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LONG-TIME DYNAMICS OF VECTORIAL VON KARMAN SYSTEM WITH NONLINEAR THERMAL EFFECTS AND FREE BOUNDARY CONDITIONS

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Abstract. This paper is concerned with long-time dynamics of a full von Karman system subject to nonlinear thermal coupling and *free* boundary conditions. In contrast with *scalar* von Karman system, vectorial full von Karman system accounts for both vertical and in plane displacements. The corresponding PDE is of critical interest in flow structure interactions where nonlinear plate/shell dynamics interacts with perturbed flows [vicid or invicid] [8, 9, 15]. In this paper it is shown that the system admits a global attractor which is also *smooth and of finite fractal dimension.* The above result is shown to hold for plates without regularizing effects of rotational inertia and without any mechanical dissipation imposed on vertical displacements. This is in contrast with the literature on this topic [15] and references therein. In order to handle highly supercritical nature of the von Karman nonlinearities, new results on "hidden" trace regularity generated by thermal effects are exploited. These lead to asymptotic compensated compactness of trajectories which then allows to use newly developed tools pertaining to *quasi stable* dynamical systems [8].

1. **Introduction.** This paper is concerned with long-time behavior and theory of global attractors associated with dynamic system of nonlinear elasticity modeled by a full *vectorial* von Karman system subject to thermal effects. This system describes nonlinear oscillations in a plate dynamics which account for *both* vertical and in plane displacements - denoted respectively by w and $u = (u_1, u_2)$ –as well as the averaged thermal stresses ϕ and θ affecting each of these displacements [17, 18, 34, 35]. The introduced mathematical model, being of interest on its own, is also a prototype for shallow shells with thermal effects [46, 52]. The latter are the building blocks for flow-structure interactions which have attracted a considerable attention in recent literature [9, 11, 15, 16]. In fact, the importance and interest in studying dynamical properties of *vectorial* Karman system can not be overstated. This is particularly pronounced for plate/shell models *without* the regularizing effects of

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rotational inertia. Indeed, stabilizing effect of the flow can only be attested for very thin plates which do not account for rotational inertia [9] and references therein. It is thus critical to be able to handle the analysis of full vectorial von Karman system *without rotational inertia*. On the other hand, mathematical treatment of such models is challenging due to severe singularities caused by nonlinear effects which are no longer mitigated by the additional regularity of vertical velocity exhibited in rotary inertial models. We shall exploit thermal effects as the carriers and propagators of partial regularity which, in turn, will allow for construction of a well posed dynamical system with a smooth and finite-dimensional long time behavior.

1.1. **The problem studied. PDE model.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two nonintersecting nonempty portions. We consider the following PDE evolutionary system

$$u_{tt} - \operatorname{div}\{\sigma[\epsilon(u) + f(\nabla w)]\} + \nabla\phi + p_1(u,w) = 0 \text{ in } \Omega\times\mathbb{R}^+, \quad (1.1)$$

$$w_{tt} + \Delta^2 w - \operatorname{div}\{\sigma[\epsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w\} + \Delta\theta + p_2(u,w) = 0 \text{ in } \Omega\times\mathbb{R}^+, \quad (1.2)$$

with Dirichlet boundary conditions on the portion of the boundary Γ_0,

$$u = 0,\ w = 0,\ \nabla w = 0 \text{ on } \Gamma_0 \times \mathbb{R}^+. \quad (1.3)$$

The boundary conditions on Γ_1 are of *free* type and given by

$$\sigma[\epsilon(u) + f(\nabla w)]\nu + \kappa u - \phi\nu + u_t = 0, \quad (1.4)$$

$$\Delta w + (1-\mu)B_1 w + \theta = 0, \quad (1.5)$$

$$\partial_\nu(\Delta w) + (1-\mu)B_2 w - \sigma[\epsilon(u) + f(\nabla w)]\nu\cdot\nabla w - \phi\,\partial_\nu w + \partial_\nu\theta = 0. \quad (1.6)$$

The average thermal stress ϕ and thermal moment θ are given by the following system of equations

$$\phi_t - \Delta\phi + \operatorname{div}\{u_t\} - \nabla w\cdot\nabla w_t = 0 \text{ in } \Omega\times(0,\infty), \quad (1.7)$$

$$\theta_t - \Delta\theta - \Delta w_t = 0 \text{ in } \Omega\times(0,\infty), \quad (1.8)$$

with boundary conditions

$$\partial_\nu\phi + \lambda_1\phi = \partial_\nu\theta + \lambda_2\theta = 0 \text{ on } \Gamma\times(0,\infty), \quad (1.9)$$

where $\lambda_1, \lambda_2 > 0$. The initial conditions are given by

$$u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, w(\cdot,0) = w_0, w_t(\cdot,0) = w_1, \phi(\cdot,0) = \phi_0, \theta(\cdot,0) = \theta_0. \quad (1.10)$$

In the system (1.1)-(1.2), $p_1(u,w) = (p_{1,1}(u,w), p_{1,2}(u,w))$ and $p_2(u,w)$ represent forces exerted by some nonlinear elastic foundation. Regarding physical parameters of the system, we have that $\mu \in (0,\frac{1}{2})$ is the Poisson's modulus and $\sigma[\,\cdot\,]$ is a tensor defined by

$$\sigma[A] = \lambda\,\operatorname{trace}[A]I + 2\eta A, \quad (1.11)$$

where $\lambda = E\mu/(1-2\mu)(1+\mu)$, $\eta = E/2(1+\mu)$ and E is the Young's modulus. The strain tensor is given by $\epsilon(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^\top\big)$, where ∇u denotes the Jacobian matrix of vector u, and the nonlinearity f is defined by $f(s) = \frac{1}{2}(s\otimes s)$, $s\in\mathbb{R}^2$. The boundary operators are given by

$$B_1 w = 2\nu_1\nu_2 w_{xw} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx},$$
$$B_2 w = \partial_\tau\big[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2\,(w_{yy} - w_{xx})\big] - lw,$$

where $\nu = (\nu_1, \nu_2)$ and $\tau = (\tau_1, \tau_2)$ represent normal and tangential directions to the boundary Γ_1, and l is a positive parameter.

We note that the presence of in-plane displacements provides for a nonlinear mixing of high energy between vertical and in plane accelerations. This very feature proves challenging in establishing uniqueness and continuous dependence on the data within finite energy framework. When rotational inertia are included (ie the term $-\gamma\Delta w_{tt}$ added to the "w" equation (1.2)), methods of weak compactness and cancellations come to rescue [28]. This is no longer valid when $\gamma = 0$ - as in the present model.

The goal of the paper is to establish existence of global attractor which captures asymptotic behavior of the nonlinear dynamics. In addition, we shall prove that such attractor is both *finite dimensional and smooth*. We note that the model neither includes *mechanical dissipation on vertical displacements* of the plate, nor accounts for rotational inertia term $\gamma\Delta w_{tt}$ which has regularizing effect on the dynamics. This is in striking contrast with the most of the literature on the topic [5, 15, 32, 33, 34, 35, 37]. In fact, since both *regularity of the dynamics* and a presence of sufficient *dissipation* have critical bearing on establishing smooth asymptotic behavior of the trajectories, proving such property for a model which has only *limited dissipation and limited regularity* is the main challenge undertaken in the present paper. Even more, $\gamma > 0$ is essential in proving uniqueness of weak solutions to a full vectorial von Karman system [28]. With $\gamma = 0$ the uniqueness and wellposedness of the corresponding dynamical system must be harvested from thermal effects.

The system described by (1.1)-(1.9) involves nonlinearly coupled thermoelastic plate with thermoelastic waves. Since thermoelastic plates are associated with analytic semigroups [41, 42, 44], we are faced with a combination of parabolic and hyperbolic like dynamics. The nonlinear effects are at the supercritical level (this means that the nonlinear terms for finite energy solutions are not bounded in a finite energy space). Indeed, finite energy displacements $u \in H^1(\Omega)$, $w \in H^2(\Omega)$ produce nonlinear stresses

$$\operatorname{div}\{f(\nabla w)\} \in H^{-\varepsilon}(\Omega) \text{ and } \operatorname{div}\{\sigma[\epsilon(u) + f(\nabla w)]\nabla w\} \in H^{-1-\varepsilon}(\Omega),\ \varepsilon > 0 \text{ (small)}.$$

Thus we are dealing with a loss of $1+\varepsilon$ derivative. This feature becomes a major difficulty in the study of Hadamard wellposednesss (uniqueness and continuous dependence on the initial data) and, above all, in obtaining the needed estimates for the existence of attractors. While parabolic like structure is typically equipped with additional regularity properties, the challenge in the present problem is the "transfer" of these beneficial effects to the hyperbolic part of the system. The carriers of propagation in the case of *free boundary conditions* are boundary traces. Thus, at the technical level, we will be concerned with "hidden" trace regularity properties which will play a role of propagators of both regularity and stability. It is well known that the analysis of free boundary conditions, in the context of thermoelasticity, is a challenging subject-even within the linear theory [41]. This is due to the fact that Kirchhoff plate with free boundary conditions does not satisfy *strong Lopatinski condition* [50]. It is well known that Lopatinski condition is responsible for "hidden" regularity of boundary traces in hyperbolic dynamics [40, 50]. In the absence of such, other tools based on microlocal calculus need to be brought to the analysis.

To our best knowledge the present paper is a first study of attractors for the dynamics described by *full* vectorial von Karman thermoelastic system with *free* boundary conditions and *with no dissipation, nor regularity imposed* on vertical displacement. This will be even more evident from the detailed review of the literature provided below.

1.2. **Discussion of the literature.** The analysis of wellposedness and of long-time behavior in nonlinear thermoelasticity has been a subject of long lasting interest [1, 19, 32, 45, 46]. Various models with different boundary conditions have been considered. However the physical interest-relevance and the degree of mathematical challenge does depend critically on the specific model and the associated *boundary conditions.* These create different configurations that require diverse mathematical treatments. The overriding desire has been to control long-time behavior of the model with a *minimal* amount of dissipation. By controlling, we mean either to steer trajectories to zero, when the external forces are absent, or driving solutions asymptotically to a pre-assigned compact set in the phase space (attractor). The structure of such attractor depends on the forcing terms $p_1(u, w)$, $p_2(u, w)$. It has been observed that thermal dissipation provides substantial damping mechanism for the oscillations so that there may be no need for mechanical dissipation. In fact, such property has been proved for the first time in a special case of a scalar linear plate equation with hinged boundary conditions [26] and in one-dimensional configuration such as thermoelastic rods [23]. However, in the case of *free* boundary conditions stabilization results in [32, 33] do require mechanical dissipation (also for thermal plates) imposed on the boundary of the plate. Only recently it has been shown that in the case of linear thermoelastic plates, uniform decay to zero of the energy can be achieved *without any mechanical dissipation*, regardless of the boundary conditions [2, 3, 41, 42]. The situation is very different when one considers vectorial structures, including thermoelastic waves. Here no longer one has smoothing property of the dynamics or *uniform* decay to zero of the energy. The best one can achieve ,without additional mechanical dissipation, is strong stability to zero with a polynomial rate [19, 27]. The problem considered in this paper falls into a category of mixed (parabolic-hyperbolic) dynamics with vectorial structure of thermal plates and waves which are nonlinear and strongly coupled. Our aim is to show that nonlinear coupling, while making estimates challenging (due to singularities of nonlinear terms), does provide beneficial mechanism in propagating thermal dissipation onto the entire system - thus forging the desired long-time behavior. The final result is that the dynamics becomes asymptotically *finite dimensional and smooth.* While this kind of result is to be expected for the dynamics with an overall smoothing effect, it is much less expected in hyperbolic type of models without strong mechanical dissipation and with highly unbounded nonlinear effects. The analysis in this paper illustrates the situation when asymptotic regularity and dissipation can be harvested from thermal effects *via boundary traces* which become the carriers of the propagation. Since the dynamics of the plate alone is hyperbolic-like and unstable, establishing the said propagation is a subtle issue-mainly due to the nature of "free" boundary conditions.

From the above discussion it follows that the combination of *free boundary conditions* in *vectorial structure* with *the lack of rotational inertia* ($\gamma = 0$) and *strong nonlinearity* induced by vectorial structure of the system are the main new features and obstacles in the analysis. The very first result addressing this problem was given in [39] where full von Karman system, without rotational inertia ($\gamma = 0$)

and with thermal effects was considered. *Uniform stabilization* to zero with free boundary conditions and boundary dissipation imposed on the in plane velocities was there established. The critical ingredient used for the analysis in [39] is partial thermal smoothing of a single unperturbed trajectory. The present paper takes this analysis to the next level, in the direction of dynamical systems and theory of attractors. This presents new set of challenges mainly due to nonlinear effects which are supercritical (with respect to finite energy space). These prevent the use of known tools in the area of attractors. Nevertheless, we shall show that this strongly nonlinear nonsmooth transient dynamics can be reduced asymptotically to a smooth and finite dimensional set. This will be achieved through a boundary frictional damping applied only to in plane displacements and *without any mechanical damping* imposed on vertical displacements. The necessity of some mechanical damping imposed on in plane displacements results from well known negative results on the lack of uniform stability in thermal linear waves whenever the dimension of the domain is greater than one [19]. At the technical level, our results critically benefit from the new quasi stability theory [8, 13] and "hidden trace regularity" harvested from thermal effects. While [39] provides preliminary road map for the needed estimates, there is a fundamental difference between the theory of attractors dealt with in the present paper and stabilization theory of [39]. While stabilization requires estimates for a single trajectory, theory of attractors require estimates for the differences of trajectories. In the case when nonlinearity of the dynamics is supercritical, single trajectory estimates may still exploit some cancellations. This is not the case with the estimates for the difference of trajectories where the mixing of nonlinearities occurs. Superlinearity does not disappear in the calculations. In order to handle this difficulty, new set of boundary trace estimates will be developed. These estimates are also of independent PDE interest.

1.3. **Main results.** We begin by introducing some notation. For the norms of standard H^s (Sobolev) and L^2 spaces we use: $\|u\|_{\alpha,\Omega} = \|u\|_{H^\alpha(\Omega)}$, $\|u\|_{\alpha,\Gamma} = \|u\|_{H^\alpha(\Gamma)}$, and the case $\alpha = 0$, which corresponds to L^2 spaces, we write $\|u\|_\Omega = \|u\|_{L^2(\Omega)}$ and $\|u\|_\Gamma = \|u\|_{L^2(\Gamma)}$. The corresponding inner products are denoted by $(u,v)_\Omega = (u,v)_{L^2(\Omega)}$ and $\langle u,v\rangle_\Gamma = \langle u,v\rangle_{L^2(\Gamma)}$. For $\alpha > 0$, the space $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, and $H^{-\alpha}(\Omega) = [H_0^\alpha(\Omega)]'$, where the duality is taken with respect to $L^2(\Omega)$ inner product. Occasionally, by the same symbol , we denote norms and inner products of n-copies of $L^2(\mathscr{O})$, where $\mathscr{O}$ is either Ω or Γ. The same is applied to $H^\alpha(\mathscr{O})$. We also consider the following Sobolev spaces

$$H^i_{\Gamma_0}(\Omega) \equiv \big\{ v \in H^i(\Omega) \mid \text{trace of } v = 0 \text{ on } \Gamma_0 \big\}, \ \ i = 1,2, \text{ with } \|v\|_{H^1_{\Gamma_0}(\Omega)} = \|\nabla v\|_\Omega,$$

The analysis for weak solutions of our system will be done on the (phase) space

$$\mathcal{H} \equiv [H^1_{\Gamma_0}(\Omega)]^2 \times [L^2(\Omega)]^2 \times H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2,$$

and the regularity of solutions will be studied in

$$\mathcal{H}_1 \equiv [H^2(\Omega)]^2 \times [H^1(\Omega)]^2 \times H^4(\Omega) \times H^2(\Omega) \times [H^2(\Omega)]^2.$$

The assumptions imposed on the external forcing terms p_1, p_2 are introduced below. Let $u = (u_1, u_2)$ and $p_1(u,w) = \big(p_{1,1}(u,w), p_{1,2}(u,w)\big)$, we assume that there exists a C^2 function $P : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\nabla P(u,w) = \big(p_{1,1}(u,w), p_{1,2}(u,w), p_2(u,w)\big), \tag{1.12}$$

and satisfies the following conditions: there exist $M, m_P \geqslant 0$ such that

$$P(u,w) \geqslant -M\big(|u_1|+|u_2|+|w|^2\big) - m_P,\ \ \forall\, u_1,u_2,w \in \mathbb{R}, \tag{1.13}$$

with

$$0 \leqslant M < M_0, \tag{1.14}$$

where M_0 is a positive constant to be defined in (2.25), dependent on σ and on the Korn and Sobolev inequalities. We also assume there exist $r \geqslant 1$ and $M_p > 0$ such that, for $i = 1, 2$,

$$|\nabla p_{1,i}(u,w)| \leqslant M_p\big(1+|u_1|^{r-1}+|u_2|^{r-1}+|w|^{r-1}\big),\ \ \forall\, u_1,u_2,w \in \mathbb{R}, \tag{1.15}$$

$$|\nabla p_2(u,w)| \leqslant M_p\big(1+|u_1|^{r-1}+|u_2|^{r-1}+|w|^{r-1}\big),\ \ \forall\, u_1,u_2,w \in \mathbb{R}. \tag{1.16}$$

Furthermore, we assume that

$$\nabla P(u,w)\cdot(u,w) - P(u,w) \geqslant -M\big(|u_1|+|u_2|+|w|^2\big) - m_P,\ \ \forall\, u_1,u_2,w \in \mathbb{R}. \tag{1.17}$$

We note that (1.15)-(1.16) imply that there exists $M_P > 0$ such that

$$P(u,w) \leqslant M_P\big(1+|u_1|^{r+1}+|u_2|^{r+1}+|w|^{r+1}\big),\ \ \forall\, u_1,u_2,w \in \mathbb{R}. \tag{1.18}$$

The wellposedness and regularity of solutions to our system are given below.

Theorem 1.1. *Assume that the forcing terms p_i satisfy (1.12)-(1.17). Then:*
(i) *Weak solutions: For any $T > 0$ and initial data $(u_0,u_1,w_0,w_1,\phi_0,\theta_0) \in \mathcal{H}$, problem (1.1)-(1.10) has a unique weak (finite energy) solution $(u,u_t,w,w_t,\phi,\theta) \in C\big([0,T];\mathcal{H}\big)$. In addition $u_t|_\Gamma \in L^2(0,T;L^2(\Gamma))$, and $\nabla\theta, \nabla\phi \in L^2(0,T;L^2(\Omega))$. Moreover, this solution depends continuously on the initial data.*
(ii) *Regular solutions: Assume that above initial data has further regularity $\mathcal{H}_1$ and appropriate compatibility with respect to the boundary. Then problem (1.1)-(1.10) has a unique regular solution $(u,u_t,w,w_t,\phi,\theta) \in C\big([0,T];\mathcal{H}_1\big)$ with $(u_{tt},w_{tt},\phi_t,\theta_t) \in C\big([0,T];[L^2(\Omega)]^5\big)$.*

In the absence of forcing terms p_1, p_2, the wellposedness of problem (1.1)-(1.10) with respect to weak and strong solutions was proved in [39, Theorem 1.1] by using nonlinear semigroup theory along with partial analyticity generated by thermal effects. Since the Nemytskii mapping associated to the forcing p_1, p_2, is locally Lipschitz on $\mathcal{H}$, the existence of a unique local solution is granted by semigroup theory (e.g. [10, Theorem 7.2]). Such solutions can be extended to any time interval $[0,T]$ by using apriori estimate (2.24) below.

