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GLOBAL SMOOTH ATTRACTORS FOR DYNAMICS OF THERMAL SHALLOW SHELLS WITHOUT VERTICAL DISSIPATION

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Abstract. Nonlinear shallow shell models with thermal effects are considered. Such models provide basic prototypes for elastic bodies appearing in the flow/fluid structure interactions. It is assumed that shells are thin and do not account for the regularizing effects of rotary inertia. The nonlinear effects in the model become supercritical, and this raises a first fundamental question of Hadamard well-posedness in the class of weak solutions. The first main result of the present paper addresses the issue of generation of a nonlinear semigroup for such a model. The second result describes longtime behavior of the resulting dynamical system. It is shown that longtime dynamics admits finite-dimensional and smooth global attractors. This result is obtained without imposing any mechanical dissipation affecting the vertical displacements of the shell where the latter satisfy *free* boundary conditions. This particular feature, along with supercritical nonlinearity, leads to substantial challenges in the analysis. The resolution of the encountered difficulties rests on recently developed mathematical tools such as

(1) maximal regularity for thermal shells with free boundary conditions,
(2) "hidden" trace regularity propagated by thermal effects,
(3) compensated compactness and related theory of quasi-stable systems derived from books by Chueshov and by Chueshov and Lasiecka.

1. Introduction

This paper is concerned with well-posedness and longtime behavior of nonlinear thermoelastic systems which arise in modeling of thin shells subjected to thermal effects. Deformations of the shell are modeled by the Marguerre–Vlasov shallow shell equations [25, 43, 76]. The principal modeling assumptions are based on the Kirchhoff hypothesis and account for both vertical and in-plane displacements with small stresses. The model under consideration accounts also for thermal effects. Shell oscillations are described by vertical and in-plane displacements—denoted by w and $\mathbf{u} = (u_1, u_2)$, respectively. Thermal effects are described by taking averaged thermal stresses ϕ and θ affecting each of these displacements [23, 25, 45, 46]. Since the shells are "thin", it is reasonable not to include in the model rotational inertia terms which are of a small order with respect to the thickness of the elastic body. On the other hand, it is known that from the mathematical point of view the

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rotational terms provide for additional regularity (one extra derivative) of velocity of vertical displacement, being, therefore, more amenable to the analysis. Well-posedness of *weak* solutions without rotational term has been an open problem [14, 46, 64, 66, 70]. Thus, the physical relevance of the model under study leads to considerable mathematical challenges and calls for development of novel tools in the partial differential equation (PDE)/dynamical system analysis. The loss of the derivative in the velocity of vertical displacement (unlike the most recent result [64], where rotational terms are present) along with nonlinearity exhibited by the model leads to delicate estimates even at the level of well-posedness of weak solutions or generation of a nonlinear semigroup.

At the level of longtime behavior, the main interest of the model is the fact that no mechanical dissipation is assumed on vertical displacements which are subject to *free* (third-order) boundary conditions. The only sources of dissipation are thermal effects and boundary dissipation affecting in-plane displacements. The latter is necessary for stability of the system even in the case of a linear system of dynamic elasticity in a dimension higher than 1 [27, 38]. This brings up a challenge of propagating thermal dissipation through the body of a shell via free boundary conditions. To accomplish this, recent tools in dynamical systems developed in [18, 19] along with sharp trace estimates for handling free boundary in the presence of thermal effects [53] will be used. The final result is a construction of global, finite-dimensional, and smooth attractor capturing asymptotically all weak solutions. A complex PDE system is reduced asymptotically to a finite-dimensional structure.

It should be remarked that general shell models display considerable modeling complexity [8, 25, 28, 63]. What we emphasize here is that, though the model considered in this work is relatively simple, it does retain the main mathematical difficulties caused by nonlinear effects and geometry—with respect to well-posedness and longtime dynamics. The supercritical nature of nonlinear terms along with restrained dissipation already makes the analysis of global solvability and longtime attractivity challenging. These challenges are more evident to display and expose on a simplified model. On the other hand, techniques and methodology introduced in this paper may pave a way to solutions of more complex situations where additional (geometric) factors come into play [8, 9, 21, 24, 25, 77].

1.1. **Shallow shell model.** We consider a dynamic thin shell defined on a two-dimensional bounded, smooth manifold with elastic deformations accounting for vertical and in-plane oscillations. The following natural hypotheses are made:

(i) The Kirchhoff–Love hypothesis, which ensures that internal deformations are determined by the deformation of a midsurface, is used.
(ii) The shell is thin and of negligible thickness.
(iii) The shell's material is isotropic and homogeneous.
(iv) Transverse deformations are relatively small with respect to other motions and have moderate rotation angles.

The deformations of the shell are subject to thermal effects which will result in additional coupling with heat transfer equations. The resulting PDE model is described by the Marguerre–Vlasov system [10, 64, 70, 76] in the variables $(\mathbf{u}, w, \theta, \phi)$ denoting, respectively, transversal displacement $\mathbf{u} = (u_1, u_2)$, vertical displacement w, and heat flux averages (θ, ϕ) in the form of thermal stress ϕ and thermal moment θ. A PDE description of the model reads as follows: let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain with smooth boundary. In $\Omega \times (0, \infty)$ we consider the following quantities [25, 45, 46, 70, 76]:

