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From the Editor's Desk

Dear Reader,

We are proud to relaunch "Nonlinear World", at the International Conference on Recent Advances in Mathematical Sciences and Applications, held from 19th-22nd December,2017, at GVP College of Engineering (A), India.

This issue takes it shape, as it was envisioned by late Prof. V. Lakshmikantham, as a compendium of advances in nonlinear research in a diverse range of fields, and to serve as a guiding torch of knowledge.

My sincere thanks to the International Federation of Nonlinear Analysts, for their continued support over the years, as well as to all the honored editors, knowledgable contributors and esteemed readers, all without whom this journal would not have been possible.

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Differential Inequalities and the Comparison Principle: the Core of Professor V. Lakshmikantham's Research

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ABSTRACT: In this article we give a brief exposition of how Professor V. Lakshmikantham used the principle of comparison in his work.

A review of the literature indicates that by the time Professor V. Lakshmikantham began his research the concepts of differential inequalities, upper and lower solutions and the idea of comparison were known. With his insight, he recognized the importance of these ideas and exploited them to open a whole new area of research. In this article we illustrate how his research was dominated by the theory of differential inequalities and the comparison principle.

In his first paper [1], which appeared in the Proceedings of the American Mathematical Society, Professor V. Lakshmikantham recognized that using differential inequalities instead of the commonly used integral inequalities to obtain qualitative results in differential equations, such as convergence of successive approximations, and uniqueness, boundedness and asymptotic behavior of solutions, one could dispense with the required monotonicity property of the functions involved.

Consider the differential system

$$\frac{dy}{dx} = f(t, x), x(t_0) = x_0, \tag{1}$$

where $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ satisfies the inequality

$$||f(t, x)|| \leq g(t, ||x||), \tag{2}$$

where $g \in C[\mathbb{R} \times \mathbb{R}^+, \mathbb{R}]$. Letting $m(t) = ||x(t)||$, it follows that

$$\left|\frac{dm(t)}{dt}\right| \leq g(t, m(t)) \text{ a.e.} \tag{3}$$

Using the theory of differential inequalities Professor V. Lakshmikantham showed that one can obtain from (3) the following estimate

$$m(t) \leq r(t; t_0, u_0), \tag{4}$$

provided $m(t_0) \leq u_0$, where $r(t; t_0, u_0)$ is the maximal solution of the scalar differential equation

$$\frac{du}{dt} = g(t, u), \qquad u(t_0) = u_0, \tag{5}$$

existing on $[t_0, \infty)$, without requiring the function g to be monotone nondecreasing. He further observed that in order to obtain the estimate (4), it is sufficient to have

$$Dm(t) \leq g(t, m(t)), \tag{6}$$

where $Dm(t)$ is any one of the Dini Derivatives. This observation led him to conclude that all that was needed is an assumption of the following type

$$||x + hf(t, x)|| \leq ||x|| + hg(t, ||x||) + o(h), \tag{7}$$

instead of (2) in order to obtain the desired inequality (6). Furthermore, he also observed that the advantage of assumption (7) is that g is not even required to be nonnegative, which results in a better estimate for $||x(t)||$.

From equations (6) and (7), he noted that instead of $\|x(t)\|$, one could use a nonnegative function with some properties of norm as follows

$$V(t+h, x+hf(t,x)) \leq V(t,x) + hg(t, V(t,x)) + o(h),$$

where $V \in C[R_+ \times R^n, R^n]$ and $V(t,x)$ is locally Lipschitz in x. Setting $m(t) = V(t, x(t))$ leads to the differential inequality (6) from which the estimate (4) is obtained.

Thus, given an initial-value problem, an important technique in the theory of differential equations consists of first obtaining a scalar differential equation, such as equation (4) and then estimating the solution of the differential system (1) in terms of the extremal solution of the scalar differential equation (4).

This development of the theory of differential and integral inequalities using norms or Lyapunov-like functions led to a number of results, such as convergence of successive approximations and stability and boundedness of solutions. The central theme in all generalizations, extensions, and refinements is the use of the comparison method in terms of Lyapunov-like functions. Professor V. Lakshmikantham recognized the advantage of using vector Lyapunov functions, which were first introduced by Bellman [2] and Matrosov [3], to investigate and refine the Lyapunov theory of stability.

This intense research activity by Professor V. Lakshmikantham resulted in a two-volume monograph [4,5], entitled Differential and Integral Inequalities, which is much more comprehensive than the books by Szarski [6] and Walter [7] on the same topic. In Volume I, for the first time, a whole chapter is devoted to the Method of Vector Lyapunov Functions.

The well-known Lyapunov Second Method deals with questions of stability. Its main characteristic is the introduction of a function, namely the Lyapunov function, which defines a generalized distance. The theory of differential inequalities together with the concept of Lyapunov function provides a very general comparison principle under much less restrictive conditions. This has resulted in its wide use not only in the theory of stability [8], but also in the investigation of various other properties of solutions of differential equations. An interesting idea was the use of two Lyapunov functions by Salvadori [9] to extend Marachkov's [10] stability result. It was therefore natural to ask whether it might be more fruitful to use several Lyapunov functions [11]. The answer is yes, and this approach, known as the Method of Vector Lyapunov Functions, offers a more flexible mechanism since each function can satisfy less rigid requirements. Using this concept, Professor V. Lakshmikantham obtained global results, which proved to be useful tools in dealing with various problems of stability and boundedness.

Analytic solutions of nonlinear problems are rarely possible and various methods are required to obtain approximate solutions of such problems. A fruitful idea of Chaplygin [12] for obtaining approximate solutions of nonlinear differential equations, such as

$$u' = f(t,u),\ u(0) = u_0, \tag{8}$$

is as follows. Suppose, we have $f_1(t,u) \leq f(t,u) \leq f_2(t,u)$ for all (t,u), where f_1 and f_2 are such that the equations $u' = f_1(t,u),\ u(0) = u_0$ and $u' = f_2(t,u),\ u(0) = u_0$ are comparatively simple to solve. Then, the solutions of these simpler differential equations can be used to bound the solutions of the original equation (8). This technique brings into play the theories of differential inequalities, upper and lower solutions, and monotone operators. Professor V. Lakshmikantham and his collaborators developed the basic theory of this method known as the Monotone Iterative Technique [13]. Such an iterative scheme, in addition to offering a flexible mechanism to find theoretical, as well as constructive existence results in a closed set, is also useful for investigating qualitative properties of solutions.

The iterates, constructed using the Monotone Iterative Technique, converge linearly. However, if the method of lower and upper solutions is combined with the method of quasilinearization, developed by Bellman and Kalaba [14], and with the idea of Newton-Fourier, it is possible to construct concurrently lower and upper bounding monotone sequences which converge quadratically. This unified methodology, developed by Professor V. Lakshmikantham and his collaborators, is known as the method of Generalized Quasilinearization. For details see the monograph [15].

Fractional Differential Equations is another area where Professor V. Lakshmikantham used the comparison principle. He and Professor Vatsala first developed the basic differential inequality result [16], parallel to Theorem 1.4.1 in [4]. With the differential inequality in place, they next developed a comparison result relating the solution of the considered fractional differential equation

$$D^q x = f(t,x),\ x(t)(t-t_0)^{1-q}|_{t=t_0} = x^0, \tag{9}$$

to the maximal solution of a scalar ordinary differential equation [17]. In the monograph, Theory of Fractional Dynamic Systems [18], the complete theory of existence, uniqueness and approximate solutions was developed using the comparison principle. We note that, due to the complexity of the problem, the variation of parameters formula for fractional differential equations is still an open problem. However, Professor V. Lakshmikantham developed a relationship between a fractional differential equation and an ordinary differential equation, and using this relation he obtained a variation of parameters formula in terms of the corresponding ordinary differential equation.

Yet another area where differential inequalities and the comparison principle play a significant role is the stability of multi-order fractional systems [18]. Vector Lyapunov functions, together with the comparison principle and differential inequalities, are employed to relate the original multi-order fractional differential system to a system of ordinary differential equations with order one. The well-known theory of non-fractional ordinary differential equations is then used to develop the corresponding theory of the original multi-order fractional system.

Differential inequalities and the comparison principle also play an important role is the area of causal set differential equations. A causal operator is a nonanticipative operator. Differential equations involving causal operators unify a variety of dynamic systems, including ordinary differential equations, delay differential equations and integro - differential equations, to name only a few. On the other hand, set differential equations are useful in the study of multivalued differential equations and multivalued differential inclusions [20]. Set differential equations include the theory of ordinary differential systems as a special case. Thus, combining these two very general and fruitful areas of research naturally results in a theory encompassing many types of dynamic systems, along with their special cases. Professor V. Lakshmikantham motivated initial research in this area, resulting in a first paper [21] on existence and uniqueness. This paper was followed by a systematic study of set differential equations involving causal operators, giving rise to publications on this topic by a number of researchers [19].

In conclusion, it is interesting to note that Professor V. Lakshmikantham always said that to him life and mathematics are the same. For instance, to him inequalities and comparison techniques pervade both mathematics and life. He observed that without comparison there is no growth, sustenance or development, as there can be no motivation without comparison. But, he understood something more. He saw a parallel between these two principles of inequality and comparison, which are central to his research, and their counterparts in real life, which could dominate someone's life and pull that person into a vicious circle of inequality and comparison. With his keen sense of perception, Professor V. Lakshmikantham was able to transcend these limitations in his life, and in some sense, in his research.

We express our immense respect and deep reverence to this great Rishi.

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References:

1. Lakshmikantham, V. On the boundedness of solutions of nonlinear differential equations. *Proc. Amer. Math. Soc.* 8, 1044-1048, 1957.

2. Bellman, R., Vector Lyapunov Function. *J. SIAM Control*, ser. A, 1, 32-34, 1962.

3. Matrosov, V. M., On the Stability of Motion, *J. Appl. Math. Mech.*, 26, 1337-1353, 1962.

4. Lakshmikantham, V.; Leela, S. Differential and Integral Inequalities: *Theory and applications. Vol. I*: Ordinary differential equations. Mathematics in Science and Engineering, Vol. 55-I. Academic Press, New York-London, 1969.

5. Lakshmikantham, V.; Leela, S. Differential and Integral Inequalities: *Theory and applications. Vol. II*: Functional,

partial, abstract, and complex differential equations. Mathematics in Science and Engineering, Vol. 55-II. Academic Press, New York-London, 1969.

6. Szarski, J., *Differential Inequalities*, PWN, Polish Sci. Pub.,Warssaw, 1965.
7. Walter, W., *Differential and Integral Inequalities*, Springer, Berlin, 1970.
8. Lakshmikantham, Vangipuram; Leela, Srinivasa; Martynyuk, Anatoly A. *Stability Analysis of Nonlinear Systems.* Second edition. Systems & Control: Foundations & Applications. Birkhäuser/Springer, Cham, 2015.
9. Salvadori, L., Some Contributions to Asymptotic Stability Theory, *Ann, Sci. Bruxelles*, Ser. I, 88, 183-194, 1974.
10. Marachkov, M., On a Theorem on Stability, *Bull. Soc. Phy. Math., Kazan*, 12, 171-174, 1940.
11. Lakshmikantham, V.; Matrosov, V. M.; Sivasundaram, S. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*. Mathematics and its Applications, 63. Kluwer Academic Publishers Group, Dordrecht, 1991.
12. Chaplygin, S. A., *Collected Papers on Mechanics and Mathematics*, Moscow, 1954.
13. Ladde, G. S.; Lakshmikantham, V.; Vatsala, A. S., *Monotone Iterative Techniques for Nonlinear Differential Equations*. Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics, 27. Pitman (Advanced Publishing Program), Boston, MA; distributed by John Wiley & Sons, Inc., New York, 1985.
14. Bellman, R.; and Kalaba, R., *Quasilinearization and Nonlinear Boundary Value Problems*, American Elsevier, New York, 1965
15. Lakshmikantham, V.; Vatsala, A. S. *Generalized Quasilinearization for Nonlinear Problems*. Mathematics and its Applications, 440. Kluwer Academic Publishers, Dordrecht, 1998.
16. Lakshmikantham, V.; Vatsala, A. S. Basic theory of fractional differential equations. *Nonlinear Anal.*69 (2008), no. 8, 2677-2682.
17. Lakshmikantham, V.; Vatsala, A. S. Theory of fractional differential inequalities and applications. *Commun. Appl. Anal.* 11 (2007), no. 3-4, 395-402.
18. Lakshmikantham, V.; Leela, S.; Devi, J. V., *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, 2009.
19. Lakshmikantham, V.; Leela, S.; Drici, Zahia; McRae, F. A. *Theory of Causal Differential Equations*. Atlantis Studies in Mathematics for Engineering and Science, 5. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.
20. Lakshmikantham, V.; Bhaskar, T. Gnana; Vasundhara Devi, J. *Theory of Set Differential Equations in Metric Spaces*. Cambridge Scientific Publishers, Cambridge, 2006.
21. McRae, F., Drici, Z., and J. Vasundhara Devi, *Set Differential Equations with Causal Operators*, Mathematical Problems in Engineering, 2, 185-194, 2005.

Fallacy: Category and Role of Environment/Assumption

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Abstract Fallacy refers to defective logic/faulty reasoning. It is an integral/inseparable part of a living being, specifically living human being and is strictly undesirable irrespective of low or high/severe impact on inference-drawing/decision-making. For a non-living being such as a digital computer, it is non-existent. If the computer depicts a fallacy, then the reason/root can be traced to the concerned human being. There are many possible categories of fallacies. However, the list of all possible categories/ types of fallacies seems to be non-exhaustive since there are indefinitely large possible defects. Nevertheless a reasonably comprehensive list of fallacies, we believe, throws light on how a fallacy pops up often without our knowledge/ready detection. We present here such a list along with the effect/role of social/cultural/environmental factors as well as assumptions when present. The list includes not only those fallacies in our everyday lives (including those concerning philosophy, psychology, and legal arena) but also those in sciences such as mathematics including computer/computational mathematics and statistics.

Keywords Assumptions, List of fallacies, Mathematics, Social/ cultural/ environmental factors, Statistics.

1. Introduction

1.1 *Definition and categories/types* A fallacy is an erroneous argument dependent upon an unsound or illogical contention. A fallacy may be defined as a deceptive, misleading, or false notion, or blind (false) belief or a combination of one or more of the foregoing terms.

A popular fallacy at one time was that the earth is flat. In other words a fallacy is an idea/reasoning/argument that many people think is true but is in fact false.

There are many possible types of fallacies. It is not possible to exactly state this number of types. This number depends from one context to another or/and from one perception to another. It could be in the range [1, indefinitely large]. Just '1' type implies 'error in any reasoning'. In fact there are numerous ways by which a fallacy may be introduced/ injected often unknowingly by a living being e.g. a living human being.

There are many fallacy examples that we can find in everyday conversations and even in serious scientific, mathematical, and legal arenas. While it is practically impossible to categorize exhaustively and precisely all (indefinitely large) the possible fallacies, the following categories will give some useful insight/idea about fallacies which may creep in the process of an argument (by a human being). Alternatively, a fallacy is anyone of the total number of defects in a reason connected with a subject or with a particular thing to be inferred.

1.2 ***Wrong inference is also a fallacy*** The outcome of a fallacy is a wrong inference. An inference is the process of deriving logical conclusions from premises/propositions known/assumed to be correct. A wrong inference also is known as a fallacy.

1.3 ***Informal logic, induction of incorrect reasoning, and formal logic*** Philosophers who study informal logic have compiled large lists of them, and cognitive psychologists have documented many biases in human reasoning, that favour/induce incorrect reasoning.

Informal logic is a logic suited for understanding and improving thinking, reasoning, and argument as they occur in real life contexts such as in public discussion and debate, in educational and intellectual exchange, in interpersonal relations, and in legal/medical professions. It intuitively refers to the principles of logicand logical thought outside of a formalsetting. It is associated with(informal) fallacies.

Formal logic also called symbolic logic, on the other hand, is the study of systems of deductive argument in which symbols are used to represent precisely categories of expressions.

Some definitions *Fallacies of definition*is used to denote the various ways in which definitions can fail to explain terms. The phrase is used to suggest an analogy with an informal fallacy"Definitions that fail to have merit because they are too broad, use obscure or ambiguous language, or contain circular reasoning are called fallacies of definition.3 major fallacies are too broad, too narrow, and mutually exclusive definitions. The 4^{th} one is incomprehensible definitions,while 1 most common fallacy is the circular definition.

(i) *Circular definition/reasoning fallacy* You may refer Circular argument under the section Categories of fallacies commonly encountered. A circular definition, to put it in another way, is one in which 1^{st} concept is defined by the 2^{nd} concept, and the 2^{nd} one is defined by the 1^{st}: Neither offers enlightenment about what one wanted to know.It is a fallacy because by using a synonym in the*definiens* (the word definiens is that which defines the definiendum in a definition; For instance, in the defining statement An **island** or isle is any piece of sub-continental land surrounded by water. , any piece of sub-continental land surrounded by water is the definiens), the reader is told nothing significantly new.

A simple example would be to define " Christian" as "a person believing in Christianity", and "Christianity" as "the religion of the Christian people", which would make "Christianity" "the religion of the people believing in Christianity" .

(ii) *Too broad and/or too narrow definition fallacy* A definition intended to describe a given set of individuals fails if its description of matching individuals is incongruous: too broad (excessively loose with parameters) or too narrow (excessively strict with parameters). For example, "a shape with four sides of equal length" is not a good definition for "square", because squares are not the only shapes that can have four sides of equal length; rhombido as well.

Likewise, defining a "rectangle" as "a shape with four perpendicular sides of equal length" is not useful because it is too narrow, as it describes only squares while excluding all other kinds of rectangles, thus being a plainly incorrect definition.

If a cow were defined as an animal with horns, this would be overly broad (including goats, for example), while if a cow were defined as a black-and-white quadruped, this would be overly narrow (excluding all-black and all-white cows, for example)and also overly broad (including Dalmatians — a breed of medium sized dog, noted for its unique black or liver spotted coat, for example).

(iii) *Mutually exclusive definition fallacy* Consider, for example, the propositions: (i) No mountains are horses, (ii) Some horses are not pets, and, therefore, (iii) Some pets are not mountains. In this example we can see that the physical difference between a horse and a mountain has no correlation to the domestication of horses. The two premises are mutually exclusive and the subsequent conclusion is nonsense, as the transpose would imply that some pets are mountains.

1.5 ***AI system produces logical/fallacy-free inferences*** An artificial intelligence (AI) system first provided automated formal logical inferences. During the later half of the 20^{th} century, this was an extraordinarily popular research topic, leading to scientific and industrial applications and is called an *expert system*and sometimes a *business rule engines* (a software systemthat executes one or morebusiness rules, that might come from legal regulation,in a runtime production environment). In course of time the formal logic became a stronger basis for **automated theorem proving**.

An inference system's job is to extend a knowledge base (KB) automatically. The knowledge base is a set of propositions that represent what the system knows about the world. *Several techniques can be used by that system to extend KB by means of valid inferences*. An additional requirement is that the conclusions the system arrives at are relevantto its task.

Example in Prolog Prolog — abbreviation of 'Programming in Logic' — is a programming language based on a subset of predicate calculus (also called by names such as predicate logic, first order logic, elementary logic, restricted functional calculus, relational calculus, and theory of quantification). Its main job is to check whether a certain proposition can be inferred from a KB using an algorithm called*backward chaining*.

We enter into our KB the following piece of Prolog code, where $:-$ denotes 'if'

```
Mortal(X):-tree(X).
```

tree(teak).

The 2 lines of the Prolog program says: all trees are mortal and that teak is a tree. Now we can ask the Prolog system about teak:

?-mortal(teak).

(where*?-*signifies a query: Can*mortal(teak).*be deduced from the KB using the rules) gives the answer "Yes" (if the KB of Prolog knows/has about teak as a tree).

On the other hand, asking the Prolog system the following:

?-mortal(banyan).

gives the answer "No".

This is because KB of Prologdoes not have any information about banyan, and hence defaults to any property about banyan being false (the so-called*closed world assumption*).

1.6 ***Undesirability of fallacy*** A fallacy of any kind is absolutely unwelcome and needs to be definitely eliminated. Else, it could/would cause a disaster in any decision-making or in determining the validity of a statement or in an inference. Only by going deepest into an argument (logical) through intense thinking and taking into account the context/environment will bring down the occurrence of fallacies to a minimum or possibly zero.

1.7 ***How to eliminate fallacies completely*: *Silence state*** At the ultimate state of mind viz. the silence (no-thought) state or, equivalently, the *Nirvikalpa Samadhi* state (the highest/deepest state of mind), the very problem of the fallacy on which the mind had focused and reached this state is resolved. No further search, no further question, and no further conflict of any kind exist. The concerned man will be able to tell whether there is a fallacy when he returns to the one- or more-thought state. In a no-thought state all body functions come to a total halt amounting to/resembling a clinical death.

In the process of the mind dropping down to deeper concentration, we have, to put it in another way, the last or, equivalently the ultimate state viz. the Samadhi state and the last but one state viz. the *Dhyana* (meditation) state. In the Dhyana state, there is the unbroken or uninterrupted flow of thought on the object of contemplation while in the Samadhi state, there is oneness with the absolute. These are concepts which cannot be described in words, but only experienced.

Entry of any prejudice, superstition, bias, and blind belief/faith in logical thought process is strictly and completely forbidden. One may have eliminated all fallacies except 1, say, occurring in a communication such as an article, a report, a speech, a conversation, and a debate. This 1 fallacy, irrespective of the degree of relative seriousness of the fallacies, is enough to mislead one to make a wrong decision/ draw a wrong inference. *It is imperative to completely root out all the fallacies from any possible communication for drawing/making a correct inference/a conclusion/a decision or establishing the validity of a statement.*

However, such a complete fallacy-free inference/statement may not be always possible nor may be always confirmed. Moreover, to prove logically/mathematically that the concerned statements/inferences are completely devoid of any fault/fallacy is not possible under all circumstances except in the situation when the intensely focussed mind (on the concerned statements has reached the silence state and then bounced back to one- or more- thought state.

In section 2, we present reasonably comprehensive categories of fallacies besides those specifically in statistics and in mathematics. The role of environment and assumptions is described with a real life incidence in section 3 while the conclusions are presented in section 4.

2. Comprehensive Categories of Fallacies Including Those in Statistics and Mathematics

2.1 ***Is it essential to be fallacy-literate*** One does not need to be well-versed/knowledgeable about various categories of fallacies (possibly indefinitely large in the number of categories) in achieving a fallacy-free argument (as long as a sound logic not marred by any kind of bias and/or belief/superstition, pervades at each and every step of the argument). In the following types of fallacies, a careful evaluation will readily demonstrate that the sound logic is undermined/ interfered.

2.2 *Categories of fallacies in Indian philosophy* According to *Navya-Nayaya* or *Neological Darśana* (view, system) viz. the new school of Indian Logic and Indian Philosophy, there are 5 categories of fallacies viz. (i) Inconstant (not constant i.e. changing often) fallacy (In contrast, the constant fallacy is a specific logical defect that is often introduced in sciences that deal with interpretation), (ii) Contradictory (Inconsistency) fallacy, (iii) Unfounded (not based on fact, and/or realistic considerations) fallacy, (iv) Counter-balanced (force or influence equally counter-acting) fallacy, and (v) Incongruous (incompatible) fallacy.

This school was founded in the 13th century CE (AD)by the philosopher Gangeśa Upādhyāya of Mithila (theMaithilispeaking region of Northern and EasternBihar, a state of India). It was a development of the classical Nyāya *Darśana*. Other influences on Navya-Nyāya were the work of earlier philosophersVācaspati Miśra(900–980 AD) andUdayana(late 10th century AD). It remained active in India through to the 18th century AD.

The pre-13th century Indian logic which was meticulously developed between 3350 BCE and 1200 CE (over 4500 years) is essentially referred to as the *old Indian (Hindu) school of philosophy and logic*.

For examples of fallacies in Indian philosophy the reader may refer [1, 2, and the references cited therein]. A student of *Nyaya*, however, should always remember that, no matter how good an English translation is, *he must be ready to do **hard thinking** for a proper understanding of the subject.*

2.3 ***Categories of fallacies with examples commonly encountered*** However, here are a few well-known categories of fallacies we might experience when making an argument:

(*a*)*Appeal to Ignorance* Appeal to ignorance happens when one individual utilizes another individual's lack of information on a specific subject as proof that his or her own particular argument is right.

Example 1 " You cannot show that there are no Martians living in caves of the mountains situated on the surface of the planet Mars. So it is sensible for me to accept that there are."

Example 2 " You cannot demonstrate there are no living beings on the surface of the planet Gleise 163 c (with 6.9 times the mass of the earth and discovered in September, 2012). So it is sensible for me to accept that there are."

Example 3 We cannot demonstrate that there are no living beings on a planet (which is 4.2 light-years or about 25 trillion miles away from earth) orbiting Proxima Centauri, the closest neighbour to our solar system; so it is sensible for us to accept there are.

(*b*)*Appeal to Authority* This sort of error is also known as " Argumentum Verecundia" (argument from modesty). Instead of concentrating on the benefits of an argument, the arguer will attempt to append their argument to an individual of power or authority in an effort to give trustworthiness to their argument.

*Example*1 " American scientist and 1954 Chemistry Noble Laureate Linus Pauling (1901-1994) trusted in the use of intravenous and oral high-dose (10 gm) vitamin C as cancer therapy for terminal patients. Do you suppose that you know more than Pauling?"

Example 2 " Well, Sir Isaac Newton had faith in Alchemy. Do you consider you are more knowledgeable than Newton?"

(*c*)*Appeal to Popular Opinion* This sort of appeal is when somebody asserts that a thought or conviction is correct since it is the thing that the general population accept.

Example Lots of people have been purchasing this tooth paste, so it must be great."

(*d*)*Association Fallacy*Sometimes called " guilt by affiliation," this happens when somebody connects a particular thought or drill to something or somebody negative so as to infer blame on another individual.

Example Adolf Hitler (1889-1945) was a veggie lover. So I do not trust vegans (vegetarians who do not use animal products and by-products)."

(*e*)*Attacking the Person*Also regarded as " Argumentumad Hominem" (argument against the man), this is a common fallacy used during debates where an individual substitutes a rebuttal with a personal insult.

Example " Do not follow Jack's advice. He is a simpleton."

(*f*)*Begging the Question*The conclusion of a contention is accepted in the statement of the inquiry itself.

Example 1 " If an outsider did not take my newspaper, then who did?" (Accept that the newspaper is really stolen.)

Example 2 " If the maid servant did not lift my ear rings from the bath room, then who did?" (Accept that the ear rings are really stolen.)

(*g*)*Circular Argument* This fallacy is also known as " Circulus in Probando" . This error is committed when an argument takes its evidence from an element inside the argument itself instead of from an outside one.

Example " A Hindu" is "the person believing in Hinduism", and "Hinduism is "the religion of the Hindu people", which would make "Hinduism" "the religion of the people believing in Hinduism" .

(*h*)*Relationship Implies Causation Fallacy*Also called " Cum Hoc Ergo Propter Hoc" , this fallacy is a deception in which the individual making the contention joins two occasions that happen consecutively and accepts that one made the other.

Example " I saw a black cat and 10 minutes after the fact, I crashed my motor-cycle, in this manner, black cats are terrible fortunes."

(*i*)*False Dilemma/Dichotomy*Sometimes called " Bifurcation" , this sort of error happens when somebody presents their argument in such a way that there are just two conceivable alternatives left.

Example " If you do not vote for this candidate, then you must be a communist."

(*j*)*Illogical conclusion*This is a fallacy wherein somebody attests a conclusion that does not follow from the suggestions.

Example All Londoners are from England. John is not a Londoner, in this way he is not English."

(*k*)*Slippery Slope*The error happens when one contends that an exceptionally minor movement will unavoidably prompt great and frequently ludicrous conclusions.

Example " If we permit gay individuals to get hitched, what is afterwards? Allowing individuals to wed their pooches (a colloquial term/slang for 'dogs')?"

(*l*)*SyllogismFallacy* This fallacy may also be used to form incorrect conclusions that are odd. Syllogism fallacy is a false argument as it implies an incorrect conclusion.

Example " All parrots are green and the bird in my cage is green. So the bird in my cage is a parrot."

2.4 ***A list of fallacies without explicit examples*** The foregoing categories are just a few of many possible categories of fallacies [3-7] that we may encounter. The following is a list of fallacies which include most of the foregoing fallacies in some form or the other. We provide the list without explicit examples. The internet/concerned literature may be referred to for specific examples. The list is considerably comprehensive but cannot be said exhaustive. We can never be sure about the total number of distinct categories of fallacies that a human can ever encounter in a reasoning/argument.

Number of categories of fallacy: How many If we talk about the number of categories of fallacy, then one may talk about just only one category viz. the category in which there is an error in any reasoning. One may also talk about 3 categories viz. (i) Formal fallacies that include Propositional, Quantification, and Formal syllogistic fallacies, (ii) Informal fallacies that comprise Faulty generalizations and Red herring fallacies, and (iii) Conditional/questionable fallacies.

Each one of these 3 categories can be further divided into a number of sub-categories. Each sub-category is also a category. In this way, there could be an indefinite number of varieties or categories as there could be an indefinite number of possible faults in a reasoning. Consequently, it is not possible to exhaustively state the total number of fallacies objectively. Based on a gross perception, however, one may categorize fallacies depending on some high level possibly exhaustively but not at the lowest possible level exhaustively.

2.4.1 Formal fallacies

(i) ***Appeal to probability fallacy*** (a statement that takes something for granted since it would probably the case),

(ii) ***Base rate fallacy*** (a conditional probability oriented statement that has not considered the influence of prior probabilities),

(iii) ***Fallacy fallacy*** (also called argument from fallacy — if an argument is fallacious, then the claim/conclusion is false)

(iv) ***Jumping to conclusion fallacy*** (the statement based on the act of taking decision without having sufficient information to be certain that they are valid), (v)***Masked-man fallacy*** (also called illegal substitution of identical fallacy — replacing a designator by an identical one in a valid statement may lead to an invalid one),

(vi) ***Conjunction fallacy*** (an assumption that an outcome satisfying several conditions at the same time is more probable than an outcome satisfying only one of the several conditions),

(vii) ***Propositional fallacy*** A compound proposition having 2 or more constituent parts should satisfy the logical connectives such as *and, or, not, if then, only if,* and *if and only if.* A propositional fallacy is an error in logic in a compound proposition. A propositional fallacy may be of 3 types viz.

(a) *Affirming the consequent fallacy* (the antecedent in an indicative conditional is claimed to be true since the consequent is true — *if* A *then* B; B, hence A),

(b) *Affirming a disjunct fallacy* (one disjunct of a logical disjunction is claimed to be false since the other disjunct is true — A *or* B; A, hence *not* B), and

(c) *Denying the antecedent* (the consequent in an indicative conditional is claimed to be false since the antecedent is false — *if* A *then* B; *not* A hence *not* B

(viii) ***Quantification fallacy*** (when the quantifier of the conclusion contradicts the quantifiers of the premises, an error in logic occurs). An *existential fallacy* viz. a fallacy in which an argument has a universal premise and a particular conclusion is a type of quantification fallacy.

(ix) ***Formal syllogistic fallacy*** (a logical fallacy in a syllogism — a kind of logical argumentthat appliesdeductive reasoningto arrive at aconclusionbased on 2 or morepropositionsthat are asserted or assumed to be true). There are several types of this fallacy:

(a) *Illegal major fallacy*(Since the major term of a categorical syllogism is not distributed in the major premise but distributed in the conclusion, the syllogism is illicit),

(b) *Illegal minor fallacy* (Since the minor term of a categorical syllogism is not distributed in the minor premise but distributed in the conclusion, the syllogism is illegal),

(c) *Illegal affirmative fallacy* viz. Negative conclusion from affirmative premises fallacy (When a categorical syllogism has a negative conclusion but affirmative premises),

(d) *Illegal negative fallacy* viz. Affirmative conclusion from a negative premise fallacy (When a categorical syllogism has a positive conclusion but one or more negative premises),

(e) *Modal fallacy* (Confusing possibility with necessity),

(f) *Distributed middle term fallacy* (The middle term in a categorical syllogism is not distributed),

(g) *Exclusive premises fallacy* (Since both of its premises are negative, the categorical syllogism is illegal),

(h) *Four terms fallacy* (A categorical syllogism that has 4 terms).

(x) ***Questionable fallacy*** **or** ***conditional fallacy*** (A fallacy which is conditional or questionable). There are several types of this fallacy:

(a) *Broken window fallacy* (a fallacy in which an argument that attaches no importance to lost costs connected with destroying property of others or associated with other ways of externalizing costs onto others — an argument that states breaking a window generates income for a window fitter, but it attaches no importance to the fact that the money spent on the new window cannot now be spent on, say, new television,

(b) *Slippery slope fallacy* or an appeal to Probability fallacy (a fallacy in which a comparatively small 1st step unavoidably leads to chain of associated events resulting in some significant event that should not occur; hence the 1st step should not happen.— if a person p does q then r would probably occur, leading to s, leading to t, leading to v, leading to g. This is related to Reductio ad absurdum or, specifically, proof by contradiction),

(c) *Definist fallacy* (a fallacy that involves confusion between 2 notions by defining one in terms of the other), (d) Appeal to nature fallacy or Naturalistic fallacy (a fallacy that is used to prove a claim about ethics by appealing to a definition of the term “ good" in terms of one or more claims about natural properties or God’s will).

(xi) ***Faulty generalization fallacy*** (Arriving at a wrong inference from weak premises) The following types of fallacy come under a faulty generalization fallacy.

(a) *Overwhelming exception fallacy* (a correct generalization which occurs with qualifications that removes many cases so that the remainder is less impressive than the initial statement which might have led one to assume),

(b) *Hasty generalization fallacy* (insufficient sample/statistics fallacy, lonely fact fallacy, jumping to a conclusion fallacy, hasty induction fallacy, converse accident fallacy, inductive fallacy are some of the alternative/closely related names; it occurs when an inference is made of premises that weakly supports it),

(c) *Incorrect analogy fallacy* (a reasoning based on an unsuitable/defective analogy),

(d) *Ignoring an exception to a generalization fallacy* (An accident fallacy i.e. an exception to a generalization is ignored or no-true Scotsman fallacy i.e. forcing a generalization true by changing the generalization that excludes a counterexample),

(e) *Misleading vividness fallacy* (an exceptional occurrence is described in vivid details to convince someone that it is the problem),

(f) *Survivorship bias fallacy* (lobbying/promoting small number of survivors/successes while completely ignoring a large number of deaths/failures; also known as cherry-picking fallacy — act of suppressing evidence or, in other words, act of pointing at individual cases or data that seem to confirm a particular position while ignoring considerable relevant cases or data that could contradict the position),

(g) *Thought-terminating cliché fallacy* (a thought-terminating stereotyped phrase, known as cliché, used to conceal lack of thought-entertainment or jump on to a different topic, in any case terminate the debate with a trite not a point).

