The Quarterly Journal of Mathematics Advance Access published on August 10, 2017
Quart. J. Math. 69 (2018), 211–222; doi:10.1093/qmath/hax035

ON COARSE EMBEDDINGS INTO $C_0(\Gamma)$

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[Received 14 March 2017. Revised 20 June 2017]

Abstract

Let λ be a large enough cardinal number (assuming the Generalized Continuum Hypothesis it suffices to let $\lambda = \aleph_\omega$). If X is a Banach space with $\mathrm{dens}(X) \geq \lambda$, which admits a coarse (or uniform) embedding into any $c_0(\Gamma)$, then X fails to have non-trivial cotype, i.e. X contains ℓ_∞^n C-uniformly for every $C > 1$. In the special case when X has a symmetric basis, we may even conclude that it is linearly isomorphic with $c_0(\mathrm{dens}\, X)$.

1. Introduction

The classical result of Aharoni states that every separable metric space (in particular every separable Banach space) can be bi-Lipschitz embedded (the definition is given below) into c_0.

The natural problem of embeddings of metric spaces into $c_0(\Gamma)$, for an arbitrary set Γ, has been treated by several authors, in particular Pelant and Swift. The characterizations that they obtained, and which play a crucial role in our argument, are described below.

Our main interest, motivated by some problems posed in [1], lies in the case of embeddings of Banach spaces into $c_0(\Gamma)$.

We now state the main results of this paper. We first define the following cardinal numbers inductively. We put $\lambda_0 = \omega_0$, and, assuming that $n \in \mathbb{N}_0$, λ_n has been defined, we put $\lambda_{n+1} = 2^{\lambda_n}$. Then we let

$$\lambda = \lim_{n\to\infty} \lambda_n. \tag{1.1}$$

It is clear that assuming the generalized continuum hypothesis (GCH) $\lambda = \aleph_\omega$.

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THEOREM 1.1 *If X is a Banach space with density* $\mathrm{dens}(X) \geq \lambda$, *which admits a coarse (or uniform) embedding into any* $c_0(\Gamma)$, *then X fails to have non-trivial cotype, that is X contains* ℓ_∞^n *C-uniformly for some $C > 1$ (equivalently, every $C > 1$).*

Our method of proof gives a much stronger result for Banach spaces with a symmetric basis. Namely, under the assumptions of Theorem 1.1, such spaces are linearly isomorphic with $c_0(\Gamma)$ (Theorem 4.2).

Theorem 1.1 will follow from the following combinatorial result which is of independent interest. For a set Λ and $n \in \mathbb{N}$, we denote by $[\Lambda]^n$ the set of subsets of Λ whose cardinality is n.

THEOREM 1.2 *Assume that Λ is a set whose cardinality is at least λ, $n \in \mathbb{N}$, and $\sigma\colon [\Lambda]^n \to \mathcal{C}$ is a map into an arbitrary set $\mathcal{C}$. Then (at least) one of the following conditions holds:*

(1) *There is a sequence $(F_j)_{j=1}^\infty$ of pairwise disjoint elements of $[\Lambda]^n$, so that $\sigma(F_i) = \sigma(F_j)$, for all $i, j \in \mathbb{N}$.*
(2) *There is an $F \in [\Lambda]^{n-1}$ so that $\sigma(\{F \cup \{\gamma\} : \gamma \in \Lambda \setminus F\})$ is infinite.*

The above Theorem 1.2 was previously deduced in [4, Lemma 4.3] from a combinatorial result of Baumgartner, provided Λ is a weakly compact cardinal number (whose existence is not provable in ZFC, as it is inaccessible [3, p. 325, 52]). The authors in [4, Question 3] pose a question whether assuming that Γ is uncountable is sufficient in Theorem 1.2.

Theorem 1.2 is used in order to obtain a scattered compact set K of height ω_0, such that $C(K)$ does not uniformly embed into $c_0(\Gamma)$. It is easy to check that our version of Theorem 1.2 implies a ZFC example of such a $C(K)$ space. It is further shown in [4] that the space $C[0, \omega_1]$ does not uniformly embed into any $c_0(\Gamma)$.

Let us point out that a special case of Theorem 1.1 was obtained by Pelant and Rödl [6, Theorem], namely it was shown there that $\ell_p(\lambda)$, $1 \leq p < \infty$, spaces (which are well known to have non-trivial cotype) do not uniformly embed into any $c_0(\Gamma)$.

The paper is organized as follows. In Section 2, we recall Pelant's [4, 5] and Swift's [8] conditions for Lipschitz, uniform and coarse embeddability into $c_0(\Gamma)$. In Section 3, we provide a proof for Theorem 1.2. Finally, in Section 4, we provide a proof of Theorem 1.1 as well as the symmetric version of the result.

All set theoretic concepts and results used in our note can be found in [3], whereas for facts concerning non-separable Banach spaces, [2] can be consulted.

