SIAM J. SCI. COMPUT.
Vol. 40, No. 4, pp. B1180–B1205

TWO-LEVEL SPECTRAL METHODS FOR NONLINEAR ELLIPTIC EQUATIONS WITH MULTIPLE SOLUTIONS*

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Abstract. The present paper provides a two-level framework based on spectral methods and homotopy continuation for solving second-order nonlinear boundary value problems exhibiting multiple solutions. Our proposed method consists of two steps: (i) solving the nonlinear problems using low-order polynomials or a small number of collocation points, and (ii) solving the corresponding linearized problems by high-order polynomials or a large number of collocation points. The resulting two-level spectral method enjoys the following merits: (i) it guarantees multiple solutions, (ii) the computational cost is relatively small, and (iii) it is of proven high-order accuracy. These claims are supported by the detailed error estimates for semilinear equations and extensive numerical experiments of both semilinear and fully nonlinear equations.

Key words. nonlinear elliptic equation, multiple solutions, two-level algorithm, spectral method, polynomial system, homotopy continuation

AMS subject classifications. 65H10, 65L10, 65N35, 65N55

DOI. 10.1137/17M113767X

1. Introduction.

In recent years, there has been a growing interest in the study of nonlinear differential equations with multiple solutions since they provide a powerful tool to quantitatively describe the important features of real-world phenomena, such as conservation, diffusion, equilibrium, motion, pattern, reaction, and so on. Applications can be found in many areas of science and engineering including astrophysics, combustion theory, differential geometry, economics, general relativity, mathematical biology, meteorology, optimal transport, and shape optimization [35, 31, 8, 18, 44, 43, 16]. Unfortunately, only very few nonlinear differential equations have known exact solutions, but many more, which are important in scientific and engineering applications, are not solvable in an explicit form. Hence, it is necessary to develop efficient and accurate numerical algorithms for nonlinear differential equations with multiple solutions.

In this work, we consider the second-order elliptic equations with nonlinearity of polynomial type, in which all nonlinearities, with respect to the solution and its derivatives, appear in a polynomial form, i.e.,

$$F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) + f(x, y) = 0, \quad (x, y) \in \Omega, \tag{1.1}$$

where $\Omega \in \mathbb{R}^2$ is an open bounded domain, $f(x, y) \in C^\infty(\Omega)$, and the function

*Submitted to the journal's Computational Methods in Science and Engineering section July 7, 2017; accepted for publication (in revised form) May 17, 2018; published electronically August 16, 2018.

http://www.siam.org/journals/sisc/40-4/M113767.html

Funding: The second author was supported by the American Heart Association (17SDG33660722) and the National Science Foundation (DMS-1818769). The third author was supported by the National Science Foundation (DMS-1555072 and DMS-1736364) and the National Natural Science Foundation of China (51728601).

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$F(X_i; 1 \leq i \leq 6) : \mathbb{R}^6 \to \mathbb{R}^1$ is a polynomial with respect to each variable X_i. A specific semilinear example is $\Delta u + g(u) = 0$, where $g(u)$ is a polynomial of u, i.e., $g(u) = \sum_{k=0}^{K} c_k(x,y)u^k$, $c_k \in C^\infty(\Omega)$, $k = 0, 1, \ldots, K$. Besides, the differential equation (1.1) could be imposed with various types of boundary conditions including Dirichlet, Neumann, Robin, or periodical. For the discussion on existence, regularity, and multiplicity of the solutions to the above problems, see [28, 7, 33, 34, 6, 25, 36, 24, 9] and the references therein.

Recent years have witnessed substantial progress in the development of numerical methods for seeking multiple solutions of semilinear elliptic equations. One is the variational-type method based on optimization algorithms, including the mountain pass algorithm [13], high linking algorithm [14], mini-max algorithm [27, 49], etc.; another is the classical numerical approach—to discretize the differential equations with proper numerical methods and then to solve the resulting nonlinear system of equations. We will adopt the latter approach in this paper.

In general, it is hard to solve the resulting nonlinear systems after numerical discretization of the original nonlinear differential equations, due to the multiplicity of the solutions. A typical approach to find nontrivial distinct solutions is to start a Newton-type iteration with many different initial guesses, with the hope of finding the solutions that lie in different basins of attraction. Unfortunately, the drawback of Newton iteration is that it is very sensitive to initial guess and becomes very expensive for calculating the inverse of the Jacobian matrix at each iteration step. Several methods have been developed to overcome these difficulties. Popular strategies are the search-extension method [11, 10, 46, 12] and the eigenfunction expansion method [50]. The basic idea is that they use the eigenfunctions of Laplacian operator as the starting points in the iterations. Another useful tool is the deflation technique [15], which enables the Newton iteration to converge to several different solutions even when starting from the same initial guess. In addition, a bootstrapping approach based on the finite difference method and homotopy continuation was proposed for computing multiple solutions [20, 21]. Most recently, Boyd proposed a degree-increasing spectral homotopy based on Chebyshev and Fourier spectral methods, and showed the numerical results for several one-dimensional problems [4, 5].

The basic idea of the two-grid method based on finite element methods is to get a rough approximation on the coarse space and use it as an initial guess on the fine grid, which has been used for solving nonsymmetric linear and semilinear elliptic and parabolic problems [47, 30, 48, 1, 23]. Here we employ two polynomial spaces in spectral methods—*coarse* space X_{N_c} and *fine* space X_{N_f} with $N_c \ll N_f$—to solve the elliptic equations with polynomial nonlinearity shown in (1.1), which has the following two steps:

1. First, the original nonlinear problem is discretized by spectral Galerkin approximation with Legendre polynomials of low degree, or by the Chebyshev collocation method with a small number of collocation points. Besides, the polynomial systems after spectral discretization are solved by homotopy continuation to obtain the numerical solutions with multiplicity.
2. Second, we solve the corresponding linearized problems using higher order Legendre polynomial approximation or a larger number of collocation points, in which the solutions obtained in the first step are chosen as starting points.

Spectral methods [37, 38, 39] are employed here to discretize the differential equations due to their optimal convergence rates, which are restricted only by the regularity of the solutions. Since the nonlinear equations considered here are algebraic, the resulting system is of polynomial type. In recent years, remarkable progress has been made in the development and implementation of efficient algorithms to numerically solve and manipulate the solutions of systems of polynomials, which is called *numerical algebraic geometry* (NAG). We employ the homotopy continuation method proposed in [42, 22, 45, 3] to solve the polynomial systems arising from spectral Galerkin or collocation discretizations of the nonlinear differential equations. The significant advantage of our proposed methods is that the multiple solutions can be obtained with high accuracy but relatively low computational cost.

The remainder of this paper is organized as follows. The detailed algorithms of our proposed two-level framework based on spectral methods are presented in section 2. In section 3, we offer a brief introduction to the numerical algebraic geometry and homotopy continuation method for polynomial systems. The spectral-type error estimates for the one-dimensional semilinear problem are provided in section 4. In section 5, we present several numerical experiments to demonstrate the accuracy, efficiency, and robustness of our proposed methods. Finally, section 6 summarizes and concludes the presented work.

2. A two-level framework of spectral methods.

We begin by considering the following one-dimensional nonlinear elliptic problems with homogeneous Dirichlet boundary conditions:

$$\begin{cases} F^{\text{simple}}(u) := (u_{xx})^q + \lambda u^p + f(x) = 0, \quad x \in I = (-1,1), \\ u(-1) = 0, \quad u(1) = 0, \end{cases} \tag{2.1}$$

where p, q are nonnegative integers, $\lambda \in \mathbb{R}$ is a known parameter, and f is a sufficiently smooth function.

The Newton iteration of the problem (2.1) can be written as

$$\begin{cases} q\left(u^{[n]}\right)^{q-1} V_{xx} + \lambda p \left(u^{[n]}\right)^{p-1} V + F^{\text{simple}}\left(u^{[n]}\right) = 0, \quad V(-1) = V(1) = 0, \\ u^{[n+1]} = u^{[n]} + V, \quad n \geq 0. \end{cases} \tag{2.2}$$

The general description of the two-level framework based on spectral methods for the problem (2.1) is shown in Algorithm 1, while the detailed techniques are shown in sections 2.1 and 2.2. It should be pointed out that although the nonlinear terms considered here are $(u_{xx})^q$ and λu^p, actually our algorithm also works for any term with polynomial nonlinearity of the form (1.1).

2.1. Legendre–Galerkin method.

Let $\mathcal{P}_N$ be the set of all polynomials of degree at most N, and define the approximation space as

$$X_N := \{p \in \mathcal{P}_N : \ p(\pm 1) = 0\}. \tag{2.3}$$

Let us denote the inner product $(u, v) = \int_{-1}^{1} u(x)v(x)\mathrm{d}x$. Then the weak formulation for the nonlinear problem (2.1) as well as its linearized problem (2.2) are as follows:

- The Legendre–Galerkin formulation for the problem (2.1) is to find $u_N \in X_N$ such that $\forall v_N \in X_N$,

$$((\partial_{xx} u_N)^q, v_N) + \lambda(u_N^p, v_N) + (f, v_N) = 0. \tag{2.4}$$

- In each iteration with given $u^{[n]}$ in the problem (2.2), the Legendre–Galerkin formulation is to find $V_N \in X_N$ such that $\forall v_N \in X_N$,

$$\left(q\left(\partial_{xx} u_N^{[n]}\right)^{q-1} \partial_{xx} V_N, v_N\right) + \lambda p\left(\left(u_N^{[n]}\right)^{p-1} V_N, v_N\right) + \left(F^{\text{simple}}(u_N^{[n]}), v_N\right) = 0, \tag{2.5}$$

where $F^{\text{simple}}(\cdot)$ is defined in (2.1).

Define the Legendre basis functions $\{\phi_k(x)\}_{k=0}^{N-2}$ as [37]

$$\phi_k(x) = L_k(x) - L_{k+2}(x), \tag{2.6}$$

where $L_k(x)$ is the Legendre polynomial of degree k. It is easy to check that $\phi_k(\pm 1) = 0\ \forall k \in \mathbb{N}$. Then we have

$$X_N = \operatorname{span}\{\phi_k(x)\}_{k=0}^{N-2}. \tag{2.7}$$

Next we consider the matrix forms of (2.4) and (2.5) after Legendre–Galerkin discretization.

First, expanding the solution $u(x)$ as the linear combination of basis functions in the coarse level yields

$$u_{N_c}(x) = \sum_{k=0}^{N_c-2} \hat{u}_k \phi_k(x), \tag{2.8}$$

where $\{\hat{u}_k\}_{k=0}^{N_c-2}$ are the unknown coefficients.