- the strain tensor given by
$$\epsilon(\mathbf{u}) \equiv \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top],$$
where ∇u denotes the Jacobian matrix of vector $\mathbf{u}$;
- the von Kármán nonlinear tensor $f(\nabla w)$, where f is defined by
$$f(s) \equiv \frac{1}{2}[s \otimes s], \qquad s \in \mathbb{R}^2,$$
so $f(\nabla w)$ becomes $f(\nabla w) = \frac{1}{2}\begin{bmatrix} w_x^2 & w_x w_y \\ w_x w_y & w_y^2 \end{bmatrix}$;
- the strain resultant $N(\mathbf{u}, w) \equiv \epsilon(\mathbf{u}) + f(\nabla w)$;
- the symmetric strain tensor A accounting for the Gaussian curvature
$$A(\mathbf{u}, w) \equiv N(\mathbf{u}, w) + J(w),$$
where $J(w) = Kw$, with $K \equiv \operatorname{diag}\{K_1, K_2\}$ and $K_i(x) > 0$ $(i = 1, 2)$ denoting Gaussian curvatures;
- the stress tensor is given by
$$\sigma[A] \equiv \lambda \operatorname{trace}[A] I + 2\eta A,$$
where $\lambda = \frac{E}{(1-2\mu)(1+\mu)}\mu$, $\eta = \frac{E}{2(1+\mu)}$, E is the Young's modulus and $\mu \in (0, 1/2)$ stands for Poisson's modulus;
- the "tangential" boundary operators which are associated with free boundary conditions
$$\begin{aligned} B_1 w &= -\partial_{\tau\tau} w - \operatorname{div}\{\nu\} \partial_\nu w, \\ B_2 w &= \partial_\tau \partial_\nu \partial_\tau w - l w, \end{aligned} \tag{1.1}$$
where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to Γ_1, $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector along Γ_1, and $l \geqslant 0$; and
- the phase (finite energy) space, denoted by H, is given by
$$H \equiv [H^1_{\Gamma_0}(\Omega)]^2 \times [L^2(\Omega)]^2 \times H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2,$$
where $H^m_{\Gamma_0}(\Omega)$ is the closure in $H^m(\Omega)$ topology of $C^\infty_{\Gamma_0}(\Omega)$ functions compactly supported near Γ_0, and $H^m(\Omega)$ is the standard Sobolev space of order m based on $L^2(\Omega)$.

Now we are in position to write down the PDE model under consideration: In $\Omega \times (0, \infty)$, with a smooth boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are disjoint and relatively open, we consider the following evolutionary system described by the independent variables $(\mathbf{u}, w, \phi, \theta)$,
$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix} \mathbf{u} \\ w \end{bmatrix} + \operatorname{div}\begin{bmatrix} -\sigma[A(\mathbf{u}, w)] \\ \nabla(\Delta w) - \sigma[A(\mathbf{u}, w)]\nabla w - \phi \nabla w \end{bmatrix} & \\ + \begin{bmatrix} \nabla \phi \\ K \cdot \sigma[A(\mathbf{u}, w)] + \Delta\theta + \beta w^3 - p_0(w) \end{bmatrix} &= 0, \end{aligned} \tag{1.2}$$
with *clamped* boundary conditions on $\Gamma_0 \times (0, \infty)$,
$$\mathbf{u} = 0, \quad w = 0, \quad \nabla w = 0, \tag{1.3}$$

and free boundary conditions on $\Gamma_1 \times (0,\infty)$,

$$\sigma[A(\mathbf{u},w)]\nu + \kappa\mathbf{u} - \phi\nu + \mathbf{u}_t = 0, \tag{1.4}$$

$$\Delta w + (1-\mu)B_1 w + \theta = 0, \tag{1.5}$$

$$\partial_\nu(\Delta w) + (1-\mu)B_2 w - \sigma[A(\mathbf{u},w)]\nu\cdot\nabla w - \phi\partial_\nu w + \partial_\nu\theta = 0. \tag{1.6}$$

The evolutionary system (1.2) is coupled with the following equations satisfied by thermal stress ϕ and thermal moment θ, also defined in $\Omega \times (0,\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\phi\\ \theta\end{bmatrix} - \mathrm{div}\begin{bmatrix}\nabla\phi\\ \nabla\theta\end{bmatrix} + \begin{bmatrix}\mathrm{div}\{\mathbf{u}_t\} - \nabla w\nabla w_t\\ -\mathrm{div}\{\nabla w_t\}\end{bmatrix} = 0, \tag{1.7}$$

subject to Robin boundary conditions on $\Gamma \times (0,\infty)$,

$$\partial_\nu\phi + \lambda_1\phi = 0, \qquad \partial_\nu\theta + \lambda_2\theta = 0, \tag{1.8}$$

where $\lambda_1, \lambda_2 > 0$.

The initial conditions defined in Ω are given by $(\mathbf{u}_0, \mathbf{u}_1, w_0, \phi_0, \theta_0) \in H$, where

$$\begin{gathered}\mathbf{u}(\cdot,0) = \mathbf{u}_0, \quad \mathbf{u}_t(\cdot,0) = \mathbf{u}_1, \quad w(\cdot,0) = w_0,\\ w_t(\cdot,0) = w_1, \quad \phi(\cdot,0) = \phi_0, \quad \theta(\cdot,0) = \theta_0.\end{gathered} \tag{1.9}$$

The force p_0 affects the vertical displacement and provides a source of potential instability. Precise assumptions imposed on p_0 are formulated in due course. The velocity of the in-plane displacement $\mathbf{u}_t$ in the boundary condition (1.4) can be interpreted as boundary feedback control acting on the edge of Γ_1 of the shell. This will be the only mechanical damping affecting the model.

Finally, by the symbol "$\cdot$" we denote the Frobenius inner product of two square matrices, and, eventually, the same notation will be used to denote the inner product in $\mathbb{R}^4$.

The *main goals* of this paper are

(i) to establish global well-posedness of the underlined PDE system by identifying the dynamics with a well-posed dynamical system defined on a phase space H corresponding to weak (finite energy) solutions,
(ii) to provide conditions under which the dynamics converges to a global attractor, and, finally,
(iii) to prove smoothness and finite dimensionality of said attractor—so that the infinite-dimensional dynamics is reduced asymptotically to a finite-dimensional and smooth structure.

1.2. **Comments.**

1. Scalar versus vectorial (full) von Kármán equations. Scalar von Kármán equation (sometimes referred to as modified) with a variety of boundary conditions have been by now well understood from the point of view of Hadamard well-posedness. This is due to the discovery of the sharp regularity of Airy's stress function [52] and related compensated compactness [26]. However, this is not the case for *vectorial* models where there is no Airy's stress function to decouple the equations [43]. Existence of weak solutions to the vectorial shell can be shown by Galerkin-type schemes as in [40, 70]. However, uniqueness and Hadamard well-posedness has been an outstanding problem. In fact, one of our contributions is handling of global

well-posedness of weak solutions to (1.2)–(1.9) without assuming the presence of regularizing effects resulting from the rotational inertia. Well-posedness results for full von Kármán models *with rotational inertia* are in [39].

