



Spectrum of Non-planar Traveling Waves

Anna Ghazaryan, Yuri Latushkin and Alin Pogan

Abstract. In this paper we prove that a class of non *self-adjoint* second order differential operators acting in cylinders $\Omega \times \mathbb{R} \subseteq \mathbb{R}^{d+1}$ have only real discrete spectrum located to the right of the right most point of the essential spectrum. We describe the essential spectrum using the limiting properties of the potential. To track the discrete spectrum we use spatial dynamics and bi-semigroups of linear operators to estimate the decay rate of eigenfunctions associated to isolated eigenvalues.

Mathematics Subject Classification. Primary 47F05; Secondary 35C05, 35P05.

Keywords. Non-planar fronts, Traveling waves, Essential spectrum, Isolated eigenvalues, Bi-semigroups of operators, Reaction–diffusion equation.

1. Introduction

Reaction–diffusion equations are used to model a variety of natural phenomena that occur as a result of interaction of spatial diffusion, convection and reaction of participating variables. In this paper, we consider the reaction–diffusion equation

$$u_t = \Delta_{x,y} u + f(u), \quad (x, y) \in \Omega \times \mathbb{R}, \quad (1.1)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^2 and the set $\Omega \subseteq \mathbb{R}^d$ is either a bounded or unbounded domain in $\mathbb{R}^d$.

In many cases, reaction–diffusion equations exhibit traveling waves, which are special solutions that preserve their shape while moving in a preferred direction. In systems posed on multi-dimensional domains, such as $\Omega \times \mathbb{R}$, a traveling wave is called planar if it is a function $\tilde{u}(t, x, y) = \tilde{u}(z)$ of the variable $z = k \cdot (x, y) - ct$, where $k = (k_1, k_2) \in \mathbb{R}^d \times \mathbb{R}$ is a constant vector, and if it is asymptotic to spatially constant steady-state solutions. Without loss of generality, one can take $k = (0, \dots, 0, 1)$, and, therefore, $\tilde{u}(t, x, y) = \tilde{u}(y - ct)$.

Partially supported by the US National Science Foundation under Grants NSF DMS-1710989 and NSF DMS-1311313, by the Research Board and Research Council of the University of Missouri, and by the Simons Foundation Award Number 402904.

On the other hand, a solution of the form $u(t, x, y) = \bar{u}(x, y - ct)$ is called a *non-planar traveling wave*. The existence of such traveling waves has been established in various cases, using methods such as center-manifold theory, topological methods, maximum principle based arguments, or by exploiting the variational structure of the equation. Detailed descriptions and specific examples may be found in [3–5, 9, 10, 18, 19, 24, 25, 29, 30] and references therein.

We now briefly describe a non trivial example of x -periodic, non-planar traveling waves. This existence result was established in [10].

Example 1.1. Assume that the function f is of class C^2 on an open interval containing $[0, 1]$, and there exists $a \in (0, 1)$ such that

$$f(0) = f(a) = f(1) = 0, \quad f(u) \neq 0 \text{ for } u \in (0, a) \cup (a, 1). \quad (1.2)$$

In addition, we assume that $u = 0$ and $u = 1$ are stable equilibria and $u = a$ is an unstable equilibrium of the diffusion free Eq. (1.1), that is,

$$f'(0) < 0, \quad f'(1) < 0, \quad f'(a) > 0. \quad (1.3)$$

If conditions (1.2) and (1.3) are satisfied, Eq. (1.1) is called bistable. A one-dimensional standing wave of (1.1), is a solutions of the form $u(t, x, y) = \bar{w}(x)$. The profile $\bar{w}$ satisfies the equation

$$\bar{w}_{xx} + f(\bar{w}) = 0. \quad (1.4)$$

Under assumptions (1.2) and (1.3) this second order equation has homoclinic (spikes), heteroclinic (layers) and periodic solutions.

It is well-known that for any $L > L_{\min} = 2\pi\sqrt{f'(a)}$ there exists a unique (up to translations) L -periodic solution of Eq. (1.4) denoted $\bar{w}_L$. In [10, Theorem 1.1] it was proved that (1.1) admits non-planar, L -periodic in x solutions connecting the standing wave $\bar{w}_L$ to the equilibrium $u = 1$ as $z \rightarrow \pm\infty$. More precisely, for each $L > L_{\min}$, there exists a minimal speed $c_L \in \mathbb{R}$ such that for any $c < c_L$ there exist solutions $\bar{u}_{c,L}(x, z)$ satisfying

- (i) $\Delta_{x,z}\bar{u}_{c,L} + c\partial_z\bar{u}_{c,L} + f(\bar{u}_{c,L}) = 0$ in $\mathbb{R}^2$;
- (ii) $\bar{u}_{c,L}(x, z) \rightarrow 1$ as $z \rightarrow +\infty$ uniformly in $x \in \mathbb{R}$;
- (iii) $\bar{u}_{c,L}(x, z) \rightarrow \bar{w}_L(x)$ as $z \rightarrow -\infty$ uniformly in $x \in \mathbb{R}$;
- (iv) $\bar{w}_L(x) < \bar{u}_{c,L}(x, z) < 1$ for any $x, z \in \mathbb{R}$;
- (v) $\bar{u}_{c,L}(x + L, z) = \bar{u}_{c,L}(x, z)$ for any $x, z \in \mathbb{R}$.

Traveling waves (planar and non-planar) are abundant in nature and human activities. In particular, Eq. (1.1) is a very natural, simple model that describes phenomena arising in chemistry and biology. In this case traveling waves solutions are ubiquitous. For traveling waves as physical phenomena an important concept is the stability of the waves which describes their resilience under perturbations. The stability analysis is based on the information about the location of the spectrum of the operator obtained by linearizing the right hand side of the reaction–diffusion Eq. (1.1) about the wave. The spectrum of the linearization consist of discrete eigenvalues of finite multiplicity and essential spectrum. This paper addresses certain properties of the spectrum of this linear operator.

Throughout this paper we assume the existence of a non-planar traveling wave $u(t, x, y) = \bar{u}(x, y - ct)$ of Eq. (1.1) traveling at speed $c \neq 0$. In the variable $z = y - ct$, Eq. (1.1) becomes

$$u_t = \Delta_{x,z} u + c \partial_z u + f(u). \quad (1.5)$$

We note that $u(t, x, z) = \bar{u}(x, z)$ is a time independent solution of (1.5). The linearization of (1.5) about $\bar{u}$ is

$$u_t = \mathcal{L}u, \quad \text{where } \mathcal{L} = \Delta_{x,z} + c \partial_z + V, \quad (1.6)$$

Here $V : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is given by $V(x, z) = f'(\bar{u}(x, z))$. We consider $\mathcal{L}$ as a closed linear operator on $L^2(\Omega \times \mathbb{R})$ with the usual domain $H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ when $\Omega \neq \mathbb{R}^d$ and $H^2(\mathbb{R}^{d+1})$ when $\Omega = \mathbb{R}^d$. For simplicity, we write $H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ in both cases, slightly abusing the notation.

Our main purpose is to describe the spectrum of the linear operator $\mathcal{L}$ defined in (1.6). The operator $\Delta_{x,z} + V$ is self-adjoint and therefore its spectrum is real. The operator $\mathcal{L}$ is not self-adjoint and its spectrum is not real. However, we prove that the discrete spectrum of $\mathcal{L}$ located to the right of the right most point of the essential spectrum is real. This result is known for the one-dimensional case when there is no x -variable, see [6, 13]. Its importance stems from the fact that in the typical situation when the essential spectrum is marginally stable, i.e., touches the imaginary axis only at 0, the eigenvalues to the right of the right most point of the essential spectrum, if any, give absolute instability of the wave $\bar{u}$, see, e.g., [27].

We consider (1.6) under the following assumptions:

Hypotheses. *The potential $V : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is bounded. Moreover, there exist two bounded functions $V_{\pm} : \Omega \rightarrow \mathbb{R}$, such that*

$$(H1) \quad \lim_{z \rightarrow \pm \infty} \|V(\cdot, z) - V_{\pm}(\cdot)\|_{L^\infty(\Omega)} = 0;$$

$$(H2) \quad z \rightarrow \|V(\cdot, z) - V_{\pm}(\cdot)\|_{L^\infty(\Omega)} \text{ belongs to } L^1(\mathbb{R}_{\pm}) \cap L^\infty(\mathbb{R}_{\pm});$$

(H3) *In the case when the domain Ω is unbounded we assume that*

$$\lim_{r \rightarrow \infty} \sup_{x \in \Omega, |x| \geq r} \|V(x, \cdot) - V_{\pm}(x)\|_{L^\infty(\mathbb{R}_{\pm})} = 0.$$

Hypotheses (H1)–(H3) are satisfied in case of the x -periodic, non-planar traveling waves $\bar{u}_{c,L}$ introduced in Example 1.1. Indeed, since f is a smooth function of class C^2 , using assertions (ii)–(iv), we immediately infer that the potential $V_{c,L}(x, z) = f'(\bar{u}_{c,L}(x, z))$ satisfies Hypothesis (H1) with $\Omega = (0, L)$, $V_{c,L,+} \equiv f'(1)$ and $V_{c,L,-}(x) = f'(\bar{w}_L(x))$. Moreover, using the results from [10, Remark 1.5, Proposition 4.1] (alternatively one can use [4, 18]) we have that the convergence of the non-planar solution $\bar{u}_{c,L}(x, z)$ at $z = \pm \infty$ is exponential in z and uniformly in $x \in (0, L)$, that is there exist $M, \alpha > 0$ such that

$$\|V_{c,L}(\cdot, z) - V_{c,L,\pm}(\cdot)\|_{L^\infty(0,L)} \leq M e^{-\alpha|z|} \quad \text{for any } z \in \mathbb{R},$$

which proves that the potential $V_{c,L}$ satisfies Hypothesis (H2).

