure.*

Proof. Fix any $\phi \in C^1(\mathbb{T}^n)$, and consider a family $\{\phi^m\} \subset C^\infty(\mathbb{T}^n)$ such that $\phi^m \to \phi$ in $C^1(\mathbb{T}^n)$ as $m \to \infty$.

Multiply the adjoint equation with ϕ^m and integrate on $\mathbb{T}^n \times [0,1]$ to imply

$$\varepsilon \int_{\mathbb{T}^n} \phi^m(x)\sigma^\varepsilon(x,0)\,dx - \varepsilon\phi^m(x_0) + \int_0^1\int_{\mathbb{T}^n} D_pH(x, Du_2^\varepsilon)\cdot D\phi^m(x)\sigma^\varepsilon(x,t)\,dxdt$$
$$= \varepsilon^4 \int_0^1\int_{\mathbb{T}^n} \Delta\phi^m(x)\sigma^\varepsilon(x,t)\,dxdt.$$

Let $\varepsilon = \varepsilon_j \to 0$ and $m \to \infty$ in this order to get

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} D_pH(x,p)\cdot D\phi(x)\,d\nu(x,p) = \iint_{\mathbb{T}^n\times\mathbb{R}^n} v\cdot D\phi(x)\,d\mu(x,v) = 0.$$

Thus, $\mu \in \mathcal{F}$.

We rewrite (2.1) as

$$\varepsilon(u_2^\varepsilon)_t + D_pH(x, Du_2^\varepsilon)\cdot Du_2^\varepsilon - \varepsilon^4\Delta u_2^\varepsilon = D_pH(x, Du_2^\varepsilon)\cdot Du_2^\varepsilon - H(x, Du_2^\varepsilon).$$

Multiply this by σ^ε and integrate on $\mathbb{T}^n \times [0,1]$ to yield

$$\varepsilon u_2^\varepsilon(x_0,1) - \varepsilon\int_{\mathbb{T}^n} u_2^\varepsilon(x,0)\sigma^\varepsilon(x,0)\,dx = \int_0^1\int_{\mathbb{T}^n} (D_pH(x, Du_2^\varepsilon)\cdot Du_2^\varepsilon - H(x, Du_2^\varepsilon))\sigma^\varepsilon\,dxdt.$$

We again let $\varepsilon = \varepsilon_j \to 0$ to achieve that

$$0 = \iint_{\mathbb{T}^n\times\mathbb{R}^n} (D_pH(x,p)\cdot p - H(x,p))\,d\nu(x,p) = \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\mu(x,v).$$

Also, note that we have

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\mu \geq 0 \qquad \text{for all } \mu\in\mathcal{F}, \tag{2.9}$$

which, together with (1.2), completes the proof. See [12, Lemma 6.12] for a proof of (2.9). □

3. Application

In this section, we always assume that (H1')–(H2) hold and that the ergodic constant $c = 0$.

Lemma 3.1. *Assume that u_0 is a viscosity subsolution of* (E)*. Then,*

$$u^\infty[u_0] = u_0 \qquad \text{on } \mathcal{M}.$$

Proof. We write u^∞ for $u^\infty[u_0]$ in the proof for simplicity.

By the usual comparison principle, we have $u(x,t) \geq u_0(x)$ for all $(x,t)\in\mathbb{T}^n\times[0,\infty)$. Hence, $u^\infty \geq u_0$ on $\mathbb{T}^n$.

Next, let ρ be a standard mollifier in $\mathbb{R}^n$. For each $\delta>0$, let $\rho^\delta(x) = \delta^{-n}\rho(\delta^{-1}x)$ for all $x\in\mathbb{R}^n$. Let $u^\delta = \rho^\delta * u$. Then due to the convexity of H in p, u^δ is a subsolution to

$$u^\delta_t + H(x, Du^\delta) \leq C\delta \qquad \text{in } \mathbb{T}^n\times(0,\infty).$$

For any Mather measure $\mu\in\widetilde{\mathcal{M}}$, by the holonomic and minimizing properties, we have

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbb{T}^n\times\mathbb{R}^n} u^\delta(x,t)\,d\mu &= \iint_{\mathbb{T}^n\times\mathbb{R}^n} (u^\delta_t + v\cdot Du^\delta - L(x,v))\,d\mu\\ &\leq \iint_{\mathbb{T}^n\times\mathbb{R}^n} u^\delta_t + H(x,Du^\delta)\,d\mu \leq C\delta.\end{aligned}$$