2. Mixing of horizontal and vertical displacements leads to supercritical effects caused by the nonlinear terms. We point out that horizontal displacements, an essential element for modeling of shallow shells [77], provide for nonlinear supercritical mixing of high energies between vertical and in-plane displacements. As a consequence, establishing Hadamard well-posedness for the model becomes a mathematical challenge. This is due to supercritical nonlinearity, as explained below. With $\mathbf{u}, w$ in the phase space the most critical nonlinear term $\operatorname{div}\{\sigma[A(\mathbf{u},w)]\nabla w + K\sigma[A(\mathbf{u},w)]\}$ belongs to $H^{-(1+\epsilon)}(\Omega)$, which represents strong unboundedness with respect to the $L^2(\Omega)$ component of the phase space. While an existence of weak solutions can be shown by taking advantage of some cancellations at the level of energy methods, the proof of uniqueness and the continuous dependence of weak solutions on the initial data are not tractable within the known methodology. Clearly, other factors must come in to play—such as the influence of the first-order evolution describing thermal effects. There is more on this below in item 4.

3. Addition of the rotational inertia alleviates supercriticality. Note that the rotational inertia term $\gamma\Delta w_{tt}$, with $\gamma > 0$, included in the w equation (1.2), where γ is proportional to the square of the thickness, changes the picture—particularly in the case of clamped or hinged boundary conditions. This is due to the fact that $[I-\Delta]^{-1}(\operatorname{div}\{\sigma[A(\mathbf{u},w)]\nabla w + K\sigma[A(\mathbf{u},w)]\}) \in H^{-\epsilon}(\Omega)$. Thus, the loss of regularity becomes "incremental" ϵ, as the loss of one full derivative has been recovered by $\gamma > 0$. In the latter case, "logarithmic compensation" [42, 70], the methods of weak compactness and energy cancellations were developed in [39]. This method has been used in [22, 64, 66] in order to establish well-posedness of dynamic shells with *rotational inertia* and clamped or *hinged* boundary conditions. When $\gamma = 0$, the situation changes drastically since the loss of derivative is no longer incremental—one loses more than one full derivative. It is here where our *first contribution* comes. We shall prove Hadamard well-posedness of finite energy solutions without regularizing effects due to the rotational terms (in contrast to [64], where well-posedness for the same system but with $\gamma > 0$ and clamped boundary conditions is established).

4. Free boundary conditions. Our *second contribution* is handling of the free boundary conditions. These are of utmost importance in applications to shells; however, they are still resistant to mathematical treatment. This is due to their algebraic complexity (pseudo-differential symbol) (1.1), (1.4), and sensitivity with respect to the geometry [29, 53]. In order to handle the supercriticality of the nonlinear term, one must exploit the effects of thermoelasticity. While in the case of standard boundary conditions imposed on the vertical displacements w—such as clamped or hinged—one can show [60] that thermal diffusion leads to a generator of an analytic semigroup—say, $\mathcal{A}$—generated by the linearization of the (w, θ)-equation. This last property helps critically in the analysis of Hadamard well-posedness. The situation is very different with free boundary conditions, which destroys the nice structure of a thermoelastic semigroup. To see this, consider just equation $\Delta w = f \in L^2(\Omega)$, with w vertical displacement subject to the free boundary conditions. This is an ill-posed problem—unlike in the case of clamped or hinged boundary conditions which add Dirichlet data to the Laplacian, making it invertible. This explains the

main difference caused by the third-order free boundary conditions where natural realization of $\mathcal{A}^{1/2}$ has nonlocal representation of the boundary condition. In order to overcome this hurdle, interpolation theory and sharp trace estimates developed in [53], along with geometric considerations, will come to the rescue.

5. Our *third contribution* is the results on longtime behavior which state that time-asymptotic dynamics is both finite dimensional and smooth. This result is obtained without any dissipation imposed on the vertical displacement. This, again, is in contrast with the literature—see [14, 62] with related references—where longtime behavior is studied in the presence of strong or viscoelastic damping added. In handling this aspect of the problem, one must obtain the estimates for asymptotic smoothness without

(i) any regularizing effects of rotational inertia and
(ii) the dissipation of vertical oscillations.

As is commonly recognized, the existence of smooth attracting sets relies on smoothing effects of the primal dynamics and the dissipation [32, 71]; see also recent contributions in the context of von Kármán models [13, 17, 35]. Both features are not present in our model.

6. In summary, resolving the three issues addressed above provides a positive answer to open questions raised in the recent literature regarding longtime behavior [22, 64, 70] and regarding Hadamard well-posedness [64]. Equally important is the fact that the resolution of these questions brings aboard new methodologies in the area of PDEs and dynamical systems which push further the relevant developments in [49] and [51] dealing with stability properties of full von Kármán systems. In fact, at the technical level critical ingredients of the proofs are new trace estimates developed for thermoelasticity with free boundary conditions and new quasi-stable estimates [14] developed for the dynamical system in hand.

Finally, it should also be noted that the shell model presented in (1.2) is only a prototype of a more general shallow shell model considered in the literature. Indeed, shell equations involve variable coefficients associated with the corresponding differential operators and accounting for geometric properties (curvature) of a shell. This is typically treated via methods of differential geometry where equations are rewritten on a Riemanian manifold [57, 77]. The presence of a variable coefficient, even in the linear case, has an essential effect on the results and the methods used in the study of stabilization and controllability, more generally inverse problems. Carleman's estimates, along with a microlocal analysis of a Riemanian manifold, became a major tool in establishing the needed observability estimates [55, 58]. We are not considering this level of generality. Our emphasis is on nonlinear effects of the coupling and their mathematical effect and treatment. Inclusion of variable coefficients, while not essential at the level of well-posedness, will lead to substantial complexity of the exposition in the context of obtaining a suitable "inverse" type of estimate. This will make it more difficult to focus on essential difficulties caused by the nonlinear effects. On the other hand, the techniques already developed for control and stabilization [56–58, 74, 77], along with the treatment of the nonlinear prototype model (which retains all of the main features), could lead to a comprehensive treatment of longtime behavior associated with shell dynamics.