To prove our result we need to use the elementary facts regarding the spectrum of the Schrödinger operator $\partial_x^2 + V_{\pm}$. In the case when $\Omega \subseteq \mathbb{R}^d$ is a bounded domain the spectrum of $\partial_x^2 + V_{\pm}$ consists of a sequence of eigenvalues

decreasing to $-\infty$. In the case when $\Omega \subseteq \mathbb{R}^d$ is unbounded there exist $\alpha_{\pm} \in \mathbb{R}$ such that $\sigma_{\text{ess}}(\partial_x^2 + V_{\pm}) = (-\infty, \alpha_{\pm}]$ and $\sigma_d(\partial_x^2 + V_{\pm})$ is a bounded set located to the right of $\alpha_{\pm}$.

Throughout this paper we denote by $\sigma(T)$ the spectrum, $\rho(T)$ the resolvent set, $\sigma_d(T)$ the set of isolated eigenvalues of finite algebraic multiplicity of a closed, densely defined linear operator $T : \text{dom}(T) \subseteq \mathbb{H} \rightarrow \mathbb{H}$ on a Hilbert space $\mathbb{H}$. The essential spectrum is defined by $\sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_d(T)$ and the essential resolvent set is given by $\rho_{\text{ess}}(T) = \mathbb{C} \setminus \sigma_{\text{ess}}(T)$. In the literature there are many different definitions of essential spectrum. We refer to the classical book by Edmunds and Evans ([8, Chapter IX, page 414]) for a list of the most common versions. One of the definitions on this list is the definition of Henry in [12, Chapter 5, page 130] that we use in this paper. On the other hand, the continuous spectrum of the operator T , denoted $\sigma_c(T)$, is defined as the set of all points $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm. This definition of continuous spectrum, used in [29, Chapter 4.3], is also listed in [8], as different from Henry's definition of the essential spectrum.

First, we describe the essential spectrum of $\mathcal{L}$ in terms of the limiting operators

$$\mathcal{L}_{\pm} : H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R}) \subset L^2(\Omega \times \mathbb{R}) \rightarrow L^2(\Omega \times \mathbb{R})$$

defined by

$$(\mathcal{L}_{\pm} u)(x, z) = \partial_x^2 u(x, z) + \partial_z^2 u(x, z) + c \partial_z u(x, z) + V_{\pm}(x) u(x, z). \quad (1.7)$$

This idea goes back to Henry ([12, Chapter 5]), for the case of the essential spectrum and $d = 0$, and to A. I. Volpert, Vi. A. Volpert, Vl. A. Volpert ([29, Chapter 4.3]) for the case of continuous spectrum and $d \geq 1$. The standard approach, c.f. [12, 27], is based on exponential dichotomies and Palmer's Theorem [2, 22, 23] and was used in many instances. To prove our first major result we use a direct argument based on sequences of approximate eigenfunctions, following the method used in [29] to characterize the continuous spectrum. Theorem 1.2(ii) is a particular case of Corollary 1 on page 208 in [29] (see Remark 2.2 below).

Theorem 1.2. *Assume Hypotheses (H1)–(H3). Then, the following assertions hold true:*

- (i) $\sigma_{\text{ess}}(\mathcal{L}_{\pm}) = \{\mu - s^2 + cis : s \in \mathbb{R}, \mu \in \sigma(\partial_x^2 + V_{\pm})\}$;
- (ii) $\sigma_{\text{ess}}(\mathcal{L}_+) \cup \sigma_{\text{ess}}(\mathcal{L}_-) \subseteq \sigma_{\text{ess}}(\mathcal{L})$;
- (iii) $\{\lambda \in \mathbb{C} : \text{Re } \lambda > \max\{\sup \sigma(\partial_x^2 + V_+), \sup \sigma(\partial_x^2 + V_-)\}\} \subseteq \rho_{\text{ess}}(\mathcal{L})$;
- (iv) $\sup \text{Re } \sigma_{\text{ess}}(\mathcal{L}) = \max\{\sup \sigma(\partial_x^2 + V_+), \sup \sigma(\partial_x^2 + V_-)\}$.

Our second result shows that adding the term $c \partial_z$ to the self-adjoint operator in (1.6) does not create non-real discrete eigenvalues to the right of the right most point of the essential spectrum. Naturally, we apply the substitution $v(x, z) = e^{cz/2} u(x, z)$ to reduce (1.6) to the self-adjoint case, c.f. [14, Sect. 2.3.1]. When z belongs to a bounded interval, obviously, the eigenvalues of $\mathcal{L}$ in (1.6) are the same as the eigenvalues of the self-adjoint operator $\Delta_{x,z} + V - c^2/4$ and therefore are real. Since in our case $z \in \mathbb{R}$, we

must verify that the potential eigenfunction $v(x, z) = e^{cz/2}u(x, z)$ of the self-adjoint operator belongs to $H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$. This requires information on the exponential rate of decay of the eigenfunctions of $\mathcal{L}$. This information is obtained by passing from the second order equation $\mathcal{L}u = \lambda u$ to the first order system $\partial_z Y = A(z)Y$, as usual in spatial dynamics [15, 20, 21, 27, 28], see (3.3). To control the decay rate of an individual solution of equation $\partial_z Y = A(z)Y$ we use a perturbation result similar to the approach of Daletski and Krein ([7]) for the case of well-posed equations. A crucial ingredient of the proof is the equality stated in Theorem 1.2 that relates the $\sup \operatorname{Re} \sigma_{\text{ess}}(\mathcal{L})$ to the spectrum of limiting, operator valued potentials, $\partial_x^2 + V_{\pm}$. In the x -independent case considered in [13] one can derive the required information on the Lyapunov exponents of this respective asymptotically autonomous system from the asymptotic systems that control the essential spectrum of $\mathcal{L}$. In the x -dependent case considered in the present paper the asymptotically autonomous equation $\partial_z Y = A(z)Y$ and the respective asymptotic equations are not well-posed, c.f. [24, 28] and [16]. The main technical point of this paper is therefore to overcome this difficulty, and to control the exponential rate of decay of the eigenfunctions of $\mathcal{L}$ via the spectrum of $\mathcal{L}_{\pm}$. This is done in Lemmas 3.2–3.8 by means of stable bi-semigroups, c.f. [1, 16] to avoid the use of Morse indices to describe the point spectrum, cf. [28, Theorem 2.8]. As a result, we arrive to the following second major result of the paper.

Theorem 1.3. *Assume Hypotheses (H1)–(H3). Then, the discrete spectrum of $\mathcal{L}$ to the right of the right most point of the essential spectrum is real, that is*

$$\{\lambda \in \sigma_d(\mathcal{L}) : \operatorname{Re} \lambda > \sup \operatorname{Re} \sigma_{\text{ess}}(\mathcal{L})\} \subseteq \mathbb{R}. \quad (1.8)$$

Remark 1.4. We emphasize that using the same techniques, it can be shown that the results of Theorem 1.2 and Theorem 1.3 hold true in the more general case of systems, where $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ provided the matrix-valued potential V is symmetric. The argument requires only minor modifications.

The difficulty in the proof of Theorem 1.3 stems from the fact that the linearization $\mathcal{L}$ is a multi-dimensional differential operator in $(x, z) \in \Omega \times \mathbb{R}$. Therefore, we can see the linearization $\mathcal{L}$ as a second order operator in $z \in \mathbb{R}$ with operator valued potentials $\partial_x^2 + V(\cdot, z)$, which is more general than the case of one-dimensional operators with matrix valued potential. In this case, the eigenvalue problem can be reduced to an infinite-dimensional, first order equation in the non-trivial space $H_0^1(\Omega) \times L^2(\Omega)$ that is not well-posed (see Sect. 3 for details).

The paper is organized as follows: In Sect. 2 we describe the essential spectrum of the linearization $\mathcal{L}$, proving Theorem 1.2. In Sect. 3 we prove that discrete eigenvalues (if any) to the right of the right most point of the essential spectrum of $\mathcal{L}$ are real, proving Theorem 1.3. In Sect. 4 we show how one can use the methods described in Sect. 3 to recover the finite dimensional result of [6, 13].

2. Essential Spectrum

In this section we describe the essential spectrum of the linear operator $\mathcal{L}$, proving Theorem 1.2. Throughout this section we assume Hypotheses (H1)–(H3). First, we compute the essential spectrum of the limiting operators $\mathcal{L}_\pm$. We use perturbation results to describe the connections between the essential spectra of $\mathcal{L}$ and $\mathcal{L}_\pm$.