Therefore, for any $T>0$,

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} u^\delta(x,T)\,d\mu \leq \iint_{\mathbb{T}^n\times\mathbb{R}^n} (u_0)^\delta(x)\,d\mu + C\delta T.$$

Let $\delta\to 0$ and $T\to\infty$ in this order to yield

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} u^\infty\,d\mu \leq \iint_{\mathbb{T}^n\times\mathbb{R}^n} u_0\,d\mu.$$

Combined with $u^\infty \geq u_0$ on $\mathbb{T}^n$, we obtain $u^\infty = u_0$ on $\mathcal{M}$, which completes the proof. □

Remark 1. Notice that we get

$$u(x,t) = u_0(x) \qquad \text{for all } x\in\mathcal{M},\ t\in[0,\infty),$$

in the above proof.

We present next the proof of Theorem 1.3. Before proceeding to the proof, it is important noticing that d has the following representation formula

$$d(x,y)=\inf\left\{\int_0^t L(\gamma(s),-\dot\gamma(s))\,ds\ :\ t>0,\gamma\in \mathrm{AC}\left([0,t],\mathbb{T}^n\right),\gamma(0)=x,\gamma(t)=y\right\}.$$

See [7] for instance.

Proof of Theorem 1.3. It is enough to give only the proof of (i). The second claim (ii) follows immediately from Corollary 1.2, claim (i) and the representation formulas of d as well as of solutions to (E).

By the definition of u_0^-, we have $u_0^-\le u_0$ on $\mathbb{T}^n$. In light of the comparison principle, $u_0^-\le u$ on $\mathbb{T}^n\times[0,\infty)$, which implies $u_0^-\le u^\infty$ on $\mathbb{T}^n$.

We prove the reverse inequality holds on $\mathcal{M}$. Fix $y\in\mathcal{M}$ and $z\in\mathbb{T}^n$. Set $w_0^z(x)=u_0(z)+d(x,z)$ for $x\in\mathbb{T}^n$. Then, note that w_0^z is a viscosity subsolution to (E). Let w be the solution to (C) with initial data w_0^z. Thanks to Lemma 3.1, we get

$$w(y,t)=w_0^z(y)=u_0(z)+d(y,z)\qquad\text{for all } t\in[0,\infty). \tag{3.1}$$

For a large $t>1$, pick $\gamma:[0,t]\to\mathbb{T}^n$ to be an optimal path such that $\gamma(0)=y$ and

$$w(y,t)=w_0^z(\gamma(t))+\int_0^t L(\gamma(s),-\dot\gamma(s))\,ds=u_0(z)+d(\gamma(t),z)+\int_0^t L(\gamma(s),-\dot\gamma(s))\,ds.$$

On the other hand, for any $\varepsilon>0$, there exists $t_\varepsilon>0$ and a path $\gamma:[t,t+t_\varepsilon]\to\mathbb{T}^n$ with $\gamma(t+t_\varepsilon)=z$ satisfying

$$d(\gamma(t),z)\ge\int_t^{t+t_\varepsilon} L(\gamma(s),-\dot\gamma(s))\,ds-\varepsilon.$$

Combine the two relations above to imply

$$w_0^z(y)+\varepsilon\ge u_0(z)+\int_0^{t+t_\varepsilon} L(\gamma(s),-\dot\gamma(s))\,ds\ge u(y,t+t_\varepsilon). \tag{3.2}$$

By letting $t\to\infty$ in (3.2), one gets

$$w_0^z(y)+\varepsilon\ge u^\infty(y).$$

Next, let $\varepsilon\to0$ to conclude that $u_0(z)+d(y,z)\ge u^\infty(y)$. Vary z to yield

$$u^\infty(y)\le\min_{z\in\mathbb{T}^n}(u_0(z)+d(y,z)).$$

Notice here that in view of the inf-stability of viscosity subsolutions to convex first order Hamilton–Jacobi equations, we have $\min_{z\in\mathbb{T}^n}(u_0(z)+d(y,z))=u_0^-(y)$, which finishes the proof. □

4. Generalization: degenerate viscous cases

In this section, we present a generalization of Theorem 1.1 to (VE) in Introduction. We need the following assumptions.