1.3. **Main results.** We shall begin by formulating the following hypotheses imposed on the source function p_0.

Assumption 1. We assume that the function $p_0 \in C^1(\mathbb{R})$ and that there exist positive constants M_0, M, m, and $r \geqslant 1$ such that

$$|p_0'(s)| \leqslant M_0(|s|^{r-1}+1), \qquad s \in \mathbb{R}. \tag{1.10}$$

Let us consider the decomposition $p_0(s) = p_0^+(s) + p_0^-(s)$, where $p_0^-(s) \leqslant 0, s \in R$. We assume the following dissipativity condition:

$$p_0^+(s) \leqslant M|s|^\alpha + m, \tag{1.11}$$

where $\alpha \leqslant 3$ when $\beta > 0$, and $\alpha \leqslant 1$ when $\beta = 0$.

Theorem 1.1 (Weak solutions)**.** *Assume that conditions* (1.10) *and* (1.11) *hold. Then, for any $T > 0$ and initial data $(\mathbf{u}_0, \mathbf{u}_1, w_0, w_1, \phi_0, \theta_0) \in H$, problem* (1.2)–(1.9) *has a unique weak* (*finite energy*) *solution $(\mathbf{u}, \mathbf{u}_t, w, w_t, \phi, \theta) \in C\big([0,T]; H\big)$. Moreover, this solution depends continuously on the initial data.*

In order to discuss smooth solutions, we introduce the space

$$H_1 \equiv [H^2(\Omega)]^2 \times [H^1(\Omega)]^2 \times H^4(\Omega) \times H^2(\Omega) \times [H^2(\Omega)]^2,$$

and we specify compatibility conditions imposed on the initial data, with $A_0 \equiv N(\mathbf{u}_0, w_0) + J(w_0)$,

- $\sigma[A_0]\nu + k\mathbf{u}_0 - \phi_0\nu + \mathbf{u}_1 = 0$ on Γ_1;
- $\Delta w_0 + (1-\mu)B_1 w_0 + \theta_0 = 0$ on Γ_1;
- $\partial_\nu(\Delta w_0) + (1-\nu)B_2 w_0 - \sigma[A_0]\nu \cdot \nabla w_0 - \phi_0 \partial_\nu w_0 + \partial_\nu \theta_0 = 0$ on Γ_1;
- $\partial_\nu \phi_0 + \lambda_1 \phi_0 = 0,\ \partial_\nu \theta_0 + \lambda_2 \theta_0 = 0$ on Γ.

Theorem 1.2 (Strong solutions)**.** *Assume that the initial data are of regularity H_1 and satisfy the above compatibility on the boundary. Then problem* (1.2)–(1.9) *has a unique regular solution $(\mathbf{u}, \mathbf{u}_t, w, w_t, \phi, \theta) \in C\big([0,T]; H_1\big)$ with $(\mathbf{u}_{tt}, w_{tt}, \phi_t, \theta_t) \in C\big([0,T]; [L^2(\Omega)]^5\big)$.*

Remark 1.1. Hadamard well-posedness for the problem with $\gamma > 0$ and without thermal effects has been proved in [39]. Hadamard well-posedness of a full von Kármán system with thermal effects, still with $\gamma > 0$, and strains accounting for shell's curvature has recently been shown in [64] by resorting to methods of [39]. Theorem 1.1 is, to our best knowledge, the first one which solves an open problem of well-posedness for weak solutions for the shell model in the supercritical case, i.e., when $\gamma = 0$, with regularizing effects absent and elastic nonlinearity at the supercritical level.

The longtime behavior of the shell model is established next. We consider the nonlinear dynamical system $(H, S(t))$, generated by the solution operator $S(t)$ given in Theorem 1.1.

Theorem 1.3 (Global and smooth attractors)**.** *Assume that Assumption* 1 *with a constant M sufficiently small is satisfied for $\alpha = 3$ with $\beta > 0$ ($\alpha = 1$ when $\beta = 0$). In addition, the following geometric condition is imposed: There exists an $x_0 \in \mathbb{R}^2$ such that*

$$(x - x_0)\cdot\nu \leqslant 0, \qquad x \in \Gamma_0. \tag{1.12}$$

Moreover, assume either that $\beta > 0$ or that the Gauss curvatures $\|K_i\|_{C(\Omega)}$, $i = 1, 2$, are sufficiently small. Then the dynamical system $(H, S(t))$ generated by problem (1.2)–(1.9) *admits a compact global attractor $\mathbf{A} \subset H$ with a finite fractal dimension. Moreover, $\mathbf{A}$ is bounded in H_1.*

Remark 1.2. The restriction that the constant M can be taken sufficiently small is automatically satisfied when $\alpha < 3$ (resp., $\alpha < 1, \beta = 0$).

A shell model interacting with a fluid has recently been considered in [21, 22]. Existence of a global attractor was shown in [22] under the assumption that $\gamma > 0$ (regularizing inertial terms) and strong mechanical damping Δw_t are added to the equations. The result presented in Theorem 1.3 shows that under suitable geometry of the shell one obtains global and smooth attractors for $\gamma = 0$, i.e., without the additional dissipation and regularity. It is well known in shell theory that in general one may have everted states [58], which prevents any notion of stability.

The presence of boundary feedback controlling longtime behavior of transversal displacements is necessary for the validity of Theorem 1.3. Indeed, it is well known [27, 38] that unless the model is one dimensional, in-plane waves cannot be uniformly stabilized by thermal effects.

Remark 1.3. One could consider more general nonlinear boundary feedback controls in (1.4). Indeed, $\mathbf{u}_t$ replaced by $g(\mathbf{u}_t)$, with appropriate conditions of monotonicity and bounds imposed on g, would lead to the same result. In fact. a related nonlinear feedback control for *linear* shells has already been studied in [55]. Here, again, in order to limit additional complications in the exposition, we restrict ourselves to the simplest possible model which retains the main characteristics of the problem.