2.4.2 ***Informal fallacies*** (fallacious argument for reasons other than formal (structural) faults and needs checking critically the contents of the argument)

The following types come under informal fallacy.

(a1) *Argument from silence fallacy* (The conclusion is based on the absence of evidence and not on the presence/existence of evidence),

(a2) *Argument from (Appeal to) ignorance fallacy* (The conclusion is based on the assumption that a claim is true since it has not been or cannot be proved false or vice versa),

(a3) *Argument from incredulity (Appeal to common sense) fallacy* (It must be false because I cannot imagine how this could be true),

(a4) *Argument from repetition fallacy* (something has been discussed so extensively that no one likes to discuss it any further (could be confused with proof by assertion)),

(a5) *Argument to moderation fallacy* (fallacy of the mean or middle ground, false compromise — assuming compromise between two positions is always correct),

(b1) *Appeal to the stone fallacy* (rejecting a claim as absurd without proving for its absurdity),

(b2) *Begging the question fallacy* (The conclusion of a contention is accepted in the statement of the inquiry itself, see the example in the earlier part of this chapter),

(b3) *Shifting the burden of proof fallacy* (I do not require to prove my claim, you must prove it is false),

(b4) *Circular argument fallacy* (when the arguer starts with what he is trying to end up with, may also be called assuming the conclusion),

(b5) *Circular cause and effect fallacy* (The effect/consequence of the phenomenon is claimed to be its root cause),

(c1) *Bald man fallacy* (also called line-drawing fallacy, continuum fallacy, fallacy of the heap or beard — improperly rejecting a claim for being imprecise),

(c2) *Correlation proves causation fallacy* (a faulty assumption that since there is a correlation between 2 variables that one caused the other),

(c3) *Suppressed correlative fallacy* (when a correlative is redefined so that one alternative is rendered impossible),

(c4) *Divine fallacy* (also called Argument from incredulity fallacy — something is so incredible that it must be outcome due to divine/paranormal power), (c5) *Ecological fallacy* (conclusions about the nature of specific individuals are on aggregate statistics obtained for the group consisting of these individuals),

(d1) *Double counting fallacy* (counting events more than once in probabilistic argument, which leads to the sum of the probabilities of all cases>1),

(d2) *Equivocation fallacy* (the use of ambiguous term with 2 or more meanings to mislead — it could be Ambiguous middle term fallacy i.e. a common ambiguity in syllogisms in which the middle term is equivocated, or it could be Definitional retreat fallacy i.e. changing the meaning of a word to deal with an objection raised against the original wording),

(d3) *Fallacy of composition* (Something that is true of part of a whole should necessarily be true of the whole),

(d4) *Fallacy of accent* (In a written passage when it is left unclear which word the emphasis was supposed to be on),

(d5) *Etymological fallacy* (Which reasons that the historical meaning of a word/phrase is necessarily similar to its actual present-day meaning),

(e1) *Fallacy of division* (Something that is true of a thing should necessarily be true of the whole or some of its parts),

(e2) *False attribution fallacy* (also Quoting out of context fallacy i.e. referring to plucked out words selectively from the original context in such a way that distorts the intended meaning — a lawyer appeals to an irrelevant source in support of an argument),

(e3) *False (Appeal to) authority fallacy* (Using one expert of controversial credentials or using only one opinion to sell a product/to lobby in favour of someone/some ideology),

(e4) *False dichotomy fallacy* (also called fallacy of bifurcation, false dilemma, or black-or-white fallacy — 2 alternative propositions are shown to be the only possible options, while there are more),

(e5) *False equivalence fallacy* (bringing in a situation of apparent equivalence while in reality there is none),

(f1) *Historian's fallacy* (happens when one assumes that decision makers of the past viewed events from the same outlook and with the same as those subsequently analyzing the decision),

(f2) *Historical fallacy* (occurs when a set of considerations is satisfied just because a completed process is inserted into the content of the process which conditions the completed one),

(f3) *Gambler's fallacy* (occurs when the false belief that independent events can influence the likelihood of another random event — if a fair coin shows tails 15 times in a row on tossing, the belief that it is due to the number of times it has previously landed on heads)

(f4) *Furtive fallacy* (occurs when outcomes are asserted to have caused by the wrongdoings of the decision makers),

(f5) *Fallacy of the single cause* (happens when it is assumed that there is one cause of an outcome whereas it might have been caused by several causes put together),

(g1) *Fallacy of presupposition* (also called fallacy of many questions — occurs when somebody asks a question that presupposes something that has not been proved/accepted by all involved; this fallacy is usually rhetorically used so that the question limits direct replies to those that serves the questioner's agenda),

(g2) *Homunculus fallacy* (occurs when an argument which accounts for a phenomenon in terms of the very phenomenon that it is meant to explain resulting an infinite regress — Phenomenon P needs to be explained, Reason R is specified. But Reason R depends on Phenomenon P),

(g3) *If-by-whiskey fallacy* (occurs when an argument that supports both sides of an issue using terms which are selectively emotionally sensitive),

(g4) *Inflation of conflict fallacy* (The experts of a particular area of knowledge disagree on a certain point, so the scholars must not know anything resulting in doubt about the legitimacy/validity of their entire area),

(g5) *Inconsistent comparison fallacy* (occurs when several methods of comparison are used, leaving us with a wrong impression of the (only one) whole comparison),

(h1) *Incomplete comparison fallacy* (occurs when information provided is insufficient for making a complete comparison), (h2) *Missing the point fallacy* (also known as irrelevant conclusion fallacy — a reasoning that may itself be correct, but does not address the issue in question),

(h3) *Intentionality fallacy* (the final meaning of a communication must be consistent with the intention of the communicator; for instance, a fiction known as an allegory must not necessarily be regarded as such if the author intended it not to be so),

(h4) *Kettle logic fallacy* (occurs when arguments with inherent inconsistencies are used to defend a position),

(h5) *Ludic fallacy* (occurs when the probabilistic games are misused to model a real-life situation),

(i1) *Quantitative fallacy* (also called McNamara fallacy — occurs when a decision is made only on quantitative observations, discounting all other considerations),

(i2) *Is-ought fallacy* (also sometimes called Naturalistic fallacy — occurs when an evaluative conclusion is made from factual premises violating fact-value distinction; inferring 'ought' from 'is' is an instance of naturalistic fallacy),

(i3) *Ought-is fallacy* (also called Moralistic fallacy — occurs when a factual conclusion is made from evaluative premises violating fact-value distinction; inferring 'is' from 'ought' is an instance of moralistic fallacy; Ought-is fallacy is the inverse of Is-ought fallacy),

(i4) *Moving the goalposts (often raising the bar) fallacy* (occurs when evidence provided in response to a specific claim is dismissed and some (often) greater evidence is demanded),

(i5) *Anti-naturalistic fallacy* (also called *Naturalistic fallacy* fallacy is an instance of argument from fallacy),

(j1) *Perfect solution fallacy* (also called Nirvana fallacy — occurs when a solution to a problem is rejected since it is not perfect),

(j2) *Faulty cause/effect fallacy* (also called correlation without causation fallacy, and coincidental correlation fallacy — the event E happened and then the event F happened; hence E caused F),

1. (j3) *Proof by repeated assertion fallacy* (a proposition is stated again and again irrespective of contradiction),

(j4) *Prosecutor's fallacy* (a fallacy of statistical reasoning, used by the prosecution to argue for the guilt of a defendant during a criminal trial),

(j5) *Proving too much fallacy* (occurs when an argument reaches the desired conclusion so that it is only aspecial caseof a larger absurd/undesirable conclusion),

(k1) *Psychologist's fallacy* (occurs when an observer assumes that his subjective experience reflects the true nature of an event),

(k2) *Referential fallacy* (occurs when the term is used to refer to the assumption that it is a necessary condition of a sign that the signifier has a referent say, a real material object or that the meaning of a sign lies purely in its referent. Such an assumption is faulty since many signifiers such as the connective 'and' do not have referents),

(k3) *Regression fallacy* (occurs when it assumes that something has returned to normal due to corrective actions taken while it was abnormal and fails to account for natural fluctuations — it is often a special kind of the post hoc fallacy),

(k4) *Fallacy of ambiguity* (occurs when it is the error of treating something as a real thing, which is not a real thing but just an idea. In other words, abstract belief is treated as if it were a physical entity or a concrete real event), (k5) *Retrospective determinism* (occurs when the argument that since an event has happened under some circumstance, the circumstance was responsible for its occurrence),

(*l*1) *Wrong direction fallacy* (occurs when the cause is said to be the effect and vice versa),

(*l*2) *Special pleading fallacy* (occurs when a person argues in favour of something by attempting to cite it as an exemption to a generally accepted rule/principle without justifying the exemption),

(*l*3) *Shotgun argumentation fallacy* (occurs when the arguer puts forth such a large number of arguments in favour of a position that the opponent cannot possibly respond to all of them),

(*l*4) *Abusive fallacy* (also called Ad hominem fallacy or personal attack fallacy — occurs when the arguer is attacked instead of his argument), the subtypes of this fallacy include *Appeal to motive fallacy* (occurs when an idea is discarded by questioning the motive of the proposer), *Poisoning the well fallacy* (occurs when adverse information is presented about a target person with the motive of discrediting everything that the target person says), *Tone argument fallacy* (occurs when emotion behind a message is focussed and not the message itself as a discrediting tactic), *Traitorous critic fallacy* (occurs when responding to the criticism by implying that the critic is biased by the affiliation to an out-group and not responding to the criticism itself),

(*l*5) *Argument from inappropriate authority fallacy* (also called Argumentum ad Verecundiam fallacy — occurs when an assertion is regarded as true because of the authority of the person asserting it),

(m1) Appeal to accomplishment/ emotion/ fear/ flattery/ pity/ ridicule/ spite/ nature/ novelty/ poverty/ tradition/ wealth/ stick/ force/ threat/ belief fallacy (occurs when any of the terms such as fear, poverty, and belief dominates rather than pure logic; as a matter of fact, the fallacy crops because of somebody's weaknesses toward another person's accomplishment or/and wealth or/and power, say).

2.5 Mathematical fallacy

Underlying any scientific/technological activity is mathematics. It is accepted by one and all in scientific/engineering community as the most powerful justification for the validity of scientific innovation. In all theoretical and applied scientific activities such as electrical, electronics, mechanical, civil, computer, biological, chemical, and physical sciences activities, mathematics is the most vital tool for their logical validation. Until such a validation is done, nothing will be accepted/given importance by the scientific community, however vital/outstanding the innovation is. Hence the concerned mathematics should be absolutely free from all kinds of fallacy.

In mathematics, certain kinds of mistaken proof are often exhibited, and sometimes collected, as illustrations of a concept of **mathematical fallacy** [8, 9]. There is a distinction between a simple*mistake*and a*mathematical fallacy*in a proof: a mistake in a proof leads to an **invalid proof** just in the same way, but in the best-known examples of mathematical fallacies, there is some concealment in the presentation of the proof. For example**, the reason validity fails may be a division by zero that is hidden by algebraic notation.** There is a striking quality of the mathematical fallacy: as typically presented, it leads not only to an absurd result, but does so in a crafty or clever way.Therefore, these fallacies, for pedagogic reasons, usually take the form of spurious proofs of obvious contradictions. Although the proofs are flawed, the errors, usually by design, are comparatively subtle, or designed to show that certain steps are conditional, and should not be applied in the cases that are the exceptions to the rules.

The traditional way of presenting a mathematical fallacy is to give an invalid step of deduction mixed in with valid steps, so that the meaning of fallacy is here slightly different from the **logical fallacy**. The latter applies normally to a form of argument that is not a genuine rule of logic, where the problematic mathematical step is typically a correct rule applied with a tacit wrong assumption. Beyond pedagogy, the **resolution of a fallacy can lead to deeper insights into a subject** (such as the introduction of Pasch's axiomof Euclidean geometry and the five color theorem of graph theory).*Pseudaria*, an ancient lost book of false proofs, is attributed to Euclid.

Mathematical fallacies exist in many branches of mathematics. In elementary algebra, typical examples may involve a step where division by zero is performed, where a root is incorrectly extracted or, more generally, where **different values of a multiple valued function are equated**. Well-known fallacies also exist in elementary Euclidean geometry and calculus.

2.5.1 ***Howler fallacy*** If an argument, however true the inference, is mathematically false, then it is called a**howler** — a fallacy. Consider for instance the calculation (anomalous cancellation):

Example 2 19/95 =1/5 by cancelling 9 from numerator as well as denominator.

Although the conclusion{ \ displaystyle \ textstyle { \ frac { 16} { 64} } ={ \ frac { 1} { 4} } } 16/64 =1/4 as well as 19/95 =1/5 are correct, there is a fallacious, invalid cancellation in the middle step. Bogus proofs, calculations, or derivations constructed to produce a correct result in spite of incorrect logic or operations were termed *howlers*by Edwin Arthur Maxwell (1907-1987).Outside the field of mathematics the term *howler* has various meanings, generally less specific.

2.5.2 ***Division-by-zero fallacy*** Division by zero explicitly or implicitly in mathematics and elsewhere is completely forbidden. It is considered an unpardonable mathematical sin. Ithas many variants. The following example uses division by zero to "prove" that{ \ displaystyle 2=1} 1=2, but can be modified to prove that any number equals any other number.

Let a and b be equal non-zero quantities.

S.1 a=b.

S.2 Multiply both sides by a: a^2=ab.

S.3 Subtract b^2 from both sides: a^2 - b^2 = ab - b^2.

S.4 Factor both sides: (a + b)(a - b)=b(a - b).

S.5 Divide both sides by (a-b): (a + b) = b. (Note that a=b: b + b = b i.e. 2b = b.)

S.6 Divide both sides by non-zero b: Hence 2=1.

The fallacy is in step S5 i.e. in “ Divide both sides by (a – b)" . Since (a – b) is 0, division by 0 is undefined. Hence the argument is wrong. In a proof (rather in an involved long proof) by a human being /mathematician/physicist, such an error (mistake) is not uncommon.

2.5.3 ***Equating 2 distinct values of multi-valued function fallacy*** Many functions do not have a unique inverse. For example squaring a real number gives a unique value, but there are two possible square roots of a positive number. The square-root is multi-valued (here 2-valued). One value can be chosen by convention as the principal value, in the case of the square-root the non-negative value is the principal value, but there is no guarantee that the square-root function given by this principal value of the square of a number will be equal to the original number, e.g. the square-root of the square of -6 is 6.

Similarly, the 4^{th} root of the 4^{th} power of -10 could be 10. Here the 4^{th} root is 4-valued (viz. 10, -10, 10i, -10i).

2.5.4 ***Fallacies in Calculus*** Calculus as the mathematical study of infinitesimal change and limit scan lead to mathematical fallacies if (i) the properties of integrals and differentials and (ii) the properties of limits are ignored. For instance, a naive use of integration by part scan be used to give a false proof that 0 = 1. { \ display style \ textstyle dv={ \ frac { dx} { x} } } We may write:

$$\int (1/(x\log x))\,dx = 5 + \int (1/(x\log x))\,dx$$

{ \ displaystyle \ int { \ frac { 1} { x\ ,\ log x} } \ ,dx=1+\ int { \ frac { 1} { x\ ,\ log x} } \ ,dx} after which the anti-derivatives may be cancelled yielding 0 = 5. The problem is that anti-derivatives are only defined up to a constant and shifting them by 5 or indeed by any number is allowed. The error really comes to light when we introduce arbitrary integration limits*a*and*b*.

{ \ displaystyle \ int _ { a} ^ { b} { \ frac { 1} { x\ ,\ log x} } \ ,dx=1| _ { a} ^ { b} +\ int _ { a} ^ { b} { \ frac { 1} { x\ ,\ log x} } \ ,dx=0+\ int _ { a} ^ { b} { \ frac { 1} { x\ ,\ log x} } \ ,dx=\ int _ { a} ^ { b} { \ frac { 1} { x\ ,\ log x} } \ ,dx} Since the difference between two values of a constant function vanishes, the same definite integral appears on both sides of the equation.

Consider, for example, computing the limit of the function

$\lim_{x\to 0}\frac{\sqrt{(1-\cos 2x)}}{x}$.

If we equate the numerator viz. $\sqrt{1-\cos 2x} = \sqrt{2}\sin x$ since $\cos 2x = 1 - 2\sin^2 x$ and then compute the limit

$\lim_{x\to 0}\frac{\sqrt{2}\sin x}{x} = \sqrt{2}\lim_{x\to 0}\frac{\sin x}{x} = \sqrt{2}$,

we go **wrong**. The limit x could tend to 0 not only from the positive side but also from the negative side (of 0). Both these limits must be the same for the limit to exist or, in other words, in the engineering sense, for the result to exist and to produce correct value of the function. The value of $\sin x$ when x tends to 0 both from the negative side (4^{th} quadrant) as well as from the positive side 1^{st} quadrant is 0. But x is negative from the negative side and positive from the positive side. Thus the foregoing limits are $-\sqrt{2}$ from the negative side and + $\sqrt{2}$ from the positive side. Since these 2 limits are different, the limit for the foregoing function does not exist. That is, the function value is non-existent at $x = 0$ (in any real-world engineering implementation; here '=' implies 'not necessarily exactly equal to'; in a continuous physical quantity, the exact quantity, though physically exist, is never known to us in exact (finite) numerical form — only best possible bounds in which the exact numerical quantity lies would be known depending on the accuracy of the measuring device).

Matlab takes care of such kinds of fallacy and compute the left-hand and right-hand limits correctly numerically. For instance, the Matlab commands

format long; x=10^ -4, r1=sqrt(1-cos(2*x))/x, x=-x, r2=sqrt(1-cos(2*x))/x, r3=abs(r1-sqrt(2))

produce r1=1.414213558075674, r2=-1.414213558075674, r3=4.297421085652786e-009 (Accuracy), while the Matlab commands

» format long; x=10^ -5,r1= sqrt(1-cos(2*x))/x, x=-x, r2=sqrt(1-cos(2*x))/x, r3=abs(r1-sqrt(2))

produce r1=1.414213620879371, r2=-1.414213620879371, r3=5.850627626813321e-008 and

» format long; x=10^ -3,r1= sqrt(1-cos(2*x))/x, x=-x, r2=sqrt(1-cos(2*x))/x, r3=abs(r1-sqrt(2))

produce r1=1.414213326683278, r2=-1.414213326683278, r3=2.356898167743537e-007.

Thus, in Matlab, a best accuracy is achieved for $x = \pm 0.0001$ for the left-hand limit as well as the right these-hand limit. Both these limits are different. Hence the required limit does not exist. In other words, if we are asked to compute the value of the function $\frac{\sqrt{(1-\cos 2x)}}{x}$ at $x = 0$ (numerical zero), then the answer will be: the value of the function does not exist. This answer is given by actually computing the foregoing limits. An analogous problem is to compute the value of the function $(x^2-4)/(x-2)$ at $x = 0$ (a numerical zero, not the mathematical zero). Then the answer will be 4 obtained by computing both the left-hand as well as the right-hand limits and observing that these two limits are the same.

For the mathematical 0, the function value is undefined because of 0/0 form. In engineering computation, it is implicit that the value of x is not exactly 0 — the difference between the exact 0 and a numerical 0 could be taken as even $\pm 10^{-500}$ times the value of an average element (a relatively extremely small value not exactly equal to 0). Since we have everything to do with numerical 0 in the physical world/engineering computation, we need to be careful not to introduce a fallacy by saying that the value of the function is undefined.

Fallacies may result due to violating elementary arithmetic rules through an incorrect manipulation of the radical.

2.5.5 Positive and negative roots Care must be taken when taking the square root of both sides of an equality. Failing to do so may result in a proof of 5=7. We can write (6 – 5)2 = (6-7)2. Taking square-root of both sides we have 6 – 5= 6 – 7. Hence 5 = 7. The fallacy is in the last but one sentence, where the square root of both sides is taken:a2=b2only implies a=b if a and b have the same sign, which is not the case here. In this case it implies a= –b and should read6 – 5 = -(6 - 7). This is 1= 1 which is correct. { \display style 5=4. }

{ \ display style 5=4.} { \ display style 5^ { 2} -5\ times 9=4^ { 2} -4\ times 9} As another example of the danger of squaring both sides of an equation, consider the fundamental identity $\cos^2 x = 1 - \sin^2 x$ { \ display style \ cos ^ { 2} x=1-\ sin ^ { 2} x} which holds as a consequence of the Pythagorean theorem. Then, by taking a square-root, $\cos x = \sqrt{1 - \sin^2 x}$ { \ display style \ cos x={ \ sqrt { 1-\ sin ^ { 2} x} } } so that $1 + \cos x = 1 + \sqrt{1 - \sin^2 x}$. For $x = \pi$, the foregoing identity gives 0=2 which is wrong.

The error in each of these examples fundamentally lies in the fact that any equation of the form $x^2 = a^2$ { \ display style x^ { 2} =a^ { 2} } has two distinct solutions viz. (i) $x = +a$ and (ii) $x = -a$, when$a \neq 0$.{ \ display style x=\ pm a}

It is imperative to check which of these solutions is relevant to the problem at hand.In the above fallacy, the square-root that allowed the second equation to be deduced from the first is valid only when cos*x*is positive. In particular, when*x*is set to π , the second equation is rendered invalid.

2.6 Fallacy in Statistics

Introduction Improper sampling or/and wrong interpretation of data generally leads to statistical fallacies [10, 11]. A statistical fallacy may also be termed as wrong interpretation of data.

Human psychology may not always be completely unbiased. Sometimes there may be a tendency on one's part to force an inference by manipulating the statistical data and inject among the audience/public a positive feeling about the health of, say, a country's economy. A prediction of rosy future for a state or a country to induce a powerful positive attitude among her citizens could be a fallacy but it has a potential to greatly improve the optimism and consequently enhanced collective/individual contribution to the society.

A completely unbiased correct interpretation out of the data, though may not be palatable and contribute to greatly improve the optimism among the people, will be definitely good to the society in the long run. The faith/dependence of the people on statistical inferences will be significantly enhanced and unshakeable.

Hence any statistician should strive for truly unbiased conclusions/inferences out of the correct interpretation of data based on proper sampling and not on any compulsion — external or/and internal . *For any wrong interpretation of data, it is not the statistics but we, the concerned people, are fully responsible and are to be blamed.*

In other words, the interpretation of data is a difficult task and needs a high degree of skill, care, judgement, and objectivity. Automation of interpretation entirely based on computer (i.e. without any live human interaction) is very difficult or almost impossible (to be programmed).

In the absence of these — skill, care, judgement, and objectivity — there is every likelihood of the data being misused to prove/interpret things that are not at all true. In reality, the largest number of mistakes are committed consciously or unconsciously (intentionally or unintentionally) while interpreting statistical data. Very often facts and figures are presented in such a manner that these are misinterpreted by majority of the readers.

The effect of a fallacy designed to mislead and the one committed unintentionally will not be distinguished. The effect is the same in both cases. However, there is an ***abuse*** of statistics in the first case while it is only a ***misuse*** in the second case.

Statistical fallacies could crop up in collection, presentation, analysis, and interpretation of data. Following are a few examples illustrating how statistics can be misinterpreted or how fallacies arise in using statistical data and statistical methods.

2.6.1 ***Inconsistency in definitions*** Improper definition of an object under consideration as well as failure of keeping the definition constant (unchanged) would result in wrong inferences. For instance, to compare the national income of 2 countries India and the USA, it is imperative that the definition of the national income is taken to be the same in both the countries. Else, if one country includes the services of homemakers in the computation of national income while the other country does not, the national income figures cannot be compared. Moreover, to felicitate the comparison over a period of time, it is essential to retain the definition unchanged. If, however, a change in the definition is unavoidable in future, then a footnote needs to be provided regarding the likely impact on comparability.

2.6.2 ***Erroneous deductions*** A general rule when applied mistakenly to a specific case, it would lead to faulty deduction. If, for instance, the profits of an industry producing 20 different items have declined in 2015-16 compared to 2014-15, it should not lead one to conclude that the industry is necessarily showing deficiency in performance. It is quite possible that it may have improved out of 20 in 19 items it is producing and has stopped the production of 1 item altogether. Let the profits of an industry in 2015-16 have come down from $ 15 million in 2014-15 to $ 12 million. The industry stops the production of item 'A' which was fetching $ 5 million/year due to the non-availability of raw materials. This implies that the industry has improved on he whole. But most people would conclude that the industry has performed badly compared to the preceding year.

2.6.3 ***Erroneous generalizations*** A common error in statistics in to rush to a conclusion or a generalization based on too small a sample or a sample not representative of the population to which the conclusion is applied.

Consider a college having, say, 3000 students. If by taking, for example, a sample of, say, 30 students from a particular section we make a generalization that on the whole the students of that college are very intelligent, then it would be faulty. There are 2 reasons. (i) the size of the sample is too small and (ii) the sample is not at all representative since the all the students have been taken from the same section and it is quite possible that this particular section may be comprised of the best students.

Consider, as another example, that 85% of those who smoke develop indigestion in the middle age. We may thus conclude that smoking leads to indigestion. This type of generalization is invalid unless we know the percentage of smokers and non-smokers in the population. If the total population of a district is 200,000 and the percentage of smokers and non-smokers is 10 and 90, respectively. This implies that out of 20,000 smokers, 80% viz. 16,000 develop indigestion in the middle age. On this basis, we cannot conclude that smoking leads to indigestion. Furthermore we must study the causes of indigestion.

From the foregoing examples, it is clear that a sweeping generalization about a population based on inadequate and individualized data needs to be avoided in the interest of a correct inference.

2.6.4 ***Faulty comparisons*** To draw inferences from the data, comparisons are needed. Comparisons between 2 items cannot be made unless they are similar. This important point is generally overlooked and comparisons are made between 2 dissimilar items, thereby leading to fallacious inferences.

If, for instance, we are comparing the wholesale price index of November, 2016 with that of November, 2017, it is imperative that (i) the number of commodities included, (ii) (their qualities, and (iii) the procedure of constructing the index are the same in both the years. Else the 2 indices cannot be compared.

If the per capita income of USA in 2017 is 4 times as high as that in 1995, then on the basis of this information we should not infer that the people are 4 times better today than what they were in 1995. We must study the behaviour of prices during this period. If there is no change in prices during this period, then only we can make the inference, else not.

2.6.5 ***Faulty use of tools of analysis such as mean, median, mode, dispersion, and correlation*** The various tools of statistical analysis are frequently misused to depict information in such a way that the information deceives the public.

Consider, for example, a company having 2010 shareholders declares that the average holding of the shares of their shareholders is 100. However, an analysis of each and every shareholder reveals that 10 persons who control the company have in all 198,000 shares. The remaining 2,000 shareholders have only 3,000 shares. This implies that an average of 1.5 shares is held by each of the 2,000 shareholders.

Likewise, the range can be used to exaggerate disparities in distribution of wages. For instance, the wages of an employee in an industry are in the range \$ 21,600-\$ 42,600 per annum. The general manager of the industry gets \$ 300,000 per annum. A report that wages of the employees of the industry range from \$ 21, 600 to \$ 42, 600 per annum is deceptive (true though). A measure of dispersion must accompany the average for a proper interpretation of the average.

Similarly a correlation procedure may also be used in such a way that it could establish a faulty conclusion. For instance one may notice that a high degree of correlation between yield of wheat per acre and rainfall in a given area. One should not infer on the basis of this correlation that the higher the rainfall is, the higher will be the yield. In fact, the excessive rains can reduce the yield drastically instead of increasing the yield.

We could have curvilinear (consisting of or bounded by curved lines) correlation with a positive relationship up to a certain point (instead of positive linear correlation) and then a negative relationship setting in after that point.

The coefficient of correlation may also be used to measure the covariation of 2 series which are truly independent of each other. We may show that increase in weight of human beings is associated with increase in income and the spread of certain disease in Florida with the increase in imports of apples. Covariation alone should not be interpreted as manifestation of functional relationship or causation.

2.6.6 ***Faulty interpretation of trend, seasonal variation, and cyclical variation*** Analysing various components of a time series (a series of data pointsindexed (or listed or graphed) in time order) incorrectly results in a wrong interpretation. A straight line, for instance, may be fitted for the complete data instead of a curve. Further, while extrapolating the trend, an assumption is made that other parameters remain unchanged. However, this assumption is rarely true particularly when projections are made over a long period of time.

In addition, at times a seasonal variation may be interpreted as a cyclical one and vice versa (resulting in a seasonal variation as a trend).

2.6.7 ***Error in technical aspects*** An incorrect conclusion for the given data results due to many types of possible technical errors committed in statistical works. Errors may be committed in the choice of an appropriate formula. For instance, the arithmetic mean is employed in a situation the harmonic mean is more appropriate.

While analyzing the data or classifying the data, arithmetic errors may be committed. Errors in units of measurement are also common. A often committed error is to confuse between 2 kinds of logarithms — natural (base e) and common (base 10), the former being 2.3026 times the later.

Another type of error that frequently prevails in statistical works is the use of percentages and ratios. A comparison of percentages without knowing the base to which they refer to would result in error and confusion.

Take, for example, 3 industries in 2016 employment. One industry expects employment 10% below normal, the 2nd industry expects employment 15% below normal while the third one expects employment 25% above normal. Can we conclude that there would be no additional unemployment since the 3rd industry would be able to absorb those displaced by other 2 industries? This type of comparison without the knowledge of the base is misleading (If the base viz. number of employees in all 3 industries are the same then it is fine, else not). If the number of employees in the 1st two industries is 20.000 and 10,000, then the number of workers unemployed would be 3,500. Suppose that th 3rd industry normally employs 12,000 people. An increase of 25% would mean 3,000 more jobs. This implies that still 500 people would remain unemployed. Hence knowledge of base is essential while comparing percentages.

Often 100 is not subtracted in figuring increases and this leads to an incorrect conclusion. The price of a commodity has increased from \$ 100 in 1996 to \$ 300 in 2016. One may say that there is 300% increase in price. However, a little thinking would show that the true increase is 200% .

2.6.8 ***Failure to comprehend total background of the data*** Often figures are interpreted without comprehending the total background of the data. This results in a wrong conclusion. For example, the mortality statistics show that the deaths from cigarette smoking (including passive smoking) have increased from 480,000 in 2015 to 576,000 in 2016 and it may be concluded that the deaths from smoking are increasing. However, for a proper interpretation of these figures, it is necessary to comprehend the total background of the data in terms of the following considerations.

(i) The collection and reporting of mortality statistics has improved over the years.

(ii) Deaths due to other diseases have been brought under control and consequently many more die of smoking now who would have died due to other diseases in earlier times.

(iii) Some deaths formerly listed as " causes unknown" are now correctly diagnosed as deaths from smoking.

(iv) Due to increasing autopsy analysis, there have been an increase in post-mortem diagnoses of smoking which show that the cause of death originally reported was not the actual cause.

Thus just on the basis of certain figures one should not rush to conclusions. One should try to interpret the figures in the light of the background of the data. The foregoing statistical fallacies are far from exhaustive.

However, statistics can neither prove nor disprove anything unlike mathematics/ computational mathematics. In fact, many persons use statistics like a blind man. Anybody who uses statistics should not be misled by bad statistics — one should not only avoid outright falsehood and must be alert to detect possible distortion of truth.

Statistics should not be condemned since it can be abused and misused. The fault lies not with statistics but with the user of statistics. Statistics are like clay of which one can make a God or a Devil as one pleases. W. Allen Wallis and Henry V. Roberts have rightly said: " He who accepts statistics indiscriminately will often be duped unnecessarily. But he who distrusts statistics indiscriminately will often be ignorant unnecessarily" .

3. Role of Environment and Assumptions in Fallacy

We specifically describe below the environmental/social/cultural influence leading to fallacy. In addition, we examine the effect of assumptions in the logical reasoning process. To which extent such an effect is not considered to generate a fallacy is discussed.

3.1 Environmental/social/cultural influence leading to fallacy

Scriptures-human values conflict fallacy One Bengali poet had written " Devatare mora atmiya jani akashe pradip jali; amaderi ei kuthire dekhechi manusher thakurali" The meaning of the foregoing 2 lines of the poem is: We know that all the gods and goddesses are our relatives and we light lamps in the sky; in our own home we have seen gods in human bodies.

This is a firm faith among Hindus. When they worship a stone/clay image of, say, Radha-Krishna (Hindu goddess and god (respectively) with a significant story of great moral value behind them in *Mahabharata*), they simply consider the image as actual live Radha and Sri Krishna as their own most beloved and revered beings.

The story [12] goes that one priest slipped while carrying this stone image/statue after giving bath to the statue at the river Ganga (also called the river Hooghly) adjacent to the Dakshineswar (North Kolkata) Kali temple. Krishna's foot got broken and, since a broken image should not be worshipped as per the scripture, pundits (learned scholars) after lot of deliberation recommended to immerse it in the Ganga (Ganges).

The Rani (Rani Rashmoni) who built the Dakshineswar Kali Temple was heartbroken because she had become very attached to this image of Krishna. To her, it was not just a mere image/stone statue. It was Sri Krishna, the god of compassion, tenderness, and love, himself in all his glory. Sri Ramakrishna (SR), the young priest of the Kali temple came to the rescue. He said, " If your son-in-law breaks his leg, will you throw him into the Ganges (Ganga)?" This very question of SR immediately made the Rani to see clearly what she should do.

Readily she rejected the pundits' recommendation. SR himself fixed this image which has been worshipped for the last over 160 years and is still being worshiped with tremendous zeal and devotion by people from various parts of India/the world.

Was the pundits' logic fallacious? If we really see from another angle (not exactly analogous to the present context), we see why the scripture suggests a broken image should not be worshiped. We should worship only a perfect unbroken image possibly imagining/attributing to it the manifestation of all the ideal qualities originating from a perfect body. This was probably due to the fact that when we get old, we discard our body like torn clothes, burn or bury the body, and assume a new body to start from where we had left in the previous birth with all the new energy and vigour. For Hindus, a rebirth is not a philosophy but a reality. At the highest state of mind (*Samadhi state*) only, one can verify the exact (fallacy-free) ultimate truth (for instance, rebirth) and convince oneself perfectly and once for all. That is precisely the concerned sages/rishis have experienced most intensely and brought the knowledge to the humanity. For them rebirth is not a belief/faith, it is the fact of life. From a lower state of mind, it may not be possible to have the first-hand experience of the truth of rebirth. One may refer the monograph " The Complete Works of Swami Vivekananda" [13] and follow the Raja-Yoga (science of mind control) way of practice to experience the truth through focusing the mind on rebirth. We may thus conclude that given the environment in which Sri Krishna was a vibrant youth (not an old or a dead man whose body is like torn clothes) of all godly qualities, should not be immersed or thrown away but should be treated for the fractured leg and cure it as we always do in our human lives.