Remark 1.1. Theorem 1.1 implies that the (weak) solution operator of problem (1.1)-(1.10) is a strongly continuous semigroup $\{S(t)\}$ on $\mathcal{H}$, which generates well-defined nonlinear dynamical system $(\mathcal{H}, S(t))$.

Remark 1.2. With $\gamma > 0$ and absence of thermal effects, Hadamard wellposedness has been proved in [28]. Wellposedness of full von Karman system with thermal effects, $\gamma = 0$ and strains accounting for shell's curvature has been recently shown in [46] by resorting to methods of [28].

To establish the existence of attractors one needs the following geometric condition imposed on the "uncontrolled" portion of the boundary Γ_0. There exists $x_0 \in \mathbb{R}^2$ such that

$$(x-x_0)\cdot\nu \leqslant 0, \quad x \in \Gamma_0. \tag{1.19}$$

Remark 1.3. We note that the geometric assumption in (1.19) is much weaker than the geometric assumption typically imposed in controllability/stabilizability theory [30, 32, 35]. It involves only uncontrolled part of the boundary Γ_0, rather than a full boundary $\partial\Omega$.

Our main result reads as follows.

Theorem 1.2. *Assume that conditions* (1.12)-(1.17) *and* (1.19) *are in force. Then, the dynamical system* $(\mathcal{H}, S(t))$ *generated by the problem* (1.1)-(1.10) *admits a compact global attractor* $\mathbf{A}\subset\mathcal{H}$. *The said attractor is of finite fractal dimension and it is bounded in a more regular space* $\mathcal{H}_1$.

Remark 1.4. By assuming additional regularity on the forcing $p_1(u,w), p_2(u,w)$ one can reiterate the proof of Theorem 1.2 in order to obtain C^∞ dynamics on the attractor $\mathbf{A}$. See for instance [13, 21].

Remark 1.5. The result stated in Theorem 1.2 is also valid with nonlinear damping imposed on u_t in (1.4). Instead of u_t one can take $g(u_t)$ with $g(s)$ monotone increasing, continuous, $g(0)=0$ and subject to: $g(s)s \leqslant Ms^2, |s|>1$ and $g(s)s \geqslant ms^2$ for $|s|<1$. The above modification will introduce additional technicalities which can be handled as in [12]. In the case when $g(s)$ has unqualified growth at the origin, only the first statement in Theorem 1.2 remains valid.

Remark 1.6. It will be interesting to see whether the result in Theorem 1.2 still holds when (1) Γ_0 has zero measure and (2) in plane displacements do not account for thermal effects ϕ. We note that in this situation uniform stability to an equilibrium for the unforced plate still holds [39]. However, when studying attractors, items (1) and (2) are needed for the proof of an appropriate unique continuation property. Whether the latter can be proved under weaker assumption, remains an open problem.

The proof of Theorem 1.2 will be given in Section 3. Here, we note, that the key of the proof relies on the following two ingredients: (1) novel abstract criterion in the area of dynamical systems which relies on *quasistability property* of the dynamical system presented in Section 2 and (2) verification of the abstract condition which depends on new PDE - boundary trace estimates for nonlinear system under consideration. The latter are presented in Section 3.

2. **Preliminary results and inequalities.**

2.1. **Energy relations.** Along a solution $y(t)=(u(t),u_t(t),w(t),w_t(t),\phi(t),\theta(t))$, $t\geqslant 0$, the energy of the system is defined by

$$\mathcal{E}_y(t)=E_y(t)+\int_\Omega P(u,w)\,\mathrm{d}\Omega, \tag{2.20}$$

where $E_y(t)=E_k(t)+E_p(t)$. Here, $E_k(\cdot)$ is the kinetic energy defined by

$$E_k(t)=\frac{1}{2}\int_\Omega\big[|u_t|^2+|w_t|^2\big]\mathrm{d}\Omega,$$

and $E_p(\cdot)$ is the potential energy given by

$$E_p(t)=\frac{1}{2}\int_\Omega\big[\sigma[N(u,w)]N(u,w)+|\phi|^2+|\theta|^2\big]\mathrm{d}\Omega+\frac{1}{2}a(w,w)+\frac{\kappa}{2}\int_{\Gamma_1}|u|^2\mathrm{d}\Gamma_1,$$

where the resultant stress $N(u,w)$ is given by $N(u,w) = \epsilon(u) + f(\nabla w)$ and

$$a(w,z)=\int_\Omega\big[w_{xx}z_{xx}+w_{yy}z_{yy}+\mu w_{xx}z_{yy}+\mu w_{yy}z_{xx}+2(1-\mu)w_{xy}z_{xy}\big]\mathrm{d}\Omega+l\int_{\Gamma_1}wz\mathrm{d}\Gamma_1.$$

It follows that the energy satisfies the identity

$$\mathcal{E}_y(t)+\int_s^t\big[\|u_t(\tau)\|^2_{\Gamma_1}+\|\nabla\phi(\tau)\|^2_\Omega+\|\nabla\theta(\tau)\|^2_\Omega+\lambda_1\|\phi(\tau)\|^2_\Gamma+\lambda_2\|\theta(\tau)\|^2_\Gamma\big]\mathrm{d}\tau=\mathcal{E}_y(s). \tag{2.21}$$

Indeed, for regular solutions, the proof of (2.21) is standard and follows from classical energy type arguments. For weak solutions the energy function satisfies the inequality. However, due to the uniqueness of weak solutions, one also shows by convexity methods [39] that actually (2.21) holds for all weak solutions.

Next we establish a relation between $\mathcal{E}_y(\cdot)$ and $E_y(\cdot)$. To this end, we note that for $u \in H^1(\Omega)$, the Korn inequality together with Sobolev embedding give

$$\|u\|^2_{1,\Omega} \leqslant M_K\big[\|N(u,v)\|^2_\Omega + \|w\|^4_{W^{1,4}(\Omega)}\big]. \tag{2.22}$$

Also, the definition of tensor $\sigma[\,\cdot\,]$ in (1.11) implies that

$$\int_\Omega \sigma[N(u,w)]N(u,w)\mathrm{d}\Omega \geqslant M_\sigma\|N(u,v)\|^2_\Omega. \tag{2.23}$$

The following lower bound is critical.

Lemma 2.1. *Let the assumption* (1.12)-(1.17) *be satisfied. Then, there exists constants* $M_E, m_E > 0$ *such that*

$$\mathcal{E}_y(t) \geqslant M_E E_y(t) - m_E, \quad \forall\, t \geqslant 0. \tag{2.24}$$

Proof. Let us define

$$M_0 = \min\left\{\frac{M_\sigma}{4M_pM_K}, \frac{M_a}{4(M_pM_KM_2+M_p)}\right\}, \tag{2.25}$$

with $M_p = \max\{M_1, M_2\} > 0$, where M_1, M_2 denote the corresponding embedding constants

$$\|u\|_\Omega \leqslant M_1\|u\|_{1,\Omega}, \ \ \|w\|^2_\Omega + \|w\|^2_{W^{1,4}(\Omega)} \leqslant M_2\|w\|^2_{2,\Omega},$$

and $M_a > 0$ is a constant such that $a(w,w) \geqslant M_a\|w\|^2_{2,\Omega}$. From inequalities (2.22)-(2.23) we obtain

$$\begin{aligned}
\mathcal{E}_y(t) &= E_k(t) + \frac{1}{2}E_p(t) + \frac{1}{4}\int_\Omega\big[\sigma[N(u,w)]N(u,w) + |\phi|^2 + |\theta|^2\big]\mathrm{d}\Omega + \frac{1}{4}a(w,w)\\
&\quad + \frac{\kappa}{4}\int_{\Gamma_1}|u|^2\mathrm{d}\Gamma_1 + \int_\Omega P(u,w)\,\mathrm{d}\Omega\\
&\geqslant CE_y(t) + \frac{M_\sigma}{4}\|N(u,v)\|^2_\Omega + \frac{M_a}{4}\|w\|^2_{2,\Omega} - MM_p\big[\|u\|_{1,\Omega} + \|w\|^2_{2,\Omega}\big] - m_P|\Omega|\\
&\geqslant CE_y(t) + \left[\frac{M_\sigma}{4} - MM_pM_K\right]\|N(u,v)\|^2_\Omega - MM_pM_k\|w\|^2_{W^{1,4}(\Omega)}\\
&\quad + \left[\frac{M_a}{4} - MM_p\right]\|w\|^2_{2,\Omega} - m_E\\
&\geqslant CE_y(t) + \left[\frac{M_\sigma}{4} - MM_pM_K\right]\|N(u,v)\|^2_\Omega\\
&\quad + \left[\frac{M_a}{4} - MM_pM_KM_2 - MM_p\right]\|w\|^2_{2,\Omega} - m_E,
\end{aligned}$$

where $m_E = m_P|\Omega| + (1/4)MM_K$. Since $M < M_0$ (assumption (1.14)), it follows that (2.24) holds. □

Remark 2.1. We note that the potential energy $E_p(\cdot)$ is topologically equivalent to space $[H^1(\Omega)]^2 \times H^2(\Omega) \times [L^2(\Omega)]^2$ and therefore $E_y(\cdot)$ is topologically equivalent to the space $\mathcal{H}$.

2.2. **Tensor identities.** In order to simplify the verification of some rather long calculations, we provide a few elementary tensor identities. Let us define the vector field: $h(x) = x - x_0$ with $x_0 \in \mathbb{R}^2$. Then we have that

$$\epsilon(h\nabla u) = \epsilon(u) + \mathscr{R}, \tag{2.26}$$

where $\mathscr{R}$ is a tensor given by

$$\mathscr{R} = \begin{bmatrix} \sum_{i=1}^{2} \frac{\partial^2 u_1}{\partial x_1 \partial x_i} h_i & \frac{1}{2}\sum_{i=1}^{2}\left[\frac{\partial^2 u_1}{\partial x_i \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_i}\right] h_i \\ \frac{1}{2}\sum_{i=1}^{2}\left[\frac{\partial^2 u_1}{\partial x_i \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_i}\right] h_i & \sum_{i=1}^{2} \frac{\partial^2 u_2}{\partial x_i \partial x_2} h_i \end{bmatrix}.$$

Given two (fourth order) tensors A, B, written as 4-vectors, we define $(A, B)_{\mathbb{R}^4}$ the inner product in $\mathbb{R}^4$. Then, if $A = [a_{i,j}]$ is a symmetric tensor, we can show that

$$(A, \mathscr{R})_{\mathbb{R}^4} = \sum_{i,j,k=1}^{2} a_{k,j} \frac{\partial^2 u_j}{\partial x_k \partial x_i} h_i. \tag{2.27}$$

Let $B = [b_{i,j}]$ be another symmetric tensor such that $a_{j,i} = \sum_{l=1}^{2} c_{j,l} b_{l,i}$, with constant and symmetric coefficients $c_{j,i}$. Then

$$\begin{aligned} \operatorname{div}\{(A, B)_{\mathbb{R}^4} h\} &= (A, B)_{\mathbb{R}^4} \operatorname{div}\{h\} + \sum_{i,j,k,l=1}^{2} c_{i,l} \frac{\partial}{\partial x_k}\left[b_{l,j} b_{j,i}\right] h_k \\ &= (A, B)_{\mathbb{R}^4} \operatorname{div}\{h\} + 2\sum_{i,j,k=1}^{2} a_{j,i} \frac{\partial b_{j,i}}{\partial x_k} h_k. \end{aligned}$$

Taking $A = \sigma[\epsilon(u)]$ and $B = \epsilon(u)$, we obtain

$$\operatorname{div}\{(\sigma[\epsilon(u)], \epsilon(u))_\Omega h\} = 2\big(\sigma[\epsilon(u)], \epsilon(u)\big)_\Omega + 2\sum_{i,j,k=1}^{2} \Big(a_{i,j}, \frac{\partial^2 u_i}{\partial x_k \partial x_j} h_k\Big)_\Omega. \tag{2.28}$$

Now, taking $A = \sigma[\epsilon(u)]$ and using identities (2.26) and (2.27), we obtain

$$\big(\sigma[\epsilon(u)], \epsilon(h\nabla u)\big)_\Omega = \big(\sigma[\epsilon(u)], \epsilon(u)\big)_\Omega + \sum_{i,j,k=1}^{2} \Big(a_{i,j}, \frac{\partial^2 u_i}{\partial x_k \partial x_j} h_k\Big)_\Omega. \tag{2.29}$$

2.3. **Dynamics of quasi-stable systems.** In this subsection we provide recent results pertaining to long time behavior of quasi–stable systems [8, 13]. These results are critical for the development, since classical approaches in dynamical system theory based on decomposition of trajectories [22, 31, 45, 51] are not applicable within the context of supercritical nonlinearities. We begin with the following classical result, cf. [4, 12, 13, 22, 31, 51].

Theorem 2.1. *Let $(H, S(t))$ be a dynamical system, dissipative and asymptotically smooth. Then it possesses a unique compact global attractor $\mathbf{A} \subset H$.*

Another type of dissipativeness is characterized by gradient systems, that is, systems possessing a strict Lyapunov function. In other words, there is a functional $\Phi : H \to \mathbb{R}$ such that
(i) the map $t \mapsto \Phi(S(t)y)$ is non-increasing for any $y \in H$,
(ii) if $\Phi(S(t)y) = \Phi(y)$ for all t, then y is a stationary point of $S(t)$.

Regarding the structure of the attractors we know that $\mathbb{M}_+(\mathcal{N}) \subset \mathbf{A}$, where $\mathcal{N}$ is the set of stationary points of $\{S(t)\}$ and $\mathbb{M}_+(\mathcal{N})$ is the unstable manifold of $y \in H$ such that there exists a full trajectory $u(t)$ satisfying

$$u(0) = y \ \text{ and } \lim_{t \to -\infty} \operatorname{dist}(u(t), \mathcal{N}) = 0.$$

For gradient systems it is possible to prove that the unstable manifold $\mathbb{M}_+(\mathcal{N})$ coincides with the attractor $\mathbf{A}$. The following result is well-known. See for instance [13, Corollary 7.5.7].

Theorem 2.2. *Let $(H, S(t))$ be an asymptotically smooth gradient system with the corresponding Lyapunov functional denoted by Φ. Suppose that*

$$\Phi(y) \to \infty \ \textit{ if and only if } \ \|y\|_H \to \infty, \tag{2.30}$$

and that the set of stationary points $\mathcal{N}$ is bounded. Then $(H, S(t))$ has a compact global attractor which coincides with the unstable manifold $\mathbb{M}_+(\mathcal{N})$.

Remark 2.2. The advantage of the theorem above is that for gradient systems an existence of global attractor does not require proving existence of an absorbing ball - a task that can be technical and cumbersome.

Our aim is to establish existence of a global attractor along with the properties such a finite-dimensionality and smoothness. In order to achieve this we shall exploit the concept of quasistability - Definition 7.9.2 in [13, Chapter 7] - which allows to prove such properties in "one shot" provided one has "good" estimates for the differences of two trajectories originating in a bounded set $B \subset H$.

Let X, Y, Z be three reflexive Banach spaces with X compactly embedded into Y, and define $H = X \times Y \times Z$. Suppose that $(H, S(t))$ is a dynamical system of the form

$$S(t)y = (u(t), u_t(t), \xi(t)), \qquad y = (u(0), u_t(0), \xi(0)) \in H, \tag{2.31}$$

where the functions u and ξ have regularity

$$u \in C([0,\infty); X) \cap C^1([0,\infty); Y), \quad \xi \in C([0,\infty); Z). \tag{2.32}$$

Then we say that $(H, S(t))$ is quasi-stable on a set $B \subset H$, if there exists a compact semi-norm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0,\infty)$, and $b(t) \in L^1(0,\infty)$ with $\lim_{t\to\infty} b(t) = 0$, such that,

$$\|S(t)y^1 - S(t)y^2\|_{\mathcal{H}}^2 \leqslant a(t)\|y^1 - y^2\|_{\mathcal{H}}^2, \tag{2.33}$$

and, for $S(t)y^i = (u^i(t), u_t^i(t), \xi^i(t))$, $i = 1, 2$,

$$\|S(t)y^1 - S(t)y^2\|_{\mathcal{H}}^2 \leqslant b(t)\|y^1 - y^2\|_{\mathcal{H}}^2 + c(t) \sup_{0 \leqslant s \leqslant t} \left[n_X(u^1(s) - u^2(s))\right]^2, \tag{2.34}$$

for any $y^1, y^2 \in B$. In this case the following result holds.

Remark 2.3. Quadratic dependence of the compact term in the inequality (2.34) is critical. In fact, achieving this quadratic dependence is one of the main technical difficulties of the problem. We note that in order to obtain just an existence of compact attractor, a much weaker form of this inequality suffices. In particular, there is no restriction on the power of compact term (could be sublinear).

Theorem 2.3. [13, Proposition 7.9.4] *Let $(H, S(t))$ be a dynamical system given by (2.31) and satisfying (2.32). Then $(H, S(t))$ is asymptotically smooth if it is quasi-stable on every bounded positively invariant set of H.*

The most useful property of quasi-stable systems is that quasistability on the attractor implies automatically smoothness and finite-dimensionality of the said attractor. This fact is stated in theorem below.

Theorem 2.4. [13, Theorems 7.9.6 and 7.9.8] *Let $(H, S(t))$ be a dynamical system given by (2.31) and satisfying (2.32). Assume that it has a global attractor $\mathbf{A}$. Then if $(H, S(t))$ is quasi-stable on $\mathbf{A}$, this global attractor has finite fractal dimension. Moreover, if $c(t)$ in (2.34) is globally bounded, its complete trajectories have additional (time) regularity*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}(S(t)y) \right\|_{L^\infty(\mathbb{R},H)} \leqslant M, \quad y \in \mathbf{A},$$

where M depends on $c(t)$.