The outline of the paper. The proof of Hadamard well-posedness is given in Section 2. The longtime behavior of the solution is discussed in Section 3. In particular, the proof of Theorem 1.3 will be given in Section 3 and relies on the following:

(1) use of partial *maximal regularity* [29, 53] generated by vertical displacement of the shell,
(2) a novel abstract criterion in the area of dynamical systems which relies on compensated compactness and the associated *quasi-stability property* of the dynamical system, and
(3) new trace estimates for vertical displacements of the shell—extending those given in [51] for a plate problem.

The appendix provides some supplementary material on formulas used throughout the paper.

2. Hadamard well-posedness

2.1. **Notation: Function spaces.** $H^s(\Omega)$ denote standard Sobolev's spaces of possibly fractional order $s \geqslant 0$. The norms are denoted by $\|u\|_{\alpha,\Omega} = \|u\|_{H^\alpha(\Omega)}$, $\|u\|_{\alpha,\Gamma} = \|u\|_{H^\alpha(\Gamma)}$, and the case $\alpha = 0$ corresponds to L^2 spaces; we write $\|u\|_\Omega = \|u\|_{L^2(\Omega)}$, $\|u\|_\Gamma = \|u\|_{L^2(\Gamma)}$. The corresponding inner products are denoted by $(u, v)_\Omega = (u, v)_{L^2(\Omega)}$ and $\langle u, v\rangle_\Gamma = \langle u, v\rangle_{L^2(\Gamma)}$. For $\alpha > 0$, the space $H^s_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$, and $H^{-\alpha}(\Omega) = [H^\alpha_0(\Omega)]'$, where the duality is taken with respect to the $L^2(\Omega)$ inner product. Occasionally, by the same symbol, we denote norms and inner products of n-copies of $L^2(\mathscr{O})$, where $\mathscr{O}$ is either Ω or Γ. The same is applied to $H^\alpha(\mathscr{O})$. We will use the notation $L^p(X)$ (resp., $C(X)$) denoting $L^p(0, T; X)$ (resp., $C((0, T]; X)$). $B_X(R)$ denotes a ball in X with a radius R. The symbol C will denote a generic constant, different at different occurrences.

2.2. **Construction, uniqueness, and continuous dependence of finite energy solutions.** The main difficulty is to prove the well-posedness of *finite energy* or weak solutions. Once well-posedness of weak solutions is established, the analysis of regularity is more routine. For this reason we shall focus on weak solutions. While Hadamard well-posedness for the vectorial von Kármán model accounting for regularizing effects of rotational inertia was shown in [39], and later the same proof was adapted to treat thermoelastic and curvature terms in [64], the case where rotational inertia are absent represents the main challenge due to the compromised regularity of the phase space. This particularly affects the uniqueness and continuous dependence on the data. Indeed, an existence of solutions can be argued as in [70] by using Galerkin-type method which depends on good a priori bounds and weak continuity—both properties available for the model under consideration. The uniqueness, however, without regularizing inertial terms, has been an open question. In [49] this issue has been successfully resolved for a simpler model by taking advantage of analytic effects propagated by analyticity of the semigroup associated with thermal plates. We will pursue this avenue and show that exploiting partial maximal regularity [29, 53], in addition to partial analyticity, resolves the problem. We shall begin with the more challenging part—uniqueness of local weak solutions.

Lemma 2.1 (Uniqueness and continuous dependence)**.** *Consider weak solutions for the system* (1.2)–(1.9) *referred to in Theorem* 1.1*. The claim is that these solutions are unique and continuously dependent on the initial data within the topology of the phase space H.*

Proof. Let $y_i = (\mathbf{u}^i, \mathbf{u}^i_t, w^i, w^i_t, \phi^i, \theta^i)$, $i = 1, 2$, be two weak local (in time) solutions of the system under consideration corresponding to some initial data $y_1(0), y_2(0) \in H$. The proof of Lemma 2.1 follows a two-step procedure. We shall first show that the needed estimate obtained for the difference of two solutions holds for slightly more regular vertical displacements. In the second step we shall establish this additional regularity valid for all weak solutions. Following this program, our first goal is to establish the following estimate.

Proposition 2.1. *Consider two solutions y_i, $i = 1, 2$, corresponding to* (1.2)–(1.9)*. Assume that $y_i \in X_T \subset C([0, T); H)$, $i = 1, 2$, where*

$$X_t \equiv C([0, t); H) \cap \{w \in H^\theta(0, t; H^{3-2\theta}(\Omega))\}, \qquad t > 0,\ \theta \in [0, 1].$$

Then there exists a $T_0 \leqslant T$ such that the following estimate holds:

$$\|y_1(t) - y_2(t)\|_H \leqslant C_{t, \|y_i\|_{X_t}} \|y_1(0) - y_2(0)\|_H, t \in [0, T_0].$$

Proof. Introduce the following notation:

$$\widetilde{y} \equiv y_1 - y_2, \quad \widetilde{P} \equiv P_1 - P_2, \quad \widetilde{Q} \equiv Q_1 - Q_2, \quad \widetilde{R} \equiv R_1 - R_2,$$

where

$$P_i = P(y_i) \equiv \sigma[f(\nabla w^i) + J(w^i)], \qquad Q_i = Q(y_i) \equiv \sigma[A(\mathbf{u}^i, w^i)]\nabla w^i + \phi^i \nabla w^i,$$

$$R_i = R(y_i) \equiv \nabla w^i \nabla w^i_t, \quad \sigma(A)_i \equiv \sigma[A(\mathbf{u}^i, w^i)], \quad \widetilde{p}_0 = p_0(w^1) - p_0(w^2).$$