Lemma 2.1. *Assume Hypotheses (H1)–(H3). Then, the essential spectrum of the linear operator $\mathcal{L}_\pm$ is given by*

$$\sigma_{\text{ess}}(\mathcal{L}_\pm) = \{\mu - s^2 + cis : s \in \mathbb{R}, \mu \in \sigma(\partial_x^2 + V_\pm)\}. \quad (2.1)$$

Proof. We note that taking Fourier Transform in $z \in \mathbb{R}$, one can readily see that the limiting operator $\mathcal{L}_\pm$ is similar to the operator $M_{\mathcal{V}_\pm}$ of multiplication by the operator valued function $\mathcal{V}_\pm(s) = \partial_x^2 + V_\pm - s^2 + cis$, $s \in \mathbb{R}$. For each $s \in \mathbb{R}$ we consider $\mathcal{V}_\pm(s)$ as a closed, densely defined linear operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$. We infer that

$$\sigma(\mathcal{L}_\pm) = \sigma(M_{\mathcal{V}_\pm}) = \{\mu - s^2 + cis : s \in \mathbb{R}, \mu \in \sigma(\partial_x^2 + V_\pm)\}. \quad (2.2)$$

Since the set above has no isolated points, we infer that the spectrum of $\mathcal{L}_\pm$ consists entirely of essential spectrum. From (2.2) we conclude that

$$\sigma_{\text{ess}}(\mathcal{L}_\pm) = \{\mu - s^2 + cis : s \in \mathbb{R}, \mu \in \sigma(\partial_x^2 + V_\pm)\},$$

proving assertion (i) of Theorem 1.2. ■

Remark 2.2. Assertion (ii) of Theorem 1.2 follows from the description of continuous spectrum given in [29] for general second order differential operators. Indeed, since $\sigma_c(\mathcal{L}) \subseteq \sigma_{\text{ess}}(\mathcal{L})$ and by [29, Chapter 4.3, Corollary 1, page 208] we have that $\sigma_{\text{ess}}(\mathcal{L}_+) \cup \sigma_{\text{ess}}(\mathcal{L}_-) \subseteq \sigma_c(\mathcal{L})$, the conclusion follows shortly. For completeness, we provide a proof of the inclusion in the essential spectrum (Lemma 2.4), that does not refer to the continuous spectrum, which is in the spirit of the proof in [29], where a more general case is treated.

To prove our theorem we first show that the essential spectrum of the linear operator $\mathcal{L}$ is equal to the essential spectrum of the linear operator

$$\mathcal{L}_\infty : H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R}) \rightarrow L^2(\Omega \times \mathbb{R})$$

defined by

$$\mathcal{L}_\infty = \partial_x^2 + \partial_z^2 + c\partial_z + V_\infty, \quad (2.3)$$

where $V_\infty : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$V_\infty(x, z) = \begin{cases} V_+(x) & \text{if } z \geq 0, \\ V_-(x) & \text{if } z < 0. \end{cases} \quad (2.4)$$

Lemma 2.3. *Assume Hypotheses (H1)–(H3). Then, $\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}_\infty)$.*

Proof. Since $\mathcal{L} = \mathcal{L}_\infty + M_{V-V_\infty}$, where M_{V-V_∞} is the operator of multiplication on $L^2(\Omega \times \mathbb{R})$ by the bounded function $V - V_\infty$, to prove the lemma it is enough to show that M_{V-V_∞} is relatively compact with respect to $\mathcal{L}$.

First, we approximate the operator M_{V-V_∞} by an operator of multiplication by a function with compact support.

Let $\{\rho_n\}_{n \geq 1}$ be a sequence of C^∞ functions satisfying the conditions: $0 \leq \rho_n \leq 1$, $\rho_n(z) = 1$ for any $z \in [-n, n]$, $\rho_n(z) = 0$ if $|z| \geq n+1$. We define the sequence of functions $K_n : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$K_n(x, z) = \rho_n(|x|)\rho_n(z)(V(x, z) - V_\infty(x, z)). \quad (2.5)$$

Since $\text{supp } K_n \subseteq \tilde{\Omega}_n := (\Omega \times \mathbb{R}) \cap (-n-1, n+1)^{d+1}$ and $H^2(\tilde{\Omega}_n)$ is compactly embedded in $L^2(\tilde{\Omega}_n)$ for any $n \geq 1$, we infer that M_{K_n} , the linear operator on $L^2(\Omega \times \mathbb{R})$ of multiplication by K_n , is relatively compact with respect to $\mathcal{L}$ for any $n \geq 1$. To finish the proof of lemma, it is enough to show that $M_{K_n} \rightarrow M_{V-V_\infty}$ as $n \rightarrow \infty$ in the operator norm. From (2.4), Hypothesis (H1) and Hypothesis (H3) it follows that

$$\begin{aligned} \lim_{z \rightarrow \pm\infty} \|V(\cdot, z) - V_\infty(\cdot, z)\|_{L^\infty(\Omega)} &= 0, \\ \lim_{r \rightarrow \infty} \sup_{x \in \Omega, |x| \geq r} \|V(x, \cdot) - V_\infty(x, \cdot)\|_{L^\infty(\mathbb{R})} &= 0. \end{aligned} \quad (2.6)$$

Using the definition of K_n in (2.5) we obtain that

$$\begin{aligned} \|M_{K_n} - M_{V-V_\infty}\|_{L^2 \rightarrow L^2} &= \|K_n - (V - V_\infty)\|_{L^\infty(\Omega \times \mathbb{R})} \\ &= \text{ess sup}_{(x,z) \in \Omega \times \mathbb{R}} (1 - \rho_n(|x|)\rho_n(z))|V(x, z) - V_\infty(x, z)| \\ &\leq \text{ess sup}_{(x,z) \in \Omega \times \mathbb{R}} (2 - \rho_n(|x|) - \rho_n(z))|V(x, z) - V_\infty(x, z)| \\ &\leq \sup_{x \in \Omega} (|1 - \rho_n(|x|)| \|V(x, \cdot) - V_\infty(x, \cdot)\|_{L^\infty(\mathbb{R})}) \\ &\quad + \sup_{z \in \mathbb{R}} (|1 - \rho_n(z)| \|V(\cdot, z) - V_\infty(\cdot, z)\|_{L^\infty(\Omega)}) \\ &\leq \sup_{x \in \Omega, |x| \geq n} (\|V(x, \cdot) - V_\infty(x, \cdot)\|_{L^\infty(\mathbb{R})}) \\ &\quad + \sup_{|z| \geq n} (\|V(\cdot, z) - V_\infty(\cdot, z)\|_{L^\infty(\Omega)}). \end{aligned} \quad (2.7)$$

From (2.6) and (2.7) we conclude that $\|M_{K_n} - M_{V-V_\infty}\|_{L^2 \rightarrow L^2} \rightarrow 0$ as $n \rightarrow \infty$, proving the lemma. $\blacksquare$

Lemma 2.4. *Assume Hypotheses (H1)–(H3). Then, $\sigma_{\text{ess}}(\mathcal{L}_+) \cup \sigma_{\text{ess}}(\mathcal{L}_-) \subseteq \sigma_{\text{ess}}(\mathcal{L})$.*

Proof. First, we prove that $\sigma_{\text{ess}}(\mathcal{L}_+) \subseteq \sigma_{\text{ess}}(\mathcal{L}_\infty)$ using sequences of approximate eigenfunctions. Similarly, we can show that $\sigma_{\text{ess}}(\mathcal{L}_-) \subseteq \sigma_{\text{ess}}(\mathcal{L}_\infty)$. From Lemma 2.1, for a fixed $\lambda \in \sigma_{\text{ess}}(\mathcal{L}_+)$ we have that there exist $a \in \mathbb{R}$ and $\mu \in \sigma(\partial_x^2 + V_+)$ such that $\lambda = \mu - a^2 + cia$. Since $\partial_x^2 + V_+$ is a self-adjoint linear operator on $L^2(\Omega)$, we obtain that there exists a sequence $\{\varphi_n\}_{n \geq 1}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ ([8, Chapter IX]) such that $\|\varphi_n\|_2 = 1$ for any $n \geq 1$ and

$$f_n := \varphi_n'' + (V_+(\cdot) - \mu)\varphi_n \rightarrow 0 \quad \text{in } L^2(\Omega) \quad \text{as } n \rightarrow \infty. \quad (2.8)$$

We then take $\psi \in C_0^\infty(\mathbb{R})$, $\psi \neq 0$, such that $\text{supp } \psi \subseteq [0, 1]$, and define $u_n : \mathbb{R} \rightarrow \mathbb{C}$ by $u_n(z) = n^{-1/2} e^{iaz} \psi(n^{-1}z - n)$. Since $\text{supp } \psi \subseteq [0, 1]$ it follows that

$$\operatorname{supp} u_n \subseteq [n^2, n^2 + n] \quad \text{for any } n \geq 1. \quad (2.9)$$

Moreover, one can readily check that

$$\|u_n\|_2^2 = \frac{1}{n} \int_{\mathbb{R}} |\psi(n^{-1}z - n)|^2 dz = \int_{\mathbb{R}} |\psi(y)|^2 dy = \|\psi\|_2^2 > 0 \quad (2.10)$$

and

$$u_n'' + cu_n' + (a^2 - cia)u_n = (c + 2ia)v_n + w_n \quad \text{for any } n \geq 1, \quad (2.11)$$

where $v_n, w_n : \mathbb{R} \rightarrow \mathbb{C}$ are defined by

$$v_n(z) = \frac{e^{iaz}}{\sqrt{n^3}} \psi'(n^{-1}z - n), \quad w_n(z) = \frac{e^{iaz}}{\sqrt{n^5}} \psi'(n^{-1}z - n). \quad (2.12)$$

Integrating, we obtain that

$$\|v_n\|_2 = \frac{1}{n} \|\psi'\|_2 \quad \text{and} \quad \|w_n\|_2 = \frac{1}{n^2} \|\psi''\|_2 \quad \text{for any } n \geq 1. \quad (2.13)$$