In order to treat the nonlinear terms in (2.4), we need to define the L_2-projection $\tilde{\Pi}_{N,p}$ from continuous function space to polynomial space $\mathcal{P}_N$, i.e.,

$$\tilde{\Pi}_{N,p}\left([v(x)]^p\right) := \sum_{k=0}^{N} \tilde{v}_k^{(p)} L_k(x) \quad \forall v \in C(I). \tag{2.9}$$

After the projections $\tilde{\Pi}_{N_c,p}$ and $\tilde{\Pi}_{N_c,q}$, the functions $u_{N_c}(x)$ and $\partial_{xx} u_{N_c}(x)$ become, respectively,

$$\tilde{\Pi}_{N_c,p}\left([u_{N_c}(x)]^p\right) = \sum_{k=0}^{N_c} \tilde{u}_k^{(p)} L_k(x), \quad \tilde{\Pi}_{N_c,q}\left([\partial_{xx} u_{N_c}(x)]^q\right) = \sum_{k=0}^{N_c} \tilde{u}_k^{(q)} L_k(x). \tag{2.10}$$

Let us rewrite $u_{N_c}(x)$ defined in (2.8) and its second derivative $\partial_{xx} u_{N_c}(x)$ as

$$u_{N_c}(x) = \sum_{k=0}^{N_c-2} \hat{u}_k \phi_k(x) = \sum_{k=0}^{N_c} \tilde{u}_k L_k(x) = \sum_{k=0}^{N_c} u_k l_k(x), \tag{2.11}$$

$$\partial_{xx} u_{N_c}(x) = \sum_{k=0}^{N_c} \tilde{u}_k'' L_k(x), \tag{2.12}$$

where $\{l_k(x)\}_{k=0}^{N_c}$ are the Lagrange basis sets corresponding to Legendre–Gauss–Lobatto nodes $\mathcal{I}_N^{leg} = \{x_k^{leg}\}_{k=0}^{N_c}$, i.e.,

$$l_k(x) = \prod_{0 \le n \le N_c, n \ne k} \frac{x - x_n^{leg}}{x_k^{leg} - x_n^{leg}}, \quad k = 0, 1, \ldots, N_c. \tag{2.13}$$

It follows that $u_k = u_{N_c}(x_k^{leg})$ for each $k = 0, 1, \ldots, N_c$. Besides, let us denote the column vectors

$$\hat{\mathbf{u}} = [\hat{u}_0, \ldots, \hat{u}_{N_c-2}]^T, \qquad \tilde{\mathbf{u}} = [\tilde{u}_0, \ldots, \tilde{u}_{N_c}]^T,$$
$$\mathbf{u} = [u_0, \ldots, u_{N_c}]^T, \qquad \bar{\mathbf{u}} = [\bar{u}_0, \ldots, \bar{u}_{N_c-2}]^T,$$
$$\tilde{\mathbf{u}}'' = [\tilde{u}_0'', \ldots, \tilde{u}_{N_c}'']^T, \qquad \tilde{\mathbf{u}}^{(p)} = [\tilde{u}_0^{(p)}, \ldots, \tilde{u}_{N_c}^{(p)}]^T, \qquad \tilde{\mathbf{u}}^{(q)} = [\tilde{u}_0^{(q)}, \ldots, \tilde{u}_{N_c}^{(q)}]^T,$$

where $\{\hat{u}_k\}_{k=0}^{N_c-2}, \{\tilde{u}_k\}_{k=0}^{N_c}, \{u_k\}_{k=0}^{N_c}$ are defined by (2.11), $\{\tilde{u}''_k\}_{k=0}^{N_c}$ are defined by (2.12), $\{\tilde{u}_k^{(p)}\}_{k=0}^{N_c}, \{\tilde{u}_k^{(q)}\}_{k=0}^{N_c}$ are defined by (2.10) and $\{\bar{u}_k\}_{k=0}^{N_c-2}$ are defined by $\bar{u}_k = (u_{N_c}(x), \phi_k(x))$. Then the transforms between these vectors are given as follows:

$$\tilde{\mathbf{u}} = \boldsymbol{B}_1\mathbf{u}, \qquad \bar{\mathbf{u}} = \boldsymbol{B}_3^T\boldsymbol{B}_2\hat{\mathbf{u}}, \qquad \tilde{\mathbf{u}} = \boldsymbol{B}_3\hat{\mathbf{u}}, \qquad \mathbf{u} = \boldsymbol{B}_4\tilde{\mathbf{u}},$$
$$\tilde{\mathbf{u}}'' = \boldsymbol{B}_5\tilde{\mathbf{u}}, \qquad \tilde{\mathbf{u}}^p = \boldsymbol{B}_1[\boldsymbol{B}_4\boldsymbol{B}_3\hat{\mathbf{u}}]_{.}^{p}, \qquad \tilde{\mathbf{u}}^q = \boldsymbol{B}_1[\boldsymbol{B}_4\boldsymbol{B}_5\boldsymbol{B}_3\hat{\mathbf{u}}]_{.}^{q},$$

where the matrices $\{\boldsymbol{B}_i\}_{i=1}^5$ are shown in Appendix A, and $[\cdot]_{.}^{p}$ means taking the power p on each entry of the corresponding vector. Now it is clear that the matrix form of the weak formulation (2.4) is a polynomial system with respect to the unknowns $\hat{\mathbf{u}}$, which reads

$$\boldsymbol{F}^{\text{galerkin}}(\hat{\mathbf{u}}) := \boldsymbol{B}_3^T\boldsymbol{B}_2\boldsymbol{B}_1[\boldsymbol{B}_4\boldsymbol{B}_5\boldsymbol{B}_3\hat{\mathbf{u}}]_{.}^{q} + \lambda\boldsymbol{B}_3^T\boldsymbol{B}_2\boldsymbol{B}_1[\boldsymbol{B}_4\boldsymbol{B}_3\hat{\mathbf{u}}]_{.}^{p} + \bar{\mathbf{f}} = 0, \tag{2.14}$$

where the vector $\bar{\mathbf{f}} = [\bar{f}_0, \ldots, \bar{f}_{N_c-2}]^T$ is defined by $\bar{f}_k = (f, \phi_k), k = 0, 1, \ldots, N_c - 2$.

We employ the homotopy method to find the multiple solutions of the polynomial system (2.14). The detailed discussion will be shown in section 3.

Suppose the number of solutions obtained from the nonlinear equation (2.14) is n_p. Now we have $\hat{\mathbf{u}}^{(i)} = [\hat{u}_0^{(i)}, \ldots, \hat{u}_{N_c-2}^{(i)}]^T$ for $i = 1, 2, \ldots, n_p$, where each $\hat{\mathbf{u}}^{(i)}$ is the coefficient of the ith solution $u_{N_c}^{(i)}(x)$ of the problem (2.4), i.e.,

$$u_{N_c}^{(i)}(x) = \sum_{k=0}^{N_c-2} \hat{u}_k^{(i)}\phi_k(x). \tag{2.15}$$

In the fine level, the starting points for the iteration (2.2) are chosen as

$$u_{N_f}^{[0],(i)}(x) = \sum_{k=0}^{N_f-2} \hat{u}_k^{[0],(i)}\phi_k(x), \quad i = 1, 2, \ldots, n_p, \tag{2.16}$$

where the coefficients $\{\hat{u}_k^{[0],(i)}\}_{k=0}^{N_f-2}$ are chosen as

$$\hat{u}_k^{[0],[i]} = \begin{cases} \hat{u}_k^{(i)}, & 0 \le k \le N_c - 2, \\ 0, & N_c - 1 \le k \le N_f - 2, \end{cases} \tag{2.17}$$

where $\{\hat{u}_k^{(i)}\}_{k=0}^{N_c-2}$ are shown in (2.15).

Given $\hat{\mathbf{u}}_i^{[0]} = [\hat{u}_0^{[0],(i)}, \ldots, \hat{u}_{N_f-2}^{[0],(i)}]^T$, the matrix form of the weak formulation (2.5) can be written as

$$\begin{aligned}\boldsymbol{B}_3^T\boldsymbol{B}_2\boldsymbol{B}_1 \Big(&q\text{diag}([\boldsymbol{B}_4\boldsymbol{B}_5\boldsymbol{B}_3\hat{\mathbf{u}}_i^{[n]}]_{.}^{q-1})\boldsymbol{B}_4\boldsymbol{B}_5 \\ &+ \lambda p\text{diag}([\boldsymbol{B}_4\boldsymbol{B}_3\hat{\mathbf{u}}_i^{[n]}]_{.}^{p-1})\boldsymbol{B}_4\Big)\boldsymbol{B}_3\boldsymbol{V}_i = \boldsymbol{F}^{\text{galerkin}}(\hat{\mathbf{u}}_i^{[n]}),\end{aligned} \tag{2.18}$$

where $\boldsymbol{F}^{\text{galerkin}}(\cdot)$ is defined by (2.14) and $\hat{\mathbf{u}}_i^{[n+1]} = \hat{\mathbf{u}}_i^{[n]} + \boldsymbol{V}_i$ for $i = 1, 2, \ldots, n_p$ and $n \ge 0$.

Note that the size of the matrices in (2.14) is $O(N_c)$, while the one in (2.18) is $O(N_f)$. Usually, N_f is much larger than N_c, which implies that we use more degrees of freedom in solving the linear problem than in solving the nonlinear problem to reduce the discretization errors.

2.2. Spectral collocation methods. Let us denote the Chebyshev–Gauss–Lobatto points as

$$\mathcal{I}_N^{cheb} = \left\{ x_j^{cheb} = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, \ldots, N \right\} \subset [-1, 1]. \tag{2.19}$$

Evaluating the functions $u(x)$ and $f(x)$ at the points $\mathcal{I}_N^{cheb}$ defined above yields the following two column vectors:

$$\mathbf{u} = (u(x_j^{cheb}))_{j=0}^N, \quad \mathbf{f} = (f(x_j^{cheb}))_{j=0}^N. \tag{2.20}$$

The Chebyshev collocation discretization of the problem (2.1) leads to the polynomial system

$$\boldsymbol{F}^{\text{collocation}}(\mathbf{u}) := \left[\boldsymbol{D}^{(2)}\mathbf{u}\right]_{.}^{q} + \lambda[\mathbf{u}]_{.}^{p} + \mathbf{f} = 0, \tag{2.21}$$

where the matrix $\boldsymbol{D}^{(2)}$ is the second-order differential matrix shown in Appendix B.

Solving (2.21) for $N = N_c$ by the homotopy method shown in section 3 gives us n_p solutions, which are denoted as

$$\mathbf{u}_{N_c}^{(i)} = [u^{(i)}(x_0^{cheb}), \ldots, u^{(i)}(x_{N_c}^{cheb})]^T, \quad i = 1, 2, \ldots, n_p. \tag{2.22}$$

The matrix form of the Chebyshev collocation method for linearized problem (2.2) is

$$\begin{cases} q\text{diag}\left(\left[\boldsymbol{D}^{(2)}\mathbf{u}_i^{[n]}\right]_{.}^{q-1}\right)\boldsymbol{D}^{(2)}\boldsymbol{V}_i + \lambda p\left(\mathbf{u}_i^{[n]}\right)^{p-1}\boldsymbol{V}_i + \boldsymbol{F}^{\text{collocation}}\left(\mathbf{u}_i^{[n]}\right) = 0, \\ \mathbf{u}_i^{[n+1]} = \mathbf{u}_i^{[n]} + \boldsymbol{V}_i, \quad n \geq 0, \end{cases} \tag{2.23}$$

where $\mathbf{u}_i^{[n]} = [u_i^{[n]}(x_0^{cheb}), \ldots, u_i^{[n]}(x_{N_f}^{cheb})]^T$, $i = 1, 2, \ldots, n_p$, and $\boldsymbol{F}^{\text{collocation}}(\cdot)$ is defined by (2.21).

The starting point in the iteration (2.23) in the fine level, denoted as $\mathbf{u}_{i,N_f}^{[0]} = [u_i^{[0]}(x_0^{cheb}), \ldots, u_i^{[n]}(x_{N_f}^{cheb})]^T$, can be chosen as the linear interpolation of the solution in the coarse level $\mathbf{u}_{N_c}^{(i)}$ for $i = 1, 2, \ldots, n_p$.