We want to finish this introductory section by thanking the anonymous referee for his or her efforts which improved the paper considerably.

2. Pelant's and Swift's criteria for Lipschitz, uniform and coarse embeddability into $c_0(\Gamma)$

In this section, we recall some of the notions and results by Pelant [4, 5] and Swift [8] about embeddings into $c_0(\Gamma)$.

For a metric space (M, d), a *cover* is a set $\mathcal{U}$ of subsets of M such that $M = \bigcup_{U \in \mathcal{U}} U$. A cover $\mathcal{U}$ of M is called *uniform* if there is an $r > 0$ so that for all $x \in M$, there is a $U \in \mathcal{U}$, so that $B_r(x) = \{x' \in M\colon d(x', x) < r\} \subset U$. It is called *uniformly bounded* if the diameters of the $U \in \mathcal{U}$ are uniformly bounded, and it is called *point finite* if every $x \in M$ lies in only finitely many $U \in \mathcal{U}$. A cover $\mathcal{V}$ of M is a *refinement* of a cover $\mathcal{U}$, if for every $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$, for which $V \subset U$.

DEFINITION 2.1 [4] A metric space (M, d) is said to have the *uniform Stone property* if every uniform cover $\mathcal{U}$ of M has a point finite uniform refinement.

DEFINITION 2.2 [8] A metric space (M, d) is said to have the *coarse Stone property* if every uniformly bounded cover is the refinement of a point finite uniformly bounded cover.

DEFINITION 2.3 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. For a map $f: M_1 \to M_2$, we define the *modulus of uniform continuity* $w_f: [0, \infty) \to [0, \infty]$, and the *modulus of expansion* $\rho: [0, \infty) \to [0, \infty]$ as follows:

$$w_f(t) = \sup\{d_2(f(x), f(y)) : x, y \in M_1, d_1(x, y) \le t\} \text{ and}$$
$$\rho_f(t) = \inf\{d_2(f(x), f(y)) : x, y \in M_1, d_1(x, y) \ge t\}.$$

The map f is called *uniformly continuous* if $\lim_{t\to 0} w_f(t) = 0$, and it is called a *uniform embedding* if, moreover, $\rho_f(t) > 0$ for every $t > 0$. It is called *coarse* if $w_f(t) < \infty$, for all $0 < t < \infty$ and it is called a *coarse embedding*, if, furthermore, $\lim_{t\to\infty} \rho_f(t) = \infty$. The map f is called *Lischitz continuous* if

$$\mathrm{Lip}(f) = \sup_{x\neq y} \frac{d_2(f(x), f(y))}{d_1(x, y)} < \infty,$$

and a *bi-Lipschitz embedding*, if, furthermore, f is injective and $\mathrm{Lip}(f^{-1})$ (being defined on the range of f) is also finite.

The following result recalls results from [4, Theorem 2.1] (for (i) $\iff$ (ii) $\iff$ (v)) and [8, Lemma 2.3, Corollary 3.11] (for (ii) $\iff$ (iii) $\iff$ (iv)).

THEOREM 2.4 *For a Banach space X, the following properties are equivalent*:

(i) *X has the uniform Stone property.*
(ii) *X is uniformly embeddable into $c_0(\Gamma)$, for some set Γ.*
(iii) *X has the coarse Stone property.*
(iv) *X is coarsely embeddable into $c_0(\Gamma)$, for some set Γ.*
(v) *X is bi-Lipschitzly embeddable into $c_0(\Gamma)$, for some set Γ.*

It is easy to see, and was noted in [4, 8], that the uniform Stone property and the coarse Stone property are inherited by subspaces. The equivalence (i) $\iff$ (ii) was used in [4] to show that $C[0, \omega_1]$ does not uniformly embed in any $c_0(\Gamma)$. It was also used to prove that certain other $C(K)$-spaces do not uniformly embed into $c_0(\Gamma)$: let Λ be any set and denote for $n \in \mathbb{N}$ by $[\Lambda]^{\le n}$ and $[\Lambda]^n$ the subsets of Λ which have cardinality at most n and exactly n, respectively. Endow $[\Lambda]^{\le n}$ with the restriction of the product topology on $\{0, 1\}^\Lambda$ (by identifying each set with its characteristic function). Then define K_Λ to be the one-point Alexandroff compactification of the topological sum of the spaces $[\Lambda]^{\le n}$, $n \in \mathbb{N}$. It was shown in [4] that if Λ satisfies Theorem 1.2, then $C(K_\Lambda)$ is not uniformly Stone, and thus does not embed uniformly into any $c_0(\Gamma)$.

3. A combinatorial argument

We start by introducing property $P(\alpha)$ for a cardinal α as follows. We say that a cardinal number satisfies $P(\alpha)$ if

$(P(\alpha))$ For every $n \in \mathbb{N}$ and any map σ: $[\alpha]^n \to \mathcal{C}$, $\mathcal{C}$ being an arbitrary set, (at least) one of the following two conditions hold:

(a) There is a sequence (F_j) of pairwise disjoint elements of $[\alpha]^n$, with $\sigma(F_i) = \sigma(F_j)$ for any $i, j \in \mathbb{N}$.