By summarizing the results stated in Theorem 2.2 which guarantees the existence of a global attractor, and Theorem 2.4, which provides finite fractal dimension and smoothness of the said attractor, we arrive at:

Corollary 2.1. *Let $(H, S(t))$ given by (2.31) and satisfying (2.32) be a quasistable, gradient system with Lyapunov function satisfying (2.30) and a bounded set of stationary points. Then $(H, S(t))$ admits a finite-dimensional global attractor $\mathbf{A}$ which is also "smooth": $\frac{\mathrm{d}}{\mathrm{d}t}\big(S(t)y\big) \in L^\infty(\mathbb{R}, H)$, for $y \in \mathbf{A}$. If, in addition, $c(t)$ in (2.34) is bounded for $t > 0$, there exists $M < \infty$ such that $\|\frac{\mathrm{d}}{\mathrm{d}t}(S(t)y)\|_H \leqslant M, t \in R$.*

3. Global attractors-proof of Theorem 1.2.

This section is devoted to the proof of the main result formulated in Theorem 1.2. This is based on the application of Corollary 2.1. To this end we must show that $(\mathcal{H}, S(t))$ is: (1) gradient system with a Lyapunov function satisfying (2.30), and (2) quasi-stable system with the appropriate bounds for $c(t)$. The property of gradient system relies on a new unique continuation property shown for the system under consideration. The property of quasi-stability is the most technical part of the proof which requires deep PDE results related to hidden regularity of the boundary traces corresponding to vectorial systems with *free boundary conditions.* These results are of independent PDE interest.

3.1. Proving quasi-stability.

In this subsection we shall prove that our problem is quasi-stable. Accordingly, we must show that the difference of two trajectories satisfies estimate (2.34). To this end one needs rather extensive background and several energy estimates. This will be established in five subsections.

In what follows we use the notations $Q = [0, T] \times \Omega$, $\quad T > 0$ and $\Sigma_\alpha = [\alpha, T - \alpha] \times \Gamma_1$, $0 < \alpha < \frac{T}{2}$. Moreover, C will denote several positive constants and, in the case it depends on some specific parameter ρ, then we write C_ρ.

3.1.1. *Comparing two trajectories.* Let B be a bounded set of $\mathcal{H}$ and consider two solutions of (1.1)-(1.10),

$$S(t)y_i = (u^i, u^i_t, w^i, w^i_t, \phi^i, \theta^i), \quad i = 1, 2, \tag{3.1}$$

with corresponding initial data $y_i(0) = (u^i_0, u^i_1, w^i_0, w^i_1, \phi^i_0, \theta^i_0) \in B$, $i = 1, 2$. Then the difference

$$(\tilde{u}, \tilde{w}, \tilde{\phi}, \tilde{\theta}) = (u^1 - u^2, w^1 - w^2, \phi^1 - \phi^2, \theta^1 - \theta^2), \tag{3.2}$$

solves the problem,

$$\tilde{u}_{tt} - \operatorname{div}\big\{\sigma[\epsilon(\tilde{u})]\big\} + \nabla\tilde{\phi} + \mathscr{P}_1(\tilde{u}, \tilde{w}) = \operatorname{div}\big\{\mathscr{N}_1\big\} \ \text{ in } \Omega \times (0, \infty), \tag{3.3}$$

$$\tilde{w}_{tt} + \Delta^2\tilde{w} + \Delta\tilde{\theta} + \mathscr{P}_2(\tilde{u}, \tilde{w}) = \operatorname{div}\big\{\mathscr{N}_2\big\} \ \text{ in } \Omega \times (0, \infty), \tag{3.4}$$

where

$$\begin{aligned}\mathscr{P}_i(\tilde{u}, \tilde{w}) &= p_i(u^1, w^1) - p_i(u^2, w^2), i = 1, 2,\\ \mathscr{N}_1 &= \sigma[f(\nabla w^1) - f(\nabla w^2)],\\ \mathscr{N}_2 &= \sigma[N(u^1, w^1)]\nabla w^1 - \sigma[N(u^2, w^2)]\nabla w^2 + \phi^1\nabla w^1 - \phi^2\nabla w^2,\end{aligned}$$

with thermal components

$$\begin{aligned}\tilde{\phi}_t - \Delta\tilde{\phi} + \operatorname{div}\{\tilde{u}_t\} - \big[\nabla w^1\cdot\nabla w^1_t - \nabla w^2\cdot\nabla w^2_t\big] = 0 \ \text{ in } \Omega \times (0, \infty),\\ \tilde{\theta}_t - \Delta\tilde{\theta} - \Delta\tilde{w}_t = 0 \ \text{ in } \Omega \times (0, \infty),\end{aligned} \tag{3.5}$$

and boundary conditions

$$\begin{aligned}\tilde{u} = 0,\ \tilde{w} = 0,\ \nabla\tilde{w} = 0 \ \text{ on } \Gamma_0 \times (0, \infty),\\ \sigma[\varepsilon(\tilde{u})]\nu + \sigma[f(\nabla w^1) - f(\nabla w^2)]\nu + \kappa\tilde{u} - \tilde{\phi}\nu + \tilde{u}_t = 0 \ \text{ on } \Gamma_1 \times (0, \infty),\\ \Delta\tilde{w} + (1 - \mu)B_1\tilde{w} + \tilde{\theta} = 0 \ \text{ on } \Gamma_1 \times (0, \infty),\\ \partial_\nu(\Delta\tilde{w}) + (1 - \mu)B_2\tilde{w} - \mathscr{N}_2\cdot\nu + \partial_\nu\tilde{\theta} = 0 \ \text{ on } \Gamma_1 \times (0, \infty),\\ \partial_\nu\tilde{\phi} + \lambda_1\tilde{\phi} = \partial_\nu\tilde{\theta} + \lambda_2\tilde{\theta} = 0 \ \text{ on } \Gamma \times (0, \infty),\end{aligned} \tag{3.6}$$

with the corresponding initial data

$$\begin{aligned}&\tilde{u}(\cdot, 0) = u^1_0 - u^2_0,\ \ \tilde{u}_t(\cdot, 0) = u^1_1 - u^2_1,\\ &\tilde{w}(\cdot, 0) = w^1_0 - w^2_0,\ \tilde{w}_t(\cdot, 0) = w^1_1 - w^2_1,\\ &\tilde{\phi}(\cdot, 0) = \phi^1_0 - \phi^2_0,\ \ \tilde{\theta}\,(\cdot, 0) = \theta^1_0 - \theta^2_0.\end{aligned} \tag{3.7}$$

The unperturbed energy of (3.3)-(3.7) is defined by

$$\tilde{E}(t) = \frac{1}{2}\int_\Omega\Big[|\tilde{u}_t|^2 + |\tilde{w}_t|^2 + \sigma[\epsilon(\tilde{u})]\epsilon(\tilde{u}) + |\tilde{\phi}|^2 + |\tilde{\theta}|^2\Big]\mathrm{d}\Omega + \frac{1}{2}a(\tilde{w}, \tilde{w}) + \frac{\kappa}{2}\int_{\Gamma_1}|\tilde{u}|^2\mathrm{d}\Gamma_1.$$

Then we have the following energy equality,

$$\tilde{E}(t) + D^t_s(\tilde{u}, \tilde{\phi}, \tilde{\theta}) = \tilde{E}(s) + \int_s^t\sum_{i=1}^4\mathscr{R}_i(\tau)\mathrm{d}\tau, \tag{3.8}$$

where

$$\mathscr{R}_1(t) = -\int_\Omega \mathscr{P}_1(\tilde{u},\tilde{w})\cdot\tilde{u}_t\,\mathrm{d}\Omega - \int_\Omega \mathscr{P}_2(\tilde{u},\tilde{w})\tilde{w}_t\,\mathrm{d}\Omega,$$

$$\mathscr{R}_2(t) = \int_\Omega \sigma[f(\nabla w^1) - f(\nabla w^2)]\tilde{u}_t\,\mathrm{d}\Omega + \int_{\Gamma_1} \sigma[f(\nabla w^1) - f(\nabla w^2)]\cdot\nu\tilde{u}_t\,\mathrm{d}\Gamma_1,$$

$$\mathscr{R}_3(t) = \int_\Omega \Big[N(u^1,w^1)\nabla w^1 - N(u^2,w^2)\nabla w^2\Big]\nabla\tilde{w}_t\,\mathrm{d}\Omega,$$

$$\mathscr{R}_4(t) = -\int_\Omega \Big[\phi^1\nabla w^1 - \phi^2\nabla w^2\Big]\nabla\tilde{w}_t\,\mathrm{d}\Omega + \int_\Omega \Big[\nabla w^1\cdot\nabla w_t^1 - \nabla w^2\cdot\nabla w_t^2\Big]\tilde{\phi}\,\mathrm{d}\Omega,$$

$$D_s^t(\tilde{u},\tilde{\phi},\tilde{\theta}) = \int_s^t \Big[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\nabla\tilde{\phi}\|_\Omega^2 + \|\nabla\tilde{\theta}\|_\Omega^2, + \lambda_1\|\tilde{\phi}\|_\Gamma^2 + \lambda_2\|\tilde{\theta}\|_\Gamma^2\Big]\mathrm{d}\tau.$$

Remark 3.1. We verify condition (2.34) by obtaining the following estimate

$$\tilde{E}(t) \leqslant C\tilde{E}(0)e^{-\beta t} + C\big[\sup_{\tau\in[0,t]}\|\tilde{u}(\tau)\|_{L^{r+1}(\Omega)}^2 + \sup_{\tau\in[0,t]}\|\tilde{u}(\tau)\|_{1-\varepsilon,\Omega}^2 + \sup_{\tau\in[0,t]}\|\tilde{w}(\tau)\|_{2-\varepsilon,\Omega}^2\big],$$

for suitable constants $C,\beta,\varepsilon > 0$. This will be achieved in Lemma 3.7.

We end this subsection with some estimates for $f(\nabla w^i)$, $i = 1,2$.

Lemma 3.1. *For every $\varepsilon\in(0,1)$ the following estimates holds:*

(i) $\displaystyle\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_\Omega^2\mathrm{d}t \leqslant C_{B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}),$

(ii) $\displaystyle\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{1,\Omega}^2\mathrm{d}t \leqslant C_B\int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t,$

where the lower orders terms are given by

$$\mathrm{l.o.t.}(\tilde{u},\tilde{w}) = \sup_{t\in[0,T]}\|\tilde{u}(t)\|_{L^{r+1}(\Omega)}^2 + \sup_{t\in[0,T]}\|\tilde{u}(t)\|_{1-\varepsilon,\Omega}^2 + \sup_{t\in[0,T]}\|\tilde{w}(t)\|_{2-\varepsilon,\Omega}^2.$$

Proof. We shall use the identity

$$f(\nabla w^1) - f(\nabla w^2) = f(\nabla\tilde{w}) + \nabla\tilde{w}\otimes\nabla w^2 + \nabla w^2\otimes\nabla\tilde{w}.$$

To prove (i), the inequality $\|u\otimes v\|_\Omega \leqslant C\|u\|_{\varepsilon,\Omega}\|v\|_{1-\varepsilon,\Omega}$ implies that

$$\begin{aligned}\int_0^T\|f(\nabla w^1) - f(\nabla w^2)\|_\Omega^2\mathrm{d}t &\leqslant C\int_0^T\big[\|\tilde{w}\|_{1+\varepsilon,\Omega}^2\|\tilde{w}\|_{2-\varepsilon,\Omega}^2 + \|w^2\|_{1+\varepsilon,\Omega}^2\|\tilde{w}\|_{2-\varepsilon,\Omega}^2\big]\mathrm{d}t\\ &\leqslant C_{B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned}$$

To prove (ii), we will use the inequality $\|u\otimes v\|_{1,\Omega} \leqslant C\|u\|_{1,\Omega}\|v\|_{1+\varepsilon,\Omega}$, and then

$$\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{1,\Omega}^2\mathrm{d}t \leqslant C_B\int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t.$$

□

3.1.2. *A first observability inequality.* Here we obtain an observability inequality that reconstructs the integral of the linear energy in terms of the dissipation, lower order terms and also boundary traces, which are not apriori bounded by the energy. The estimate will be obtained by multipliers method applied to all three components of the system [2, 39]. In order to control these boundary terms, more subtle estimates will be needed which invoke partially regularizing effect of thermoelasticity as well as micro local estimates applied to a hyperbolic component represented by u. This will be done in Subsection 3.1.3.

Lemma 3.2. *Let $(\tilde{u}, \tilde{u}_t, \tilde{w}, \tilde{w}_t, \tilde{\phi}, \tilde{\theta})$ be a solution of the system (3.3)-(3.7). Then there exists $T > 0$ large enough, such that for any $\varepsilon \in (0, \frac{1}{4})$, the following estimate holds.*

$$\int_0^T \tilde{E}(t)\mathrm{d}t \leqslant C\big[\tilde{E}(0)+\tilde{E}(T)\big]+C_B\int_0^T\big[\|\tilde{u}_t\|_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}+\|\tilde{\theta}\|^2_{1,\Omega}\big]\mathrm{d}t+C\int_0^T \|\nabla\tilde{u}\|^2_{\Gamma_1}\mathrm{d}t$$
$$+ C_B\int_0^T\big[\|\Delta\tilde{w}\|^2_{-\frac{1}{2},\Gamma_0}+\|\tilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1}+\|\tilde{w}\|^2_{2+\varepsilon,\Omega}\big]\mathrm{d}t+C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

Proof. The proof of this lemma is divided into several steps. The geometric condition (1.19) will be used.

Step 1. Estimate for kinetic energy of in-plane displacement: Multiplying both sides of equation (3.3) by $h\nabla\tilde{u}$, where $h(x) = x - x_0$, and integrate in time and space. We find

$$\int_0^T \big(\tilde{u}_{tt} - \operatorname{div}\big\{\sigma[\epsilon(\tilde{u})] + \sigma[f(\nabla w^1) - f(\nabla w^2)]\big\}+\nabla\tilde{\phi} + \mathscr{P}_1(\tilde{u},\tilde{w}), h\nabla\tilde{u}\big)_\Omega\mathrm{d}t = 0. \tag{3.9}$$

Integrating by parts in time and using divergence theorem yield

$$\int_0^T \big(\tilde{u}_{tt}, h\nabla\tilde{u}\big)_\Omega\mathrm{d}t = \Big[\big(\tilde{u}_t, h\nabla\tilde{u}\big)_\Omega\Big]_0^T + \int_Q |\tilde{u}_t|^2\mathrm{d}Q - \frac{1}{2}\int_{\Sigma_1} |\tilde{u}_t|^2 h\cdot\nu\,\mathrm{d}\Sigma_1. \tag{3.10}$$

Applying the divergence and Gauss theorems in the second term of (3.9), we obtain

$$\int_0^T \big(\operatorname{div}\{\sigma[\epsilon(\tilde{u})]\}, h\nabla\tilde{u}\big)_\Omega\mathrm{d}t = \int_0^T \big\langle\sigma[\epsilon(\tilde{u})]\nu, h\nabla\tilde{u}\big\rangle_\Gamma\mathrm{d}t - \int_0^T \big(\sigma[\epsilon(\tilde{u})], \nabla(h\nabla\tilde{u})\big)_\Omega\mathrm{d}t. \tag{3.11}$$

Note that

$$\int_0^T \big\langle\sigma[\epsilon(\tilde{u})]\nu, h\nabla\tilde{u}\big\rangle_\Gamma\mathrm{d}t = \int_0^T \big\langle\sigma[\epsilon(\tilde{u})]\nu, h\nabla\tilde{u}\big\rangle_{\Gamma_0}\mathrm{d}t + \int_0^T \big\langle\sigma[\epsilon(\tilde{u})]\nu, h\nabla\tilde{u}\big\rangle_{\Gamma_1}\mathrm{d}t.$$

Then, the identity

$$\big\langle\sigma[\epsilon(\tilde{u})]\nu, h\nabla\tilde{u}\big\rangle_{\Gamma_0}= \big\langle\sigma[\epsilon(\tilde{u})], \epsilon(\tilde{u})h\cdot\nu\big\rangle_{\Gamma_0},$$

together with boundary condition $(3.6)_2$ imply that

$$\int_0^T \big\langle\sigma[\epsilon(\tilde{u})]\nu, h\nabla\tilde{u}\big\rangle_\Gamma\mathrm{d}t = \int_0^T \big\langle\sigma[\epsilon(\tilde{u})], \epsilon(\tilde{u})h\cdot\nu\big\rangle_{\Gamma_0}\mathrm{d}t$$
$$- \int_0^T \big\langle\sigma[f(\nabla w^1) - f(\nabla w^2)]\nu+\kappa\tilde{u}-\tilde{\phi}\nu-\tilde{u}_t, h\nabla\tilde{u}\big\rangle_{\Gamma_1}\mathrm{d}t. \tag{3.12}$$

It follows from identity (2.29) that

$$\int_0^T \big(\sigma[\epsilon(\tilde{u})], \nabla(h\nabla\tilde{u})\big)_\Omega\mathrm{d}t = \int_0^T \big(\sigma[\epsilon(\tilde{u})], \epsilon(h\nabla\tilde{u})\big)_\Omega\mathrm{d}t$$
$$= \int_0^T \big(\sigma[\epsilon(\tilde{u})], \epsilon(\tilde{u})\big)_\Omega\mathrm{d}t + \sum_{i,j,k=1}^{2}\int_0^T \big(a_{i,j}, \frac{\partial^2\tilde{u}_i}{\partial x_k\partial x_j}h_k\big)_\Omega\mathrm{d}t,$$

which combined with (2.28) and with Gauss theorem implies that

$$\begin{aligned}\int_0^T \big(\sigma[\epsilon(\tilde u)],\epsilon(\tilde u)\big)_\Omega \mathrm{d}t + \sum_{i,j,k=1}^{2}\int_0^T \big(a_{i,j},\frac{\partial^2 \tilde u_i}{\partial x_k\partial x_j}h_k\big)_\Omega \mathrm{d}t \\ = \frac12\int_0^T \big\langle \sigma[\epsilon(\tilde u)],\epsilon(\tilde u)h\cdot\nu\big\rangle_{\Gamma_0}\,\mathrm{d}t + \frac12\int_0^T \big\langle \sigma[\epsilon(\tilde u)],\epsilon(\tilde u)h\cdot\nu\big\rangle_{\Gamma_1}\,\mathrm{d}t.\end{aligned} \tag{3.13}$$

Consequently from (3.11)-(3.13) we find that

$$\begin{aligned}\int_0^T \big(\mathrm{div}\{\sigma[\epsilon(\tilde u)]\},h\nabla\tilde u\big)_\Omega \mathrm{d}t = \frac12\int_0^T \big[\big\langle \sigma[\epsilon(\tilde u)],\epsilon(\tilde u)h\cdot\nu\big\rangle_{\Gamma_0} - \big\langle \sigma[\epsilon(\tilde u)],\epsilon(\tilde u)h\cdot\nu\big\rangle_{\Gamma_1}\big]\mathrm{d}t \\ - \int_0^T \big\langle \sigma[f(\nabla w^1)-f(\nabla w^2)]\nu+\kappa\tilde u-\tilde\phi\nu+\tilde u_t, h\nabla\tilde u\big\rangle_{\Gamma_1}\mathrm{d}t.\end{aligned} \tag{3.14}$$

