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A rigidity result for effective Hamiltonians with 3-mode periodic potentials [☆]

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A B S T R A C T

We continue studying an inverse problem in the theory of periodic homogenization of Hamilton–Jacobi equations proposed in [14]. Let $V_1, V_2 \in C(\mathbb{R}^n)$ be two given potentials which are $\mathbb{Z}^n$-periodic, and $\overline{H}_1, \overline{H}_2$ be the effective Hamiltonians associated with the Hamiltonians $\frac{1}{2}|p|^2+V_1$, $\frac{1}{2}|p|^2+V_2$, respectively. A main result in this paper is that, if the dimension $n=2$, and each of V_1, V_2 contains exactly 3 mutually non-parallel Fourier modes, then

$$\overline{H}_1 \equiv \overline{H}_2 \iff V_1(x) = V_2\left(\frac{x}{c}+x_0\right) \quad \text{for all } x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

for some $c \in \mathbb{Q} \setminus \{0\}$ and $x_0 \in \mathbb{T}^2$. When $n \geq 3$, the scenario is slightly more subtle, and a complete description is provided for any dimension. These resolve partially a conjecture stated in [14]. Some other related results and open problems are also discussed.

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1. Introduction

1.1. Periodic homogenization and the inverse problem

We first describe the theory of periodic homogenization of Hamilton–Jacobi equations. For each length scale $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to

$$\begin{cases} u^\varepsilon_t + H(Du^\varepsilon) + V\left(\frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, the Hamiltonian $H(p) - V(x)$ is of separable form with $H \in C(\mathbb{R}^n)$, which is coercive (i.e., $\lim_{|p|\to\infty} H(p) = +\infty$), and $V \in C(\mathbb{R}^n)$, which is $\mathbb{Z}^n$-periodic. The initial data $g \in \mathrm{BUC}\,(\mathbb{R}^n)$, the set of bounded, uniformly continuous functions on $\mathbb{R}^n$.

It was shown in [13] that, in the limit as the length scale ε tends to zero, u^ε converges to u locally uniformly on $\mathbb{R}^n \times [0,\infty)$, and u solves the effective equation

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The effective Hamiltonian $\overline{H} \in C(\mathbb{R}^n)$ is determined in a nonlinear way by H and V through the cell problems as follows. For each $p \in \mathbb{R}^n$, it was derived in [13] that there exists a unique constant $c \in \mathbb{R}$ such that the following cell problem has a continuous viscosity solution

$$H(p + Dv) + V(x) = c \quad \text{in } \mathbb{T}^n, \tag{1.3}$$

where $\mathbb{T}^n$ is the n-dimensional flat torus $\mathbb{R}^n/\mathbb{Z}^n$. We then denote by $\overline{H}(p) := c$.

During past decades, there have been tremendous progress and vast literature about the validity of homogenization and the well-posedness of cell problems in various generalized settings. Nevertheless, understanding theoretically how $\overline{H}$ depends on the potential V remains a very challenging and still largely open problem even for the most basic case $H(p) = \frac{1}{2}|p|^2$. For a smooth periodic potential V, a deep result in [4] asserts that when $n = 2$ and $H(p) = \frac{1}{2}|p|^2$, each non-minimum level curve of $\overline{H}$ associated with $\frac{1}{2}|p|^2 - V$ must contain line segments unless V is constant. Its proof relies on delicate analysis based on detailed structure of Aubry–Mather sets in two dimensions and a rigidity result in Riemannian geometry (the Hopf conjecture). Besides, due to the highly nonlinear nature of the problem, efficient numerical schemes to compute $\overline{H}$ have yet to be found. We refer to [1–3,5–11,15] and the references therein for recent progress.

In this paper, we aim to investigate the relation between V and $\overline{H}$ from the perspective of the following inverse problem first formulated in [14].

Question 1. *Let $H \in C(\mathbb{R}^n)$ be a given coercive function, that is, $\lim_{|p|\to\infty} H(p) = +\infty$. Let $V_1, V_2 \in C(\mathbb{R}^n)$ be two given potential energy functions which are $\mathbb{Z}^n$-periodic. Let*

$\overline{H}_1, \overline{H}_2$ be the effective Hamiltonians corresponding to the Hamiltonians $H(p) + V_1(x)$, $H(p) + V_2(x)$, respectively. If

$$\overline{H}_1 \equiv \overline{H}_2,$$

then what can we conclude about the relations between V_1 and V_2?

When $n = 1$, a complete answer was provided in [14] for a general class of convex H. It was shown that

$$\overline{H}_1 \equiv \overline{H}_2 \quad \iff \quad V_1 \text{ and } V_2 \text{ have same distributions,}$$

that is, $\int_0^1 f(V_1(x))\,dx = \int_0^1 f(V_2(x))\,dx$ for all $f \in C(\mathbb{R})$.

In case $n \geq 2$, the only known $\overline{H}$-invariant transformations are translation and scaling, i.e., for some $c \in \mathbb{Q} \cap (0, +\infty)$ and $x_0 \in \mathbb{T}^n$,

$$V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \quad \text{for all } x \in \mathbb{T}^n \quad \Longrightarrow \quad \overline{H}_1 \equiv \overline{H}_2.$$

If H is convex and even, the rescaling factor c could also be negative. However, if H is even but nonconvex, c has to be positive due to some pathological phenomena associated with nonconvexity (loss of evenness [15]). It is natural to investigate the following converse question. Throughout this paper, we focus on the mechanical Hamiltonian case, that is, the case where $H(p) = \frac{1}{2}|p|^2$ for $p \in \mathbb{R}^n$.

Question 2. *Assume that $n \geq 2$, and $H(p) = \frac{1}{2}|p|^2$ for $p \in \mathbb{R}^n$. Let $V_1, V_2 \in C(\mathbb{R}^n)$ be two given potential energy functions which are $\mathbb{Z}^n$-periodic. Let $\overline{H}_1, \overline{H}_2$ be the effective Hamiltonians corresponding to the Hamiltonians $H(p) - V_1(x)$, $H(p) - V_2(x)$, respectively. If*

$$\overline{H}_1 \equiv \overline{H}_2,$$

then can we conclude that

$$V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \quad \textit{for all } x \in \mathbb{T}^n,$$

for some $c \in \mathbb{Q} \setminus \{0\}$ and $x_0 \in \mathbb{T}^n$?

Some results related to this question were established in [14]. For example, if V_1 is constant, then the conclusion of Question 2 holds, that is, V_2 must be the same constant (see [14, Theorem 1.1]). In the general setting where $V_1, V_2 \in C^\infty(\mathbb{T}^n)$, by [14, Theorem 1.2], $\overline{H}_1 = \overline{H}_2$ implies that

$$\int_{\mathbb{T}^n} V_1\,dx = \int_{\mathbb{T}^n} V_2\,dx,$$

and under an extra decay condition of the Fourier coefficients of V_1, V_2, we also have

$$\int_{\mathbb{T}^n} V_1^2\,dx = \int_{\mathbb{T}^n} V_2^2\,dx.$$

It was conjectured in [14, Remark 1.1] that under the settings of Question 2 and some further reasonable assumptions on V_1, V_2, if $\overline{H}_1 = \overline{H}_2$, then V_1 and V_2 have the same distribution. Clearly, this conjecture is weaker than the conclusion of Question 2. We address more about this point at the end of Subsection 1.2.

It is natural to study the above questions in the case that V_1 and V_2 are trigonometric polynomials with m mutually non-parallel Fourier modes. In this paper, as a preliminary step, we settle Question 2 when number of modes $m = 3$. When $m \le 2$, the analysis is much simpler. We give the main results in the following subsection.

