STUDIA MATHEMATICA 245 (2) (2019)

Generalization of Zippin's theorem on perturbing Banach spaces with separable dual

by

P. Hájek (Praha), Th. Schlumprecht (College Station, TX, and Praha) and A. Zsák (Cambridge)

Abstract. We generalize a result on Banach spaces with separable dual which was first shown by Zippin, and was explicitly formulated by Benyamini. We prove that there is a class of Asplund spaces, which includes all spaces with separable dual, whose members can be perturbed inside a suitable ambient space to be contained in the space of continuous functions on a well-founded compact tree.

1. Introduction. In 1977, Zippin [10, Theorem 1.2] proved a result, which was later reformulated by Benyamini [2] as follows. For a Banach space X and $\varepsilon > 0$ we denote by $\mathrm{Sz}(X,\varepsilon)$ the ε-Szlenk index of X whose definition will be recalled at the end of Section 3.

Theorem 1.1 ([2, p. 27]). *Let X be a Banach space with separable dual. Let $\varepsilon > 0$, and let F be a w^*-closed totally disconnected, $(1-\varepsilon)$-norming subset of B_{X^*}, the unit ball of X^*. Then there is a countable ordinal $\alpha < \omega^{\mathrm{Sz}(X,\varepsilon/8)+1}$, and a subspace Y of $C(F)$, which is isometrically isomorphic to $C[0,\alpha]$, such that for every $x \in X$ there exists $y \in Y$ with $\|i_F(x) - y\| \le \varepsilon(1-\varepsilon)^{-1}\|i_F(x)\|$, where $i_F\colon X \to C(F)$ denotes the embedding defined by $i_F(x)(f) = f(x)$ for $x \in X$ and $f \in F$.*

Our goal is to prove a generalization of this theorem which includes non-separable Banach spaces; at the same time we provide a more conceptual proof of Zippin's result (see Theorem 5.1 and Corollary 5.2).

The paper is organized as follows. In Section 2 we recall the definition of trees, and introduce the tree topology defined on the set $[T]$ of all branches of a tree T. This topology is generated by a basis consisting of clopen sets, and if T has finitely many roots, $[T]$ with this topology is compact. Frag-

2010 *Mathematics Subject Classification*: Primary 46B20; Secondary 46B26.
Key words and phrases: Zippin, embedding, Asplund, tree.
Received 19 June 2017; revised 28 November 2017.
Published online 20 July 2018.

DOI: 10.4064/sm170619-30-11

mentation indices of topological spaces with respect to a pseudo-metric will be recalled in Section 3. As a particular example we define the Szlenk index of a Banach space at the end of Section 3. Section 4 represents the heart of the proof of our main result (Theorem 4.2), which might be interesting in its own right. It is shown that if T is a tree and d is a pseudo-metric on $[T]$ such that $[T]$ is fragmentable with respect to d, then for every $\varepsilon > 0$ there is a subset $\mathcal{N}$ of the basis of the topology of $[T]$ which forms a well-founded tree with respect to containment so that $\mathcal{N}$ (in this case equal to $[\mathcal{N}]$) with its tree topology is a quotient of $[T]$, and for all $M \in \mathcal{N}$ the d-diameter of $M \setminus \bigcup_{N \subsetneq M,\, N \in \mathcal{N}} N$ is smaller than ε. In Section 5 we present and prove the generalization of Zippin's theorem.

2. Trees and the tree topology. Let T be a *tree*, which means that T is a set with a reflexive partial order denoted by $\preccurlyeq$ (we also write $x \prec y \Leftrightarrow x \preccurlyeq y$ and $x \neq y$), with the property that for each $t \in T$, the *set of predecessors* of t,

$$b_t = \{s \in T : s \preccurlyeq t\},$$

is finite and linearly ordered. An *initial node* of T is a minimal element of T, i.e., an element $t \in T$ for which $b_t = \{t\}$. If $t \in T$ is not an initial node, then $b_t \setminus \{t\}$ is not empty and we call its maximum the *direct predecessor* of t. A *successor* of $u \in T$ is an element $w \in T$ such that $u \prec w$, and a *direct successor* of u if moreover there is no $v \in T$ with $u \prec v \prec w$, in other words if u is the direct predecessor of w. The set of all direct successors of u is denoted by S_u.

A *branch* of T is a non-empty, linearly ordered subset of T which is closed under taking predecessors.

Remark 2.1. If b is a finite branch of T, then there is a unique $t \in T$ such that $b = b_t = \{s \in T : s \preccurlyeq t\}$ (take t to be the maximal element of b). The infinite branches are maximal linearly ordered subsets of T. Indeed, if b is infinite, then we can find $t_j \in b$, $j \in \mathbb{N}$, with $t_1 \prec t_2 \prec t_3 \prec \cdots$ (choose $t_1 \in b$ arbitrarily, then $t_2 \in b \setminus b_{t_1}$, then $t_3 \in b \setminus b_{t_2}$, etc.). Since b is closed under taking predecessors, we have $\bigcup_{j=1}^{\infty} b_{t_j} \subset b$. On the other hand, if for some $t \in T \setminus \bigcup_{j=1}^{\infty} b_{t_j}$ the set $\bigcup_{j=1}^{\infty} b_{t_j} \cup \{t\}$ were linearly ordered, we would have $t \succ t_j$ for all $j \in \mathbb{N}$, which would contradict the assumption that b_t is finite. It follows that $b = \bigcup_{j=1}^{\infty} b_{t_j}$ and b is a maximal linearly ordered subset of T.

We will identify a finite branch b of T with the element $t \in T$ such that $b = b_t$, and hence identify the tree T with the set of all its finite branches. We call the set of all infinite branches the *boundary* of T and denote it by ∂T. If $s \in T$ and $b \in \partial T$, we also write $s \prec b$ if $s \in b$. The set of all branches of

T is denoted by $[T]$. Thus, $[T] = T \cup \partial T$. The tree T is called *well-founded* if $\partial T = \emptyset$.