To recognize/detect such a fallacy as the foregoing one and take an appropriate course of action/decision clearly needs intense concentration with uncommon critical analyzing power.

3.2 ***Role of assumptions in fallacy*** Assumptions are also not desirable since these distort to a varying degree the real physical problem. Then why do we bring assumptions into real-world problem-solving? If the real-world problem can be solved computationally sufficiently meaningfully then we will not or definitely should not resort to make/introduce assumptions.

The nature of an axiom and that of an assumption are similar. While an axiom is the perceived (self-evident) truth and does not distort the truth, an assumption introduces errors and distorts the truth [14].

However, there are many situations in problem solving, where assumptions are necessitated to get a solution of the problem. But for the assumption(s), a solution may be intractable/impossible within the means/tools (computational) available at that time and in the concerned environment. This solution is that of the distorted problem rather than that of the exact real-world problem.

To know to which extent the problem is distorted due to introduction of an assumption is important. Error in the solution is always injected due to an assumption. Verification of the solution against the experimental result or the result obtained by some other means tells us whether the solution is acceptable within the specified/permitted range in the context.

If the solution is acceptable, then the assumption will be considered fallacy-free and only acceptable error component is present. Else, it may be considered fallacious in the given context. Such a fallacious assumption is not only undesirable but also should be removed/ rejected. Instead, a new assumption with reduced distortion effect should be sought and introduced for a new solution. The process may be continued till a fallacy-free (not error-free since error is an inseparable part of all numerical solutions) solution is obtained. In case we are not able to get a fallacy-free assumption given the available resources, then the resulting solution should be rejected. Consequently the problem remains unsolved.

4. Conclusions

4.1 ***Root of fallacy is a living being*** A fallacy originates from a living being such as a living human being and not from a non-living being such as a computer. If we discover a fallacy in a computer database, then we should go backward and search the cause/origin. Invariably we will find the origin viz. a fallacious statement introduced by a living human being. Such a statement needs to be rectified readily. In fact, a fallacy/mistake and a living being are inseparable. The proverb " To err (mistake/fallacy) is human" is eternally true. It is impossible for any human being including the greatest prodigy in any field, who can say that he does not commit any mistake/fallacy at all. However, anything/any statement or realization originating from the ultimate state of mind (Samadhi or, equivalently, no-thought state) of a human being is fallacy-free. Any sayings/actions of Ramakrishna Paramahamsa, an outstanding modern mystic saint of 19^{th} century India has not been found/proved to be fallacious.

4.2 ***Living being and proof*** Any being never starts life with a clean slate. He is endowed with certain inherent physical and mental qualities from the very first day of his birth. He is born not with just a living body and nothing else such as a feeling of fear, love, hunger, which and how to eat and drink, a faculty for recognizing the environment, and the knowledge of a language. Certain faculty such as swimming and producing a particular sound implying certain message/feeling (to attract attention to get help from his mother/others) comes along with the birth of some living beings viz. waterfowls, dolphins, tigers, and elephants.

There are trades such as hunting, saving oneself from predators through running (for life)/dodging/hiding/camouflaging, and adapting the environment are either learnt and/or improved as he grows. He also grows in experience with time subject to his learning abilities which vary from one living being to another of his kind.

This experience implicitly/explicitly is connected with the 'proof' of a statement/knowledge. It is intimately linked with cause and effect (action and reaction). As instances, we may cite the following.

i. Smoking cigarettes, John got lung cancer.

ii. The city received 21 cm of rain in 5 hours. — The low-lying roads were flooded.

iii. Seeing animals such as dogs, cats, and goats, Mathew stopped brushing his teeth. — He developed 3 cavities in 6 months.

iv. Illegal hunting killed many Royal Bengal tigers in Sundarban, West Bengal (India) and Bangladesh. — The tigers almost became extinct.

It is not far to realize that we, the human beings — quite like other living beings and more — implicitly or explicitly, consciously or unconsciously or subconsciously, look for a proof of the effect of any event or any statement. This immensely helps us to take decision and/or build further valid statements that are useful and may not be readily comprehended.

It may be remarked that a scientist — and for that matter any logical human being — will not or should not accept any statement without fully convincing himself i.e. without getting a proof of the validity of the statement. If he does so i.e. if he accepts any statement without proof, then he may be called a pseudo-scientist (or a pseudo-logician). As a matter of fact, a 'faith' — unproved, might be valid though — should not be advised to be practised by a scientist or a logician.

However, sometimes a statement already proved/experienced by our forefathers/rishis as valid/proved and if the statement is not easy to be proved by us or see/experience its validity, we may still assume its validity in our quest to prove further statements. This implies that these further statements, if proved, are valid based on this assumption. In case these further statements become invalid, then either our proof is faulty or the assumption is wrong or both are defective.

We do emphasize this 'assumption' fact clearly. If, in future, some effect contradicts the 'further statements' which is unlikely though, then one should fall back on the 'assumption' and make necessary corrections and also critically check the proof to avoid/overcome all defects.

4.3 ***Human being — the final judge to verify proof but could be fallible*** It may be remarked that every 'proof' is always essentially done and verified by a human being. No human being in the universe and, for that matter, any living being anywhere can ever be free from error. " To err (mistake) is human (living being)" is a proverb which is eternally true assuming that the action/ saying of the human being has not sprung up from the ultimate (the highest super-conscious) state of his mind. Occurrence of such an action/a saying is, however, a numerical zero in our current (21^{st} century) environment.

On the other hand, only negligibly few 'proofs' can be performed by a computer (non-living being) subject to the adequate axioms and assumptions supplied along with the requisite proof procedures.

4.4 ***Is there an absolutely fail-proof way to check the validity of a proof always?*** A living being, a living human being for example, can never vouch that he never goes wrong. Even the help rendered by a computer cannot be always bug-free/error-free since the program executed by the computer has been written by a human (who can never be non-error-prone. Even before this question, comes the question: Who is the final authority to check the correctness of a proof? The answer of the later question is '*human(s)*'. The answer to the former question is '*No*'. Checking the validity of a proof can be only done by human(s) or by a combination of human(s) and a computer. There exists no other way that can be scientifically/logically accepted by a human being.

The foregoing precise meaning/definition of " Proof" is simple enough, but the insurmountable problems are that this checking has to be done by (i) a human only or by (ii) a human assisted by a computer (which always executes a program (error-freely subject to the precision of the computer, which is always finite and never infinite) written by a human). While " To err (meaning mistake here) is human (implying any living being including a human being here)" is eternally true, " Not to err is computer (modern)" is also true. But the program written by a human may not be bug-free (error-free).

What if the person (or a group of persons), say, Person 1 (assisted/not assisted by a computer) checking a proof for correctness makes a mistake and thinks that a step which is logically correct is in fact incorrect or vice versa? Obviously another person, say Person 2 (assisted/not assisted by a computer) will need to check that the Person 1 doing the checking did not commit any mistake. Again somebody else, say, Person 3 will need to check Person 2, and so on. Eventually we may run out of people who could check the proof and it is possible not only theoretically but also practically that they all could have made a mistake! The probability of committing a mistake might come down by successive checking. It cannot, however, be ascertained that it is exactly 0 in every situation.

4.5 ***Digital computer versus natural computer*** While man-made computers are all finite-precision (finite word-length) machines, the natural computers (computations carried out by nature always eternally, parallely and error-freely without any machine failure and without any maintenance) are infinite precision machines.

The real quantities in the natural computation have exact real (physical) existence but cannot be exactly captured by us (in the form of an exact finite number), so are the outputs of any natural computation. While 'error' is ever existent, impossible to be removed, in any computation by a man-made computer — digital, analog, as well as hybrid, it is completely non-existent in every natural computation. Also, the inputs of a natural computer is exact while these are never so in a digital computer.

In every natural computation, not a single law of nature — known to us or not — is violated. This may be viewed as an axiom (self-evident truth). We have not experienced any violation of any law of nature as we go on observing natural activities such as an earth-quake, a tsunami, and a hurricane. As a matter of fact, we are ignorant of most laws of nature and even if we know a negligibly few, these laws are not always exactly known to us.

It has happened, rarely though, that a flower plant which has been producing red flowers rather eternally might have produced a white one in the same plant. This might lead one to believe that a law of nature has been violated. In fact, believing and actually establishing the real reason are different. " **Belief" has no place in scientific investigation.**

4.6 ***Truths/statements arising out of subtle insight versus blind faith*** However, the great sages and *rishis* (spiritual/materials scientists) of history possessed extraordinarily subtle insight as well as unique experiences, which led them to establish certain truths which are well-established time-tested empirical laws/generalizations/theories about the properties (of objects) derived by examining many instances and observing that these conform, without exception, to a single general pattern.

One accepts such theories because of his total faith in the wisdom of these sages, or because he believes this to be the primary (or the sole) path to knowledge. These are like axioms and are used for further conclusions/inferences. Accepting these as valid will not be construed as blind faith.

However, to have the first hand experience of these truths/ theories one really needs to have the state of mind similar to that of a rishi so that he develops extraordinarily subtle insight and also unique experiences/critical analyzing power. These evidently require extraordinary mind control that needs sincere practice (of, say, *Raja-Yoga*) and consequent time. On the other hand, much easier is to accept these truths as axioms and proceed to develop/innovate further truths/theories.

4.7 ***Laws of nature has never been violated*: *An eternal axiom*** Consider the foregoing white flower in a red flower plant. There must be some solid reason for such a rare happening viz. the red-flower plant producing a white flower. Our ignorance of most laws of nature and also our knowledge of a very few laws of nature often quite approximately (i.e. in a macro-level) should be the reason of our failure for a plausible explanation of such a happening. We have not critically observed any violation of a law of nature.

Consider the tossing of a coin. It is not possible to say whether we get a head in the next toss. If the same coin is tossed 1000 times **exactly under the identical conditions**, then we should definitely get all the 1000 heads if the first one was a head. However, it is practically impossible to retain identical conditions for every toss. The laws of nature here too seem to have not been violated. At least nobody has ever demonstrated such a violation.

Another instance is the Brownian motion of particles in a colloidal solution. We say that the particles move randomly between any 2 collisions. Are these really random not following a law of nature? Or, are these our failure of capturing/determining exactly the path of the particle between any 2 successive collisions based on the several parameters such as angle of collision, strength of the impact between 2 particles, and other environmental parameters related to colloidal solutions?

Whenever any parameter is highly sensitive (a slightest change in the parameter causes very large change in the system), we have the inadequacy (in tools) of capturing the parameters sufficiently accurately to get an acceptable deterministic solution. Further, if the number of collisions is in billions, then the solution of the problem becomes intractable. Under these circumstances, we use a probabilistic approach such as a randomized algorithm, to obtain a probabilistic solution. A deterministic solution is practically impossible and would continue to remain impossible perhaps eternally.

4.8 ***Proof originating from ultimate state of mind is fallacy-free*** Although " **Not to err (mistake) is computer (non-living being)**" , it is only the error-prone human beings who are the final judges in deciding the validity of a proof. Probably at the ultimate state of mind (state of Nirvikalpa Samadhi), when attained, a man will be completely error-free (and exact) in his judgment.

To appreciate the validity of such a judgment, one needs to focus his mind intensely on the judgment and critically analyze it. Ideally he should pass through the same procedure as the man did to achieve the Samadhi (a no-thought or, equivalently, a state of silence that results from the 1-thought (viz. the intense thought of the correct judgment) state.

While it is easy to talk about this silence state implying complete control over the flickering mind, it is difficult but not impossible for a common man to achieve this control, say, through Raja-yoga [Complete Works of Swami Vivekananda, Vol. 1]. One may correctly say in a positive sense that **the greater the control over mind is, the greater is the man**.

In an elementary state of mind, mind is random, moving from one thought to another (usually unconnected) randomly. Just now I am thinking about my mother's ill-health. In the next moment I will be thinking about the basket ball match between our university and university of Southern California and at the following moment I will be thinking about tomorrow's class test.

When we read a book with concentration, we will have a chain of connected thoughts. These thoughts are not random. This is a higher state of mind.

If we keep our mind focussed intensely on one thought, say, a thought of a beautiful scenery or a thought of an extraordinary soothing music or a thought of how to discover a medicine based on herbs for epilepsy or a thought of solving a non-linear optimization problem in polynomial-time, then this one thought condition is still a higher state of mind. Most of the revelations/discoveries of a scientist spring from this 1-thought state of mind.

From this 1-thought state, mind sometimes drops down to silence i.e. no-thought or, equivalently, 0-thought state. In this state, the heart (a small amazingly powerful pumping system), the lung, and all other organs stop functioning. There is nobody to pump our blood that circulates in our body 4 times in 1 minute and nobody to digest our food. The body becomes cold like a dead body. This is the highest state of mind. When one reaches this state, he has known everything i.e. he is " all-knowing" . He has no question to ask. This state is called " Nirvikalpa Samadhi — the highest (ultimate) state of mind which is underlying the 1-thought state" . There can never be a state higher than this 0-thought state (highest potential energy state of mind).

Any living being — human or not — in any planet or any region of the universe can never be superior than a living being (on earth) who has reached the ultimate state of mind. The fictional stories of extra-terrestrial living beings depicting their superiority over the counter-parts on earth are nothing but just fond stories based on our imagination.

These stories should not be viewed as the ones having some substance for us to believe or sincerely learn. As a matter of fact, **we, the humans, are the greatest in the universe**. No living being at any location of the universe, maybe 5 billion light-years away, can ever be superior to us. They may be at best equal to us.

The 0-thought state of mind is analogous to the " absolute zero" viz. 0 Kelvin temperature state of matter. At this state the volume of matter turns out to be " numerical zero" . There is no physical state of matter below 0 Kelvin — the lowest limit of temperature.

The highest limit of temperature is not yet known or possibly there is none (no highest limit). This highest temperature (rather limitless/unbounded) is analogous to lowest elementary state of mind where the thoughts are most random — impossible to find any pattern. The ultimate lowest state of mind possibly is non-existent or, in other words, like " the highest temperature" , it is boundless/limitless.

When one reaches this " silence' (ultimate) state (or possibly the next lower state i.e. " 1-thought" state), fallacy i.e. logical defect in reasoning ceases. Any proof that crops up from this 0-thought state is supposed to be ever valid.

A proof of a statement is always based on one or more axioms (statements which are accepted without proof) explicitly or implicitly used and possibly also on one or more assumptions (statements which distort the corresponding natural problem to a varying extent so that a solution can be obtained/computed within the available means).

All said and done, as long as a human being is to certify the validity of the statement through critically verifying/checking the proof given by a person or a group of persons, there could still remain a non-zero probability of an error/mistake in the proof. This is based on the assumption that the person or the group of persons is or are checking the proof not from 1-thought or 0-thought mind but from lower (2- or more thought) state of mind. This is particularly so if the proof is involved and borders around the limit of common human comprehension.

4.9 ***Various kinds of fallacy: Another classification*** A fallacy is (the use of) a faulty reasoning as stated earlier. The faulty reasoning could arise from adeceptive, misleading, or false notion/blind faith/ deeply rooted conviction which has never even prompted the concerned man to critically analyze the validity of the conviction. There are several proof procedures. A proof should definitely avoid every possible fallacy which could crop up in the garb of a valid/correct statement thus leading to a wrong/disastrous conclusion/inference/consequence. Else the proof will become invalid and consequently the resulting statement is prohibited to be used.

In literature/internet we have observed many possible types of fallacy that might crop up in reasoning put forth by a human being. Although it is not possible to exhaustively enumerate all types of fallacy, these types definitely provide enough information to anyone regarding the possible pitfalls in reasoning one may encounter. Knowledge of these fallacies would definitely minimize or even eliminate a fallacy in a proof procedure.

However, if one has been told none of the possible many fallacies discussed here but is guided by his pure conscience without any bias due to environmental factors, then he will be using consciously fallacy-free reasoning. In fact, it is not essential to have extensive knowledge of numerous fallacies to discover a fallacy, if any, inside a proof. An intense logical thinking (free from faculties such as a superstition, an emotion, a deeply rooted conviction, and a blind faith), in the concerned statement whether it is valid or not (i.e. whether its proof is free from fallacy or not) is definitely helpful.

4.10 ***Limitation of common human comprehension*** A common human being comprehends 7 ± 2 things at a time. This limitation has lot to do in understanding a long/complicated proof. Consequently, it is desirable to construct/develop the proof procedure into parts of reasonable sizes (sizes neither too physically large nor too physically small so that it can be easily understood and at the same time not too simple/obvious). These parts are logically connected to one another and constitute step-by-step development. Such a part is termed in mathematics as a lemma whose proof can be easily understood. **If a long complicated proof is described without several logically connected structures (lemmas), comprehending/grasping the whole of the proof correctly could be difficult for a common man/scientist/mathematician. Such common human beings constitute majority of the population including the scientific ones.**

4.11 ***Non-limitation of computer comprehension*** So far as the non-living computer is concerned, there is no such limitation as the comprehension. A computer can be made to comprehend lots of things correctly artificially always. As a matter of fact, in the current (21st century) scenario a combination of man and machine (computer) could be a better alternative in a proof procedure than just the man.

However, such a combination does not necessarily imply that a fallacy (error in reasoning) will definitely be non-existent. It could possibly make our lives relatively easy.

4.12 ***Use of data and facts to avoid bias*** An important need for one is to be guided by data and facts to arrive at a valid conclusion. This guidance will help him avoid being biased/opinionated. One who makes such a guidance a regular habit and practise all his life is less likely to make a faulty reasoning and introduce a fallacy.

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References

1. Visvanatha-Nyaya-Pancanana, Bhaasa Pariccheda with Siddhanta-Muktavali (translated by Swami Madhavananda from Sanskrit to English), Advaita Ashrama, 5 Dehi Entally Road, Kolkata 700014, India, 1940.

2. Tarka Samgraha (with the Dipika of Annambhatta, and Notes)Paperback– October 2, 1980 translated by Swami Virupakshananda (Author).

3. George Dvorsky, 8 Logical fallacies that fuel Anti-science Sentiments, February, 2015 (Humanity stands a much better chance of success when we apply reason and scientific method.

4. Bustamente, Thomas; Dahlman, Christian, eds.,*Argument types and fallacies in legal argumentation*. Heidelberg: Springer International Publishing. ISBN978-3-319-16147-1, 2015.

5. "Informal Fallacies, Northern Kentucky University". *Retrieved2013-09-10*.

6. Dowden, Bradley."Fallacy".Internet Encyclopedia of Philosophy. Retrieved17 February2016.

7. "Aristotle's original 13 fallacies". The Non Sequitur. Retrieved2013-05-28.

8. Edwin Arthur Maxwell, Fallacies in Mathematics, Cambridge University Press, Great Britain, 1959.

9. Bryan,Bunch, Mathematical fallacies and paradoxes, New York:Dover Publications,ISBN978-0-486-29664-7,1997.

10. S. P. Gupta, Statistical Methods (44th Revised Edition), Sultan Chand & Sons, New Delhi, 2014.

11. W. Allen Wallis and Henry V. Roberts, The Nature of Statistics (page 29, Chapter 2), Courier Corporation, USA, 2014.

12. A Short Life of Sri Ramakrishna, Advaita Ashrama, Published by *Adhyaksha*, Advaita Ashrama, Uttara-khand (India) from its publication Department, Kolkata.

13. Swmi Vivekananda (SV), The Complete Works of Swami Vivekananda, Advaita Ashrama, Mayavati, Uttarakhand, India, 1963 (Birth Centenary Year of SV Volumes, specifically Vol. 1 — Raja-Yoga).

14. S.K. Sen and R.P. Agarwal, Zero: A Landmark Discovery, the Dreadful Void, and the Ultimate Mind, Academic Press, New York, 2016.

Nonlinear Flow-Structure Interaction: A Survey

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ABSTRACT: We discuss the interaction between a flow and a nonlinear oscillating structures through various PDE models. The goal is to study structural instability (*flutter*) induced by a fluid flow, and how it can be suppressed or eliminated. The analysis provided focuses on effects brought about by different plate and fluid boundary conditions. We discuss an assortment of models related to reductions of the full flow-structure coupled systems, and discuss the axial flow configuration of recent interest.

Dedication

The authors would like to dedicate this work to: (1) Professor A.V. Balakrishnan, whose pioneering work on flow-structure interactions served as a major motivation for much of what is reviewed here and to (2) Professor V. Lakshmikantham, whose seminal work in nonlinear analysis has provided a road map to study the qualitative properties of nonlinear phenomena under consideration.

0.1 INTRODUCTION

In this paper we provide a mathematical survey of various PDE models for fluttering plates [5, 14, 42, 49, 57, 90] and associated results, as well as discuss some compelling open problems. Some of the exposition here focuses on *modeling concerns*, and we provide a presentation of mathematical results and their relationship with known physical results regarding the flutter phenomenon. We shall give descriptions of results and a sense for why they hold. In addition we mention a handful of mathematical models which are of recent interest in aeroelasticity *and also* represent fertile ground from the point of view of mathematical analysis.

The *flutter phenomenon* is of interest across many fields; flutter is a sustained, self-excitation instability that occurs as a feedback between an elastic structure and a fluid flow, when the natural structural modes “couple" with the fluid's dynamic loading. For certain flow velocities a bifurcation may occur in the dynamics of the coupled *flow-structure* system, and stable dynamics may become oscillatory (limit cycle oscillations, or LCO) or chaotic [97]. A static bifurcation may also occur, known as *divergence* or buckling (depending on the structural boundary conditions).

The above phenomena can occur in a multitude of applications: buildings and bridges in strong winds, panel and flap structures on vehicles, and in the human respiratory system (snoring and sleep apnea [61]). Recently, flutter resulting from *axial flow* (which can be achieved for *low flow velocities*) has been studied from the point of view of energy harvesting [52]. Flutter considerations are paramount in the supersonic and transonic regime, with the renewed interest in supersonic flight. Here we consider flow-plate dynamics corresponding to both *subsonic* and *supersonic* flow regimes and a wide array of structural boundary conditions and flow-plate coupling conditions.

Flow-structure models have attracted considerable attention in the past mathematical literature, see, e.g., [6, 7, 15, 16, 17, 23, 31, 32, 42, 96] and the references therein. However, the majority of the work (predominantly in the engineering literature) that has been done on flow-structure interactions has been devoted to numerical and experimental studies, see, for instance, [5, 14, 42, 49, 55], and also the survey [78]. Many mathematical studies have been based on linear, two

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dimensional, plate models with specific geometries, where the primary goal was to determine the *flutter point* (i.e., the flow speed at which flutter occurs) [5, 14, 49, 55, 78]. See also [6, 8, 9, 88] for the recent studies of linear models with a one dimensional flag-type structure (beams). This line of work has focused primarily on spectral properties of the system, with particular emphasis on identifying aeroelastic *eigenmodes* corresponding to the associated Possio integral equation (addressed classically by [93]). We emphasize that these investigations have been *linear*, as their primary goal is to predict the flutter phenomenon and isolate aeroelastic modes. Given the difficulty of modeling coupled PDEs at an interface [71, 73], theoretical results have been sparse. On the other hand, it is known that post-flutter dynamics are inherently nonlinear; although the flutter point (the flow velocity for which the transition to periodic or chaotic behavior occurs) can be ascertained within the realm of linear theory, predicting the magnitude of the post-onset instability requires a nonlinear model of the structure (and potentially for the flow as well) [42, 43, 44, 45].

The challenges in the analysis involve (i) the mismatch of regularity between two types of dynamics: the flow and the structure which are strongly coupled, (ii) the presence of unbounded or ill-defined terms (boundary traces in the coupling conditions), and (iii) intrinsically non-dissipative generators of dynamics, even in the linear case. For flow-structure dynamics, the type of instability, whether static (divergence) or dynamic (LCO), depends both on the plate boundary conditions and on the free-stream (or unperturbed) flow velocity. For example, one observes that if a (two-dimensional) panel is simply supported or clamped on the leading *and* trailing edge it undergoes divergence in subsonic flow but flutters in supersonic flow; conversely, a cantilevered panel clamped at one end and free along the others flutters in subsonic flow and may undergo divergence in supersonic flows [65].

This paper primarily addresses the interactive dynamics between a nonlinear plate and a surrounding potential flow [14, 42]. The physical description of flutter and divergence translates into mathematical questions related to existence of nonlinear semigroups representing a dynamical system, asymptotic stability of trajectories, and convergence to attracting sets. Interestingly enough, different model configurations lead to an array of diverse mathematical issues that involve not only classical PDEs, but subtle questions in non-smooth elliptic theory, harmonic analysis, and singular operator theory. For more details concerning the mathematical theory developed for the flutter models discussed below, see [35, 29].

0.2 FLOW STRUCTURE INTERACTIONS-PDE MODEL

The principal model under consideration involves the interaction of a plate with a flow above it. To describe the behavior of the gas (inviscid fluid), the theory of potential flows [14, 42, 49] is utilized. The dynamics of the plate are governed by plate equations with the von Karman (vK) nonlinearity [31, 40, 66]. This model is appropriate for plate dynamics with "large" displacements (characteristic displacements on the order of a few plate thicknesses), and therefore applicable for the flexible structures of interest [14, 40, 42, 49, 66]. There will be configurations in the sequel where vK theory is not valid (e.g., in a flag-like, axial flow configuration), but this will be indicated. Including nonlinearity in the structure of the model is critical, not only for the sake of model accuracy, but also because nonlinear effects play a major mathematical role in controlling ground states; most of the mathematical results reported in this paper are no longer valid for linearized structures.

For generality, let us first simply consider the interaction between a nonlinear plate and some fluid flow above it. In this case, take the domain $\mathcal{O} \subseteq \mathbb{R}^3_+ \equiv \{\mathbf{x} \ : \ z > 0\}$ to be filled with fluid whose dynamics are governed by the *compressible* Navier–Stokes system which are written for the density $\tilde{\rho}$, velocity $\tilde{\mathbf{u}}$ and pressure $\tilde{p}$. Let $T = \{T^{ij}\}_{i,j=1}^3$ be the *total stress* tensor of the fluid,

$$T^{ij} \equiv T^{ij}(\mathbf{u}, p) = \nu\left(v^i_{x_j} + v^j_{x_i}\right) + \left[\lambda \operatorname{div} v - p\right]\delta_{ij}, \quad i, j = 1, 2, 3,$$

with the dynamic viscosity of the fluid given by ν, and λ is Lamé's first parameter (both of which vanish in the case of inviscid fluid). Linearize with respect to a reference state $\{\rho_*; \mathbf{U}; p_*\}$ and suppose that unperturbed flow $\mathbf{U}$ represents an ambient fluid flow, and ρ_*, p_* are constant in time. For small perturbations $\{\rho; \mathbf{u}; p\}$ of this state, assuming that the fluid is isothermal and taking $\tilde{p} = \tilde{\rho}$ with $p_* = \rho_* = 1$; then:

$$(\partial_t + \mathbf{U} \cdot \nabla)p + \text{div } \mathbf{u} + (\text{div } \mathbf{U})p = F \quad \text{in } \mathcal{O} \times \mathbb{R}_+, \tag{0.2.1a}$$

$$(\partial_t + \mathbf{U} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} - (\nu + \lambda)\gamma\text{div } \mathbf{u} + \gamma p + \mathbf{u} \cdot \nabla\mathbf{U} + (\mathbf{U} \cdot \nabla\mathbf{U})p = \mathbf{F} \quad \text{in } \mathcal{O} \times \mathbb{R}_+, \tag{0.2.1b}$$

In this case we need to also supply the fluid equation with appropriate boundary conditions. We consider Ω to be the interactive portion of the boundary, and take Ω to be flat and embedded in the plane $\{z = 0\}$. In line with classic [14] and more recent references [24, 25], we choose impermeable boundary conditions, which look as follows:

$$T\mathbf{n} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\mathcal{O}, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O} \setminus \overline{\Omega}, \quad \mathbf{u} \cdot \mathbf{n} = [u_t + Uu_x] \text{ on } \Omega,$$

where $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{O}$, $\boldsymbol{\tau}$ is a unit tangent direction to $\partial\mathcal{O}$, u is the deflection of a flexible, flat portion Ω of the boundary.

Remark 0.2.1. *The boundary condition* (0.2.2) *is the general analog of the so called* Kutta-Joukowski *condition, described below. The latter* (0.2.3) *is the* adherence *condition.*

$$Tn = 0 \ \textit{on} \ S; \ \ \mathbf{u} \cdot \mathbf{n} = w_t + Uu_{x_1} \ \ \textit{on} \ \Omega; \tag{0.2.2}$$

$$\mathbf{u} = 0 \ \textit{on} \ S; \ \ \mathbf{u} \cdot \mathbf{n} = w_t + Uu_{x_1} \ \ \textit{on} \ \Omega, \tag{0.2.3}$$

where, again, T^{ij} *is the total stress tensor of the fluid.*

For more discussion of the compressible, viscous model above, including semi-group well-posedness in a simplified case, see [3], as well as [24] when $\mathbf{U} \equiv 0$, and [25] for the inviscid, incompressible case. The survey [38] provides a nice overview. (Each of these references takes $\mathcal{O}$ to be bounded.)

The environment we now consider is $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \, : \, z > 0\}$. The thin plate has negligible thickness in the z-direction. The unperturbed flow field moves in the x-direction at the fixed velocity U (i.e., $\mathbf{U} = \langle U, 0, 0\rangle$). The physical constants have been normalized (corresponding to the linear potential flow) such that $U = 1$ corresponds to Mach 1 (to simplify the models and to emphasize the mathematical properties of interest). The plate is modeled by a bounded domain $\Omega \subset \partial\mathbb{R}^3_+$ with smooth boundary $\partial\Omega = \Gamma$.

0.2.1 Flow of Gas

Assuming the flow, in addition to being compressible, is irrotational, we invoke the theory of potential flows [14, 42]. In this case, $\mathbf{u} = \nabla\phi$, where the scalar function $\phi : \mathbb{R}^3_+ \times \mathbb{R} \to \mathbb{R}$ is the linear *flow potential* and satisfies:

$$\begin{cases} (\partial_t + U\partial_x)^2\phi = \Delta_{x,y,z}\phi \text{ in } \mathbb{R}^3_+ \\ FC(\phi), \end{cases} \tag{0.2.4}$$

with appropriate initial data ϕ_0 and ϕ_1. We also note that in this case the dynamic pressure $p(x, t)$ can be explicitly solved for in terms of the *acceleration potential of the flow*: $p(x, t) = \phi_t + U\phi_x$. The term $FC(\phi)$ represents the *flow boundary conditions* or *interface conditions*. We will consider two primary flow conditions, written in terms of the transverse (Lagrangian) displacement $u(x, y; t)$ on Ω:

(NC) The standard flow boundary conditions—the *full Neumann condition*—are henceforth denoted (NC). This is typically utilized when a majority of the plate boundary is clamped or hinged. It takes the form:

$$\partial_z\phi\big|_{z=0} = (\partial_t + U\partial_x)u_{\text{ext}}, \ \text{ on } \mathbb{R}^2 = \partial\mathbb{R}^3_+, \tag{0.2.5}$$

where the subscript 'ext' indicates the extension by zero for functions defined on Ω to all of $\mathbb{R}^2$.

(KJC) The *Kutta-Joukowsky* flow condition, henceforth denoted (KJC), is a mixed (Zaremba-type [85]) condition which will be discussed below. Then the *Kutta-Joukowsky* flow condition on $\mathbb{R}^2$ is

$$\partial_z\phi = (\partial_t + U\partial_x)u \ \text{ on } \ \Omega; \ \ \psi = 0 \text{ on } \ \mathbb{R}^2 \setminus \Omega. \tag{0.2.6}$$

In both cases above the interface coupling on the surface of the plate occurs in a Neumann type boundary condition known as the *downwash* of the flow. The choice of flow conditions is itself dependent upon the plate boundary conditions, and this is determined by application.

0.2.2 Structure

The scalar function $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ represents the displacement of the plate in the z-direction at the point (x, y) at the moment t. The plate is taken with general boundary conditions $BC(u)$, which will be specified later. The dynamics are thus given by:

$$\begin{cases} u_{tt} + \Delta^2_{x,y} u + k_0 u_t + f(u) = p(\mathbf{x}, t) \text{ in } \Omega \\ BC(u) \text{ on } \partial\Omega, \end{cases} \tag{0.2.7}$$

with appropriate initial data $u_0 = u(0),\ u_1 = u_t(0)$. The quantity $p(\mathbf{x}, t)$ corresponds to the aerodynamic pressure of the flow on the top of the plate and, in the standard configuration, is given in terms of the flow:

$$p(\mathbf{x}, t) = p_0(\mathbf{x}) + (\partial_t + U\partial_x)tr[\phi]\big|_{\Omega}, \ \ \mathbf{x} \in \Omega. \tag{0.2.8}$$

The quantity p_0 represents static pressure applied on the surface of the plate. The coefficient k_0 captures structural viscous damping or a feedback controller. The vK nonlinearity is given by:

$$f(u) = -[u, v(u) + F_0],$$

where F_0 is a given *in-plane load* (of sufficient regularity). The vK bracket above corresponds to

$$[u, w] = u_{xx} w_{yy} + u_{yy} w_{xx} - 2u_{xy} w_{xy},$$

and the Airy stress function $v(u_1, u_2)$ solves the elliptic problem

$$\Delta^2 v(u_1, u_2) + [u_1, u_2] = 0 \ \text{ in } \ \Omega, \quad \partial_\nu v(u_1, u_2) = v(u_1, u_2) = 0 \ \text{ on } \ \Gamma,$$

(the notation $v(u) = v(u, u)$ is employed).

Remark 0.2.2 (Rotational Inertia)**.** *In line with the conventions of aeroelasticity [14, 49], we omit the structural rotational inertia parameter, which would be given as $\alpha > 0$ in the plate equation:*

$$(1 - \alpha\Delta)u_{tt} + \Delta^2_{x,y} u + k_0(1 - \alpha\Delta)u_t + f(u) = p(\mathbf{x}, t) \tag{0.2.9}$$

Much of the early mathematical work on the flow-plate interactions utilized this term [15, 16], owing to its regularizing effects on the plate velocity $u_t \in H^1(\Omega)$ (rather than $L_2(\Omega)$), and the impact this has on the ability to study the flow-structure problem component-wise (see Section **??***). Recent work [60] on the flutter of cantilever beams has also discussed this inertia term extensively.*

The following types of boundary conditions for the displacement u are typical for flow-structure interaction models.