1.2. Main results

For $l = 1, 2$, set

$$\text{(A)}\quad \begin{cases} V_l(x) = a_{l0} + \sum_{j=1}^m (\lambda_{lj} e^{i2\pi k_{lj}\cdot x} + \overline{\lambda_{lj}} e^{-i2\pi k_{lj}\cdot x}), \\ \text{where } a_{l0} \in \mathbb{R},\ \{\lambda_{lj}\}_{j=1}^m \subset \mathbb{C} \text{ and } \{k_{lj}\}_{j=1}^m \subset \mathbb{Z}^n \setminus \{0\} \text{ such that} \\ \text{each pair of the } m \text{ vectors } \{k_{lj}\}_{j=1}^m \text{ are not parallel.} \end{cases}$$

Here, $\overline{\lambda_{1j}}$ is the complex conjugate of λ_{1j} for $1 \le j \le m$. The following are our main results.

Theorem 1.1. *Assume that $m = 3$, $n = 2$, $H(p) = \frac{1}{2}|p|^2$ for all $p \in \mathbb{R}^2$, and (A) holds. Assume that*

$$\overline{H}_1(p) = \overline{H}_2(p) \qquad \textit{for all } p \in \mathbb{R}^2.$$

Then there exist $c \in \mathbb{Q} \setminus \{0\}$ and $x_0 \in \mathbb{T}^2$ such that

$$V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \qquad \textit{for all } x \in \mathbb{T}^2.$$

Theorem 1.2. *Assume that $m = 3$, $n \ge 3$, $H(p) = \frac{1}{2}|p|^2$ for all $p \in \mathbb{R}^n$, and (A) holds. There are three cases as follows.*

(1) *If $\{k_{1j}\}_{j=1}^3$ are mutually orthogonal, then $\overline{H}_1 = \overline{H}_2$ if and only if for $1 \le j \le 3$,*

$$k_{1j} \parallel k_{2j} \quad \text{and} \quad |\lambda_{1j}| = |\lambda_{2j}|.$$

(2) *If* $k_{11} \perp k_{12}$ *and* $k_{11} \perp k_{13}$, *but* $k_{12} \not\perp k_{13}$, *then* $\overline{H}_1 = \overline{H}_2$ *if and only if*

$$k_{11} \parallel k_{21}, \quad ck_{12} = k_{22}, \; ck_{13} = k_{23} \quad \textit{for some } c \in \mathbb{Q} \setminus \{0\},$$

and for $1 \leq j \leq 3$,

$$|\lambda_{1j}| = |\lambda_{2j}|.$$

(3) *If* $\{k_{1j}\}_{j=1}^{3}$ *do not satisfy* (1) *and* (2) *after permutations, then*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Longleftrightarrow \quad V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \quad \textit{for all } x \in \mathbb{T}^n,$$

for some $c \in \mathbb{Q} \setminus \{0\}$ *and* $x_0 \in \mathbb{T}^n$.

Remark 1. Theorem 1.1 can actually be viewed as a special case of (3) in Theorem 1.2. Nevertheless, a major part of this paper is devoted to proving this two dimensional result, and hence, it is worth stating it as a separate theorem.

For completeness, we also present the case when $m \leq 2$.

Theorem 1.3. *Assume that* $m \leq 2$, $H(p) = \frac{1}{2}|p|^2$ *for all* $p \in \mathbb{R}^n$, *and* (A) *holds. Then*

(1) *If* $m = 1$, *then*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Longleftrightarrow \quad V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \quad \textit{for all } x \in \mathbb{T}^n,$$

for some $c \in \mathbb{Q} \setminus \{0\}$ *and* $x_0 \in \mathbb{T}^n$.

(2) *If* $m = 2$, *then there are two cases.*

(i) *If* $k_{11} \perp k_{12}$, *then* $\overline{H}_1 \equiv \overline{H}_2$ *if and only if for* $j = 1, 2$,

$$k_{1j} \parallel k_{2j} \quad \text{and} \quad |\lambda_{1j}| = |\lambda_{2j}|.$$

(ii) *If* k_{11} *is not perpendicular to* k_{12}, *then*

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Longleftrightarrow \quad V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \quad \textit{for all } x \in \mathbb{T}^n,$$

for some $c \in \mathbb{Q} \setminus \{0\}$ *and* $x_0 \in \mathbb{T}^n$.

Theorems 1.1–1.3 settle the conjecture stated in [14, Remark 1.1] completely in case $m \leq 3$. Of course, the case $m > 3$ is still open.

We believe that the rigidity property should hold for “generic” periodic potentials in any dimension. More precisely, we formulate the following conjecture.

Conjecture 1. *We conjecture that*

(1) *Theorem 1.1 holds when $m \geq 3$, $n = 2$.*
(2) *If $n \geq 3$, then the result of Theorem 1.1 is valid provided that V_1, V_2 belong to a dense open set of smooth periodic functions. In particular, the case that $m, n \geq 3$ should be fully characterized.*

It is clear that this conjecture is stronger than that in [14, Remark 1.1], but under the caveat that we require a generic assumption on V_1, V_2. Otherwise, it does not hold true (see parts (1)–(2) of Theorem 1.2 and part (2)(i) of Theorem 1.3 above).

1.3. Outline of the paper

In Section 2, we give a quick review of the method of asymptotic expansions of $\overline{H}_1, \overline{H}_2$ at infinity introduced in [14] (see also [12]). This is our main tool in studying the inverse problem. The proofs of our results will be given in Sections 3 and 4. They involve delicate analysis combining plane geometry, linear algebra and trigonometric functions.

2. Preliminary: asymptotic expansions of $\overline{H}_1, \overline{H}_2$ at infinity

2.1. Settings

For $x \in \mathbb{R}^n$, we write $x = (x_1, x_2, ..., x_n)$.

Assume there exists $m \in \mathbb{N}$ such that (A) holds. Let us only perform calculations with respect to $\overline{H}_1$. In light of (A), V_1 satisfies that

$$\begin{cases} V_1(x) = a_{10} + \sum_{j=1}^m (\lambda_{1j} e^{i2\pi k_{1j}\cdot x} + \overline{\lambda_{1j}} e^{-i2\pi k_{1j}\cdot x}), \\ \text{where } a_{10} \in \mathbb{R},\ \{\lambda_{1j}\}_{j=1}^m \subset \mathbb{C} \text{ and } \{k_{1j}\}_{j=1}^m \subset \mathbb{Z}^n \setminus \{0\} \text{ such that} \\ \text{each pair of the } m \text{ vectors } \{k_{1j}\}_{j=1}^m \text{ are not parallel.} \end{cases}$$

Recall that $\overline{\lambda_{1j}}$ is the complex conjugate of λ_{1j} for $1 \leq j \leq m$.