Combining (3.10) and (3.14) with (3.9) we obtain

$$\begin{aligned}\int_Q |\tilde u_t|^2\mathrm{d}Q = &- \Big[\big(\tilde u_t, h\nabla\tilde u\big)_\Omega\Big]_0^T + \frac12\int_{\Sigma_1}|\tilde u_t|^2 h\cdot\nu\mathrm{d}\Sigma_1 \\ &+ \frac12\int_0^T \big\langle \sigma[\epsilon(\tilde u)],\epsilon(\tilde u)h\cdot\nu\big\rangle_{\Gamma_0}\,\mathrm{d}t - \frac12\int_0^T \big\langle \sigma[\epsilon(\tilde u)],\epsilon(\tilde u)h\cdot\nu\big\rangle_{\Gamma_1}\,\mathrm{d}t \\ &- \int_0^T \big\langle \sigma[f(\nabla w^1) - f(\nabla w^2)]\nu + \kappa\tilde u - \tilde\phi\nu + \tilde u_t, h\nabla\tilde u\big\rangle_{\Gamma_1}\,\mathrm{d}t \\ &+ \int_0^T \big(\mathrm{div}\{\sigma[f(\nabla w^1)-f(\nabla w^2)]\} - \nabla\tilde\phi - \mathscr{P}_1(\tilde u,\tilde w), h\nabla\tilde u\big)_\Omega \mathrm{d}t.\end{aligned} \tag{3.15}$$

Let us estimate the nonlinear term $\mathscr{P}_1(\tilde u,\tilde w)$. Using the assumption (1.15) we find that

$$|p_{1,i}(u^1,w^1) - p_{1,i}(u^2,w^2)| \leqslant C(\nabla p)\big(|\tilde u| + |\tilde w|\big),\ \ i=1,2,$$

where

$$C(\nabla p) = C\big(1 + |u_1^1|^{r-1} + |u_1^2|^{r-1} + |u_2^1|^{r-1} + |u_2^2|^{r-1} + |w^1|^{r-1} + |w^2|^{r-1}\big).$$

Then Hölder's inequality with $\frac{(r-1)}{2(r+1)} + \frac{1}{r+1} + \frac12 = 1$ implies that

$$\int_0^T \big(\mathscr{P}_1(\tilde u,\tilde w), h\nabla\tilde u\big)_\Omega \mathrm{d}t \leqslant C_{\delta,B,T}\mathrm{l.o.t.}(\tilde u,\tilde w) + \delta\int_Q |\nabla\tilde u|^2\mathrm{d}Q. \tag{3.16}$$

The geometric condition (1.19) implies that $\frac12\int_0^T \big\langle\sigma[\epsilon(\tilde u)],\epsilon(\tilde u)\big\rangle_{\Gamma_0} h\cdot\nu\mathrm{d}t \leqslant 0$. Finally, using these inequalities and Lemma 3.1 in (3.15) we obtain

$$\begin{aligned}\int_Q |\tilde u_t|^2\mathrm{d}Q \leqslant\ & C\big[\tilde E(0) + \tilde E(T)\big] + C_\delta\int_{\Sigma_1}\big[|\tilde u_t|^2 + |\nabla\tilde u|^2\big]\mathrm{d}\Sigma_1 + C_\delta\int_0^T \|\tilde\phi\|_{1,\Omega}^2\mathrm{d}t \\ &+\delta\kappa\int_{\Sigma_1}|\tilde u|^2\mathrm{d}\Sigma_1 + \delta\int_Q|\nabla\tilde u|^2\mathrm{d}Q + C_{B,\delta}\int_0^T\|\tilde w\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\mathrm{l.o.t.}(\tilde u,\tilde w).\end{aligned} \tag{3.17}$$

Here we used the fact that $\int_0^T \langle \sigma[\epsilon(\tilde{u})], \epsilon(\tilde{u}) h\cdot\nu\rangle_{\Gamma_1} \,\mathrm{d}t \leqslant C \int_{\Sigma_1} |\nabla \tilde{u}|^2 \mathrm{d}\Sigma_1$.

Step 2. Estimate for the difference of potential and kinetic energies: Multiply both sides of equation (3.3) by $\tilde{u}$ and integrate in time and space

$$\int_0^T \big(\tilde{u}_{tt} - \mathrm{div}\big\{\sigma[\epsilon(\tilde{u})] + \sigma[f(\nabla w^1) - f(\nabla w^2)]\big\} + \nabla\tilde{\phi} + \mathscr{P}_1(\tilde{u},\tilde{w}), \tilde{u}\big)_\Omega \mathrm{d}t = 0. \tag{3.18}$$

Using Gauss theorem in the second term of (3.18) we find

$$\int_0^T \big(\mathrm{div}\{\sigma[\epsilon(\tilde{u})]\}, u\big)_\Omega \mathrm{d}t = \int_0^T \big\langle\, \sigma[\epsilon(\tilde{u})]\nu,\ \tilde{u}\,\big\rangle_\Gamma \mathrm{d}t - \int_0^T \big(\sigma[\epsilon(\tilde{u})], \epsilon(\tilde{u})\big)_\Omega \mathrm{d}t.$$

Boundary conditions $(3.6)_1$ and $(3.6)_2$ imply that

$$\int_0^T \big\langle \sigma[\epsilon(u)]\nu, u\big\rangle_\Gamma \mathrm{d}t = -\int_0^T \big\langle \sigma[f(\nabla w^1) - f(\nabla w^2)]\nu + \kappa\tilde{u} - \tilde{\phi}\nu + \tilde{u}_t, \tilde{u}\big\rangle_{\Gamma_1} \mathrm{d}t.$$

These identities in (3.18) imply in the following equality

$$\begin{aligned} -\int_0^T\int_\Omega |\tilde{u}_t|^2 \mathrm{d}\Omega \mathrm{d}t + \int_0^T\int_\Omega \sigma[\epsilon(\tilde{u})]\epsilon(\tilde{u})\mathrm{d}\Omega\mathrm{d}t + \kappa\int_0^T\int_{\Gamma_1} |\tilde{u}|^2 \mathrm{d}\Gamma_1 \mathrm{d}t \\ = -\Big[\big(\tilde{u}_t,\tilde{u}\big)_\Omega\Big]_0^T - \int_0^T \big\langle\sigma[f(\nabla w^1) - f(\nabla w^2)]\nu + \kappa\tilde{u} - \tilde{\phi}\nu + \tilde{u}_t, \tilde{u}\big\rangle_{\Gamma_1} \mathrm{d}t \\ + \int_0^T \big(\,\mathrm{div}\{\sigma[f(\nabla w^1) - f(\nabla w^2)]\} - \nabla\tilde{\phi} - \mathscr{P}_1(\tilde{u},\tilde{w}), \tilde{u}\,\big)_\Omega \,\mathrm{d}t. \end{aligned}$$

Proceeding as in (3.16) we obtain the following estimate $\int_0^T \big(\mathscr{P}_1(\tilde{u},\tilde{w}),\tilde{u}\big)_\Omega \mathrm{d}t \leqslant C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w})$. This inequality and Trace theorem imply that

$$\begin{aligned} -\int_0^T\int_\Omega |\tilde{u}_t|^2 \mathrm{d}\Omega \mathrm{d}t + \int_0^T\int_\Omega \sigma[\epsilon(\tilde{u})]\epsilon(\tilde{u})\mathrm{d}\Omega\mathrm{d}t + \kappa\int_0^T\int_{\Gamma_1} |\tilde{u}|^2 \mathrm{d}\Gamma_1 \mathrm{d}t \\ \leqslant C[\tilde{E}(0) + \tilde{E}(T)] + C_\delta \int_0^T \big[\|f(\nabla w^1) - f(\nabla w^2)\|^2_{1,\Omega} + \|\tilde{u}_t\|^2_{\Gamma_1}\big]\mathrm{d}t \\ + C_\delta \int_0^T \|\tilde{\phi}\|^2_{1,\Omega}\mathrm{d}t \ + \ \delta\int_0^T \|\tilde{u}\|^2_{1,\Omega}\mathrm{d}t \ + \ C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \end{aligned}$$

Choosing $\delta > 0$ small enough and using Lemma 3.1 we find

$$\begin{aligned} -\int_0^T\int_\Omega |\tilde{u}_t|^2 \mathrm{d}\Omega \mathrm{d}t + \int_0^T\int_\Omega \sigma[\epsilon(\tilde{u})]\epsilon(\tilde{u})\mathrm{d}\Omega\mathrm{d}t + \kappa\int_0^T\int_{\Gamma_1} |\tilde{u}|^2 \mathrm{d}\Gamma_1 \mathrm{d}t \\ \leqslant C[\tilde{E}(0) + \tilde{E}(T)] + C\int_0^T \Big[\|\tilde{u}_t\|^2_{\Gamma_1} + \|\tilde{\phi}\|^2_{1,\Omega}\Big]\mathrm{d}t \\ + C_B \int_0^T \|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t \ + \ C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \end{aligned} \tag{3.19}$$

Now, multiplying both sides of equation (3.4) by $\tilde{w}$ and integrating in time and space,

$$\int_0^T \big(\tilde{w}_{tt} + \Delta^2\tilde{w} - \mathrm{div}\{\mathscr{N}_2\} + \Delta\tilde{\theta} + \mathscr{P}_2(\tilde{u},\tilde{w}), \tilde{w}\big)_\Omega \mathrm{d}t = 0. \tag{3.20}$$

To handle the second term in (3.20) we use the following identity

$$\begin{aligned}\big(\Delta^2\tilde{w},\psi\big)_\Omega &= a(\tilde{w},\psi)+\int_{\Gamma_1}[\partial_\nu(\Delta\tilde{w})+(1-\mu)B_2\tilde{w}]\psi\,\mathrm{d}\Gamma_1\\ &\quad-\int_{\Gamma_1}\big[\Delta\tilde{w}+(1-\mu)B_1\tilde{w}\big]\partial_\nu\psi\mathrm{d}\Gamma_1+\int_{\Gamma_0}[\partial_\nu(\Delta\tilde{w})\,\psi-\Delta\tilde{w}\partial_\nu\psi]\mathrm{d}\Gamma_0,\end{aligned}$$

where $\psi\in H^2(\Omega)$. Taking $\psi=\tilde{w}$ and using boundary conditions $(3.6)_1$, $(3.6)_3$,$(3.6)_4$ we find

$$\int_0^T\big(\Delta^2\tilde{w},\tilde{w}\big)_\Omega\mathrm{d}t=\int_0^T a(\tilde{w},\tilde{w})\mathrm{d}t+\int_0^T\langle\mathscr{N}_2\cdot\nu-\partial_\nu\tilde{\theta},\tilde{w}\rangle_{\Gamma_1}\mathrm{d}t+\int_{\Sigma_1}\tilde{\theta}\partial_\nu\tilde{w}\mathrm{d}\Sigma_1. \tag{3.21}$$

Using Gauss theorem we can rewrite the third term of (3.20) as

$$\int_0^T\big(\mathrm{div}\{\mathscr{N}_2\},\tilde{w}\big)_\Omega\mathrm{d}t=-\int_0^T\big(\mathscr{N}_2,\nabla\tilde{w}\big)_\Omega\mathrm{d}t+\int_0^T\big\langle\mathscr{N}_2\cdot\nu,\tilde{w}\big\rangle_{\Gamma_1}\mathrm{d}t. \tag{3.22}$$

Combining (3.21) and (3.22) with (3.20) we obtain

$$-\int_Q|\tilde{w}_t|^2\mathrm{d}Q+\int_0^T a(\tilde{w},\tilde{w})\mathrm{d}t=-\Big[\big(\tilde{w}_t,\tilde{w}\big)_\Omega\Big]_0^T-\int_0^T\Big[\langle\tilde{\theta},\partial_\nu\tilde{w}\rangle_{\Gamma_1}-\big(\nabla\tilde{\theta},\nabla\tilde{w}\big)_\Omega\Big]\mathrm{d}t-\mathscr{R}, \tag{3.23}$$

where

$$\begin{aligned}\mathscr{R}&=\int_0^T\big(\sigma[N(u^1,w^1)]\,\nabla w^1-\sigma[N(u^2,w^2)]\,\nabla w^2,\nabla\tilde{w}\big)_\Omega\,\mathrm{d}t\\ &\quad+\int_0^T\big(\phi^1\nabla w^1-\phi^2\nabla w^2,\nabla\tilde{w}\big)_\Omega\mathrm{d}t+\int_0^T\big(\mathscr{P}_2(\tilde{u},\tilde{w}),\tilde{w}\big)_\Omega\,\mathrm{d}t.\end{aligned}$$

Next, we estimate the integrals on the right-hand side of (3.23). Trace theorem provides

$$\langle\theta,\partial_\nu\tilde{w}\rangle_{\Gamma_1}\leqslant\|\theta\|_{\Gamma_1}\,\|\partial_\nu\tilde{w}\|_{\Gamma_1}\leqslant C_\delta\|\theta\|^2_{1,\Omega}+\delta\|w\|^2_{2,\Omega}. \tag{3.24}$$

Let us estimate $\mathscr{R}$. Using the definition of stress $N(\cdot,\cdot)$ we find

$$\begin{aligned}&\int_0^T\big(\sigma[N(u^1,w^1)]\nabla w^1-\sigma[N(u^2,w^2)]\nabla w^2,\nabla\tilde{w}\big)_\Omega\mathrm{d}t\\ &\quad=\int_Q\sigma[\epsilon(\tilde{u})]\cdot(\nabla w^2\otimes\nabla\tilde{w})\mathrm{d}Q+\int_Q\sigma\big[f(\nabla w^1)-f(\nabla w^2)\big]\cdot(\nabla w^2\otimes\nabla\tilde{w})\mathrm{d}Q\\ &\qquad+\int_Q\sigma\big[\epsilon(u^1)+f(\nabla w^1)\big]\cdot(\nabla\tilde{w}\otimes\nabla\tilde{w})\mathrm{d}Q.\end{aligned}$$

The inequality $\|u\otimes v\|_\Omega\leqslant C\|u\|_{\varepsilon,\Omega}\|v\|_{1-\varepsilon,\Omega}$, which holds for $\varepsilon\in(0,1)$, implies that

$$\int_Q\sigma\big[\epsilon(\tilde{u})\big]\cdot(\nabla w^2\otimes\nabla\tilde{w})\mathrm{d}Q\leqslant\delta\int_0^T\|\sigma[\epsilon(\tilde{u})]\|^2_\Omega\mathrm{d}t+C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}),$$

$$\begin{aligned}&\int_Q\sigma\big[f(\nabla w^1)-f(\nabla w^2)\big]\cdot(\nabla w^2\otimes\nabla\tilde{w})\mathrm{d}Q\\ &\qquad\leqslant C\int_0^T\|f(\nabla w^1)-f(\nabla w^2)\|^2_\Omega\mathrm{d}t+C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}),\end{aligned}$$

$$\int_Q[\sigma\big[\epsilon(u^1)+f(\nabla w^1)\big]\cdot(\nabla\tilde{w}\otimes\nabla\tilde{w})\mathrm{d}Q\leqslant C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}),$$

$$\int_0^T \big(\phi^1\nabla w^1 - \phi^2\nabla w^2, \nabla\tilde{w}\big)_\Omega \mathrm{d}t \leqslant C\int_0^T \|\tilde{\phi}\|_\Omega^2 \mathrm{d}t + C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w})$$

and $\int_0^T \big(\mathscr{P}_2(\tilde{u},\tilde{w}),\tilde{w}\big)_\Omega \mathrm{d}t \leqslant C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w})$. These estimates and Lemma 3.1 imply that

$$\mathscr{R} \leqslant \delta\int_0^T \|\sigma[\epsilon(\tilde{u})]\|_\Omega^2 \mathrm{d}t + C\int_0^T \|\tilde{\phi}\|_\Omega^2 \mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.25}$$

Inserting this and (3.24) into (3.23), we obtain

$$-\int_Q |\tilde{w}_t|^2 \mathrm{d}Q + \int_0^T a(\tilde{w},\tilde{w})\mathrm{d}t \leqslant C[\tilde{E}(0)+\tilde{E}(T)] + C_B\int_0^T \big[\|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{\theta}\|_{1,\Omega}^2\big]\mathrm{d}t$$
$$+\,\delta\int_0^T \|\sigma[\epsilon(\tilde{u})]\|_\Omega^2 \,\mathrm{d}t \,+\, C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

This estimate and (3.19), for $\delta > 0$ small enough, show that

$$\int_0^T \big[\tilde{E}_p(t) - \tilde{E}_k(t)\big]\mathrm{d}t \leqslant C[\tilde{E}(0)+\tilde{E}(T)] + C_B\int_0^T \big[\|\tilde{u}\|_{1,\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{\theta}\|_{1,\Omega}^2\big]\mathrm{d}t$$
$$+\, C_B\int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 \mathrm{d}t + C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.26}$$

Step 3. Estimate for kinetic energy of vertical displacement: Let us consider the following operators.