(b) There is an $F \in [\alpha]^{n-1}$, so that $\sigma(\{F \cup \{\gamma\} : \gamma \in \alpha \backslash F\})$ is infinite.

As remarked in Section 2, if κ is an uncountable weakly compact cardinal number, then $P(\kappa)$ holds. But the existence of weakly compact cardinal numbers requires further set theoretic axioms, beyond ZFC [**3**]. In [**4**, Question 3], the authors ask if $P(\omega_1)$ is true.

Theorem 3.1 *For λ defined by* (1.1), *$P(\lambda)$ holds.*

For our proof of Theorem 3.1, it will be more convenient to reformulate it into a statement about n-tuples, instead of sets of cardinality n. We will first introduce some notation.

Let $n \in \mathbb{N}$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be sets of infinite cardinality, and put $\Gamma = \prod_{i=1}^n \Gamma_i$. For $a \in \Gamma$ and $1 \le i \le n$, we denote the ith coordinate of a by $a(i)$. We say that two points a and b in Γ are *diagonal*, if $a(i) \ne b(i)$, for all $i \in \{1, 2, \ldots, n\}$.

Let $a \in \Gamma$ and $i \in \{1, 2 \ldots, n\}$. We call the set

$$H(a, i) = \{(b_1, b_2, \ldots, b_{i-1}, a(i), b_{i+1}, \ldots, b_n): b_j \in \Gamma_j, \text{ for } j \in \{1, \ldots, n\} \backslash \{i\}\},$$

the *hyperplane through the point a orthogonal to i.* We call the set

$$L(a, i) = \{(a(1), \ldots, a(i-1), b_i, a(i+1), \ldots, a(n)): b_i \in \Gamma_i\},$$

the *line through the point a in direction of i.*

For a cardinal number γ, we define recursively the following sequence of cardinal numbers $(\exp_+(\gamma, n): n \in \mathbb{N}_0)$: $\exp_+(\gamma, 0) = \gamma$, and, assuming $\exp_+(\gamma, n)$ has been defined for some $n \in \mathbb{N}_0$, we put

$$\exp_+(\gamma, n+1) = (2^{\exp_+(\gamma,n)^+})^+.$$

Here γ^+ denotes the *successor cardinal*, for a cardinal γ, that is the smallest cardinal number γ' with $\gamma' > \gamma$. Note that since $\exp_+(\gamma, 1) \le 2^{2^{2^\gamma}}$, it follows for the above-defined cardinal number λ, that

$$\lambda = \lim_{n \to \infty} \exp_+(\omega_0, n).$$

Secondly, successor cardinals are regular [**3**], and thus every infinite set of cardinality γ, with γ being a successor cardinal, can be partitioned for $n \in \mathbb{N}$ into n disjoint sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, all of them having also cardinality γ, and the map $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n \to [\bigcup_{i=1}^n \Gamma_i]^n$, $(a_1, a_2, \ldots, a_n) \mapsto \{a_1, a_2, \ldots, a_n\}$, is injective. We, therefore, deduce that the following statement implies Theorem 3.1.

THEOREM 3.2 *Let $n \in \mathbb{N}$ and a assume that the sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ have cardinality at least $\exp_+(\omega_1, n^2)$. For any function,*

$$\sigma\colon \Gamma := \prod_{i=1}^{n} \Gamma_i \to \mathcal{C},$$

where $\mathcal{C}$ is an arbitrary set, at least one of the following two conditions hold:

(a) *There is a sequence $(a^{(j)})_{j=1}^{\infty}$, of pairwise diagonal elements in Γ, so that $\sigma(a^{(i)}) = \sigma(a^{(j)})$, for any $i, j \in \mathbb{N}$.*
(b) *There is a line $L \subset \Gamma$, for which $\sigma(L)$ is infinite.*

Before proving Theorem 3.2. We need the following observation.

LEMMA 3.3 *Let $n \in \mathbb{N}$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be non-empty sets. Let*

$$\sigma\colon \Gamma := \prod_{i=1}^{n} \Gamma_i \to \mathcal{C},$$

be a function that fails both conditions (a) and (b) in the definition $(P(\alpha))$.