We define the *ordinal index* $\mathrm{o}(T)$ of a well-founded tree T as follows. For every subset S of T we set $S' = \{s \in S : s \text{ is not maximal in } S\}$. Note that since T is well-founded, if S is not empty then $S' \subsetneq S$. Then we set $T^{(0)} = T$, and by transfinite induction for any ordinal α we define

$$T^{(\alpha)} = \begin{cases} (T^{(\gamma)})' & \text{if } \alpha = \gamma + 1 \text{ for some } \gamma, \\ \bigcap_{\gamma<\alpha} T^{(\gamma)} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Since T is assumed to be well-founded, it follows that

$$\mathrm{o}(T) = \min\{\alpha \in \mathrm{Ord} : T^{(\alpha)} = \emptyset\}$$

exists. Note that since the sets $T^{(\alpha)} \setminus T^{(\alpha+1)}$, $\alpha < \mathrm{o}(T)$, are non-empty and pairwise disjoint, it follows that if T is countable, then so is $\mathrm{o}(T)$.

Let T be an arbitrary tree. We define a locally compact topology on $[T]$ as follows. For $t \in T$ we set

$$U_t = \{b \in [T] : t \in b\}$$

and we let the *tree topology* be generated by the set

$$\big\{U_t, [T] \setminus U_t : t \in T\big\}.$$

We call $[T]$ with its tree topology the *tree space* of T.

Note that for $s, t \in T$, either $U_s \subset U_t$, or $U_t \subset U_s$, or $U_t \cap U_s = \emptyset$. Indeed, using the properties of trees and the definition of branches, it follows that if $s \preccurlyeq t$, then $U_t \subset U_s$, if $t \preccurlyeq s$, then $U_s \subset U_t$, and if s and t are incomparable then $U_t \cap U_s = \emptyset$. Consequently,

$$\mathcal{B} = \Big\{U_t \setminus \bigcup_{j=1}^{n} U_{s_j} : t \in T,\ n \in \mathbb{N}_0,\ s_1, \ldots, s_n \in S_t\Big\} \cup \{\emptyset\}$$

is stable under taking finite intersections, and thus is a basis of the tree topology consisting of clopen sets. We note that for $b \in \partial T$,

$$\mathcal{U}_b = \{U_t : t \in b\}$$

is a neighbourhood basis of b, and for a finite branch $b = b_t$,

$$\mathcal{U}_b = \Big\{U_t \setminus \bigcup_{s \in F} U_s : F \subset S_t \text{ finite}\Big\}$$

is a neighbourhood basis of b. In particular, if t has only finitely many direct successors, then the singleton $\{b_t\}$ is clopen.

Remark 2.2. Branches of T are subsets of T. Thus we can think of the tree space $[T]$ as a subset of $\{0,1\}^T$. The tree topology of $[T]$ is simply the restriction to $[T]$ of the product topology of $\{0,1\}^T$. It is easy to see that $[T] \cup \{\emptyset\}$ is closed in $\{0,1\}^T$. The next result follows.

Proposition 2.3. *$[T]$ is a locally compact Hausdorff space, and $[T]$ is compact if and only if T has finitely many initial nodes.*

We shall call a tree T *compact* if the corresponding tree space $[T]$ is compact, i.e., when T has finitely many initial nodes.

We next recall the definition of the Cantor–Bendixson index of a compact topological space K. For a closed set $F \subset K$ we set

$$d(F) = \{\xi \in F : \xi \text{ is not an isolated point in } F\}.$$

We let $d_0(K) = K$. By transfinite induction we define $d_\alpha(K)$ for ordinals α as follows: $d_\alpha(K) = d(d_\gamma(K))$ if $\alpha = \gamma+1$ for some γ, and $d_\alpha(K) = \bigcap_{\gamma<\alpha} d_\gamma(K)$ if α is a limit ordinal. It follows that there must be an ordinal α_0 for which $d_{\alpha_0}(K) = d_{\alpha_0+1}(K) = d_{\alpha_0+2}(K) = \cdots$, and if in that case $d_{\alpha_0}(K) = \emptyset$, we define the *Cantor–Bendixson index of K* to be

$$\mathrm{CB}(K) = \min\{\alpha \in \mathrm{Ord} : d_\alpha(K) = \emptyset\}\ ,$$

otherwise we set $\mathrm{CB}(K) = \infty$.

Now we assume that T is a well-founded tree with finitely many initial nodes, and we want to compare $\mathrm{o}(T)$ with $\mathrm{CB}(T)$. Since every maximal element in any subset S of T is isolated in S, we have

$$\mathrm{CB}(T) \le \mathrm{o}(T). \tag{2.1}$$

It follows from general topology that if K is a non-empty, countable, compact Hausdorff space, then $\mathrm{CB}(K) = \beta + 1$ for a countable ordinal β, and $|d_\beta(K)| = n$ for some $0 < n < \omega$, and moreover K is homeomorphic to the ordinal interval $[0, \omega^\beta \cdot n]$. Another description of countable compact spaces is provided by well-founded, countable, compact trees with the tree topology. More generally, the following holds.

Theorem 2.4. *Given a countable, well-founded tree $(T, \preccurlyeq)$, there is an ordinal $\beta \le \omega^{\mathrm{o}(T)}$ such that T with its tree topology is homeomorphic to the ordinal interval $[0, \beta)$. Conversely, given a countable ordinal β, the interval $[0, \beta)$ is homeomorphic to a countable, well-founded tree.*

Proof. For the first part, we may assume by adjoining a root if necessary that T has one initial node. For $t \in T$ let $\alpha(t)$ be the ordinal $\alpha < \mathrm{o}(T)$ such that $t \in [T]^{(\alpha)} \setminus [T]^{(\alpha+1)}$. A routine induction on $\alpha(t)$ shows that for each $t \in T$ there is an ordinal $\beta \le \omega^{\alpha(t)}$ and a homeomorphism $\varphi\colon U_t \to [0, \beta]$ with $\varphi(t) = \beta$. This completes the proof of the first implication. The converse also follows by induction. As we shall not need the converse, we leave the details to the reader. ∎

Remarks 2.5. Since for $\beta > 0$ the interval $[0, \beta)$ is compact if and only if β is a successor ordinal, it follows from Theorem 2.4 that a non-empty, countable, well-founded, compact tree T is homeomorphic to $[0,\beta]$ for some ordinal $\beta < \omega^{o(T)}$.

We next give an example of an uncountable compact space that can also be realized as a tree space. This example will be important later.

EXAMPLE 2.6. Let D be the *Cantor set*, i.e., the set $D = \{0,1\}^{\mathbb{N}}$ endowed with the product topology of the discrete topology on $\{0,1\}$. Denote by $[\mathbb{N}]$, $[\mathbb{N}]^{<\omega}$ and $[\mathbb{N}]^{\omega}$ the subsets of $\mathbb{N}$, the finite subsets of $\mathbb{N}$, and the infinite subsets of $\mathbb{N}$, respectively. Identifying a subset of $\mathbb{N}$ with its indicator function defines a one-to-one correspondence between $[\mathbb{N}]$ and D. Via this identification, $[\mathbb{N}]$ becomes a compact, metrizable space. The topology of $[\mathbb{N}]$ is generated by clopen sets, which we now describe.