We note that assertions (2.10), (2.12), and (2.13) show that $\{u_n\}_{n \geq 1}$ is a sequence of approximate eigenvalues of the constant coefficient operator $\partial_z^2 + c\partial_z$. From (2.9) we have that $\operatorname{supp} u_n \subseteq \mathbb{R}_+$, therefore

$$u_n(z)V_\infty(x, z) = u_n(z)V_+(x) \quad \text{for any } n \geq 1, x \in \Omega, z \in \mathbb{R}. \quad (2.14)$$

We introduce $\phi_n : \Omega \times \mathbb{R} \rightarrow \mathbb{C}$ by $\phi_n(x, z) = \varphi_n(x)u_n(z)$. From (2.11) and (2.14) it follows that for any $n \geq 1, x \in \Omega$ and $z \in \mathbb{R}$,

$$\begin{aligned} ((\mathcal{L}_\infty - \lambda)\phi_n)(x, z) &= \varphi_n''(x)u_n(z) + \varphi_n(x)u_n''(z) + c\varphi_n(x)u_n'(z) \\ &\quad + V_\infty(x, z)\varphi_n(x)u_n(z) - \lambda\varphi_n(x)u_n(z) \\ &= \varphi_n(x)(u_n''(z) + cu_n'(z) + (a^2 - cia)u_n(z)) \\ &\quad + u_n(z)(\varphi_n''(x) + (V_+(x) - \mu)\varphi_n(x)) \\ &= f_n(x)u_n(z) + ((c + 2ia)v_n(z) + w_n(z))\varphi_n(x). \end{aligned} \quad (2.15)$$

Since $\|\varphi_n\|_2 = 1$ for any $n \geq 1$ from (2.10) we infer that

$$\|\phi_n\|_2 = \|\varphi_n\|_2 \|u_n\|_2 = \|\psi\|_2 > 0 \quad \text{for any } n \geq 1. \quad (2.16)$$

In addition, from (2.8) and (2.13) we have that $f_n \rightarrow 0$ in $L^2(\Omega)$, $v_n \rightarrow 0$ and $w_n \rightarrow 0$ in $L^2(\mathbb{R})$ as $n \rightarrow \infty$. From (2.15) we conclude that

$$(\mathcal{L}_\infty - \lambda)\phi_n \rightarrow 0 \quad \text{in } L^2(\Omega \times \mathbb{R}) \quad \text{as } n \rightarrow \infty. \quad (2.17)$$

From (2.16) and (2.17) it follows that $\lambda \in \sigma(\mathcal{L}_\infty)$. Summarizing, we have that

$$\sigma_{\text{ess}}(\mathcal{L}_+) = \{\mu - s^2 + cis : s \in \mathbb{R}, \mu \in \sigma(\partial_x^2 + V_+)\} \subseteq \sigma(\mathcal{L}_\infty). \quad (2.18)$$

Since $\sigma_{\text{ess}}(\mathcal{L}_+)$ does not have isolated points and $\sigma_{\text{ess}}(\mathcal{L}_\infty) = \sigma_{\text{ess}}(\mathcal{L})$, we conclude that $\sigma_{\text{ess}}(\mathcal{L}_+) \subseteq \sigma_{\text{ess}}(\mathcal{L})$, thus proving the lemma. $\blacksquare$

Next, we prove that the resolvent set of the linear operator $\mathcal{L}_\infty$ defined in (2.3) contains a right half-plane.

Lemma 2.5. *Assume Hypotheses (H1)–(H3). Then, the following inclusion holds true:*

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \max\{\sup \sigma(\partial_x^2 + V_+), \sup \sigma(\partial_x^2 + V_-)\}\} \subseteq \rho(\mathcal{L}_\infty).$$

Proof. Fix $\lambda_0 \in \mathbb{C}$ such that $\operatorname{Re} \lambda_0 > \sup \sigma(\partial_x^2 + V_\pm)$. Since the linear operators $\partial_x^2 + V_\pm$ are self-adjoint, from the min-max principle (see, e.g. [26]), we have that there exists $\varepsilon_0 > 0$ such that $\partial_x^2 + V_\pm \leq (\operatorname{Re} \lambda_0 - \varepsilon_0)I$, that is

$$\langle (\partial_x^2 + V_\pm - \operatorname{Re} \lambda_0)g, g \rangle_{L^2(\Omega)} \leq -\varepsilon_0 \|g\|_2^2 \quad \text{for any } g \in H^2(\Omega) \cap H_0^1(\Omega). \quad (2.19)$$

We define the operator valued function

$$F_\infty(z) = \begin{cases} \partial_x^2 + V_+ & \text{if } z \geq 0, \\ \partial_x^2 + V_- & \text{if } z < 0. \end{cases} \quad (2.20)$$

For each $z \in \mathbb{R}$, we consider $F_\infty(z)$ as a closed, densely defined linear operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$. The inequality (2.19) is equivalent to

$$\langle (F_\infty(z) - \operatorname{Re} \lambda_0)g, g \rangle_{L^2(\Omega)} \leq -\varepsilon_0 \|g\|_2^2 \quad \text{for any } z \in \mathbb{R}, g \in H^2(\Omega) \cap H_0^1(\Omega). \quad (2.21)$$

The linear operator $\mathcal{L}_\infty$ admits the representation

$$\mathcal{L}_\infty = \partial_z^2 + c\partial_z + F_\infty. \quad (2.22)$$

Since $F_\infty(z)$ is self-adjoint for any $z \in \mathbb{R}$ it follows that $\operatorname{dom}(\mathcal{L}_\infty^*) = \operatorname{dom}(\mathcal{L}_\infty)$. Moreover, from (2.21) we obtain that

$$\begin{aligned} \operatorname{Re} \langle (\lambda_0 - \mathcal{L}_\infty)v, v \rangle_{L^2(\Omega \times \mathbb{R})} &= \langle (\operatorname{Re} \lambda_0 - \operatorname{Re} \mathcal{L}_\infty)v, v \rangle_{L^2(\Omega \times \mathbb{R})} \\ &= \operatorname{Re} \lambda_0 \|v\|_2^2 + \|\partial_z v\|_2^2 - \langle F_\infty(\cdot)v, v \rangle_{L^2(\Omega \times \mathbb{R})} \\ &\geq \langle (\operatorname{Re} \lambda_0 - F_\infty(\cdot))v, v \rangle_{L^2(\Omega \times \mathbb{R})} \\ &= \int_{\mathbb{R}} \langle (\operatorname{Re} \lambda_0 - F_\infty(z))v(\cdot, z), v(\cdot, z) \rangle_{L^2(\Omega)} dz \\ &\geq \varepsilon_0 \int_{\mathbb{R}} \langle v(\cdot, z), v(\cdot, z) \rangle_{L^2(\Omega)} dz = \varepsilon_0 \|v\|_2^2 \end{aligned} \quad (2.23)$$

for any $v \in \operatorname{dom}(\mathcal{L}_\infty) = \operatorname{dom}(\mathcal{L}_\infty^*)$. From (2.23) we conclude that $\ker(\bar{\lambda}_0 - \mathcal{L}_\infty^*) = \{0\}$ and

$$\|(\lambda_0 - \mathcal{L}_\infty)v\|_{L^2(\Omega \times \mathbb{R})} \geq \varepsilon_0 \|v\|_{L^2(\Omega \times \mathbb{R})} \quad \text{for any } v \in \operatorname{dom}(\mathcal{L}_\infty), \quad (2.24)$$

which implies that $\lambda_0 - \mathcal{L}_\infty$ is invertible, proving the lemma. $\blacksquare$

Proof of Theorem 1.2. We have now all the ingredients needed to describe the essential spectrum of $\mathcal{L}$. Assertions (i) and (ii) were proved in Lemma 2.1 and Remark 2.2 or Lemma 2.4, respectively. Since $\sigma_{\text{ess}}(\mathcal{L}_\infty) = \sigma_{\text{ess}}(\mathcal{L})$ by Lemma 2.3, assertion (iii) follows shortly from Lemma 2.5. Assertion (iv) is a direct consequence of (ii) and (iii). $\blacksquare$

3. Bi-semigroups and Eigenfunction Decay Rate

In this section we prove that the eigenvalues to the right of the right most point of the essential spectrum of $\mathcal{L}$ must be real. The main idea of the proof is to make a change of variables in the eigenvalue equation and then prove that the eigenvalues of $\mathcal{L}$ to the right of the right most point of the essential spectrum are eigenvalues of a second order, self-adjoint linear operator. To achieve this goal we need to estimate the decay rate of eigenfunctions by reducing the second order eigenvalue equation to a first order linear differential equation on a suitable Hilbert space, and then use perturbation results to prove Theorem 1.3.