Then there is a set $\tilde{\mathcal{C}}$ and a function

$$\tilde{\sigma}\colon \Gamma := \prod_{i=1}^{n} \Gamma_i \to \tilde{\mathcal{C}},$$

that fails both, (a) and (b), and, moreover, has

(c) *for all $c \in \tilde{\mathcal{C}}$, there is a hyperplane $H_c \subset \Gamma$ so that $\{b \in \Gamma\colon \tilde{\sigma}(b) = c\} \subset H_c$.*

Proof. We may assume without loss of generality that σ is surjective. Since (a) is not satisfied, for each $c \in \mathcal{C}$, there exists an $m(c) \in \mathbb{N}$ and a (finite) sequence $(a^{(c,j)})_{j=1}^{m(c)} \subset \sigma^{-1}(\{c\})$, which is pairwise diagonal, and maximal. Hence

$$\sigma^{-1}(\{c\}) \subset \bigcup_{j=1}^{m(c)} \bigcup_{i=1}^{n} H(a^{(c,j)}, i).$$

Indeed, from the maximality of $(a^{(c,j)})_{j=1}^{m(c)} \subset \sigma^{-1}(\{c\})$, it follows that each $b \in \sigma^{-1}(\{c\})$ must have at least one coordinate in common with at least one element of $(a^{(c,j)})_{j=1}^{m(c)}$.

We define

$$\tilde{\mathcal{C}} = \bigcup_{c \in \mathcal{C}} (\{1, 2, \ldots, m(c)\} \times \{1, 2 \ldots n\} \times \{c\}),$$

and $\tilde{\sigma}\colon \Gamma \to \tilde{\mathcal{C}}, \quad b \mapsto (j, i, c)$, where

$$c = \sigma(b), j = \min\left\{ j' : b \in \bigcup_{i'=1}^{n} H(a^{(c,j')}, i') \right\}, \text{ and } i = \min\{ i' : b \in H(a^{(c,j)}, i') \}.$$

It is clear that $\tilde{\sigma}$ satisfies (c). Since $\tilde{\sigma}(j, i, b) = \tilde{\sigma}(j', i', d)$ implies $\sigma(b) = \sigma(d)$ for $(j, i, b), (j', i', d) \in \tilde{\Gamma}$, (a) fails for $\tilde{\sigma}$. In order to verify that (b) is not satisfied, assume $L \subset \Gamma$ is a line, and let $\{c_1, c_2, \ldots, c_p\}$ be the image of L under σ. By construction,

$$\tilde{\sigma}(L) \subset \{ (j, i, c_k), k \leq p, j \leq m(c_k), i \leq n \},$$

which is also finite. □

Proof of Theorem 3.2. We assume that $\sigma: \Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n \to \mathcal{C}$ is a map which fails both (a) and (b). By Lemma 3.3, we may also assume that σ satisfies (c). For each $a \in \Gamma$, we fix a number $h(a) \in \{1, 2, \ldots, n\}$ so that $\sigma^{-1}(\{\sigma(a)\}) \subset H(a, h(a))$. Thus, $h(a)$ is the direction, for which all $b \in \Gamma$, with $\sigma(b) = \sigma(a)$, lie in the hyperplane through a orthogonal to $h(a)$. It is important to note that since (b) is not satisfied, it follows that each line L, whose direction is some $j \in \{1, 2, \ldots, n\}$, can only have finitely many elements b for which $h(b) = j$. Indeed, if $h(b) = j$, then b is uniquely determined by the value $\sigma(b)$. To continue with the proof, the following *Reduction Lemma* will be essential. □

LEMMA 3.4 *Let β be an uncountable regular cardinal. Assume that $\tilde{\Gamma}_1 \subset \Gamma_1, \tilde{\Gamma}_2 \subset \Gamma_2, \ldots, \tilde{\Gamma}_n \subset \Gamma_n$ are such that $|\tilde{\Gamma}_i| \geq \exp_+(\beta, n)$, for all $i \in \{1, 2, \ldots, n\}$. Then, for any $i \in \{1, 2 \ldots, n\}$, there are a number $K_i \in \mathbb{N}$, and subsets $\Gamma'_1 \subset \tilde{\Gamma}_1, \Gamma'_2 \subset \tilde{\Gamma}_2, \ldots, \Gamma'_n \subset \tilde{\Gamma}_n$, with $|\Gamma'_i| \geq \beta$, so that*

$$\forall (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \prod_{j=1, j\neq i}^{n} \Gamma'_i$$
$$|\{ \alpha \in \Gamma'_i : h(a_1, a_2, \ldots, a_{i-1}, \alpha, a_{i+1}, \ldots, a_n) = i \}| \leq K_i. \quad (3.1)$$

Proof. We assume without loss of generality that $i = n$. Abbreviate $\beta_j = \exp_+(\beta, j)$, for $j = 1, 2, \ldots, n$. We first choose (arbitrary) subsets $\tilde{\Gamma}^{(0)}_j \subset \tilde{\Gamma}_j$, for which $|\tilde{\Gamma}^{(0)}_j| = \beta_{n+1-j}$.