For $A = \{a_1,\dots,a_m\} \in [\mathbb{N}]^{<\omega}$ and $B = \{b_1,b_2,\dots\} \in [\mathbb{N}]$, both sets written in increasing order, we say that B *is an extension of* A, or that A *is an initial segment of* B, and write $A \prec B$, if $|B| > m$ and $a_i = b_i$ for $i = 1,\dots,m$. The topology of $[\mathbb{N}]$ is then generated by the clopen sets $N_A = \{B \in [\mathbb{N}] : A \preccurlyeq B\}$, $A \in [\mathbb{N}]^{<\omega}$.

We will now show that the topology on $[\mathbb{N}]$, and hence the product topology on D, is identical with the tree topology on the branches of a tree. Indeed, let $T = [\mathbb{N}]^{<\omega}$. Extension defines a partial order on T and turns T into a tree whose only initial node is $\emptyset$. It is easy to see that $[T] = [\mathbb{N}]$ and $\partial T = [\mathbb{N}]^{\omega}$. Here we identify any $A = \{a_1,a_2,\dots\} \in [\mathbb{N}]^{\omega}$ (written in increasing order) with the branch $b = \{\emptyset\} \cup \{\{a_1,\dots,a_n\} : n = 1,2,\dots\}$. Under the identification of $[T]$ with $[\mathbb{N}]$, given $A \in T$, the clopen set N_A in $[\mathbb{N}]$ becomes the clopen set U_A of $[T]$ as defined for general trees. Thus the topologies of $[\mathbb{N}]$ and $[T]$ have the same generating sets, and they coincide.

We conclude this section with a well-known result in topology. For the convenience of the reader we include the proof.

LEMMA 2.7. *Let K be a compact Hausdorff space, $\varepsilon > 0$, and $f_1 : K \to \mathbb{R}$ a function such that every point of K has a neighbourhood on which the oscillation of f_1 is at most ε. Then there is a continuous function $f : K \to \mathbb{R}$ such that $|f(x) - f_1(x)| \le \varepsilon$ for all $x \in K$.*

Proof. By the assumption, the family of open subsets of K on which the oscillation of f_1 is at most ε is an open cover for K, and hence contains a finite subcover $U_1,\dots,U_n$. Let $\varphi_1,\dots,\varphi_n$ be a partition of unity subordinate to the cover $U_1,\dots,U_n$. Thus, each $\varphi_j : K \to [0,1]$ is a continuous function with support contained in U_j and such that $\sum_{j=1}^{n} \varphi_j(x) = 1$ for all $x \in K$. For each $j = 1,\dots,n$ fix $x_j \in U_j$, and define $f : K \to \mathbb{R}$ by setting $f(x) =$

$\sum_{j=1}^{n} f_1(x_j)\varphi_j(x)$ for $x \in K$. Then f is continuous and

$$|f(x) - f_1(x)| \leq \sum_{j=1}^{n} \varphi_j(x)|f_1(x_j) - f_1(x)| \leq \varepsilon \quad \text{for all } x \in K,$$

as required. ∎

3. Fragmentation indices. In this section we recall some well known notation and results on fragmentation indices. All of the results below, and much more, may be found in books on topology and descriptive set theory (for example [4]). Nevertheless, for better reading, we recall the results we will need. We also do this because we consider fragmentations of topological spaces with respect to pseudo-metrics, and not only metrics.

DEFINITION 3.1. Let $(X, \mathcal{T})$ be a topological space and $d(\cdot,\cdot)$ a pseudo-metric on X. We say that $(X, \mathcal{T})$ is *d-fragmentable* if for all non-empty closed subsets F of X and all $\varepsilon > 0$ there is an open set $U \subset X$ such that $U \cap F \neq \emptyset$ and d-$\mathrm{diam}(U \cap F) < \varepsilon$.

DEFINITION 3.2. Let $(X, \mathcal{T})$ be a topological space and $d(\cdot,\cdot)$ a pseudo-metric on X. For a closed set $F \subset X$ and $\varepsilon > 0$ we define the *ε-derivative* of F by

$$\begin{aligned} F'_\varepsilon &= F \setminus \bigcup \{U \in \mathcal{T} : d\text{-}\mathrm{diam}(U \cap F) < \varepsilon\} \\ &= \{\xi \in F : d\text{-}\mathrm{diam}(U \cap F) \geq \varepsilon \text{ for all } U \in \mathcal{U}_\xi\}, \end{aligned}$$

where $\mathcal{U}_\xi$ is the set of all $\mathcal{T}$-open neighbourhoods of ξ. For every ordinal α we define the *ε-derivative of F of order α*, denoted $F_\varepsilon^{(\alpha)}$, by transfinite induction: we first set $F_\varepsilon^{(0)} = F$, and then

$$F_\varepsilon^{(\alpha)} = \begin{cases} (F_\varepsilon^{(\gamma)})'_\varepsilon & \text{if } \alpha = \gamma + 1 \text{ for some } \gamma, \\ \bigcap_{\gamma<\alpha} F_\varepsilon^{(\gamma)} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Let $(X, \mathcal{T})$ be a topological space, $d(\cdot,\cdot)$ a pseudo-metric on X, F be a $\mathcal{T}$-closed subset of X, and $\varepsilon > 0$. First we note that if $F_\varepsilon^{(\alpha)} = F_\varepsilon^{(\alpha+1)}$, then $F_\varepsilon^{(\alpha)} = F_\varepsilon^{(\beta)}$ for all $\beta > \alpha$. It follows that if $F_\varepsilon^{(\alpha)} \neq F_\varepsilon^{(\alpha+1)}$, then $\beta \mapsto F_\varepsilon^{(\beta)} \setminus F_\varepsilon^{(\beta+1)}$ defines an injection on α into the power set of F, which is not possible for α sufficiently large. Therefore there must be a minimal ordinal α_0 for which

$$F_\varepsilon^{(\alpha_0)} = F_\varepsilon^{(\alpha_0+1)} = F_\varepsilon^{(\alpha_0+2)} = \cdots .$$

We set $F_\varepsilon^{(\infty)} = F_\varepsilon^{(\alpha_0)}$. If $(X, \mathcal{T})$ is d-fragmentable, then $F_\varepsilon^{(\infty)} = \emptyset$.