We assume $\lambda_0 \in \sigma_d(\mathcal{L})$ is such that $\operatorname{Re} \lambda_0 > \sup \operatorname{Re} \sigma_{\text{ess}}(\mathcal{L})$. Therefore, there exists an eigenfunction $u_0 \in H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ satisfying

$$\partial_x^2 u_0 + \partial_z^2 u_0 + c \partial_z u_0 + V(x, z) u_0 = \lambda_0 u_0. \quad (3.1)$$

Next, we introduce the function $v_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{C}$ by $v_0(x, z) = e^{cz/2} u_0(x, z)$. One can readily check that $v_0 \in H_{\text{loc}}^2(\Omega \times \mathbb{R})$ and

$$\partial_x^2 v_0 + \partial_z^2 v_0 + \left(V(x, z) - \frac{c^2}{4} \right) v_0 = \lambda_0 v_0. \quad (3.2)$$

To prove our result we need to show that v_0 is a genuine eigenfunction, that it belongs to $H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$. To do this, we study its decay rate using the spatial dynamics method by treating $z \in \mathbb{R}$ as time in (3.2). First, we introduce $w_0 = \partial_z v_0 \in H_{\text{loc}}^1(\Omega \times \mathbb{R})$. The pair of functions $(v_0, w_0)^T$ satisfies the first order differential equation

$$\partial_z \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = A(z) \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}, \quad (3.3)$$

where for each $z \in \mathbb{R}$ the linear operator $A(z)$ acting in the space $H_0^1(\Omega) \times L^2(\Omega)$ is defined by

$$A(z) = \begin{bmatrix} 0 & I \\ \lambda_0 + \frac{c^2}{4} - V(\cdot, z) - \partial_x^2 & 0 \end{bmatrix} \quad (3.4)$$

with domain $\operatorname{dom}(A(z)) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. We note that for each $z \in \mathbb{R}$ the linear operator $A(z)$ can be obtained as the bounded perturbation of the linear operator acting in the space $H_0^1(\Omega) \times L^2(\Omega)$ and defined by

$$A_{\pm} = \begin{bmatrix} 0 & I \\ \lambda_0 + \frac{c^2}{4} - V_{\pm}(\cdot) - \partial_x^2 & 0 \end{bmatrix}. \quad (3.5)$$

with domain $\operatorname{dom}(A_{\pm}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. The spectrum of the linear operator $A_{\pm}$ can be computed in terms of the spectrum of the self-adjoint operator $\partial_x^2 + V_{\pm}$ as follows:

$$\sigma(A_{\pm}) = \left\{ \pm \sqrt{\lambda_0 + c^2/4 - \mu} : \mu \in \sigma(\partial_x^2 + V_{\pm}) \right\}. \quad (3.6)$$

Hence, the real part of the spectrum of $A_{\pm}$ is unbounded from below and from above, therefore it cannot generate a C^0 -semigroup. We conclude that Eq. (3.3) is not well-posed.

Remark 3.1. In the case of the not well-posed first order differential Eq. (3.3) one cannot immediately infer the existence of backward nor forward propagators. Hence, when studying exponential dichotomy of (3.3), in [16, 24] the following backward-forward uniqueness was assumed:

- (i) If Y is a solution of (3.3) on $\mathbb{R}$ and $Y(z_0) = 0$ for some $z_0 \in \mathbb{R}$, then $Y \equiv 0$;
- (ii) If Z is a solution of the adjoint equation $Z' = -A(z)^*Z$ on $\mathbb{R}$ and $Z(z_0) = 0$ for some $z_0 \in \mathbb{R}$, then $Z \equiv 0$.

This assumption was used to prove exponential dichotomies, more precisely, to prove the existence and uniqueness of solutions on the semi-lines $(-\infty, a]$ and $[b, \infty)$ that decay exponentially in backward and forward time, respectively. It is proved in [24, Sect. 6.2] that Eq. (3.3) satisfies the backward-forward uniqueness properties above. In our case we are particularly interested in the solution $(v_0, \partial_z v_0)^T$ of (3.3). This solution exists because λ_0 is a discrete eigenvalue of $\mathcal{L}$ and therefore we do not need any general results to construct dichotomies.

Our first task is to prove that $A_{\pm}$ generates a bi-semigroup on $H_0^1(\Omega) \times L^2(\Omega)$. Here we recall that a closed, densely defined linear operator G generates a bi-semigroup on a Hilbert space $\mathbb{H}$, if there exist $\mathbb{H}_s$ and $\mathbb{H}_u$ two closed subspaces of $\mathbb{H}$ invariant under G such that $\mathbb{H} = \mathbb{H}_s \oplus \mathbb{H}_u$ (here $\oplus$ is a direct sum, not necessarily orthogonal) and $G|_{\mathbb{H}_s}$ and $-G|_{\mathbb{H}_u}$ generate C^0 semigroups on $\mathbb{H}_s$ and $\mathbb{H}_u$, denoted by $\{T_s(z)\}_{z \geq 0}$ and $\{T_u(z)\}_{z \geq 0}$, respectively. We say that the bi-semigroup has decay rate $-\nu < 0$ if there exists $C > 0$ such that

$$\|T_s(z)\| \leq Ce^{-\nu z} \quad \text{and} \quad \|T_u(z)\| \leq Ce^{-\nu z} \quad \text{for any } z \geq 0. \quad (3.7)$$

To prove that $A_{\pm}$ generates a bi-semigroup on $\mathbb{H} = H_0^1(\Omega) \times L^2(\Omega)$ we use the following abstract result.

Lemma 3.2. *Assume $\mathbb{H}$ is a Hilbert space, $\beta \in \mathbb{R}$ and $T : \text{dom}(T) \subset \mathbb{H} \rightarrow \mathbb{H}$ is a closed, densely defined, self-adjoint linear operator on $\mathbb{H}$ satisfying the condition*

$$\sigma(T) \subseteq [\alpha, \infty) \quad \text{for some } \alpha > 0. \quad (3.8)$$

If $\mathbb{Y} = \text{dom}(|T|^{1/2}) \times \mathbb{H}$, then the linear operator $G : \text{dom}(G) \subset \mathbb{Y} \rightarrow \mathbb{Y}$ defined by

$$\text{dom}(G) = \text{dom}(T) \times \text{dom}(|T|^{1/2}), \quad G = \begin{bmatrix} 0 & I \\ i\beta + T & 0 \end{bmatrix} \quad (3.9)$$

generates an exponentially stable bi-semigroup on $\mathbb{Y}$ having decay rate $-\nu$, satisfying the condition $\nu \geq \sqrt{\alpha}$.

Proof. Since T is a closed, densely defined, positive self-adjoint linear operator on $\mathbb{H}$, by the Spectral Theorem we have that there exists (Γ, μ) , a measure space, $\gamma : \Gamma \rightarrow \mathbb{R}$, a μ -measurable function, and $U : \mathbb{H} \rightarrow L^2(\Gamma, \mu)$, a unitary operator, such that

$$T = U^* M_{\gamma^2} U. \quad (3.10)$$

Here M_{γ^2} denotes the operator of multiplication on $L^2(\Gamma, \mu)$ by the function γ^2 . Using this representation one can readily check that

$$\begin{aligned}\operatorname{dom}(|T|) &= \{h \in \mathbb{H} : \gamma^2 U h \in L^2(\Gamma, \mu)\}, \\ \operatorname{dom}(|T|^{1/2}) &= \{h \in \mathbb{H} : |\gamma| U h \in L^2(\Gamma, \mu)\}.\end{aligned}\quad (3.11)$$

From (3.8) and (3.10) we obtain that $\sigma(M_{\gamma^2}) = \sigma(T) = [\alpha, \infty)$. Since the spectrum of multiplication operators is given by their essential range, we infer that

$$|\gamma(\omega)| \geq \sqrt{\alpha} \text{ for } \mu - \text{almost all } \omega \in \Gamma. \quad (3.12)$$

Let $S : \operatorname{dom}(M_{|\gamma|}) \rightarrow L^2(\Gamma, \mu)$ be the multiplication operator by the complex valued function $\sqrt{\gamma^2 + i\beta}$. Here $\sqrt{\lambda}$ denotes the principal branch of the complex square root of $\lambda \in \mathbb{C}$. Using (3.10) we have that

$$T + i\beta = U^* S^2 U. \quad (3.13)$$

From (3.12) it follows that

$$\operatorname{Re} \sqrt{\gamma^2(\omega) + i\beta} = \sqrt{\frac{\gamma^2(\omega) + \sqrt{\gamma^4(\omega) + \beta^2}}{2}} \geq |\gamma(\omega)| \geq \sqrt{\alpha} \quad (3.14)$$

for μ -almost all $\omega \in \Gamma$, which implies that $S = M_{\sqrt{\gamma^2 + i\beta}}$ is invertible with bounded inverse. We define $W : \operatorname{dom}(M_{|\gamma|}) \times L^2(\Gamma, \mu) \rightarrow L^2(\Gamma, \mu) \times L^2(\Gamma, \mu)$ by

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} S & -I \\ S & I \end{bmatrix}. \quad (3.15)$$

Since S is invertible with bounded inverse, we infer that W is invertible with bounded inverse and

$$W^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} S^{-1} & S^{-1} \\ -I & I \end{bmatrix}. \quad (3.16)$$

Let $\tilde{G} : \operatorname{dom}(M_{|\gamma|}) \times \operatorname{dom}(M_{|\gamma|}) \rightarrow L^2(\Gamma, \mu) \times L^2(\Gamma, \mu)$ be the linear operator defined by

$$\tilde{G} = \frac{1}{\sqrt{2}} \begin{bmatrix} -S & 0 \\ 0 & S \end{bmatrix}. \quad (3.17)$$