(H2') There exist $\gamma > 1$ and $C > 0$ such that, for all $(x,p) \in \mathbb{T}^n \times \mathbb{R}^n$,

$$\begin{cases} \frac{1}{C}|p|^\gamma - C \le H(x,p) \le C(|p|^\gamma + 1), \\ |D_x H(x,p)| \le C(1+|p|^\gamma), \\ |D_p H(x,p)| \le C(1+|p|^{\gamma-1}). \end{cases}$$

(H3) $A(x) = (a_{ij}(x))_{1\le i,j\le n} \in \mathbb{M}^{n\times n}_{\rm sym}$ with $A \ge 0$, and $a_{ij} \in C^2(\mathbb{T}^n)$ for all $1 \le i,j \le n$.

By normalization, we always assume that $c = 0$ in this section. In fact, under assumptions (H1), (H2') and (H3), for any $w \in C(\mathbb{T}^n)$ solving (VE), $w \in \mathrm{Lip}\,(\mathbb{T}^n)$ (see [1, Theorem 3.1]).

Definition 2. *Let $\widetilde{\mathcal{M}}_V$ be the set of all minimizers of the minimizing problem*

$$\min_{\mu\in\mathcal{F}} \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\mu(x,v), \tag{4.1}$$

where

$$\mathcal{F}_V = \left\{\mu \in \mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n) \,:\, \iint_{\mathbb{T}^n\times\mathbb{R}^n} v\cdot D\phi - a_{ij}\phi_{x_ix_j}\,d\mu(x,v) = 0 \ \text{ for all } \phi \in C^2(\mathbb{T}^n)\right\}.$$

Each measure in $\widetilde{\mathcal{M}}_V$ is called a generalized Mather measure.

The notion of generalized Mather measures was first introduced and analyzed in [9, 10]. Because of normalization that $c = 0$, as in the first order case, one has that

$$\min_{\mu\in\mathcal{F}_V} \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\mu(x,v) = 0. \tag{4.2}$$

The proof of this claim follows [12, Lemma 6.12]. To be more precise, [12, Lemma 6.12] deals with the special case $A(x) = a(x)I_n$ where $a \in C^2(\mathbb{T}^n, [0,\infty))$ and I_n is the identity matrix of size n. For general diffusion matrix A satisfying (H3), we perform first inf-sup convolutions, and additionally a convolution by using a standard mollifier for a solution w of (VE). See also [11] for a form of (4.2) in fully nonlinear, degenerate elliptic PDE settings.

Theorem 4.1. *Assume* (H1), (H2'), (H3). *Let w_1, w_2 be any continuous viscosity solutions of ergodic problem* (E). *Assume further that*

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} w_1(x)\,d\mu(x,v) \le \iint_{\mathbb{T}^n\times\mathbb{R}^n} w_2(x)\,d\mu(x,v) \quad \text{for all } \mu \in \widetilde{\mathcal{M}}_V.$$

Then $w_1 \le w_2$ in $\mathbb{T}^n$.

Proof. We basically repeat the proof of Theorem 1.1.

For each $k = 1,2$ and each $\varepsilon > 0$, let u_k^ε be the solution to the Cauchy problem

$$\begin{cases} \varepsilon(u_k^\varepsilon)_t + H(x, Du_k^\varepsilon) = a_{ij}(u_k^\varepsilon)_{x_ix_j} + \varepsilon^4\Delta u_k^\varepsilon & \text{in } \mathbb{T}^n\times(0,1), \\ u_k^\varepsilon(x,0) = w_k(x) & \text{on } \mathbb{T}^n. \end{cases}$$

Without the viscosity $\varepsilon^4 \Delta u_k^\varepsilon$, (2.1) becomes

$$\begin{cases} \varepsilon (u_k)_t + H(x, Du_k) = a_{ij}(u_k)_{x_i x_j} & \text{in } \mathbb{T}^n \times (0,1), \\ u_k(x,0) = w_k(x) & \text{on } \mathbb{T}^n, \end{cases} \tag{4.3}$$

It is clear that the unique viscosity solution to (4.3) is $u_k(x,t) = w_k(x)$ for all $(x,t) \in \mathbb{T}^n \times [0,1)$ because of the fact that w_k is a solution to (VE). Thanks to (H2') (see [12, Theorem 4.5] for instance), there exists $C > 0$ independent of ε such that

$$\|Du_i^\varepsilon\|_{L^\infty(\mathbb{T}^n \times (0,1))} \le C \quad \text{and} \quad \|u_i^\varepsilon - w_i\|_{L^\infty(\mathbb{T}^n \times (0,1))} \le C\varepsilon. \tag{4.4}$$

As above, we use $u_1^\varepsilon, u_2^\varepsilon$ to deduce the conclusion as $\varepsilon \to 0$.