We define the *ε-fragmentation index of F with respect to d* by

$$\mathrm{Frag}(d,F,\varepsilon)=\mathrm{Frag}(F,\varepsilon)=\begin{cases}\min\{\beta\in\mathrm{Ord}: F_\varepsilon^{(\beta)}=\emptyset\} & \text{if } F_\varepsilon^{(\infty)}=\emptyset,\\ \infty & \text{if not.}\end{cases}$$

Here we consider "∞" to be outside of the class of ordinals. Secondly, we define the *fragmentation index of F with respect to d* by

$$\mathrm{Frag}(d,F)=\mathrm{Frag}(F)=\sup_{\varepsilon>0}\mathrm{Frag}(F,\varepsilon)$$

with $\mathrm{Frag}(F)=\infty$ if for some $\varepsilon>0$ we have $\mathrm{Frag}(F,\varepsilon)=\infty$.

Remark 3.3. Assume that $(X,\mathcal{T})$ is d-fragmentable and $F\subset X$ is compact. Let $\varepsilon>0$. If α is a limit ordinal for which $F_\varepsilon^{(\gamma)}\neq\emptyset$ whenever $\gamma<\alpha$, then also $F_\varepsilon^{(\alpha)}\neq\emptyset$. Therefore $\mathrm{Frag}(d,F,\varepsilon)$ will always be a successor ordinal.

If $(X,\mathcal{T})$ is second countable and d-fragmentable, F is a closed subset of X and $\varepsilon>0$, then the fact that $F_\varepsilon^{(\beta)}\subsetneq F_\varepsilon^{(\alpha)}$ for $\alpha<\beta\le\mathrm{Frag}(F,\varepsilon)$ and [6, Theorem 6.9] imply that $\mathrm{Frag}(F,\varepsilon)<\omega_1$, where ω_1 denotes the first uncountable ordinal, and thus also $\mathrm{Frag}(F)<\omega_1$.

As an important example we consider the Szlenk index of a Banach space, which we will introduce for general Bananch spaces, not only for separable ones. We call a Banach space X an *Asplund space* if every separable subspace of X has separable dual. This is not Asplund's original definition [1], but known to be equivalent to it. The following equivalence is stated in [5] and gathers the results from [1, 7, 8]. For a separable space, this is a consequence of the Baire Category Theorem.

Theorem 3.4 ([5, Theorem 11.8, p. 486]). *Let $(X,\|\cdot\|)$ be a Banach space. Then the following assertions are equivalent:*

(i) *X is an Asplund space.*
(ii) *B_{X^*} with the w^*-topology is $\|\cdot\|$-fragmentable.*

Here $\|\cdot\|=\|\cdot\|_{X^*}$ denotes the dual norm on X^*, and $\|\cdot\|$-fragmentable means d-fragmentable, where d is the induced metric defined by $d(x^*,y^*)=\|x^*-y^*\|$ for $x^*,y^*\in X^*$.

Assume that X is an arbitrary Banach space. For a w^*-closed subset F of B_{X^*} and $\varepsilon>0$ we denote the ε-fragmentation index of F with respect to $\|\cdot\|_{X^*}$ by $\mathrm{Sz}(F,\varepsilon)$ and call it the *ε-Szlenk index* of F. The *Szlenk index* of F is $\mathrm{Sz}(F)=\sup_{\varepsilon>0}\mathrm{Sz}(F,\varepsilon)$. The *$\varepsilon$-Szlenk index* of X is then defined to be $\mathrm{Sz}(B_{X^*},\varepsilon)$ and denoted by $\mathrm{Sz}(X,\varepsilon)$, and the *Szlenk index* of X is $\mathrm{Sz}(X)=\sup_{\varepsilon>0}\mathrm{Sz}(X,\varepsilon)=\mathrm{Sz}(B_{X^*})$. Note that by Theorem 3.4 above, X is an Asplund space if and only if all these indices are ordinal numbers.

Remark 3.5. Let K be a compact topological space. By identifying an element of K with the corresponding Dirac measure, we can think of K

as a compact subset of $B_{C(K)^*}$ which 1-norms the elements of $C(K)$. It is then easy to see that $\mathrm{CB}(K) = \mathrm{Sz}(K, \varepsilon) = \mathrm{Sz}(K)$ for all $0 < \varepsilon < 2$. Hence $\mathrm{CB}(K) = \mathrm{Sz}(K) \le \mathrm{Sz}(C(K))$. In general it is not true that $\mathrm{Sz}(K) = \mathrm{Sz}(C(K))$. Nevertheless, in [9, Theorem C] for the case of separable dual, and in [3, Theorem 1.1] for the general case, it was shown that if X is a Banach space and $B \subset B_{X^*}$ is compact and 1-norming for X, then

$$\mathrm{Sz}(X) = \min\{\omega^\alpha : \omega^\alpha \ge \mathrm{Sz}(B)\} \tag{3.1}$$

if X is an Asplund space, and $\mathrm{Sz}(X) = \mathrm{Sz}(B) = \infty$ otherwise.

4. Fragmentation of $[T]$**.** Throughout this section we fix a tree T and a pseudo-metric $d(\cdot,\cdot)$ on its tree space $[T]$ which, we recall, is the set of all branches of T equipped with the tree topology. We assume that T is compact, i.e., it has finitely many initial nodes, or equivalently the tree space $[T]$ is compact. We also assume that $[T]$ is d-fragmentable. This situation arises in the following important example which we will use later.

Example 4.1. Consider the space $C([T])$ of continuous functions on $[T]$ for our compact tree T. Let X be a closed subspace of $C([T])$ and assume that X is an Asplund space. For $b_1, b_2 \in [T]$ set

$$d(b_1, b_2) = \sup_{x \in B_X} |x(b_1) - x(b_2)|.$$

Then $d(\cdot,\cdot)$ is a pseudo-metric on $[T]$ and the map sending $b \in [T]$ to the Dirac measure at b restricted to X is an isometry of $([T], d)$ into $(B_{X^*}, \|\cdot\|)$. It follows from Theorem 3.4 above that $[T]$ is d-fragmentable.