(C) *Clamped boundary conditions* (corresponding to a panel element) take the form

$$u = \partial_\nu u = 0 \ \text{ on } \Gamma.$$

In this case, the plate is considered to be embedded in or affixed to a large rigid body.

(CF) Let the boundary be partitioned in two pieces: Γ_0 and Γ_1. *Clamped-free boundary conditions* (possibly corresponding to a flap, flag, or cantilevered airfoil) are given by

$$\begin{cases} u = \partial_\nu = 0 \ \text{ on } \ \Gamma_0 \\ \Delta u + (1 - \mu)B_1 u = D_1(\partial_\nu u_t) \ \text{ on } \ \Gamma_1, \\ \partial_\nu \Delta u + (1 - \mu)B_2 u - \mu_1 u = D_2(u, u_t) + \alpha \partial_\nu u_{tt} \ \text{ on } \ \Gamma_1, \end{cases}$$

where the boundary operators B_1 and B_2 are given by [66]:

$$\begin{aligned} B_1 u &= 2\nu_1 \nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} = -\partial_\tau^2 u - \nabla \cdot \nu(\mathbf{x})\partial_\nu u \ , \\ B_2 u &= \partial_\tau \left[\left(\nu_1^2 - \nu_2^2\right) u_{xy} + \nu_1 \nu_2 \left(u_{yy} - u_{xx}\right) \right] \ = \partial_\tau \partial_\nu \partial_\tau u; \end{aligned} \tag{0.2.10}$$

The parameter μ_1 is nonnegative and $0 < \mu < 1$ is the Poisson modulus. The operators D_i for $i = 1, 2$ are possible energy damping/dissipation functions.

0.3 ULTIMATE MODEL: COUPLING OF FLOW AND STRUCTURE

Our primary interest here is the following PDE system:

$$\begin{cases} u_{tt}+\Delta^2 u+k_0u_t+f(u)=p_0+\big(\partial_t+U\partial_x\big)tr[\phi] & \text{in } \Omega\times(0,T),\\ u(0)=u_0;\;\; u_t(0)=u_1, & \\ u=\partial_\nu u=0 & \text{on } \partial\Omega\times(0,T),\\ (\partial_t+U\partial_x)^2\phi=\Delta\phi & \text{in } \mathbb{R}^3_+\times(0,T),\\ \phi(0)=\phi_0;\;\; \phi_t(0)=\phi_1 & \text{in } \mathbb{R}^3_+\\ \partial_z\phi=\big[(\partial_t+U\partial_x)u(\mathbf{x})\big]_{\text{ext}} & \text{on } \mathbb{R}^2_{\{(x,y)\}}\times(0,T). \end{cases} \tag{0.3.1}$$

We focus on this panel flutter model as a starting point for the analysis of coupled flow-plate systems. After providing results for this model (a majority of extant mathematical results hold for this model), we describe extensions to accommodate other configurations.

0.3.1 Energies and Topology of Solutions

Finite energy constraints (as dictated by the physics of the model) manifest themselves in the natural topological requirements on the solutions ϕ and u to (0.3.1). Letting $\mathcal{H}$ correspond to the state space for plate displacements $u(t)$ (depending on boundary conditions). Solutions we consider here will have the properties: $u\in C(0,T;\mathcal{H})\cap C^1(0,T;L_2(\Omega))$; $\phi\in C(0,T;H^1(\mathbb{R}^3_+))\cap C^1(0,T;L^2(\mathbb{R}^3_+))$. To set up the model in a dynamical systems framework, the principal state space is taken to be

$$Y=Y_{fl}\times Y_{pl}\equiv\big(W_1(\mathbb{R}^3_+)\times L_2(\mathbb{R}^3_+)\big)\times\big(\mathcal{H}\times L_2(\Omega)\big)$$

where $W_1(\mathbb{R}^3_+)$ will denote the homogeneous Sobolev space of order 1. We will also consider a stronger space:

$$Y^s\equiv H^1(\mathbb{R}^3_+)\times L_2(\mathbb{R}^3_+)\times\mathcal{H}\times L_2^\alpha(\Omega).$$

Remark 0.3.1 (Parameters)**.** *The central parameters in the PDE analysis of the general models above are the rotational inertia parameter $\alpha\geq 0$ and the flow velocity U. First, α is critical in that the model of primary physical interest in the theory of large deflections takes $\alpha=0$, yet the essential nonlinearity* and *interactive flow terms are* not of a compact nature *in this case. Indeed, the presence of rotational terms provides an additional regularizing effect on the transverse velocity of the plate, u_t [15, 17, 31]; this, in turn, leads to several desirable mathematical properties such as compactness of trajectories, gain of derivatives, etc. It is also clear that, with respect to U, when $U>1$, there is a loss of spatial ellipticity of the principal part of the flow operator $\partial_t^2-\Delta+U^2\partial_x^2$, and near $U=1$ there is full x degeneracy in the model.*

Remark 0.3.2. *For the remainder of the text the case of clamped plate boundary conditions (C) with standard Neumann flow conditions (NC) is referred to as the* **standard panel** *configuration.*

The subsonic case, $U\in[0,1)$, is initially considered. The following "energies" are observed with the standard *velocity multipliers* (we suppress the dependence on t in the expressions below):

$$E_{pl}(u)=\frac{1}{2}[||u_t||^2_{L_2^\alpha(\Omega)}+||\Delta u||^2_{0,\Omega}+\frac{1}{2}||\Delta v(u)||^2_{0,\Omega}]-\langle F_0,[u,u]\rangle_\Omega+\langle p,u\rangle_\Omega \tag{0.3.2}$$

$$E_{fl}(\phi)=\frac{1}{2}[||\phi_t||^2_{0,\mathbb{R}^3_+}+||\nabla\phi||^2_{0,\mathbb{R}^3_+}-U^2||\phi_x||^2_{0,\mathbb{R}^3_+}] \tag{0.3.3}$$

$$E_{int}(u,\phi)=U\langle\phi,u_x\rangle_\Omega;\qquad \mathcal{E}=E_{pl}+E_{fl}+E_{int} \tag{0.3.4}$$

where E_{int} represents the (not necessarily positive) *interactive* "energy".

In the discussion below, we will encounter strong (classical), generalized (mild), and weak (variational) solutions. In obtaining existence and uniqueness of solutions, semigroup theory is utilized; this necessitates the use of *generalized* solutions. These are strong limits of strong solutions and satisfy an integral formulation of (0.3.1) (they are called *mild* by some authors). We note that generalized solutions are also weak solutions and satisfy a variational formulation of the flow-plate system (see, e.g., [31, Section 6.5.5] and [96]). For the exact definitions of strong and generalized solutions we refer to [31, 32, 34, 96].

0.3.2 Hadamard Well-Posedness of Nonlinear Panel Model

Mathematical wellposedness of solutions is a fundamental issue which need to be dealt with whenever discussing qualitative properties of the physical model. Particularly important, and also quite delicate, are issues of stability/robustness with respect to the initial conditions and the parameters of the model. This area has attracted enormous attention in mathematical literature with seminal and very general works going back to [68]. In what follows we shall focus on the past literature dealing with wellposedness of the model under consideration. Well-posedness results in the past literature deal mainly with the dynamics possessing some regularizing effects. This has been accomplished by either accounting for non-negligible rotational inertia effects through the mass operator $(1-\alpha\Delta)$ acting on u_{tt} [15, 17, 31] and strong damping of the form $-\alpha\Delta u_t$, or by incorporating helpful thermal effects into the structural model [82, 83]. In the cases listed above, the natural structure of the dynamics dictate that the plate velocity has the property $u_t \in H^1(\Omega)$, which provides the needed regularity for the applicability of many standard tools in nonlinear analysis. One is still faced with the low regularity of boundary traces, due to the failure of the Lopatinski conditions [84].

The first contribution to the mathematical analysis of the problem is [16, 17] (see also [31, Section 6.6]), where the case $\alpha > 0$ (rotational) is fully treated. The method employed in [16, 17, 31] relies on sharp microlocal estimates for the solution to the wave equation driven by $H^{1/2}(\Omega)$ Neumann boundary data given by $u_t + Uu_x$. This gives $\phi_t|_\Omega \in L_2(0,T;H^{-1/2}(\Omega))$ [80]. Along with an explicit solver for the three dimensional flow equation (the Kirchhoff formula), and a Galerkin approximation for the structure, one may construct a fixed point for the appropriate solution map. This method works well for all values of $U \neq 1$.

Related ideas were used more recently in the thermoelastic flow-plate interactions in [82, 83]; in this case the dynamics enjoy $H^1(\Omega)$ regularity of the velocity u_t (independent of α) due to the analytic regularity induced by thermoelasticity [73]. When $\alpha = 0$, and we take no additional smoothing in the model, the estimates corresponding to the above approaches become singular, destroying the applicability of the previous techniques. The main mathematical difficulty of this problem is the presence of the boundary terms: $(\phi_t + U\phi_x)|_\Omega$ acting as the aerodynamic pressure on the plate. When $U = 0$, the corresponding boundary terms exhibit monotone behavior with respect to the natural energy inner product (see [31, Section 6.2] and [71]), which is topologically equivalent to the topology of the state space Y. The latter enables the use of monotone operator theory [31] (and references therein). However, when $U > 0$ this is no longer true, and other techniques must be employed to obtain solutions [32, 96].

In [5, 6, 87, 88] (and references therein) a linear *airfoil* immersed in a subsonic flow is considered. The wing is taken to have a high aspect ratio thereby allowing for the suppression of the span variable, and reducing the analysis to individual chords normal to the span. By reducing the problem to a one dimensional analysis, many technical hangups are avoided, and Fourier-Laplace analysis is effective. The analysis of two dimensional structures requires very different tools. subsubsectionGeneration of Nonlinear Semigroup. We begin with an overview of the well-posedness results for the standard panel model which simultaneously covers both subsonic and supersonic cases (and thus is a weaker result than what appears in the sequel). However, in both cases we are able to establish generation of nonlinear semigroups describing flow-structure interaction. This allows to consider the problem within the realm of dynamical systems.

Theorem 0.3.1. *Consider problem in (0.3.1) with* $U \neq 1$, $p \in L_2(\Omega), F_0 \in H^{3+\delta}(\Omega)$. *Let* $T > 0$ *and*

$$(\phi_0, \phi_1; u_0, u_1) \in Y = Y_{fl} \times Y_{pl} \equiv \big(W_1(\mathbb{R}^3_+) \times L_2(\mathbb{R}^3_+)\big) \times \big(H^2_0(\Omega) \times L_2(\Omega)\big). \tag{0.3.5}$$

Then there exists unique generalized solution

$$(\phi(t), \phi_t(t); u(t), u_t(t)) \in C([0,T], Y). \tag{0.3.6}$$

This solution is also weak and generates a nonlinear continuous semigroup $S_t : Y \to Y$.

The proof of the above theorem in the general (supersonic) case makes use of a change of state variable: the dynamic variable $\psi = \phi_t + U\phi_x$ is considered as the second flow state. In this case the adapted energies are:

$$\widehat{E}_{fl} = \frac{1}{2}[\|\nabla\phi\|^2_{0,\mathbb{R}^3_+}] + \|\psi\|^2_{0,\mathbb{R}^3_+}, \quad \widehat{E}_{int} = 0, \quad \widehat{\mathcal{E}} = E_{pl} + \widehat{E}_{fl} + \widehat{E}_{int}. \tag{0.3.7}$$

Generalized solutions satisfy the identity is given by:

$$\widehat{\mathcal{E}}(t) + \int_s^t \int_\Omega k_0 u_t^2 + U\int_0^t \langle u_x, \psi\rangle_\Omega d\tau = \widehat{\mathcal{E}}(s). \tag{0.3.8}$$

Remark 0.3.3. *Note that when $U > 1$ the equation (0.3.1) displays degeneracy of ellipticity of the stationary problem. This results in the loss of dissipativity in the energy relation (0.3.8). In addition, the boundary term $\psi|_\Omega$ is not well defined on the energy space. Handling of these obstacles requires the development of an appropriate "hidden" trace regularity theory for the aeroelastic potential ψ, which plays pivotal role in the proof of Theorem 0.3.1.*

In the subsonic case, instead, there is no spatial "degeneracy" in the flow equation, leading to a stronger well-posedness result:

Theorem 0.3.2. *Suppose $U < 1$, $p \in L_2(\Omega)$ and $F_0 \in H^{3+\delta}(\Omega)$. Assume that initial data sytisfy* (0.3.5)*. Then there exists unique generalized solution to* (0.3.1)

$$(\phi(t), \phi_t(t); u(t), u_t(t)) \in C([0,T], Y). \tag{0.3.9}$$

This solution is also weak and generates a nonlinear continuous semigroup $S_t : Y \to Y$. Every generalized solution is also weak. Moreover these solutions to (0.3.1) *generates a nonlinear continuous semigroup (S_t, Y). Any generalized (and hence weak) solution to* (0.3.1)*, satisfies the global-in-time bound*

$$\sup_{t\geq 0}\left\{\|u_t\|^2_{0,\Omega} + \|\Delta u\|^2_\Omega + \|\phi_t\|^2_{\mathbb{R}^3_+} + \|\nabla\phi\|^2_{\mathbb{R}^3_+}\right\} < +\infty. \tag{0.3.10}$$

and thus (S_t, Y) is uniformly bounded in time, i.e., there exists a constant C such that

$$\|S_t(y_0)\|_Y \leq C(\|y_0\|_Y), \quad t > 0.$$

The dynamics satisfy the energy identity

$$\mathcal{E}(t) + \int_s^t \int_\Omega k_0 u_t^2 = \mathcal{E}(s), \quad t \geq s, \tag{0.3.11}$$

Remark 0.3.4. *This theorem remains true if the von Karman nonlinearity is replaced by any nonlinearity $f : H^2(\Omega) \cap H^1_0(\Omega) \to L_2(\Omega)$ which is (a) locally Lipschitz, and (b) the nonlinear trajectories possess a uniform bound in time which typically holds when nonlinearity satisfies some coercivity properties, see [30].*

The proof of Theorem 0.3.2 given in [31] and [96] relies on three principal ingredients: (i) renormalization of the state space which yields shifted dissipativity for the dynamics operator (which is nondissipative in the standard norm on the state space); (ii) the sharp regularity of Airy's stress function, which converts a supercritical nonlinearity into a critical one [31] (and references therein):

$$\|v(u)\|_{W^{2,\infty}(\Omega)} \leq C\|u\|^2_{H^2(\Omega)}; \tag{0.3.12}$$

and (iii) control of low frequencies for the system by the nonlinear source, represented by the inequality

$$\|u\|^2_{L_2(\Omega)} \leq \epsilon[\|u\|^2_{H^2(\Omega)^2} + \|\Delta v(u)\|^2_{L_2(\Omega)}] + C_\epsilon. \tag{0.3.13}$$

where $\epsilon > 0$ can be taken arbitrary small. An alternative proof based on a *viscosity method* can be found in [32].

In comparing the results obtained for supersonic and subsonic cases, there are two major differences at the qualitative level, in addition to an obvious fact that E_{fl} is no longer positive when $U > 1$: *First*, the regularity of strong solutions obtained in the subsonic case [32, 96] coincides with regularity expected for classical solutions. In the supersonic case, there is a loss of differentiability in the flow in the tangential x direction, which then propagates to the loss of differentiability in the structural variable u. As a consequence, strong solutions do not exhibit sufficient regularity in order to perform the needed calculations. To cope with the problem a special regularization procedure was introduced in [34], where strong solutions are approximated by sufficiently regular functions, though not solutions to the given PDE. The limit passage allows one to obtain the needed estimates valid for the original solutions [34]. *Secondly*, in the subsonic case, one shows that the solutions are *bounded* in time, see [31, Proposition 6.5.7] and also [32, 96]. This property cannot be shown in the supersonic analysis. The leak of the energy in the energy relation can not be compensated by the nonlinear terms. In fact, while in the subsonic case one deals with gradient system, in the supersonic regime, the resulting dynamical system is not dissipative.

0.3.3 Flow-Structure Model—Dynamical System Properties

It is known [49] (and references therein) that, although the flow coupling may introduce potential instability, it can also help bound the plate dynamics and may assist in the dissipation of plate energy associated to high frequencies [44, 45]. The work in [33] is an attempt to address these observations rigorously, via the result in Theorem 0.3.4 below, which reduces the full flow-plate dynamics to a plate equation with a delay term. This result is implemented and the problem is cast within the realm of PDE with delay (and hence our state space for the plate contains a delay component which encapsulates the flow contribution to the dynamics). One would like to show that such dynamics, as described above, are stable in the sense that trajectories converge asymptotically to a "nice" set. Unlike parabolic dynamics, there is no a priori reason to expect that hyperbolic-like dynamics can be asymptotically reduced to a truly finite dimensional dynamics. By showing that the PDE dynamics converges to a finite dimensional (compact) *attractor* it effectively allows the asymptotic-in-time reduction of the analysis of the infinite dimensional, unstable model to a finite dimensional set upon which classical control theory can be applied. We note that owing to the non-dissipative terms present in flutter models, global attractors associated to plate's flutter dynamics (as well as the set of equilibria points for the full flow-plate system) can have non-trivial structure. See [48] and references cited therein, as well as [50, 97, 98] for the engineering discussion of these properties.

The main point of the treatment in [33] is to demonstrate the existence and finite-dimensionality of a global attractor for the reduced plate dynamics (see Theorem 0.3.4) *in the absence of any imposed damping mechanism* and in the absence of any regularizing mechanism generated by the dynamics itself. Imposing no mechanical damping, taking $\alpha = 0$, and assuming the flow data are compactly supported we obtain the primary theorem in [33]:

Theorem 0.3.3. *Suppose* $0 \le U \ne 1$, $\alpha = 0$, $F_0 \in H^{3+\delta}(\Omega)$ *and* $p_0 \in L_2(\Omega)$. *Then there exists a compact set* $\mathcal{U} \subset Y_{pl}$ *of finite fractal dimension such that*

$$\lim_{t\to\infty} d_{Y_{pl}}\big((u(t), u_t(t)), \mathcal{U}\big) = \lim_{t\to\infty} \inf_{(\nu_0,\nu_1)\in\mathcal{U}} \big(\|u(t)-\nu_0\|_2^2 + \|u_t(t)-\nu_1\|^2\big) = 0$$

for any weak solution $(u, u_t; \phi, \phi_t)$ *to* (0.3.1) *with initial data*

$$(u_0, u_1; \phi_0, \phi_1) \in Y, \quad Y \equiv H_0^2(\Omega) \times L_2(\Omega) \times W_1(\mathbb{R}^3_+) \times L_2(\mathbb{R}^3_+),$$

which are localized in $\mathbb{R}^3_+$ *(i.e.,* $\phi_0(\mathbf{x}) = \phi_1(\mathbf{x}) = 0$ *for* $|\mathbf{x}| > R$ *for some* $R > 0$*). Additionally, there is the extra regularity* $\mathcal{U} \subset \big(H^4(\Omega) \cap H_0^2(\Omega)\big) \times H^2(\Omega)$.

A key to obtaining attracting sets is the representation of the flow on the structure via a delay potential (see Section 3.3 in [31]). Reducing this full flow-plate problem to a delayed von Karman plate is the primary motivation for our main result and permits a starting-point for long-time behavior analysis of the flow-plate system, which is considerably more difficult otherwise. The exact statement of this key reduction is given in the following assertion:

Theorem 0.3.4. *Let the hypotheses of Theorem 0.3.3 be in force, and* $(u_0, u_1; \phi_0, \phi_1) \in \mathcal{H} \times L_2(\Omega) \times H^1(\mathbb{R}^3_+) \times L_2(\mathbb{R}^3_+)$. *Assume that there exists an* R *such that* $\phi_0(\mathbf{x}) = \phi_1(\mathbf{x}) = 0$ *for* $|\mathbf{x}| > R$. *Then the there exists a time* $t^{\#}(R, U, \Omega) > 0$ *such that for all* $t > t^{\#}$ *the weak solution* $u(t)$ *to (0.3.1) also satisfies the following equation:*

$$u_{tt} + \Delta^2 u - [u, v(u) + F_0] = p_0 - (\partial_t + U\partial_x)u - q^u(t) \tag{0.3.14}$$

with

$$q^u(t) = \frac{1}{2\pi}\int_0^{t^*} ds \int_0^{2\pi} d\theta [M_\theta^2 u_{ext}](x - (U + \sin\theta)s, y - s\cos\theta, t - s). \tag{0.3.15}$$

Here, $M_\theta = \sin\theta\partial_x + \cos\theta\partial_y$ *and*

$$t^* = \inf\{t \ : \ \mathbf{x}(U,\theta,s) \notin \Omega \textit{ for all } \mathbf{x} \in \Omega, \ \theta \in [0,2\pi], \textit{ and } s > t\} \tag{0.3.16}$$

with $\mathbf{x}(U,\theta,s) = (x-(U+\sin\theta)s, y - s\cos\theta) \subset \mathbb{R}^2$.

Thus, after some time, the behavior of the flow can be captured by the aerodynamical pressure term $p(t)$ in the form of a delayed forcing. Theorem 0.3.4 allows us to suppress the dependence of the problem on the flow variable ϕ. The flow state variables (ϕ, ϕ_t) manifest themselves in our rewritten system via the delayed character of the problem; they appear in the initial data for the delayed component of the plate, namely $u^t\big|_{(-t^*,0)}$. Here we emphasize that the structure of aerodynamical pressure posited in the hypotheses leads to the velocity term $-u_t$. We may utilize this as natural damping appearing in the structure of the reduced flow pressure. One should also note that the main tool used for the derivation of the reduced formula *Huygen's Principle* and the associated three-dimensional nature of the problem. The effect of the flow is dispersed after some time.

The reduced model displays the following features: (i) it does not have gradient structure (due to dispersive and delay terms), (ii) the delay term appears at the critical level of regularity. However, despite of the lack of gradient structure, compensated compactness methods allow to "harvest" some compactness properties from the reduction, so the ultimate dynamics does admit global attracting set.

The proof of Theorem 0.3.3 requires modern tools and new long-time behavior technologies applied within this delay framework. Specifically, the approach mentioned above (and utilized in [31, 82, 83]) *does not apply* in this case. A relatively new technique [30, 31] allows one to address the asymptotic compactness property for the associated dynamical system without making reference to any gradient structure of the dynamics (not available in this model, owing to the dispersive flow term). In addition, extra regularity and finite dimensionality of the attractor is demonstrated via a quasi-stability approach [31] (see also [26] for recent developments of quasi-stability techniques). The criticality of the nonlinearity and the lack of gradient structure prevents one from using a powerful technique of *backward smoothness of trajectories* [31], where smoothness is obtained near the equilibria points and propagated forward. Without a gradient structure for this model, the associated attractor may have complicated structure (not being strictly characterized by the equilibria points).

In the presence of additional damping imposed on the structure, it is reasonable to expect that the entire evolution (both plate and flow) is strongly stable in the sense of strong convergence to equilibrium states. Such results are shown for the model above (in the same references) only for the case of *subsonic flows*, $U < 1$. The key to obtaining such results is a *viable energy relation* and a priori bounds on solutions which yield *finiteness of the dissipation integral*. When $U > 1$, recent calculations reported in [94] indicate that such results are false, even for low supersonic speeds. The following result provides a stabilization of the full subsonic flow-plate dynamics (as opposed to the result above we obtain for all flow velocities, but only the structural dynamics).

For this we recall the global bound in (0.3.10) which along with the energy identity in (0.3.11) allows us to obtain the following:

Corollary 0.3.5. *Let* $k_0 > 0$, $0 \le U < 1$ *and* $\alpha = 0$. *Then the dissipation integral is finite. Namely, for a generalized solution to* (0.3.1) *we have*

$$\int_0^\infty \|u_t(t)\|^2_{0,\Omega} dt \le K_{u,k_0} < \infty.$$

From this finiteness, we can show the following result [75] for *smooth initial data*. Let $\mathcal{N}$ denote the set of stationary solutions to (0.3.1) (for the existence and properties see [31]).

Theorem 0.3.6. *Let* $0 \le U < 1$ *and* $\alpha = 0$. *Assume* $p_0 \in L_2(\Omega)$ *and* $F_0 \in H^4(\Omega)$. *Assume that* $\mathcal{N}$ *is* discrete[1] *Then for all* $k_0 > 0$, *any solution* $(u(t), u_t(t); \phi(t), \phi_t(t))$ *to the flow-plate system* (0.3.1) *with initial data*

$$(u_0, u_1; \phi_0, \phi_1) \in (H_0^2 \cap H^4)(\Omega) \times H_0^2(\Omega) \times H^2(\mathbb{R}^3_+) \times H^1(\mathbb{R}^3_+).$$

that are spatially localized in the flow component (i.e., there exists a $\rho_0 > 0$ *so that for* $|\mathbf{x}| \ge \rho_0$ *we have* $\phi_0(\mathbf{x}) =$

[1]This is generically true; see [31, 75] for a more detailed discussion.

$\phi_1(\mathbf{x}) = 0$*) has the property that*

$$\lim_{t\to\infty}\left\{\|u(t)-\hat{u}\|^2_{H^2(\Omega)} + \|u_t(t)\|^2_{L_2(\Omega)} + \|\phi(t)-\hat{\phi}\|^2_{H^1(K_\rho)} + \|\phi_t(t)\|^2_{L_2(K_\rho)}\right\} = 0$$

for some $(\hat{u},\hat{\phi}) \in \mathcal{N}$*, and for any* $\rho > 0$*, where* $K_\rho = \{\mathbf{x} \in \mathbb{R}^3_+ : \|\mathbf{x}\| \leq \rho\}$.

The above result remains true for finite energy initial data (only in Y) with convergence in a *weak sense*. Additionally, for the system with rotational inertia present $\alpha > 0$ (and corresponding strong damping) [23, 31] *or* with thermal effects present [82, 83] the analogous *strong* stabilization result holds for finite energy initial data

$$(u_0, u_1; \phi_0, \phi_1) \in Y = H^2_0(\Omega) \times L^\alpha_2(\Omega) \times W_1(\mathbb{R}^3_+) \times L_2(\mathbb{R}^3_+).$$

The fact that $k_0 > 0$ is needed in order to obtain convergence to equilibria is corroborated by the counterexample [27], which shows that, with $k_0 = 0$, periodic solutions may remain in the limiting dynamics. This example indicates the necessity of introducing mechanical damping in the plate model *if one expects a strong convergence to equilibria* of the full flow-structure system. In fact, taking $k_0 > 0$ allows one to prove that the entire flow-structure system is a gradient-type system.

In view of the above result—Theorem 0.3.6—with smooth initial data, we see that *any damping* imposed on the structure seems to eliminate the flutter in the subsonic case. (This is consistent with experiment for clamped plates [47].) From a physical point of view, the aerodynamic damping for a plate in subsonic flow is much lower than in supersonic flow, so the plate tends to oscillate in a neutrally stable state in the absence of aerodynamic damping or structural damping [48].

We also note that when the vK nonlinearity is replaced by Berger's nonlinearity, and the damping coefficient $k_0 > 0$ is taken sufficiently large, it has been shown [76] that all *finite energy* trajectories converge strongly to equilibria, and hence the flutter is eliminated. Though we have Theorem 0.3.6 for *smooth initial data*, the analogous result for the vK nonlinearity is still unknown. In [75] we consider finite energy data and the vK or Berger nonlinearity; when large static *and* large viscous damping is active in the model, we can show strong convergence of the entire flow-plate system to equilibria.

0.4 PISTON–THEORETIC PLATES

In the analysis of the full flow-plate model one considers the pressure acting on the surface of the plate, $p(x,t)$, as a function of the flow dynamics restricted to the interface. In classical piston theory [14, 42, 63] one replaces flow effects driving the plate dynamics (the *dynamic pressure across the plate*) with a nonlinear function of the so called *downwash*:

$$p(x,t) = p_0(x) + D\Big([u_t + Uu_x]\Big),$$

where, $D(\cdot)$ can be any number of (potentially nonlinear [2, 77, 28, 29]) expressions. However, the most common piston theoretic model (utilized for the sake of simplicity, and which is sufficiently accurate here [94, 95]) is linear. Thus, standard, linear piston theory (or law of plane sections [14, 42, 63]) replaces the *acceleration potential of the flow*, $\psi = tr[\phi_t + U\phi_x]$ with $-[u_t + Uu_x]$ in the RHS of the reduced plate equation (0.3.14). Thus we arrive at the following *piston-theoretic* model (with imposed viscous damping $k_0 \geq 0$):

$$\begin{cases} u_{tt} + k_0 u_t + \Delta^2 u + f(u) = p_0(x) - \beta[u_t + Uu_x] \quad \text{in } \Omega \times (0,T), \\ u = \partial_\nu u = 0 \quad \text{on } \partial\Omega \times (0,T). \end{cases} \tag{0.4.1}$$

Here $U > 0$ is the *unperturbed flow velocity* and $\beta > 0$ measures the strength of the interaction between the fluid and the structure (and typically depends on, for instance, free-stream density [95], or can be written as $\beta(U)$).

Remark 0.4.1. *Formally, we could arrive at the aforementioned, standard (linear) piston theory model by utilizing the reduction result in Theorem 0.3.4 and neglecting the delay potential in the case of large speeds* U.

We note that the effects of the flow are two-fold and conflicting: (i) a linear damping term with "good sign", fully-supported on the interior of the plate, and (ii) a non-dissipative (and destabilizing) term, scaled by the flow velocity parameter U. In some sense the interaction between these two terms is the key driver of interest in this plate problem (and in the general flutter phenomenon). Such a non-dissipative, piston-theoretic model has been intensively studied in the literature for different types of boundary conditions and resistive damping forces, see [30, 22, 31]. It is natural to ask what results (on long-time behavior) can be obtained for given $\beta > 0$ and $k_0 = 0$ (no imposed damping), and perhaps, what further can be said in the case of imposed damping $k_0 > 0$ (and perhaps large).

This model was intensively studied in the literature for different types of boundary conditions and frictional damping forces, see [31] and also [30] for related second order abstract models. In the recent work [59], an $\alpha = 0$ abstract piston-theoretic panel is thoroughly investigated in the presence of control-theoretic interior damping ($\beta > 0$ fixed, large k_0).

Theorem 0.4.1. *The equations* (0.4.1) *generates a dynamical system in the state space* $H_0^2(\Omega) \times L_2(\Omega)$*, possessing a compact global attractor of finite fractal dimension which is also more regular than the state space, in the sense of a gain of two derivatives in each component.*

Existence of compact and finite dimensional attractor for the structure yields an important reduction of an infinite dimensional, hyperbolic-like PDE system to a finite dimensional dynamics (ODE) system in the asymptotic limit. Such a result can not to be (a priori) predicted by a specific or numerical experiment. In addition, the rigorous mathematical validation of this fact is far from obvious, since the reduced plate system *is not a gradient dynamical systems*, and there is no natural compactness built in into the dynamics. In fact, one of the key roles is played by the nonlinearity, whose presence is critical for proving an existence of the absorbing set.

Many of the steps in [59] mirror those presented in the analysis of the delay plate in (0.3.16), and compact global attractors are obtained (exploiting the recent quasi-stability theory [26, 31]). A stronger class of result is then given in [59]: a (fractal) exponential attractor [26, 31]. The technique melds the quasi-stability analysis of [31, 26] with the *transitivity of exponential attraction* [53]. A key point in all of the analysis in [59] is the *size of the damping*; results on attractors are given in the form of: (i) "for any damping coefficient $k > 0$..." or (ii) "for $k > k_c$ sufficiently large". Finally, the obtained attractors are then studied numerically by investigating a large class of trajectories for various parameters in the 1-D case. (We note that in this work, the Berger plate nonlinearity and the vK nonlinearity are studied side by side, with *different* results based on the results of differentiating the nonlinearities.)

Remark 0.4.2. *Free plate boundary conditions have also been studied in [13] in the context of piston theory with boundary (and/or geometrically constrained) dissipation.*

Remark 0.4.3. *We note that, as mentioned above, some classical piston-theoretic analyses [12] retain polynomial terms in* $-[u_t+Uu_x]$ *up to the third order. Adding third powers of* $-[u_t+Uu_x]$ *will lead to a model with nonlinear, monotone (cubic-type) damping in* u_t *(see also some discussion in [30, p.161]). Such damping has already been considered in wave and plate models [31], and is known to induce stronger stability properties in mitigating effects of nonconservative terms in the equation. It would be interesting to study this situation in the context of the piston model under consideration.*

0.5 KUTTA-JOUKOWSKY FLOW WITH A CLAMPED-FREE PLATE

As one might surmise, allowing a portion of the panel boundary to be free is a scenario of great physical applicability (airfoils and cantilevers), as well as of great mathematical interest. Although the crux of the problem occurs in 1-D spatial dynamics, we describe the problem in full generality below with a 2-D structure and 3-D flow.

A general configuration that supports multiple models of interest is now described; the principal distinction of this configuration is the existence of large portion of the plate boundary $\Gamma_2 \equiv \partial\Omega_2$ that is free. Taking a clamped-free plate boundary allows one to consider (in generality) the situation of an *airfoil*, as well as a *mostly clamped panel*, or a flap/flag in so called *axial flow*. When one considers a free plate coupled to fluid flow (unlike the clamped case) the key modeling issues correspond to (i) the validity of the structural nonlinearity and (ii) the aerodynamic theory near the free plate boundary.

In these configurations, the natural *flow* interface boundary condition is the aforementioned *Kutta-Joukowsky flow condition* KJC [54]. Simply put, the KJC corresponds to a zero pressure jump off the wing and at the trailing edge [6, 49]. The KJC has been implemented in numerical aeroelasticity as mechanism for "...removal of a velocity singularity at some distinguished point on a body in unsteady flow" [41]. From an engineering standpoint, the KJC is required to provide a unique solution for the potential flow model for a lifting surface, and gives results in correspondence with experiment, i.e.

the pressure difference across the trailing edge is zero and the pressure difference at the leading edge is a near maximum. Studies of viscous flow models in the limit of very high Reynolds numbers lend support to the KJC.