In addition, for the problem defined on $[0, 2\pi)$ with periodical boundary conditions, we should choose the Fourier collocation points

$$\mathcal{I}_N^{fourier} = \left\{ x_j^{fourier} = j\frac{2\pi}{N} \quad j = 0, \ldots, N-1 \right\} \subset [0, 2\pi). \tag{2.24}$$

The above procedure based on the Chebyshev collocation method could be easily applied to the Fourier case.

Remark 1. Several additional remarks about the implementation of Algorithm 1 are listed as follows.

- For simplicity, we consider the one-dimensional problem with Dirichlet boundary conditions here. In section 5, we will show the numerical results for the problems with Dirichlet, Neumann, periodical, and mixed boundary conditions in both one- and two-dimensional spaces.

Algorithm 1 A two-level spectral algorithm for semilinear problem (2.1).

Input:

1. N_c: degree of freedom in the coarse level;
2. N_f: degree of freedom in the fine level;
3. τ and n_{max}: tolerance and maximum number of Newton iterations in the fine level.

Stage I: **Solving the nonlinear problem (2.1) in the coarse level.** Let n_p denote the number of numerical solutions obtained in the coarse level. More precisely, we need to solve $\{u_{N_c}^{(i)}(x)\}_{i=1}^{n_p}$ in terms of Legendre expansion coefficients from (2.14) or in terms of function values at collocation points from (2.21) using the homotopy continuation method.

Stage II: **Solving the linearized problem (2.2) in the fine level.** More precisely, we do the following iterations:

For $i = 1, 2, \ldots, n_p$,

Setting $u_{N_f}^{[0],(i)}(x) = u_{N_c}^{(i)}(x)$,

While $n < n_{max}$ and $\tau_{n,i} > \tau$,

1. Given $u^{[n],(i)}$, solving V_i from (2.18) in the Legendre–Galerkin method or (2.23) in the collocation method.
2. $\tau_{n,i} = \|V_i\|_{L^2}$.
3. $u_{N_f}^{[n+1],(i)} = u_{N_f}^{[n],(i)} + V_i$.

EndWhile

EndFor

Output: The set of solutions $\left\{u_{N_f}^{[n],(i)}\right\}_{i=1}^{n_p}$.

- We show the detailed algorithm of Legendre, Galerkin, and Chebyshev collocation methods, which can be easily generalized to the Jacobi–Galerkin methods and Jacobi-collocation method [39]. Besides, the error estimates shown in section 4 is based on the Galerkin or collocation formulation.
- The above two-level algorithm is based on either Galerkin or collocation formulation. Actually, one can easily derive the *hybrid* version within this flexible framework. For instance, one can choose the Galerkin method in Stage I and switch to the collocation method in Stage II in Algorithm 1.
- In the fine level (Stage II), in order to obtain stable solutions, one should choose a series of grids and repeat the iterations successively until the solutions converge. For example, in the coarse level we choose $N_c = 8$; then in the fine level we can choose $N_{f_1} = 16$, $N_{f_2} = 32$, $N_{f_3} = 64$, and the solutions from the grid N_{f_k} would be used as the initial guess for the iteration in the grid $N_{f_{k+1}}$ for $k = 1, 2, 3$. The divergent solutions would be removed.
- It is well known that dense matrices are frequently encountered in spectral methods, while low-order methods, such as finite elements and finite differences, usually lead to sparse or banded matrices. However, in Stage I, the computational cost for finding multiple solutions of polynomial systems is always $O(d^N)$, where $d = \max(p, q)$, no matter whether matrices are dense or sparse (see section 3). In Stage II, we can use the fast structured direct solver with nearly linear cost to solve the dense linear systems [40, 41]. In short, we proposed a framework of high-order methods, while the computational costs are almost the same as those for other low-order methods.

- It is well known that, in comparison with Galerkin formulations, the collocation methods are very flexible in dealing with variable coefficients and nonlinear problems. However, differentiation matrices are ill-conditioned; more precisely, the condition number of kth derivative matrix $\boldsymbol{D}^{(k)}$ grows like $O(N^{2k})$. We are trying to design a fast structured spectral collocation method based on the idea shown in [40, 41] to circumvent these difficulties.
- We restricted our attention to one-dimensional problems in the above algorithm. Actually, the framework presented here can be extended to multi-dimensions in a straightforward way. The numerical results in both one and two dimensions will be shown in section 5.

3. Homotopy continuation method for solving discretized polynomial systems. In this section, we briefly show the homotopy continuation method to compute the multiple solutions of the discretized polynomial systems (2.14) and (2.21). Homotopy is one of the main numerical approaches to compute the isolated roots of polynomial systems [42, 45, 21]. We consider a general polynomial system

$$\boldsymbol{F}(\mathbf{u}) = 0, \tag{3.1}$$

where the function $\boldsymbol{F}(\mathbf{u}) = [f_1(\mathbf{u}), \ldots, f_N(\mathbf{u})] : \mathbb{C}^N \to \mathbb{C}^N$ and each f_i is a polynomial with respect to the variables $\mathbf{u} = [u_1, \ldots, u_N]^T$. By denoting the degree of f_k as $\deg(f_k) = d_k$, we then construct the homotopy function

$$\boldsymbol{H}(\mathbf{u}, t) = (1-t)\boldsymbol{F}(\mathbf{u}) + \gamma t G(\mathbf{u}), \tag{3.2}$$

where $\boldsymbol{G}(\mathbf{u}) = [g_1(\mathbf{u}), \ldots, g_N(\mathbf{u})]^T : \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial system with known solutions, $t \in [0,1]$ is a homotopy parameter, and γ is a random complex number. In addition, each g_i has the same degree as f_i for $i = 1, \ldots, N$. When $t = 1$, we have known solutions to $\boldsymbol{G}(\mathbf{u}) = 0$ or, equivalently, $\boldsymbol{H}(\mathbf{u}, 1) = 0$.

We can choose a specific system for $\boldsymbol{G}(\mathbf{u})$, namely, $g_k = u_k^{d_k} - 1$ with solutions

$$u_k = e^{2\pi i/(d_k)}, \qquad i = 0, 1, \ldots, d_k - 1\,. \tag{3.3}$$

The known solutions to $\boldsymbol{G}(\mathbf{u}) = 0$ are called *start points*, and the system $\boldsymbol{H}(\mathbf{u}, 1) = 0$ is called the *start system*. Furthermore, such a start system is called the *total degree start system*, since $\deg(g_k) = \deg(f_k) = d_k$ for each k and the number of solutions $\boldsymbol{H}(\mathbf{u}, 1) = 0$ is equal to $\prod_{k=1}^{n} d_k$. Finally, choosing a total degree start system and a random complex number γ guarantees finding all the solutions, which is called the γ-trick [2].

There are several public software packages available for solving polynomial systems using homotopy continuation methods. In the numerical experiments, we use *Bertini,* developed by Bates et al. [3]. Suppose $d = \max(d_k)$; then the total cost for solving the polynomial system (3.1) is $O(d^N)$. In practice, the solvers for finding all roots of the polynomial system are limited to small size N, typically $N \leq 20$, due to both operation flops and memory storage. Besides, in our proposed two-level Algorithm 1, the cost of Stage I is $O(d^{N_c})$, where d is the degree of the polynomial in the differential equation, while the cost of Stage II is at most $O(N_f^3)$.

4. Error estimates for one-dimensional semilinear problems. In this section, we carry out the error estimates of our proposed two-level spectral Galerkin method for the following one-dimensional semilinear problem with Dirichlet boundary conditions:

$$\begin{cases} \mathcal{F}(u) := -u_{xx} + \lambda u^p + f(x) = 0, \quad x \in I = (-1,1), \\ u(-1) = 0, \quad u(1) = 0, \end{cases} \tag{4.1}$$

where p is a nonnegative integer, $\lambda \in \mathbb{R} \setminus \{0\}$ is a nonzero parameter, and $f(x)$ is a known sufficiently smooth function.

Recall the Jacobi weight

$$\omega^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1,\ x \in I. \tag{4.2}$$

We consider the following Jacobi weighed spaces and related norms:

$$L^2_{\omega^{\alpha,\beta}}(I) := \{u : \|u\|_{\omega^{\alpha,\beta}} < \infty\} \quad \text{with norm } \|u\|_{\omega^{\alpha,\beta}} = (u,u)^{1/2}_{\omega^{\alpha,\beta}}; \tag{4.3}$$

$$H^1_{\omega^{\alpha,\beta}}(I) := \left\{u : \|u\|_{1,\omega^{\alpha,\beta}} < \infty\right\} \quad \text{with norm and seminorm} \tag{4.4}$$

$$\|u\|^2_{1,\omega^{\alpha,\beta}} = |u|^2_{1,\omega^{\alpha,\beta}} + \|u\|^2_{\omega^{\alpha,\beta}}, \quad |u|_{1,\omega^{\alpha,\beta}} = \|u'\|_{\omega^{\alpha,\beta}};$$

$$H^1_{0,\omega^{\alpha,\beta}}(I) := \left\{u \in H^1_{\omega^{\alpha,\beta}} : u(\pm 1) = 0\right\}. \tag{4.5}$$

Then the Jacobi weighted weak formulation for the problem (4.1) is to find $u \in H^1_{0,\omega^{\alpha,\beta}}(I)$ such that

$$A(u,\psi) := a^{\alpha,\beta}(u,\psi) + \lambda(u^p,\psi)_{\omega^{\alpha,\beta}} + (f,\psi)_{\omega^{\alpha,\beta}} = 0 \quad \forall \psi \in H^1_{0,\omega^{\alpha,\beta}}(I), \tag{4.6}$$

where the Jacobi weighted inner product $(\cdot,\cdot)_{\omega^{\alpha,\beta}}$ is defined by

$$(u,v)_{\omega^{\alpha,\beta}} := \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)\mathrm{d}x \quad \forall u,v, \in L^2_{\omega^{\alpha,\beta}}(I), \tag{4.7}$$

and the bilinear form $a^{\alpha,\beta}(u,v)$ is defined by

$$a^{\alpha,\beta}(u,v) = \int_{-1}^{1} [\partial_x u] \left[\partial_x(\omega^{\alpha,\beta} v)\right] \mathrm{d}x \quad \forall u,v, \in H^1_{0,\omega^{\alpha,\beta}}(I). \tag{4.8}$$

Hereafter, $D_u\mathcal{F}[w]$ denotes the linearized operator of $\mathcal{F}$ at w, namely, the Fréchet derivative with respect to u of $\mathcal{F}$ computed at the point w, i.e.,

$$D_u\mathcal{F}[w]v := -v_{xx} + \lambda p w^{p-1} v \quad \forall v. \tag{4.9}$$

The Jacobi weighted bilinear form $A'(w;v,\psi)$ introduced by $D_u\mathcal{F}[w]$ is

$$A'(w;v,\psi) := a^{\alpha,\beta}(v,\psi) + \lambda p(w^{p-1}v,\psi)_{\omega^{\alpha,\beta}} \quad \forall v,\psi \in H^1_{0,\omega^{\alpha,\beta}}(I). \tag{4.10}$$

The boundedness of $A'(w;v,\psi)$ is shown in Lemma 4.8.