We now fix an $\varepsilon > 0$, and let η be the ordinal such that $\mathrm{Frag}(d, [T], \varepsilon) = \eta + 1$. We abbreviate $[T]^{(\alpha)} = [T]^{(\alpha)}_\varepsilon$ for $\alpha \in \mathrm{Ord}$. Let $\mathcal{B}$ be the family of basic open subsets of $[T]$, i.e., sets of the form $N = U_t \setminus \bigcup_{s \in F} U_s$, where $t \in T$ and F is a finite (possibly empty) subset of S_t. Note that t and F are uniquely determined by N. Indeed, t is the least element of N, and then F is the complement in S_t of the set of minimal elements of $N \setminus \{t\}$. We say N is of *type I* if $F = \emptyset$; otherwise it is of *type II*. Note that $\mathcal{B}$ is partially ordered by containment: $M \preccurlyeq N$ if and only if $M \supseteq N$. However, in general, $\mathcal{B}$ is not a tree. The following theorem is the main result of this section.

Theorem 4.2. *Let T, d, ε, η and $\mathcal{B}$ be as above. Then there exists a subset $\mathcal{N}$ of $\mathcal{B}$ which is a well-founded tree under containment such that*

$$d\text{-diam}\Big(M \setminus \bigcup_{N \in S_M} N\Big) < \varepsilon$$

for each $M \in \mathcal{N}$ (where, as before, S_M denotes the set of direct successors of M in the tree $(\mathcal{N}, \supset)$), and the ordinal index $\mathrm{o}(\mathcal{N})$ of $\mathcal{N}$ satisfies $\mathrm{o}(\mathcal{N}) \le$

$\lambda + 2n + 2$, *where* $\eta = \lambda + n$, λ *is a limit ordinal and* $n < \omega$. *Moreover,* $\bigcup \mathcal{N} = [T]$, *and* $\mathcal{N}$ *has finitely many initial nodes.*

Proof. For $b \in [T]$ let $\alpha(b)$ be the ordinal $\alpha \leq \eta$ such that $b \in [T]^{(\alpha)} \setminus [T]^{(\alpha+1)}$. Then b has a neighbourhood whose intersection with $[T]^{(\alpha)}$ has d-diameter less than ε. If there exists a type I neighbourhood of b with that property (which is the case if $b \in \partial T$), then there exists a least $s \in b$ such that $d\text{-diam}(U_s \cap [T]^{(\alpha)}) < \varepsilon$, and in this case we set $N_b = U_s$. Otherwise b is necessarily a finite branch b_t for some $t \in T$ and $d\text{-diam}(U_t \cap [T]^{(\alpha)}) \geq \varepsilon$. In this case there is a minimal (with respect to inclusion), finite, non-empty subset F of S_t such that $d\text{-diam}((U_t \setminus \bigcup_{s \in F} U_s) \cap [T]^{(\alpha)}) < \varepsilon$. Note that F is not necessarily unique; we simply choose one such F and set $N_b = U_t \setminus \bigcup_{s \in F} U_s$. We do this for every $b \in [T]$ and set $\mathcal{N} = \{N_b : b \in [T]\}$. For $N \in \mathcal{N}$ we let $\alpha(N) = \alpha(b)$ where $b \in [T]$ is such that $N = N_b$. Note that this definition does not depend on the choice of b. Indeed,

$$\begin{aligned} \alpha(N) &= \max\{\beta \leq \eta : N \cap [T]^{(\beta)} \neq \emptyset\} \\ &= \min\{\beta : d\text{-diam}(N \cap [T]^{(\beta)}) < \varepsilon\}. \end{aligned} \tag{4.1}$$

We now prove two simple facts. Recall that we identify $t \in T$ with the finite branch b_t. So we will sometimes write N_t instead of N_{b_t}.

LEMMA 4.3. *Let* $M_1, M_2 \in \mathcal{N}$. *Then either* $M_1 \subset M_2$ *or* $M_1 \supset M_2$ *or* $M_1 \cap M_2 = \emptyset$.

Proof. First note that if $N \in \mathcal{N}$ is of type II, and thus of the form $N = U_t \setminus \bigcup_{s \in F} U_s$ for a unique $t \in T$ and finite, non-empty $F \subset S_t$, then for $b \in [T]$ we have $N = N_b$ if and only if $b = b_t$. It follows that if N_1 and N_2 in $\mathcal{N}$ are both of type II and of the form $N_1 = U_t \setminus \bigcup_{s \in F_1} U_s$ and $N_2 = U_t \setminus \bigcup_{s \in F_2} U_s$, then $N_1 = N_{b_t} = N_2$.

For each $i = 1, 2$, choose $t_i \in T$ and finite $F_i \subset S_{t_i}$ such that $M_i = U_{t_i} \setminus \bigcup_{s \in F_i} U_s$ (where the F_i could be empty, and thus M_i be of type I). If t_1 and t_2 are incomparable, then $M_1 \cap M_2 \subset U_{t_1} \cap U_{t_2} = \emptyset$. If $t_1 = t_2$ and one of F_1 and F_2 is empty, then $M_1 \subset M_2$ or $M_1 \supset M_2$. If $t_1 = t_2$ and both F_1 and F_2 are non-empty, then M_1 and M_2 are type II neighbourhoods, and hence, by the remark at the beginning of the proof, $b = b_{t_1} = b_{t_2}$ is the unique branch such that $M_1 = M_2 = N_b$.

Finally, assume that t_1 and t_2 are comparable and distinct. We may without loss of generality assume that $t_1 \prec t_2$. Let s be the unique direct successor of t_1 with $s \preccurlyeq t_2$. Then either $s \in F_1$, and thus $M_1 \cap M_2 = \emptyset$, or $s \notin F_1$, and then $M_1 \supset M_2$. ∎

Before the next lemma, we observe the following consequence of (4.1): If $M, N \in \mathcal{N}$ and $M \supset N$, then $\alpha(M) \geq \alpha(N)$.

Lemma 4.4. *Let $M, N \in \mathcal{N}$. Assume that $M \supsetneq N$ and $\alpha(M) = \alpha(N)$. Then M is of type II and N is of type I.*

Proof. Set $\alpha = \alpha(M) = \alpha(N)$, and choose branches $b, c \in [T]$ such that $M = N_b$ and $N = N_c$.

Assume for a contradiction that M is of type I. Then $M = U_t$ for some $t \in b$, and so $d\text{-diam}(U_t \cap [T]^{(\alpha)}) < \varepsilon$. Since $M \supset N$, we have $t \in c$, and thus by the definition of N_c, we have $N_c = U_s$ with $s = \min\{r \in c : d\text{-diam}(U_r \cap [T]^{\alpha}) < \varepsilon\}$. But this implies that $s = t$ and we must have $M = U_t = N$, which is a contradiction.