2.2. Asymptotic expansion at infinity

For a given vector $Q \neq 0$ and $\varepsilon > 0$, set $p = \frac{Q}{\sqrt{\varepsilon}}$. The cell problem for this vector p is

$$\frac{1}{2}\left|\frac{Q}{\sqrt{\varepsilon}} + Dv_1^\varepsilon\right|^2 + V_1(x) = \overline{H}_1\left(\frac{Q}{\sqrt{\varepsilon}}\right) \quad \text{in } \mathbb{T}^n.$$

Here, $v_1^\varepsilon \in C(\mathbb{T}^n)$ is a solution to the above. Multiply both sides by ε to yield

$$\frac{1}{2}|Q + \sqrt{\varepsilon}Dv_1^\varepsilon|^2 + \varepsilon V_1(x) = \varepsilon\overline{H}_1\left(\frac{Q}{\sqrt{\varepsilon}}\right) =: \overline{H}^\varepsilon(Q) \quad \text{in } \mathbb{T}^n. \tag{2.1}$$

Let us first use a formal asymptotic expansion to do computations. We use the following ansatz

$$\begin{cases} \sqrt{\varepsilon} v_1^\varepsilon(x) = \varepsilon v_{11}(x) + \varepsilon^2 v_{12}(x) + \varepsilon^3 v_{13}(x) + \cdots, \\ \overline{H}^\varepsilon(Q) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3 + \cdots. \end{cases}$$

Plug these into (2.1) to imply

$$\frac{1}{2}|Q + \varepsilon Dv_{11} + \varepsilon^2 Dv_{12} + \cdots|^2 + \varepsilon V_1 = \overline{H}^\varepsilon(Q) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots \quad \text{in } \mathbb{T}^n.$$

We first compare the $O(1)$ terms in both sides of the above equality to get

$$a_0 = \frac{1}{2}|Q|^2.$$

By using $O(\varepsilon)$, we get

$$Q \cdot Dv_{11} + V_1 = a_1 \quad \text{in } \mathbb{T}^n.$$

Hence, $a_1 = \int_{\mathbb{T}^n} V_1\, dx = a_{10}$ and

$$Dv_{11} = -\sum_{j=1}^m (\lambda_{1j} e^{i2\pi k_{1j}\cdot x} + \overline{\lambda_{1j}} e^{-i2\pi k_{1j}\cdot x}) \frac{k_{1j}}{k_{1j}\cdot Q}. \tag{2.2}$$

Next, using $O(\varepsilon^2)$, we achieve that

$$a_2 = \sum_{j=1}^m \frac{|\lambda_{1j}|^2 |k_{1j}|^2}{|k_{1j}\cdot Q|^2}, \tag{2.3}$$

and furthermore,

$$\begin{aligned} Q \cdot Dv_{12} &= a_2 - \frac{1}{2}|Dv_{11}|^2 \\ &= -\frac{1}{2} \sum_{\pm k_{1j} \pm k_{1l} \neq 0} \frac{\lambda_{1j}^\pm \lambda_{1l}^\pm k_{1j}\cdot k_{1l}}{(k_{1j}\cdot Q)(k_{1l}\cdot Q)} e^{i2\pi(\pm k_{1j} \pm k_{1l})\cdot x}. \end{aligned}$$

Here for convenience, for $1 \le j \le m$, we denote by

$$\lambda_{1j}^+ = \lambda_{1j} \quad \text{and} \quad \lambda_{1j}^- = \overline{\lambda_{1j}}.$$

Thus,

$$Dv_{12} = -\frac{1}{2} \sum_{\pm k_{1j} \pm k_{1l} \neq 0} \frac{\lambda_{1j}^\pm \lambda_{1l}^\pm k_{1j}\cdot k_{1l}}{(k_{1j}\cdot Q)(k_{1l}\cdot Q)} e^{i2\pi(\pm k_{1j} \pm k_{1l})\cdot x} \frac{\pm k_{1j} \pm k_{1l}}{(\pm k_{1j} \pm k_{1l})\cdot Q}.$$

Let us now switch to a symbolic way of writing to keep track with all terms. Denote by $\sum_G$ to be a good sum where all terms are well-defined, that is, all denominators of the fractions in the sum are not zero. We have

$$Dv_{12} = -\frac{1}{2}\sum_G \frac{\lambda_{1j_1}^{\pm}\lambda_{1j_2}^{\pm}k_{1j_1}\cdot k_{1j_2}}{(k_{1j_1}\cdot Q)(k_{1j_2}\cdot Q)} e^{i2\pi(\pm k_{1j_1}\pm k_{1j_2})\cdot x}\frac{\pm k_{1j_1}\pm k_{1j_2}}{(\pm k_{1j_1}\pm k_{1j_2})\cdot Q}. \tag{2.4}$$

Let us now look at $O(\varepsilon^3)$:

$$Q\cdot Dv_{13} = a_3 - Dv_{11}\cdot Dv_{12}.$$

Hence,

$$a_3 = \int_{\mathbb{T}^n} Dv_{11}\cdot Dv_{12}\,dx,$$

and

$$Dv_{13} = -\frac{1}{2}\sum_G \frac{\lambda_{1j_1}^{\pm}\lambda_{1j_2}^{\pm}\lambda_{1j_3}^{\pm}(k_{1j_1}\cdot k_{1j_2})(\pm k_{1j_1}\pm k_{1j_2})\cdot k_{1j_3}}{(k_{1j_1}\cdot Q)(k_{1j_2}\cdot Q)(k_{1j_3}\cdot Q)(\pm k_{1j_1}\pm k_{1j_2})\cdot Q}\times$$
$$\times\, e^{i2\pi(\pm k_{1j_1}\pm k_{1j_2}\pm k_{1j_3})\cdot x}\frac{\pm k_{1j_1}\pm k_{1j_2}\pm k_{1j_3}}{(\pm k_{1j_1}\pm k_{1j_2}\pm k_{1j_3})\cdot Q}. \tag{2.5}$$

The $O(\varepsilon^4)$ term yields

$$Dv_{11}\cdot Dv_{13} + \frac{1}{2}|Dv_{12}|^2 + Q\cdot Dv_{14} = a_4.$$

Integrate to get

$$a_4 = \frac{1}{2}\int_{\mathbb{T}^2} |Dv_{12}|^2\,dx + \int_{\mathbb{T}^2} Dv_{11}\cdot Dv_{13}\,dx.$$

The first integral in the formula of a_4 contains terms like $I(j_1,j_2) + I(j_3,j_4) + II(j_1,j_2,j_3,j_4)$ with

$$I(j_1,j_2) = \frac{1}{8}\frac{|\lambda_{1j_1}|^2|\lambda_{1j_2}|^2|k_{1j_1}\cdot k_{1j_2}|^2|\pm k_{1j_1}\pm k_{1j_2}|^2}{|k_{1j_1}\cdot Q|^2|k_{1j_2}\cdot Q|^2|(\pm k_{1j_1}\pm k_{1j_2})\cdot Q|^2},$$

and $I(j_3,j_4)$ is of the exact same form with (j_3,j_4) in place of (j_1,j_2). Besides,

$$II(j_1,j_2,j_3,j_4) = \frac{1}{8}\frac{\lambda_{1j_1}^{\pm}\lambda_{1j_2}^{\pm}\lambda_{1j_3}^{\pm}\lambda_{1j_4}^{\pm}(k_{1j_1}\cdot k_{1j_2})(k_{1j_3}\cdot k_{1j_4})}{(k_{1j_1}\cdot Q)(k_{1j_2}\cdot Q)(k_{1j_3}\cdot Q)(k_{1j_4}\cdot Q)}\cdot\frac{|\pm k_{1j_1}\pm k_{1j_2}|^2}{|(\pm k_{1j_1}\pm k_{1j_2})\cdot Q|^2}$$

provided that $(j_1,j_2)\neq(j_3,j_4)$ and $\pm k_{1j_1}\pm k_{1j_2}\pm k_{1j_3}\pm k_{1j_4}=0$.