From (3.14) we obtain that the linear operator $-S = M_{-\sqrt{\gamma^2 + i\beta}}$ generates a C^0 -semigroup having decay rate $-\nu$, for some $\nu \geq \sqrt{\alpha}$. Hence, $\tilde{G}$ generates a stable bi-semigroup having decay rate $-\nu$. From (3.13) and (3.15)–(3.17) we conclude that

$$G = \begin{bmatrix} 0 & I \\ i\beta + T & 0 \end{bmatrix} = \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix} W^{-1} \tilde{G} W \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}, \quad (3.18)$$

proving the lemma. ■

Lemma 3.3. *Assume Hypotheses (H1)–(H3) and $\operatorname{Re} \lambda_0 > \sup \operatorname{Re} \sigma_{\operatorname{ess}}(\mathcal{L})$. Then, the linear operator $A_{\pm}$ defined in (3.5) generates an exponentially stable bi-semigroup on the Hilbert space $H_0^1(\Omega) \times L^2(\Omega)$ having decay rate $-\nu_{\pm}$ satisfying the condition $\nu_{\pm} > |c|/2$.*

Proof. The lemma follows from Lemma 3.2 above and Theorem 1.2(iv), describing the connection between $\sigma_{\text{ess}}(\mathcal{L})$ and $\sigma(\partial_x^2 + V_{\pm})$. Indeed,

$$T_{\pm} = \text{Re } \lambda_0 + \frac{c^2}{4} - \partial_x^2 - V_{\pm}(x) \quad (3.19)$$

is a closed, densely defined, self-adjoint linear operator on $L^2(\Omega)$ having domain $\text{dom}(T_{\pm}) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, since $\text{Re } \lambda_0 > \sup \text{Re } \sigma_{\text{ess}}(\mathcal{L}) = \max\{\max \sigma(\partial_x^2 + V_+), \max \sigma(\partial_x^2 + V_-)\}$, we infer that

$$\sigma(T_{\pm}) \subseteq [\alpha_*, \infty), \quad \text{where } \alpha_* = \text{Re } \lambda_0 - \sup \text{Re } \sigma_{\text{ess}}(\mathcal{L}) + \frac{c^2}{4}. \quad (3.20)$$

In addition, one can readily check that $\text{dom}(|T_{\pm}|^{1/2}) = H_0^1(\Omega)$. Applying Lemma 3.2, we conclude that

$$A_{\pm} = \begin{bmatrix} 0 & I \\ i\text{Im } \lambda_0 + T_{\pm} & 0 \end{bmatrix} \quad (3.21)$$

generates an exponentially stable bi-semigroup on $H_0^1(\Omega) \times L^2(\Omega)$ having decay rate $-\nu_{\pm}$. From (3.20) and since $\text{Re } \lambda_0 > \sup \text{Re } \sigma_{\text{ess}}(\mathcal{L})$ it follows that $\nu_{\pm} \geq \sqrt{\alpha_*} > |c|/2$. ■

We turn our attention to equation $Y' = A(z)Y$. We note that it is equivalent to

$$Y' = (A_{\pm} + B_{\pm}(z))Y, \quad (3.22)$$

where the operator valued function $B_{\pm} : \mathbb{R} \rightarrow \mathcal{B}(H_0^1(\Omega) \times L^2(\Omega))$ is defined by

$$B_{\pm}(z) = \begin{bmatrix} 0 & 0 \\ V_{\pm}(\cdot) - V(\cdot, z) & 0 \end{bmatrix}. \quad (3.23)$$

Lemma 3.4. *Assume Hypotheses (H1)–(H3). Then, the following assertions hold true:*

- (i) $B_{\pm}$ is strongly continuous;
- (ii) $\|B_{\pm}(\cdot)\|_{\mathcal{B}(H_0^1(\Omega) \times L^2(\Omega))} \in L^1(\mathbb{R}_{\pm}) \cap L^{\infty}(\mathbb{R}_{\pm})$.

Proof. Let $f \in H_0^1(\Omega)$ and $g \in L^2(\Omega)$. From (3.23) we have that

$$(B_{\pm}(z_1) - B_{\pm}(z_2))(f, g)^T = (0, V(\cdot, z_1)f - V(\cdot, z_2)f)^T \text{ for any } z_1, z_2 \in \mathbb{R}.$$

Assertion (i) follows from Lebesgue Dominated Convergence Theorem and the fact that $V(\cdot, \cdot)$ is bounded and continuous on $\Omega \times \mathbb{R}$. Moreover,

$$\begin{aligned} \|B_{\pm}(z)(f, g)^T\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 &= \int_{\mathbb{R}} |V(x, z) - V_{\pm}(x)|^2 |f(x)|^2 dx \\ &\leq \|f\|_{L^2(\Omega)}^2 \|V(\cdot, z) - V_{\pm}(\cdot)\|_{L^{\infty}(\Omega)}^2 \\ &\leq \|f\|_{H^1(\Omega)}^2 \|V(\cdot, z) - V_{\pm}(\cdot)\|_{L^{\infty}(\Omega)}^2 \end{aligned}$$

which implies that $\|B_{\pm}(z)\| \leq \|V(\cdot, z) - V_{\pm}(\cdot)\|_{L^{\infty}(\Omega)}$ for any $z \in \mathbb{R}$. From Hypothesis (H2), we immediately infer that $\|B_{\pm}(\cdot)\| \in L^1(\mathbb{R}_{\pm}) \cap L^{\infty}(\mathbb{R}_{\pm})$. ■

In what follows, we denote by $\{T_{s/u}^\pm(z)\}_{z \geq 0}$ the semigroups generated by $A_\pm$ on its invariant stable/unstable subspaces $\mathbb{H}_{s/u}^\pm$. Furthermore, we denote by $P_{s/u}^\pm$ the projection into $\mathbb{H}_{s/u}^\pm$ parallel to $\mathbb{H}_{u/s}^\pm$. We recall from [24] the definition of mild solutions of the not well-posed Eq. (3.22).

Definition 3.5. We say that a continuous function $Y : [a, b] \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ is a *mild solution* of (3.22) on $[a, b]$ if the following equation holds:

$$\begin{aligned} Y(z) = & T_s^\pm(z-a)P_s^\pm Y(a) + \int_a^z T_s^\pm(z-\zeta)P_s^\pm B_\pm(\zeta)Y(\zeta)d\zeta \\ & + T_u^\pm(b-z)P_u^\pm Y(b) - \int_z^b T_u^\pm(\zeta-z)P_u^\pm B_\pm(\zeta)Y(\zeta)d\zeta \end{aligned} \quad (3.24)$$

for any $z \in [a, b]$.

Remark 3.6. Using a frequency domain reformulation, it was shown in [16] that (3.24) is equivalent to

$$((\mathcal{F} - M_{R_\pm} \mathcal{F} M_{B_\pm})Y|_{[a,b]})(\xi) = R_\pm(\xi)(e^{-2\pi i \xi a} Y(a) - e^{-2\pi i \xi b} Y(b)) \quad (3.25)$$

for any $\xi \in \mathbb{R}$, where $R_\pm : \mathbb{R} \rightarrow \mathcal{B}(H_0^1(\Omega) \times L^2(\Omega))$ is defined by $R_\pm(\xi) = (2\pi i \xi - A_\pm)^{-1}$, $M_{R_\pm}$ and $M_{B_\pm}$ denote the operators of multiplication on $L^2(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$ by the operator valued functions $R_\pm$ and $B_\pm$, respectively.

Remark 3.7. Going back to the eigenvalue problems (3.1)–(3.2), since $u_0 \in H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ and $v_0(x, z) = e^{cz/2} u_0(x, z)$, we can immediately infer that the function $Y_0 : \mathbb{R} \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ defined by

$$Y_0(z) = (v_0(\cdot, z), \partial_z v_0(\cdot, z))^T \quad (3.26)$$

is continuous. Taking Fourier transform in (3.2), we infer that Y_0 satisfies Eq. (3.25) for any $a < b$. From Remark 3.6 conclude that Y_0 is a mild solution of equation $Y' = (A_\pm + B_\pm(z))Y$ on $[a, b]$ for any $a < b$.

Next, we study the decay rates of mild solutions of Eq. (3.22). We use the following abstract lemma that extends the results of Daletskii and Krein ([7]) from the case of well-posed equations.

Lemma 3.8. Assume G is the generator of an exponentially stable bi-semigroup $\{T_{s/u}(z)\}_{z \geq 0}$ on a Hilbert space $\mathbb{H}$ having decay rate $-\nu$ for some $\nu > 0$, $F : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{H})$ is a piecewise strongly continuous operator valued function such that $\|F(\cdot)\|_{\mathcal{B}(\mathbb{H})} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. If u is a mild solution of equation $u' = (G + F(z))u$ on $[a, \infty)$ satisfying the condition $\|u(z)\| \leq Me^{\theta z}$ for any $z \geq a$, for some $\theta < \nu$, then for any $\delta \in (0, \nu)$ there exists $N > 0$ such that $\|u(z)\| \leq Ne^{-\delta z}$ for any $z \geq a$.