Thus b is a finite branch b_t, say, and $M = U_t \setminus \bigcup_{s \in F} U_s$ for some nonempty, finite set $F \subset S_t$. Since $M \supsetneq N$, there must be an $s \in S_t \setminus F$ such that $c \in U_s$. Since $U_s \subset M$, it follows that $d\text{-diam}(U_s \cap [T]^{(\alpha)}) < \varepsilon$. Hence $N = U_s$, and so N is of type I. ∎

We shall make use of the following consequence of Lemma 4.4: Given $M, N, P \in \mathcal{N}$, if $M \supsetneq N \supsetneq P$, then $\alpha(M) > \alpha(P)$.

Continuation of the proof of Theorem 4.2. It follows from Lemma 4.3 that for $M \in \mathcal{N}$ the set $b_M = \{N \in \mathcal{N} : N \supset M\}$ is linearly ordered. Write M as $M = U_t \setminus \bigcup_{s \in F} U_s$ with $t \in T$ and $F \subset S_t$ finite. To see that b_M is finite, observe that if $N_1 \supsetneq N_2$ in $\mathcal{N}$ with $N_i = U_{t_i} \setminus \bigcup_{s \in F_i} U_s$ for some $t_i \in T$ and finite $F_i \subset S_{t_i}$ $(i = 1, 2)$, then either $t_1 \prec t_2$, or $t_1 = t_2$ and $F_1 = \emptyset \neq F_2$. This shows that the cardinality of b_M is at most twice the cardinality of the set of predecessors of t, and thus $\mathcal{N}$ is a tree.

We next verify that $\mathcal{N}$ is well-founded. Assume that there is an infinite sequence $N_1 \supsetneq N_2 \supsetneq \cdots$ in $\mathcal{N}$. By (4.1), we have $\alpha(N_1) \geq \alpha(N_2) \geq \cdots$, and hence this sequence of ordinals is eventually constant. Lemma 4.4 shows that this is not possible.

We will now prove the stated upper bound on $\mathrm{o}(\mathcal{N})$. Fix a limit ordinal α and assume that

$$\mathcal{N}^{(\alpha)} \subset \{N \in \mathcal{N} : \alpha(N) \geq \alpha\}. \tag{4.2}$$

We show by induction that $\mathcal{N}^{(\alpha+2m)} \subset \{N \in \mathcal{N} : \alpha(N) \geq \alpha + m\}$ for all $m < \omega$. The base case $m = 0$ is our assumption. Now let $M \in \mathcal{N}^{(\alpha+2m+2)}$. Then $M \supsetneq N \supsetneq P$ for some $N \in \mathcal{N}^{(\alpha+2m+1)}$ and $P \in \mathcal{N}^{(\alpha+2m)}$. By the induction hypothesis we have $\alpha(P) \geq \alpha + m$, and hence, by Lemma 4.4, we have $\alpha(M) \geq \alpha + m + 1$. It remains to show that (4.2) in fact holds for all limit ordinals α. This can be done by an easy induction argument. When $\alpha = \beta + \omega$ for a limit ordinal β, we use the previous fact about finite ordinals to obtain

$$\mathcal{N}^{(\alpha)} = \bigcap_{m<\omega} \mathcal{N}^{(\beta+2m)} \subset \bigcap_{m<\omega} \{N \in \mathcal{N} : \alpha(N) \geq \beta + m\}$$
$$= \{N \in \mathcal{N} : \alpha(N) \geq \alpha\}.$$

If $\alpha = \sup I$, where I is the set of limit ordinals strictly smaller than α, then

$$\mathcal{N}^{(\alpha)} = \bigcap_{\gamma\in I} \mathcal{N}^{(\gamma)} \subset \bigcap_{\gamma\in I} \{N \in \mathcal{N} : \alpha(N) \geq \gamma\} = \{N \in \mathcal{N} : \alpha(N) \geq \alpha\}.$$

We next establish the statement concerning d-diameters. Fix $M \in \mathcal{N}$ and let $\alpha = \alpha(M)$. If $b \in M \setminus [T]^{(\alpha)}$, then $\alpha(b) < \alpha$. Thus $\alpha(N_b) < \alpha(M)$, and since $N_b \cap M$ contains b, we must have $N_b \subsetneq M$ using Lemma 4.3 and (4.1). It follows that $b \in N$ for some $N \in S_M$. We have prove that $M \setminus \bigcup_{N\in S_M} N \subset M \cap [T]^{(\alpha)}$, which shows that $d\text{-diam}(M \setminus \bigcup_{N\in S_M} N) < \varepsilon$.

For the "moreover" part observe that $b \in N_b$ for all $b \in [T]$, and thus $[T] = \bigcup \mathcal{N}$. Since $[T]$ is assumed to be compact, there is a finite cover of $[T]$ by some elements $N_1, \ldots, N_k$ of $\mathcal{N}$. By Lemma 4.3, there can be at most k initial nodes of $\mathcal{N}$. ∎

Lemma 4.5. *Let $\mathcal{N}$ be defined as in the proof of Theorem 4.2. For $M \in \mathcal{N}$ set $\widetilde{M} = M \setminus \bigcup_{N\in S_M} N$. Then the sets $\widetilde{M}$ for all $M \in \mathcal{N}$, are pairwise disjoint, and for each $b \in [T]$ there is an $M \in \mathcal{N}$ such that $b \in \widetilde{M}$. Thus $\{\widetilde{M} : M \in \mathcal{N}\}$ is a partition of $[T]$.*

Proof. Let $M, N \in \mathcal{N}$ with $\widetilde{M} \neq \widetilde{N}$. If $M \cap N = \emptyset$, then it is clear that $\widetilde{M} \cap \widetilde{N} = \emptyset$. Thus, by Lemma 4.3, we may assume that $N \subsetneq M$. Since $\mathcal{N}$ is a tree, this means that there is an $M' \in S_M$ with $N \subset M'$, which yields our first claim.

Since $[T] = \bigcup \mathcal{N}$, and since $\mathcal{N}$ is a well-founded tree, there is a smallest (with respect to inclusion) $M \in \mathcal{N}$ such that $b \in M$. This means that $b \notin N$ for any $N \in S_M$, which implies our second claim. ∎

By Lemma 4.5 we can define a map $q \colon [T] \to \mathcal{N}$ by letting $q(b)$ be the unique $M \in \mathcal{N}$ such that $b \in \widetilde{M}$.