- The Laplace operator: $\mathcal{A}_D : L^2(\Omega) \to L^2(\Omega)$, where $\mathcal{A}_D = -\Delta$, equipped with Dirichlet boundary condition and domain $D(\mathcal{A}_D) = H^2(\Omega)\cap H_0^1(\Omega)$.
- The elliptic operator $\mathcal{D}$: $\mathcal{D}h = v \Leftrightarrow \begin{cases} \Delta v = 0 \text{ in } \Omega \\ v = h \text{ on } \Gamma. \end{cases}$ Classical elliptic regularity [43] provides

$$\|\mathcal{A}_D^{-1}v\|_{2,\Omega} \leqslant C\|v\|_\Omega \quad v \in L^2(\Omega),$$

and

$$\mathcal{D} \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{1}{2}}(\Omega)), \quad s \in \mathbb{R}.$$

For $v \in H^2(\Omega)$ we have that

$$-v + \mathcal{D}(v|_\Gamma) \in D(\mathcal{A}_D) \ \text{ and } \ \mathcal{A}_D^{-1}\Delta v = -v + \mathcal{D}(v|_\Gamma). \tag{3.27}$$

Now, multiply both sides of equation (3.4) by $\mathcal{A}_D^{-1}\theta$ and integrating in time and space,

$$\int_0^T \big(\tilde{w}_{tt} + \Delta^2\tilde{w} - \mathrm{div}\{\mathscr{N}_2\} + \Delta\tilde{\theta} + \mathscr{P}_2(\tilde{u},\tilde{w}), \mathcal{A}_D^{-1}\tilde{\theta}\big)_\Omega \mathrm{d}t = 0.$$

Proceeding as before, we obtain

$$\int_0^T \big(\tilde{w}_{tt}, \mathcal{A}_D^{-1}\tilde{\theta}\big)_\Omega \mathrm{d}t + \int_0^T \big[a(\tilde{w}, \mathcal{A}_D^{-1}\tilde{\theta}) + \big(\mathscr{N}_2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega + \big(\mathscr{P}_2(\tilde{u},\tilde{w}), \mathcal{A}_D^{-1}\tilde{\theta}\big)_\Omega\big]\mathrm{d}t$$
$$= \int_0^T \big[-\big\langle\tilde{\theta}, \partial_\nu(\mathcal{A}_D^{-1}\tilde{\theta})\big\rangle_{\Gamma_1} + \big\langle\Delta\tilde{w}, \partial_\nu(\mathcal{A}_D^{-1}\tilde{\theta})\big\rangle_{\Gamma_0} + \big(\nabla\tilde{\theta}, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega\big]\mathrm{d}t. \tag{3.28}$$

Using relation (3.27) we obtain

$$\int_0^T (\tilde{w}_{tt}, \mathcal{A}_D^{-1}\tilde{\theta})_\Omega \mathrm{d}t = \Big[(\tilde{w}_t, \mathcal{A}_D^{-1}\tilde{\theta})_\Omega\Big]_0^T + \int_0^T \|\tilde{w}_t\|_\Omega^2 - \big(\tilde{w}_t, \mathcal{D}(\tilde{w}_t|_\Gamma) - \tilde{\theta} + \mathcal{D}(\tilde{\theta}|_\Gamma)\big)_\Omega \mathrm{d}t.$$

On the other hand, for every $\delta, \delta_0 > 0$, there exist constants $C_\delta, C_{\delta_0} > 0$ such that

$$\begin{aligned}\int_0^T \big(\tilde{w}_t, \mathcal{D}(\tilde{w}_t|_\Gamma)\big)_\Omega \mathrm{d}t &\leqslant \int_0^T \|\tilde{w}_t\|_{-(1-\varepsilon),\Omega} \|\mathcal{D}(\tilde{w}_t|_\Gamma)\|_{1-\varepsilon,\Omega}\, \mathrm{d}t \\ &\leqslant \delta_0 \int_0^T \|\tilde{w}_t\|_\Omega^2 \mathrm{d}t + C_{\delta_0} \int_0^T \|\tilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1} \mathrm{d}t,\end{aligned} \tag{3.29}$$

$$\int_0^T \big(\tilde{w}_t, \tilde{\theta} - \mathcal{D}(\tilde{\theta}|_\Gamma)\big)_\Omega \mathrm{d}t \leqslant \delta_0 \int_0^T \|\tilde{w}_t\|_\Omega^2 \mathrm{d}t + C_{\delta_0} \int_0^T \|\tilde{\theta}\|^2_{1,\Omega} \mathrm{d}t, \tag{3.30}$$

$$\int_0^T a(\tilde{w}, \mathcal{A}_D^{-1}\tilde{\theta}) \mathrm{d}t \leqslant \delta \int_0^T \|\tilde{w}\|^2_{2,\Omega} \mathrm{d}t + C_\delta \int_0^T \|\tilde{\theta}\|^2_\Omega \mathrm{d}t, \tag{3.31}$$

$$\int_0^T \big(\mathscr{P}_2(\tilde{u}, \tilde{w}), \mathcal{A}_D^{-1}\tilde{\theta}\big)_\Omega \mathrm{d}t \leqslant C_{B,T} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}) + C \int_0^T \|\tilde{\theta}\|^2_\Omega \mathrm{d}t, \tag{3.32}$$

$$\int_0^T \Big[\big(\nabla\tilde{\theta}, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega - \big\langle \tilde{\theta}, \partial_\nu(\mathcal{A}_D^{-1}\tilde{\theta})\big\rangle_{\Gamma_1}\Big] \mathrm{d}t \leqslant C \int_0^T \|\tilde{\theta}\|^2_{1,\Omega} \mathrm{d}t, \tag{3.33}$$

$$\int_0^T \big\langle \Delta\tilde{w}, \partial_\nu(\mathcal{A}_D^{-1}\tilde{\theta})\big\rangle_{\Gamma_0} \mathrm{d}t \leqslant C \int_0^T \|\Delta\tilde{w}\|^2_{-\frac{1}{2},\Gamma_0} \mathrm{d}t + C \int_0^T \|\tilde{\theta}\|^2_\Omega \mathrm{d}t. \tag{3.34}$$

It remains to estimate the nonlinear terms in (3.28). For this, considering the definition of $\mathscr{N}_2$, we find

$$\begin{aligned}\int_0^T \big(\mathscr{N}_2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega \mathrm{d}t &= \int_0^T \big(\sigma[N(u^1, w^1)]\nabla w^1 - \sigma[N(u^2, w^2)]\nabla w^2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega \mathrm{d}t \\ &\quad + \int_0^T \big(\phi^1 \nabla w^1 - \phi^2 \nabla w^2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega \mathrm{d}t.\end{aligned} \tag{3.35}$$

Let us estimate the integrals on the right-side of (3.35). Proceeding as in (3.25) we obtain

$$\begin{aligned}&\int_0^T \big(\sigma[N(u^1, w^1)]\nabla w^1 - \sigma[N(u^2, w^2)]\nabla w^2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega \mathrm{d}t \\ &\qquad \leqslant \delta \int_0^T \|\sigma[\epsilon(\tilde{u})]\|^2_\Omega \mathrm{d}t + C_{B,\delta} \int_0^T \|\tilde{\theta}\|^2_\Omega \mathrm{d}t + C_{B,T,\delta} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}),\end{aligned}$$

$$\int_0^T \big(\phi^1 \nabla w^1 - \phi^2 \nabla w^2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega \mathrm{d}t \leqslant C_B \int_0^T \big[\|\tilde{\phi}\|^2_{1,\Omega} + \|\tilde{\theta}\|^2_{1,\Omega}\big] \mathrm{d}t + C_{B,T} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}).$$

Then we have

$$\begin{aligned}&\int_0^T \big(\mathscr{N}_2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta})\big)_\Omega \mathrm{d}t \\ &\qquad \leqslant \delta \int_0^T \|\sigma[\epsilon(\tilde{u})]\|^2_\Omega \mathrm{d}t + C_{B,\delta} \int_0^T \Big[\|\tilde{\phi}\|^2_{1,\Omega} + \|\tilde{\theta}\|^2_{1,\Omega}\Big] \mathrm{d}t + C_{B,T,\delta} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}).\end{aligned} \tag{3.36}$$

Therefore the estimates (3.29)-(3.34) and (3.36) applied in (3.28), for $\delta_0 > 0$ small enough, yield

$$\begin{aligned}\int_0^T \|\tilde{w}_t\|_\Omega^2 \mathrm{d}t \\ \leqslant C[\tilde{E}(0)+\tilde{E}(T)]+\delta\int_0^T \|\sigma[\epsilon(\tilde{u})]\|_\Omega^2\mathrm{d}t + C_{B,\delta}\int_0^T \big[\|\tilde{\theta}\|_{1,\Omega}^2+\|\tilde{\phi}\|_{1,\Omega}^2\big]\mathrm{d}t \\ + C_B\int_0^T\big[\|\Delta\tilde{w}\|_{-\frac{1}{2},\Gamma_0}^2+\|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon,\Gamma_1}^2+\|\tilde{w}\|_{2+\varepsilon,\Omega}^2\big]\mathrm{d}t+C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned} \tag{3.37}$$

Step 4. Completion of the proof: Combining the inequalities (3.17), (3.26), (3.37) and selecting suitable $\delta > 0$ small we obtain the conclusion. □

3.1.3. *Trace regularity and analytic estimates.* In order to control boundary terms in estimate given in Lemma 3.2, more subtle estimates are needed, including trace regularity and analytic estimates. They are essential to prove the quasistability inequality. Our result is based on the corresponding trace estimate for the linear model of dynamic elasticity [24]. The analytic estimates rely on the analyticity of the semigroup generator associated with the linear thermoelastic plate.

Lemma 3.3. *Let $(\tilde{u},\tilde{u}_t,\tilde{w},\tilde{w}_t,\tilde{\phi},\tilde{\theta})$ be a regular solution of the system (3.3)-(3.7). Then for any $\varepsilon\in(0,\frac{1}{4})$ and $\alpha\in(0,\frac{T}{2})$ the following trace regularity is valid.*

$$\int_{\Sigma_\alpha}|\nabla\tilde{u}|^2\mathrm{d}\Sigma_\alpha \leqslant C_\alpha\int_0^T\big[\|\tilde{u}_t\|_{\Gamma_1}^2+\|\tilde{\phi}\|_{1,\Omega}^2\big]\mathrm{d}t + C_{\alpha,B}\int_0^T\|\tilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{\alpha,B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

Proof. The proof is divided into several steps.

Step 1. Trace regularity for the linear model: Consider

$$\tilde{F} = \mathrm{div}\{\sigma[f(\nabla w^1)-f(\nabla w^2)]\} - \nabla\tilde{\phi} - \mathscr{P}_1(\tilde{u},\tilde{w}). \tag{3.38}$$

Then the solution $\tilde{u}=\tilde{u}(x,y,t)$ satisfies the problem

$$\tilde{u}_{tt} - \mathrm{div}\{\sigma[\epsilon(\tilde{u})]\} = \tilde{F}.$$

By using the trace regularity stated in [24] we obtain the estimate

$$\int_{\Sigma_\alpha}|\nabla\tilde{u}\,\tau|^2\mathrm{d}\Sigma_\alpha \leqslant C_\alpha\int_0^T\big[\|\tilde{u}_t\|_{\Gamma_1}^2+\|\tilde{F}\|_{-\frac{1}{2},\Omega}^2+\|\sigma[\epsilon(\tilde{u})]\nu\|_{\Gamma_1}^2+\|\tilde{u}\|_{1-\varepsilon,\Omega}^2\big]\mathrm{d}t, \tag{3.39}$$

where we used the inequality $\|\tilde{u}\|_{\frac{1}{2}+\varepsilon,\Omega}^2 \leqslant C\|\tilde{u}\|_{1-\varepsilon,\Omega}^2$.

Step 2. Estimate for $\tilde{F}$ defined in (3.38): For $\varepsilon\in(0,\frac{1}{2})$, we have

$$\|\tilde{F}(t)\|_{-\frac{1}{2},\Omega}^2 \leqslant C_B\|\tilde{w}\|_{2+\epsilon,\Omega}^2 + C\|\tilde{\phi}\|_{1,\Omega}^2 + C_B\mathrm{l.o.t.}(\tilde{u},\tilde{w}), \quad \forall\, t\geqslant 0. \tag{3.40}$$

To prove this, consider $\psi\in H^{\frac{1}{2}}(\Omega)$. Then

$$\begin{aligned}&\big(\mathrm{div}\{\sigma[f(\nabla w^1)-f(\nabla w^2)]\},\psi\big)_\Omega \\ &\qquad\leqslant C\big[\|\tilde{w}\|_{2,\Omega}\|\nabla\tilde{w}\cdot\psi\|_\Omega+\|\tilde{w}\|_{2,\Omega}\|\nabla w^2\cdot\psi\|_\Omega+\|w^2\|_{2,\Omega}\|\nabla\tilde{w}\cdot\psi\|_\Omega\big].\end{aligned}$$

Hölder inequality and Sobolev embeddings $H^{2-\varepsilon}(\Omega)\subset W^{1,4}(\Omega)$ and $H^{\frac{1}{2}}(\Omega)\subset L^4(\Omega)$ imply that

$$\|\tilde{w}\|_{2,\Omega}\|\nabla\tilde{w}\cdot\psi\|_\Omega \leqslant C_B\|\tilde{w}\|_{W^{1,4}(\Omega)}\|\psi\|_{L^4(\Omega)} \leqslant C_B\|\tilde{w}\|_{2-\varepsilon,\Omega}\|\psi\|_{\frac{1}{2},\Omega},$$

$$\|\tilde{w}\|_{2,\Omega}\|\nabla w^2\cdot\psi\|_\Omega \leqslant C\|\tilde{w}\|_{2,\Omega}\|w^2\|_{W^{1,4}(\Omega)}\|\psi\|_{L^4(\Omega)} \leqslant C_B\|\tilde{w}\|_{2,\Omega}\|\psi\|_{\frac{1}{2},\Omega},$$

$$\|w^2\|_{2,\Omega}\|\nabla\tilde{w}\cdot\psi\|_\Omega \leqslant C_B\|\tilde{w}\|_{W^{1,4}(\Omega)}\|\psi\|_{L^4(\Omega)} \leqslant C_B\|\tilde{w}\|_{2-\varepsilon,\Omega}\|\psi\|_{\frac{1}{2},\Omega}.$$

These inequalities and Sobolev embedding $H^{2+\varepsilon}(\Omega)\subset H^2(\Omega)$ imply that

$$\|\mathrm{div}\{\sigma[f(\nabla w^1)-f(\nabla w^2)]\}\|^2_{-\frac{1}{2},\Omega} \leqslant C_B\|\tilde{w}\|^2_{2+\varepsilon,\Omega} + C_B\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.41}$$

Using Hölder's inequality and Sobolev embedding we find that

$$\big(\nabla\tilde{\phi},\psi\big)_\Omega \leqslant \|\nabla\tilde{\phi}\|_\Omega\|\psi\|_\Omega \leqslant C\|\tilde{\phi}\|_{1,\Omega}\|\psi\|_{\frac{1}{2},\Omega}, \tag{3.42}$$

which with (1.15) lead to

$$\big(\mathscr{P}_1(\tilde{u},\tilde{w}),\psi\big)_\Omega \leqslant C_B\|\tilde{u}\|_{L^{r+1}(\Omega)}\|\psi\|_\Omega \leqslant C_B\|\tilde{u}\|_{L^{r+1}(\Omega)}\|\psi\|_{\frac{1}{2},\Omega}. \tag{3.43}$$

Therefore (3.41) together with (3.42) and (3.43) shows the estimate (3.40) holds.

Step 3. Estimate for the stress tensor: For $\varepsilon\in(0,\frac{1}{2})$, we have

$$\int_{\Sigma_\alpha}|\sigma[\epsilon(\tilde{u})]|^2\mathrm{d}\Sigma_\alpha \leqslant C\int_0^T\big[\|f(\nabla w^1)-f(\nabla w^2)\|^2_{\Gamma_1}+\|\tilde{u}\|^2_{1-\varepsilon,\Omega}+\|\tilde{\phi}\|^2_{1,\Omega}+\|\tilde{u}_t\|^2_{\Gamma_1}\big]\mathrm{d}t. \tag{3.44}$$

Indeed, the boundary condition $(3.6)_2$ implies that

$$\|\sigma[\epsilon(\tilde{u})]\|^2_{\Sigma_1} \leqslant C\big[\|f(\nabla w^1)-f(\nabla w^2)\|^2_{\Sigma_1}+\|\tilde{u}\|^2_{\Sigma_1}+\|\tilde{\phi}\|^2_{\Sigma_1}+\|\tilde{u}_t\|^2_{\Sigma_1}\big].$$

Then using inequalities $\|\tilde{u}\|^2_{\Sigma_1}\leqslant C\|\tilde{u}\|^2_{1-\varepsilon,\Omega}$ and $\|\tilde{\phi}\|^2_{\Sigma_1}\leqslant C\|\tilde{\phi}\|^2_{1,\Omega}$, we obtain (3.44).

Step 4. Estimate for $|\nabla\tilde{u}|$: We have

$$|\nabla\tilde{u}| \leqslant C\Big[|\nabla\tilde{u}\ \tau| + |\sigma[\tilde{u}]\ \nu|\Big], \tag{3.45}$$

where $\nu=(\nu_1,\nu_2)$ and $\tau=(\tau_1,\tau_2)=(-\nu_2,\nu_1)$ denote, respectively, the outward unit normal and the unit tangential vectors, at a point of Γ. To prove this, let us denote $\nabla\tilde{u}$ as a 4-vector $(\nabla\tilde{u})=(\tilde{u}_{1,x},\tilde{u}_{1,y},\tilde{u}_{2,x},\tilde{u}_{2,y})$. Then we obtain the algebraic system

$$A(\nabla\tilde{u})^\top = (\nabla\tilde{u}\ \tau,\sigma[\tilde{u}]\ \nu)^\top,$$

where A is a 4×4 matrix with constant determinant over Γ. Note that the right hand side of above identity denotes indeed a 4-vector. Then we obtain

$$(\nabla\tilde{u})^\top = A^{-1}(\nabla\tilde{u}\ \tau,\sigma[\tilde{u}]\ \nu)^\top,$$

and this implies (3.45).

Step 5. Conclusion: Integrating in time and space the inequality (3.45) we obtain

$$\int_{\Sigma_\alpha}|\nabla\tilde{u}|^2\,\mathrm{d}\Sigma_\alpha \leqslant C\int_{\Sigma_\alpha}|\nabla\tilde{u}\ \tau|^2\,\mathrm{d}\Sigma_\alpha + C\int_{\Sigma_\alpha}|\sigma[\tilde{u}]\ \nu|^2\mathrm{d}\Sigma_\alpha.$$

Inequalities (3.39), (3.40) and (3.44) imply

$$\begin{aligned}\int_{\Sigma_\alpha}|\nabla\tilde{u}|^2\,\mathrm{d}\Sigma_\alpha \leqslant{}& C_\alpha\int_0^T\|f(\nabla w^2)-f(\nabla w^2)\|^2_{\Gamma_1}\mathrm{d}t + C_\alpha\int_0^T\|\tilde{u}_t\|^2_{\Gamma_1}\mathrm{d}t\\ &+C_\alpha\int_0^T\big[\|\tilde{\phi}\|^2_{1,\Omega}+\|\tilde{w}\|^2_{2+\varepsilon,\Omega}\big]\mathrm{d}t + C_{\alpha,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned}$$

Then, inequality (ii) of Lemma 3.1 implies in the assertion of Lemma 3.3. $\square$

Next we prove an improved regularity for the vertical displacement $\tilde{w}$. This is done by exploiting the analyticity of the thermoelastic semigroup.

Lemma 3.4. *Let* $(\tilde{u}, \tilde{u}_t, \tilde{w}, \tilde{w}_t, \tilde{\phi}, \tilde{\theta})$ *be a regular solution of the system* (3.3)-(3.7). *Then for any* $\varepsilon \in (0, \frac{1}{2})$,

$$\int_0^T \left[\|\tilde{w}\|^2_{3-\varepsilon,\Omega} + \|\tilde{w}_t\|^2_{1-\varepsilon,\Omega} + \|\tilde{\theta}\|^2_{1-\varepsilon,\Omega}\right] \mathrm{d}t \leqslant C\tilde{E}(0) + C_B \int_0^T \|\tilde{\phi}\|^2_{1,\Omega} \mathrm{d}t + C_{B,T} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}).$$

Proof. The proof of the lemma is divided into three parts.

Step 1. Abstract setting: We rewrite the original problem via variation of parameters. To accomplish this we introduce the following operators.