The Newton iteration of the problem (4.6) is, for each n, to find $v \in H^1_{0,\omega^{\alpha,\beta}}(I)$ such that

$$\begin{cases} A'\left(u^{[n]};v,\psi\right) + A\left(u^{[n]},\psi\right) = 0 \quad \forall \psi \in H^1_{0,\omega^{\alpha,\beta}}(I), \\ u^{[n+1]} = u^{[n]} + v, \quad n \geq 0. \end{cases} \tag{4.11}$$

The detailed error estimates in the coarse level for the nonlinear problem (4.6) are shown in section 4.1, while the convergence results in the fine level for the linearized problem (4.11) are shown in section 4.2.

4.1. Jacobi weighted error estimates for semilinear problems. To this end, we assume that $-1 < \alpha,\ \beta < 1$, and p is a fixed integer. First, we consider some useful embedding results.

LEMMA 4.1. *Some embedding results about the Jacobi weighted spaces are as follows:*

(1) *The embedding $H^1_{0,\omega^{\alpha,\beta}}(I) \subset L^2_{\omega^{\alpha,\beta}}(I)$ is compact.*

(2) $H^1_{\omega^{\alpha,\beta}}(I) \subset L^\infty(I)$.

Proof. For part (1), since the embedding $H^1_0(I) \subset L^2(I)$ is compact, it is sufficient to prove the following:

(a) $u \in L^2_{\omega^{\alpha,\beta}}(I) \to u(\omega^{\alpha,\beta})^{1/2} \in L^2(I)$ is an isomorphism.

(b) $u \in H^1_{0,\omega^{\alpha,\beta}}(I) \to u(\omega^{\alpha,\beta})^{1/2} \in H^1_0(I)$ is a continuous mapping.

Property (a) holds trivially. For $u \in H^1_{0,\omega^{\alpha,\beta}}(I)$, we have

$$\partial_x \left(u(\omega^{\alpha,\beta})^{1/2} \right) = u_x(\omega^{\alpha,\beta})^{1/2} + \frac{-(\alpha+\beta)x + (\beta-\alpha)}{2} u\omega^{\alpha/2-1,\beta/2-1}. \tag{4.12}$$

Obviously, $u_x(\omega^{\alpha,\beta})^{1/2} \in L^2(I)$. Besides, by Lemma 3.7 in [19], for any $v \in H^1_{0,\omega^{\alpha,\beta}}(I)$ and $-1 < \alpha, \beta < 1$,

$$\|v\|_{\omega^{\alpha-2,\beta-2}} \lesssim \|v\|_{1,\omega^{\alpha,\beta}}. \tag{4.13}$$

It implies that $u\omega^{\alpha/2-1,\beta/2-1} \in L^2(I)$. Thus, property (b) holds.

Part (2) is followed by Lemma 3.5 in [19]. ∎

For $-1 < \alpha, \beta < 1$, one can verify the Poincaré-type inequality as well as the continuity and coercivity of the bilinear form $a^{\alpha,\beta}(\cdot,\cdot)$ defined by (4.8) in the space $H^1_{0,\omega^{\alpha,\beta}}(I)$.

LEMMA 4.2 (Lemma 3.5 and Lemma B.7 in [39]). *If $-1 < \alpha, \beta < 1$, then $\forall u, v \in H^1_{0,\omega^{\alpha,\beta}}(I)$, there exist three positive constants C_1, C_2, C_3, independent of u and v, such that $\forall u, v \in H^1_{0,\omega^{\alpha,\beta}}(I)$,*

$$|a^{\alpha,\beta}(u,v)| \le C_1 |u|_{1,\omega^{\alpha,\beta}}\ |v|_{1,\omega^{\alpha,\beta}}, \tag{4.14}$$

$$a^{\alpha,\beta}(v,v) \ge C_2 |v|^2_{1,\omega^{\alpha,\beta}}, \tag{4.15}$$

$$\|u\|_{\omega^{\alpha,\beta}} \le C_3 |u|_{1,\omega^{\alpha,\beta}}. \tag{4.16}$$

Define the orthogonal projection $\Pi^{1,0}_{N,\alpha,\beta} : H^1_{0,\omega^{\alpha,\beta}}(I) \to X_N$ by

$$a^{\alpha,\beta}\left(u - \Pi^{1,0}_{N,\alpha,\beta} u, v \right) = 0 \quad \forall v \in X_N. \tag{4.17}$$

Recall the error estimate for orthogonal projection $\Pi^{1,0}_{N,\alpha,\beta}$ defined in (4.17) (cf. Theorem 3.39 in [39]): for $1 \le m \le N$ and $\mu = 0, 1$,

$$\|u - \Pi^{1,0}_{N,\alpha,\beta} u\|_{\mu,\omega^{\alpha,\beta}} \lesssim N^{\mu-m} \|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}} \tag{4.18}$$

for any $u \in H^1_{0,\omega^{\alpha,\beta}}(I) \cap B^m_{\omega^{\alpha,\beta}}(I)$.

In addition, in order to deal with the nonlinear term u^p, where $p \in \mathbb{N}$, we introduce the following differential operators:

$$D_x^{0,p} u := u^p, \tag{4.19}$$

$$D_x^{1,p}u := \frac{\partial}{\partial x}(u^p) = pu^{p-1}\partial_x u, \tag{4.20}$$

$$D_x^{2,p}u := \frac{\partial^2}{\partial x^2}(u^p) = p(p-1)u^{p-2}(\partial_x u)^2 + pu^{p-1}\partial_{xx}u, \tag{4.21}$$

$$\dots$$

$$D_x^{k,p}u := \frac{\partial^k}{\partial x^k}(u^p), \quad k = 0, 1, 2, \dots. \tag{4.22}$$

Note that in the above definitions, the powers of u should be nonnegative, i.e., $u^q \equiv 0$ for $q < 0$. We consider two more spaces,

(4.23)

$B^m_{\omega^{\alpha,\beta}}(I) := \left\{u : \partial_x^k u \in L^2_{\omega^{\alpha+k,\beta+k}}, 0 \le k \le m\right\}$, $m \in \mathbb{N}$, with norm and seminorm

$$\|u\|_{B^m_{\omega^{\alpha,\beta}}} = \left(\sum_{k=0}^m \|\partial_x^k u\|^2_{\omega^{\alpha+k,\beta+k}}\right)^{1/2}, \ |u|_{B^m_{\omega^{\alpha,\beta}}} = \|\partial_x^m u\|_{\omega^{\alpha+k,\beta+k}};$$

(4.24)

$B^{m,p}_{\omega^{\alpha,\beta}}(I) := \left\{u : D_x^{k,p} u \in L^2_{\omega^{\alpha+k,\beta+k}}, 0 \le k \le m\right\}$, $m \in \mathbb{N}$, with norm and seminorm

$$\|u\|_{B^{m,p}_{\omega^{\alpha,\beta}}} = \left(\sum_{k=0}^m \|D_x^{k,p} u\|^2_{\omega^{\alpha+k,\beta+k}}\right)^{1/2}, \ |u|_{B^{m,p}_{\omega^{\alpha,\beta}}} = \|D_x^{m,p} u\|_{\omega^{\alpha+k,\beta+k}}.$$

It follows that $u \in B^{m,p}_{\omega^{\alpha,\beta}}(I)$ implies $u^p \in B^m_{\omega^{\alpha,\beta}}(I)$. Then for the projection $\Pi^{1,0}_{N,\alpha,\beta}$ defined in (4.17), we have

$$\|u^p - \Pi^{1,0}_{N,\alpha,\beta}u^p\|_{\mu,\omega^{\alpha,\beta}} \lesssim N^{\mu-m}\|D_x^{m,p}u\|_{\omega^{\alpha+m-1,\beta+m-1}} \tag{4.25}$$

for any $u \in H^1_{0,\omega^{\alpha,\beta}}(I) \cap B^{m,p}_{\omega^{\alpha,\beta}}(I)$.

For the convenience of notation, let us denote $\omega = \omega^{\alpha,\beta}, V = H^1_{0,\omega^{\alpha,\beta}}(I), W = V'$, $\Pi_N = \Pi^{1,0}_{N,\alpha,\beta}$ defined by (4.17) and $V_N = X_N$ defined in (2.3).

Let us define the linear operator $\mathcal{T} : W \to V$ by

$$a^{\alpha,\beta}(\mathcal{T}g, \psi) = \langle g, \psi\rangle \quad \forall \psi \in V, \tag{4.26}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. It follows from (4.14) and (4.15) that $\mathcal{T}$ is a bounded operator. Moreover, we have the following lemma.

LEMMA 4.3. *For any $s \ge 0$, $\mathcal{T}$ is a linear, bounded, continuous operator from $H^s_\omega(I)$ into $H^1_{0,\omega}(I) \cap H^{s+2}_\omega(I)$. For any $s \in [-1, 0)$, $\mathcal{T}$ is continuous from $(H^{-s}_{0,\omega}(I))'$ into $H^1_{0,\omega}(I) \cap H^{s+2}_\omega(I)$.*

Proof. The linearity is obvious by definition. Let us check the continuity. If s is a positive integer, then integrating by parts in (4.26) gives

$$(-\mathcal{T}g)_{xx} = g, \quad x \in I, \tag{4.27}$$

which implies that

$$\|\mathcal{T}g\|_{s+2,\omega} \lesssim \|g\|_{s,\omega}. \tag{4.28}$$

For noninteger $s > 0$ and $-1 \le s < 0$, the same result follows by space interpolation. $\square$

The following result is a direct consequence of Lemmas 4.1 and 4.3.

LEMMA 4.4. *$\mathcal{T}$ is a compact operator from W to V.*

For fixed integer p, the mapping $\mathcal{G} : \mathbb{R} \times V \to L^2_\omega(I)$ is defined by

$$\mathcal{G}(\lambda, u; p) = \lambda u^p + f. \tag{4.29}$$

It is easy to see that the mapping $\mathcal{G}(\lambda, u; p)$ defined above has the following properties:

(1) $\mathcal{G}$ is a C^∞ mapping.

(2) For any $k \in \mathbb{N}$, $D^k\mathcal{G}$ is bounded on any bounded subset of the space $\mathbb{R} \times H^1_{0,\omega}(I)$.

(3) Since W contains topologically L^2_ω, we have

$$\|\mathcal{G}(\lambda, u; p)\|_W \lesssim \|\mathcal{G}(\lambda, u; p)\|_\omega. \tag{4.30}$$

Throughout this section, we make the following assumption on the exact solution u_λ to the problem (4.6):

(H-I) For fixed p, there exists a branch $\{(\lambda, u), \lambda \in \Lambda\}$ such that $u_\lambda \in \tilde{V}^m := H^1_{0,\omega}(I) \cap B^m_\omega(I) \cap B^{m,p}_\omega(I)$, where $m \geq 0$.

By the above assumption and property (2) in Lemma 4.1, we have

$$\|\mathcal{G}(\lambda, u; p)\|_W \lesssim \lambda \|u\|^p_\infty + \|f\|_\omega. \tag{4.31}$$

Consider the operator $\tilde{\mathcal{F}} : \mathbb{R} \times V \to V$, defined by

$$\tilde{\mathcal{F}}(\lambda, u; p) := u + \mathcal{T}\mathcal{G}(\lambda, u; p), \tag{4.32}$$

and its Fréchet derivative $D_u\tilde{\mathcal{F}}$ defined by

$$D_u\tilde{\mathcal{F}}[\lambda, w]v = (Id + \mathcal{T} D_u\mathcal{G}[\lambda, w])v, \tag{4.33}$$

where Id denotes the identity operator. It is easy to check that the problem (4.6) can be written equivalently as follows: find $u \in V$ such that

$$\tilde{\mathcal{F}}(\lambda, u; p) = 0. \tag{4.34}$$

Now we make the following additional assumption on the exact solution u_λ:

(H-II) There exists a branch $\{(\lambda, u), \lambda \in \Lambda\}$ such that the problem (4.32) admits at least one isolated nonzero weak solution u_λ such that

$$\forall \lambda \in \Lambda,\ \forall v \in V, \quad \|(D_u\tilde{\mathcal{F}}[\lambda, u_\lambda]v)\|_V \geq C_0\|v\|_V, \tag{4.35}$$

where C_0 is independent of v.