Proposition 4.6. *The map $q \colon [T] \to \mathcal{N}$ defined above is onto. The quotient topology on $\mathcal{N}$ induced by q coincides with the tree topology of $\mathcal{N}$.*

Proof. Let $M \in \mathcal{N}$. We need to show that $\widetilde{M} \neq \emptyset$. Choose $b \in [T]$ with $M = N_b$, and set $\alpha = \alpha(b) = \alpha(M)$. Let $N \in S_M$. Then $\alpha(N) \leq \alpha$ by (4.1). If $\alpha(N) < \alpha$, then N is disjoint from $[T]^{(\alpha)}$, and hence $b \notin N$. If $\alpha(N) = \alpha$, then M is of type II and N is of type I by Lemma 4.4. It follows that $b = b_t$ for some $t \in T$, and $N \subset U_s$ for some $s \in S_t$. But this means that $t \notin U_s$, and thus again $b = b_t \notin N$. This shows that $b \in \widetilde{M}$, and so $M = q(b)$ is in the image of q.

We next observe that q is continuous when $\mathcal{N}$ is given the tree topology. Indeed, fix $M \in \mathcal{N}$ and $b \in [T]$, and set $N = q(b)$. Then $b \in M$ if and only if $M \supset N$. Thus the inverse image under q of the basic clopen set U_M (in the tree topology of $(\mathcal{N}, \supset)$, i.e., $U_M = \{N \in \mathcal{N} : N \subset M\}$) in $\mathcal{N}$ is the clopen subset M of $[T]$. Hence the quotient topology of $\mathcal{N}$ is finer than the tree topology. Since the quotient topology is compact and the tree topology is Hausdorff, the two topologies coincide, as claimed. ∎

5. Zippin's theorem. We now present our main result.

THEOREM 5.1. *Let X be an Asplund space, let $(T, \preccurlyeq)$ be a tree with finitely many initial nodes, and let $i\colon X \to C([T])$ be an isometric embedding. Then for all $\varepsilon > 0$ there exist a well-founded, compact tree S with $\mathrm{o}(S) < \mathrm{Sz}(X, \varepsilon/2) + \omega$ and an isometric copy Y of $C(S)$ in $C([T])$ such that for all $x \in X$ there exists $y \in Y$ with $\|i(x) - y\| \le \varepsilon\|x\|$.*

Proof. Consider the pseudo-metric d on $[T]$ defined as follows:

$$d(b,c) = \sup_{x \in B_X} |i(x)(b) - i(x)(c)|, \qquad b, c \in [T].$$

We identify $b \in [T]$ with its Dirac measure δ_b. Note that the dual map i^* sends $[T]$ onto a w^*-closed, 1-norming subset of B_{X^*}, and

$$\|i^*(\delta_b) - i^*(\delta_c)\|_{X^*} = \sup_{x \in B_X} |i(x)(b) - i(x)(c)| = d(b,c) \qquad \text{for all } b, c \in [T].$$

Fix $\varepsilon > 0$. It follows from above that the $\varepsilon/2$-fragmentation index of $[T]$ with respect to d is equal to $\mathrm{Sz}(i^*([T]), \varepsilon/2) \le \mathrm{Sz}(X, \varepsilon/2)$. (See Example 4.1.)

Theorem 4.2, applied to $\varepsilon/2$, provides us with a well-founded, compact tree $\mathcal{N}$ of basic clopen subsets of $[T]$ with $\mathrm{o}(\mathcal{N}) < \mathrm{Frag}(d, [T], \varepsilon/2) + \omega = \mathrm{Sz}(i^*([T]), \varepsilon/2) + \omega \le \mathrm{Sz}(X, \varepsilon/2) + \omega$ such that

$$d\text{-diam}\Big(M \setminus \bigcup_{N \in S_M} N\Big) < \varepsilon/2$$

for all $M \in \mathcal{N}$. By Proposition 4.6, we also have a quotient map $q\colon [T] \to \mathcal{N}$ which is continuous with respect to the tree topologies of $[T]$ and $\mathcal{N}$. Thus, we have an isometric embedding $q^*\colon C(\mathcal{N}) \to C([T])$ given by $f \mapsto f \circ q$. Let Y be the image of q^*. We will now show that Y is ε-close to $i(X)$, which will prove the theorem with $S = \mathcal{N}$.

Let $x \in B_X$ and $g = i(x)$. Then g is a continuous function on $[T]$ whose oscillation on $\widetilde{M} = M \setminus \bigcup_{N \in S_M} N$ is less than $\varepsilon/2$ for all $M \in \mathcal{N}$. Indeed, for all $M \in \mathcal{N}$ and all $b, c \in \widetilde{M}$, we have

$$|g(b) - g(c)| = |i(x)(b) - i(x)(c)| \le d(b,c) < \varepsilon/2.$$

For each $M \in \mathcal{N}$ fix $x_M \in \widetilde{M}$ and set $f_1(M) = g(x_M)$. This defines a function

$f_1 : \mathcal{N} \to \mathbb{R}$. We will now show that f_1 is not far from being continuous, and hence it is not far from a continuous function.

Fix $M \in \mathcal{N}$. Since the oscillation of g on $\widetilde{M}$ is smaller than $\varepsilon/2$, we have

$$\{(b,c) \in M \times M : |g(b) - g(c)| \geq \varepsilon/2\} \subset \bigcup_{N \in S_M} (M \times N \cup N \times M),$$

and thus by compactness of the left-hand set, there is a finite set $F_M \subset S_M$ such that

$$\{(b,c) \in M \times M : |g(b) - g(c)| \geq \varepsilon/2\} \subset \bigcup_{N \in F_M} (M \times N \cup N \times M),$$

and hence

$$|g(b) - g(c)| < \varepsilon/2 \quad \text{for all } b, c \in M \setminus \bigcup_{N \in F_M} N. \tag{5.1}$$

Since for $P \in \mathcal{N}$, if $P \in U_M \setminus \bigcup_{N \in F_M} U_N$, then $x_P \in \widetilde{P} \subset M \setminus \bigcup_{N \in F_M} N$, it follows from (5.1) above that for all $P, Q \in U_M \setminus \bigcup_{N \in F_M} U_N$, we have

$$|f_1(P) - f_1(Q)| = |g(x_P) - g(x_Q)| < \varepsilon/2.$$

We have shown that in the compact space $\mathcal{N}$ every point has a neighbourhood on which the oscillation of f_1 is at most $\varepsilon/2$. An application of Lemma 2.7 now yields a continuous function $f : \mathcal{N} \to \mathbb{R}$ which is $\varepsilon/2$-close to f_1. We complete the proof by showing that $y = q^*(f)$ is ε-close to $g = i(x)$. Indeed, given $b \in [T]$, for $M = q(b)$ we have $b, x_M \in \widetilde{M}$, and hence

$$|y(b) - g(b)| = |f(M) - g(b)| \leq |f(M) - f_1(M)| + |g(x_M) - g(b)| \leq \varepsilon,$$

as required. ∎

From Theorem 5.1 we can deduce Zippin's theorem. Recall that D denotes the Cantor set.