The configuration below represents an attempt to model oscillations of a plate which is *mostly free*. The dynamic nature of the flow conditions corresponds to the fact that the interaction of the plate and flow is no longer static at the free edge. In this case we take the free-clamped plate boundary conditions, and the mixed flow boundary conditions:

$$\begin{cases} u = \partial_\nu u = 0, & \text{on } \partial\Omega_1 \times (0,T) \\ B_1 u = 0, B_2 u = 0, & \text{on } \partial\Omega_2 \times (0,T) \\ \partial_z \phi = (\partial_t + U\partial_x)u, & \text{on } \Omega \times (0,T) \\ \partial_z \phi = 0, & \text{on } \Theta_1 \times (0,T) \\ \psi = \phi_t + U\phi_x = 0, & \text{on } \Theta_2 \times (0,T) \end{cases} \tag{0.5.1}$$

where $\partial\Omega_i$, $i = 1, 2$ are complementary parts of the boundary $\partial\Omega$, and B_1, B_2 represent moments and shear forces, given by [66] (and earlier here) and, Θ_i extend in the natural way into the remainder of the $x - y$ plane)

Remark 0.5.1. *The configuration above arises in the study of airfoils. In this case, we refer to* normal flow *(along the* x*-axis). Another interesting configuration is referred to as* axial flow*, which takes the flow occurring along* y *axis and reduces the structural theory to a 1-D beam. Physically, the orientation of the flow can have a dramatic effect on the occurrence and magnitude of the oscillations associated with the flow-structure coupling. This manifests itself, specifically, in the choice of nonlinearity modeling the plate (or possibly beam) equation.*

From the mathematical point of view, the difficulty arises at the level of linear theory when one attempts to construct "smooth" solutions of the corresponding evolution. The typical procedure of extending plate solutions by zero outside Ω leads to the jump in the Neumann boundary conditions imposed on the flow. In order to contend with this issue, regularization procedures are needed in order obtain smooth approximations of the original solutions. While some regularizations have been already introduced in [34], more work is needed in order to demonstrate the effectiveness of this "smoothing" for the large array of problems described in this work.

Remark 0.5.2. *With regard to the above, we note that when we are in the case of* normal flow *(as opposed to* axial flow *(as in Remark 0.5.1) the scalar von Karman model is largely still viable [49] (based on extensibility—the dominant nonlinear effect is that of stretching on bending [66]). Additionally, any configuration where the free portion of the plate boundary conditions is "small" with respect to the clamped portion will satisfy the hypotheses for the theory of large deflections, and hence, von Karman theory is applicable. However, in the context of* axial flow*, a complete reformation of the appropriate* inextensible *theory is needed. See Section* **??** *below.*

0.5.1 Mathematical Treatment of (KJC)

Arguably, the Kutta-Joukowsky boundary conditions for the flow (0.2.6) are the *most important* when modeling an airfoil immersed in a flow of gas [4, 5]. Not surprisingly, these boundary conditions are also the most challenging from mathematical stand point. This was recently brought to the fore in an extensive treatise [5]. Various aspects of the problem—in both subsonic and supersonic regime—have been discussed in [5] in the context of mostly one dimensional structures.

The aim of this section is to address the mathematical problem posed by (KJC), by putting them within the framework of modern harmonic analysis. We consider the "toy" problem of (KJC) conditions coupled with clamped plate boundary conditions below. Indeed, the recently studies [35, 74] show how the flow condition (KJC) interacts with the clamped plate in subsonic flows *in order to develop a suitable abstract theory* for this particular flow condition. The resulting papers [35, 74] give well-posedness of this fluid-structure interaction configuration. Though the analysis is subsonic for (KJC), utilizing the flow energy from the supersonic panel (as above in Theorem 0.3.1) is effective in the abstract setup of this problem. In fact, even in the subsonic case, the analysis of semigroup generation proceeds through the technicalities developed earlier for supersonic case [34]. The key distinction from the analysis of the clamped flow-plate interaction (owing to the dynamic nature of the boundary conditions) is that a *dynamic flow-to-Neumann map* must be utilized to incorporate the boundary conditions into the abstract setup. The regularity properties of this map are critical in admitting techniques from abstract boundary control [73], and are determined from the Zaremba elliptic problem [85]. The necessary trace regularity hinges upon the invertibility of an operator which is analogous (in two dimensions) to the finite Hilbert transform [5, 6, 93]. And this is a critical additional element of the challenging harmonic analysis brought about by the (KJC). When the problem is reduced in dimensionality to a beam structural model, this property can be demonstrated and our analysis has parallels with that in [5, 6] (and older references which are featured therein). Specifically, one must invert the finite Hilbert transform in $L_p(\Omega)$ for $p \neq 2$; in higher dimensions, this brings about nontrivial (open) problems in harmonic analysis and the theory of singular integrals.

From the mathematical point of view the difficulty lies, again, at the level of the linear theory. In order to deal with the effects of the *unbounded* traces $tr[\psi]$ in the energy relation microlocal calculus is necessary. This has been successfully accomplished in [34] where clamped boundary conditions in the supersonic case were considered. However, in the case of (KJC) there is an additional difficulty that involves "invertibility" of *finite* Hilbert (resp. Riesz) transforms. This latter property is known to fail within L_2 framework, thus it is necessary to build the L_p theory, $p \neq 2$. This was for the first time observed in [6] and successfully resolved in the one dimensional case. However, any progress to higher dimensions depends on the validity of the corresponding harmonic analysis result developed for finite Riesz transforms.

The full system with (0.2.6) can be written in terms of aeroelastic potential variable as:

$$\begin{cases} u_{tt}+\Delta^2 u + f(u) = p_0 + tr[\psi] & \text{in } \Omega \times (0,T), \\ u(0)=u_0;\;\; u_t(0)=u_1, & \\ u=\partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ (\partial_t + U\partial_x)\phi = \psi & \text{in } \mathbb{R}^3_+ \times (0,T), \\ (\partial_t + U\partial_x)\psi = \Delta\phi & \text{in } \mathbb{R}^3_+ \times (0,T), \\ \partial_\nu \phi = -(\partial_t + U\partial_x)u & \text{on } \Omega \times (0,T), \\ (\partial_t + U\partial_x)\phi = 0 & \text{on } \mathbb{R}^2 \backslash \Omega \times (0,T), \end{cases} \tag{0.5.2}$$

with the given initial data $(u_0, u_1, \phi_0, \psi_0)$. We make use of the flow acceleration multiplier $(\partial_t + U\partial_x)\phi \equiv \psi$ for the *subsonic flow*, taken with (KJC). Thus for the flow dynamics, instead of $(\phi;\phi_t)$ we again have the state variables $(\phi;\psi)$ (as in the supersonic panel case) which then leads to a *non dissipative* energy balance.

Our result is formulated under the following regularity condition (to be discussed later).

Condition 1 (**Flow Trace Regularity).** *Assume that* $\phi(\mathbf{x},t)$ *satisfies* (0.2.4) *and the Kutta-Joukowsky condition (KJC), and* $\partial_t tr[\phi],\ \partial_x tr[\phi] \in L_2(0,T;H^{-1/2-\epsilon}(\mathbb{R}^2))\quad \forall T>0$. *In addition,* $\psi = \phi_t + U\phi_x$ *satisfies the estimate*

$$\int_0^T \|tr[\psi](t)\|^2_{H^{-1/2-\epsilon}(\mathbb{R}^2)} dt \le C_T \left(E_{fl}(0) + \int_0^T \|\partial_\nu \phi(t)\|^2_\Omega dt \right) \tag{0.5.3}$$

We note that the regularity required by Condition 1 is exactly the one that will make energy relation meaningful. To wit: $u_x(t) \in H^1_0(\Omega)$ and $tr[\psi] \in L_2(0,T;H^{-\theta}),\ \theta > 1/2$ defines the correct duality pairing in the tangential direction on $\mathbb{R}^2$. It is also at this point where we use the fact that u satisfies $u=0$ on $\partial\Omega$. Thus simply supported and clamped boundary conditions imposed on the structure fully cooperate with this regularity. The principal result of [74] (see also [35]) reads as follows:

Theorem 0.5.1. *With reference to the model (0.5.2), with* $0 \le U < 1$*: Assuming the trace regularity Condition 1 holds for the aeroelastic potential* $\psi = (\phi_t + U\phi_x)$*, there exists a unique finite energy solution which exists on any* $[0,T]$*. This is to say, for any* $T>0$*,* $(\phi,\phi_t,u,u_t) \in C(0,T;Y)$ *for all initial data* $(\phi_0,\phi_1,u_0,u_1) \in Y$*. This solution depends continuously on the initial data.*

The proof of Theorem 0.5.1 follows the technology developed for the supersonic case in [34] and is given in [74]. In view of Theorem 0.5.1 the final conclusion on generation of nonlinear semigroup is pending upon verification of the flow regularity Condition. While the complete solution to this question is still unavailable, and pending further progress in the theory of finite Riesz transforms, we can provide positive answer in the case when Ω is one dimensional. In fact, this positive assertion also follows a-posteriori from direct calculations given in [4], and based on the analysis of finite Hilbert transforms. The arguments given [74] are independent and more general, with potential adaptability to multidimensional cases. The corresponding result is formulated below.

Theorem 0.5.2. *Assume $\Omega = (-1, 1)$ (suppressing the span variable y). Then the flow trace regularity in Condition 1 holds for ϕ. In this case the semiflow defined by (0.5.2) taken with (KJC) generates a continuous semigroup.*

Remark 0.5.3. *As discussed above, the generation of semigroups for an arbitrary three dimensional flow is subjected to the validity of the trace Condition 1. While it is believed that this property should be generically true, at the present stage this appears to be an open question in the analysis of singular integrals and depends critically on the geometry of Ω in two dimensions.*

Remark 0.5.4. *We note that the regularity of the aeroelastic potential required by Condition 1 in one dimension, follows from the analysis in [5, 6], where the author proves that aeroelastic potential $\psi \in L_2(0, T, L_q(\Omega))$ for $q < 4/3$. Since for $p > 4$ there exists $\epsilon > 0$ such that*

$$H^{1/2+\epsilon}(\Omega) \subset L_p(\Omega), \ \ p > 4, \ \ \textit{dim} \ \ \Omega \leq 2,$$

and one then obtains that $L_q(\Omega) \subset H^{-1/2-\epsilon}(\Omega)$ with $q < 4/3$.

Remark 0.5.5. *The loss of $1/2$ derivative in the characteristic region was already observed and used in the analysis of regularity of the aeroelastic potential for the Neumann problem with supersonic velocities U [34]. However, in the case of (KJC) there is an additional loss, due to the necessity of inverting finite Hilbert transform which forces to work with L_p theory for $p < 2$. This is due to the fact that finite Hilbert transform is invertible on L_p, $p < 2$, rather than $p = 2$.*

0.6 AXIAL FLOW

In the case of *axial flow*, mentioned in Remark 0.5.1, a beam or plate is clamped on the *leading edge* and free elsewhere is described. To provide a clear picture of the dynamics, consider the figure at the beginning of Section **??** with the over-body flow occurring in the y-*direction*, as opposed to being in the direction of the plate's chord—x-direction. This axial configuration, owing to LCO response to low flow velocities [5], is that which has been considered from the point of view of energy harvesting. This would be accomplished by configuring an axially oriented flap on land or air vehicles with a piezo device which could generate current as it flutters [52]. From [90] the appropriate structural equations of motion here correspond to those of a thin pipe conveying fluid, and many aspects of such dynamics mirror those we are investigating [81, 90]. Expressions for the kinetic and potential energy of the beam and plate are given in [89, 91]. Note that structural nonlinearities occur in both the inertia (kinetic energy) and stiffness (potential energy) terms for the structural equations of motion.

The recent theoretical and experimental work of Tang and Dowell [89, 91] has been encouraging. In this work a new nonlinear structural model has been used based upon the inextensibility assumption and the comparison between theory and experiment for the LCO response has been much improved over earlier results [90]. The study of a linear aerodynamic model, combined with the new nonlinear structural model, is worthy of more rigorous mathematical attention.

In the case of *axial flow*, mentioned in Remark 0.5.1, a beam or plate is clamped on the *leading edge* and free elsewhere, with the flow directed along the longer axis of the structure. To provide a clear picture of the dynamics, consider the figure below (0.5.1) with the over-body flow occurring in the y-*direction*. This axial configuration, owing to LCO response to low flow velocities [5], is that which has been considered from the point of view of piezoelectric energy harvesting [52] (as well as snoring and sleep apnea [61]). In this case, it is typical to reduce the structural dynamics to that of a beam. The key point in modeling is that such a beam has a predominant nonlinear effect due to an inextensibility constraint, rather than a typical von Karman type extensible theory [51, 91].

From [90] the appropriate structural equations of motion here correspond to those of a thin pipe conveying fluid, and many aspects of such dynamics mirror those we have discussed above [90]. Expressions for the kinetic and potential energy of the beam and plate are given in [51]. Structural nonlinearities will occur in both the inertia (kinetic energy) and stiffness (potential energy) terms for the structural equations of motion. The recent theoretical and experimental [89, 91] has validated the modeling done based upon the inextensibility assumption and the comparison between theory and experiment for the limit cycle response has been much improved over earlier results [90]. The study of a linear aerodynamic model, combined with the new nonlinear structural model, is worthy of more rigorous mathematical attention.

Below, we will discuss three approaches to structural modeling in this axial flow scenario, of increasing complexity.

The most important distinction to make in a cantilever model is the handling of *the extensibility property* [51, 66, 86]. In the case of *extensible* beams, transverse deflection *necessarily* leads to local stretching, which is the principal contributor for the elastic restoring force; the *very recent* engineering literature indicates that dominant nonlinear effects should instead be *inextensible* [51, 79, 86, 91].

The property of inextensibility is characterized as local arc length preservation *throughout deflection* (with u out-of-plane, and w in-plane, Lagrangian deflections of the beam):

$$1 \equiv \left[(1+w_x)^2 + u_x^2\right]^{1/2} \quad \text{linearized to} \quad 0 \equiv w_x + (1/2)[u_x]^2. \tag{0.6.1}$$

This linearization is referred to as the *inextensibility constraint*. Note that if in-plane u dynamics are prohibited at $x = L$, then the model must be extensible. elasticity [66], a quadratic strain-displacement law is invoked. The coefficient $\alpha > 0$ is required to be positive in (0.6.2). The terms $D_1,\ D_2 > 0$ are elastic coefficients related by $D_2 = \alpha D_1$, though they are often considered independent. We exclude damping terms for now.

$$\begin{cases} w_{tt} - D_1\left[w_x + \frac{1}{2}(u_x)^2\right]_x = 0; \qquad (1-\alpha\partial_x^2)u_{tt} + D_2\partial_x^4 u - D_1\left[u_x(w_x + \frac{1}{2}u_x^2)\right]_x = p(x,t) \\ u(t=0) = u_0;\ \ u_t(t=0) = u_1;\ \ w(t=0) = w_0;\ \ w_t(t=0) = w_1 \\ w(0) = 0; \qquad D_1\left[w_x(L) + \frac{1}{2}u_x^2(L)\right] = 0; \qquad u(0) = u_x(0) = 0; \qquad D_2 u_{xx}(L) = 0; \\ -\alpha\partial_x u_{tt}(L) + D_2\partial_x^3 u(L) = 0. \end{cases} \tag{0.6.2}$$

As above, one can arrive at a scalar equation by assuming in-plane accelerations are negligible, i.e., $w_{tt} \approx 0$. Integrating the w equation over the beam, and imposing that $w(L)$ is “small"—a contentious assumption—we obtain the toy model studied in [60]:

$$\begin{cases} (1-\alpha\partial_x^2)u_{tt} + D\partial_x^4 u + (b - \|u_x\|^2)u_{xx} = p(x,t) \\ u(t=0) = u_0;\ \ u_t(t=0) = u_1;\ \ u(0) = u_x(0) = 0 \\ \partial_x^2 u(L) = 0; \quad -\alpha\partial_x u_{tt}(L) + D\partial_x^3 u(L) + (b - \|u_x\|^2)u_x(L) = 0. \end{cases} \tag{0.6.3}$$

Above, we have permitted all values $\alpha \geq 0$, a point discussed in [60]; $b \in \mathbb{R}$ represents equilibrium in-plane tension or compression. This models appears to be a standard nonlinear beam equation of Krieger type, *with a nonstandard (nonlinear) boundary condition*. The physical meaning of this boundary condition is vague, but such models have been previously studied (see [60] and references therein). From the point of view here, the analysis of this model is preliminary to that of the more sophisticated (0.6.2) and (0.6.4).

In contrast to the extensible theory above, one may invoke the inextensibility condition (0.6.1) at the variational level to derive the nonlinear potential and kinetic energies for u and w [51, 79, 86]. Doing so and invoking Hamilton's principle, one obtains “new" equations of motion for cantilever large deflections:

$$u_{tt} - D\partial_x\left[(u_{xx})^2 u_x\right] + D\partial_{xx}\left[u_{xx}\big(1 + (u_x)^2\big)\right] + \partial_x\left[u_x\int_x^L w_{tt}(\xi)d\xi\right] = p(x,t); \tag{0.6.4}$$

$$w(x) = -\frac{1}{2}\int_0^x [u_x(\xi)]^2 d\xi. \tag{0.6.5}$$

Unlike (0.6.3), one recovers the standard clamped-free boundary conditions for u. Since u and w are related through the constraint, we may express the dynamics entirely in u via

$$\partial_x\left[u_x\int_x^L w_{tt}(\xi)d\xi\right] = \tag{0.6.6}$$

$$-u_x\Big[\int_0^x \Big((u_{\xi t})^2 + u_{\xi tt}u_\xi\Big)d\xi\Big] + u_{xx}\int_x^L \Big[\int_0^{\xi_2}\big((u_{\xi t})^2)u_{\xi tt}u_\xi\big)d\xi_1\Big]d\xi_2. \tag{0.6.7}$$

Enforcing inextensibility contributes both *nonlinear inertial and stiffness* terms not present in the equations of extensible elasticity. Additionally, the inertial terms are nonlocal and prevent the dynamics from being written as a traditional second order evolution in (u, w).

In the very recent manuscript [60], the free boundary condition (for a beam) is explored in the context of energy harvesting. A piston-theoretic cantilever model is investigated from the point of view described above. A key distinction is uncovered in the presence of a free boundary condition: rotational inertia is indispensable for von Karman type nonlinearities. With $\alpha > 0$, compact global attractors are constructed for $f = f_B$ (albeit in 1-D), and extensive numerical simulations are performed. Virtually no mathematical analyses have been performed (in this aeroelastic context) with the structural models (0.6.2) and (0.6.4).

We conclude by noting a key representative model in the case of axial flow, omitting initial conditions, in the linear flow-cantilever configuration, implementing the KJC at the free end:

$$\begin{cases} u_{tt} + D\partial_x^4 u = [(\partial_t + U\partial_x)\phi]\big|_{(0,L)} & \text{in } (0,L) \\ u = u_x = 0 & \text{in } \{0\} \times (0,T) \\ u_{xx}(L) = 0; \;\; u_{xxx}(L) = 0 \end{cases} \qquad \begin{cases} (\partial_t + U\partial_x)^2\phi = \Delta_{x,z}\phi & \text{in } \mathbb{R}^2_+ \\ \partial_z\phi = \begin{cases} (\partial_t + U\partial_x)u & \text{on } (0,L) \\ 0 & (-\infty, 0] \end{cases} \\ (\phi_t + U\phi_x) = 0 & \text{on } [L,\infty) \end{cases}$$

These dynamics can be addressed by the theory described in this treatment, though in the nonlinear, post-flutter regime there are many modeling problems, which do not necessarily translate mathematically in this description: viscous effects, non-zero vorticity, large deflections of the flow-structure interface, etc.

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REFERENCES

[1] A. Babin and M. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.

[2] H. Ashley and G. Zartarian, Piston theory: A new aerodynamic tool for the aeroelastician, *J. Aero. Sci.*, 23(12), 1956, 1109–1118.

[3] G. Avalos, P.G. Geredeli, and J.T. Webster, Semigroup Well-posedness of A Linearized, Compressible Fluid with An Elastic Boundary, *Discr. Cont. Dyn. Sys.: B*, to appear, 2018.

[4] A.V. Balakrishnan and K.W. Iliff, A continuum aeroelastic model for inviscid subsonic wing flutter, *J. of Aerospace Engineering*, 20.3 (2007), pp. 152–164.

[5] A.V. Balakrishnan, Aeroelasticity—Continuum Theory, *Springer-Verlag*, 2012.

[6] A.V. Balakrishnan, Nonlinear aeroelasticity, continuum theory, flutter/divergence speed, plate wing model, Free and Moving Boundaries, Lecture Notes, *Pure. A. Math.*, 252, *Chapman & Hall*, FL, 2007, pp. 223–244.

[7] A.V. Balakrishnan and M.A. Shubov, Asymptotic behaviour of the aeroelastic modes for an aircraft wing model in a subsonic air flow, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460 (2004), pp.1057–1091.

[8] A.V. Balakrishnan, M. Shubov, and C. Peterson, Spectral analysis of coupled Euler-Bernoulli and Timoshenko beam model, *ZAMM Z. Angew. Math. Mech.*, 84 (2004), no. 5, pp. 291–ÃÂ313.

[9] A.V. Balakrishnan and M.A. Shubov, Asymptotic and spectral properties of operator-valued functions generated by aircraft wing model, *Math. Methods Appl. Sci.*, 27 (2004), no. 3, pp. 329–Â362.

[10] O.O. Bendiken and G.A. Davis, Nonlinear traveling wave flutter in panels with transonic flow. *AIAA Paper 95-1496*. 1995.

[11] O.O. Bendiksen, G. Seber, Fluid-structure interactions with both structural and fluid nonlinearities, *J. Sound and Vibration*, 315 (2008), pp. 664–684.

[12] R. Bisplinghoff, H. Ashley, Principles of Aeroelasticity. *Wiley*, 1962; also *Dover*, New York, 1975.

[13] L. Bociu and D. Toundykov. Attractors for non-dissipative irrotational von Karman plates with boundary damping. *J. Differential Equations*, 253 (2012), pp 3568–3609.

[14] V.V. Bolotin, Nonconservative problems of elastic stability, *Pergamon Press*, Oxford, 1963.

[15] A. Boutet de Monvel and I. Chueshov, The problem of interaction of von Karman plate with subsonic flow gas, *Math. Methods in Appl. Sc.*, 22 (1999), pp. 801–810.

[16] L. Boutet de Monvel and I. Chueshov, Non-linear oscillations of a plate in a flow of gas, C.R. Acad. Sci. Paris, Ser.I, 322 (1996), pp. 1001–1006.

[17] L. Boutet de Monvel and I. Chueshov, Oscillation of von Karman's plate in a potential flow of gas, *Izvestiya RAN: Ser. Mat.* 63 (1999), pp. 219–244.

[18] L. Boutet de Monvel, I. Chueshov, and A. Rezounenko, Long-time behaviour of strong solutions of retarded nonlinear PDEs, *Comm. PDEs*, 22 (1997), pp. 1453–1474.

[19] A.J. Chorin, J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, 3rd ed., Springer, New York, 1993.

[20] I. Chueshov, Strong solutions and the attractors for von Karman equations, *Math USSR-Sb*, 69 (1991), 25-36.

[21] I.D. Chueshov, Asymptotic behavior of the solutions of a problem on the aeroelastic oscillations of a shell in hypersonic Limit, *Teor. Funktsii Funktsional. Anal. i Prilozhen.* 51, (1989), 137–141 (in Russian); tranaslation in *J. Mathematical Sci.*, 52 (1990), pp. 3545–3548.

[22] I. Chueshov, Introduction to the Theory of Infinite Dimensional Dissipative Systems, *Acta*, Kharkov, 1999, in Russian; English translation: *Acta*, Kharkov, 2002;
see also http://www.emis.de/monographs/Chueshov/

[23] I. Chueshov, Dynamics of von Karman plate in a potential flow of gas: rigorous results and unsolved problems, Proceedings of the 16th IMACS World Congress, Lausanne (Switzerland), 2000, pp. 1–6.

[24] I. Chueshov, Dynamics of a nonlinear elastic plate interacting with a linearized compressible viscous fluid, *Nonlinear Anal., TMA*, 95 (2014), 650–665.

[25] I. Chueshov, Interaction of an elastic plate with a linearized inviscid incompressible fluid, *Commun. Pure Appl. Anal.*, 13 (2014), 1759–1778.

[26] I.Chueshov, Dynamics of Quasi-Stable Dissipative systems, Springer, New York, 2015.

[27] I. Chueshov, Remark on an elastic plate interacting with a gas in a semi-infinite tube: periodic solutions, preprint, 2015.

[28] I. Chueshov, E. H. Dowell, I. Lasiecka, and J.T. Webster, Mathematical aeroelasticity: A survey, *Mathematics in Engineering, Science & Aerospace (MESA)*, 7, 2016, 5–29.

[29] I. Chueshov, E.H. Dowell, I. Lasiecka, and J.T. Webster, Nonlinear Elastic Plate in a Flow of Gas: Recent Results and Conjectures, *App. Mathem. Optim.*, 73(3), 2016, 475–500.

[30] I. Chueshov and I. Lasiecka, Long-time behavior of second-order evolutions with nonlinear damping, *Memoires of AMS*, v. 195, 2008.

[31] I. Chueshov and I. Lasiecka, Von Karman Evolution Equations, Wellposedness and Long Time Behavior, Monographs, *Springer-Verlag*, 2010.

[32] I. Chueshov and I. Lasiecka, Generation of a semigroup and hidden regularity in nonlinear subsonic flow-structure interactions with absorbing boundary conditions. *Jour. Abstr. Differ. Equ. Appl.* 3 (2012), pp. 1–27.

[33] I. Chueshov, I. Lasiecka and J. T. Webster, Attractors for delayed, non-rotational von Karman plates with applications to flow-structure interactions without any damping, *Comm. in PDE*, **39** (2014), pp. 1965–1997.

[34] I. Chueshov, I. Lasiecka and J. T. Webster, Evolution semigroups for supersonic flow-plate interactions, *J. of Diff. Eqs.*, 254 (2013), pp. 1741–1773.

[35] I. Chueshov, I. Lasiecka and J. T. Webster, Flow-plate interactions: Well-posedness and long-time behavior, *Discrete Contin. Dyn. Syst. Ser. S, Special Volume: New Developments in Mathematical Theory of Fluid Mechanics*, **7** (2014), pp. 925–965.

[36] I. Chueshov and I. Ryzhkova, A global attractor for a fluid-plate interaction model, Comm. Pure Appl. Anal. **12** (2013), 1635–1656.

[37] I. Chueshov and I. Ryzhkova, Unsteady interaction of a viscous fluid with an elastic shell modeled by full von Karman equations, J. Diff. Eqs. **254** (2013), 1833–1862.

[38] I. Chueshov and I. Ryzhkova, Well-posedness and long time behavior for a class of fluid-plate interaction models, in: *IFIP Advances in Information and Communication Technology*, vol.391, (25th IFIP TC7 Conference, Berlin, Sept.2011), D. Hömberg and F. Tröltzsch (Eds.), Springer, Berlin, 2013, pp.328–337.

[39] I. Chueshov and I. Ryzhkova, On interaction of an elastic wall with a Poiseuille type flow. Ukrainian Math. J., **65** (2013), 158–177.

[40] P. Ciarlet and P. Rabier, Les Equations de Von Karman, *Springer*, 1980.

[41] D.G. Crighton, The Kutta condition in unsteady flow, *Ann. Rev. Fluid Mech.*, 17, 1985, 411–445.

[42] E. Dowell, Nonlinear Oscillations of a Fluttering Plate, I and II, AIAA J., 4, (1966), pp. 1267–1275; and 5, (1967), pp. 1857–1862.

[43] E. Dowell, Panel flutter–A review of the aeroelastic stability of plates and shells. AIAA Journal, 8 (1970), pp. 385–399.

[44] E. Dowell, O. Bendiksen, J. Edwards, T. Strganac, Transonic Nonlinear Aeroelasticity, Enc. of Aerospace Engin., *Wiley*, 2010.

[45] E.H. Dowell and O. Bendiksen, Panel Flutter, Enc. of Aerospace Engin., *Wiley*, 2010.

[46] E. Dowell and D.B. Bliss, New Look at Unsteady Supersonic Potential Flow Aerodynamics and Piston Theory, *AIAA J.*, 51.9 (2013), pp. 2278–2281.

[47] E.H. Dowell, Some recent advances in nonlinear aeroelasticity: FSI in the 21st century, Proceedings of the 51st AIAA/ASME/ASCE/AHS/ASC Structures, *AIAA*, 3137, 2010.

[48] E. Dowell, Aeroelasticity of Plates and Shells, *Nordhoff*, Leyden, 1975.

[49] E. Dowell, A Modern Course in Aeroelasticity, *Kluwer Academic Publishers*, 2004.

[50] E.H. Dowell, Flutter of a buckled plate as an example of chaotic motion of a deterministic autonomous system, *J. Sound and Vibration*, 85 (1982), pp. 333–344.

[51] E. Dowell and K. McHugh. Equations of Motion for an Inextensible Beam Undergoing Large Deflections, *J. Appl. Mech.*, 83(5), 2016.

[52] E. Erturk and D. Inman, Piezoelectric Energy Harvesting, *John Wiley and Sons*, United Kingdom, 2011.

[53] P. Fabrie, C. Galusinski, A. Miranville, and S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation. *Discr. Contin. Dyn. Sys.*, 10(1/2), pp. 211–238, 2004.

[54] W. Frederiks, H.C.J. Hilbering, and J.A. Sparenberg, On the Kutta condition for the flow along a semi-infinite elastic plate, *J. Engin. Math.*, 20, 1986, 27–50.

[55] D.H. Hodges, G.A. Pierce, Introduction to Structural Dynamics and Aeroelasticity, *Cambridge Univ. Press*, 2002.

[56] S.C. Gibbs and E.H. Dowell, Membrane Paradox for Solar Sails, *AIAA J.*, 52 (2014), pp. 2904–2906.

[57] S.C. Gibbs, I. Wang, E. Dowell, Theory and experiment for flutter of a rectangular plate with a fixed leading edge in three-dimensional axial air flow, J. Fluids Structures, 34 (2012), pp. 68–83.

[58] S.C. Gibbs, I. Wang, E. Dowell, Stability of Rectangular Plates in Subsonic Flow with Various Boundary Conditions, *J. Aircraft*, 52 (2015), pp. 429–451.

[59] J.S. Howell, I. Lasiecka, and J.T. Webster, Quasi-stability and exponential attractors for a non-gradient system—Applications to piston-theoretic plates with internal damping, *Evol. Equns. Contr. Theory*, 5(4), 2016, 567–603.

[60] J. Howell, D. Toundykov, and J.T. Webster, A Cantilevered Extensible Beam in Axial Flow: Semigroup Solutions and Post-flutter Regimes, SIAM J. Math. Analys., to appear, 2018.

[61] L. Huang, Flutter of cantilevered plates in axial flow, *J. Fluids Struct.*, 9, 1995, 127–147.

[62] R. A. Ibrahim, Liquid Sloshing Dynamics: Theory and Applications, *Cambridge University Press*, 2005.

[63] A.A. Il'ushin, The plane sections law in aerodynamics of large supersonic speeds, *Prikladnaya Matem. Mech.*, 20 (1956), no.6, pp. 733–755 (in Russian).

[64] H. Koch and I. Lasiecka, Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity - full von Karman systems. *Evolution Equations, Semigroup and Functional Analysis*, vol 50, Birkhauser, 2002, 197–212.

[65] A. Kornecki, E.H. Dowell, and J. O'Brien, On the aeroelastic instability of two-dimensional panes in uniform incompressible flow, 47, 1976, 163–178.

[66] J. Lagnese, Boundary Stabilization of Thin Plates, *SIAM*, 1989.

[67] J.E. Lagnese and G. Leugering, Uniform stabilization of a nonlinear beam by nonlinear boundary feedback. *J. Diff. Equns.*, 91(2), 1991, 355–388.

[68] Lakshmikantham, V.; Leela, S. Differential and integral inequalities: Theory and applications. Vol. II: Functional, partial, abstract, and complex differential equations. Mathematics in Science and Engineering, Vol. 55-II. Academic Press, New York-London, 1969. ix+319 pp.

[69] Lakshmikantham, V.; Bhaskar, T. Gnana; Vasundhara Devi, J. Theory of set differential equations in metric spaces. Cambridge Scientific Publishers, Cambridge, 2006. x+204 pp.

[70] Lakshmikantham, V.; Leela, S.; Vasundhara Devi, J. Stability theory for set differential equations. Special issues in honor of Professor Siljak's 70th birthday. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 11 (2004), no. 2-3, 181Ð189.

[71] I. Lasiecka, Mathematical Control Theory of Coupled PDE's, CMBS-NSF Lecture Notes, *SIAM*, 2002.

[72] I. Lasiecka, Uniform stabilizability of a full von Karman system with nonlinear boundary feedback, *SIAM J. Control*, 36, 1998, 1376–1422.

[73] I. Lasiecka and R. Triggiani, Control Theory for Partial Differential Equations, vol. I, II, *Cambridge University Press*, 2000.

[74] I. Lasiecka and J.T. Webster, Kutta-Joukowski flow conditions in flow-plate interactions: subsonic case, *Nonlinear Anal. B*, 7 (2014), pp. 171–191.

[75] I. Lasiecka and J. T. Webster, Eliminating flutter for clamped von Karman plates immersed in subsonic flows, *Comm. Pure Appl. Math.*, 13 (2014), pp.1935–1969, Updated version (May, 2015): http://arxiv.org/abs/1409.3308.

[76] I. Lasiecka and J. T. Webster, Feedback stabilization of a fluttering panel in an inviscid subsonic potential flow, submitted, June 2015, Preprint: http://arxiv.org/abs/1506.05704.

[77] M.J. Lighthill, Oscillating airfoils at high mach number, *J. Aero. Sci.*, 20(6), 1953, 402–406.

[78] E. Livne, Future of Airplane Aeroelasticity, J. of Aircraft, 40 (2003), pp. 1066–1092.

[79] J.L. Lopez, M.P. Païdoussis, and C. Semler, Linear and nonlinear dynamics of cantilevered cylinders in axial flow. Part 2: The equations of motion, *J. Fluids Struct.*, 16, 2002, 715–737.

[80] S. Miyatake, Mixed problem for hyperbolic equation of second order, *J. Math. Kyoto Univ.*, 13 (1973), pp. 435–487.

[81] Paidoussis, M.P. Paidoussis, Fluid-Structure Interactions: Slender Structures and Axial Flow, 1, *Academic Press*, San Diego, 1998.

[82] I. Ryzhkova, Stabilization of a von Karman plate in the presence of thermal effects in a subsonic potential flow of gas, *J. Math. Anal. and Appl.*, 294 (2004), pp. 462–481.

[83] I. Ryzhkova, Dynamics of a thermoelastic von Karman plate in a subsonic gas flow, *Zeitschrift Ang. Math. Phys.*, 58 (2007), pp. 246–261.

[84] R. Sakamoto, Mixed problems for hyperbolic equations, *J. Math. Kyoto Univ*, 2 (1970), pp. 349–373.

[85] G. Savare, Regularity and perturbations results for mixed second order elliptic problems *Comm. PDEs*, 22 (1997), nos. 5–6, pp. 869–900.