It is more important noticing that the terms that are not vanished in the above second integral of a_4 are the ones that have $\pm k_{1j_1} \pm k_{1j_2} \pm k_{1j_3} \pm k_{1j_4} = 0$. Hence, $\pm k_{1j_1} \pm k_{1j_2} \pm k_{1j_3} = \mp k_{1j_4}$ and these terms look like

$$\frac{\lambda^{\pm}_{1j_1}\lambda^{\pm}_{1j_2}\lambda^{\pm}_{1j_3}\lambda^{\pm}_{1j_4}(k_{1j_1}\cdot k_{1j_2})[(\pm k_{1j_1}\pm k_{1j_2})\cdot k_{1j_3}]|k_{1j_4}|^2}{(k_{1j_1}\cdot Q)(k_{1j_2}\cdot Q)(k_{1j_3}\cdot Q)[(\pm k_{1j_1}\pm k_{1j_2})\cdot Q]|k_{1j_4}\cdot Q|^2}. \tag{2.6}$$

Of course, v_{14} satisfies

$$Q\cdot Dv_{14} = a_4 - Dv_{11}\cdot Dv_{13} - \frac{1}{2}|Dv_{12}|^2. \tag{2.7}$$

By computing in an iterative way, we can get formulas of a_l and v_{1l} for all $l\in\mathbb{N}$. It turns out that this formal asymptotic expansion of $\overline{H}^{\varepsilon}(Q)$ holds true rigorously. For our purpose here, we only need the first five terms in the expansion.

Proposition 2.1. *Assume that $H(p)=\frac{1}{2}|p|^2$ for all $p\in\mathbb{R}^n$ and* (A) *holds. Let $\overline{H}_1$ be the effective Hamiltonian corresponding to the Hamiltonian $H(p)+V_1(x)$. Let $Q\neq 0$ be a vector in $\mathbb{R}^n$ such that Q is not perpendicular to each nonzero vector of $k_{1j_1}, \pm k_{1j_1}\pm k_{1j_2}, \pm k_{1j_1}\pm k_{1j_2}\pm k_{1j_3}$ and $\pm k_{1j_1}\pm k_{1j_2}\pm k_{1j_3}\pm k_{1j_4}$ for $1\le j_1,j_2,j_3,j_4\le m$.*

For $\varepsilon>0$, set $\overline{H}^{\varepsilon}(Q)=\varepsilon\overline{H}_1\left(\frac{Q}{\sqrt{\varepsilon}}\right)$. Then we have that, as $\varepsilon\to 0$,

$$\overline{H}^{\varepsilon}(Q)=\frac{1}{2}|Q|^2+\varepsilon a_1+\varepsilon^2 a_2+\varepsilon^3 a_3+\varepsilon^4 a_4+O(\varepsilon^5).$$

Here the error term satisfies $|O(\varepsilon^5)|\le K\varepsilon^5$ for some K depending only on Q, $\{\lambda_{1j}\}_{j=1}^m$ and $\{k_{1j}\}_{j=1}^m$.

Let us present the proof of this proposition here for the sake of completeness. A version of this was presented in [14, Proof of Theorem 1.2 (Part 3)]. See also [12, Lemma 3.1].

Proof. Let $v_{11}, v_{12}, v_{13}, v_{14}$ be solutions to (2.2), (2.4), (2.5), (2.7), respectively. Let $\phi=\varepsilon v_{11}+\varepsilon^2 v_{12}+\varepsilon^3 v_{13}+\varepsilon^4 v_{14}$, then ϕ satisfies

$$\frac{1}{2}|Q+D\phi|^2+\varepsilon V_1=\frac{1}{2}|Q|^2+\varepsilon a_1+\varepsilon^2 a_2+\varepsilon^3 a_3+\varepsilon^4 a_4+O(\varepsilon^5)\quad \text{in } \mathbb{T}^n.$$

Recall that $w=\sqrt{\varepsilon}v_1^{\varepsilon}$ is a solution to (2.1). By looking at the places where $w-\phi$ attains its maximum and minimum and using the definition of viscosity solutions, we arrive at the conclusion. □

We prepare some further definitions. Denote

$$\begin{cases} A_1=\{\pm k_{1j},\ \pm k_{1j}\pm k_{1l}\ :\ 1\le j,l\le m \text{ and } k_{1j}\cdot k_{1l}\neq 0\}, \\ A_2=\{\pm k_{2j},\ \pm k_{2j}\pm k_{2l}\ :\ 1\le j,l\le m \text{ and } k_{2j}\cdot k_{2l}\neq 0\}. \end{cases}$$

In other words, if $k_{1j} \cdot k_{2j} = 0$ for some $i, j \in \{1, \dots, m\}$, then we do not collect $\pm k_{1j} \pm k_{2j}$ in A_1.

Definition 1 *(Sole vectors).* A vector $\alpha k_{1j_1} + \beta k_{1j_2}$, where $\alpha, \beta \in \{-1, 1\}$ and $1 \leq j_1, j_2 \leq m$, is called a sole vector from A_1 if it is in A_1 and is not equal to any other vectors in A_1.

A vector $\alpha k_{2j_1} + \beta k_{2j_2}$, where $\alpha, \beta \in \{-1, 1\}$ and $1 \leq j_1, j_2 \leq m$, is called a sole vector from A_2 if it is in A_2 and is not equal to any other vectors in A_2.

Remark 2. If $\alpha k_{1j_1} + \beta k_{1j_2}$ is a sole vector from A_1, then

$$\frac{1}{4} \frac{|\lambda_{1j_1}|^2 |\lambda_{1j_2}|^2 |k_{1j_1} \cdot k_{1j_2}|^2 |\alpha k_{1j_1} + \beta k_{1j_2}|^2}{|k_{1j_1} \cdot Q|^2 |k_{1j_2} \cdot Q|^2 |(\alpha k_{1j_1} + \beta k_{1j_2}) \cdot Q|^2}$$

is the only term in a_4 containing $\frac{1}{|(\alpha k_{1j_1} + \beta k_{1j_2}) \cdot Q|^2}$.

Definition 2. Let A_1 and A_2 be two sets of vectors in $\mathbb{R}^n$. We write

$$A_1 \prec A_2$$

if for any $u \in A_1 \setminus \{0\}$, there exists $v \in A_2 \setminus \{0\}$ such that $u \parallel v$.

Remark 3. If $\overline{H}_1 \equiv \overline{H}_2$, then Remark 2, together with Proposition 2.1, implies that

$$\begin{cases} \{\text{Sole vectors from } A_1\} \prec A_2 \\ \{\text{Sole vectors from } A_2\} \prec A_1. \end{cases}$$

Heuristically, this could lead to an over-determined linear system, which plays a key role in proving our rigidity results.

3. Proof of Theorem 1.1

In this section, we always assume that the settings in Theorem 1.1 are in force. In particular, we have $n = 2$ and $m = 3$. Without loss of generality, we assume further that for $l = 1, 2$,

(H) $k_{l1},\ k_{l2},\ k_{l3}$ are aligned in the counter-clockwise order on the upper half plane $\{x = (x_1, x_2) : x_2 \geq 0\}$.

See Fig. 3.1 below.

We proceed to prove Theorem 1.1 via the following lemmas.

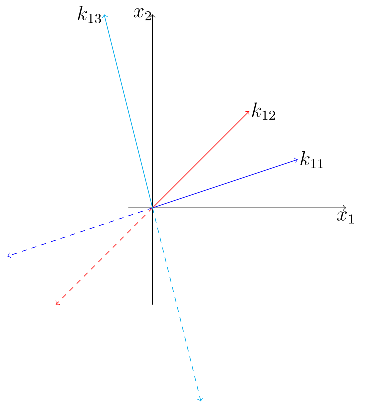

Fig. 3.1. The vectors $\{k_{1j}\}_{j=1}^3$.

Lemma 3.1. *Assume that the settings in Theorem 1.1 hold. Then $a_{10} = a_{20}$ and, for all $1 \le j \le 3$,*

$$|\lambda_{1j}| = |\lambda_{2j}| \quad \textit{and} \quad \frac{k_{1j}}{|k_{1j}|} = \frac{k_{2j}}{|k_{2j}|}.$$

Proof. We use the asymptotic expansion of $\overline{H}^\varepsilon(Q)$ in Proposition 2.1 and compare the coefficients to get the conclusion. Firstly, by comparing a_1, we imply $a_{10} = a_{20}$ immediately.