For any $x_0 \in \mathbb{T}^n$, let σ^ε be the solution to

$$\begin{cases} -\varepsilon \sigma_t^\varepsilon - \operatorname{div}(D_p H(x, Du_2^\varepsilon)\sigma^\varepsilon) = (a_{ij}\sigma^\varepsilon)_{x_i x_j} + \varepsilon^4 \Delta \sigma^\varepsilon & \text{in } \mathbb{T}^n \times (0,1), \\ \sigma^\varepsilon(x,1) = \delta_{x_0} & \text{on } \mathbb{T}^n. \end{cases}$$

Here δ_{x_0} is the Dirac delta mass at x_0.

By convexity of H, we have

$$\varepsilon (u_1^\varepsilon - u_2^\varepsilon)_t + D_p H(x, Du_2^\varepsilon) \cdot D(u_1^\varepsilon - u_2^\varepsilon) \le a_{ij}(u_1^\varepsilon - u_2^\varepsilon)_{x_i x_j} + \varepsilon^4 \Delta (u_1^\varepsilon - u_2^\varepsilon).$$

Multiply this by σ^ε and integrate on $\mathbb{T}^n$ to yield

$$\frac{d}{dt} \int_{\mathbb{T}^n} (u_1^\varepsilon - u_2^\varepsilon)\sigma^\varepsilon \, dx \le 0.$$

Hence,

$$(u_1^\varepsilon - u_2^\varepsilon)(x_0, 1) \le \int_0^1 \int_{\mathbb{T}^n} (u_1^\varepsilon - u_2^\varepsilon)\sigma^\varepsilon \, dx dt. \tag{4.5}$$

Let $\nu^\varepsilon \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ be the measure satisfying

$$\iint_{\mathbb{T}^n \times \mathbb{R}^n} \varphi(x,p) \, d\nu^\varepsilon(x,p) = \int_0^1 \int_{\mathbb{T}^n} \varphi(x, Du_2^\varepsilon)\sigma^\varepsilon \, dx dt \quad \text{for all } \varphi \in C_c(\mathbb{T}^n \times \mathbb{R}^n).$$

Then, (4.5) becomes

$$(u_1^\varepsilon - u_2^\varepsilon)(x_0, 1) \le \iint_{\mathbb{T}^n \times \mathbb{R}^n} (u_1^\varepsilon - u_2^\varepsilon) \, d\nu^\varepsilon(x,p). \tag{4.6}$$

Thanks to (4.4), we have that $\operatorname{supp}(\nu^\varepsilon) \subset \mathbb{T}^n \times \overline{B}(0,C)$. There exists $\{\varepsilon_j\} \to 0$ such that $\nu^{\varepsilon_j} \rightharpoonup \nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ as $j \to \infty$ weakly in the sense of measures. We set $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ be such that

$$\iint_{\mathbb{T}^n \times \mathbb{R}^n} \varphi(x,p) d\nu(x,p) = \iint_{\mathbb{T}^n \times \mathbb{R}^n} \varphi(x, D_v L(x,v)) \, d\mu(x,v).$$

Note that μ is a generalized Mather measure defined in Definition 2. We refer to [17, Proposition 2.3] or [12, Proposition 6.11] for the details.

Sending $j \to \infty$ in (4.6) and using (4.4) to yield

$$w_1(x_0) - w_2(x_0) \le \iint_{\mathbb{T}^n \times \mathbb{R}^n} (w_1 - w_2) \, d\mu(x,v) \le 0. \qquad \square$$

Let $\mathcal{M}_V$ be the generalized projected Mather set on $\mathbb{T}^n$, that is,

$$\mathcal{M}_V = \overline{\bigcup_{\mu \in \widetilde{\mathcal{M}}_V} \operatorname{supp}(\operatorname{proj}_{\mathbb{T}^n} \mu)}.$$

Theorem 4.1 gives the following straightforward result.

Corollary 4.2. *Assume* (H1), (H2'), (H3). *Let w_1, w_2 be any continuous viscosity solutions of ergodic problem* (VE). *Assume further that $w_1 \le w_2$ on $\mathcal{M}_V$. Then $w_1 \le w_2$ in $\mathbb{T}^n$.*

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