Consider the following system,

$$
\begin{aligned}
\widetilde{\mathbf{u}}_{tt} - \operatorname{div}\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\} + \nabla\widetilde{\phi} &= \operatorname{div}\{\tilde{P}\},\\
\widetilde{w}_{tt} + \Delta^2\widetilde{w} + \Delta\widetilde{\theta} + \beta((w^1)^3 - (w^2)^3) &= \widetilde{p}_0 + \operatorname{div}\{\widetilde{Q}\} - K(\sigma(A)_1 - \sigma(A)_2),\\
\widetilde{\phi}_t - \Delta\widetilde{\phi} + \operatorname{div}\{\widetilde{\mathbf{u}}_t\} &= \widetilde{R},\\
\widetilde{\theta}_t - \Delta\widetilde{\theta} - \Delta\widetilde{w}_t &= 0,
\end{aligned} \tag{2.1}
$$

with clamped boundary conditions on $\Gamma_0 \times (0,\infty)$ and with the following conditions on $\Gamma_1 \times (0,\infty)$,

$$
\begin{aligned}
\sigma[\epsilon(\widetilde{\mathbf{u}})] - \widetilde{\phi}\nu + k\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}_t &= -\widetilde{P}\nu,\\
\Delta\widetilde{w} + (1-\nu)B_1\widetilde{w} + \widetilde{\theta} &= 0,\\
\partial_\nu(\Delta\widetilde{w}) + (1-\nu)B_2\widetilde{w} + \partial_\nu\widetilde{\theta} &= \widetilde{Q}\nu,
\end{aligned} \tag{2.2}
$$

and Robin boundary conditions imposed on $(\widetilde{\phi},\widetilde{\theta})$.

We note that (2.1) and (2.2) with $\widetilde{P}=0$, $\widetilde{Q}=0$, $\widetilde{R}=0$ is represented by a nonlinear semigroup continuous on H. This follows from a standard perturbation argument applied within the context of monotone operator theory.

In particular, considering the correspondent equations for $(\mathbf{u},\phi)$ in (2.1), which represent classical system of thermoelasticity forced by $\operatorname{div}\{\widetilde{P}\}$ and $\widetilde{R}$, with boundary conditions forced by $\widetilde{P}\nu$, we can apply standard energy estimates provided that the forcing terms are in the finite energy space $H_{u,\phi} = [H^1(\Omega)]^2 \times [L^2(\Omega)]^2 \times L^2(\Omega)$. This, in turn, requires that

$$
\operatorname{div}\{\widetilde{P}\} \in L^1(0,T;L^2(\Omega)), \qquad \widetilde{R} \in L^2(0,T;[H^1(\Omega)]').
$$

For every $y \in X_T$ such regularity can be established. Indeed,

$$
\|\operatorname{div}\{P_i\}\|_\Omega \leqslant C\big(\|w^i\|_{2,\Omega}\|\nabla w^i\|_{L^\infty(\Omega)} + |K|\|w\|_{1,\Omega}\big) \leqslant C\big(\|w^i\|_{2,\Omega}\|w^i\|_{2+\epsilon,\Omega} + \|w\|_{1,\Omega}\big).
$$

To bound $\widetilde{R}$, we use a combination of interpolation and Young's inequality. Indeed, with $\epsilon < 1/2$ (so that $H_0^\epsilon(\Omega)$ coincides with $H^\epsilon(\Omega)$) and for $v \in H^1(\Omega)$ we write

$$
|(\nabla w_t^i \nabla w^i, v)_\Omega| \leqslant \|\nabla w_t^i\|_{-\epsilon,\Omega}\|\nabla w^i v\|_{\epsilon,\Omega} \leqslant C\|w_t^i\|_{1-\epsilon,\Omega}\|w^i\|_{1+\epsilon+\epsilon_0}\|v\|_{1,\Omega}, \qquad \epsilon_0 > 0.
$$

Thus, for any $\epsilon < 1/2$ and $\epsilon_0 > 0$ $\|R_i\|_{[H^1(\Omega)]'} \leqslant C\|w_t^i\|_{1-\epsilon,\Omega}\|w^i\|_{1+\epsilon+\epsilon_0,\Omega}$. Thus, for some $r_1 > 0$

$$
\begin{aligned}
\int_0^T \|R_i\|^2_{[H^1(\Omega)]'}\mathrm{d}t &\leqslant C\|w^i\|^2_{C(H^2)}\int_0^T \|w_t^i\|_{1,\Omega}^{2(1-\epsilon)}\|w_t^i\|^{2\epsilon}\mathrm{d}t\\
&\leqslant CT^{r_1}\|w^i\|^2_{C(H^2)}\big(\|w_t^i\|^2_{C(L_2)} + \int_0^T \|w_t^i\|^2_{1,\Omega}\mathrm{d}t\big),
\end{aligned} \tag{2.3}
$$

We are now in position to apply an energy estimate valid for the forced thermoelasticity system (u,ϕ). With $t \leqslant T$,

$$
\begin{aligned}
&\|\widetilde{y}_{u,\phi}(t)\|^2_{H_{u,\phi}} + 2\int_0^t \big(\|\widetilde{\mathbf{u}}_t(t)\|^2_{\Gamma_1} + \|\nabla\widetilde{\phi}(t)\|^2_\Omega + \lambda_1\|\widetilde{\phi}(t)\|^2_\Gamma\big)\mathrm{d}t\\
&\quad = \|\widetilde{y}_{u,\phi}(0)\|^2_{H_{u,\phi}} + 2\int_0^t \big((\operatorname{div}\{\widetilde{P}\},\widetilde{\mathbf{u}}_t)_\Omega - \langle\widetilde{P}\nu,\widetilde{\mathbf{u}}_t\rangle_{\Gamma_1} + (\widetilde{R},\widetilde{\phi})_\Omega\big)\mathrm{d}t.
\end{aligned} \tag{2.4}
$$

Note that

$$
\begin{aligned}
(2.5)\qquad \int_0^t (\operatorname{div}\{\widetilde{P}\}, \widetilde{\mathbf{u}}_t)_\Omega \mathrm{d}s - \int_0^t \langle \widetilde{P}\nu, \widetilde{\mathbf{u}}_t\rangle_{\Gamma_1} \mathrm{d}s &= -\int_0^t (\widetilde{P}, \epsilon(\widetilde{\mathbf{u}}_t))_\Omega \mathrm{d}s \\
&= -(\widetilde{P}, \epsilon(\widetilde{\mathbf{u}}))_\Omega\Big|_0^t + \int_0^t (\widetilde{P}_t, \epsilon(\widetilde{\mathbf{u}}))_\Omega \mathrm{d}s.
\end{aligned}
$$