Corollary 5.2. *Let X be a Banach space with separable dual and $i : X \to C(D)$ an isometric embedding (which always exists). Then for all $\varepsilon > 0$ there is a countable ordinal $\alpha < \omega^{\mathrm{Sz}(X,\varepsilon/2)+\omega}$ and a subspace Y of $C(D)$ isometric to $C[0,\alpha]$ such that for all $x \in X$ there exists $y \in Y$ with $\|i(x) - y\| \leq \varepsilon\|x\|$.*

Proof. Since X is separable, we can think of it as a subspace of $C(D)$. As explained in Example 2.6, D can be seen as the set of all branches of a tree T endowed with the tree topology. Applying now Theorem 5.1, we obtain a well-founded, compact tree S with ordinal index $\mathrm{o}(S) < \mathrm{Sz}(X, \varepsilon/2) + \omega$ and an isometric copy Y of $C(S)$ in $C(D)$ such that for all $x \in X$ there exists $y \in Y$ with $\|i(x) - y\| \leq \varepsilon\|x\|$. It follows from the proof of Theorem 5.1 that S is a subset of the basis $\mathcal{B}$ of D consisting of clopen sets, so in particular S is countable. By Theorem 2.4 and Remark 2.5, there is a countable ordinal

α such that S is homeomorphic to $[0, \alpha]$, and moreover

$$\alpha < \omega^{\mathrm{o}(S)} < \omega^{\mathrm{Sz}(X,\varepsilon/2)+\omega},$$

as claimed. ∎

REMARK 5.3. Compared to Theorem 1.1, here we have a fixed ambient space $C(D)$ that works for all spaces with separable dual. The bound on α is essentially the same in both statements. Choosing ordinals η, λ, n such that $\mathrm{Sz}(X, \varepsilon/2) = \eta + 1$, $\eta = \lambda + n$, λ is a limit and $n < \omega$, a careful inspection of the application of Theorem 4.2 in the proof of Theorem 5.1 actually yields the bound $\alpha < \omega^{\mathrm{Sz}(X,\varepsilon/2)+n+1}$. Of course, $\mathrm{Sz}(X, \varepsilon/8) \geq \mathrm{Sz}(X, \varepsilon/2)$.

We note the following corollary of Theorem 5.1.

COROLLARY 5.4. *Let T be a tree with finitely many initial nodes. If T is well-founded, then $C(T)$ is an Asplund space. Conversely, if $C([T])$ is an Asplund space, then there is a well-founded, compact tree S homeomorphic to $[T]$.*

Proof. Assume that T is well-founded. It follows that the ordinal index $\mathrm{o}(T)$ of T exists. As explained in Remark 3.5, we can identify each $t \in T$ with the Dirac measure δ_t, and hence view T as a w^*-closed, 1-norming subset of $B_{C(T)^*}$. Since $\|\delta_s - \delta_t\| = 2$ for $s \neq t$ in T, it follows from (2.1) for $0 < \varepsilon < 2$ that $\mathrm{Sz}(T) = \mathrm{Sz}(T, \varepsilon) = \mathrm{CB}(T) \leq \mathrm{o}(T)$. Thus, in particular, $\mathrm{Sz}(T) \neq \infty$, and hence it follows from (3.1) that $C(T)$ is Asplund.

Now assume that $C([T])$ is Asplund. Then we apply Theorem 5.1 to $X = C([T]) \subset C([T])$ and $\varepsilon = 1/2$ and find a well-founded, compact tree S and a closed subspace Y of $C([T])$ as in the statement of the theorem. Since now every element of $C([T])$ has to be close to an element of Y, it follows that $Y = C([T])$. Since Y is isometric to $C(S)$, the Banach–Stone theorem implies that $[T]$ is homeomorphic to S. ∎

Acknowledgements. The first named author was supported by GAČR 16-07378S and RVO: 67985840. The second named author was supported by the National Science Foundation under the Grant Number DMS-1464713.

References

[1] E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. 121 (1968), 31–47.

[2] Y. Benyamini, *An extension theorem for separable Banach spaces*, Israel J. Math. 29 (1978), 24–30.

[3] R. M. Causey, *The Szlenk index of injective tensor products and convex hulls*, J. Funct. Anal. 272 (2017), 3375–3409.

[4] P. Dodos, *Banach Spaces and Descriptive Set Theory: Selected Topics*, Lecture Notes in Math. 1993, Springer, Berlin, 2010.

[5] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2011.

[6] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, New York, 1995.

[7] I. Namioka and R. R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. 42 (1975), 735–750.

[8] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Math. 1364, Springer, Berlin, 1993.

[9] Th. Schlumprecht, *On Zippin's embedding theorem of Banach spaces into Banach spaces with bases*, Adv. Math. 274 (2015), 833–880.

[10] M. Zippin, *The separable extension problem*, Israel J. Math. 26 (1977), 372–387.

P. Hájek
Institute of Mathematics
Czech Academy of Sciences
Žitná 25
115 67 Praha 1, Czech Republic
and
Faculty of Electrical Engineering
Czech Technical University in Prague
Zikova 4
166 27 Praha, Czech Republic
E-mail: hajek@math.cas.cz

Th. Schlumprecht
Department of Mathematics
Texas A&M University
College Station, TX 77843, U.S.A.
and
Faculty of Electrical Engineering
Czech Technical University in Prague
Zikova 4
166 27 Praha, Czech Republic
E-mail: schlump@math.tamu.edu

A. Zsák
Peterhouse
Cambridge, CB2 1RD, UK
E-mail: a.zsak@dpmms.cam.ac.uk