Since the β_j's are regular, it follows for each $j = 1, 2 \ldots, n-2$ that

$$\begin{aligned} |\tilde{\Gamma}^{(0)}_j| &= \beta_{n+1-j} \\ &> 2^{\beta_{n-j}} \\ &= 2^{|\tilde{\Gamma}^{(0)}_{j+1} \times \tilde{\Gamma}^{(0)}_{j+2} \times \cdots \times \tilde{\Gamma}^{(0)}_{n-1}|} \\ &= |\{ f : \tilde{\Gamma}^{(0)}_{j+1} \times \tilde{\Gamma}^{(0)}_{j+2} \times \cdots \times \tilde{\Gamma}^{(0)}_{n-1} \to \mathbb{N} \}|. \end{aligned}$$

Here we are using for the second equality the assumption that $|\tilde{\Gamma}^{(0)}_n| < |\tilde{\Gamma}^{(0)}_{n-1}| < \cdots < |\tilde{\Gamma}^{(0)}_{j+1}| = \beta_{n-j}$, and the assumption that β_{n-j} is regular. For the third equality, we are using the fact that $|\{f: \Lambda \to \mathbb{N}\}| = |\{f: \Lambda \to \{0, 1\}\}| = 2^{|\Lambda|}$ for infinite sets Λ. Abbreviate for $j = 1, \ldots, n-1$:

$$\mathcal{F}_j = \{ f : \tilde{\Gamma}^{(0)}_j \times \tilde{\Gamma}^{(0)}_{j+1} \times \cdots \times \tilde{\Gamma}^{(0)}_{n-1} \to \mathbb{N} \}$$

and note that by above estimates $|\mathcal{F}_{j+1}| < |\tilde{\Gamma}_j^{(0)}|$, for $j = 1, 2, \ldots, n-1$. We consider the function

$$\phi_1\colon \prod_{j=1}^{n-1} \tilde{\Gamma}_j^{(0)} \to \mathbb{N},\ (a_1, a_2, \ldots, a_{n-1}) \mapsto |\{\alpha \in \tilde{\Gamma}_n^{(0)}\colon h(a_1, a_2, \ldots, a_{n-1}, \alpha) = n\}|.$$

For fixed $a_1 \in \tilde{\Gamma}_1^{(0)}$, $\phi_1(a_1, \cdot) \in \mathcal{F}_2$, and the cardinality of $\mathcal{F}_2$ is by the above estimates smaller than β_n, the cardinality of $\tilde{\Gamma}_1^{(0)}$, which is regular. Therefore, we can find a function $\phi_2 \in \mathcal{F}_2$ and a subset $\Gamma'_1 \subset \tilde{\Gamma}_1^{(0)}$ of cardinality β_n so that $\phi_1(a_1, \cdot) = \phi_2$ for all $a_1 \in \Gamma'_1$. Now we can apply the same argument to the function $\phi_2\colon \prod_{j=2}^{n-1} \tilde{\Gamma}_j^{(0)} \to \mathbb{N}$ and obtain a function $\phi_3 \in \mathcal{F}_3$ and $\Gamma'_2 \subset \tilde{\Gamma}_2^{(0)}$ of cardinality β_{n-1} so that $\phi_2(a_2, \cdot) = \phi_3$ for all $a_2 \in \Gamma'_2$. We continue the process and find $\Gamma'_j \subset \tilde{\Gamma}_j^{(0)}$, for $j = 1, 2, \ldots, n-2$ of cardinality β_{n+1-j} and functions $\phi_j \in F_j$, for $j = 1, 2, \ldots, n-1$, so that for all $(a_1, a_2, \ldots, a_{n-2}) \in \prod_{j=1}^{n-2} \Gamma'_j$ and $a_{n-1} \in \tilde{\Gamma}_{n-1}^{(0)}$ we have

$$\phi_1(a_1, a_2, \ldots, a_{n-1}) = \phi_2(a_2, \ldots, a_{n-1}) = \ldots = \phi_{n-1}(a_{n-1}). \tag{3.2}$$

Then, since ϕ_{n-1} is $\mathbb{N}$ valued, we can finally choose $K_n \in \mathbb{N}$ and a subset $\Gamma'_{n-1} \subset \tilde{\Gamma}_{n-1}^{(0)}$, of cardinality at least β, so that $\phi_{n-1}(a_{n-1}) \le K_n$, for all $a_{n-1} \in \Gamma'_{n-1}$, which finishes our argument.

Continuation of the proof of Theorem 3.2. We apply Lemma 3.4 successively to all $i \in \{1, 2, \ldots, n\}$, and the cardinals $\beta^{(i)} = \exp_+(\omega_1, n(n-i))$. We obtain numbers $K_1, K_2, \ldots, K_n$ in $\mathbb{N}$ and infinite sets $\Lambda_j \subset \Gamma_j$, for $j = 1, 2, \ldots, n$, so that for all $i \in \{1, 2, \ldots, n\}$ and all $a = (a_j\colon j \in \{1, 2, \ldots, n\} \setminus \{i\}) \in \prod_{j=1, j\neq i}^{n} \Lambda_j$

$$|\{\alpha \in \Lambda_i\colon h(a_1, a_2, \ldots, a_{i-1}, \alpha, a_{i+1}, \ldots, a_n) = i\}| \le K_i.$$