[86] C. Semler, G.X. Li, and M.P. Païdoussis, The non-linear equations of motion of pipes conveying fluid, *J. Sound Vibrat.*, 169, 1994, 577–599.

[87] M. Shubov, Riesz basis property of mode shapes for aircraft wing model (subsonic case). Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 462 (2006), 607–646.

[88] M. Shubov, Solvability of reduced Possio integral equation in theoretical aeroelasticity, *Adv. Differential Equations*, 15 (2010), pp. 801–828.

[89] D. Tang, S.C. Gibbs, and E.H. Dowell, Nonlinear Aeroelastic Analysis with Inextensible Plate Theory Including Correlation with Experiment, *AIAA J.*, 53 (2015), pp. 1299–1308.

[90] D.M. Tang, H. Yamamoto, E.H. Dowell, Flutter and limit cycle oscillations of two dimensional panels in three-dimensional axial flow, *J. Fluids and Structures*, 17 (2003), pp. 225–242.

[91] D. Tang, M. Zhao, and E.H. Dowell, Inextensible Beam and Plate Theory: Computational Analysis and Comparison with Experiment, *J. Appl. Mech.*, 81.6 (2014), 061009.

[92] D. Tataru and M. Tucsnak, On the Cauchy problem for the full von Karman system, *NODEA-Nonlinear Diff.*, 4, 1997, 325–340.

[93] F.G. Tricomi, Integral Equations, *Interscience Publishers Inc.*, New York, 1957.

[94] V. Vedeneev, Effect of damping on flutter of simply supported and clamped panels at low supersonic speeds, *J. of Fluids and Structures*, 40 (2013), pp. 366–372.

[95] V.V. Vedeneev, Panel flutter at low supersonic speeds, *J. Fluids and Structures*, 29 (2012), pp. 79–96.

[96] J.T. Webster, Weak and strong solutions of a nonlinear subsonic flow-structure interaction: semigroup approach, *Nonlinear Anal. A*, 74 (2011), pp. 3123–3136.

[97] D. Xie, M. Xu, H. Dai, and E.H. Dowell, Observation and evolution of chaos for a cantilever plate in supersonic flow, *J. Fluids and Structures*, 50 (2014), pp. 271–291.

[98] D. Xie, M. Xu, H. Dai, and E.H. Dowell, Proper orthogonal decomposition method for analysis of nonlinear panel flutter with thermal effects in supersonic flow, *J. Sound and Vibration*, 337 (2015), pp. 263–283.

Stability in Two Measures with Applications to Impulsive Control Systems

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ABSTRACT: This paper addresses the problem of stability in terms of two measures for impulsive systems on time scales. By establishing a new comparison result, it establishes stability criteria in terms of two measures. As an application, nonlinear impulsive control problems of continuous and discrete chaotic systems are discussed, respectively. Some less conservative impulsive stabilization conditions are obtained where both nonuniform and uniform impulsive intervals are considered. Some examples are provided to illustrate the effectiveness of the obtained results and the proposed impulsive control schemes.

Index terms—

Stability in terms of two measures, Impulsive systems, Time scales, Impulsive control, Continuous and discrete chaotic systems

0.1 Introduction

As special classes of hybrid dynamical systems, impulsive differential equations and systems on time scales have been studied extensively over the past three decades [1-12]. The theory of impulsive differential equations provides a general framework for mathematical modeling of many important dynamic process. For example, it serves as an adequate mathematical tool to study evolution processes that are subjected to abrupt changes in their states [1,2]. On the other hand, the theory on time scales, initialed by Stefan Hilger in his PhD thesis [6] in 1988, has provided a unified framework to study continuous and discrete dynamic systems simultaneously. Due to its two main features of unification and extension, this theory has tremendous potential for application in control theory [7-9], economics [10], geometric analysis [11], neural networks [12,13] and so on. For example, it can model the insect populations that are continuous in season and die out in winter, while their eggs are dormant, and then hatch in a new season [4].

Recently, various work has been done on the stability problem of dynamic systems on time scales [14-17]. The study of stability problem of impulsive systems on time scales was initialed by Lakshmikantham in [18], but the corresponding results of this problem are not much. In [19-21], the authors have considered the stability of impulsive systems on time scales by using comparison method and Lyapunov functions. In [22,23], the ϕ_0-stability of impulsive systems on time scales have been investigated. Most of the existed results are based on the assumption that all the impulsive instants are right-dense points (which will be introduced in section 3), which means that the corresponding results can not be used to the discrete impulsive systems. To the best of our knowledge, the stability properties in terms of two measures for nonlinear impulsive systems on time scales have not been investigated by comparison method.

Motivated by the above discussion, in this paper, we consider the stability in terms of two measures for nonlinear impulsive systems on time scales. We firstly establish a new comparison theorem about (uniform, uniform asymptotic) stability in terms of two measures for impulsive systems on time scales with impulses at fixed time. These results are used to design impulsive controllers for continuous and discrete chaotic systems. We hope to emphasize that, in our results, both uniform and nonuniform impulsive intervals are used, and nonlinear impulsive control schemes are obtained, which are less conservative than the results in [24-26]. Moreover, since there are many other time scales than just the set of real numbers or the set of integers, our results are more general.

The rest of this paper is organized as follows. In Section 2, we introduce some basic knowledge of dynamic systems on time scales. In section 3, we formulate the problem and present several definitions of stability in terms of two measures. In Section 4, a new comparison result is constructed which plays an important role in this paper, then several comparison theorems and (h_0, h)-stability criteria are established, which are used to analyze the asymptotic stability of a class of nonlinear impulsive systems on time scales. These results are applied to obtain sufficient conditions for impulsively stabilization of continuous and discrete chaotic systems in Section 5 and Section 6, respectively. For illustration of our results, four examples are shown in Section 7.

0.2 Preliminaries

In this section, we introduce some notations and definitions for later use.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$, $\mathbb{N} = \{1, 2, ...\}$, and $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ the set of all real matrices. The superscripts 'T' stands for the transpose of a matrix; $E \in \mathbb{R}^{n\times n}$ is an identity matrix. For $A \in R^{n\times n}$, let $||A||$ denotes the norm of A by Euclidean norm, i.e., $||A|| = \sqrt{\lambda_{max}(A^T A)}$, where $\lambda_{max}(P)$ denotes the largest eigenvalue of a symmetric matrix P.

Let $\mathbb{T}$ be an arbitrary nonempty closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ is a topological space with relative topology induced from $\mathbb{R}$. Then, $\mathbb{T}$ is called a time scale.

Definition 0.2.1. *The mappings* $\sigma, \theta : \mathbb{T} \to \mathbb{T}$ *defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

$$\theta(t) = \sup\{s \in \mathbb{T} : s < t\}$$

are called forward and backward jump operators, respectively.

A nonmaximal element $t \in \mathbb{T}$ is called right-scattered (rs) if $\sigma(t) > t$ and right-dense (rd) if $\sigma(t) = t$. A nonminimal element $t \in \mathbb{T}$ is called left-scattered (ls) if $\theta(t) < t$ and left-dense (ld) if $\theta(t) = t$. If $\mathbb{T}$ has a ls maximum m, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 0.2.2. *The graininess function* $:\mathbb{T} \to \mathbb{R}^+$ *is defined by* $(t) = \sigma(t) - t$.

Definition 0.2.3. *For* $y : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^k$*, we define the delta derivative* $y^{\Delta}(t)$ *of* $y(t)$*, to be the number (when it exists) with the property that for any* $\varepsilon > 0$*, there is a neigborhood* U *of* t *(i.e.,* $U = (t - \delta, t + \delta) \bigcap \mathbb{T}$ *for some* $\delta > 0$*) such that*

$$|y(\sigma(t)) - y(s) - y^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|, \ \text{ for all } s \in U.$$

If $a, b \in \mathbb{T}$, we then define the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] := \{t \in \mathbb{T} : a \le t \le b\}$. Open intervals and half-open intervals etc. are defined accordingly.

Definition 0.2.4. *For* $x : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^k$*, we define the upper right-hand Dini delta derivative of* $x(t)$ *by*

$$D^+x^{\Delta}(t) = \begin{cases} \frac{x(\sigma(t))-x(t)}{(t)}, \sigma(t) > t \\ \limsup_{s\to t^+} \frac{x(s)-x(t)}{s-t}, \quad \sigma(t) = t, \end{cases}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at rd points in $\mathbb{T}$ and its left-side limits exist at ld points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$. If f is continuous at each rd point and each ld point, f is said to be continuous function on $\mathbb{T}$.

Definition 0.2.5. *Let* $f \in C_{rd}$*. A function* $g : \mathbb{T} \to \mathbb{R}$ *is called the antiderivative of* f *on* $\mathbb{T}$ *if it is differentiable on* $\mathbb{T}$ *and satisfies* $g(t) = f(t)$ *for all* $t \in \mathbb{T}$*. In this case, we define*

$$\int_a^t f(s)s = g(t) - g(a), \ \ a, t \in \mathbb{T}.$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + (t)p(t) \neq 0$ for all $t \in \mathbb{T}$ holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted in this paper by $C_{rd}\mathcal{R} = C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, and the set of all positively regressive elements of $C_{rd}\mathcal{R}$ is denoted by $C_{rd}\mathcal{R}^+ = C_{rd}\mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in C_{rd}\mathcal{R} : 1 + (t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Definition 0.2.6. *If* $p \in C_{rd}\mathcal{R}$*, then we define the exponential function on time scales* $\mathbb{T}$ *by*

$$e_p(t, s) = \exp\Big(\int_s^t \xi_{(\times)}(p(\times)) \times\Big), \ \ \textit{for} \ \ t, s \in \mathbb{T}$$

where the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & h \neq 0 \\ z, & h = 0, \end{cases}$$

where Log *is the principal logrithm function.*

It is known that $x(t) = e_p(t, t_0)$ is the unique solution of the initial value problem $x(t) = p(t)x(t)$, $x(t_0) = 1$.

Remark 0.2.1. *Let* $\alpha \in C_{rd}\mathcal{R}$ *be constant. If* $\mathbb{T} = \mathbb{Z}$*, then* $e_\alpha(t, t_0) = (1 + \alpha)^{t-t_0}$ *for all* $t \in \mathbb{T}$*. If* $\mathbb{T} = \mathbb{R}$*, then* $e_\alpha(t, t_0) = e^{\alpha(t-t_0)}$ *for all* $t \in \mathbb{T}$.

0.3 Problem Formulation

Consider the following impulsive system on time scale

$$\begin{cases} x(t) = f(t, x), \ \ t \in \mathbb{T}, \ \ t \neq t_k, \\ x(t) = I_k(x(t)), \ \ \ t = t_k, \ \ k = 1, 2, ... \\ x(t_0^+) = x_0 \end{cases} \tag{0.3.1}$$

where

a) $\mathbb{T}$ is a time scale with $t_0 \geq 0$ as its minimum element and no maximal element.

b) $\{t_k\} \in \mathbb{T}$, $t_0 < t_1 < t_2 < ... < t_k < ...$ and $\lim_{k\to\infty} t_k = \infty$.

c) $x \in \mathbb{R}^n$ and $x(t_k) = x(t_k^+) - x(t_k^-)$. If t_k is rd point, $x(t_k^+)$ denotes the right limit of x at t_k; if t_k is rs point, $x(t_k^+)$ denotes the state of x at t_k with the impulse. If t_k is ld point, $x(t_k^-)$ denotes the left limit of x at t_k with $x(t_k^-) = x(t_k)$ if t_k is rd point. Here, we assume that $x(t_k^-) = x(t_k)$.

d) $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^n$ for $k = 1, 2, ...$, $f(t, 0) = 0$ and for each $x \in \mathbb{R}^n$, $k = 1, 2, ...$, $\lim_{(t,y)\to(t_k^+,x)} f(t, y) = f(t_k^+, x)$;

e) $I_k : \mathbb{R}^n \to \mathbb{R}^n$ and $I_k(0) = 0$.

Let $x(t) = x(t; t_0, x_0)$ denote the solution of system (0.3.1) satisfying initial condition $x(t_0^+) = x_0$. Obviously, system (0.3.1) admits the trivial solution. Moreover, f is assumed to satisfy necessary assumptions so that the following initial value problems:

$$\begin{cases} x = f(t, x), & t \in [t_0, t_1], \\ x(t_0) = x_0 \end{cases}$$

and

$$\begin{cases} x = f(t, x), & t \in (t_{k-1}, t_k], \\ x(t_k^+) = x_{k-1}(t_k) + I_{k-1}(x_{k-1}(t_k)) \end{cases}$$

have unique solutions $x_0(t)$, $t \in [t_0, t_1]$, and $x_k(t)$, $t \in (t_{k-1}, t_k]$, respectively (e.g., see [3] for existence and uniqueness results for dynamical systems on time scales.). Thus, if we define

$$x(t; t_0, x_0) = \begin{cases} x_0(t), & t \in [t_0, t_1] \\ x_1(t), & t \in (t_1, t_2] \\ \vdots & \vdots \\ x_k(t), & t \in (t_k, t_{k+1}] \\ \vdots & \vdots \end{cases}$$

then it is easy to see that $x(t; t_0, x_0)$ is the unique solution of (0.3.1) for $t \in \mathbb{T}$.

Let us list the classes of functions and definitions for convenience.

$PC = \{\delta : \mathbb{T} \to \mathbb{R}^+,$ continuous on $(t_{k-1}, t_k]$ and $\lim_{t \to t_k^+} \delta(t) = \delta(t_k^+)$ exists$\}$;

$\mathcal{K} = \{\delta \in C(\mathbb{R}^+, \mathbb{R}^+),$ strictly increasing and $\delta(0) = 0\}$;

$PC\mathcal{K} = \{\delta : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+,\ \delta(\cdot, u) \in PC$ for each $u \in \mathbb{R}^+$ and $\delta(t, \cdot) \in \mathcal{K}$ for each $t \in \mathbb{T}\}$;

$\Gamma = \{h : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^+,\ h(\cdot, x) \in PC$ for each $x \in \mathbb{R}^n,\ h(t, \cdot) \in C(\mathbb{R}^n, \mathbb{R}^+)$ for each $t \in \mathbb{T}$ and $\inf h(t, x) = 0\}$;

$\nu_0 = \{V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^+,$ continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and $\lim_{(t,y) \to (t_k, x),\ t > t_k} V(t, y) = \nu(t_k^+, x)$ exists$\}$.

For $V \in \nu_0$, $(t, x) \in (t_{k-1}, t_k] \times \mathbb{R}^n$, we define the upper right-hand Dini delta derivative of $V(t, x)$ relative to (0.3.1) as follows:

$$D^+V(t, x) = \begin{cases} \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{(t)}, & \sigma(t) > t \\ \limsup_{s \to t^+} \frac{V(s, x(t) + (s-t)f(t, x(t))) - V(t, x(t))}{s - t}, & \sigma(t) = t \end{cases}$$

Definition 0.3.1. *Let $h_0, h \in \Gamma$. Then we say that*

(i) h_0 is finer than h if there exists a constant $\rho > 0$ and a function $\varphi \in \mathcal{K}$ such that $h_0(t, x) < \rho$ implies $h(t, x) \leq \varphi(h_0(t, x))$;

(ii) h_0 is weakly finer than h if there exists a constant $\rho > 0$ and a function $\varphi \in PC\mathcal{K}$ such that $h_0(t, x) < \rho$ implies $h(t, x) \leq \varphi(t, h_0(t, x))$.

Definition 0.3.2. *Let $V \in \nu_0$ and $h_0, h \in \Gamma$. Then $V(t, x)$ is said to be*

(i) h-positive definite if there exist a $\rho > 0$ and a function $b \in \mathcal{K}$ such that $h(t, x) < \rho$ implies $b(h(t, x)) \leq V(t, x)$;

(ii) h_0-decrescent if there exist a $\rho > 0$ and a function $a \in \mathcal{K}$ such that $h_0(t, x) < \rho$ implies $V(t, x) \leq a(h_0(t, x))$;

(iii) h_0-weakly decrescent if there exist a $\rho > 0$ and a function $a \in PC\mathcal{K}$ such that $h_0(t, x) < \rho$ implies $V(t, x) \leq a(t, h_0(t, x))$.

Definition 0.3.3. *The impulsive differential system (0.3.1) is said to be*

*(S_1) (h_o, h)-**stable**, if for each $\varepsilon > 0$, $t_0 \in \mathbb{T}$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0$ for any solution $x(t) = x(t; t_0, x_0)$ of (0.3.1);*

*(S_2) (h_0, h)-**uniformly stable** if the δ in (S_1) is independent of t_0;*

*(S_3) (h_o, h)-**attractive**, if for each $\varepsilon > 0$, $t_0 \in \mathbb{T}$, there exist two positive constant $\delta = \delta(t_0, \varepsilon)$ and $T = T(t_0, \varepsilon)$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0 + T$;*

*(S_4) (h_0, h)-**uniformly attractive** if (S_3) holds with δ and T being independent of t_0;*

*(S_5) (h_0, h)-**asymptotic stable** if (S_1) and (S_3) hold simultaneously;*

*(S_6) (h_0, h)-**uniformly asymptotic stable** if (S_2) and (S_4) hold together;*

*(S_7) (h_0, h)-**unstable** if (S_1) fails to hold.*

0.4 (h_0, h)-Stability Criteria

In this section, we shall present several stability criteria about (h_0, h)-stability of impulsive systems on time scales by comparison method.

Firstly, we need the following comparison result which will be useful in our investigation.

Theorem 0.4.1. *Assume that*

(i) $V \in \nu_0$, $K : \mathbb{R}^+ \to (0, \infty)$ and satisfy the following inequalities:

$$\begin{cases} K(\sigma(t))D^+V(t, x) + V(t, x)D^+K(t) \leq g(t, K(t)V(t, x)), & t \neq t_k \\ K(t_k^+)V(t_k^+, x(t_k) + I_k(x(t_k))) \leq'_k (K(t_k)V(t_k, x(t_k))), & k \in \mathbb{N}, \\ K(t_0)V(t_0, x_0) \leq u_0 \end{cases} \tag{0.4.1}$$

where $g : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$, continuous on $(t_{k-1}, t_k] \times \mathbb{R}^+$, $k = 1, 2, ...$, $\lim_{(t,\nu) \to (t_k, u)\ t > t_k} g(t, \nu) = g(t_k^+, u)$ exists and $'_k : \mathbb{R}^+ \to \mathbb{R}^+$, nondecreasing;

(ii) $g(t,u)(t)+u$ *is nondecreasing in* u *for each* $t\in\mathbb{T}$*;*

(iii) $r(t)=r(t;t_0,u_0)$ *is the maximal solution of the following scalar impulsive differential equation*

$$\begin{cases} u(t)=g(t,u), \;\; t\in\mathbb{T}, \;\; t\neq t_k, \\ u(t_k^+)=\psi_k'(u(t_k)), \quad k\in\mathbb{N}, \\ u(t_0^+)=u_0 \end{cases} \tag{0.4.2}$$

existing on $\mathbb{T}$ *for* $t\geq t_0$.

Then we have

$$K(t)V(t,x(t))\leq r(t), \;\; t\in\mathbb{T}, \; t\geq t_0.$$

Remark 0.4.1. *We give two remarks about Theorem 0.4.1.*
(1) In case $\mathbb{T}=\mathbb{R}^+$*, the above theorem reduces to the Lemma in [28]. In case* $\mathbb{T}=\mathbb{Z}$*, the above theorem reduces to the corresponding discrete result which, as far as we know, is brand new.*
(2) If $K(t)\equiv 1$*, Theorem 0.4.1 reduces to the Lemma in [21]. Furthermore, if* $\psi_k(x)=x$ *for* $k=1,2,...$ *in Theorem 0.4.1, the corresponding result can be found in [3].*

We are now in the position to establish some comparison theorems for (h_0,h)-stability of impulsive differential system (0.3.1). Let

$$S(h,\rho)=\{(t,x)\in\mathbb{T}\times\mathbb{R}^n: \; h(t,x)<\rho\}.$$

Theorem 0.4.2. *Assume that*

(i) $h_0,h\in\Gamma$ *and* h_0 *is finer that* h*;*

(ii) $V\in\nu_0$, $V(t,x)$ *is* h*-positive definite on* $S(h,\rho)$*, locally Lipschitz in* x *for each* $t\in\mathbb{T}$ *which is rd,* $K(t)V(t,x)$ *is* h_0*-decrescent, and*

$$K(\sigma(t))D^+V(t,x)+V(t,x)D^+K(t)\leq g(t,K(t)V(t,x)), \;\; t\neq t_k, \; (t,x)\in S(h,\rho)$$

where $g(t,u)$*,* $K(t)$ *are the same as described in Theorem 0.4.1 and in addition* $g(t,0)\equiv 0$*,* $K(t)\geq m>0$ *(here,* m *is a constant);*

(iii) $K(t_k^+)V(t_k^+,x_k+I_k(x_k))\leq\psi_k'(K(t_k)V(t_k,x_k))$*,* $(t_k,x_k)\in S(h,\rho)$*, where* $x_k=x(t_k)$*, and* $\psi_k':\mathbb{R}^+\to\mathbb{R}^+$ *is nondecreasing,* $k\in\mathbb{N}$*;*

(iv) $g(t,u)(t)+u$ *is nondecreasing in* u *for each* $t\in\mathbb{T}$*;*

(v) there exists a ρ_1*,* $0<\rho_1<\rho$*, such that* $h(t,x(t))<\rho_1$ *implies* $h(\sigma(t),x(\sigma(t)))<\rho$*;*

(vi) there exists a ρ_0*,* $0<\rho_0<\rho_1$*, such that* $h(t_k,x_k)<\rho_0$ *implies that* $h(t_k^+,x_k+I_k(x_k))<\rho_1$.

Then, the stability properties of the trivial solution of system (0.4.2) imply the corresponding (h_0,h)*-stability properties of system (0.3.1)*

Remark 0.4.2.

(1) If $K(t)\equiv 1$*, the continuous version of Theorem 4.2 can be found in [33].*

(2) If $K(t)\equiv 1$ *and* $h(t,x)=h_0(t,x)=||x||$*, then Theorem 0.4.2 reduces to Theorem 3.1 in [20] with* $\rho=\infty$.

(3) If $K(t)\equiv 1$*,* $\mathbb{T}=\mathbb{R}^+$ *and* $h(t,x)=h_0(t,x)=||x||$*, then Theorem 0.4.2 reduces to Theorem 1 in [28].*

In Theorem 0.4.2, stronger requirement on KV has been assumed to unify all the stability criteria in one theorem. Therefore, to obtain only nonuniform stability criteria, we could weaken the assumption about KV, h, and h_0 of Theorem 0.4.2, as in the following result. We omit the details of the proof which is analogous to the proof of Theorem 0.4.2.

Theorem 0.4.3. *Assume that conditions (i)-(vi) of Theorem 0.4.2 hold with the following change:*

(i^*) h_0 *is weakly finer than* h*;*

(ii^*) $K(t)V(t,x)$ *is* h_0*-weakly decrescent.*

Then the uniform and nonuniform stability properties of the trivial solution of (0.4.2) imply the corresponding nonuniform (h_0,h)*-stability properties of system (0.3.1).*

In Theorem 0.4.2 and 0.4.3, the comparison system (0.4.2) may have special forms. In case when $g(t,u)\equiv 0$ and $\psi_k(r)=d_kr$, $k\in\mathbb{N}$, we have the following $(h_0,h)-$stability results.

Corollary 0.4.1. *Assume that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is finer than* h*;*

(ii) $V \in \nu_0$, $V(t,x)$ *is* h*-positive definite on* $S(h,\rho)$*, locally Lipschtiz in* x *for each* $t \in \mathbb{T}$ *which is rd,* $K(t)V(t,x)$ *is* h_0*-decrescent, and*

$$K(\sigma(t))D^+V(t,x) + V(t,x)D^+K(t) \leq 0, \ \ t \neq t_k, \ (t,x) \in S(h,\rho),$$

where $g(t,u)$, $K(t)$ *are the same as described in Theorem 0.4.1 and in addition* $g(t,0) \equiv 0$, $K(t) \geq m > 0$ *(here,* m *is a constant);*

(iii) $K(t_k^+)V(t_k^+, x_k + I_k(x_k)) - K(t_k)V(t_k,x_k) \leq d_k K(t_k)V(t_k,x_k)$, $(t_k,x_k) \in S(h,\rho)$*, where* $d_k \geq 0$ *and* $\sum_{i=1}^{\infty} d_i < \infty$ *for* $k \in \mathbb{N}$*;*

(iv) there exists a ρ_1, $0 < \rho_1 < \rho$*, such that* $h(t,x(t)) < \rho_1$ *implies* $h(\sigma(t), x(\sigma(t))) < \rho$*;*

(v) there exists a ρ_0, $0 < \rho_0 < \rho_1$*, such that* $h(t_k,x_k) < \rho_0$ *implies that* $h(t_k^+, x_k + I_k(x_k)) < \rho_1$.
Then, system (0.3.1) is (h_0,h)*-uniformly stable.*

Proof: Consider the following comparison system

$$\begin{cases} u(t) = 0, \ \ t \in \mathbb{T}, \ \ t \neq t_k, \\ u(t_k^+) = (1+d_k)u(t_k), \ \ k \in \mathbb{N}, \\ u(t_0^+) = u_0 \end{cases} \tag{0.4.3}$$

Then, for $t \in (t_k, t_{k+1}]$, we have

$$u(t) = \prod_{i=1}^{k}(1+d_i)u_0 < Mu_0 \tag{0.4.4}$$

where $M = \prod_{i=1}^{\infty}(1+d_i) < \infty$, since $\sum_{i=1}^{\infty} d_i < \infty$.

For any $\varepsilon > 0$, we choose $\delta = \frac{\varepsilon}{M}$. Then, form (0.4.4), we know that $|u_0| < \delta$ implies $|u(t)| < \varepsilon$, i.e., $u = 0$ of system (0.4.3) is uniformly stable. Hence, it follows from Theorem 4.2 that system (0.3.1) is $(h_0,h)-$uniformly stable and the proof is complete. $\square$

Corollary 0.4.2. *Assume that conditions (i)-(v) of Corollary 0.4.1 hold with the following change:*

(i^**)*h_0 *is weakly finer than* h*;*

(ii^**)* $K(t)V(t,x)$ *is* h_0*-weakly decrescent.*
Then, system (0.3.1) is (h_0,h)*-stable.*

Remark 0.4.3. *If* $K(t) \equiv 1$, $d_k = 0$ *for* $k = 1, 2, ...$ *and* $\mathbb{T} = \mathbb{R}^+$ *in Corollary 4.1 and 4.2, then we can obtain Theorem 3.1 in [34].*

In case when $g(t,u) = p(t)u(t)$, and $'_k(r) = d_k r$, $k \in \mathbb{N}$, then we have the following (h_0,h)-asymptotic stability result.

Corollary 0.4.3. *System (0.3.1) is (*h_0,h*)-asymptotically stable if the conditions of Theorem 0.4.3 and the following conditions hold:*
(i) $p \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$.
(ii) $\sup_{k \in \mathbb{N}}\{d_{2k-1}e_p(t_{2k}, t_{2k-1})\} = \xi < \infty$*, where* $d_k > 0$, $k \in \mathbb{N}$.
(iii) There exists a constant $\gamma > 1$ *such that*

$$\gamma d_{2k-1} d_{2k} e_p(t_{2k+1}, t_{2k-1}) \leq 1, \ k \in \mathbb{N}.$$

Proof: Consider the following comparison system

$$\begin{cases} \omega(t) = p(t)\omega(t), \ \ t \neq t_k \\ \omega(t_k^+) = d_k\omega(t_k), \quad t = t_k, \ k \in \mathbb{N}, \\ \omega(t_0^+) = \omega_0 \end{cases} \tag{0.4.5}$$

It can be seen that the solution of system (0.4.5) is

$$\omega(t, t_0, \omega_0) = \omega_0 \prod_{t_0 < t_k < t} d_k e_p(t, t_0)$$

We shall show that

$$\omega(t, t_0, \omega_0) \le \max\{1, \xi\}\omega_0 e_p(t_1, t_0) \tag{0.4.6}$$

To do this, there are three cases to consider.
Case 1. If $t \in [t_0, t_1]$, then

$$\omega(t, t_0, \omega_0) = \omega_0 e_p(t, t_0) \le \omega_0 e_p(t_1, t_0) \le \max\{1, \xi\}\omega_0 e_p(t_1, t_0)$$

Case 2. If $t \in (t_{2k-1}, t_{2k}]$ and $k > 0$, then by condition (ii) and (iii),

$$\begin{aligned}
\omega(t, t_0, \omega_0) &= \omega_0 \prod_{i=1}^{2k-1} d_i e_p(t, t_1) e_p(t_1, t_0) \\
&\le \omega_0 \prod_{i=1}^{2k-1} d_i e_p(t_{2k}, t_1) e_p(t_1, t_0) \\
&= \omega_0 e_p(t_1, t_0) \cdot d_1 d_2 e_p(t_3, t_1) \cdot \ldots \cdot d_{2k-3} d_{2k-2} e_p(t_{2k-1}, t_{2k-3}) \\
&\quad \times d_{2k-1} e_p(t_{2k}, t_{2k-1}) \\
&\le \xi\omega_0 \frac{1}{\gamma^{k-1}} e_p(t_1, t_0) \\
&\le \max\{1, \xi\}\omega_0 e_p(t_1, t_0)
\end{aligned}$$

Case 3. If $t \in (t_{2k}, t_{2k+1}]$ and $k > 0$, then by condition (iii),

$$\begin{aligned}
\omega(t, t_0, \omega_0) &= \omega_0 \prod_{i=1}^{2k} d_i e_p(t, t_1) e_p(t_1, t_0) \\
&\le \omega_0 \prod_{i=1}^{2k} d_i e_p(t_{2k+1}, t_1) e_p(t_1, t_0) \\
&= \omega_0 e_p(t_1, t_0) \cdot d_1 d_2 e_p(t_3, t_1) \cdot \ldots \cdot d_{2k-1} d_{2k} e_p(t_{2k+1}, t_{2k-1}) \\
&\le \omega_0 \frac{1}{\gamma^{k}} e_p(t_1, t_0) \\
&\le \max\{1, \xi\}\omega_0 e_p(t_1, t_0)
\end{aligned}$$

Therefor (0.4.6) holds.

For a given $\epsilon > 0$ we can choose $\delta = \frac{\epsilon}{2\max\{1,\xi\}e_p(t_1,t_0)}$ such that $0 \le \omega_0 < \delta$ implies $\omega_0(t, t_0, \omega_0) \le \epsilon/2 < \epsilon$. Hence the trivial solution of system (0.4.5) is stable.

Note that $t \to \infty$ as $k \to \infty$. From Case 2. and Case 3., we have

$$\lim_{t\to\infty} \omega(t, t_0, \omega_0) = 0.$$

Hence we proved that the trivial solution of system (0.4.5) is asymptotically stable. Followed from Theorem 0.4.3 we know that system (0.3.1) is (h_0, h)-asymptotically stable. □

The next theorem is a generalization of Corollary 0.4.3.

Corollary 0.4.4. *System (0.3.1) is* (h_0, h)*-asymptotically stable if the conditions of Theorem 0.4.3 and the following conditions hold:*
(i) $p \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$.
(ii) There exists a constant $\gamma > 1$ *and some* $i (i \ge 2)$ *such that the following conditions hold*

$$\gamma \prod_{j=0}^{i-1} d_{ik-j} e_p(t_{ik+1}, t_{ik-(i-1)}) \le 1,$$

and

$$\begin{cases} \sup_{k\in\mathbb{N}}\{d_{ik-(i-1)}e_p(t_{ik-(i-2)}, t_{ik-(i-1)})\} = \xi_1 < \infty \\ \sup_{k\in\mathbb{N}}\{d_{ik-(i-1)}d_{ik-(i-2)}e_p(t_{ik-(i-3)}, t_{ik-(i-1)})\} = \xi_2 < \infty \\ \dots\dots \\ \sup_{k\in\mathbb{N}}\{\prod_{j=1}^{i-1} d_{ik-j}e_p(t_{ik}, t_{ik-(i-1)})\} = \xi_{i-1} < \infty \end{cases}$$

where $d_k > 0$, $k \in \mathbb{N}$.

It is obvious that Corollary 0.4.3 is a special case of Corollary 0.4.4 when $i = 2$.

Next, we shall apply the previous results to investigate asymptotic stable property of a special case of system (0.3.1).

Consider the impulsive system of the form

$$\begin{cases} x(t) = Ax + \phi(t, x), \ \ t \in \mathbb{T}, \ \ t \neq t_k, \\ x(t) = I_k(x(t)), \quad t = t_k, \ \ k \in \mathbb{N}, \\ x(t_0^+) = x_0 \end{cases} \tag{0.4.7}$$

where $A \in \mathbb{R}^n$, $\phi \in \nu_0$, and the following assumptions are satisfied.

(A1) There exists a positive scalar L such that

$$\|\phi(t, x)\| \leq L\|x\|, \ \ \text{for any } t \in \mathbb{T}, \ x \in \mathbb{R}^n.$$

(A2) There exist constants $l_k > 0$, $k = 1, 2, \dots$ such that

$$\|x + I_k(x)\| \leq l_k\|x\|, \quad x \in \mathbb{R}^n.$$

Theorem 0.4.4. *Suppose that assumptions (A1) and (A2) hold and there exists a constant* $\gamma > 1$ *such that, for* $k \in \mathbb{N}$,

$$\sup_{k\in\mathbb{N}}\{l_{2k-1}^2 e_q(t_{2k}, t_{2k-1})\} = \xi < \infty, \tag{0.4.8}$$

and

$$\gamma l_{2k-1}^2 l_{2k}^2 e_q(t_{2k+1}, t_{2k-1}) \leq 1, \tag{0.4.9}$$

where

$$q = \alpha + 2L\sqrt{\alpha_0} + (t)(L^2 - 1) \geq 0,$$

$$\text{and}$$

$$\alpha_0 = \sup_{t\in\mathbb{T}}\{\lambda_{max}(E + A^T)^T(E + A^T)\},$$
$$\alpha = \sup_{t\in\mathbb{T}}\{\lambda_{max}(A^T + A + A^TA + E)\}.$$

Then the trivial solution of system (0.4.7) is asymptotically stable.