Proof. First, we recall the following standard notations: throughout this lemma we denote by $\mathbb{H}_{s/u}$ the stable/unstable subspaces of G and by $P_{s/u}$ the projections into $\mathbb{H}_{s/u}$ parallel to $\mathbb{H}_{u/s}$. Next, we note that

$$T_u(\cdot - z)P_u F(\cdot)u(\cdot) \in L^1([z, \infty), \mathbb{H}) \quad \text{for any } z \geq a. \quad (3.27)$$

Indeed, one can readily check that

$$\|T_u(\zeta - z)P_u F(\zeta)u(\zeta)\| \leq M e^{-\nu(\zeta-z)} e^{\theta\zeta} = (M e^{\nu z}) e^{-(\nu-\theta)\zeta} \quad (3.28)$$

for any $\zeta \geq z \geq a$. Since $\theta < \nu$ assertion (3.27) follows shortly. Since u is a mild solution of equation $u' = (G + F(z))u$ on $[a, b]$ for any $b > a$, we have that

$$\begin{aligned} u(z) &= T_s(z-a)P_s u(a) + T_u(b-z)P_u u(b) + \int_a^z T_s(z-\zeta)P_s F(\zeta)u(\zeta)d\zeta \\ &\quad - \int_z^b T_u(\zeta-z)P_u F(\zeta)u(\zeta)d\zeta \quad \text{for any } z \in [a, b]. \end{aligned} \quad (3.29)$$

From (3.28) it follows that $\lim_{b \rightarrow \infty} T_u(b-z)P_u u(b) = 0$ for any $z \geq a$. Passing to the limit in (3.29), from (3.27) we obtain that

$$\begin{aligned} u(z) &= T_s(z-a)P_s u(a) + \int_a^z T_s(z-\zeta)P_s F(\zeta)u(\zeta)d\zeta \\ &\quad - \int_z^\infty T_u(\zeta-z)P_u F(\zeta)u(\zeta)d\zeta \quad \text{for any } z \geq a. \end{aligned} \quad (3.30)$$

We conclude that

$$\|u(z)\| \leq M e^{-\nu(z-a)} + M \int_a^\infty e^{-\nu|z-\zeta|} \|F(\zeta)\|_{\mathcal{B}(\mathbb{H})} \|u(\zeta)\| d\zeta \quad \text{for any } z \geq a. \quad (3.31)$$

Since $\|F(\cdot)\|_{\mathcal{B}(\mathbb{H})} \in L^1(\mathbb{R}_+)$ from [7, Chapter III, Lemma 2.2] we infer that for any $\delta \in (0, \nu)$ there exists $N > 0$ such that $\|u(z)\| \leq N e^{-\delta z}$ for any $z \geq a$. $\blacksquare$

Next, we prove that the function Y_0 decays exponentially, which allows us to conclude that v_0 is a genuine eigenfunction.

Lemma 3.9. *Assume Hypotheses (H1)–(H3), $\operatorname{Re} \lambda_0 > \sup \operatorname{Re} \sigma_{\text{ess}}(\mathcal{L})$. We recall the definition of the function v_0 introduced in (3.2). Then, the following assertions hold true:*

(i) *There exist $M, \delta > 0$ such that*

$$\|v_0(\cdot, z)\|_{H_0^1(\Omega)}^2 + \|\partial_z v_0(\cdot, z)\|_{L^2(\Omega)}^2 \leq M^2 e^{-2\delta|z|} \quad \text{for any } z \in \mathbb{R}; \quad (3.32)$$

(ii) *The function v_0 belongs to $H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ and therefore is an eigenfunction of the linear operator $\Delta_{x,z} + V - c^2/4$.*

Proof. (i) From Lemma 3.3 we have that the linear operator $A_\pm$ generates an exponentially stable bi-semigroups on $H_0^1(\Omega) \times L^2(\Omega)$ having decay rate $-\nu_\pm$ such that $\nu_\pm > |c|/2$. Therefore, we can choose $\delta > 0$ such that $\nu_\pm > \delta > |c|/2$. Since $Y_0(z) = (v_0(\cdot, z), \partial_z v_0(\cdot, z))^T$ is a mild solution of equation $Y_0' = (A_+ + B_+(z))Y_0$ by Remark 3.7, from Lemma 3.4 and Lemma 3.8 we infer that there exists $M > 0$

$$\|Y_0(z)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq M e^{-\delta z} \quad \text{for any } z \geq 0. \quad (3.33)$$

From the definition of bi-semigroups, one can readily check that $-A_-$ generates an exponentially stable bi-semigroup on $H_0^1(\Omega) \times L^2(\Omega)$. Making the change of variables $z \rightarrow -z$ in equation $Y_0' = (A_- + B_-(z))Y_0$, it follows that

$Y_0(\cdot)$ is a mild solution of equation $Y' = (-A_- - B_-(-z))Y$. Using again Lemma 3.4, Remark 3.7 and Lemma 3.8 we conclude that

$$\|Y_0(-z)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq M e^{-\delta z} \quad \text{for any } z \geq 0. \quad (3.34)$$

The estimate (3.32) follows shortly from (3.33) and (3.34).

(ii) Since $u_0 \in H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ and $v_0(x, z) = e^{cz/2} u_0(x, z)$ for any $x \in \Omega$ and $z \in \mathbb{R}$, from (3.32) we immediately conclude that $v_0 \in H_0^1(\Omega \times \mathbb{R})$. To prove (ii) we use that v_0 satisfies Eq. (3.2). We introduce $\kappa_n \in C_0^\infty(\mathbb{R})$ such that $0 \leq \kappa_n \leq 1$, $\kappa_n(z) = 1$ for any $z \in [-n, n]$, $\text{supp } \kappa_n$ is compact and the derivatives of κ_n satisfy

$$\sup_{n \geq 1} \|\kappa_n^{(j)}\|_\infty < \infty \quad \text{for } j = 1, 2. \quad (3.35)$$

Let $v_n : \Omega \times \mathbb{R} \rightarrow \mathbb{C}$ be the function defined by $v_n(x, z) = \kappa_n(z) v_0(x, z)$. Since $v_n(x, z) = e^{cz/2} \kappa_n(z) u_0(x, z)$ for any $n \geq 1$, $x \in \Omega$, $z \in \mathbb{R}$, $\kappa_n \in C_0^\infty(\mathbb{R})$ for any $n \geq 1$ and $u_0 \in H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ we obtain that $v_n \in H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ and

$$(\Delta v_n)(x, z) = 2\kappa_n'(z) \partial_z v_0(x, z) + \left(\kappa_n''(z) - \left(\frac{c^2}{4} + \lambda_0 - V(x, z) \right) \kappa_n(z) \right) v_0(x, z) \quad (3.36)$$

for any $n \geq 1$, $x, z \in \mathbb{R}$. Since $v_0 \in H_0^1(\Omega \times \mathbb{R})$, from (i), (3.35) and (3.36) it follows that $\sup_{n \geq 1} \|(\Delta_{x,z} - I)v_n\|_{L^2(\Omega \times \mathbb{R})} < \infty$. Using that $\Delta_{x,z} - I$ is invertible with bounded inverse from $H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$ to $L^2(\Omega \times \mathbb{R})$ we obtain that $\sup_{n \geq 1} \|v_n\|_{H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})} < \infty$. Since $\kappa_n(z) = 1$ for any $n \geq 1$ and $z \in [-n, n]$, we conclude that $v_0 \in H^2(\Omega \times \mathbb{R}) \cap H_0^1(\Omega \times \mathbb{R})$, proving (ii) and the lemma. ■

Proof of Theorem 1.3. Fix $\lambda_0 \in \sigma_d(\mathcal{L})$ such that $\text{Re } \lambda_0 > \sup \text{Re } \sigma_{\text{ess}}(\mathcal{L})$. From (3.2) and Lemma 3.9(ii) we have that $\lambda_0 \in \sigma_d(\Delta_{x,z} + V - c^2/4)$. Since the linear operator $\Delta_{x,z} + V - c^2/4$ is self-adjoint we conclude that $\lambda_0 \in \mathbb{R}$, proving the theorem. ■

4. Discussion: The Finite-Dimensional Case

In this section we describe how to use the methods from this paper to recover the results from [6, 13] similar to Theorem 1.3. Let $\bar{u}(y - ct)$ be a traveling wave of the Allen–Cahn equation

$$u_t + F'(u) = u_{yy}, \quad t \geq 0, \quad y \in \mathbb{R}, \quad (4.1)$$

and assume its profile is such that $u_\pm = \lim_{z \rightarrow \pm \infty} \bar{u}(z)$ exists. The linearization along the wave is given by $\mathcal{L}_* = \partial_z^2 + c \partial_z - F''(\bar{u}(z))$ in the moving frame $z = y - ct$.

Theorem 4.1. ([6, 13]) *Assume that the profile $\bar{u}$ satisfies the condition*

$$\int_{-\infty}^s (1 + |x|) |\bar{u}(x) - u_-| dx < \infty \quad \text{and} \quad \int_s^\infty (1 + |x|) |\bar{u}(x) - u_+| dx < \infty \quad (4.2)$$

for any $s \in \mathbb{R}$. Then, the discrete spectrum of $\mathcal{L}_*$ to the right of the right most point of its essential spectrum is real.

To prove this result, the authors make the change of variables $v_*(z) = e^{-c/2z}u_*(z)$ in the eigenvalue equation $\mathcal{L}_*u_* = \lambda_0u_*$, where $\lambda_0 \in \sigma_d(\mathcal{L}_*)$ with $\operatorname{Re}\lambda_0 > \sup \operatorname{Re} \sigma_{\text{ess}}(\mathcal{L}_*)$. Then, they control the decay rate of the eigenfunction u_* by computing the decay rate of the Jost solutions associated to the eigenvalue problem. This argument allows then to conclude that v_* is a genuine eigenfunction of the self-adjoint operator $\partial_z^2 - c^2/4 - F''(\bar{u}(\cdot))$ associated to the eigenvalue λ_0 , which proves the statement.