In order to deduce a contradiction, choose for each $j = 1, \ldots, n$ a subset A_j of Λ_j of cardinality $l_j = (n+1)K_j$. Then it follows that

$$\begin{aligned}\prod_{j=1}^{n} l_j &= \left|\prod_{j=1}^{n} A_j\right| \\ &= \sum_{i=1}^{n} \sum_{a \in \prod_{j=1, j\neq i}^{n} A_j} |\{\alpha \in A_i\colon h(a_1, a_2, \ldots, a_{i-1}, \alpha, a_{i+1}, \ldots, a_n) = i\}| \\ &\le \sum_{i=1}^{n} K_i \prod_{j=1, j\neq i}^{n} l_j \le \frac{n}{n+1} \prod_{j=1}^{n} l_j\end{aligned}$$

which is a contradiction and finishes the proof of the Theorem. □

We can now state the ZFC version of [4, Theorem 4.1], in which it was shown that for weakly compact cardinalities κ_0 the space $C(K_{\kappa_0})$, where K_{κ_0} was defined at the end of Section 2, cannot be uniformly (or coarsely) embedded into any $c_0(\Gamma)$, where Γ has any cardinality. Since the only property of κ_0, which is needed in [4], is the fact that $P(\kappa_0)$ holds, we deduce

COROLLARY 3.5 *$C(K_\lambda)$ does not coarsely (or uniformly) embed into $c_0(\Gamma)$, for any cardinality Γ.*

4. Proof of Theorem 1.1

In this section, we use our combinatorial Theorem 1.2 from Section 3 to show Theorem 1.1.

Recall that a long Schauder basis of a Banach space X is a transfinite sequence $\{e_\gamma\}_{\gamma=0}^{\Gamma}$ such that for every $x \in X$, there exists a unique transfinite sequence of scalars $\{a_\gamma\}_{\gamma=0}^{\Gamma}$ such that $x = \sum_{\gamma=0}^{\Gamma} a_\gamma e_\gamma$. Similarly, a long Schauder basic sequence in a Banach space X is a transfinite sequence $\{e_\gamma\}_{\gamma=0}^{\Gamma}$ which is a long Schauder basis of its closed linear span. Recall that the $w^* - \text{dens}(X^*)$ is the smallest cardinal such that there exists a w^*-dense subset of X^*. Analogously to the classical Mazur construction of a Schauder basic sequence in a separable Banach space, we have the following result, proved for example in [2, p. 135] (the fact that the basis is normalized, that is $\|e_\gamma\| = 1$, is not a part of the statement in [2], but it is easy to get it by normalizing the existing basis).

THEOREM 4.1 *Let X be a Banach space with $\Gamma = w^* - \text{dens}(X^*) > \omega_0$. Then X contains a monotone normalized long Schauder basic sequence of length Γ.*

Proof of Theorem 1.1. Using the Hahn–Banach theorem, it is easy to see that $w^* - \text{dens}(X^*) \le \text{dens}(X)$. On the other hand, since every $x \in X$ is uniquely determined by its values on a w^*-dense subset of X^*, it is clear that

$$\lambda \le \text{dens}(X) \le \text{card}(X) \le 2^{w^* - \text{dens}(X^*)}. \tag{4.1}$$

But this implies that $w^* - \text{dens}(X^*) \ge \lambda$. Indeed, otherwise we had $w^* - \text{dens}(X^*) < \lambda$, and thus $w^* - \text{dens}(X^*) \le \lambda_n$, for some $n \in \mathbb{N}$ (λ_n was defined before the statement of Theorem 1.1), and thus $2^{w^* - \text{dens}(X^*)} \le 2^{\lambda_n} = \lambda_{n+1} < \lambda$, which contradicts (4.1).

In order to prove Theorem 1.1, we may assume without loss of generality that X has a long normalized and monotone Schauder basis $(e_\mu)_{\mu<\lambda}$, of length λ.

Suppose that $F = \{\gamma_1, \ldots, \gamma_n\} \in [\lambda]^n$ where $\gamma_1 < \cdots < \gamma_n$ is arranged in an increasing order. Consider the corresponding finite set

$$M_F = \left\{ \sum_{i=1}^{n} \varepsilon_i e_{\gamma_i} : \varepsilon_i \in \{-1, 1\} \right\},$$

containing 2^n distinct vectors of X, and put a linear order $\prec$ on this set according to the arrangement of the signs ε_i, setting

$$\sum_{i=1}^{n} \varepsilon_i e_{\gamma_i} \prec \sum_{i=1}^{n} \tilde{\varepsilon}_i e_{\gamma_i}$$

if and only if for the minimal i, such that $\varepsilon_i \neq \tilde{\varepsilon}_i$, it holds $\varepsilon_i < \tilde{\varepsilon}_i$. In order to prove Theorem 1.1, it suffices to show that if $M = \cup_{F \in [\lambda]^n, n \in \mathbb{N}} M_F \subset X$ has the coarse Stone property, then X fails to have non-trivial cotype. To this end, starting with $\mathcal{U} = \{B_2(x) : x \in M\}$, we find a uniformly bounded cover $\mathcal{V}$, which is point finite and so that $\mathcal{U}$ refines $\mathcal{V}$, and we fix for all $x \in M$ a $V_x \in \mathcal{V}$ with $B_2(x) \subset V_x$. Let $r > 0$ be such that each $V \in \mathcal{V}$ is a subset of a ball of radius r.