Proof: Let the Lyapunov function be in the form of

$$V(t, x) = x^Tx.$$

Choose $h_0(t, x) = h(t, x) = \|x\|$ and $K(t) \equiv 1$, then condition (i) and (v) of Theorem 0.4.2 are satisfied. For $t \neq t_k$, we have

$$
\begin{aligned}
D^{+}V(t,x) &= (x^{T})x^{\sigma} + x^{T}x \\
&= (x^{T})(x+x) + x^{T}x \ \ (where\ \ x^{\sigma} = x + x) \\
&= [Ax + \phi(t,x)]^{T}\{x + [Ax + \phi(t,x)]\} + x^{T}[Ax + \phi(t,x)] \\
&= x^{T}(A^{T} + A + A^{T}A)x + 2x^{T}(E + A^{T})\phi(t,x) \\
&= x^{T}(A^{T} + A + A^{T}A + E)x + 2x^{T}(E + A^{T})\phi(t,x) \\
&\leq [\alpha + 2L\sqrt{\alpha_0} + (t)(L^{2} - 1)]V(x) \\
&= qV(x). \qquad (0.4.10)
\end{aligned}
$$

Hence, condition (ii) of Theorem 0.4.2 is satisfied with $g(t,u) = qu$. Then $g(t,u)(t) + u = (q+1)u$ is nondecreasing in u for each $t \in \mathbb{T}$, which implies that condition (iv) of Theorem 0.4.2 is satisfied.

In addition, when $t = t_k$, $k \in \mathbb{N}$, we have

$$
\begin{aligned}
V(t_k^{+}, x(t_k^{+})) &= x^{T}(t_k^{+})x(t_k^{+}) \\
&= [x(t_k) + I_k(x(t_k))]^{T}[x(t_k) + I_k(x(t_k))] \\
&= ||x(t_k) + I_k(x(t_k))||^{2} \\
&\leq l_k^{2}||x(t_k)||^{2} \\
&= l_k^{2}V(t_k, x(t_k)). \qquad (0.4.11)
\end{aligned}
$$

Thus, condition (iii) of Theorem 0.4.2 is also satisfied with $'_k(\omega) = l_k^{2}\omega$.

Based on Theorem 0.4.2, the asymptotic stability of system (0.4.7) is implied by that of the following comparison system

$$
\begin{cases}
\omega(t) = q\omega(t), \ \ t \neq t_k \\
\omega(t_k^{+}) = l_k^{2}\omega(t_k), \ \ k \in \mathbb{N}, \\
\omega(t_0^{+}) = \omega_0 \leq 0
\end{cases} \qquad (0.4.12)
$$

It can be seen that (0.4.8) and (0.4.9) implies that condition (ii) and (iii) of Corollary 0.4.3 are satisfied. Therefore, all conditions of Corollary 0.4.3 are satisfied, and the trivial solution of system (0.4.7) is asymptotically stable. □

By Corollary 0.4.4, we extend Theorem 0.4.4 to a more general theorem.

Theorem 0.4.5. *Assume that assumptions (A1) and (A2) hold and there exists a constant* $\gamma > 1$ *and some* $i(i \geq 2)$ *such that the following conditions hold*

$$
\gamma \prod_{j=0}^{i-1} l_{ik-j}^{2} e_q(t_{ik+1}, t_{ik-(i-1)}) \leq 1, \ \ k \in \mathbb{N},
$$

and

$$
\begin{cases}
\sup_{k\in\mathbb{N}}\{l_{ik-(i-1)}^{2} e_q(t_{ik-(i-2)}, t_{ik-(i-1)})\} = \xi_1 < \infty \\
\sup_{k\in\mathbb{N}}\{l_{ik-(i-1)}^{2} l_{ik-(i-2)}^{2} e_q(t_{ik-(i-3)}, t_{ik-(i-1)})\} = \xi_2 < \infty \\
\dots \dots \\
\sup_{k\in\mathbb{N}}\{\prod_{j=1}^{i-1} l_{ik-j}^{2} e_q(t_{ik}, t_{ik-(i-1)})\} = \xi_{i-1} < \infty
\end{cases}
$$

Then the origin of system (0.4.7) is asymptotically stable.

0.5 Impulsive Control of Continuous Chaotic Systems

In this section, we study the impulsive control problem of the following class of continuous chaotic systems by applying theories presented in the previous section:

$$
\dot{x}(t) = Ax(t) + \phi(x(t)) \qquad (0.5.1)
$$

where $x \in \mathbb{R}^n$ is the state vector, A is an $n \times n$ constant matrix, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function, $\phi(0) = 0$, and $||\phi(x)|| \le L||x||$, where $L > 0$ is a constant.

The impulsive control system of (0.5.1) is given by

$$\begin{cases} \dot{x} = Ax + \phi(x), \ \ t \neq t_k \\ x(t) = I_k(x(t)), \ \ t = t_k, \ k \in \mathbb{N}, \end{cases} \tag{0.5.2}$$

where $x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{t \to t_k+} x(t)$, $x(t_k^-) = \lim_{t \to t_k-} x(t) = x(t)$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $I_k(0) = 0$, for $k \in \mathbb{N}$, and the impulsive sequence $\{t_k\}$ satisfy

$$0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...; \ \ t_k \to \infty \ \text{ as } \ k \to \infty.$$

We have the following theorems which guarantee the impulsive control system (0.5.2) be asymptotically stable at the origin.

Theorem 0.5.1. *Let α be the largest eigenvalue of $A^T + A$. Then the origin of system (0.5.2) is asymptotically stable if the following conditions hold:*
(i) There exist constants $l_k > 0, \ k \in \mathbb{N}$, such that

$$||x + I_k(x)|| \le l_k||x||, x \in \mathbb{R}^n \tag{0.5.3}$$

(ii) There exists a constant $\gamma > 1$ such that

$$\ln(\gamma l_{2k-1}^2 l_{2k}^2) + (\alpha + 2L)(t_{2k+1} - t_{2k-1}) \le 0, \ \ k \in \mathbb{N}, \tag{0.5.4}$$

and $\sup_{k\in\mathbb{N}}\{l_{2k-1}^2 e^{(\alpha+2L)(t_{2k}-t_{2k-1})}\} = \xi < \infty$.

Proof: Letting $\mathbb{T} = \mathbb{R}$, it can be seen that system (0.5.2) is a special case of system (0.4.7) with $(t) \equiv 0$. Then, we have $\alpha_0 = 1, \ \ \alpha = \lambda_{max}\{A^T + A\}$ and $q = \alpha + 2L$.

By Remark 0.2.1 and (0.5.4), we obtain

$$\gamma l_{2k-1}^2 l_{2k}^2 e^{(\alpha+2L)(t_{2k+1}-t_{2k-1})} = \gamma l_{2k-1}^2 l_{2k}^2 e_q(t_{2k+1}, t_{2k-1}) \le 1$$

and

$$l_{2k-1}^2 e^{(\alpha+2L)(t_{2k}-t_{2k-1})} = l_{2k-1}^2 e_q(t_{2k}, t_{2k-1}) = \ln \xi \le \infty$$

which implies that condition (0.4.8) and (0.4.9) of Theorem 0.4.4 is satisfied. Coupled with condition (i), we know that all the conditions of Theorem 0.4.4 are satisfied. Hence, the impulsive control system (0.5.2) is asymptotically stable at the origin. □

Theorem 0.5.2. *Let α be the largest eigenvalue of $A^T + A$. Then the origin of system (0.5.2) is asymptotically stable if the following conditions hold:*
(i) There exist constants $l_k \ge 0, k \in \mathbb{N}$, such that

$$||x + I_k(x)|| \le l_k||x||, x \in \mathbb{R}^n \tag{0.5.5}$$

(ii) There exists a constant $\gamma > 1$ and some $i(i \ge 2)$ such that the following conditions hold

$$\ln(\gamma \prod_{j=0}^{i-1} l_{ik-j}^2) + (\alpha + 2L)(t_{ik+1} - t_{ik-(i-1)}) \le 0, \ \ k \in \mathbb{N},$$

and

$$\begin{cases} \sup_{k\in\mathbb{N}}\{l_{ik-(i-1)}^2 e^{(\alpha+2l))(t_{ik-(i-2)}-t_{ik-(i-1)})}\} = \xi_1 < \infty \\ \sup_{k\in\mathbb{N}}\{l_{ik-(i-1)}^2 l_{ik-(i-2)}^2 e^{(\alpha+2l))(t_{ik-(i-3)}-t_{ik-(i-1)})}\} = \xi_2 < \infty \\ \dots \dots \\ \sup_{k\in\mathbb{N}}\{\prod_{j=1}^{i-1} l_{ik-j}^2 e^{(\alpha+2l))(t_{ik}-t_{ik-(i-1)})}\} = \xi_{i-1} < \infty \end{cases}$$

Then the origin of system (0.5.2) is asymptotically stable.

The proof is similar to that of Theorem 0.5.1 and is thus omitted.

For convenience, we may choose a uniform impulsive controller such as $I_k(x) = I(x)(k \in \mathbb{N})$ and uniform impulsive intervals. In this case, the conditions of Theorem 0.5.1 become much simpler.

Corollary 0.5.1. *Assume* $t_{2k+1} - t_{2k-1} = \times > 0$, $I_k = I(k \in \mathbb{N})$ *and* $||x + I(x)|| \leq l||x||$, *where* $l \geq 0$ *is a constant. If there exists a constant* $\gamma > 1$ *such that*

$$\ln(\gamma l^4) + (\alpha + 2L)\times \leq 0, \tag{0.5.6}$$

then the origin of system (0.5.2) is asymptotically stable.

Remark 0.5.1. *By Theorem 0.5.1, we can design a nonlinear impulsive controller to stabilize the continuous chaotic system (0.5.1). In the special case of* $I_k(x) = B_k x(k \in \mathbb{N})$, *where* B_k *is a* $n \times n$ *constant matrix, inequalities (0.5.5) are always hold. It means that we can also use Theorem 0.5.1 to design linear impulsive controller which has been considered in [24,25,29]. Hence, our results are more general and comprehensive.*

Remark 0.5.2. *From condition (ii) of Theorem 0.5.1, we need only to choose the odd impulsive sequence* $\{t_{2k-1}\}$ *instead of the whole impulsive sequence* $\{t_k\}$ *which has been discussed in [24,29]. In Section 7, we shall give numerical simulations to show this remark.*

Remark 0.5.3. *From Corollary 0.5.1, the estimate of upper bound on uniform impulsive interval* $\times$ *of impulsive system (0.5.2) is obtained as follows:*

$$0 < \times \leq \frac{-\ln\left(\gamma l^4\right)}{\alpha + 2L}.$$

Similarly, in the case of uniform impulsive controller and uniform impulsive interval, we have the following corollary.

Corollary 0.5.2. *Assume* $t_{ik+1} - t_{ik-(i-1)} = \times > 0$, $I_k = I(k = 1, 2, ...)$ *and* $||x + I(x)|| \leq l||x||$, *where* $l \geq 0$ *is a constant. If there exists a constant* $\gamma > 1$ *such that*

$$\ln(\gamma l^{2i}) + (\alpha + 2L)\times \leq 0, \tag{0.5.7}$$

then the origin of system (0.5.2) is asymptotically stable.

0.6 Impulsive Control of Discrete Chaotic Systems

In this section, we study the impulsive control problem of the following class of discrete chaotic systems by applying theories presented in section 4:

$$x(n+1) = \bar{A}x(n) + \phi(x(n)) \tag{0.6.1}$$

where $x \in \mathbb{R}^m$ is the state vector, $\bar{A}$ is an $m \times m$ constant matrix, and $\phi : \mathbb{R}^m \to \mathbb{R}^m$, $\phi(0) = 0$, and $||\phi(x)|| \leq L||x||$, where $L > 0$ is a constant.

The impulsive control system of (0.6.1) is given by

$$\begin{cases} x(n+1) = \bar{A}x(n) + \phi(x(n)), & n \neq n_k \\ x(n) = I_k(x(n)), & n = n_k \end{cases} \tag{0.6.2}$$

where $x(n_k) = x(n_k^+) - x(n_k)$. $x(n_k^+)$ and $x(n_k)$ denote the state x at n_k without and with impulse, respectively, which are also considered in [30,31]. $I_k : \mathbb{R}^m \to \mathbb{R}^m$, $I_k(0) = 0$, for $k = 1, 2, ...$, and the impulsive sequence $\{n_k\} \subset \mathbb{N}$ satisfy

$$1 < n_1 < n_2 < ... < n_k < n_{k+1} < ...; \ \ n_k \to \infty \ \text{ as } \ k \to \infty.$$

We have the following theorems which guarantee the impulsive control system (0.6.2) be asymptotically stable at the origin.

Theorem 0.6.1. *Let* α *be the largest eigenvalue of* $\bar{A}^T\bar{A}$. *Then the origin of system (0.6.2) is asymptotically stable if the following conditions hold:*

(i) There exist constants $l_k \geq 0 (k \in \mathbb{N})$ *such that*

$$||x + I_k(x)|| \leq l_k||x||, x \in \mathbb{R}^n \tag{0.6.3}$$

(ii) There exists a constant $\gamma > 1$ *such that*

$$\ln(\gamma l_{2k-1}^2 l_{2k}^2) + 2(t_{2k+1} - t_{2k-1}) \ln(\sqrt{\alpha} + L) \leq 0, \quad k \in \mathbb{N}, \tag{0.6.4}$$

and $\sup_{k\in\mathbb{N}}\{\ln(l_{2k-1}^2) + 2(t_{2k} - t_{2k-1}) \ln(\sqrt{\alpha} + L)\} = \xi < \infty$.

Proof: Let $\mathbb{T} = \mathbb{Z}^+$ and $A = \bar{A} - E$, then we have

$$\begin{aligned} x(n) &= x(n+1) - x(n) \\ &= \bar{A}x(n) + \phi(x(n)) - x(n) \\ &= (\bar{A} - E)x(n) + \phi(x(n)) \\ &= Ax(n) + \phi(x(n)) \end{aligned}$$

which implies that system (0.6.2) can be rewritten in the form of (0.4.7).

When $\mathbb{T} = \mathbb{N}$, we get $(n) \equiv 1$, then

$$\begin{aligned} &\alpha = \alpha_0 = \lambda_{max}\{(E + A)(E + A^T)\}, \\ &q = \alpha + 2L\sqrt{\alpha_0} + L^2 - 1 = (\sqrt{\alpha} + L)^2 - 1. \end{aligned}$$

By Remark 0.2.1 and (0.6.4), we obtain

$$\begin{aligned} \gamma l_{2k-1}^2 l_{2k}^2 \left(\sqrt{\alpha} + L\right)^{2(t_{2k+1}-t_{2k-1})} &= \gamma l_{2k-1}^2 l_{2k}^2 (1 + q)^{t_{2k+1}-t_{2k-1}} \\ &= \gamma l_{2k-1}^2 l_{2k}^2 e_q(t_{2k+1}, t_{2k-1}) \\ &\leq 1 \end{aligned} \tag{0.6.5}$$

and

$$l_{2k-1}^2 \left(\sqrt{\alpha} + L\right)^{2(t_{2k+1}-t_{2k-1})} = l_{2k}^2 e_q(t_{2k+1}, t_{2k-1}) = \ln \xi \leq \infty$$

then the condition (0.4.8) and (0.4.9) of Theorem 0.4.4 is satisfied. Coupled with condition (i), we know that all the conditions of Theorem 0.4.4 are satisfied. Hence, the impulsive control system (0.6.2) is asymptotically stable at the origin. □

Theorem 0.6.2. *Let* α *be the largest eigenvalue of* $\bar{A}^T\bar{A}$. *Then the origin of system (0.6.2) is asymptotically stable if the following conditions hold:*

(i) There exist constants $l_k \geq 0 (k \in \mathbb{N})$ *such that*

$$||x + I_k(x)|| \leq l_k||x||, x \in \mathbb{R}^n \tag{0.6.6}$$

(ii) There exists a constant $\gamma > 1$ *and some* $i (i \geq 2)$ *such that the following conditions hold*

$$\ln\left(\gamma \prod_{j=0}^{i-1} l_{ik-j}^2\right) + 2\left(t_{ik+1} - t_{ik-(i-1)}\right) \ln\left(\sqrt{\alpha} + L\right) \leq 0, \quad k \in \mathbb{N},$$

and

$$\begin{cases} \sup_{k\in\mathbb{N}}\{\ln l_{ik-(i-1)}^2 + 2(t_{ik-(i-2)} - t_{ik-(i-1)}) \ln(\sqrt{\alpha} + L))\} = \xi_1 < \infty \\ \sup_{k\in\mathbb{N}}\{\ln\left(l_{ik-(i-1)}^2 l_{ik-(i-2)}^2\right) + 2(t_{ik-(i-3)} - t_{ik-(i-1)}) \ln(\sqrt{\alpha} + L))\} = \xi_2 < \infty \\ \cdots\cdots \\ \sup_{k\in\mathbb{N}}\{\ln\left(\prod_{j=1}^{i-1} l_{ik-j}^2\right) + 2(t_{ik} - t_{ik-(i-1)}) \ln(\sqrt{\alpha} + L))\} = \xi_{i-1} < \infty \end{cases}$$

Then the origin of system (0.6.2) is asymptotically stable.

Similar to the discussion in previous section, we obtain the following two corollaries.

Corollary 0.6.1. *Assume* $t_{2k+1} - t_{2k-1} = \times > 0$, $I_k = I(k \in \mathbb{N})$ *and* $\|x + I(x)\| \leq l\|x\|$, *where* $l \geq 0$ *is a constant. If there exists a constant* $\gamma > 1$ *such that*

$$\ln(\gamma l^4) + 2 \times \ln(\sqrt{\alpha} + L) \leq 0, \tag{0.6.7}$$

then the origin of system (0.6.2) is asymptotically stable.

Remark 0.6.1. *Form corollary 0.6.1, we get the estimate of the upper bound of impulsive sampling interval as follows:*

$$0 < \times \leq \left[\frac{-\ln(\gamma l^4)}{2\ln(\sqrt{\alpha} + L)} \right],$$

where $[\chi]$ *denotes the largest integer less than* χ ($\chi \in \mathbb{R}^+$).

Corollary 0.6.2. *Assume* $t_{ik+1} - t_{ik-(i-1)} = \times > 0$, $I_k = I(k \in \mathbb{N})$ *and* $\|x + I(x)\| \leq l\|x\|$, *where* $l \geq 0$ *is a constant. If there exists a constant* $\gamma > 1$ *such that*

$$\ln(\gamma l^{2i}) + 2 \times \ln(\sqrt{\alpha} + L) \leq 0, \tag{0.6.8}$$

then the origin of system (0.6.2) is asymptotically stable.

0.7 Numerical Examples

In this section, four examples are presented to evaluate the results obtained in previous sections.

Example 0.7.1. *Consider the following system [21]*

$$\begin{cases} x_1 = -(e^{-t} + 1)x_1 - \frac{x_1 x_2^2}{1+x_2^2}, & t \neq t_k \\ x_2 = -(e^{-t} + 1)x_2 - \frac{x_1^2 x_2}{1+x_1^2}, & t \neq t_k \\ x_1 = \frac{1}{2}x_1, & t = t_k \\ x_2 = \frac{1}{2}x_2, & t = t_k \end{cases} \tag{0.7.1}$$

on time scale $\mathbb{T}$ *with* $\mu(t) \leq \frac{1}{9}$.

Let $V(t, x) = x_1^2 + x_2^2$ and $K(t) = e^{-t} + 1$, then we get

$$\begin{aligned} V(t, x) &= x_1 x_1 + x_1^\sigma x_1 + x_2 x_2 + x_2^\sigma x_2 \\ &= x_1 x_1 + (x_1 + \mu x_1^\delta)x_1 + x_2 x_2 + (x_2 + \mu x_2^\delta)x_2 \\ &= 2x_1 x_1 + 2x_2 x_2 + \mu[(x_1)^2 + (x_2)^2] \end{aligned}$$

where $x_i^\sigma = x_i(\sigma(t))$, $i = 1, 2$.

As discussed in [21], we have the following estimates

$$V(t, x) \leq -2(e^{-t} + 1)V(t, x) + 9\mu(t)V(t, x) \leq (-2e^{-t} - 1)V(t, x).$$

When t is rd, then we have $\sigma(t) = t$ and

$$\begin{aligned} D^+K(t)V(t, x) + K(\sigma(t))D^+V(t, x) &\leq -e^{-t}V(t, x) + (e^{-t} + 1)(-2e^{-t} - 1)V(t, x) \\ &\leq (-2e^{-t} - 1)V(t, x) \leq -K(t)V(t, x) \end{aligned}$$

When t is rs, then we have $\sigma(t) > t$ and

$$\begin{aligned} D^+K(t)V(t, x) + K(\sigma(t))D^+V(t, x) &\leq \frac{e^{-\sigma(t)} - e^{-t}}{\mu(t)}V(t, x) + (e^{-\sigma(t)} + 1)(-2e^{-t} - 1)V(t, x) \\ &\leq (-2e^{-t} - 1)V(t, x) \leq -K(t)V(t, x) \end{aligned}$$

Thus,

$$D^+K(t)V(t, x) + K(\sigma(t))D^+V(t, x) \leq -K(t)V(t, x), \quad t \neq t_k, \quad k \in \mathbb{N}.$$

Next, we consider comparison system as follows:

$$\begin{cases} u = -u, \ \ t \neq t_k \\ u(t_k^+) = \frac{1}{2}u(t_k), \ \ t = t_k, \ k \in \mathbb{N}, \\ u(t_0) = u_0. \end{cases} \tag{0.7.2}$$

It is easy to see that $u(t)$ is decreasing on $\mathbb{T}$, $t \geq t_0$, and

$$u(t) \leq \frac{1}{2^k}u(t_0), \ \ t \in (t_k, t_{k-1}].$$

Therefore, the trivial solution of (0.7.2) is uniformly asymptotically stable.

If $h_0(t,x) = 2(x_1^2 + x_2^2)$, and $h(t,x) = 2|x_1x_2|$, then all the conditions of Theorem 0.4.2 are satisfied. Hence, system (0.7.1) is (h_0, h)-uniformly asymptotically stable.

Example 0.7.2. *Consider the following impulsive systems on time scale* $\mathbb{T}$

$$\begin{cases} x_1 = ax_1 + \sqrt{|\sin x_2^2|}, & t \neq t_k \\ x_2 = ax_2 + \sqrt{|\sin x_1^2|}, & t \neq t_k \\ x_1 = bx_2 - x_1, & t = t_k \\ x_2 = bx_1 - x_2, & t = t_k \\ x_1(0) = x_{10} \\ x_2(0) = x_{20} \end{cases} \tag{0.7.3}$$

where $a = 0.5$, $b = 0.195$*, and*

$$\mathbb{T} = \left(\bigcup_{i=0}^{\infty}[3i, 3i+1]\right) \bigcup \left(\bigcup_{j=0}^{\infty}\{3j+2\}\right),$$

$$t_k = \frac{3}{2}k + 2, \quad k = 1, 2, ...$$

We claim that the trivial solution of system (0.7.3) is asymptotically stable.

First, we rewrite system (0.7.3) in the following form

$$\begin{cases} x = Ax + \phi(x), & t \neq t_k \\ x = I_k(x), & t = t_k \\ x(0) = x_0 \end{cases}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \ A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \ \ \phi(x) = \begin{bmatrix} \sqrt{|\sin x_2^2|} \\ \sqrt{|\sin x_1^2|} \end{bmatrix}, \ \ I_k(x) = \begin{bmatrix} bx_2 - x_1 \\ bx_1 - x_2 \end{bmatrix}.$$

It is easy to see that $\alpha = \lambda_{max}\{A^T + A\} = 1$.

Next, we make the following estimation

$$||\phi(x)|| = \sqrt{|\sin x_2^2| + |\sin x_1^2|} \leq \sqrt{x_1^2 + x_2^2} = ||x||$$

then we can set constant $L = 1$.

Apparently, we have $||x + I_k(x)|| = |b| \cdot ||x||$, then l_k can be chosen as $l_k = |b|$.

By Remark 0.2.1 and the properties of the given time scale $\mathbb{T}$, we have

$$l_{2k-1}^2 e_q(t_{2k}, t_{2k-1}) = b^2\left((\sqrt{\alpha} + L)^2 + e^{\alpha+2L}\right) \approx 0.965 < \infty, \ \ k \in \mathbb{N},$$

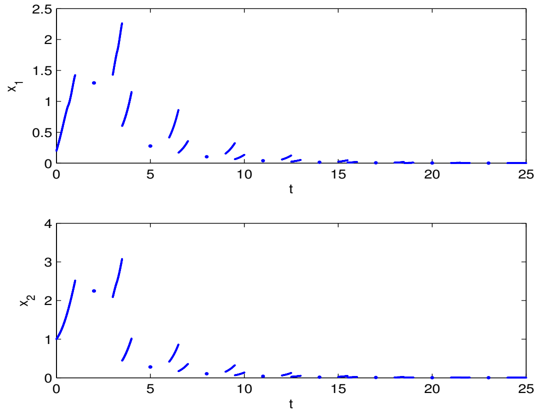

Figure 1: State trajectories of impulsive system (0.7.3) on time scale $\mathbb{T}$.

and

$$\gamma l_{2k-1}^2 l_{2k}^2 e_q(t_{2k+1}, t_{2k-1}) = \gamma l_{2k}^4 e_q^2(t_{2k+1}, t_{2k}) = \gamma b^4 \left((\sqrt{\alpha} + L)^2 + e^{\alpha+2L}\right)^2 = 0.94 \leq 1$$

where $\gamma = 1.01$. Thus, all the conditions of Theorem 0.4.4 are satisfied which means that system (0.7.3) is asymptotically stable. Fig.1 shows the simulation results with initial value $[x_{10}, x_{20}]^T = [0.2, 1]^T$.

Example 0.7.3. *Consider the following continuous chaotic system [29]*

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) + x_2 x_3 \\ \dot{x_2} = c x_1 - x_2 - x_1 x_3 \\ \dot{x_3} = x_1 x_2 - b x_3 \end{cases} \tag{0.7.4}$$

where the parameters $a = 35$, $b = 8/3$, $c = 20$ *with the initial condition* $[x_1(0), x_2(0), x_3(0)]^T = [3, 4, 5]^T$ *which ensure the existence of chaotic attractor shown in Fig.2.*

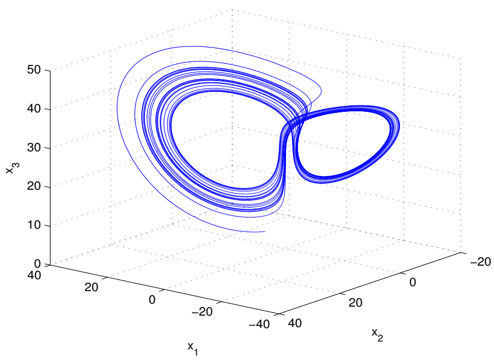

Figure 2: Chaotic attractor of continuous chaotic system (0.7.4).

System (0.7.4) can be rewritten as

$$\dot{x} = Ax + \phi(x)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad \phi(x) = \begin{bmatrix} x_2 x_3 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix},$$

then

$$A^T + A = \begin{bmatrix} -2a & a+c & 0 \\ a+c & -2 & 0 \\ 0 & 0 & -2b \end{bmatrix} = \begin{bmatrix} -70 & 60 & 0 \\ 60 & -2 & 0 \\ 0 & 0 & -\frac{16}{3} \end{bmatrix}$$

The eigenvalues of this matrix are -104.9634, -5.3333 and 32.9638. Thus $\alpha = \lambda_{max}\{A^T + A\} = 32.9638$.

Next we have the following estimation

$$\begin{aligned} \|\phi(x)\| &= \left(x_2^2x_3^2 + x_1^2x_3^2 + x_1^2x_2^2\right)^{\frac{1}{2}} \\ &= \left[(x_1^2 + x_2^2)x_3^2 + \frac{1}{2}x_1^4 + \frac{1}{2}x_2^4\right]^{\frac{1}{2}} \\ &\leq \sqrt{(x_1^2 + x_2^2)(x_1^2 + x_2^2 + x_3^2)} \\ &\leq \max_{t\in\mathbb{R}^+}\{\sqrt{x_1^2 + x_2^2}\}\|x\| \end{aligned}$$

From the chaotic attractor of system (0.7.4), we find $-30 \leq x_1 \leq 38$ and $-20 \leq x_2 \leq 25$, which lead to $\max_{t\in\mathbb{R}^+}\{\sqrt{x_1^2 + x_2^2}\} \leq 45.4863$. Hence, the constant L can be chosen to be $L = 45.4863$.

We design a nonlinear impulsive control function as follows

$$I_k(x) = \begin{bmatrix} e^{-1}\sqrt{\frac{1}{2}(x_2^2 + x_3^2)} - x_1 \\ e^{-1}\sqrt{\frac{1}{2}(x_1^2 + x_3^2)} - x_2 \\ e^{-1}\sqrt{\frac{1}{2}(x_1^2 + x_2^2)} - x_3 \end{bmatrix}, \quad k \in \mathbb{N}. \tag{0.7.5}$$

It is easy to see that $l_k = l = e^{-1}$. Letting $\gamma = 1.1$, the estimate bound of stable region is given by

$$0 < \times \leq \frac{-\ln\gamma - 4\ln l}{\alpha + L} \approx 0.0498,$$

where $\times = t_{2k+1} - t_{2k-1} (k \in \mathbb{N})$, i.e., if $0 < \times \leq 0.0498$, chaotic system (0.7.4) can be asymptotically stabilized by the nonlinear impulsive controller (0.7.5). Without loss of generality, we choose $\times = 0.04$ in this example.

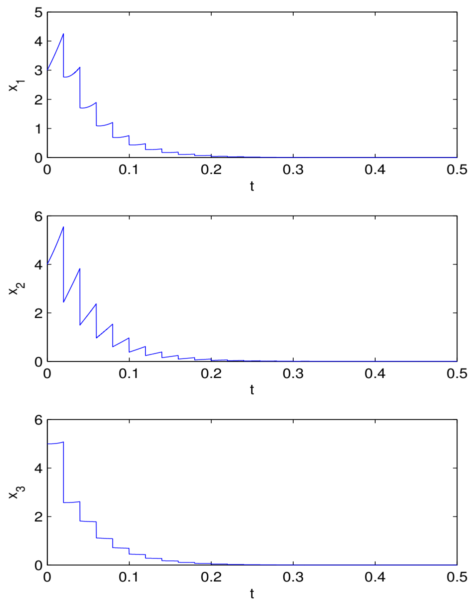

Figure 3: Stable results with uniform interval $t_{k+1} - t_k = 0.02$.

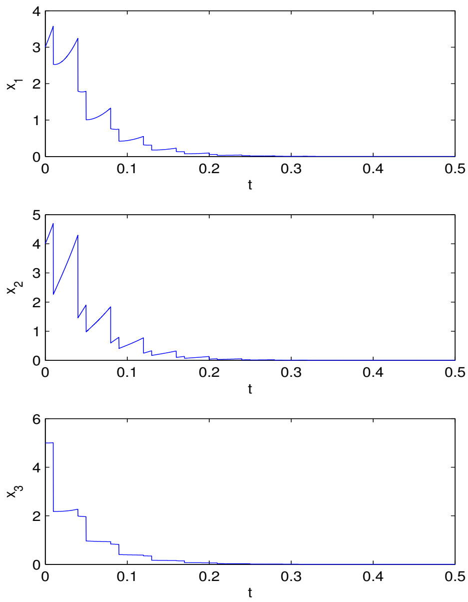

Figure 4: Stable results with $\times = 0.04$ and $t_{2k} - t_{2k-1} = 0.03$.

The numerical simulation results with uniform interval $t_{k+1} - t_k = \frac{1}{2} \times \ (k \in \mathbb{N})$ are shown in Fig.3.

By Remark 0.5.1 and Corollary 0.5.1, we only need to choose odd impulsive sequence $\{t_{2k-1}\}$, and with different choices of the even impulsive sequence $\{t_{2k}\}$, chaotic system (0.7.4) can also be stabilized by impulsive controller (0.7.5). Then, we have the following two numerical simulations: Fig.4 shows the state trajectory of system (0.7.4) under the nonlinear impulsive control with $t_{2k} - t_{2k-1} = \frac{3}{4} \times \ (k \in \mathbb{N})$; Fig.5 shows the results with $t_{2k} - t_{2k-1} = \frac{1}{4} \times \ (k \in \mathbb{N})$.

Remark 0.7.1. *We have investigate the problem of stabilization of continuous chaotic system (0.7.4) via nonlinear impulsive control. If we choose the impulsive controller as* $I_k(x) = B_k x$*, then our results reduce to the stability criteria in [29]. Simulation results show that our designed nonlinear impulsive controller are highly effective for asymptotically stabilizing the continuous chaotic system (0.7.4).*

Example 0.7.4. *Consider the following discrete chaotic system [32]*

$$\begin{cases} x_1(n+1) = 1.9x_1(n) - x_1^3(n) + x_2(n) \\ x_2(n+1) = 0.5x_1(n) \end{cases} \tag{0.7.6}$$

The initial value is given as $[x_1(0), x_2(0)]^T = [0.6, 0.3]^T$. *Chaotic attractor of system (0.7.6) is shown in Fig.6.*

System (0.7.6) can be rewritten as

$$x(n+1) = \bar{A}x(n) + \phi(x(n))$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 1.9 & 1 \\ 0.5 & 0 \end{bmatrix}, \quad \phi(x) = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix}.$$

It is easy to see that $\alpha = \lambda_{max}\{\bar{A}^T\bar{A}\} = 3.6780$.

Next we have the following estimation

$$||\phi(x)|| = \sqrt{x_1^6} \leq \sqrt{x_1^4(x_1^2 + x_2^2)} \leq \max_{n \in \mathbb{Z}^+}\{x_1^2\}||x||$$

By Fig.6, we find that $-1.7 \leq x_1 \leq 1.7$ which implies that $\max_{n\in\mathbb{Z}^+}\{x_1^2\} \leq 2.89$. Hence, the constant L can be chosen

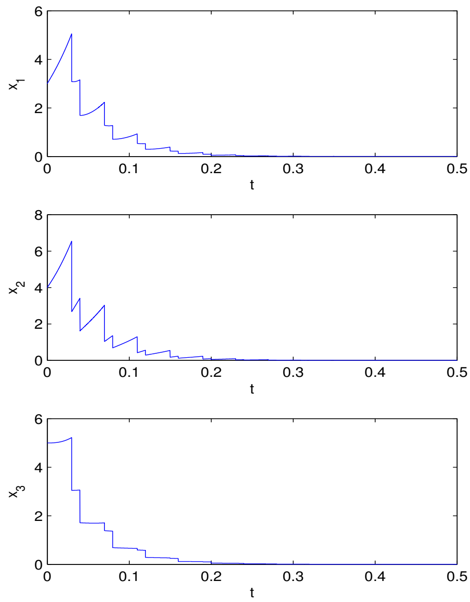

Figure 5: Stable results with $\times = 0.04$ and $t_{2k} - t_{2k-1} = 0.01$.

to be $L = 2.89$.