Secondly, we use the formula of a_2 given in (2.3) to get

$$\sum_{j=1}^{3} \frac{|\lambda_{1j}|^2 |k_{1j}|^2}{|k_{1j} \cdot Q|^2} = \sum_{j=1}^{3} \frac{|\lambda_{2j}|^2 |k_{2j}|^2}{|k_{2j} \cdot Q|^2}.$$

Fix $j \in \{1, 2, 3\}$. By letting $Q \to k_{1j}^\perp$, we use (H) to conclude

$$\frac{k_{1j}}{|k_{1j}|} = \frac{k_{2j}}{|k_{2j}|} \quad \text{and} \quad |\lambda_{1j}| = |\lambda_{2j}|. \quad \square \tag{3.1}$$

Thanks to Lemma 3.1, for $1 \le j \le 3$, there exists $\alpha_j > 0$ such that

$$k_{1j} = \alpha_j k_{2j}.$$

The following is a result in linear algebra (or plane geometry), which we believe is of independent interest.

Lemma 3.2. *For $j = 1, 2, 3$, let $\alpha_j > 0$ be a given number. Let u_1, u_2 and u_3 be non-parallel vectors on the upper half plane $\{x = (x_1, x_2) : x_2 \ge 0\}$, which are aligned in the counter-clockwise order. Set*

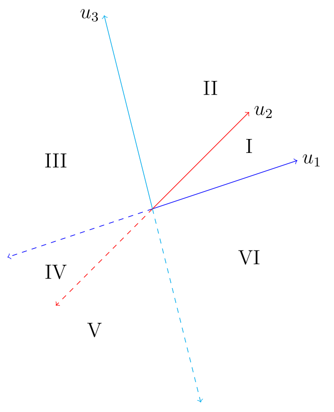

Fig. 3.2. Vectors $\{u_j\}_{j=1}^3$ and six regions I–VI.

$$\begin{cases} S_1 = \{\pm u_i,\ \pm u_i \pm u_j\ :\ 1 \le i, j \le 3 \text{ and } u_i \cdot u_j \ne 0\}, \\ S_2 = \{\pm u_i,\ \pm \alpha_i u_i \pm \alpha_j u_j\ :\ 1 \le i < j \le 3 \text{ and } u_i \cdot u_j \ne 0\}. \end{cases}$$

If

$$\begin{cases} \{\text{Sole vectors from } S_1\} \prec S_2, \\ \{\text{Sole vectors from } S_2\} \prec S_1, \end{cases}$$

then $\alpha_1 = \alpha_2 = \alpha_3$.

Proof. We may normalize $\alpha_1 = 1$. For $1 \le i, j \le 3$, denote

$$a_{ij} = u_i \times u_j = \det[u_i, u_j].$$

Set $\vec{a} = (a_{12}, a_{13}, a_{23}) \in \mathbb{R}^3$.

We prove by contradiction by assuming that α_2 and α_3 are not both 1. This is rather a lengthy proof and we divide it into steps in order to keep track with the key points easily. The directions $\{\pm u_j\}_{j=1}^3$ divide $\mathbb{R}^2$ into six regions named I–VI as in Fig. 3.2.

Part I: Non-orthogonal case. We first assume that u_1, u_2, u_3 are mutually non-orthogonal. Then it is easy to see that

$$\{\pm(u_1 + u_2),\ \pm(u_2 + u_3), \pm(u_3 - u_1)\} \subseteq \{\text{Sole vectors from } S_1\}$$

and

$$\{\pm(u_1 + \alpha_2 u_2),\ \pm(\alpha_2 u_2 + \alpha_3 u_3),\ \pm(\alpha_3 u_3 - u_1)\} \subseteq \{\text{Sole vectors from } S_2\}.$$

Step 1. Assume that $\alpha_2 = 1$ but $\alpha_3 \neq 1$. Then we have that

$$\begin{cases} u_2 + u_3 \parallel u_1 + \alpha_3 u_3 \text{ or } u_1 - u_2, \\ u_3 - u_1 \parallel u_1 - u_2 \text{ or } \alpha_3 u_3 - u_2, \end{cases}$$

and

$$\begin{cases} u_2 + \alpha_3 u_3 \parallel u_1 + u_3 \text{ or } u_1 - u_2, \\ \alpha_3 u_3 - u_1 \parallel u_1 - u_2 \text{ or } u_3 - u_2. \end{cases}$$

Since $u_2 + u_3$, $u_3 - u_1$, $u_2 + \alpha_3 u_3$ and $\alpha_3 u_3 - u_1$ are mutually non-parallel, there are only two possibilities.

Case 1.1. None of these four vectors is parallel to $u_1 - u_2$. Then

$$\begin{cases} u_2 + u_3 \parallel u_1 + \alpha_3 u_3, \\ u_3 - u_1 \parallel \alpha_3 u_3 - u_2, \\ u_2 + \alpha_3 u_3 \parallel u_1 + u_3, \\ \alpha_3 u_3 - u_1 \parallel u_3 - u_2. \end{cases}$$

We use the fact that $u \times \hat{u} = 0$ provided $u \parallel \hat{u}$ to yield

$$\vec{a} \cdot w_k = 0 \quad \text{for all } k = 1, 2, 3, 4.$$

Here

$$w_1 = (-1, -1, \alpha_3),\ w_2 = (1, -\alpha_3, 1),\ w_3 = (-1, -\alpha_3, 1),\ w_4 = (1, -1, \alpha_3).$$

Therefore, the dimension of $V = \operatorname{span}\{w_1, w_2, w_3, w_4\}$ is at most 2. Therefore $\det|w_1, w_2, w_4| = 0$, which leads to $\alpha_3 = 1$. This is a contradiction.

Case 1.2. One and only one of these four vectors is parallel to $u_1 - u_2$. As the roles of u_3 and $\alpha_3 u_3$ are the same, we only need to consider two situations. Either

$$\begin{cases} u_2 + u_3 \parallel u_1 - u_2, \\ u_3 - u_1 \parallel \alpha_3 u_3 - u_2, \\ u_2 + \alpha_3 u_3 \parallel u_1 + u_3, \\ \alpha_3 u_3 - u_1 \parallel u_3 - u_2, \end{cases} \quad \text{or} \quad \begin{cases} u_2 + u_3 \parallel u_1 + \alpha_3 u_3, \\ u_3 - u_1 \parallel u_1 - u_2, \\ u_2 + \alpha_3 u_3 \parallel u_1 + u_3, \\ \alpha_3 u_3 - u_1 \parallel u_3 - u_2. \end{cases}$$

Then we have either the dimension of $\operatorname{span}\{\hat{w}_1, w_2, w_3, w_4\}$ is 2 or the dimension of $\operatorname{span}\{w_1, \hat{w}_2, w_3, w_4\}$ is 2. Here $\hat{w}_1 = (-1, -1, 1)$ and $\hat{w}_2 = (1, -1, 1)$. Both cases lead to the same conclusion that $\alpha_3 = 1$. This is a contradiction.

Step 2. Either $\alpha_2 \neq \alpha_3 = 1$ or $\alpha_2 = \alpha_3 \neq 1$. This case can be transformed back to the previous case by suitable rotations, reflections and normalizations.