The discrete weak formulation related to (4.6) is to find $u_N \in X_N$ such that

$$A(u_N, \psi_N) := a^{\alpha,\beta}(u_N, \psi_N) + \lambda(u^p_N, \psi_N)_\omega + (f, \psi_N)_\omega = 0 \quad \forall \psi_N \in X_N, \tag{4.36}$$

where X_N is defined in (2.3).

We define $\mathcal{T}_N : W \to V_N$ by

$$\mathcal{T}_N = \Pi^{1,0}_{N,\alpha,\beta} \circ \mathcal{T}. \tag{4.37}$$

Thanks to the definition of $\mathcal{T}$ and $\Pi^{1,0}_{N,\alpha,\beta}$, for any $g \in W$, it follows that

$$a^{\alpha,\beta}(\mathcal{T}_N g, \psi) = \langle g, \psi \rangle \quad \forall \psi \in V_N. \tag{4.38}$$

Now we are ready to define $\tilde{\mathcal{F}}_N : \Lambda \times V_N \to V_N$ and its Fréchet derivative $D_u\tilde{\mathcal{F}}_N$ by

$$\tilde{\mathcal{F}}_N(\lambda, v; p) := v + \mathcal{T}_N \mathcal{G}(\lambda, v; p), \tag{4.39}$$

$$D_u\tilde{\mathcal{F}}_N[\lambda, v; p] := Id + \mathcal{T}_N D_u\mathcal{G}[\lambda, v; p]. \tag{4.40}$$

Then the discrete formulation of the problem (4.34) is to find $u_N \in V_N$ such that

$$\tilde{\mathcal{F}}_N(\lambda, u_N; p) = 0, \tag{4.41}$$

which is equivalent to (4.36).

For the reader's convenience, let us recall the following results.

LEMMA 4.5 (Lemma 2.2 in [29]). *Assume the following are true.*

(H-1.1) *Assume that $\mathcal{G}^{m+1}, m \in \mathbb{N}$, is a C^{m+1} mapping from $\Lambda \times V$ into V', and $D^{m+1}\mathcal{G}$ is bounded on any bounded subset of $\Lambda \times V$.*

(H-1.2) *Assume that $\Pi_N : V \to V_N$ is a continuous operator satisfying*

$$\lim_{N\to\infty} \|v - \Pi_N v\| = 0 \quad \forall v \in V. \tag{4.42}$$

(H-1.3) *Assume that $\mathcal{T} \in \mathscr{L}(V', V)$ and $\mathcal{T}_N \in \mathscr{L}(V', V_N)$ satisfy*

$$\lim_{N\to\infty} \|\mathcal{T} - \mathcal{T}_N\|_{\mathscr{L}(V',V)} = 0. \tag{4.43}$$

Then there exist a neighborhood θ of the origin in V and, for large enough N, a unique C^{m+1} mapping $\lambda \in \Lambda \to u_{N,\lambda} \in V_N$ such that

$$\forall \lambda \in \Lambda, \quad \tilde{\mathcal{F}}_N(\lambda, u_N(\lambda)) = 0, \quad u_{N,\lambda} - u_\lambda \in \theta, \tag{4.44}$$

where $\tilde{\mathcal{F}}_N$ is shown in (4.39)*. Furthermore, there exists a positive constant K independent of λ and N such that*

$$\|u_{N,\lambda} - u_\lambda\|_{1,\omega} \le K \left(\|u_\lambda - \Pi_N u_\lambda\|_{1,\omega} + \|(\mathcal{T} - \mathcal{T}_N)\mathcal{G}(\lambda, u_\lambda)\|_{1,\omega} \right). \tag{4.45}$$

LEMMA 4.6 (Lemma 2.3 in [29]). *Assume* (H-1.1)–(H-1.3) *in Lemma* 4.5 *hold. Additionally, we assume the following:*

(H-2.1) *The mapping $v \in V \to D_u\mathcal{G}[\lambda, v] \in \mathscr{L}(L^2_\omega(I), V')$ is continuous.*

(H-2.2) *$\mathcal{T} \in \mathscr{L}(V', L^2_\omega(I))$ and $\mathcal{T}_N \in \mathscr{L}(V', L^2_\omega(I))$ satisfy*

$$\lim_{N\to\infty} \|\mathcal{T} - \mathcal{T}_N\|_{\mathscr{L}(V',L^2_\omega(I))} = 0. \tag{4.46}$$

(H-2.3) *If $v \in L^2_\omega(I)$ satisfies*

$$v + \mathcal{T} D_u\mathcal{G}[\lambda, u_\lambda]v = 0, \tag{4.47}$$

then $v \in V$.

Then for N large enough, we have

$$\|u_{N,\lambda} - u_\lambda\|_\omega \lesssim \|\tilde{\mathcal{F}}_N(\lambda, u_\lambda)\|_\omega \quad \forall \lambda \in \Lambda. \tag{4.48}$$

Now we can present our main result on the Jacobi weighted approximation (4.36) to the semilinear problem (4.6).

THEOREM 4.7. *Assume that the hypotheses* (H-I) *and* (H-II) *hold. There exist a neighborhood θ of the origin in V and, for $N \geq N_0$ large enough, a unique $\lambda \in \Lambda \to u_{N,\lambda}(x) \in X_N$ such that for any $\lambda \in \Lambda$, $u_{N,\lambda}(x)$ solves* (4.41).

Furthermore, assuming that the mapping $\lambda \in \Lambda \to u_\lambda(x) \in \tilde{V}^m$ is continuous for a suitable m, then for any $\lambda \in \Lambda$, the following estimate holds:

$$\|u_{N,\lambda}-u_\lambda\|_{\mu,\omega^{\alpha,\beta}} \lesssim N^{\mu-m}\left(\|\partial_x^m u_\lambda\|_{\omega^{\alpha+m-1,\beta+m-1}} + \|D_x^{m,p}u_\lambda\|_{\omega^{\alpha+m-1,\beta+m-1}}\right), \quad (4.49)$$

where $\mu = 0, 1$.

Proof. Assumption (H-1.1) is true by the definition of $\mathcal{G}(\lambda, u)$, and (H-1.2) holds according to the estimate (4.18). Besides, $\forall g \in W$, we have

$$(\mathcal{T} - \mathcal{T}_N)g = (Id - \Pi_N)\mathcal{T}g. \quad (4.50)$$

By (4.50), (4.18), and Lemma 4.3, for any $g \in W$, we have

$$\|(\mathcal{T} - \mathcal{T}_N)g\|_{1,\omega} = \|(Id - \Pi_N)\mathcal{T}g\|_{1,\omega} \lesssim N^{-1}\|\partial_{xx}(\mathcal{T}g)\|_\omega \lesssim N^{-1}\|g\|_\omega, \quad (4.51)$$

which implies (H-2.3). Applying Lemma 4.5 gives the first part of this theorem.

Furthermore, assume u_λ and $u_{\lambda,N}$ solve (4.34) and (4.41), respectively. Then by (4.25) we have

$$\|(\mathcal{T} - \mathcal{T}_N)\mathcal{G}(\lambda, u_\lambda)\|_{1,\omega} = \|u_\lambda^p - \Pi_N u_\lambda^p\|_{1,\omega} \lesssim N^{-m}\|D_x^{m,p}u_\lambda\|_{\omega^{\alpha+m-1,\beta+m-1}}. \quad (4.52)$$

Plugging (4.52) into (4.45) and combining the projection error estimate (4.18), we obtain the desired estimate (4.49) for the case with $\mu = 1$.

To obtain the L^2_ω estimate, we need to check (H-2.1)–(H-2.3). First, by definition and property (2) of Lemma 4.1, we have

$$\|D_u\mathcal{G}[\lambda, v]w\|_\omega = \|\lambda p v^{p-1}w\|_\omega \lesssim \|v\|_\infty^{p-1}\|w\|_\omega, \quad (4.53)$$

which implies (H-2.1). Second, (H-2.2) holds followed by Lemma 4.3 and estimate (4.25). Third, (H-2.3) is a consequence of (H-2.1) and Lemma 4.3. Now we proved that the estimate (4.48) holds.

Finally, by the definition of $\tilde{\mathcal{F}}_N$, we have

$$\begin{aligned}\|\tilde{\mathcal{F}}_N(\lambda, u_\lambda)\|_\omega \lesssim\ & \|u_\lambda - \Pi_N u_\lambda\|_\omega + \|(\mathcal{T} - \mathcal{T}_N)\mathcal{G}(\lambda, u_\lambda)\|_\omega \\ =\ & \|u_\lambda - \Pi_N u_\lambda\|_\omega + \|u_\lambda^p - \Pi_N u_\lambda^p\|_\omega.\end{aligned}$$

By (4.18) , (4.25), and (4.48), we can conclude (4.49) for the case with $\mu = 0$. ∎

4.2. Error estimates for the linearized problem by the Newton method. Let us fix λ, p, and N in the problem (4.41) and choose the initial guess $u^{[0]} \in X_N$. Then the Newton iterations for solving the nonlinear problems (4.34) and (4.41) are, respectively,

- to find $u^{[n+1]} \in V$ such that

$$D_u\tilde{\mathcal{F}}[\lambda, u^{[n]}]\left(u^{[n+1]} - u^{[n]}\right) + \tilde{\mathcal{F}}(\lambda, u^{[n]}) = 0, \quad n \geq 0, \quad (4.54)$$

  where $\tilde{\mathcal{F}}$ and $D_u\tilde{\mathcal{F}}$ are defined by (4.32) and (4.33), respectively;
- to find $u_N^{[n+1]} \in X_N$ such that

$$D_u\tilde{\mathcal{F}}_N[\lambda, u_N^{[n]}]\left(u_N^{[n+1]} - u_N^{[n]}\right) + \tilde{\mathcal{F}}_N(\lambda, u_N^{[n]}) = 0, \quad n \geq 0, \quad (4.55)$$

  where $\tilde{\mathcal{F}}_N$ and $D_u\tilde{\mathcal{F}}_N$ are defined by (4.39) and (4.40), respectively.

Note that the formulation (4.54) is equivalent to (4.11), and (4.55) is the discretized version.

By Lemma 4.2, one can verify the ellipticity of the operator $D_u\tilde{\mathcal{F}}[\lambda, u_\lambda]$ (equivalently, $A'(u_\lambda; v, \psi)$) and the boundedness of the operator $\mathcal{T}D_{uu}\mathcal{G}[\lambda, u_\lambda]$.