We design a nonlinear impulsive control function as follows

$$I_k(x) = \begin{bmatrix} a\sin x_2 - x_1 \\ a\sin x_1 - x_2 \end{bmatrix}, \quad k \in \mathbb{N}, \tag{0.7.7}$$

where we choose $a = 0.04$.

Since

$$\|x + I_k(x)\| = \sqrt{a^2(\sin^2 x_1 + \sin^2 x_2)} \le |a|\sqrt{x_1^2 + x_2^2} = |a| \cdot \|x\|,$$

then we choose $l_k = l = a$. By Remark 0.6.1, the estimate of the bound of stable region is given by

$$0 < 2\delta = \times \le \left[\frac{-\ln\gamma - 4\ln l}{2\ln\sqrt{\alpha} + L}\right] = [4.0976] = 4,$$

where $\delta = t_{k+1} - t_k (k \in \mathbb{N})$ and $\gamma = 1.01$.

In the following simulation, we choose $\delta = 2$, then all the conditions of Corollary 0.6.1 are satisfied, i.e. the impulsive controlled discrete chaotic system is asymptotically stable. Stable results are shown in Fig.7.

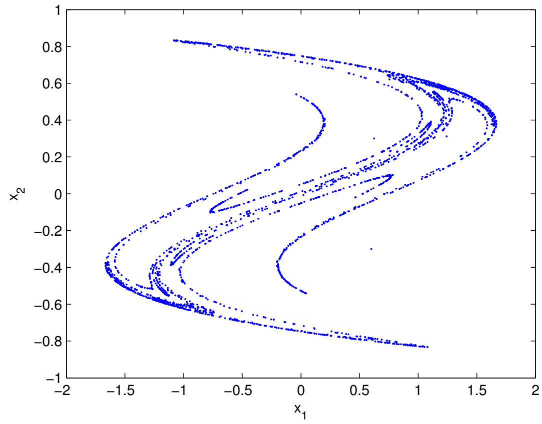

Figure 6: Chaotic attractor of discrete chaotic system (0.7.6).

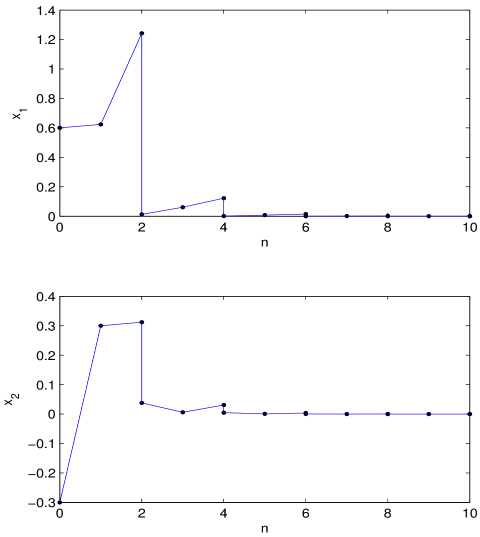

Figure 7: Stable results with uniform interval $n_k - n_{k-1} = 2$.

References

REFERENCES

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of impulsive differential eqations, Series in modern applied mathemaitcs, Singapore and Teaneck, Word Scientific, 1989.

[2] A.M. Samoilenko, N.A. Peretyuk, Impulsive differential equations, World Scientific, Singapore, 1995.

[3] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamic Systems on Measure Chains, Kluwer Academic Publishers, Boston, 1996.

[4] M. Bohner, A. Peterson, Dynamic Equation on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[5] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Seales, Birkhäuser, Boston, 2003.

[6] S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD. Thesis, Universität Würzburg, 1988 (in German).

[7] Z. Bartosiewicz, E. Pawluszewicz, Realizations of linear control systems on time scales, Control and Cybernetics, 35 (4) (2006) 769-786.

[8] Z. Bartosiewicz, E. Pawluszewicz, Realizations of nonlinear control systems on time scales, IEEE Trans. Automat. Contr. 53 (2) (2008) 571-575.

[9] Z. Zhan, W. Wei, On existence of optimal control governed by a class of the first-order linear dynamic systems on time scales, Appl. Math. Comput. 215 (2009) 2070-2081.

[10] F.M. Atici, D.C. Biles, A. Lebedinsky, An application of time scales to economics, Math. Comput. Model. 43 (2006) 718-726.

[11] G.S. Guiseinov, E. Ozyylmaz, Tangent lins of generalized regular curves parameterized by time scales, Turk. J. Math. 25 (2001) 553-562.

[12] A. Chen, D. Du, Global exponential stability of delayed BAM networks on time scales, Neurocomputing, 71 (2008) 3582-3588.

[13] Y, Li, X. Chen, L. Zhao, Stability and existence of periodic solutions to delayed Chhen-Grossberg BAM neural networks with impulses on time scales, Neurocomputing 72 (2008) 1621-1630.

[14] C. P¡§ozsche, S. Siegmund, F. Wirth, A spectral chatacterization of exponential stability for linear time-invariant systems on time scales, Discrete Contin. Dyn. Syst., 9 (2003) 1223- 1241.

[15] Choi S.K., Koo N.J., On the stability of linear dynamic systems on time scales, J. Difference Equ. Appl., 15 (2) (2009) 167-183.

[16] T.S. Doana, A. Kalauch, S. Siegmunda, F.R. Wirthb, Stability radii for positive linear time-invariant systems on time scales, Syst. Control Lett. 59 (2010) 173-179.

[17] S. Hong, Stability criteria for set dynamic equations on time scales, Comput. Math. Appl., 59 (2010) 3444-3457.

[18] V. Lakshmikantham, A.S. Vatsala, Hybrid systems on time scales, J. Comput. Appl. Math., 141 (2002) 227-235.

[19] Y.J. Ma, J.T. Sun, Stability criteria of delayed impulsive systems on time scales, Nonlinear Anal., 67 (2007) 1181-1189.

[20] Y.J. Ma, J.T. Sun, Stability criteria of impulsive systems on time scales, J. Comput. Appl. Math., 213 (2008) 400-407.

[21] Y.J. Ma, J.T. Sun, Uniform eventual Lipschtiz stability impulsive systems on time scales, Appl. Math. Comput., 211 (2009) 246-250.

[22] P.G. Wang, M. Wu, Practical ϕ_0- stability of impulsive dynamic systems on time scales, Appl. Math. Lett., 20 (2007) 651-658.

[23] P.G. Wang, M. Wu, On the ϕ_0- stability of impulsive dynamic system on time scales, Electonic Journal of Differential Equations, 2005 (2005) 128: 1-7.

[24] T.Yang, L.O.Chua,Impulsive stabilizaton of control and synchronization of chaotic systems: theory and application to secure communication, IEEE Trans.Circuits Syst.I, 44 (1997) 976-988.

[25] H. Zhang, D. Liu, Z. Wang, Controlling Chaos · Suppression, Synchronization and Chaotification, Springer, London 2009.

[26] Y. Zheng, Y. Nian, Z. Liu, Impulsive control for the stabilization of discrete chaotic systems, Chin. Phys. Lett, 19 (2002) 1251-1253.

[27] J. Sun, Stability criteria of impulsive differential systems, Appl. Math. Comput., 156 (2004) 85-91.

[28] J. Sun, Y. Zhang, Q. Wu, Less conservative conditions for asymptotic stability of impulsive control systems, IEEE Trans. Automat. Contr. 48 (5) (2008) 829-831.

[29] G. Qi, G. Chen, S. Du, Z. Chen, Z. Yuan, Analysis of a new chaotic system, Physica A, 352 (2005) 295-308.

[30] Z. Zhang, X. Liu, Robust stability of uncertain discrete impulsive switching systems, Comput.Math.Appl., 58 (2009) 380-389.

[31] Z. Zhang, Rubost $H\infty$ control of a class of discrete impulsive switched systems, Nonlinear Anal., 71 (2009) e2790-e2796.

[32] T. Ushio, Chaotic synchronization and controlling chaos based on contraction mappings, Phys. Lett. A, 198 (1995) 14.

[33] V. Lakshmikantham, X. Liu, Stability criteria for impulsive differential equations in terms of two measures, J. Math. Anal. Appl., 137 (1989) 591-604.

[34] V. Lakshmikantham, X. Liu, Stability for impulsive systems in terms of two measures, Appl. Math. Comput., 29 (1989) 89-98.

A Shrinking Projection Method for a Common Zero of a Finite Family of Monotone Operators

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ABSTRACT: The purpose of this paper is to introduce some strong convergence theorems for the problem of finding a common zero of a finite family of monotone operators and also, we give some similar methods for the problem of finding a common fixed point of a finite family of nonexpansive mappings in Hilbert spaces by shrinking projection method.

0.1 INTRODUCTION

Numerous problems in mathematics and physical sciences can be recast in terms of the problem finding a zero for monotone operator. For instance, when the maximal monotone operator is the subdifferential of a proper, lower semicontinuous and convex function, it gives an approximation to the solutions of a minimization problem for the convex function.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We use the symbols $\rightharpoonup$ and $\rightarrow$ to denote the weak convergence and strong convergence, respectively.

We consider the following problem: Finding an element $x \in H$ such that for all $i = 1, 2, ..., N$,

$$0 \in A_i(x). \tag{0.1.1}$$

where $A_i : D(A_i) \subset H \longrightarrow 2^H$ are monotone operators. We denote the set of solution of this problem by S, that is

$$S := \{x \in H : 0 \in A_i(x), \ \forall i = 1, 2, ..., N\}.$$

One of the classical methods for solving equation $0 \in A(x)$ with A is a maximal monotone operator in Hilbert space H, is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in H$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J^A_{r_n}(x_n), \tag{0.1.2}$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence of positive real numbers and

$$J^A_{r_n} = (I + r_n A)^{-1}$$

is the resolvent of A. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (0.1.2) and others proved strong convergence theorems by imposing assumptions on A.

Note that, algorithm (0.1.2), can be rewritten as

$$x_{n+1} - x_n + r_n A(x_{n+1}) \ni 0, \tag{0.1.3}$$

for all $n \in \mathbb{N}$, This algorithm was first introduced by Martinet [14]. If $\psi : H \to \mathbb{R} \cup \{\infty\}$ is proper lower semicontinuous convex function, the algorithm reduces to

$$x_{n+1} = \operatorname{argmin}_{y \in H} \left\{ \psi(y) + \frac{1}{2c} \|x_n - y\|^2 \right\},$$

for all $n \in \mathbb{N}$. Moreover, Rockafellar [16] has given a more practical method which is an inexact variant of the method:

$$x_n + e_n \ni x_{n+1} + c_n A x_{n+1}, \tag{0.1.4}$$

for all $n \in \mathbb{N}$, where $\{e_n\}$ is regarded as an error sequence and $\{c_n\}$ is a sequence of positive regularization parameters.

Note that the algorithm (0.1.4) can be rewritten as

$$x_{n+1} = J^A_{r_n}(x_n + e_n), \tag{0.1.5}$$

for all $n \in \mathbb{N}$. This method is called inexact proximal point algorithm. It was shown in Rockafellar [16] that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in Az$.

Further, Rockafellar [16] posed an open question of whether the sequence generated by (0.1.2) converges strongly or not. In 1991, Güler [6] gave an example showing that Rockafellar's proximal point algorithm does not converges strongly. An example of the authors Bauschke, Matoušková and Reich [1] also showed that the proximal algorithm only converges weakly but not in norm.

In 2000, Solodov and Svaiter [18] proposed the following algorithm: Choose any $x_0 \in H$ and $\sigma \in [0, 1)$. At iteration n, having x_n, choose $\mu_n > 0$ and find (y_n, v_n) an inexact solution of

$$0 \in A(x) + \mu_n(x - x_n),$$

with tolerance σ. Define

$$C_n = \{z \in H : \langle z - y_n, v_n\rangle \le 0\}$$

and

$$Q_n = \{z \in H : \langle z - x_n, x_0 - x_n\rangle \le 0\}.$$

Take

$$x_{n+1} = P_{C_n \cap Q_n} x_0.$$

They prove that if the sequence of the regularization parameters μ_n is bounded from above, then $\{x_n\}$ converges strongly to $x^* \in A^{-1}0$.

To find a fixed point of a nonexpansive mapping T on the closed and convex subset C of H, that is, find an element $p \in F(T) = \{x \in C : Tx = x\}$, motivated by result of Solodov and Svaiter, in 2008, Takahashi et al. [15] introduced the following iterative algorithm:

$$\begin{cases} C_0 = C,\ x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 0, \end{cases} \tag{0.1.6}$$

and they proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $\{\alpha_n\} \subset [0, a)$, with $a \in [0, 1)$. Moreover, they also gave a similar iterative sequence to find zero of a maximal monotone in the following form:

$$\begin{cases} C_0 = C,\ x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)J^A_{r_n} x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 0. \end{cases} \tag{0.1.7}$$

They showed that if $\{\alpha_n\} \subset [0, a)$, with $a \in [0, 1)$ and $r_n \to \infty$, then the sequence $\{x_n\}$ generated by (0.1.7) converges strongly to $P_{A^{-1}0}x_0$.

Further, some generalized hybrid projection methods have been introduced for families of relatively or weak relatively nonexpansive mappings (see, [3, 8, 19, 21, 22]), or equilibrium problems and fixed point problems (see, [2, 4, 9, 11, 13, 17, 20]).

Recently, Kim et al. [12] introduced a new method which is a combination of proximal point algorithm, viscosity approximation method and alternating resolvent method to find a common zero of two accretive operators in Banach spaces.

In this paper, base on iterative methods (0.1.6) and (0.1.7) of Takahashi et al. [15], we introduce some new iterative methods to find a common zero of a finite family of monotone operators and a common fixed point of a finite family of nonexpansive mappings in a real Hilbert space (cf.[10]).

0.2 PRELIMINARIES

Let C be a nonempty, closed and convex subset of a real Hilbert space H. We know that for each $x \in H$, there is unique $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u \in C} \|x - u\|, \tag{0.2.1}$$

and the mapping $P_C : H \longrightarrow C$ define by (0.2.1) is called metric projection from H onto C. Moreover, we have

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{0.2.2}$$

for all $x \in H,\ y \in C$. Recall that, a mapping $T : C \longrightarrow C$ is said to be nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote the set of fixed point of T by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.

For an operator $A : H \longrightarrow 2^H$, we define its domain, range and graph as follows:

$$D(A) = \{x \in H : Ax \neq \emptyset\},$$
$$R(A) = \cup\{Az : z \in D(A)\},$$

and

$$G(A) = \{(x, y) \in H \times H : x \in D(A),\ y \in Ax\},$$

respectively. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y, \text{ if and only if } y \in Ax.$$

The operator A is said to be monotone if, for each $x, y \in D(A)$, we have

$$\langle u - v, x - y\rangle \geq 0$$

for all $u \in Ax$ and $v \in Ay$. We denote by I the identity operator on H. A monotone operator A is said to be maximal monotone if there is no proper monotone extension of A or $R(I + \lambda A) = H$ for all $\lambda > 0$. If A is monotone, then we can define, for each $\lambda > 0$, a nonexpansive s ingle-valued mapping $J_\lambda^A : R(I + \lambda A) \longrightarrow D(A)$ by

$$J_\lambda^A = (I + \lambda A)^{-1},$$

it is called the resolvent of A. A monotone operator A is said to satisfy the range condition if

$$\overline{D(A)} \subset R(I + \lambda A)$$

for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. We know that for a monotone operator A which satisfies the range condition, $A^{-1}0 = F(J_\lambda^A)$ for all $\lambda > 0$.

Remark 0.2.1. If A is a maximal monotone, then A satisfies the range condition.

The following lemmas will be needed in the sequel for the proof of main results in this paper.

Lemma 0.2.1. [7] *Let* H *be a real Hilbert space. Then for all* $x, y \in H$ *and* $t \in [0,1]$*, we have*

$$\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2.$$

Lemma 0.2.2. *Let* H *be a real Hilbert space and let* C *be a nonempty, closed and convex subset of* H*. Then, for all* $x \in H$ *and* $y \in C$*, we have*

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \le \|x - y\|^2,$$

where P_C *is the metric projection of* H *onto* C.

Proof: *We have*

$$\begin{aligned}\|x-y\|^2 &= \|(x - P_C x) - (y - P_C y)\|^2 \\ &= \|x - P_C x\|^2 + \|y - P_C x\|^2 - 2\langle x - P_C x, y - P_C y\rangle.\end{aligned}$$

From (0.2.2)*, we get the proof of this lemma.*

Lemma 0.2.3. [5] *Let* $A : D(A) \longrightarrow 2^H$ *be a monotone operator. For* $r \ge s > 0$*, we have*

$$\|x - J_s^A x\| \le 2\|x - J_r^A x\|,$$

for all $x \in R(I + rA) \cap R(I + sA)$.

0.3 MAIN RESULTS

First, we define the following iterative sequences for finding a common zero of a finite family of monotone operators in a Hilbert space:

Let H be a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let $A_i : D(A_i) \subset H \longrightarrow 2^H$, $i = 1, 2, ..., N$, be monotone operators with

$$S := \bigcap_{i=1}^{N} A_i^{-1} 0 \neq \emptyset$$

and $\overline{D(A_i)} \subset C \subset \cap_{r>0} R(I + rA_i)$ for all $i = 1, 2, ..., N$. Let $\{\beta_n^i\}$ and $\{r_n^i\}$, $i = 1, 2, ..., N$ be sequences of positive real numbers such that $\{\beta_n^i\} \subset (\alpha, \beta)$, with $\alpha,\ \beta \in (0,1)$ and

$$\min_{1 \le i \le N} \left\{ \inf_n \{r_n^i\} \right\} \ge r > 0.$$

Then, we define the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} C_0 = C, \\ y_n^0 = x_n, \\ y_n^i = \beta_n^i y_n^{i-1} + (1 - \beta_n^i) J_{i,n} y_n^{i-1},\ J_{i,n} = J_{r_n^i}^{A_i},\ i = 1, 2, ..., N \\ C_{n+1} = \{z \in C_n :\ \|y_n^N - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 0, \end{cases} \tag{0.3.1}$$

Lemma 0.3.1. *In (3.1), we know that for all* $n \ge 0$*,* $S \subset C_n$.

Proof: *We show the result by induction on* n*. Obviously,* $S \subset C_0 = C$*. Suppose that* $S \subset C_n$ *for some* $n \ge 0$*. Then, for*

$u \in S \subset C_n$, *we have*

$$\begin{aligned}\|y_n^N - u\| &= \|\beta_n^N y_n^{N-1} + (1-\beta_n^N)J_{N,n}y_n^{N-1} - u\| \\ &\le \beta_n^N\|y_n^{N-1} - u\| + (1-\beta_n^N)\|J_{N,n}y_n^{N-1} - u\| \\ &\le \beta_n^N\|y_n^{N-1} - u\| + (1-\beta_n^N)\|y_n^{N-1} - u\| \\ &= \|y_n^{N-1} - u\| \\ &\ \ \vdots \\ &\le \|y_n^0 - u\| \\ &= \|x_n - u\|.\end{aligned}$$

This implies that $u \in C_{n+1}$. *Hence we have* $S \subset C_n$ *for all* $n \ge 0$.

Lemma 0.3.2. *In (3.1), we know that* C_n *is closed and convex for each* $n \ge 0$. *And also, this implies that* $\{x_n\}$ *is well-defined.*

Proof: *It is clear that* $C_0 = C$ *is closed and convex. Suppose that* C_n *is a closed and convex subset of* C. *From the inequality* $\|y_n^N - z\| \le \|x_n - z\|$, *we obtain*

$$\langle x_n - y_n^N, z\rangle \le \frac{1}{2}\Big(\|x_n\|^2 - \|y_n^N\|^2\Big).$$

Therefore,

$$C_{n+1} = C_n \bigcap \Big\{z \in H :\ \langle x_n - y_n^N, z\rangle \le \frac{1}{2}(\|x_n\|^2 - \|y_n^N\|^2)\Big\},$$

which implies that C_{n+1} *is closed and convex. Thus,* C_n *is closed and convex subset of* C *for each* $n \ge 0$. *This implies that* $\{x_n\}$ *is well-defined.*

Lemma 0.3.3. *In (3.1), the sequence* $\{x_n\}$ *is bounded and there exists the finite limit* $\lim_{n\to\infty}\|x_n - x_0\|$.

Proof: *Since* S *is closed and convex, there exists the metric projection* P_S *such that* $x^* = P_S x_0 \in S$. *Since* $S \subset C_n$ *from Lemma 3.1,* $x^* \in C_n$. *Thus, from* $x_n = P_{C_n}x_0$, *we have*

$$\|x_n - x_0\| \le \|x_0 - x^*\|, \tag{0.3.2}$$

this implies that $\{x_n\}$ *is bounded. Next, we have* $C_{n+1} \subset C_n$ *from the definition of* C_n. *So,* $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$. *Now taking into account* $x_n = P_{C_n}x_0 \in C_n$ *and using Lemma 0.2.2, we get*

$$\|x_n - x_0\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2.$$

This implies that $\{\|x_n - x_0\|\}$ *is nondecreasing. And by the boundedness of* $\{x_n\}$, *the limit of* $\{\|x_n - x_0\|\}$ *exists.*

Now we are in a position to prove the problem to find a common zero of a finite family of monotone operators in Hilbert spaces with suitable conditions:

Theorem 0.3.4. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $A_i :\ D(A_i) \subset H \longrightarrow 2^H$ *be monotone operators with*

$$S := \bigcap_{i=1}^{N} A_i^{-1}0 \ne \emptyset$$

and $\overline{D(A_i)} \subset C \subset \cap_{r>0}R(I + rA_i)$ *for all* $i = 1, 2, ..., N$. *Let* $\{\beta_n^i\}$ *and* $\{r_n^i\}$, $i = 1, 2, ..., N$ *be sequences of positive real numbers such that* $\{\beta_n^i\} \subset (\alpha, \beta)$, *with* $\alpha,\ \beta \in (0, 1)$ *and* $\min_{i=1,2,...,N}\{\inf_n\{r_n^i\}\} \ge r > 0$. *Then the sequence* $\{x_n\}$ *generated by (3.1) converges strongly to* $P_S x_0$. **Proof:** *In order to prove the theorem, we divide three steps.*

Step 1. *First, the sequence* $\{x_n\}$ *converges strongly to some point* $p \in C$. *Indeed, for all* $m \ge n$, *we have* $C_m \subset C_n$. *Thus,* $x_m \in C_n$. *By Lemma 0.2.2, we have*

$$\|x_m - x_n\|^2 \le \|x_m - x_0\|^2 - \|x_n - x_0\|^2 \to 0,$$

as $m, n \to \infty$. *So,* $\{x_n\}$ *is Cauchy sequence. Since* C *is closed subset of* H*, there exists a* $p \in C$ *such that* $\lim_{n\to\infty} x_n = p$.

Step 2. *Next, we prove that* $\lim_{n\to\infty} \|x_n - y_n^N\| = 0$*. From Step 1, we have*

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - p\| + \|x_n - p\| \to 0,$$

which implies that $\|x_{n+1} - x_n\| \to 0$ *as* $n \to \infty$*. By* $x_{n+1} \in C_{n+1}$*, we obtain that*

$$\|x_{n+1} - y_n^N\| \leq \|x_{n+1} - x_n\| \to 0.$$

So, $\|x_{n+1} - y_n^N\| \to 0$*, as* $n \to \infty$*. From the following estimate*

$$\|x_n - y_n^N\| \leq \|x_{n+1} - y_n^N\| + \|x_{n+1} - x_n\|,$$

we get $\lim_{n\to\infty} \|x_n - y_n^N\| = 0$.

Step 3. *Next, we prove that* $\|x_n - J_{i,n}x_n\| \to 0$ *and* $\|x_n - J_r^{A_i}x_n\| \to 0$ *for all* $i = 1, 2, ..., N$*, as* $n \to \infty$*. Since, for* $x^* = P_S x_0$*,*

$$\begin{aligned}
\|x_n - x^*\| &\leq \|x_n - y_n^N\| + \|y_n^N - x^*\| \\
&\leq \|x_n - y_n^N\| + \|y_n^{N-1} - x^*\| \\
&\ \ \vdots \\
&\leq \|x_n - y_n^N\| + \|y_n^0 - x^*\|
\end{aligned}$$

and so, the both sides of the above inequality go to $\|p - x^*\|$*, as* $n \to \infty$*, it implies that*

$$\lim_{n\to\infty} \|y_n^{i-1} - x^*\| = \|p - x^*\|, \tag{0.3.3}$$

for all $i = 1, 2, ..., N$*. From Lemma 0.2.1, we have*

$$\begin{aligned}
\|y_n^i - x^*\|^2 &= \beta_n^i \|y_n^{i-1} - x^*\|^2 + (1 - \beta_n^i)\|J_{i,n}y_n^{i-1} - x^*\|^2 \\
&\quad - 2\beta_n^i(1 - \beta_n^i)\|y_n^{i-1} - J_{i,n}y_n^{i-1}\|^2 \\
&\leq \beta_n^i \|y_n^{i-1} - x^*\|^2 + (1 - \beta_n^i)\|y_n^{i-1} - x^*\|^2 \\
&\quad - 2\beta_n^i(1 - \beta_n^i)\|y_n^{i-1} - J_{i,n}y_n^{i-1}\|^2 \\
&= \|y_n^{i-1} - x^*\|^2 - 2\beta_n^i(1 - \beta_n^i)\|y_n^{i-1} - J_{i,n}y_n^{i-1}\|^2,
\end{aligned}$$

this implies that

$$2\alpha(1 - \beta)\|y_n^{i-1} - J_{i,n}y_n^{i-1}\|^2 \leq \|y_n^{i-1} - x^*\|^2 - \|y_n^i - x^*\|^2 \to 0,$$

as $n \to \infty$ *for all* $i = 1, 2, ..., N$*. Hence,*

$$\|y_n^{i-1} - J_{i,n}y_n^{i-1}\| \to 0, \tag{0.3.4}$$

for all $i = 1, 2, ..., N$.

Hence, for the case $i = 1$*, we have* $\|x_n - J_{1,n}x_n\| = \|y_n^0 - J_{1,n}y_n^0\| \to 0$*. In the case* $i = 2$*, also we have*

$$\begin{aligned}
\|x_n - J_{2,n}x_n\| &\leq \|x_n - y_n^1\| + \|y_n^1 - J_{2,n}y_n^1\| + \|J_{2,n}y_n^1 - J_{2,n}x_n\| \\
&\leq 2\|x_n - y_n^1\| + \|y_n^1 - J_{2,n}y_n^1\| \\
&\leq 2\|y_n^0 - J_{1,n}y_n^0\| + \|y_n^1 - J_{2,n}y_n^1\| \\
&\to 0,
\end{aligned}$$

it implies that $\|x_n - J_{2,n}x_n\| \to 0$*. Similarly, we obtain that* $\|x_n - J_{i,n}x_n\| \to 0$ *for all* $i = 3, 4, ..., N$*. Consequently, we have* $\|x_n - J_{i,n}x_n\| \to 0$ *for all* $i = 1, 2, ..., N$.

On the other hand, from Lemma 0.2.3, we have

$$\|x_n - J_r^{A_i} x_n\| \leq 2\|x_n - J_{i,n} x_n\|.$$

So, $\|x_n - J_r^{A_i} x_n\| \to 0$ *for all* $i = 1, 2, ..., N$.

Step 4. *Finally, we prove that* $p = x^* = P_S x_0$. *Since, from Step 1,* $x_n \to p$, *letting* $n \to \infty$ *in above inequality, then we get*

$$p \in \bigcap_{i=1}^{N} F(J_r^{A_i}) = S.$$

In (0.3.2), *letting* $n \to \infty$, *we get*

$$\|x_0 - p\| \leq \|x_0 - x^*\|,$$

and by the uniqueness of x^*, *we obtain that* $p = x^*$. *This completes the proof.*

Remark 0.3.1. Every maximal monotone operators in a Hilbert space has the range condition, so Theorem 0.3.4 holds still without range condition.

Remark 0.3.2. If $N = 1$, we can choose the sequence $\{\beta_n\} \subset [0, a)$, with $a \in [0, 1)$. So, we have the following corollary.

Corollary 0.3.5. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $A : D(A) \subset H \longrightarrow 2^H$ *be a monotone operator such that*

$$S := A^{-1}0 \neq \emptyset$$

and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. *Let* $\{\beta_n\}$ *and* $\{r_n\}$ *be sequences of positive real numbers such that* $\{\beta_n\} \subset [0, a)$, *with* $a \in [0, 1)$ *and* $\inf_n\{r_n\} \geq r > 0$. *Then the sequence* $\{x_n\}$ *generated by* $x_0 \in C$ *and*

$$\begin{cases} C_0 = C, \\ y_n = \beta_n x_n + (1 - \beta_n) J_{r_n}^A x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0, \end{cases} \tag{0.3.5}$$

converges strongly to $P_S x_0$.

Remark 0.3.3. The Corollary 0.3.5 is more general than the result of Takahashi et al. in [15] (see, Theorem 4.5).

Next, we give strong convergence theorems to find a common fixed point of nonexpansive mapping. By the careful analysis of the proof of Theorem 0.3.4, we can obtain the following results for the problem of finding a common fixed point of a finite family of nonexpasive mappings. Because its proof is much simpler than that of Theorem 0.3.4, we omit the proof.

Theorem 0.3.6. *Let* C *be a nonempty closed and convex subset of areal Hilbert space* H. *Let* $T_i : C \longrightarrow C$, $i = 1, 2, ..., N$, *be nonexpansive mappings from* C *into itself with*

$$\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset.$$

Let $\{\beta_n^i\}$ *be sequences of positive real numbers such that* $\{\beta_n^i\} \subset (\alpha, \beta)$ *with* $\alpha, \ \beta \in (0, 1)$ *for all* $i = 1, 2, ..., N$. *Then* *the sequence* $\{x_n\}$ *generated by* $x_0 \in C$ *and*

$$\begin{cases} C_0 = C, \\ y_n^0 = x_n, \\ y_n^i = \beta_n^i y_n^{i-1} + (1-\beta_n^i)T_i y_n^{i-1}, \ i = 1, 2, ..., N \\ C_{n+1} = \{z \in C_n : \ \|y_n^N - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \ge 0, \end{cases} \tag{0.3.6}$$

converges strongly to $P_{\mathcal{F}} x_0$.

Corollary 0.3.7 (Theorem 4.1, in [15])**.** *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $T : C \longrightarrow C$ *be a nonexpansive mapping from* C *into itself with* $F(T) \neq \emptyset$. *Let* $\{\beta_n\}$ *be a sequence of positive real numbers such that* $\{\beta_n\} \subset [0, a)$*, with* $a \in [0, 1)$. *Then the sequence* $\{x_n\}$ *generated by* $x_0 \in C$ *and*

$$\begin{cases} C_0 = C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \ \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \ge 0, \end{cases} \tag{0.3.7}$$

converges strongly to $P_{F(T)} x_0$.

Remark 0.3.4. The Theorem 0.3.6 is more general than the result of Takahashi et al. in [15] (see, Theorem 4.1).

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REFERENCES

[1] H. H. Bauschke, E. Matoušková, S. Reich, Projection and proximal point methods convergence results and counterexamples, Nonlinear Anal., 56 (2004), 715-738.

[2] S.S. Chang , H.W. Joseph Lee, C.K. Chan, A new hybrid method for solving generalized equilibrium problem variational inequality and common fixed point in Banach spaces with applications, Nonlinear Anal., 73 (2010), 2260-2270.

[3] L.C. Ceng, A. Petruael, S.Y. Wu, On hybrid proximal-type algorithms in Banach spaces, Taiwan. J. Math., 12 (8) (2008), 2009-2029.

[4] L.C. Ceng, S.M. Guu, H.Y. Hu, J.C. Yao, Hybrid shrinking projection method for a generalized equilibrium problem, a maximal monotone operator and a countable family of relatively nonexpansive mappings, Comput. and Math. with Appl., 61 (2011), 2468-2479.

[5] H. Cui, M. Su, On sufficient conditions ensuring the norm convergence of an iterative sequence to zeros of accretive operators, Appl. Math. Comput., 258 (2015), 67-71.

[6] O. Güler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Contr. Optim., 29 (1991), 403-419.

[7] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336-346.

[8] J. Kang, Y. Su and X. Zhang, Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications, Nonlinear Anal. Hybrid Syst., 4 (2010), 755-765.

[9] J.K. Kim, Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-φ- nonexpansive mappings, Fixed Point Theory Appl. 2011:10 (2011).

[10] J.K. Kim and Ng. Buong, Convergence rates in regularization for a system of nonlinear Ill-posed equations with m-accretive operator, Jour. of Inequ. and Appl., 2014:440 (2014).

[11] J.K. Kim and Ng. Buong, A new iterative method for equilibrium problems and fixed point problems for infinite family of nonself strictly pseudocontractive mappings, Fixed Point Theory and Appl., 2013:286 (2013).

[12] J.K. Kim and T.M. Tuyen, Approximation common zero of two accretive operators in Banach spaces, Applied Math. and Comput., 283(2016), 265-281.

[13] J.K. Kim and T.M. Tuyen, On the some regularization methods for common fixed point of a finite family of nonexpansive mappings, Jour. of Non. and Convex Anal., 17(1) (2016), 99-104.

[14] B. Martinet, Regularisation dinequations variationnelles par approximations successives, Rev. FranMc-aise Informat, Recherche Operationnalle, 4 (1970), 154-158.

[15] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 341 (2008), 276-286.

[16] R.T. Rockaffelar, Monotone operators and proximal point algorithm, SIAM J. on Contr. and Optim., 14 (1976), 887-897.

[17] S. Saewan, P. Kumam, The shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-φ-nonexpansive mappings, Fixed Point Theory and Appl., 2011: 9 (2011).

[18] M.V. Solodov, B.F. Svaiter, Forcing strong convergence of proximal point iterations in Hilbert space, Math. Progr. 87 (2000), 189-202.

[19] Y.F. Su, Z.M. Wang, H.K. Xu, Strong convergence theorems for a common fixed point of two hemi-relatively nonexpansive mappings, Nonlinear Anal. 71 (2009), 5616-5628.

[20] Z.M. Wang, Convergence theorems based on the shrinking projection method for hemi-relatively nonexpansive mappings, variational inequalities and equilibrium problems, Nonlinear Funct. Anal. and Appl., 22 (2017), 459-483.

[21] Z.M. Wang, P. Kumam, Hybrid projection algorithm for two countable families of hemirelatively nonexpansive mappings and applications, J. Appl. Math. 2013 (2013), Article ID 524795.

[22] K. Wattanawitoon, P. Kumam and U.W. Humphries, Strong convergence theorem by the shrinking projection method for hemi-relatively nonexpansive mappings, Thai Jour. of Math. 7 (2009), 329-337.

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