• The biharmonic operator: Let $\mathcal{A}_M$ be a positive and self-adjoint operator on $L^2(\Omega)$ given by $\mathcal{A}_M v = \Delta^2 v$ with domain

$$D(\mathcal{A}_M) = \left\{ v \in H^4(\Omega) \;\middle|\; \begin{array}{l} v = 0,\ \nabla v = 0 \text{ on } \Gamma_0, \\ \left[\Delta v + (1-\mu) B_1 v\right]\big|_{\Gamma_1} = 0, \\ \left[\frac{\partial}{\partial \nu}(\Delta v) + (1-\mu) B_2 v\right]\big|_{\Gamma_1} = 0. \end{array} \right\}.$$

• The Green's operators: Let G_i, $i = 1, 2$, be the operators corresponding to the mechanical boundary conditions defined by

$$G_1 g = v \Leftrightarrow \left\{ \begin{array}{l} \Delta^2 v = 0 \text{ in } \Omega, \\ \left\{ \begin{array}{l} v = 0,\ \nabla v = 0 \text{ on } \Gamma_0, \\ \left[\Delta v + (1-\mu) B_1 v\right]\big|_{\Gamma_1} = g, \\ \left[\dfrac{\partial}{\partial \nu}(\Delta v) + (1-\mu) B_2 v\right]\big|_{\Gamma_1} = 0, \end{array} \right. \end{array} \right.$$

and

$$G_2 g = v \Leftrightarrow \left\{ \begin{array}{l} \Delta^2 v = 0 \text{ in } \Omega, \\ \left\{ \begin{array}{l} v = 0,\ \nabla v = 0 \text{ on } \Gamma_0, \\ \left[\Delta v + (1-\mu) B_1 v\right]\big|_{\Gamma_1} = 0, \\ \left[\dfrac{\partial}{\partial \nu}(\Delta v) + (1-\mu) B_2 v\right]\bigg|_{\Gamma_1} = g. \end{array} \right. \end{array} \right. .$$

Elliptic regularity (e.g. [43]) gives $G_1 : L^2(\Gamma_1) \to H^{\frac{5}{2}}(\Omega) \subset H^{\frac{5}{2}-4\varepsilon}(\Omega) \equiv D(\mathcal{A}_M^{\frac{5}{8}-\varepsilon})$ and $G_2 : L^2(\Gamma_1) \to H^{\frac{7}{2}}(\Omega) \subset H^{\frac{7}{2}-4\varepsilon}(\Omega) \equiv D(\mathcal{A}_M^{\frac{7}{8}-\varepsilon})$, $\varepsilon > 0$. By application of Green's formula (e.g. [42, 43]) we get, for $v \in D(\mathcal{A}_M)$,

$$G_1^* \mathcal{A}_M\, v = -\frac{\partial}{\partial \nu}(v|_{\Gamma_1}),\ G_2^* \mathcal{A}_M\, v = -v|_{\Gamma_1}. \tag{3.46}$$

We have that

$$\mathbf{w} := \left[\tilde{w} + G_1(\tilde{\theta}|_{\Gamma_1}) + G_2\left(\frac{\partial \tilde{\theta}}{\partial \nu}\right)\right] \in D(\mathcal{A}_M).$$

Then from the definition of operators $\mathcal{A}_M$, G_1, G_2 and $\mathcal{A}$, we have, (cf. [41])

$$\begin{aligned} \tilde{w}_{tt} + \mathcal{A}_M \mathbf{w} - \operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} - \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi}) \cdot \nu) + \Delta\tilde{\theta} + \mathscr{P}_2(\tilde{u}, \tilde{w}) = 0, \\ \tilde{\theta}_t - \Delta\tilde{\theta} - \Delta\tilde{w}_t = 0, \end{aligned}$$

where

$$F(\tilde{u}, \tilde{w}, \tilde{\phi}) = \sigma[N(u^1, w^1)]\nabla w^1 - \sigma[N(u^2, w^2)]\nabla w^2 + \phi^1 \nabla w^1 - \phi^2 \nabla w^2.$$

Therefore, we can rewrite the problem for $(\tilde{w}, \tilde{\theta})$ in the following form

$$\begin{bmatrix} \tilde{w}_t \\ \tilde{w}_{tt} \\ \tilde{\theta}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} + \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi})\cdot\nu) - \mathscr{P}_2(\tilde{u}, \tilde{w}) \\ 0 \end{bmatrix}, \tag{3.47}$$

where $\mathcal{A} : H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \to H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ with domain

$$D(\mathcal{A}) = \left\{ (w, v, \theta) \in H^4(\Omega) \times [H^2(\Omega)]^2 \;\middle|\; \begin{array}{l} w = 0, \nabla w = 0 \text{ on } \Gamma_0, \\ \Delta w + (1-\mu)B_1 w + \theta|_{\Gamma_1} = 0, \\ \partial_\nu(\Delta w) + (1-\mu)B_2 w + \partial_\nu \theta\big|_{\Gamma_1} = 0, \\ \partial_\nu \theta + \lambda_2 \theta = 0 \text{ on } \Gamma. \end{array} \right\}$$

and defined by

$$\mathcal{A} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} \tilde{w}_t \\ -\mathcal{A}_M \tilde{w} - \mathcal{A}_M(G_1 \tilde{\theta}|_{\Gamma_1}) - \mathcal{A}_M G_2(\frac{\partial \tilde{\theta}}{\partial \nu}) - \Delta \tilde{\theta} \\ \Delta \tilde{\theta} + \Delta \tilde{w}_t \end{bmatrix}.$$

The operator $\mathcal{A}$ generates an analytic and exponentially stable semigroup on the space

$$H = H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

cf. [41]. Moreover $\mathcal{A}$ is m-dissipative and $\mathcal{A}^{-1}$ is bounded in H. Therefore, from [6, Proposition 6.1] we infer that, for $\alpha \in (0,1)$,

$$D(\mathcal{A}^\alpha) = [D(\mathcal{A}), H]_{1-\alpha} \subset H^{2(1+\alpha)}(\Omega) \times H^{2\alpha}(\Omega) \times H^{2\alpha}(\Omega). \tag{3.48}$$

Then, for $\varepsilon < \frac{1}{2}$, we can rewrite the solution of (3.47) using variation of parameters formula,

$$\mathcal{A}^{\frac{1-\varepsilon}{2}} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} = \mathcal{A}^{\frac{1-\varepsilon}{2}} e^{\mathcal{A}t} \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} + \int_0^t \mathcal{A} e^{\mathcal{A}(t-s)} \mathcal{A}^{-\frac{1+\varepsilon}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) \\ 0 \end{bmatrix} \mathrm{d}s, \tag{3.49}$$

where $\mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) = \operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} + \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi})\cdot\nu) - \mathscr{P}_2(\tilde{u}, \tilde{w})$.

Step 2. Some estimates: Since $\mathcal{A}$ is m-dissipative, invertible and generates an analytic semigroup which is exponentially stable, the following estimates are valid, for $\alpha \leqslant \frac{1}{2}$,

$$\begin{aligned} \left\| \int_0^t \mathcal{A} e^{(t-s)\mathcal{A}} f(s) \mathrm{d}s \right\|_{L^2(0,T;H)} + \left\| \int_0^t \mathcal{A}^\alpha e^{(t-s)\mathcal{A}} f(s) \mathrm{d}s \right\|_H &\leqslant C \|f\|_{L^2(0,T;H)}, \\ \left\| \mathcal{A}^\alpha e^{t\mathcal{A}} x \right\|_{L^2(0,T;H)} &\leqslant C \|x\|_H. \end{aligned} \tag{3.50}$$

Inserting inequalities (3.50) in (3.49), for $\varepsilon \in (0, \frac{1}{2})$, shows that

$$\left\| \mathcal{A}^{\frac{1-\varepsilon}{2}} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} \right\|_{L^2(0,T;H)} \leqslant C \left\| \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} \right\|_H + C \left\| \mathcal{A}^{-\frac{1+\varepsilon}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) \\ 0 \end{bmatrix} \right\|_{L^2(0,T;H)}. \tag{3.51}$$

Using the characterization (3.48), with $\alpha = \frac{1+\varepsilon}{2}$, and by duality we find that

$$\left\| \mathcal{A}^{-\frac{1+\varepsilon}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(\tilde{u},\tilde{w},\tilde{\phi}) \\ 0 \end{bmatrix} \right\|_H \leqslant C \|\mathcal{F}(\tilde{u},\tilde{w},\tilde{\phi})\|_{-(1+\varepsilon),\Omega}. \tag{3.52}$$

Let us estimate $\|\mathcal{F}(\tilde{u},\tilde{w},\tilde{\phi})\|_{-(1+\varepsilon),\Omega}$. The equalities (3.46) imply, for every $\psi \in H^{1+\varepsilon}(\Omega)$,

$$\begin{aligned}(\mathcal{F}(\tilde{u},\tilde{w},\tilde{\phi}),\psi)_\Omega &= \big(\mathrm{div}\{F(\tilde{u},\tilde{w},\tilde{\phi})\} + \mathcal{A}_M G_2(F(\tilde{u},\tilde{w},\tilde{\phi})\cdot\nu),\psi\big)_\Omega - (\mathscr{P}_2(\tilde{u},\tilde{w}),\psi)_\Omega \\ &= -(F(\tilde{u},\tilde{w},\tilde{\phi}),\nabla\psi)_\Omega - (\mathscr{P}_2(\tilde{u},\tilde{w}),\psi)_\Omega.\end{aligned} \tag{3.53}$$

First, using the definition of F we obtain

$$\begin{aligned}\big(F(\tilde{u},\tilde{w},\tilde{\phi}),\nabla\psi\big)_\Omega &= \big(\sigma[\epsilon(\tilde{u})]\nabla w^2,\nabla\psi\big)_\Omega + \big(\sigma[f(\nabla w^1)-f(\nabla w^2)]\nabla w^2,\nabla\psi\big)_\Omega \\ &\quad + \big(\sigma[\epsilon(u^1)+f(\nabla w^1)]\nabla\tilde{w},\nabla\psi\big)_\Omega + \big(\tilde{\phi}\nabla w^2 + \phi^1\nabla\tilde{w},\nabla\psi\big)_\Omega.\end{aligned} \tag{3.54}$$

Let us estimate the inner products in (3.54). For this we recall the following inequalities (cf. [13])

$$\begin{aligned}\big(uv,w\big)_\Omega &\leqslant C\|uv\|_{\eta,\Omega}\|w\|_{-\eta,\Omega},\\ \|uv\|_{\eta,\Omega} &\leqslant C\|u\|_{1,\Omega}\|v\|_{\eta+\eta_0,\Omega},\\ \|uv\|_{-\eta,\Omega} &\leqslant C\|u\|_{1-\eta,\Omega}\|v\|_\Omega,\end{aligned}$$

where $\eta < \frac{1}{2}$ and $\eta_0 \in \mathbb{R}$. Consider $\eta < \varepsilon$. Then

$$\begin{aligned}\big(\sigma[\epsilon(\tilde{u})]\nabla w^2,\nabla\psi\big)_\Omega &\leqslant C\|\epsilon(\tilde{u})\|_{-\eta,\Omega}\|\nabla w^2\cdot\nabla\psi\|_{\eta,\Omega}\\ &\leqslant C\|\epsilon(\tilde{u})\|_{-\eta,\Omega}\|\nabla w^2\|_{1,\Omega}\|\nabla\psi\|_{\eta+\eta_0,\Omega},\end{aligned}$$

and taking $\eta_0 \in \mathbb{R}$ such that $\eta_0 + \eta \leqslant \varepsilon$, we obtain

$$\big(\sigma[\epsilon(\tilde{u})]\nabla w^2,\nabla\psi\big)_\Omega \leqslant C\|\tilde{u}\|_{1-\eta,\Omega}\|w^2\|_{2,\Omega}\|\psi\|_{1+\varepsilon,\Omega}.$$

To the others terms, we use Hölder inequality and Sobolev embedding, so we find

$$\begin{aligned}\big(\sigma[f(\nabla w^1)-f(\nabla w^2)]\nabla w^2,\nabla\psi\big)_\Omega &\leqslant C\|f(\nabla w^1)-f(\nabla w^2)\|_\Omega\|w^2\|_{2+\epsilon,\Omega}\|\psi\|_{1+\varepsilon,\Omega}\\ &\leqslant C_B\|\tilde{w}\|_{2-\varepsilon,\Omega}\|w^2\|_{2+\varepsilon,\Omega}\|\psi\|_{1+\varepsilon,\Omega},\end{aligned}$$

$$\begin{aligned}\big(\sigma[\epsilon(u^1)+f(\nabla w^1)]\nabla\tilde{w},\nabla\psi\big)_\Omega &\leqslant C_B\|\tilde{w}\|_\Omega^{\frac{1-2\varepsilon}{3-\varepsilon}}\|\tilde{w}\|_{3-\varepsilon,\Omega}^{\frac{2+\varepsilon}{3-\varepsilon}}\|\psi\|_{1+\varepsilon,\Omega}\\ &\leqslant \big[\delta\|\tilde{w}\|_{3-\varepsilon,\Omega} + C_{B,\delta}\|\tilde{w}\|_\Omega\big]\|\psi\|_{1+\varepsilon,\Omega},\end{aligned}$$

$$\begin{aligned}\big(\tilde{\phi}\nabla w^2 + \phi^1\nabla\tilde{w},\nabla\psi\big)_\Omega &\leqslant C\big[\|\tilde{\phi}\nabla w^2\|_{-\varepsilon,\Omega} + \|\phi^1\nabla\tilde{w}\|_{-\varepsilon,\Omega}\big]\|\nabla\psi\|_{\varepsilon,\Omega}\\ &\leqslant C\big[\|\tilde{\phi}\|_\Omega\|w^2\|_{2-\varepsilon,\Omega} + \|\phi^1\|_\Omega\|\tilde{w}\|_{2-\varepsilon,\Omega}\big]\|\psi\|_{1+\varepsilon,\Omega}.\end{aligned}$$

Finally, by (1.16) we obtain

$$(\mathscr{P}_2(\tilde{u},\tilde{w}),\psi)_\Omega \leqslant C_B\big[\|\tilde{u}\|_{L^{r+1}} + \|\tilde{w}\|_{L^{r+1}}\big]\|\psi\|_\Omega.$$

Combining these estimates with (3.53) we obtain

$$\|\mathcal{F}(\tilde{u},\tilde{w},\tilde{\phi})\|^2_{-(1+\varepsilon),\Omega} \leqslant C_B\|\tilde{\phi}\|^2_\Omega + \delta\|\tilde{w}\|^2_{3-\varepsilon,\Omega} + C_{B,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.55}$$

Step 3. Conclusion: Combining the estimate (3.55) together with (3.51)-(3.52), and the characterization of $D(\mathcal{A}^{\frac{1-\varepsilon}{2}})$, it follows that

$$\int_0^T \big[\|\tilde{w}\|^2_{3-\varepsilon,\Omega} + \|\tilde{w}_t\|^2_{1-\varepsilon,\Omega} + \|\tilde{\theta}\|^2_{1-\varepsilon,\Omega}\big]\mathrm{d}t$$
$$\leqslant C\tilde{E}(0) + C_B \int_0^T \big[\|\tilde{\phi}\|^2_{\Omega} + \delta\|\tilde{w}\|^2_{3-\varepsilon,\Omega}\big]\mathrm{d}t + C_{\delta,B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

Taking $\delta > 0$ small enough we obtain the main result of Lemma 3.4. $\square$

Remark 3.2. The estimate in the Lemma 3.4 and the Trace Theorem imply that

$$\int_0^T \big[\|\Delta\tilde{w}\|^2_{\Gamma_0} + \|\tilde{w}_t\|^2_{\frac{1}{2}-\epsilon,\Gamma_1}\big]\mathrm{d}t \leqslant C\tilde{E}(0) + C_B \int_0^T \|\tilde{\phi}\|^2_{1,\Omega}\mathrm{d}t + C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}),$$

holds for $\epsilon \in (0, \frac{1}{2})$.

3.1.4. *A second observability inequality.*

Lemma 3.5. *Let $(\tilde{u},\tilde{u}_t,\tilde{w},\tilde{w}_t,\tilde{\phi},\tilde{\theta})$ be a solution of the system (3.3)-(3.7). Then for $\alpha \in (0, \frac{T}{2})$, there exist positive constants $C_\alpha, C_{\alpha,B}, C_{\alpha,B,T}$, such that*

$$T\tilde{E}(T) + \int_0^T \tilde{E}(t)\mathrm{d}t + \int_0^T \big[\|\tilde{u}_t\|^2_{\Gamma_1} + \|\tilde{\phi}\|^2_{1,\Omega}\big]\mathrm{d}t + \Big[\sup_{t\in[0,T]} \tilde{E}^{\frac{1}{2}}(t)\Big]^2$$
$$\leqslant C\tilde{E}(0) + [C+2\alpha]\tilde{E}(T) + C_{\alpha,B}\, D_0^T\,(\tilde{u},\tilde{\phi},\tilde{\theta}) + C_\alpha\int_0^T \left|\sum_{i=1}^4 \mathscr{R}_i(t)\right|\mathrm{d}t$$
$$+ \int_0^T\int_s^T \left|\sum_{i=1}^4 \mathscr{R}_i(t)\right|\mathrm{d}t\mathrm{d}s + C_{\alpha,B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.56}$$

Proof. The Lemma 3.2 applied to the interval $[\alpha, T-\alpha]$ and estimate in the Lemma 3.3 imply that

$$\int_\alpha^{T-\alpha} \tilde{E}(t)\mathrm{d}t$$
$$\leqslant C\big[\tilde{E}(\alpha) + \tilde{E}(T-\alpha)\big] + C_{\alpha,B}\int_\alpha^{T-\alpha}\Big[\|\tilde{u}_t\|^2_{\Gamma_1} + \|\tilde{\theta}\|^2_{1,\Omega} + \|\tilde{\phi}\|^2_{1,\Omega}\Big]\mathrm{d}t$$
$$+ C_{\alpha,B}\int_\alpha^{T-\alpha}\Big[\|\Delta\tilde{w}\|^2_{-\frac{1}{2},\Gamma_0} + \|\tilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1} + \|\tilde{w}\|^2_{2+\varepsilon,\Omega}\Big]\mathrm{d}t + C_{\alpha,B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

Interpolation inequality $\|w\|_{2+\varepsilon,\Omega} \leqslant C\|w\|_{\Omega}^{\frac{1-2\varepsilon}{3-\epsilon}}\,\|w\|_{3-\varepsilon,\Omega}^{\frac{2+\varepsilon}{3-\varepsilon}}$, together with Lemma 3.4 imply that

$$\int_\alpha^{T-\alpha}\tilde{E}(t)\mathrm{d}t \leqslant C\big[\tilde{E}(0)+\tilde{E}(\alpha)+\tilde{E}(T-\alpha)\big] + C_{\alpha,B}D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta}) + C_{\alpha,B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.57}$$