LEMMA 4.8. *Assume that the hypotheses* (H-I) *and* (H-II) *hold. There exist two positive constants* $\tilde{C}_1, \tilde{C}_2$, *independent of* u *and* v, *such that* $\forall u, v \in H^1_{0,\omega}(I)$,

$$|A'(u_\lambda; u, v)| \le \tilde{C}_1 |u|_{1,\omega}\ |v|_{1,\omega}, \tag{4.56}$$

$$|A'(u_\lambda; v, v)| \ge \tilde{C}_2 |v|^2_{1,\omega}. \tag{4.57}$$

Moreover, the operator $\mathcal{T}D_{uu}\mathcal{G}[\lambda, u_\lambda]$ *is uniformly bounded, i.e.,*

$$\|\mathcal{T}D_{uu}[\lambda, u_\lambda]\|_{\mathscr{L}(V,V)} \le \delta. \tag{4.58}$$

Proof. Equations (4.14) and (4.53) imply (4.56), and (4.35) indicates (4.57). Besides, since $D_{uu}\mathcal{G}[\lambda, u_\lambda] = \lambda p(p-1)u^{p-2}$, we can get (4.58) thanks to assumptions (H-I) and (H-II).

By standard argument for Newton iteration [48, 23], one can show the following convergence results.

THEOREM 4.9. *Assume* $u_{\lambda,N}$ *solves* (4.41) *and* $u_N^{[0]}$ *is the initial guess in the Newton iteration* (4.55). *Choose* $\rho \in (0, 1)$. *We make two more assumptions.*

(H-III) *N is fixed and large enough such that*

$$\|u_{N,\lambda} - \Pi_N u_\lambda\|_{1,\omega} < \rho. \tag{4.59}$$

(H-IV) *The initial guess* $u_N^{[0]}$ *satisfies*

$$\|u_N^{[0]} - u_{N,\lambda}\|_{1,\omega} < \rho. \tag{4.60}$$

Then $u_N^{[n]}$ *converges quadratically to* $u_{N,\lambda}$ *in the Newton iteration* (4.54) *as* $n \to \infty$; *i.e.,* $\exists n_0$ *such that* $\forall n > n_0$,

$$\|u_N^{[n]} - u_{N,\lambda}\|_{1,\omega} \lesssim \rho^{2^n}. \tag{4.61}$$

5. Numerical experiments.

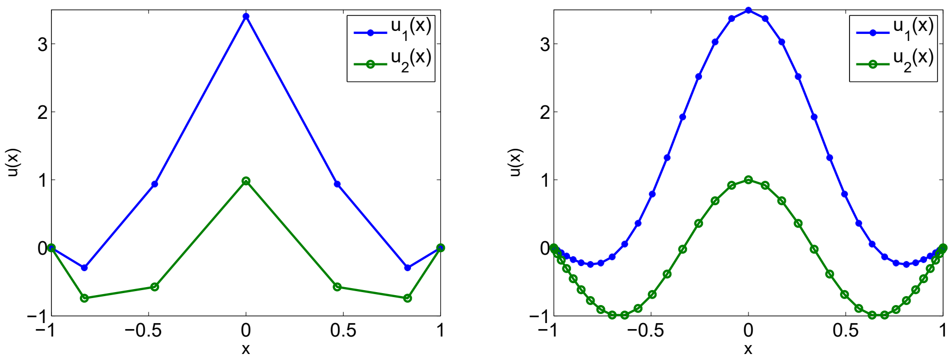
In this section, we present several numerical experiments to illustrate the efficiency and accuracy of the proposed two-level spectral methods for one-dimensional semilinear problems with Dirichlet boundary conditions (Examples 1 and 3) and mixed boundary conditions (Example 2), the fully nonlinear problems in one dimension (Example 4), and the two-dimensional semilinear system with Dirichlet boundary conditions (Example 5) and periodical boundary conditions (Example 6).

Example 1. Consider the semilinear problem (4.1) in which $f(x)$ is chosen as

$$f(x) = \frac{9\pi^2}{4}\cos\left(\frac{3\pi}{2}x\right) - \lambda\left[\cos\left(\frac{3\pi}{2}x\right)\right]^p. \tag{5.1}$$

It is easy to verify that one of the exact solutions is $u(x) = \cos\left(\frac{3\pi}{2}x\right)$.

Let us consider the case with $p = 2, \lambda = 1$. In our two-level spectral methods, we choose $N_c = 6$ in the coarse level and $N_f = 36$ in the fine level. The tolerance in the



(i) Stage one ($N = 6$) (ii) Stage two ($N = 36$)

FIG. 5.1. *Example* 1*: numerical solutions from the Legendre–Galerkin method.*

linearized iteration is chosen as $\tau = 1e - 15$. The numerical solutions obtained from the Legendre–Galerkin method and the Chebyshev collocation method are shown in Figure 5.1 and Figure 5.2, respectively. In addition, the numerical results obtained from the shooting method are shown in part (i) of Figure 5.3, in order to verify the correctness of the numerical solutions obtained by our method. The discussion on the accuracy and efficiency is listed below.

1. In order to the show the spectral accuracy of our methods, we can compute the errors in the L_2-norm for the semilinear problem (4.1) with exact solution $u(x) = \cos\left(\frac{3\pi}{2}x\right)$. We observe from part (ii) of Figure 5.3 that the errors decay exponentially like $O(\mathrm{e}^{-rN})$, for both Legendre–Galerkin and Chebyshev collocation methods, which verifies the error estimates in section 4.
2. In order to show the efficiency of our methods, we compare the results of the one-level method (only using the homotopy continuation method) and the proposed two-level methods. In Table 5.1, we can observe that the errors decay exponentially, but the computational costs grow exponentially in the coarse level. The numerical results in the fine level (after solving the problem in the coarse level with $N_c = 6$) are shown in Table 5.2, in which n_{it} means the number of Newton iterations in the fine level. We can observe that the errors decay exponentially, while the computational costs in the fine level are much less than the computational costs in the coarse level.

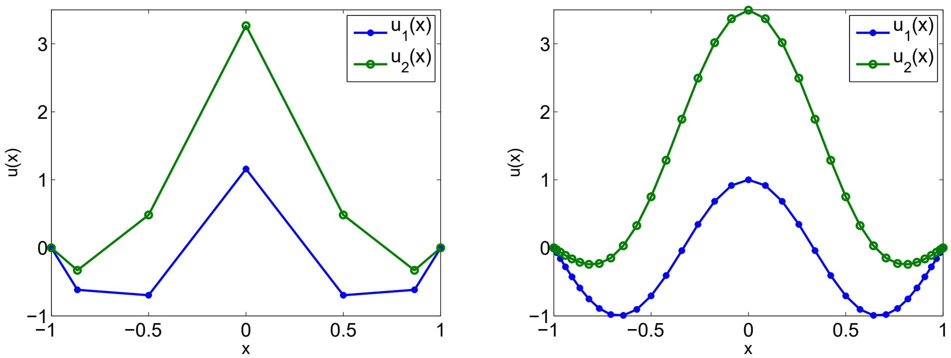
Example 2. Consider the semilinear equation with mixed boundary conditions [20]

$$\begin{cases} u_{xx} = \lambda(1 + u^p), \quad x \in (0, 1), \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{5.2}$$

where $\lambda \in \mathbb{R}^+, p \in \mathbb{N}$. After the linear map from $x \in [0, 1]$ to $\tilde{x} \in [-1, 1]$, we employ the Legendre–Galerkin method to solve this problem. Due to the mixed boundary conditions in (5.2), instead of using the basis functions (2.6), we need to design a new basis set $\{\psi_k(x)\}_{k=0}^{N-2}$ in which [39]

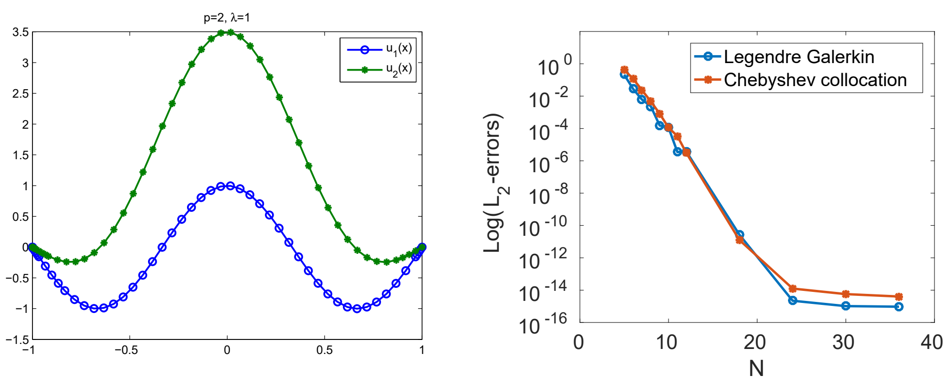
$$\psi_k(x) = L_k(x) - \frac{2k+3}{(k+2)^2} L_k(x) - \frac{(k+1)^2}{(k+2)^2} L_{k+2}(x), \tag{5.3}$$

which satisfies $\psi_k'(-1) = \psi_k(1) = 0 \ \forall k$.



(i) Stage one ($N = 6$) (ii) Stage two ($N = 36$)

FIG. 5.2. *Example* 1: *numerical solutions from the Chebyshev collocation method.*



(i) Numerical solutions of shooting method (ii) Exponential convergence rates of spectral methods

FIG. 5.3. *Example* 1: *spectral accuracy.*

The explicit form of the exact solutions for the problem (5.2) is not available. But it is known that for $p = 4$ there exists a critical value of λ, say $\lambda^* \approx 1.30107$, such that (1) if $0 < \lambda < \lambda^*$, there are two solutions; (2) if $\lambda = \lambda^*$, the two solutions merge; and (3) if $\lambda > \lambda^*$, there are no solutions.

First, let us consider the case with $p = 4$ and $\lambda = 1.2$, in which there are two solutions. We compare the numerical results between the one-level and two-level methods in Table 5.3. In the one-level method, we solve the nonlinear system of size N with the homotopy continuation method. In the two-level method, we choose $N_c = 3$ in the coarse level and various N_f in the fine level. The tolerance of the iteration in Stage II is chosen as $\tau = 1e - 14$. Besides, the last two columns in Table 5.3 show the L_2-norm of the differences between the numerical results obtained from the one-level and two-level methods. We observe that the two-level method is much faster than the one-level method, while the numerical solutions obtained from these two methods are almost the same.

In addition, the numerical solutions of the two-level method for the cases with $p = 4$ and $\lambda = 1.2$, 1.30107 are shown in Figures 5.4 and 5.5, respectively, which

TABLE 5.1
One-level spectral methods: Example 1 ($p = 2, \lambda = 1$).

N_c	Legendre–Galerkin		Chebyshev collocation	
	L_2 errors	Time (s)	L_2 errors	Time (s)
5	2.1218e-01	6.16e-01	4.2475e-01	7.21e-01
6	2.8077e-02	1.39e+00	1.1857e-01	1.50e+00
7	6.1128e-03	3.96e+00	2.3677e-02	2.52e+00
8	2.3606e-03	1.04e+01	4.6959e-03	2.96e+00
9	1.5291e-04	3.43e+01	8.2189e-04	7.02e+00
10	1.1356e-04	9.99e+01	1.1204e-04	1.61e+01
11	3.4722e-06	3.47e+02	3.3799e-05	3.73e+01
12	3.7841e-06	8.35e+02	3.0905e-06	8.46e+01

TABLE 5.2
Two-level spectral methods: Example 1 ($p = 2, \lambda = 1$ and $N_c = 6$).

N_f	Legendre–Galerkin			Chebyshev collocation		
	n_{it}	Time (s)	L_2 errors	n_{it}	Time (s)	L_2 errors
12	5	1.06e-02	3.7841e-06	5	9.54e-04	3.0905e-06
18	4	4.66e-03	2.6682e-11	7	6.17e-04	1.2624e-11
24	5	2.67e-03	2.2178e-15	12	1.04e-03	1.2551e-14
30	4	4.84e-03	1.0328e-15	23	2.16e-03	5.7133e-15
36	4	3.41e-03	9.4913e-16	29	3.89e-03	4.0461e-15

TABLE 5.3
Numerical results of Example 2 ($p = 4, \lambda = 1.2$).