Let $\mathcal{C}$ be the set consisting of all finite tuples of $\mathcal{V}$. We now define the function σ: $[\lambda]^n \to \mathcal{C}$ as follows. If $F \in [\lambda]^n$, $F = \{\gamma_1, \ldots, \gamma_n\}$ where $\gamma_1 < \cdots < \gamma_n$, we let

$$\sigma(F) = (V_{y_1}, \ldots, V_{y_{2^n}}), \tag{4.2}$$

where $y_1 \prec \cdots \prec y_{2^n}$ are the elements of M_F arranged in the increasing order defined above. Applying Theorem 1.2 to the function σ, for a fixed $n \in \mathbb{N}$, yields only two possibilities:

Case 1: The set $\sigma(\{F \cup \{\tau\} : \tau \in \lambda \backslash F\})$ is infinite for some $F = \{\gamma_1, \ldots, \gamma_{n-1}\}$, where $\gamma_1 < \cdots < \gamma_{n-1}$.

In this case, pick an infinite sequence of distinct $\{\tau_j\}_{j=1}^{\infty}$ witnessing the desired property. By passing to a subsequence, we may assume without loss of generality that either there exists k, $1 \leq k \leq n-1$, so that for all $j \in \mathbb{N}$, $\gamma_k < \tau_j < \gamma_{k+1}$, or $\tau_j < \gamma_1$ for all $j \in \mathbb{N}$, or $\gamma_{n-1} < \tau_j$ for all $j \in \mathbb{N}$. For simplicity of notation, assume the last case, that is $\gamma_1 < \cdots < \gamma_{n-1} < \tau_j$ holds for all $j \in \mathbb{N}$. Denoting $F^j = \{\gamma_1, \ldots, \gamma_{n-1}, \tau_j\}$, we conclude that there exists a fixed selection of signs $\varepsilon_1, \ldots, \varepsilon_n$ such that the set

$$B = \left\{ V_y : y = \sum_{i=1}^{n-1} \varepsilon_i e_{\gamma_i} + \varepsilon_n e_{\tau_j}, j \in \mathbb{N} \right\}$$

is infinite. Indeed, otherwise the set of values $\{\sigma(\{\gamma_1, \ldots, \gamma_{n-1}, \tau_j\}), j \in \mathbb{N}\}$, which are determined by the definition (4.2), would have only a finite set of options for each coordinate, and would, therefore, have to be finite. This is a contradiction with the point finiteness of the system $\mathcal{V}$, because

$$\sum_{i=1}^{n-1} \varepsilon_i e_{\gamma_i} \in V_y, \quad \text{for all } V_y \in B.$$

Case 2: there is a sequence (F_j) of pairwise disjoint elements of $[\lambda]^n$, with $\sigma(F_i) = \sigma(F_j)$, for any $i, j \in \mathbb{N}$.

In fact, it suffices to choose just a pair of such disjoint elements (written in an increasing order of ordinals) $F = \{\gamma_1, \ldots, \gamma_n\}$, $G = \{\beta_1, \ldots, \beta_n\}$, such that $\sigma(F) = \sigma(G)$. This means, in particular, that for every fixed selection of signs $\varepsilon_1, \ldots, \varepsilon_n$,

$$V_{\sum_{i=1}^n \varepsilon_i e_{\gamma_i}} = V_{\sum_{i=1}^n \varepsilon_i e_{\beta_i}}.$$

By our assumption, the elements of $\mathcal{V}$ are contained in a ball of radius r, hence

$$\left\| \sum_{i=1}^n \varepsilon_i e_{\gamma_i} - \sum_{i=1}^n \varepsilon_i e_{\beta_i} \right\| \leq 2r \tag{4.3}$$

holds for any selection of signs $\varepsilon_1, \ldots, \varepsilon_n$. Let $u_j = e_{\gamma_j} - e_{\beta_j}$, $j \in \{1, \ldots, n\}$. Because $\{e_\gamma\}$ is a monotone normalized long Schauder basis, we have the trivial estimate $1 \leq \|u_j\| \leq 2$. Equation (4.3) means that

$$1 \leq \left\| \sum_{i=1}^{n} \varepsilon_i u_i \right\| \leq 2r \tag{4.4}$$

holds for any selection of signs $\varepsilon_1, \ldots, \varepsilon_n$. Since norm functions are convex, this means that for the unit vector ball B_E of $E = \operatorname{span}(u_i\colon i \leq n)$ it follows that

$$\left\{ \sum_{j=1}^{n} a_j u_j : |a_j| \leq \frac{1}{2r} \right\} \subset B_E \subset \left\{ \sum_{j=1}^{n} a_j u_j : |a_j| \leq 2 \right\},$$

which means that $(u_j)_{j=1}^n$ is $4r$-equivalent to the unit vector basis of ℓ_∞^n. □