Step 3: Now we consider the case $1 \neq \alpha_2 \neq \alpha_3 \neq 1$. Then we must have that for $i, j \in \{1, 2, 3\}$

$$\begin{cases} u_i + u_j \nparallel \alpha_i u_i + \alpha_j u_j, \\ u_i - u_j \nparallel \alpha_i u_i - \alpha_j u_j. \end{cases}$$

Accordingly,

$$\begin{cases} u_1 + u_2 \parallel u_1 + \alpha_3 u_3 \text{ or } \alpha_3 u_3 - \alpha_2 u_2, \\ u_2 + u_3 \parallel u_1 + \alpha_3 u_3 \text{ or } \alpha_2 u_2 - u_1, \\ u_3 - u_1 \parallel \ \alpha_2 u_2 - u_1 \text{ or } \alpha_3 u_3 - \alpha_2 u_2. \end{cases} \tag{3.2}$$

Since $u_1 + u_2$, $u_2 + u_3$ and $u_3 - u_1$ are mutually linearly independent, we have only two scenarios

$$\begin{cases} u_1 + u_2 \parallel u_1 + \alpha_3 u_3, \\ u_2 + u_3 \parallel \alpha_2 u_2 - u_1, \\ u_3 - u_1 \parallel \alpha_3 u_3 - \alpha_2 u_2, \end{cases} \quad \text{or} \quad \begin{cases} u_1 + u_2 \parallel \alpha_3 u_3 - \alpha_2 u_2, \\ u_2 + u_3 \parallel u_1 + \alpha_3 u_3, \\ u_3 - u_1 \parallel \alpha_2 u_2 - u_1. \end{cases} \tag{3.3}$$

Similarly, there are two other cases to be considered for $u_1 + \alpha_2 u_2$, $\alpha_2 u_2 + \alpha_3 u_3$, $\alpha_3 u_3 - u_1$

$$\begin{cases} u_1 + \alpha_2 u_2 \parallel u_1 + u_3, \\ \alpha_2 u_2 + \alpha_3 u_3 \parallel u_2 - u_1, \\ \alpha_3 u_3 - u_1 \parallel u_3 - u_2, \end{cases} \quad \text{or} \quad \begin{cases} u_1 + \alpha_2 u_2 \parallel u_3 - u_2, \\ \alpha_2 u_2 + \alpha_3 u_3 \parallel u_1 + u_3, \\ \alpha_3 u_3 - u_1 \parallel u_2 - u_1. \end{cases} \tag{3.4}$$

In total, there are four cases to be studied.

Case 3.1. Assume that

$$\begin{cases} u_1 + u_2 \parallel u_1 + \alpha_3 u_3, \\ u_2 + u_3 \parallel \alpha_2 u_2 - u_1, \\ u_3 - u_1 \parallel \alpha_3 u_3 - \alpha_2 u_2, \end{cases} \quad \text{and} \quad \begin{cases} u_1 + \alpha_2 u_2 \parallel u_1 + u_3, \\ \alpha_2 u_2 + \alpha_3 u_3 \parallel u_2 - u_1, \\ \alpha_3 u_3 - u_1 \parallel u_3 - u_2. \end{cases} \tag{3.5}$$

Considering cross product between parallel vectors, we get that

$$\vec{a} \cdot v_i = 0,$$

for (here we write $\alpha = \alpha_2$ and $\beta = \alpha_3$)

$$v_1 = (-1, \beta, \beta), \ v_2 = (1, 1, -\alpha), \ v_3 = (\alpha, -\beta, \alpha)$$

and

$$v_4 = (-\alpha, 1, \alpha),\ v_5 = (\alpha, \beta, -\beta),\ v_6 = (1, -1, \beta).$$

Clearly, the dimension of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is 2. By noting that $v_2 + v_3 = (1 + \alpha, 1 - \beta, 0)$ and $v_5 + v_6 = (\alpha + 1, \beta - 1, 0)$, we imply $v_2 + v_3$ and $v_5 + v_6$ are linearly dependent. Otherwise, $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq \{x_3 = 0\}$, which is impossible. Hence we obtain that $1 - \beta = \beta - 1$, that is, $\beta = 1$. This is a contradiction.

Case 3.2. We have that

$$\begin{cases} u_1 + u_2 \parallel u_1 + \alpha_3 u_3, \\ u_2 + u_3 \parallel \alpha_2 u_2 - u_1, \\ u_3 - u_1 \parallel \alpha_3 u_3 - \alpha_2 u_2, \end{cases} \quad \text{and} \quad \begin{cases} u_1 + \alpha_2 u_2 \parallel u_2 - u_3, \\ \alpha_2 u_2 + \alpha_3 u_3 \parallel u_1 + u_3, \\ \alpha_3 u_3 - u_1 \parallel u_2 - u_1. \end{cases}$$

Set

$$\tilde{v}_4 = (1, -1, -\alpha),\ \tilde{v}_5 = (-\alpha, -\beta, \alpha),\ \tilde{v}_6 = (-1, \beta, -\beta).$$

Similarly, the rank of $\{v_1, v_2, v_3, \tilde{v}_4, \tilde{v}_5, \tilde{v}_6\}$ is 2. Note that $v_2 + v_3 = (1 + \alpha, 1 - \beta, 0)$ and $v_2 + \tilde{v}_5 = (1 - \alpha, 1 - \beta, 0)$. By the same argument as above, $v_2 + v_3$ and $v_2 + \tilde{v}_5$ are linearly dependent, which leads to $\beta = 1$. We again arrive at a contradiction.

Case 3.3. We have that

$$\begin{cases} u_1 + u_2 \parallel \alpha_3 u_3 - \alpha_2 u_2, \\ u_2 + u_3 \parallel u_1 + \alpha_3 u_3, \\ u_3 - u_1 \parallel \alpha_2 u_2 - u_1, \end{cases} \quad \text{and} \quad \begin{cases} u_1 + \alpha_2 u_2 \parallel u_2 - u_3, \\ \alpha_2 u_2 + \alpha_3 u_3 \parallel u_1 + u_3, \\ \alpha_3 u_3 - u_1 \parallel u_2 - u_1. \end{cases}$$

Set

$$\hat{v}_1 = (-\alpha, \beta, \beta),\ \hat{v}_2 = (-1, -1, \beta),\ \hat{v}_3 = (-\alpha, 1, -\alpha).$$

Again, the rank of $\{\hat{v}_1, \hat{v}_2, \hat{v}_3, \tilde{v}_4, \tilde{v}_5, \tilde{v}_6\}$ is 2. Note that $\tilde{v}_4 + \tilde{v}_5 = (1 - \alpha, -1 - \beta, 0)$ and $\hat{v}_1 - \hat{v}_2 = (1 - \alpha, \beta + 1, 0)$. Similar to the above, $\tilde{v}_4 + \tilde{v}_5$ and $\hat{v}_1 - \hat{v}_2$ must be linearly dependent, which leads to $\alpha = 1$. This is again a contradiction.

Due to the symmetry, the remaining case is essentially the same as Case 3.2. We omit the proof.

Part II: Orthogonal case. Without loss of generality, we assume the $u_1 \perp u_3$. The other two situations ($u_1 \perp u_2$ or $u_2 \perp u_3$) can be converted into this case by suitable reflections and rotations. For this case,

$$\{\pm(u_1 + u_2),\ \pm(u_2 + u_3)\}$$

and

$$\{\pm(u_1+\alpha_2 u_2),\ \pm(\alpha_2 u_2+\alpha_3 u_3)\}$$

are still sole vectors of S_1 and S_2, respectively. Also, it is important to note that, by definitions,

$$\pm u_1 \pm u_3 \notin S_1 \quad \text{and} \quad \pm u_1 \pm \alpha_3 u_3 \notin S_2.$$

We consider two cases.

Case II.1. Assume that $1=\alpha_2\neq\alpha_3$. Then $\alpha_1 u_1+\alpha_2 u_2=u_1+u_2$. By the assumption

$$\begin{cases} u_2+u_3 \parallel u_1-u_2, \\ u_2+\alpha_3 u_3 \parallel u_1-u_2. \end{cases} \tag{3.6}$$

This leads to $u_2+u_3 \parallel u_2+\alpha_3 u_3$, which is absurd.