One-level method		Two-level method ($N_c = 3$)			Errors between them	
N	Time (s)	N_f	n_{it}	Time (s)	$u_1(x)$	$u_2(x)$
3	3.83e-01					
4	1.55e+00	4	5	5.33e-01	1.5701e-16	0.0000e+00
5	7.69e+00	5	5	5.42e-01	0.0000e+00	7.8505e-17
6	5.29e+01	6	5	3.50e-01	1.5701e-16	7.8505e-17
7	3.82e+02	7	5	4.17e-01	1.5701e-16	0.0000e+00

verifies the conclusion about the number of solutions of this problem.

Example 3. Consider the problem with sign-changing even nonlinearity [11]

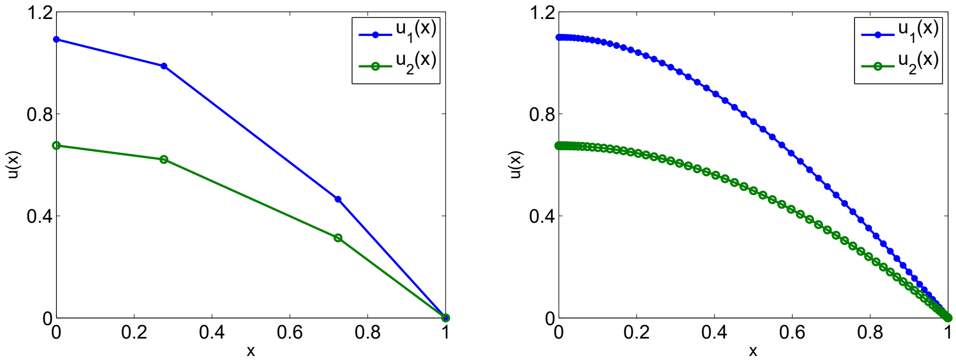
$$\begin{cases} u_{xx} + u^2(u^2 - p) = 0, \quad x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{5.4}$$

where $p \geq 0$ is a real parameter.

We use the Legendre–Galerkin method with coarse level $N_c = 8$ and a series of fine levels $N_f = 16, 32, 64, 128$ to solve the problems with $p = 2, 4, 6, 8$. The number of solutions we obtained in coarse and fine levels, the maximum number of Newton iterations in the fine level, and the maximum residual of the final numerical solutions are shown in Table 5.4. The nonzero numerical solutions are shown in Figures 5.6–5.9.

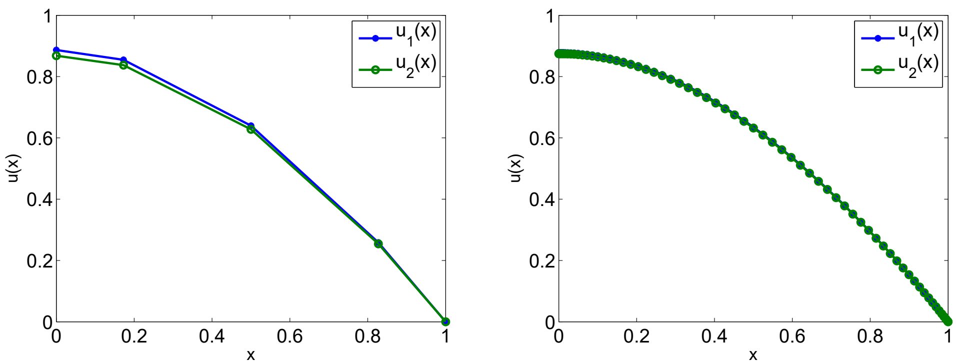
Example 4. Consider the fully nonlinear problem [17]

$$\begin{cases} -u_{xx}^2 + 1 = 0, \quad x \in (0, 1), \\ u(0) = 0, \quad u(1) = \dfrac{1}{2}, \end{cases} \tag{5.5}$$



(i) Nonlinear problem ($N_c = 3$) (ii) Linearized problem ($N_f = 64$)

FIG. 5.4. *Numerical solutions of Example* 2 *(*$p = 4, \lambda = 1.2$*).*



(i) Nonlinear problem ($N_c = 4$) (ii) Linearized problem ($N_f = 64$)

FIG. 5.5. *Numerical solutions of Example* 2 *(*$p = 4, \lambda = 1.30107$*).*

TABLE 5.4
Numerical results of Example 3.

p	No. of solutions		On the fine level	
	$N_c = 8$	$N_f = 128$	n_{it}	Residuals
2	4	4	2	1.4627e-14
4	8	4	3	3.2857e-14
6	32	6	12	2.5624e-13
8	54	6	40	1.1273e-13

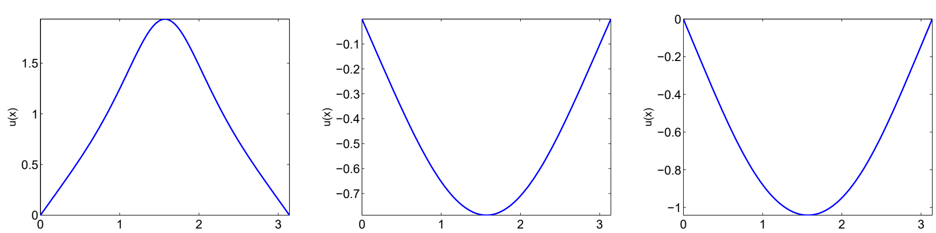


FIG. 5.6. *Numerical solutions of Example* 3: $p = 2$.

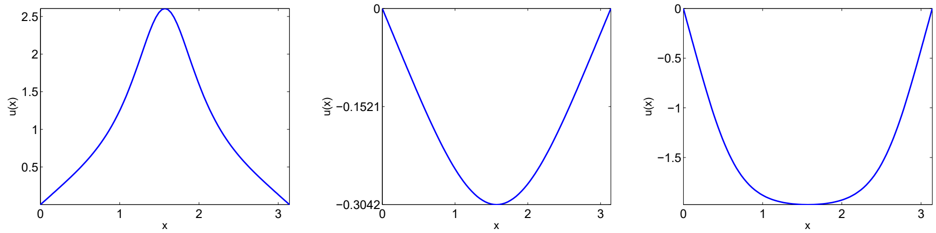


FIG. 5.7. *Numerical solutions of Example* 3: $p = 4$.

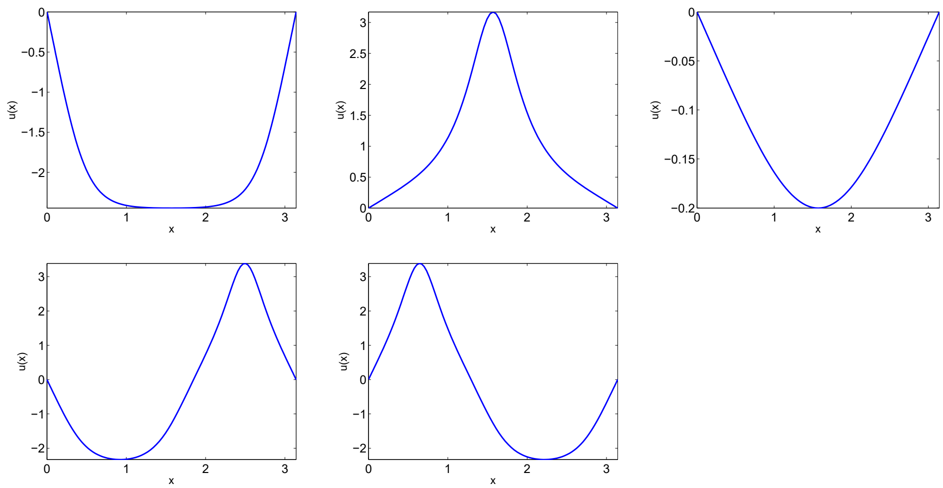


FIG. 5.8. *Numerical solutions of Example* 3: $p = 6$.

with two exact solutions

$$u^+(x) = \frac{x^2}{2}, \quad u^-(x) = -\frac{x^2}{2} + x. \tag{5.6}$$

We use the Legendre–Galerkin method shown in section 2.1 to solve this problem, in which we choose $N_c = 8$ to solve the nonlinear problem and $N_f = 16$ to solve the linearized problem. We obtain 64 numerical solutions on the coarse level (see part (i) of Figure 5.10). After the iteration on the fine level, there are only two stable numerical solutions which perfectly match the exact ones, as shown in part (ii) of Figure 5.10.

Example 5. Consider the two-dimensional steady Allen–Cahn equation on the unit square with Dirichlet boundary conditions [15]

$$\begin{cases} -\varepsilon^2 \Delta u + u^3 - u = 0, \quad (x, y) \in \Omega = (0, 1)^2, \\ u(0, y) = u(1, y) = 1, \quad y \in (0, 1), \\ u(x, 0) = u(x, 1) = -1, \quad x \in (0, 1). \end{cases} \tag{5.7}$$

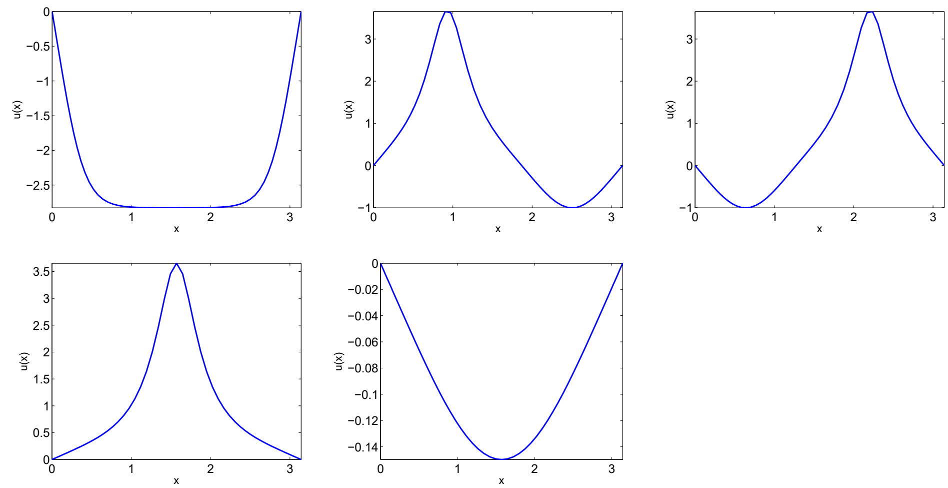
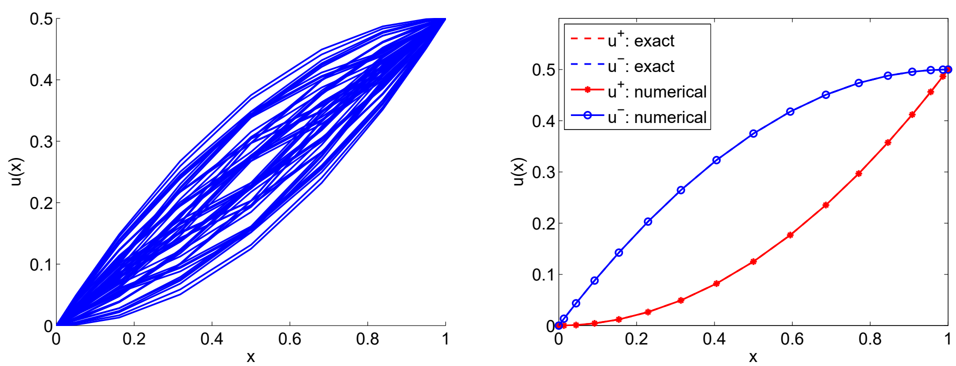


FIG. 5.9. *Numerical solutions of Example* 3: $p = 8$.