In fact, our proof gives a much stronger condition than just failing cotype, because our copies of ℓ_∞^k are formed by vectors of the type $e_\alpha - e_\beta$. This fact can be used to obtain much stronger structural results for spaces with special bases. Recall that a long Schauder basis $\{e_\gamma\}_{\gamma=1}^{\Lambda}$ is said to be symmetric if

$$\left\| \sum_{i=1}^{n} a_i e_{\gamma_i} \right\| = \left\| \sum_{i=1}^{n} a_i e_{\beta_i} \right\|$$

for any selection of $a_i \in \mathbb{R}$, and any pair of sets $\{\gamma_i\}_{i=1}^n \subset [1, \Lambda)$, $\{\beta_i\}_{i=1}^n \subset [1, \Lambda)$. It is well-known (cf. [7, Proposition II.22.2]) that each symmetric basis is automatically unconditional, that is there exists $K > 0$ such that

$$\frac{1}{K} \left\| \sum_{i=1}^{n} |a_i| e_{\gamma_i} \right\| \leq \left\| \sum_{i=1}^{n} a_i e_{\gamma_i} \right\| \leq K \left\| \sum_{i=1}^{n} |a_i| e_{\gamma_i} \right\|.$$

In particular,

$$\frac{1}{K} \left\| \sum_{i \in A} a_i e_{\gamma_i} \right\| \leq \left\| \sum_{i \in B} a_i e_{\gamma_i} \right\|$$

whenever $A \subset B$.

THEOREM 4.2 *Let X be a Banach space of density $\Lambda \geq \lambda$, with a symmetric basis $\{e_\gamma\}_{\gamma=1}^{\Lambda}$, which coarsely (or uniformly) embeds into some $c_0(\Gamma)$. Then X is linearly isomorphic with $c_0(\Lambda)$.*

Proof. By the proof of the above results, if X embeds into $c_0(\Gamma)$, there exists a $C > 0$, such that for each $k \in \mathbb{N}$, there are some vectors $\{v_i\}_{i=1}^k$ of the form $v_i = e_{\gamma_i} - e_{\beta_i}$ satisfying the conditions

$$\frac{1}{2} \max_j |a_i| \leq \left\| \sum_{i=1}^{k} a_i v_i \right\| \leq C \max_j |a_j|. \tag{4.5}$$

Using the fact that the basis $\{e_\gamma\}$ is unconditional and symmetric, we obtain that there exist some constants $A, B > 0$ such that

$$A\left\|\sum_{i=1}^{k} a_i e_{\gamma_i}\right\| \leq \left\|\sum_{i=1}^{k} a_i v_i\right\| \leq B\left\|\sum_{i=1}^{k} a_i e_{\gamma_i}\right\|. \tag{4.6}$$

Combining (4.5) and (4.6), we finally obtain that for some $D \geq 1$, and any $k \in \mathbb{N}$,

$$\frac{1}{D}\max_j |a_i| \leq \left\|\sum_{i=1}^{k} a_i e_{\beta_i}\right\| \leq D \max_j |a_j| \tag{4.7}$$

for all $\{\beta_1, \dots, \beta_k\} \subset [1, \Lambda)$, which proves our claim. □

5. Final comments and open problems

Let us mention in this final section some problems of interest.

First of all, we do not know whether or not Theorem 1.1 is true if we replace λ by smaller cardinal numbers.

PROBLEM 5.1 *Assume that X is a Banach space with* $\operatorname{dens}(X) \geq \omega_1$, *and assume that X coarsely embeds into $c_0(\Gamma)$ for some cardinal number Γ. Does X have trivial co-type? If, moreover, X has a symmetric basis, must it be isomorphic to $c_0(\omega_1)$?*

Of course, Problem 5.1 would have a positive answer if the following is true.

PROBLEM 5.2 *Is Theorem 1.2 true for ω_1?*

Connected to Problems 5.1 and 5.2 is the following.

PROBLEM 5.3 *Does ℓ_∞ coarsely embed into $c_0(\kappa)$ for some uncountable cardinal number κ?*

Another line of interesting problems asks which isomorphic properties do non-separable Banach spaces have which coarsely embed into $c_0(\Gamma)$.

PROBLEM 5.4 *Does a non-separable Banach space which coarsely embeds into some $c_0(\Gamma)$, Γ being uncountable, contain copies of c_0, or even $c_0(\omega_1)$?*

Funding

The first named author was supported by GAČR 16-07378S and RVO: 67985840. The second named author was supported by the National Science Foundation under the Grant no. DMS-1464713.

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