By the Hölder and Young inequalities we estimate the inner products on the right-side hand of (2.5):

$$
|(\widetilde{P}, \epsilon(\widetilde{\mathbf{u}}))_\Omega\Big|_0^t| \leqslant C_\delta \|\widetilde{P}\|^2_{L^\infty(L^2)} + \delta\|\epsilon(\widetilde{\mathbf{u}})\|^2_{L^\infty(L^2)},
$$

$$
|\int_0^t (\widetilde{P}_t, \epsilon(\widetilde{\mathbf{u}}))_\Omega \ \mathrm{d}s| \leqslant C_\delta \|\widetilde{P}_t\|^2_{L^1(L^2)} + \delta\|\epsilon(\widetilde{\mathbf{u}})\|^2_{L^\infty(L^2)}.
$$

Thus, inserting these inequalities into the right-hand side of (2.5) and using the inequality $\|\widetilde{P}\|_{L^\infty(L^2)} \leqslant C\|\widetilde{P}_t\|_{L^1(L^2)}$, we obtain

$$
\int_0^t (\operatorname{div}\{\widetilde{P}\}, \widetilde{\mathbf{u}}_t)_\Omega \ \mathrm{d}s - \int_0^t \langle \widetilde{P}\nu, \widetilde{\mathbf{u}}_t\rangle_{\Gamma_1} \mathrm{d}s \leqslant C_\delta \|\widetilde{P}_t\|^2_{L^1(L^2)} + \delta\|\epsilon(\widetilde{\mathbf{u}})\|^2_{L^\infty(L^2)}.
$$

Combining this estimate with (2.4) and taking $\delta > 0$ small enough, we find that

$$
\begin{aligned}
\|\widetilde{y}_{u,\phi}(t)\|^2_{H_{u,\phi}} + C\int_0^t \Big(\|\widetilde{\mathbf{u}}_t(s)\|^2_{\Gamma_1} + \|\nabla\widetilde{\phi}(s)\|^2_\Omega \ + \ \lambda_1\|\widetilde{\phi}(s)\|^2_\Omega \Big)\mathrm{d}s \\
\leqslant C\|\widetilde{y}_{u,\phi}(0)\|^2_{H_{u,\phi}} + C\|\widetilde{P}_t\|^2_{L^1(L^2)} + C\|\tilde{R}\|^2_{L^2(H^{-1})}.
\end{aligned}
$$

Since

$$
\|\widetilde{P}_t\|_\Omega \leqslant C \sup_{i=1,2} \left\{ \|\nabla\widetilde{w_t} \otimes \nabla w^i\|_\Omega + \|\nabla\widetilde{w} \otimes \nabla w^i_t\|_\Omega \right\} + C\|\widetilde{w}_t\|_\Omega,
$$

the estimate for $\|\widetilde{P}_t\|^2_{L^1(0,T;L^2(\Omega))}$ will be concluded using the estimates

$$
\begin{aligned}
\left\|\nabla\widetilde{w_t} \otimes \nabla w^i\right\|_{L^1(L^2)} &\leqslant C\left(\int_0^T \|\widetilde{w_t}\|^2_{1,\Omega}\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \|w^i\|^2_{2+\varepsilon,\Omega}\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leqslant C(\|y_i\|_{X_T})T^{r_2}\left(\int_0^T \|\widetilde{w_t}\|^2_{1,\Omega}\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\left\|\nabla\tilde{w} \otimes \nabla w^i_t\right\|_{L^1(L^2)} \leqslant C(\|y_i\|_{X_T})T^{r_2}\left(\int_0^T \|\widetilde{w}\|^2_{3,\Omega}\mathrm{d}t\right)^{\frac{1}{2}}
$$

for some $r_2 > 0$. Here, we have used $H^{1+\epsilon}(\Omega) \subset L^\infty(\Omega)$, interpolation, and Young inequalities. This gives

$$
\|\widetilde{P}_t\|_{L^1(L^2)} \leqslant C_{\|y_i\|_{X_T}} T^{r_2}\left(\left[\int_0^T \|\tilde{w}_t\|^2_{1,\Omega}\mathrm{d}t\right]^{\frac{1}{2}} + \left[\int_0^T \|\widetilde{w}\|^2_{3,\Omega}\mathrm{d}t\right]^{\frac{1}{2}}\right).
$$

For the term $\widetilde{R}$ we apply the estimate in (2.3). Thus, for all $t \in [0,T]$ we have

$$
\begin{aligned}
&\|\widetilde{y}_{u,\phi}(t)\|^2_{H_{u,\phi}} + C\int_0^t \Big(\|\widetilde{\mathbf{u}}_t(s)\|^2_{\Gamma_1} \ + \ \|\nabla\widetilde{\phi}(s)\|^2_{\Omega} + \lambda_1\|\widetilde{\phi}(s)\|^2_{\Omega} \Big) \mathrm{d}s \\
&\qquad \leqslant C\|\widetilde{y}_{u,\phi}(0)\|^2_{H_{u,\phi}} + C_{\|y_i\|_{X_T}} T^{r_1}\|\widetilde{w}\|^2_{C(H^2)} \\
&\qquad\quad + C_{\|y_i\|_{X_T}}(T^{r_1}+T^{r_2})\int_0^T \big(\|\widetilde{w}\|^2_{3,\Omega}+\|\widetilde{w_t}\|^2_{1,\Omega}\big)\mathrm{d}t.
\end{aligned} \tag{2.6}
$$