We shall extend the integral on the left-hand side to the interval $(0,T)$. To this end, using the energy equality (3.8) we find that

$$\int_0^\alpha \tilde{E}(t)\mathrm{d}t \leqslant \alpha\tilde{E}(\alpha) + \int_0^\alpha\int_0^t \sum_{i=1}^4 \mathscr{R}_i(s)\mathrm{d}s\mathrm{d}t,$$

$$
\begin{aligned}
&\int_{T-\alpha}^{T} \tilde{E}(t)\mathrm{d}t \leqslant \alpha\tilde{E}(T-\alpha) + \int_{T-\alpha}^{T}\int_0^t \sum_{i=1}^4 \mathscr{R}_i(s)\mathrm{d}s\mathrm{d}t,\\
&\tilde{E}(\alpha) \leqslant \tilde{E}(T) + D_\alpha^T(\tilde{u},\tilde{\phi},\tilde{\theta}) + \left|\int_\alpha^T \sum_{i=1}^4 \mathscr{R}_i(t)\mathrm{d}t\right|,\\
&\tilde{E}(T-\alpha) \leqslant \tilde{E}(T) + D_{T-\alpha}^T(\tilde{u},\tilde{\phi},\tilde{\theta}) + \left|\int_{T-\alpha}^T \sum_{i=1}^4 \mathscr{R}_i(t)\mathrm{d}t\right|,
\end{aligned}
$$

and thus

$$
\int_0^\alpha \tilde{E}(t)\mathrm{d}t + \int_{T-\alpha}^T \tilde{E}(T)\,\mathrm{d}t \leqslant 2\alpha\tilde{E}(T) + 2\alpha D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta}) + 2(\alpha+1)\int_0^T\left|\sum_{i=1}^4 \mathscr{R}_i(t)\right|\mathrm{d}t. \tag{3.58}
$$

The energy identity (3.8) also implies that

$$
T\tilde{E}(T) \leqslant \int_0^T \tilde{E}(s)\mathrm{d}s + \int_0^T\int_s^T \left|\sum_{i=1}^4 \mathscr{R}_i(t)\right|\mathrm{d}t\mathrm{d}s. \tag{3.59}
$$

Combining (3.58) and (3.59) with (3.57) we obtain the inequality

$$
\begin{aligned}
T\tilde{E}(T) &+ \int_0^T \tilde{E}(t)\mathrm{d}t\\
&\leqslant C\tilde{E}(0) \;+\; [C+2\alpha]\tilde{E}(T) \;+\; C_{\alpha,B}\, D_0^T\,(\tilde{u},\tilde{\phi},\tilde{\theta}) + C_{\alpha,B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w})\\
&\quad + C_\alpha\int_0^T\left|\sum_{i=1}^4 \mathscr{R}_i(t)\right|\mathrm{d}t + \int_0^T\int_s^T\left|\sum_{i=1}^4 \mathscr{R}_i(t)\right|\mathrm{d}t\mathrm{d}s.
\end{aligned}\tag{3.60}
$$

In order to absorb some terms that will come from the nonlinearities, we shall add $\int_0^T\left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2\right]\mathrm{d}t + \left[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\right]^2$ in both sides of (3.60). Then using the energy equality (3.8) we obtain

$$
\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2 \leqslant \tilde{E}(0) + D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta}) + \left|\int_0^T\sum_{i=1}^4 \mathscr{R}_i(t)\mathrm{d}t\right|.
$$

Therefore we obtain (3.56). This completes the proof of Lemma 3.5. ☐

3.1.5. *Quasistability inequality.* We begin with estimates for $\mathscr{R}_i(t)$, $1\leqslant i\leqslant 4$.

Lemma 3.6. *One has*

$$
\begin{aligned}
&\max\Big\{\int_0^T\int_s^T\Big|\sum_{i=1}^4 \mathscr{R}_i(t)\Big|\mathrm{d}t\mathrm{d}s\,,\int_0^T\Big|\sum_{i=1}^4 \mathscr{R}_i(t)\Big|\mathrm{d}t\Big\}\\
&\quad\leqslant \delta\int_0^T\left[\|\sigma[\epsilon(\tilde{u})]\|_\Omega^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_\Omega^2 + \|\tilde{w}_t\|_\Omega^2 + \|\tilde{\phi}\|_{1,\Omega}^2\right]\mathrm{d}t + \delta\tilde{E}(T)\\
&\qquad + \delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2 + C_{B,T,\delta}\int_0^T\|\tilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).
\end{aligned}\tag{3.61}
$$

Proof. The inequalities (1.15) and (1.16) imply that

$$
\int_0^T\int_s^T \mathscr{R}_1(t)\,\mathrm{d}t\mathrm{d}s \leqslant C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}) + \delta\int_0^T\Big[\|\tilde{u}_t\|_\Omega^2 + \|\tilde{w}_t\|_\Omega^2\Big]\mathrm{d}t. \tag{3.62}
$$

Using estimates of Lemma 3.1 we find that

$$\int_0^T \int_s^T \mathscr{R}_2(t)\,\mathrm{d}t\mathrm{d}s \leqslant C_{B,T,\delta}\int_0^T \|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t + \delta\int_0^T \Big[\|\tilde{u}_t\|^2_\Omega + \|\tilde{u}_t\|^2_{\Gamma_1}\Big]\mathrm{d}t. \quad (3.63)$$

Now,

$$\begin{aligned}\int_0^T \int_s^T \mathscr{R}_3(t)\mathrm{d}t\mathrm{d}s = &\int_0^T\int_s^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla \tilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s\\ &+ \int_0^T\int_s^T \big(\sigma\big[f(\nabla w^1) - f(\nabla w^2)\big]\nabla w^2, \nabla\tilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s \quad (3.64)\\ &+ \int_0^T\int_s^T \big(\sigma\big[\epsilon(u^1) + f(\nabla w^1)\big]\nabla\tilde{w}, \nabla w_t\big)_\Omega \mathrm{d}t\mathrm{d}s.\end{aligned}$$

We estimate the three integrals in (3.64). Integrating by parts in time we obtain

$$\begin{aligned}\int_0^T\int_s^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s = &\int_0^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}\big)_\Omega\Big|_s^T \mathrm{d}s\\ &- \int_0^T\int_s^T \big(\sigma[\epsilon(\tilde{u}_t)]\nabla w^2 - \sigma[\epsilon(\tilde{u})]\nabla w^2_t, \nabla\tilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s.\end{aligned} \quad (3.65)$$

But,

$$\begin{aligned}&\int_0^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}\big)_\Omega\Big|_s^T \mathrm{d}s\\ &\qquad = T\big(\sigma[\epsilon(\tilde{u}(T))]\nabla w^2(T), \nabla\tilde{w}(T)\big)_\Omega - \int_0^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}\big)_\Omega \mathrm{d}t.\end{aligned}$$

Using Sobolev embedding $H^{2-\varepsilon}(\Omega) \subset W^{1,4}(\Omega)$, we can show that

$$\begin{aligned}&T\big(\sigma[\epsilon(\tilde{u}(T))]\nabla w^2(T), \nabla\tilde{w}(T)\big)_\Omega \leqslant \delta\tilde{E}(T) + C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}),\\ &\int_0^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}\big)_\Omega \mathrm{d}t \leqslant \delta\int_0^T \|\sigma[\epsilon(\tilde{u})]\|^2_\Omega \mathrm{d}t + C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned}$$

Therefore we conclude that

$$\int_0^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}\big)_\Omega\Big|_s^T \mathrm{d}s \leqslant \delta\tilde{E}(T) + \delta\int_0^T \|\sigma[\epsilon(\tilde{u})]\|^2_\Omega \mathrm{d}t + C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

Integration by parts in space variable and the Trace Theorem imply that

$$\int_0^T\int_s^T \big(\sigma[\epsilon(\tilde{u}_t)]\nabla w^2, \nabla\tilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s \leqslant \delta\int_0^T [\|\tilde{u}_t\|^2_{\Gamma_1} + \|\tilde{u}_t\|^2_\Omega]\mathrm{d}t + C_{B,T,\delta}\int_0^T \|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t.$$

Hölder inequality and Sobolev embedding imply that

$$\int_0^T\int_s^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2_t, \nabla\tilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s \leqslant \delta\Big[\sup_{t\in[0,T]} \tilde{E}^{\frac{1}{2}}(t)\Big]^2 + C_{B,T,\delta}\int_0^T \|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t.$$

Inserting these estimates in (3.65) we obtain

$$\begin{aligned}&\int_0^T\int_s^T \big(\sigma[\epsilon(\tilde{u})]\nabla w^2, \nabla\tilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s\\ &\qquad \leqslant \delta\tilde{E}(T) + \delta\int_0^T \big[\|\sigma[\epsilon(\tilde{u})]\|^2_\Omega + \|\tilde{u}_t\|^2_{\Gamma_1} + \|\tilde{u}_t\|^2_\Omega\big]\mathrm{d}t\end{aligned}$$

$$+\delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2+C_{B,T,\delta}\int_0^T\|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t+C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.66}$$

Let us estimate the second integral in (3.64). Taking $\tilde{\varepsilon}<1-2\varepsilon$, we have that $H^{2-\varepsilon}(\Omega)\subset H^{1+\varepsilon+\tilde{\varepsilon}}(\Omega)$, and then

$$\begin{aligned}\int_0^T\int_s^T&\big(\sigma\big[f(\nabla w^1)-f(\nabla w^2)\big]\nabla w^2,\nabla\tilde{w}_t\big)_\Omega\mathrm{d}t\mathrm{d}s\\&\leqslant C_T\int_0^T\|f(\nabla w^1)-f(\nabla w^2)\|_{1,\Omega}\|\nabla w^2\|_{\varepsilon+\tilde{\varepsilon},\Omega}\|\nabla\tilde{w}_t\|_{-\varepsilon,\Omega}\mathrm{d}t\\&\leqslant C_{B,T,\delta}\int_0^T\|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t+\delta\int_0^T\|\tilde{w}_t\|^2_{1-\varepsilon,\Omega}\mathrm{d}t.\end{aligned}\tag{3.67}$$

To conclude, we have to estimate the third integral in (3.64). Integration by parts in time and space, and the fact that $\||\nabla\tilde{w}|^2\|_{1,\Omega}=\|\nabla\tilde{w}\cdot\nabla\tilde{w}\|_{1,\Omega}\leqslant C\|\tilde{w}\|_{1,\Omega}\|\tilde{w}\|_{2+\varepsilon,\Omega}$, we obtain

$$\begin{aligned}\int_0^T\int_s^T&\big(\sigma\big[\epsilon(u^1)\big]\nabla\tilde{w},\nabla\tilde{w}_t\big)_\Omega\mathrm{d}t\mathrm{d}s\\&\leqslant C_T\|\epsilon(u^1(T))\|_\Omega\||\nabla\tilde{w}(T)|^2\|_\Omega+C_T\int_0^T\|\epsilon(u^1)\|_\Omega\||\nabla\tilde{w}|^2\|_\Omega\mathrm{d}t\\&\quad+C_T\int_0^T\|u^1_t\|_{\Gamma_1}\||\nabla\tilde{w}|^2\|_{\Gamma_1}\mathrm{d}t+C_T\int_0^T\|u^1_t\|_\Omega\||\nabla\tilde{w}|^2\|_{1,\Omega}\mathrm{d}t\\&\leqslant C_{B,T}\int_0^T\|\tilde{w}\|^2_{2+\varepsilon,\Omega}+C_{B,T}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned}\tag{3.68}$$

As before, from the fact that $H^{2-\varepsilon}(\Omega)\subset H^{1+\varepsilon+\tilde{\varepsilon}}(\Omega)$ we see that

$$\begin{aligned}\int_0^T\int_s^T&\big(\sigma\big[f(\nabla w^1)\big]\nabla\tilde{w},\nabla\tilde{w}_t\big)_\Omega\mathrm{d}t\mathrm{d}s\\&\leqslant C_T\int_0^T\|\sigma\big[f(\nabla w^1)\big]\|_{1,\Omega}\|\nabla\tilde{w}\|_{\varepsilon+\tilde{\epsilon}}\|\tilde{w}_t\|_{1-\varepsilon,\Omega}\mathrm{d}t\\&\leqslant\delta\int_0^T\|\tilde{w}_t\|^2_{1-\varepsilon,\Omega}+C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned}\tag{3.69}$$

Inserting the estimates (3.66)-(3.69) into (3.64) implies that

$$\begin{aligned}\int_0^T\int_s^T&\mathscr{R}_3(t)\mathrm{d}t\mathrm{d}s\\&\leqslant\delta\tilde{E}(T)+\delta\int_0^T\Big[\|\sigma[\epsilon(\tilde{u})]\|^2_\Omega+\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{u}_t\|^2_\Omega+\|\tilde{w}_t\|^2_{1-\varepsilon,\Omega}\Big]\mathrm{d}t\\&\quad+\delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2+C_{B,T,\delta}\int_0^T\|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t+C_{B,T,\delta}\mathrm{l.o.t.}(\tilde{u},\tilde{w}).\end{aligned}\tag{3.70}$$

Finally we estimate $\int_0^T\int_s^T\mathscr{R}_4(t)\mathrm{d}t\mathrm{d}s$. Taking $\tilde{\varepsilon}<1-2\varepsilon$, as in 3.67 we see that

$$\begin{aligned}\int_0^T\int_s^T&\mathscr{R}_4(t)\mathrm{d}t\mathrm{d}s\\&=\int_0^T\int_s^T\Big[-\big(\phi^1\nabla\tilde{w},\nabla\tilde{w}_t\big)_\Omega+\big(\nabla\tilde{w}\cdot\nabla w^2_t,\tilde{\phi}\big)_\Omega+\big(\tilde{\phi}\nabla\tilde{w},\nabla\tilde{w}_t\big)_\Omega\Big]\mathrm{d}t\mathrm{d}s\end{aligned}$$

$$\leqslant \delta \int_0^T \big[\|\tilde{\phi}\|^2_{1,\Omega}+\|\tilde{w}_t\|^2_{1-\varepsilon,\Omega}\big]\mathrm{d}t+\delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2$$
$$+ C_{B,T,\delta}\int_0^T \|\tilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.71}$$

Combining (3.62), (3.70)-(3.71) and observing that $\|\tilde{w}_t\|^2_{1-\varepsilon,\Omega}$ was estimated in the Lemma 3.4, we conclude that $\int_0^T\int_s^T|\sum\mathscr{R}_i(t)|\,\mathrm{d}t\mathrm{d}s$ satisfies the desired estimate. Analogous argument shows that $\int_0^T|\sum\mathscr{R}_i(t)|\,\mathrm{d}t$ also satisfies the same estimate. □

Lemma 3.7. (Stabilizability estimate) *Under hypotheses of Theorem 1.2, let B be a bounded set of $\mathcal{H}$. Then in the context of (3.1)-(3.2), for $\varepsilon\in(0,\frac{1}{4})$, there exist constants $\beta>0$ and $C_1, C_2>0$, depending only on B, such that*

$$\tilde{E}(t)\leqslant C_1\tilde{E}(0)e^{-\beta t}+C_2\big[\sup_{\tau\in[0,t]}\big\{\|\tilde{u}(\tau)\|^2_{L^{r+1}(\Omega)}+\|\tilde{u}(\tau)\|^2_{1-\varepsilon,\Omega}\big\}+\sup_{\tau\in[0,t]}\|\tilde{w}(\tau)\|^2_{2-\varepsilon,\Omega}\big].$$

Proof. Inserting estimate from Lemma 3.6 into (3.56) and using interpolation inequality we find that

$$T\tilde{E}(T)+\int_0^T\tilde{E}(t)\mathrm{d}t+\int_0^T\big[\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}\big]\mathrm{d}t+\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2$$
$$\leqslant C\tilde{E}(0)+\big[C_\delta+2\alpha\big]\tilde{E}(T)+\delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2+C_{B,\alpha}D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta})+\delta\int_0^T E(t)\mathrm{d}t$$
$$+\delta\int_0^T\big[\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}+\|\tilde{w}_t\|^2_{1-\varepsilon,\Omega}+\|\tilde{w}\|^2_{3-\varepsilon,\Omega}\big]\mathrm{d}t+C_{\alpha,B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.72}$$

This together with (3.72) and estimate from Lemma 3.4 imply that

$$T\tilde{E}(T)+\int_0^T\tilde{E}(t)\mathrm{d}t+\int_0^T\big[\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}\big]\mathrm{d}t+\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2$$
$$\leqslant C\tilde{E}(0)+\big[C_\delta+2\alpha\big]\tilde{E}(T)+C_{B,\alpha}\,D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta})+C_{\alpha,B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w})$$
$$+\delta\int_0^T\tilde{E}(t)\mathrm{d}t+\delta\int_0^T\Big[\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}\Big]\mathrm{d}t+\delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2.$$

Now let $\delta>0$ be small enough. For $T>4C_\delta=T_0$ and $\alpha=C_\delta<\frac{T}{2}$ we have that

$$\tilde{E}(T)+\int_0^T\tilde{E}(t)\mathrm{d}t+\int_0^T\big[\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}\big]\mathrm{d}t+\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2$$
$$\leqslant\; C_T\tilde{E}(0)+C_{B,T}D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta})+C_{B,T}\mathrm{l.o.t.}(\tilde{u},\tilde{w}). \tag{3.73}$$

Next, we estimate the damping term $D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta})$. Energy equality (3.8) and estimate from Lemma 3.6 imply that

$$D_0^T(\tilde{u},\tilde{\phi},\tilde{\theta})\leqslant\tilde{E}(0)-\tilde{E}(T)+\int_\alpha^T\left|\sum_{i=1}^4\mathscr{R}_i(t)\right|\mathrm{d}t$$
$$\leqslant\tilde{E}(0)-\tilde{E}(T)+\delta\tilde{E}(T)+\delta\int_0^T\tilde{E}(t)\mathrm{d}t+\delta\int_0^T\big[\|\tilde{u}_t\|^2_{\Gamma_1}+\|\tilde{\phi}\|^2_{1,\Omega}\big]\mathrm{d}t$$
$$+\delta\Big[\sup_{t\in[0,T]}\tilde{E}^{\frac{1}{2}}(t)\Big]^2+C_{B,T,\delta}\,\mathrm{l.o.t.}(\tilde{u},\tilde{w}).$$

This inequality together with (3.73) give, for δ small enough, the inequality

$$\tilde{E}(T) \leqslant C_{B,T}\tilde{E}(0) - C_{B,T}\tilde{E}(T) + C_{B,T}\,\text{l.o.t.}(\tilde{u},\tilde{w})$$

and therefore

$$\tilde{E}(T) \leqslant \frac{C_{B,T}}{1+C_{B,T}}\tilde{E}(0) + C_{B,T}\text{l.o.t.}(\tilde{u},\tilde{w}).$$

Repeating this argument on the interval $I_m = [mT,(m+1)T]$, $m \in \mathbb{N}$, we obtain

$$\tilde{E}((m+1)T) \leqslant \frac{C_{B,T}}{1+C_{B,T}}\tilde{E}(mT) + C_{B,T}\text{l.o.t.}_m(\tilde{u},\tilde{w}),$$

for fixed $T > T_0$, where

$$\text{l.o.t.}_m(\tilde{u},\tilde{w}) = \sup_{t\in \mathrm{I}_m}\|\tilde{u}\|^2_{L^{r+1}(\Omega)} + \sup_{t\in \mathrm{I}_m}\|\tilde{u}\|^2_{1-\varepsilon,\Omega} + \sup_{t\in \mathrm{I}_m}\|\tilde{w}\|^2_{2-\varepsilon,\Omega}.$$