Case II. 2. Assume that $\alpha_2\neq 1$. By the assumption, we must that

$$\begin{cases} u_1+u_2 \parallel \alpha_3 u_3-\alpha_2 u_2, \\ \alpha_3 u_3+\alpha_2 u_2 \parallel u_2-u_1. \end{cases}$$

This is equivalent to

$$\begin{cases} u_1+u_2 \parallel u_3-\frac{\alpha_2}{\alpha_3}u_2, \\ u_2-u_1 \parallel u_3+\frac{\alpha_2}{\alpha_3}u_2. \end{cases}$$

Then $-ra_{12}+a_{13}+a_{23}=-ra_{12}-a_{13}+a_{23}=0$ for $r=\frac{\alpha_2}{\alpha_3}$. This implies that $a_{13}=0$, i.e., $u_1 \parallel u_3$, which is again absurd. The proof is complete. □

Combining Remark 3 and the above Lemma 3.2, we obtain that there exists $c\in\mathbb{Q}$ such that for $j=1,2,3$,

$$k_{2j}=ck_{1j}. \tag{3.7}$$

Without loss of generality, we set $c=1$. Note however that Lemma 3.1 only gives us that $|\lambda_{1j}|=|\lambda_{2j}|$ for $1\le j\le 3$, which is not yet enough to conclude Theorem 1.1. To finish the proof, we need one more relation between $\{\lambda_{1j}\}_{j=1}^3$ and $\{\lambda_{2j}\}_{j=1}^3$.

Since $\overline{H}_1=\overline{H}_2$, we get that

$$\max_{\mathbb{T}^2} V_1=\overline{H}_1(0)=\overline{H}_2(0)=\max_{\mathbb{T}^2} V_2. \tag{3.8}$$

We use this relation to get the final piece of information. Before doing so, we need some preparations.

Definition 3. Given $r_1, r_2, r_3 > 0$ and $\alpha_1, \alpha_2 \in \mathbb{Q}$, denote

$$M(t) = \max_{\theta_1,\theta_2\in\mathbb{R}} \{r_1 \cos\theta_1 + r_2 \cos\theta_2 + r_3 \cos(\alpha_1\theta_1 + \alpha_2\theta_2 + t)\} \quad \text{for } t \in \mathbb{R}.$$

Of course $M(t)$ depends on the parameters $r_1, r_2, r_3, \alpha_1, \alpha_2$, but we do not write down this dependence explicitly unless there is some confusion.

It is easy to see that $\max_{\mathbb{R}} M = r_1 + r_2 + r_3$, and the maximum is attained when

$$t = 2m\pi + 2m_1\alpha_1\pi + 2m_2\alpha_2\pi$$

for $m, m_1, m_2 \in \mathbb{Z}$. Note that the function $x \mapsto \cos x$ does not have non-global local maximum. We now show that this fact is also true for $M(t)$.

Lemma 3.3. *Every local maximum of M is a global maximum.*

Proof. Suppose that t_0 is a local maximum of M. Assume that

$$M(t_0) = r_1 \cos\theta_{1,0} + r_2 \cos\theta_{2,0} + r_3 \cos(\alpha_1\theta_{1,0} + \alpha_2\theta_{2,0} + t_0),$$

for some $\theta_{1,0}, \theta_{2,0} \in \mathbb{R}$. Then we must have that

$$\cos\theta_{1,0} = \cos\theta_{2,0} = \cos(\alpha_1\theta_{1,0} + \alpha_2\theta_{2,0} + t_0) = 1.$$

Otherwise, we can easily perturb $\theta_{1,0}, \theta_{2,0}$ and t_0 a bit to get a greater value of M near t_0. □

Now set

$$l = \min\{|m\pi + m_1\alpha_1\pi + m_2\alpha_2\pi| \, : \, |m\pi + m_1\alpha_1\pi + m_2\alpha_2\pi| > 0, \; m, m_1, m_2 \in \mathbb{Z}\}.$$

Clearly, $l > 0$ and, for all $t \in \mathbb{R}$,

$$M(t) = M(2l + t) = M(-t) = M(2l - t). \tag{3.9}$$

Proposition 3.4. *The function M is strictly decreasing on $[0, l]$, and is strictly increasing on $[l, 2l]$.*

Proof. Thanks to Lemma 3.3 and the choice of l, M has no local maximum in $(0, 2l)$. Besides, (3.9) gives that $M(t) = M(2l - t)$ for all $t \in (0, 2l)$, and thus, M cannot have any local minimum in $(0, l)$. The proof is complete. □

The following is an immediate implication from Proposition 3.4 and (3.9).

Corollary 3.5. *For $t_1, t_2 \in \mathbb{R}$,*

$$M(t_1) = M(t_2)$$

if and only if $t_1 = t_2 + 2kl$ or $t_1 = 2kl - t_2$ for some $k \in \mathbb{Z}$.

We are now ready to prove the main result.

Proof of Theorem 1.1. Thanks to (3.7) and the normalization that $c = 1$, we have $k_{1j} = k_{2j}$ for all $1 \le j \le 3$. We now write $k_j = k_{1j} = k_{2j}$ for simplicity for all $1 \le j \le 3$. Then

$$V_1(x) = a_1 + \sum_{j=1}^{3} \left(\lambda_{1j} e^{i2\pi k_j \cdot x} + \overline{\lambda_{1j}} e^{-i2\pi k_j \cdot x}\right).$$

Since k_1, k_2, k_3 are mutually non-parallel, by translation (i.e., $x \mapsto x + x_0$ for suitable x_0), we may assume that

$$V_2(x) = a_1 + \sum_{j=1}^{2} \left(\lambda_{1j} e^{i2\pi k_j \cdot x} + \overline{\lambda_{1j}} e^{-i2\pi k_j \cdot x}\right) + \tilde{\lambda}_{23} e^{i2\pi k_3 \cdot x} + \overline{\tilde{\lambda}_{23}} e^{-i2\pi k_3 \cdot x}.$$

Denote $\lambda_{1j} = r_j e^{i\omega_j}$ for $1 \le j \le 3$ and $\tilde{\lambda}_{23} = r_3 e^{i\tilde{\omega}_3}$, where $r_j > 0$ and $\omega_j, \tilde{\omega}_3 \in [0, 2\pi)$ for $1 \le j \le 3$. Then

$$V_1(x) = a_1 + r_1 \cos(2\pi k_1 \cdot x + \omega_1) + r_2 \cos(2\pi k_2 \cdot x + \omega_2) + r_3 \cos(2\pi k_3 \cdot x + \omega_3)$$

and

$$V_2(x) = a_1 + r_1 \cos(2\pi k_1 \cdot x + \omega_1) + r_2 \cos(2\pi k_2 \cdot x + \omega_2) + r_3 \cos(2\pi k_3 \cdot x + \tilde{\omega}_3).$$

Again by translations, we may further assume that $\omega_1 = \omega_2 = 0$. We write $k_3 = \alpha_1 k_1 + \alpha_2 k_2$ for some $\alpha_1, \alpha_2 \in \mathbb{Q}$. Then it is clear from the definition of $M(\cdot)$ that

$$\max_{\mathbb{T}^2} V_1 = a_1 + M(\omega_3) \quad \text{and} \quad \max_{\mathbb{T}^2} V_2 = a_1 + M(\tilde{\omega}_3).$$

In light of (3.8), we get $M(\omega_3) = M(\tilde{\omega}_3)$. Assume that

$$l = m\pi + m_1\alpha_1\pi + m_2\alpha_2\pi,$$

for some $m, m_1, m_2 \in \mathbb{Z}$. Accordingly, by Corollary 3.5, we have two cases.