The key part of this argument is to control the decay rate of the eigenfunction u_* . In the infinite-dimensional case its not straitforward to construct the Jost solutions of the not well-posed, first order Eq. (3.3), nor to conclude from here that the decay rate of the eigenfunction u_0 introduced in (3.1) can be evaluated using the decay rates of the (infinitely many) Jost solutions. In fact, in [17] it is shown that the construction of Jost solutions associated to (3.3) requires a significant effort in this infinite-dimensional, not well-posed case. The method we used to prove Theorem 1.3 can be applied to control the growth rate of the eigenfunction u_* by recasting the second order equation

$$v_*'' - \left(\frac{c^2}{4} + F''(\bar{u}(z)) \right) v_* = \lambda_0 v_* \quad (4.3)$$

as a first order, linear differential equation on some finite-dimensional space, whose decay rates can be estimated using Lemma 3.8. Moreover, we can prove the result by assuming that $\bar{u} - u_{\pm} \in L^1(\mathbb{R}_{\pm}) \cap L^{\infty}(\mathbb{R}_{\pm})$, thus relaxing assumption (4.2). Indeed, as shown in [13], if $w_* = v'_*$ then the pair (v_*, w_*) satisfies the equation

$$\frac{d}{dz} \begin{pmatrix} v_* \\ w_* \end{pmatrix} = A_*(z) \begin{pmatrix} v_* \\ w_* \end{pmatrix}, \quad (4.4)$$

where $A_*(z)$ is the matrix-valued function defined by

$$A_*(z) = \begin{bmatrix} 0 & I \\ \lambda_0 + \frac{c^2}{4} + F''(\bar{u}(z)) & 0 \end{bmatrix}. \quad (4.5)$$

Similar to the infinite-dimensional case (3.3), we have the decomposition $A_*(z) = A_{*,\pm} + B_{*,\pm}(z)$, where

$$A_{*,\pm} = \begin{bmatrix} 0 & I \\ \lambda_0 + \frac{c^2}{4} + F''(u_{\pm}) & 0 \end{bmatrix}, \quad B_{*,\pm}(z) = \begin{bmatrix} 0 & 0 \\ F''(u_{\pm}) - F''(\bar{u}(z)) & 0 \end{bmatrix}. \quad (4.6)$$

Similar to the proof of Lemma 3.3, using Lemma 3.2 and the condition $\operatorname{Re}\lambda_0 > \sup \operatorname{Re} \sigma_{\text{ess}}(\mathcal{L}_*)$, we can show that $A_{*,\pm}$ is a hyperbolic matrix, hence it generates a bi-semigroup. Since F is a smooth function (typically one assumes that F is of class C^2) and $\bar{u} - u_{\pm} \in L^1(\mathbb{R}_{\pm}) \cap L^{\infty}(\mathbb{R}_{\pm})$ it follows that $\|B_{*,\pm}(\cdot)\| \in L^1(\mathbb{R}_{\pm}) \cap L^{\infty}(\mathbb{R}_{\pm})$. Next, we can apply Lemma 3.8 to prove that $\|(v_*(z), w_*(z))^T\| \leq Me^{-\delta|z|}$ for any $z \in \mathbb{R}$, for some $M, \delta > 0$. From here we can immediately conclude that v_* is a genuine eigenfunction of the

self-adjoint operator $\partial_z^2 - c^2/4 - F''(\bar{u}(\cdot))$ associated to the eigenvalue λ_0 , proving that λ_0 is real.

Acknowledgements

The authors gratefully thank the Referee for the constructive comments and recommendations.

References

- [1] Bart, H., Gohberg, I., Kaashoek, M.A.: Wiener–Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators. *J. Funct. Anal.* **68**(1), 1–42 (1986)
- [2] Ben-Artzi, A., Gohberg, I.: Dichotomy of systems and invertibility of linear ordinary differential operators. In: Gohberg, I. (ed.) *Time-Variant Systems and Interpolation, Operator Theory: Advances and Applications*, vol. 56, pp. 91–119. Birkhäuser, Basel (1992)
- [3] Berestycki, H., Larrouturou, B., Lions, P.L.: Multi-dimensional travelling-wave solutions of a flame propagation model. *Arch. Ration. Mech. Anal.* **111**, 33–49 (1990)
- [4] Berestycki, H., Nirenberg, L.: Some Qualitative Properties of Solutions of Semilinear Elliptic Equations in Cylindrical Domains, in *Analysis, et cetera*, pp. 115–164. Academic Press, Boston (1990)
- [5] Berestycki, H.: The influence of advection on the propagation of fronts in reaction-diffusion equations. In: Berestycki, H., Pomeau, Y. (eds.) *Nonlinear PDEs in Condensed Matter and Reactive Flows*. NATO Science Series C, vol. 569. Kluwer Academic, Dordrecht (2003)
- [6] Bose, A., Jones, C.K.R.T.: Stability of the in-phase travelling wave solution in a pair of coupled nerve fibers. *Indiana Univ. Math. J.* **44**(1), 189–220 (1995)
- [7] Daletskii, Y.L., Krein, M.G.: *Stability of Solutions of Differential Equations in Banach Spaces*. AMS Translation, Providence (1974)
- [8] Edmunds, D.E., Evans, W.D.: *Spectral Theory and Differential Operators*, Oxford Mathematical Monographs. Oxford University Press, New York (1987)
- [9] Fiedler, B., Scheel, A., Vishik, M.I.: Large patterns of elliptic systems in infinite cylinders. *J. Math. Pures Appl.* **77**, 879–907 (1998)
- [10] Hamel, F., Roquejoffre, J.M.: Heteroclinic connections for multidimensional bistable reaction-diffusion equations. *Discrete Contin. Dyn. Syst.* **4**, 101–123 (2011)
- [11] Haragus, M., Scheel, A.: A bifurcation approach to non-planar traveling waves in reaction-diffusion systems. *GAMM-Mitteilungen* **30**, 75–95 (2007)
- [12] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840. Springer, New York (1981)
- [13] Howard, P., Latushkin, Y., Sukhtayev, A.: The Maslov and Morse indices for Schrödinger operators on $\mathbb{R}$. *Indiana Univ. Math. J.* **(to appear)**
- [14] Kapitula, T., Promislow, K.: *Spectral and Dynamical Stability of Nonlinear Waves*. With a foreword by Christopher K. R. T. Jones. Applied Mathematical Sciences, vol. 185. Springer, New York (2013)

- [15] Kirchgasner, K.: Wave-solutions of reversible systems and applications. *J. Differ. Equ.* **45**, 113–127 (1982)
- [16] Latushkin, Y., Pogan, A.: The dichotomy theorem for evolution bi-families. *J. Differ. Equ.* **245**(8), 2267–2306 (2008)
- [17] Latushkin, Y., Pogan, A.: The infinite dimensional Evans function. *J. Funct. Anal.* **268**(6), 1509–1586 (2015)
- [18] Lord, G.J., Peterhof, D., Sandstede, B., Scheel, A.: Numerical computation of solitary waves in infinite cylindrical domains. *SIAM J. Numer. Anal.* **37**(5), 1420–1454 (2000)
- [19] Morita, Y., Ninomiya, H.: Monostable-type traveling waves of bistable reaction-diffusion equations in the multi-dimensional space. *Bull. Inst. Math. Acad. Sin. (N.S.)* **3**(4), 567–584 (2008)
- [20] Mielke, A.: A reduction principle for nonautonomous systems in infinite-dimensional spaces. *J. Differ. Equ.* **65**, 68–88 (1986)
- [21] Mielke, A.: *Hamiltonian and Lagrangian Flows on Center Manifolds with Applications to Elliptic Variational Problems*, Lecture Notes in Mathematics 1489. Springer, Berlin (1991)
- [22] Palmer, K.J.: Exponential dichotomies and transversal homoclinic points. *J. Differ. Equ.* **55**, 225–256 (1984)
- [23] Palmer, K.J.: Exponential dichotomies and Fredholm operators. *Proc. Am. Math. Soc.* **104**, 149–156 (1988)
- [24] Peterhof, D., Sandstede, B., Scheel, A.: Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders. *J. Differ. Equ.* **140**(2), 266–308 (1997)
- [25] Rabinowitz, P.H.: Solutions of heteroclinic type for some classes of semilinear elliptic partial differential equations. *J. Math. Sci. Univ. Tokyo* **1**, 525–550 (1994)
- [26] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics, IV. Analysis of operators*. Academic Press, New York-London (1978)
- [27] Sandstede, B.: Stability of Traveling Waves, *Handbook of Dynamical Systems*, vol. 2, pp. 983–1055. North-Holland, Amsterdam (2002)
- [28] Sandstede, B., Scheel, A.: On the structure of spectra of modulated travelling waves. *Math. Nachr.* **232**, 39–93 (2001)
- [29] Volpert, A.I., Volpert, V.A., Volpert, V.A.: *Traveling Wave Solutions of Parabolic Systems (Mathematical Monographs Trans.)*, vol. 140. American Mathematical Society, Providence (1994)
- [30] Xin, J.: Analysis and modelling of front propagation in heterogeneous media. *SIAM Rev.* **42**, 161–230 (2000)

Anna Ghazaryan and Alin Pogan (✉)

Department of Mathematics

Miami University

Oxford, OH 45056

USA

e-mail: pogana@miamioh.edu

URL: <http://www.users.miamioh.edu/pogana/>

Anna Ghazaryan

e-mail: ghazarar@miamioh.edu

URL: <http://www.users.miamioh.edu/ghazarar/>

Yuri Latushkin

Department of Mathematics

University of Missouri

Columbia, MO 65211

USA

e-mail: latushkiny@missouri.edu

URL: <http://www.math.missouri.edu/personnel/faculty/latushkiny.html>

Received: October 2, 2017.

Revised: March 9, 2018.