Denoting $\gamma_B = \frac{C_{B,T}}{1+C_{B,T}} < 1$, we can show, by induction, that

$$\tilde{E}(nT) \leqslant \gamma_B^n\tilde{E}(0) + C_{B,T}\sum_{k=1}^{n}\gamma_B^{n-k}\,\text{l.o.t.}_{k-1}(\tilde{u},\tilde{w}), \quad \forall n \in \mathbb{N}. \tag{3.74}$$

Using the energy equality we can prove that

$$\tilde{E}(t) \leqslant C^1_{B,T}\tilde{E}(nT)e^{\omega T} \text{ for all } nT \leqslant t \leqslant (n+1)T, \tag{3.75}$$

where the constant ω depends on B. Let $\beta = \frac{1}{T}\ln\frac{1}{\gamma_B}$, then, for $t = nT + m$ with $m < T$, we have

$$\gamma_B^n \leqslant \exp(-\beta t)\gamma_B^{-1}$$

and for $k \leqslant n$

$$\gamma_B^{n-k} = \exp(-\beta(n-k)T) \leqslant 1.$$

These facts combined with (3.74) and (3.75) imply the desired conclusion with $C_1 = C^1_{B,T}\gamma_B^{-1}e^{\omega T}$ and $C_2 = C^1_{B,T}C_{B,T}e^{\omega T}$. □

Lemma 3.8. *Under hypotheses of Theorem 1.2, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on every bounded forward invariant set.*

Proof. By using an isomorphism, we can reorder the components of a trajectory as $(u, w, u_t, w_t, \phi, \theta)$. That is, we can assume $S(t) : \mathcal{H} \to \mathcal{H}$, with $\mathcal{H} = X \times Y \times Z$, where

$$X = [H^1_{\Gamma_0}(\Omega)]^2 \times H^2_{\Gamma_0}(\Omega), \;\; Y = [L^2(\Omega)]^3, \;\; Z = [L^2(\Omega)]^2.$$

Then conditions (2.31), (2.32) and (2.33) are clearly satisfied. Let show that (2.34) also holds. To this end, we consider a X-seminorm defined by,

$$n_X(u,w)^2 = \|u\|^2_{L^{r+1}(\Omega)} + \|u\|^2_{1-\epsilon,\Omega} + \|w\|^2_{2-\epsilon,\Omega}.$$

This is compact on X since the embedding $[H^1_{\Gamma_0}(\Omega)]^2 \subset [L^{r+1}(\Omega)]^2$, $[H^1_{\Gamma_0}(\Omega)]^2 \subset [H^{1-\epsilon}(\Omega)]^2$ and $H^2_{\Gamma_0}(\Omega) \subset H^{2-\epsilon}(\Omega)$ are compact. Therefore, given a bounded forward set $B \in \mathcal{H}$, using Lemma 3.7, we can write

$$\|S(t)y^1 - S(t)y^2\|^2_{\mathcal{H}} \leqslant b(t)\|y^1 - y^2\|^2_{\mathcal{H}} + c(t)\sup_{\tau\in[0,t]}\Big[n_X\big(u^1(\tau)-u^2(\tau), w^1(\tau)-w^2(\tau)\big)\Big]^2,$$

where $b(t) = C_1e^{-\beta t}$ and $c(t) = C_2$. This proves that our system is quasi-stable on B with the $c(t)$ independent on time $t > 0$ -as desired. □

3.2. **Gradient systems and completion of the proof of the Theorem 1.2.** The proof of Theorem 1.2 will follow from Theorem 2.2. To accomplish this we need to establish gradient structure of system $(\mathcal{H}, S(t))$. We shall take the energy functional $\mathcal{E}_y$ as a Lyapunov function $\Phi(y)$, where $\mathcal{E}_y$ corresponds to the energy at the point y defined by (2.20). Thus $\Phi(S(t)y) = \mathcal{E}_{S(t)y}$ of the trajectory $S(t)y$ with a given initial data $y \in \mathcal{H}$. From (2.21) it follows that $t \to \Phi(S(t)y)$ is decreasing for any $y \in \mathcal{H}$. The fact that Lyapunov function is strict follows from the following Unique Continuation Property formulated in the Lemma below.

Lemma 3.9. *Let $\Gamma_0 \neq \emptyset$. The following property holds*

$$\mathcal{E}_{y(t)} = \mathcal{E}_{y(0)}, \forall t > 0 \Rightarrow S(t)y = y, \forall t > 0.$$

Proof. Weak solutions under consideration satisfy

$$\begin{gathered}(u, u_t, w, w_t, \theta, \phi) \in C([0,T], \mathcal{H}),\\ u_t|_\Gamma \in L^2(0,T; L^2(\Gamma)),\ \sigma[\epsilon(u)]\nu \in L^2(0,T; L^2(\Gamma)),\end{gathered}$$

where the last boundary regularity follows from $f(\nabla w)|_{\Gamma_1} \in C([0,T]; L^2(\Gamma_1))$ on the strength $\nabla w|_\Gamma \in C([0,T], H^{1/2}(\Gamma))$. By the assumption and the energy relation (2.21) we have that

$$u_t|_\Gamma = 0 \text{ in } L^2((0,T) \times \Gamma),\ \phi = 0 \text{ in } L^2(0,T; H^1(\Omega)),\ \theta = 0 \text{ in } L^2(0,T; H^1(\Omega)).$$

Hence, distributionally, $\Delta w_t = 0$ and also $w_t = 0$, $\nabla w_t = 0$ on Γ_0. Since $\text{meas}(\Gamma_0) > 0$, by the elliptic unique continuation property we infer that $w_t \equiv 0$. Since $\phi \equiv 0$, we read off from ϕ equation that $\text{div}\{u_t\} \equiv 0$. Feeding this information back to (1.1) and (1.2) we find

$$u_{tt} - \text{div}\{\sigma[\epsilon(u) + f(\nabla w)]\} + p_1(u,w) = 0 \text{ in } \Omega \times (0,\infty), \tag{3.76}$$

with the overdetermined boundary conditions

$$u_t = 0 \text{ on } \Gamma_0,\ u_t = 0 \text{ on } \Gamma,\ \sigma[\epsilon(u) + f(\nabla w)]\nu + \kappa u = 0 \text{ on } \Gamma_1, \tag{3.77}$$

and the elliptic problem

$$\Delta^2 w - \text{div}\{\sigma[\epsilon(u) + f(\nabla w)]\nabla w\} + p_2(u,w) = 0 \text{ in } \Omega \times (0,\infty). \tag{3.78}$$

with boundary conditions on the portion of the boundary Γ_0 given by

$$u = 0,\ w = 0,\ \nabla w = 0 \text{ on } \Gamma_0 \times \mathbb{R}^+, \tag{3.79}$$

$$\Delta w + (1-\mu)B_1 w = 0 \text{ on } \Gamma_1 \times \mathbb{R}^+, \tag{3.80}$$

$$\partial_\nu(\Delta w) + (1-\mu)B_2 w - \sigma[\epsilon(u) + f(\nabla w)]\nu \cdot \nabla w = 0 \text{ on } \Gamma_1 \times \mathbb{R}^+. \tag{3.81}$$

Since the system (3.76)-(3.81) is overdetermined, it is expected that the corresponding weak solutions are more regular. As always, this is a general property derived from Observability Estimates (often Carleman's based) obtained for finite energy solutions [38]. In our case, this follows from Observability Estimates obtained for a difference of two solutions in Lemma 3.5 along with energy estimate (3.8) and the estimate in Lemma 3.6. Indeed, the second observability inequality applied to the overdetermined problem along with energy estimate yields

$$T\tilde{E}(T) + \int_0^T \tilde{E}(t)\mathrm{d}t + \Big[\sup_{t\in[0,T]} \tilde{E}^{\frac{1}{2}}(t)\Big]^2$$

$$\leqslant C\tilde{E}(T) + C_\alpha \int_0^T \left| \sum_{i=1}^4 \mathscr{R}_i(t) \right| \mathrm{d}t + \int_0^T \int_s^T \left| \sum_{i=1}^4 \mathscr{R}_i(t) \right| \mathrm{d}t\mathrm{d}s + C_{\alpha,B,T} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}). \tag{3.82}$$

Applying the estimate in Lemma 3.6 where $\|\tilde{w}\|^2_{2+\epsilon}$ is estimated via analytic bound in Lemma 3.4 and interpolation, and taking δ sufficiently small gives

$$(T-C)\tilde{E}(T) + \int_0^T \tilde{E}(t)\mathrm{d}t + \Big[\sup_{t\in[0,T]} \tilde{E}^{\frac{1}{2}}(t) \Big]^2 \leqslant C_{\alpha,B,T} \mathrm{l.o.t.}(\tilde{u}, \tilde{w}). \tag{3.83}$$

Taking T large enough (note C does not depend on T) allows us to deduce that the energy of the difference of two finite energy solutions can be estimated by lower order terms which have quadratic dependence with respect to lower order norms. The latter implies, in a standard way, that the time derivatives of weak solutions are of finite energy as well - see [12, p.101] or [13, p. 386]. This additional regularity of the overdetermined solutions allows to consider $\bar{u} \equiv u_t \in C([0,T], H^1(\Omega))$. Then (3.76) and (3.85) with $w_t \equiv 0$ lead to

$$\bar{u}_{tt} - \mathrm{div}\{\sigma[\epsilon(\bar{u})]\} = -\Big(\frac{\mathrm{d}}{\mathrm{d}u^1} p_{1,1}(u,w)\bar{u}^1, \frac{\mathrm{d}}{\mathrm{d}u^2} p_{1,2}(u,w)\bar{u}^2 \Big) \text{ in } \Omega \times (0,\infty) \tag{3.84}$$

with the overdetermined boundary conditions

$$\bar{u} = 0 \text{ on } \Gamma, \ \ \sigma[\epsilon(\bar{u})]\nu = 0 \text{ on } \Gamma_1. \tag{3.85}$$

We note that for divergence free vectors u one has $\mathrm{div}[\sigma(\epsilon(u))] = \eta(\Delta u_1, \Delta u_2)$. Moreover for $u = 0$ on Γ, we have $\epsilon(u)\cdot\nu = M\frac{\partial \bar{u}}{\partial \nu}$, where the determinant of the matrix M is equal to 1/2. Indeed, let $\nu = (\nu_1, \nu_2)$ with $\nu_1^2 + \nu_2^2 = 1$. In addition we have that $u_\tau = 0$ on Γ_1, where τ denotes tangential direction to the boundary. Then the algebraic formulas in [42, p. 299] give

$$u_x = \nu_1 u_\nu, \ \ u_y = \nu_2 u_\nu. \tag{3.86}$$

This gives the following representation of stress tensor

$$\epsilon(u)\nu = \begin{bmatrix} \nu_1 u_\nu^1 & 1/2(\nu_1 u_\nu^2 + \nu_2 u_\nu^1) \\ 1/2(\nu_1 u_\nu^2 + \nu_2 u_\nu^1) & \nu_2 u_\nu^2 \end{bmatrix} \nu = M u_\nu,$$

where the matrix M has the form

$$M \equiv \begin{bmatrix} \nu_1^2 + 1/2\nu_2^2 & 1/2\nu_2\nu_1 \\ 1/2\nu_1\nu_2 & \nu_2^2 + 1/2\nu_1^2 \end{bmatrix}.$$

The determinant of M is equal to $1/2(\nu_1^2+\nu_2^2)^2 = 1/2$. Since $\bar{u} = 0$ *on* Γ and $\mathrm{div}\bar{u} = 0$ we obtain that $\bar{u}^i$, for $i = 1,2$, satisfy

$$\bar{u}^i_{tt} - \eta\Delta\bar{u}^i = V_i \bar{u}^1, \ \bar{u} \in C([0,T]; L^2(\Omega)), \tag{3.87}$$

$$\bar{u}^i = 0 \text{ on } \Gamma, \ \frac{\partial \bar{u}^i}{\partial \nu} = 0 \text{ on } \Gamma_1. \tag{3.88}$$

where the potential

$$V_i = V_i(u,w) = -\frac{\mathrm{d}}{\mathrm{d}u^i}(p_{1,i}(u,w))$$

Thus, we are in a position to apply UCP principle applicable to wave equations $\bar{u}^i, i = 1,2$ with overdetermined Cauchy data on the boundary. Since, in particular, $\bar{u}^i \in L^2(0,T; L^2(\Omega))$, we can apply unique continuation principle in [47], provided the potential $V_i \in L^n(\Omega \times (0,T))$. Let us determine the regularity of the potential

$V_i(u,w) = \frac{\mathrm{d}}{\mathrm{d}u^i}(p_{1,i}(u,w))$. Since $u \in C([0,T];H^1(\Omega))$ and $w \in C([0,T];H^2(\Omega))$ the growth condition imposed on p implies that

$$V_1, V_2 \in L^\infty(0,T;L^q(\Omega)),\ \forall q < \infty. \tag{3.89}$$

Thus (3.87) is overdetermined on the boundary wave equation with a potential satisfying regularity assumption in (3.89). We appeal now to [47] [or Theorem 1.2 in [25] and [20]] to claim that $\bar{u} = u_t \equiv 0$ in Ω. Thus, the dynamics has been reduced to a stationary elliptic problem. □

Remark 3.3. It is interesting to note the role played by the second thermal variable ϕ. In the previous calculations [leading to quasistability] the additional dissipation due to ϕ did not play any major role. However, when dealing with weak solutions which are overdetermined on the boundary, the condition $\operatorname{div}\{u_t\} = 0$ − resulting from the dissipation in ϕ variable, allows to reduce system of dynamic elasticity to a classical wave equation (3.87) with Cauchy zero data on Γ_1. For such equation UCP property has been shown [47] for just L^2 solutions (as in our case). Otherwise, one would need to introduce appropriate approximations of overdetermined problems (as in Proposition 2.1 in [38], or [25]) which would allow to deduce additional regularity of the overdetermined problem. This, however, will make the analysis much more technical -see [37], Section 6).

Lemma 3.10. *Under the hypotheses of Theorem 1.2, the set of equilibrium points $\mathcal{N}$ is bounded in $\mathcal{H}$.*

Proof. If $y \in \mathcal{N}$, we have that $y = (u,0,w,0,0,0)$ and satisfies the stationary problem

$$-\operatorname{div}\{\sigma[\epsilon(u) + f(\nabla w)]\} + p_1(u,w) = 0 \text{ in } \Omega, \tag{3.90}$$

$$\Delta^2 w - \operatorname{div}\{\sigma[\epsilon(u) + f(\nabla w)]\nabla w\} + p_2(u,w) = 0 \text{ in } \Omega, \tag{3.91}$$

with clamped boundary condition on Γ_0 and

$$\begin{aligned}
\sigma[\epsilon(u) + f(\nabla w)]\nu + \kappa u = 0 \text{ on } \Gamma_1,\\
\Delta w + (1-\mu)B_1 w = 0 \text{ on } \Gamma_1,\\
\partial_\nu(\Delta w) + (1-\mu)B_2 w - \sigma[\epsilon(u) + f(\nabla w)]\nu\cdot\nabla w = 0 \text{ on } \Gamma_1.
\end{aligned}$$

Multiplying (3.90), (3.91) by u, w, respectively, and integrating over Ω, we obtain

$$\frac{1}{2}\int_\Omega \sigma[N(u,w)]N(u,w)\mathrm{d}\Omega + \frac{1}{2}a(w,w) + \frac{\kappa}{2}\int_{\Gamma_1}|u|^2\mathrm{d}\Gamma_1 = -\int_\Omega \nabla P(u,w)\cdot(u,w)\,\mathrm{d}\Omega.$$

Using inequalities (1.17), (2.22) and (2.23) we find that

$$\begin{aligned}
&-\int_\Omega \nabla P(u,w)\cdot(u,w)\,\mathrm{d}\Omega\\
&\leqslant 2MM_pM_K\left[\|N(u,v)\|_\Omega + \|w\|^2_{W^{1,4}(\Omega)}\right] + 2MM_p\|w\|^2_{2,\Omega} + 2m_E\\
&\leqslant 2\frac{MM_pM_K}{M_\sigma}\int_\Omega \sigma[N(u,w)]N(u,w)\mathrm{d}\Omega + 2\frac{M(M_pM_KM_2 + M_p)}{M_a}a(w,w) + 2m_E.
\end{aligned}$$

Since $M < M_0$ (see (2.25)), we conclude that for some $\delta_p \in (0, \frac{1}{2})$,

$$-\int_\Omega \nabla P(u,w)\cdot(u,w)\,\mathrm{d}\Omega \leqslant \delta_p\int_\Omega \sigma[N(u,w)]N(u,w)\mathrm{d}\Omega + \delta_p\, a(w,w) + 2m_E.$$

This shows that $\mathcal{N}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 1.2. From Lemmas 3.8 and 3.10 we know that $(\mathcal{H}, S(t))$ is an asymptotically compact gradient system with bounded set of stationary points. To apply Corollary 2.1 it remains to show condition (2.30). To see this, from (1.18) and (2.20) we have that

$$\mathcal{E}_y(t) \leqslant \|y(t)\|^2_{\mathcal{H}} + C(1 + \|y(t)\|^{r+1}_{\mathcal{H}}).$$

Then $\mathcal{E}_y(t) \to \infty$ implies that $\|y(t)\|_{\mathcal{H}} \to \infty, \quad t \geqslant 0$. On the other hand, the inequality (2.24) implies that $E_y(t) \leqslant \frac{1}{M_E}(\mathcal{E}_y(t) + m_P|\Omega|)$, and then $\|y(t)\|_{\mathcal{H}} \to \infty$ implies that $\mathcal{E}_y(t) \to \infty, \quad t \geqslant 0$. Then condition (2.30) is satisfied. Therefore system $(\mathcal{H}, S(t))$ has a global attractor $\mathbf{A}$.

From Theorem 2.4, $\mathbf{A}$ has finite fractal dimension and further "time" regularity,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} S(t) y_0 \right\|_{\mathcal{H}} \leqslant C, \quad \forall\, t \in \mathbb{R}, \quad \forall\, y_0 \in \mathbf{A}. \tag{3.92}$$

The improved spatial regularity follows from updating problem (1.1)-(1.10) with (3.92) and then applying elliptic regularity. The argument is analogous to the one in [36, Section 2]. Therefore we conclude that

$$\|u(t)\|_{2,\Omega} + \|w(t)\|_{4,\Omega} + \|\theta(t)\|_{2,\Omega} + \|\phi(t)\|_{2,\Omega} \leqslant C, \quad t \in \mathbb{R}.$$

This ends the proof of Theorem 1.2. □

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