Case 1. $\omega_3 = \tilde{\omega}_3 + 2kl$ for some $k \in \mathbb{Z}$. Choose x_0 such that

$$\begin{cases} k_1 \cdot x_0 = km_1, \\ k_2 \cdot x_0 = km_2. \end{cases}$$

Then $k_3 \cdot x_0 = k\alpha_1 m_1 + k\alpha_2 m_2$ and

$$V_1(x) = V_2(x + x_0) \quad \text{for all } x \in \mathbb{T}^2.$$

Case 2. $\omega_3 = 2kl - \tilde{\omega}_3$ for some $k \in \mathbb{Z}$. Choose x_0 such that

$$\begin{cases} k_1 \cdot x_0 = km_1, \\ k_2 \cdot x_0 = km_2. \end{cases}$$

Then $k_3 \cdot x_0 = k\alpha_1 m_1 + k\alpha_2 m_2$ and

$$V_1(x) = V_2(-x - x_0) \quad \text{for all } x \in \mathbb{T}^2. \quad \square$$

Remark 4. It is natural to try using more the coefficients $\{a_j\}_{j\in\mathbb{N}}$ in the asymptotic expansion of $\overline{H}^{\varepsilon}$ instead of (3.8) to prove the last step above. It is, however, quite hard to implement this idea. Let us still mention it here.

Choose $(m_1, m_2, m_3) \in \mathbb{N}^3$ such that the $\gcd(m_1, m_2, m_3) = 1$ and

$$m_2 k_2 = m_1 k_1 + m_3 k_3.$$

Let $L = m_1 + m_2 + m_3$. It is easy to see that a_L is the first coefficient that provides us information about $\{\lambda_{1j}\}_{j=1}^3, \{\lambda_{2j}\}_{j=1}^3$ further than Lemma 3.1. For $r_j = |\lambda_{1j}| = |\lambda_{2j}|$ for $1 \le j \le 3$, we have

$$a_L = P(r_j,\ k_j,\ Q:\ 1 \le j \le 3) + J(k_1, k_2, k_3, Q)\mathrm{Re}\left(\lambda_{11}^{m_1} \left(\overline{\lambda_{12}}\right)^{m_2} \lambda_{13}^{m_3}\right).$$

Here P is a real valued function depending only on $\{r_j,\ k_j,\ Q:\ 1 \le j \le 3\}$ and J a real valued function depending only on $\{k_1, k_2, k_3, Q\}$. It will be done if we can manage to show that $J(k_1, k_2, k_3, Q)$ is not zero for some $Q \in \mathbb{R}^2$. However, it is not clear to us how to verify this since the expression of J is too complicated.

4. Proofs of Theorems 1.2 and 1.3

We first provide the proof of Theorem 1.2.

Proof of Theorem 1.2. We consider each case separately.

(1) The sufficiency part follows immediately from Lemma 3.1. Let us prove the converse. Since $\{k_{lj}\}_{j=1}^3$ is linearly independent, by suitable translations $(x \mapsto x + x_{0l})$, we may assume that

$$V_1(x) = \sum_{j=1}^{3} r_j \cos(2\pi k_{1j} \cdot x)$$

and for $c_j > 0$,

$$V_2(x) = \sum_{j=1}^{3} r_j \cos(c_j 2\pi k_{1j} \cdot x).$$

Then the conclusion follows from Lemma 4.1 and changing of variables.

(2) Let us first prove the sufficiency part. Clearly, $k_{12} + k_{13}$ and $k_{22} + k_{23}$ are sole vectors. Since $\{k_{lj}\}_{j=1}^3$ is linearly independent, due to Lemma 3.1 and Remark 3, we must have

$$k_{12} + k_{13} \parallel k_{22} + k_{23}.$$

Hence there exists $c \in \mathbb{Q}$ such that $k_{22} = ck_{12}$ and $k_{23} = ck_{13}$.

We now prove the converse. By suitable translations, we may assume that

$$V_1(x) = r_1 \cos(2\pi k_{11} \cdot x) + r_2 \cos(2\pi k_{12} \cdot x) + r_3 \cos(2\pi k_{13} \cdot x)$$

and for $c_1 > 0$,

$$V_2(x) = r_1 \cos(c_1 2\pi k_{11} \cdot x) + r_2 \cos(c 2\pi k_{12} \cdot x) + r_3 \cos(c 2\pi k_{13} \cdot x).$$

We then use Lemma 4.1 and changing of variables to get the conclusion.

(3) The necessity part is obvious. Let us prove the sufficiency. Due to Lemma 3.1, there are two cases.

Case 1. $\{k_{1j}\}_{j=1}^3$ is linearly independent. Due to symmetry, we may assume that k_{11} is not perpendicular to k_{12} and k_{13}. Then similar to (2), we have that

$$k_{11} + k_{12} \parallel k_{21} + k_{22} \quad \text{and} \quad k_{11} + k_{13} \parallel k_{21} + k_{23}.$$

Hence there exists $c \in \mathbb{Q}$ such that for $j = 1, 2, 3$,

$$k_{2j} = ck_{1j}.$$

Since $\{k_{1j}\}_{j=1}^3$ is linearly independent, it is easy to see that we can find $x_0 \in \mathbb{R}^n$ such that

$$V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \qquad \text{for all } x \in \mathbb{T}^n.$$

Case 2. $\{k_{1j}\}_{j=1}^3$ is linearly dependent. The situation is essentially reduced to the 2-dimensional case and the conclusion follows from Theorem 1.1. $\square$

Next, let us prove Theorem 1.3.

Proof of Theorem 1.3. We consider each situation separately.

(1) follows immediately from Lemma 3.1.

(2) The proof of part (i) is similar to (1) of Theorem 1.2, and is omitted. Let us now consider part (ii). Since k_{11} and k_{12} are linearly independent and non-orthogonal, due to Lemma 3.1 and Remark 3, we get that

$$k_{11} + k_{12} \parallel k_{21} + k_{22}.$$

So there exists $c \in \mathbb{Q} \setminus \{0\}$ such that, for $j = 1, 2$.

$$k_{2j} = ck_{1j}.$$

Accordingly, it is easy to see that we can find x_0 such that

$$V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \quad \text{for all } x \in \mathbb{T}^n. \quad \square$$

The following is a simple lemma which should be well known to experts. We leave its proof as an exercise to the interested readers.

Lemma 4.1. *Let $n, n_1, n_2 \in \mathbb{N}$ be such that $n = n_1 + n_2$. For $x \in \mathbb{R}^n$, we write $x = (x_1, x_2, \dots, x_n) = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $x' = (x_1, x_2, ..., x_{n_1})$ and $x'' = (x_{n_1+1}, ..., x_n)$. Similarly, for $p \in \mathbb{R}^n$, we write $p = (p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.*

Let $W_j \in C(\mathbb{T}^{n_j})$ be a given potential energy and $c_j \in \mathbb{R} \setminus \{0\}$ be a given constant for $j = 1, 2$. Assume that $\overline{H}_1(p'), \overline{H}_2(p''), \overline{H}(p)$ are the effective Hamiltonians associated with the Hamiltonians $\frac{1}{2}|p'|^2 + W_1(x')$, $\frac{1}{2}|p''|^2 + W_2(x'')$, $\frac{1}{2}|p|^2 + W_1(c_1x') + W_2(c_2x'')$, respectively. Then

$$\overline{H}(p) = \overline{H}_1(p') + \overline{H}_2(p'') \quad \text{for all } p = (p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

In particular, $\overline{H}$ is independent of c_1 and c_2.

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