(i) Numerical solutions on the coarse level ($N_c = 8$)

(ii) Numerical and exact solutions on the fine level ($N_f = 16$)

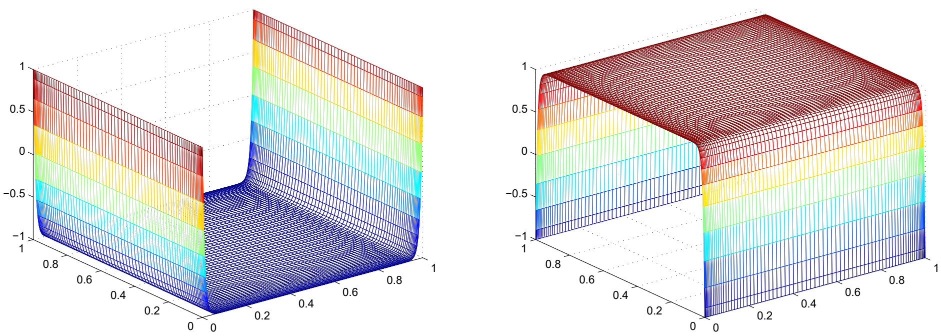
FIG. 5.10. *Numerical and exact solutions of Example* 4.

The Allen–Cahn equation is used to describe the motion of antiphase boundaries in crystalline solids, in which u represents the concentration of one of the two metallic components of the alloy. In the above equation, $u = +1$ corresponds to one material and $u = -1$ to the other.

There are no exact solutions available for this problem. We use the Legendre–Galerkin method to solve this problem with $\varepsilon = 0.04$. We choose $N_c = 16$ in the coarse level and a series of $N_f = 32, 64, 128$ in the fine level. We find two stable solutions, which are shown in Figure 5.11.

Example 6. Consider the following coupled system arising from chemical sciences and biology:

$$\begin{cases} \varepsilon_u \Delta u - uv^2 + F(1-u) = 0, \\ \varepsilon_v \Delta v + uv^2 - (F+k)v = 0 \end{cases} \tag{5.8}$$



(i) Negative solution (ii) Positive solution

FIG. 5.11. *Numerical solutions of Example* 5.

on the domain $[0, 2.5]^2$ with periodical boundary conditions, where $\varepsilon_u, \varepsilon_v, u, v$ are unknowns and F, k are parameters.

This reaction-diffusion system can be used to model the chemical reaction with two components [32] and irregular patterns [35]. We choose the parameters $\varepsilon_u = 2 \times 10^{-5}, \varepsilon_v = 10^{-5}, F = 0.1, k = 0.02$. We use the Fourier collocation method with $N_c = 3$ in the coarse level and $N_f = 6, 12, 24$ in the fine level to solve this problem. Actually, we obtained more than 2000 different patterns, some of which are shown in Figures 5.12–5.14.

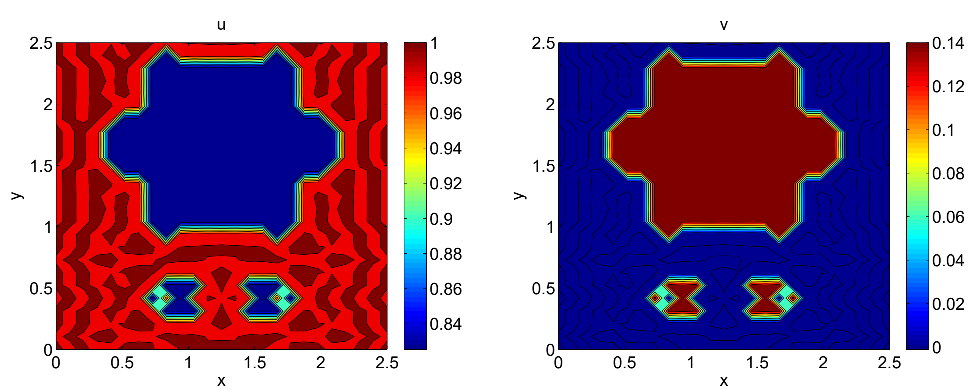


FIG. 5.12. *Numerical solutions of Example* 6.

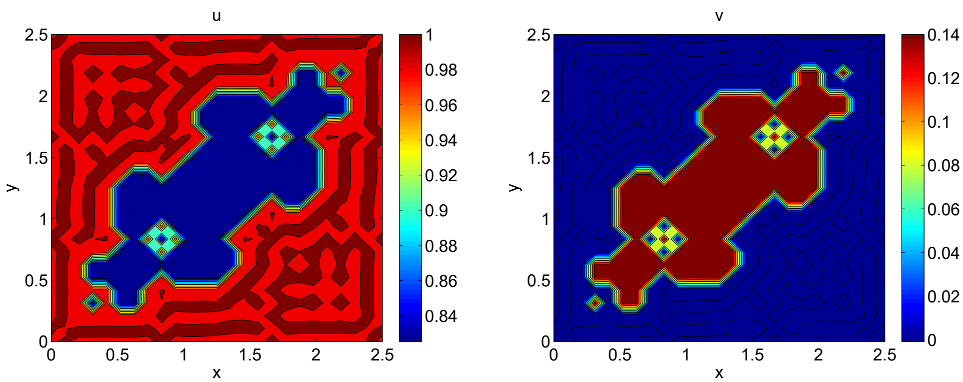


FIG. 5.13. *Numerical solutions of Example* 6.

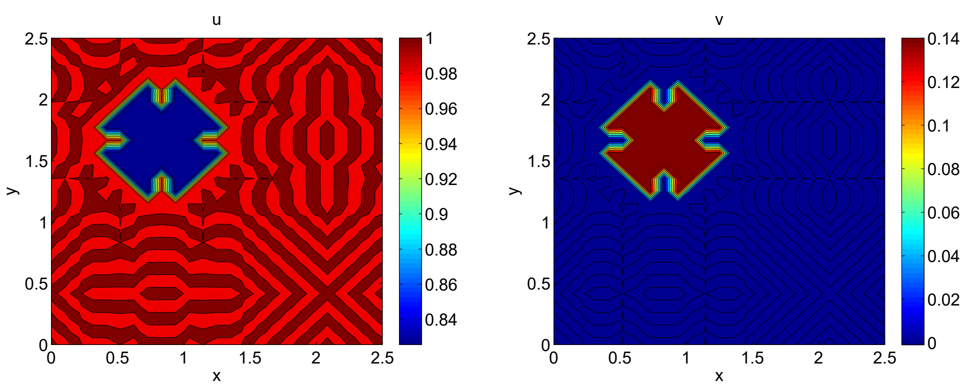


FIG. 5.14. *Numerical solutions of Example* 6.

6. Concluding remarks. We have developed a two-level framework of spectral methods for solving a class of semilinear and fully nonlinear elliptic problems with multiple solutions. The key to the success of the proposed method is the combination of the spectral discretization of differential equations, which enjoys high-order accuracy, and the homotopy continuation method for solving nonlinear systems, which guarantees the multiple solutions. We gain efficiency from the coarse level as well as accuracy from the fine level. The detailed algorithm shows that our method is very easy to implement and ready for parallel computing. We also derived the optimal error estimates for the Jacobi–Galerkin method and Newton iterations for semilinear problems. The error estimates for the fully nonlinear case will be addressed in a future work. Numerical experiments illustrate that our proposed methods, which enjoy high accuracy and efficiency in both semilinear and fully nonlinear problems, are obviously good alternatives to other numerical methods available in the literature for elliptic problems with polynomial nonlinearity.

To further reduce the computational cost, one can make use of the symmetric property [26, 51]. The results presented in this paper indicate that the gain in numerical efficiency in the two-level spectral methods should allow the consideration of more complicated problems in nontrivial stationary elliptic-type solutions of dynamical models.

Appendix A. Matrices in the Legendre–Galerkin method. $\boldsymbol{B}_1$ and $\boldsymbol{B}_4$ are the forward and backward Legendre transforms, respectively.

Let $\boldsymbol{B}_2 = (b^2_{n,k})^N_{n,k=0}, \boldsymbol{B}_3 = (b^3_{n,k})_{0\le n\le N, 0\le k\le N-2}, \boldsymbol{B}_5 = (b^5_{n,k})^N_{n,k=0}$, and $\boldsymbol{M} = (m_{n,k})^{N-2}_{n,k=0}, \boldsymbol{S} = (s_{n,k})^{N-2}_{n,k=0}$. Then their entries are defined by

$$b^2_{n,k} = \begin{cases} \dfrac{1}{n+1/2}, & n=k, \\ 0 & \text{otherwise}, \end{cases}$$

$$b^3_{n,k} = \begin{cases} 1, & n=k \text{ and } 0\le k,n\le N-2, \\ -1, & n=k+2 \text{ and } 0\le k\le N-2, 0\le n\le N, \\ 0 & \text{otherwise}, \end{cases}$$

$$b^5_{n,k} = \begin{cases} (n+1/2)(k(k+1)-n(n+1)), & (n+k) \text{ even and } n\le k-2, \\ 0 & \text{otherwise}, \end{cases}$$

$$m_{n,k} = m_{k,n} = \begin{cases} \dfrac{2}{2n+1} + \dfrac{2}{2n+5}, & n=k, \\ -\dfrac{2}{2n+3}, & n=k+2, \\ 0 & \text{otherwise}, \end{cases}$$

$$s_{n,k} = s_{k,n} = \begin{cases} 4n+6, & n=k, \\ 0 & \text{otherwise}. \end{cases}$$

Appendix B. Matrices in the Chebyshev collocation method. The second-order differentiation matrices $\boldsymbol{D}^{(2)} = \left(d^{(2)}_{k,j}\right)^N_{k,j=0}$ are defined by

$$d^{(2)}_{jl} = \begin{cases} \frac{(-1)^{j+l}}{\tilde{c}_l}\frac{x_j^2+x_jx_l-2}{(1-x_j^2)(x_j-x_l)^2}, & 1\le j\le N-1, 0\le l\le N, j\ne l, \\ -\frac{(N^2-1)(1-x_j^2)+3}{3(1-x_j^2)^2}, & 1\le j=l\le N-1, \\ \frac{2(-1)^l}{3\tilde{c}_l}\frac{(2N^2+1)(1-x_l)-6}{(1-x_l)^2}, & j=0, 1\le l\le N, \\ \frac{2(-1)^{l+N}}{3\tilde{c}_l}\frac{(2N^2+1)(1-x_l)-6}{(1+x_l)^2}, & j=N, 0\le l\le N-1, \\ \frac{N^4-1}{15}, & j=l=0, j=l=N, \end{cases} \tag{B.1}$$

where the $\{x_k\}^N_{k=0}$ in (B.1) are the Chebyshev–Gauss–Lobatto points defined in (2.19).

Acknowledgments. We would like to thank Prof. Jie Shen and Prof. Bei Hu for their helpful advice. We would like to thank the editors and anonymous referees for their careful reading and valuable suggestions, which allowed us to improve the presentation of the results.

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