The above estimate indicates a need for *higher-norm estimates* of vertical displacements w. Here, the idea is to use maximal regularity established for thermoelastic plates [53]—see also [29] for related results. The $\widetilde{w}$ problem is given as follows:

$$
\begin{aligned}
\widetilde{w}_{tt} + \Delta^2\widetilde{w} + \Delta\widetilde{\theta} &= F, \\
\widetilde{\theta}_t - \nabla\widetilde{\theta} - \Delta\widetilde{w}_t &= 0,
\end{aligned} \tag{2.7}
$$

where $F \equiv \operatorname{div}\{\widetilde{Q}\} - K(\sigma(A)_1+\sigma(A)_2) - \beta((w^1)^3-(w^2)^3)+\widetilde{p}_0(w)$ with the boundary conditions

$$
\begin{aligned}
\widetilde{w} = 0, \nabla\widetilde{w} &= 0 \text{ on } \Gamma_0\times(0,\infty), \\
\Delta\widetilde{w} + (1-\nu)B_1\widetilde{w} + \widetilde{\theta} &= 0 \text{ on } \Gamma_1\times(0,\infty), \\
\partial_\nu(\Delta\widetilde{w}) + (1-\nu)B_2\widetilde{w} + \partial_\nu\widetilde{\theta} - \widetilde{Q}\nu &= 0 \text{ on } \Gamma_1\times(0,\infty), \\
\partial_\nu\widetilde{\theta} + \lambda\widetilde{\theta} &= 0 \text{ on } \Gamma\times(0,\infty).
\end{aligned} \tag{2.8}
$$

We shall need a semigroup representation of the nonhomogeneous on the boundary system (2.7) and (2.8). To accomplish this, we introduce a classical generator $\mathcal{A}$ corresponding to a biharmonic operator with *clamped-free* boundary conditions and a biharmonic extension G of clamped-free boundary conditions. These are given by

- $\mathcal{A}(v) = \Delta^2 v$ for $v \in D(\mathcal{A})$ with the domain
$$
D(\mathcal{A}) = \left\{ v \in H^2_{\Gamma_0}(\Omega) \;\middle|\; \begin{array}{l} \Delta^2 v \in L^2(\Omega), \\ \Delta v + (1-\mu)B_1 v = 0 \text{ on } \Gamma_1, \\ \partial_\nu(\Delta v) + (1-\mu)B_2 v = 0 \text{ on } \Gamma_1. \end{array} \right\};
$$
- $G_2(g) \equiv v$ iff $\Delta^2 v = 0$ in Ω,
$v = 0, \nabla v = 0$ on Γ_0, $\Delta w+(1-\nu)B_1 w = 0, \partial_\nu(\Delta w)+(1-\nu)B_2 w = g$ on Γ_1;
- $G_1(g) \equiv v$ iff $\Delta^2 v = 0$ in Ω,
$v = 0, \nabla v = 0$ on Γ_0, $\Delta w+(1-\nu)B_1 w = g, \partial_\nu(\Delta w)+(1-\nu)B_2 w = 0$ on Γ_1.

It is known [53, 54] that

- $D(\mathcal{A}) \subset H^4(\Omega)$ since Γ_1 and Γ_0 are separated;
- $G_2 : L^2(\Gamma_1) \to H^{7/2}(\Omega) \subset D(\mathcal{A}^{7/8-\epsilon})$, $G_1 : L^2(\Gamma_1) \to H^{5/2}(\Omega) \subset D(\mathcal{A}^{5/8-\epsilon})$ continuously;
- $G_1^*\mathcal{A}v = \frac{\partial}{\partial\nu}v|_{\Gamma_1}, v \in H^2_{\Gamma_0}(\Omega)$; $G_2^*\mathcal{A}v = -v|_{\Gamma_1}, v \in H^2_{\Gamma_0}(\Omega)$.

The system consisting of equations (2.7) with $F = 0$ and equipped with boundary conditions (2.8) with $\widetilde{Q} = 0$ corresponds to the evolution of the thermoelastic plate [43] evolving in the phase space

$$
H_{w,\theta} \equiv H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Omega).
$$

In fact, this system generates a strongly continuous semigroup of contractions [43, 46]. With the notation introduced above one can represent the thermoelastic generator A as

$$A : D(A) \subset H_{w,\theta} \to H_{w,\theta},$$

$$A \begin{pmatrix} w \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ -\mathcal{A}[w - G_1\theta|_{\Gamma_1} - G_2(\partial_\nu\theta)] - \Delta\theta \\ \Delta\theta + \Delta v \end{pmatrix},$$

$$D(A) = \big\{(w, v, \theta) \in H_{w,\theta} \cap [H^4(\Omega) \times (H^2_{\Gamma_0}(\Omega))^2) \text{ and (2.8) holds with } \widetilde{Q} = 0\big\}.$$

A critical piece of information is that, in addition to the fact that A generates a C_0-semigroup of contractions on $H_{w,\theta}$, this semigroup is *analytic* [53]. Moreover, A is m-dissipative and invertible and enjoys maximal regularity [7, 29] with $\mathrm{tr}[D(A), H_{w,\theta}]_{1/2,2} = D(A^{1/2})$. The latter statement translates into the estimate: For all $f \in L^2(H_{w,\theta})$ and $x \in H_{w,\theta}$ we have

$$\|A^{1/2} \int_0^t e^{A(t-s)} f(s)\mathrm{d}s\|_{L^2(H_{w,\theta})} \leqslant C\|A^{-1/2} f\|_{L^2(H_{w,\theta})}, \tag{2.9}$$

$$\| \int_0^t e^{A(t-s)} f(s)\mathrm{d}s\|_{C(H_{w,\theta})} \leqslant C\|A^{-1/2} f\|_{L^2(H_{w,\theta})}, \tag{2.10}$$

and

$$\|A^{1/2} e^{At} x\|_{L^2(H_{w,\theta})} \leqslant C\|x\|_{H_{w,\theta}}. \tag{2.11}$$

The above allows us to represent system (2.7) and (2.8) with $\widetilde{y}_{w,\theta} \equiv (\widetilde{w}, \widetilde{w}_t, \widetilde{\theta})$ via the semigroup formula

$$\widetilde{y}_{w,\theta}(t) = e^{At}\widetilde{y}_{w,\theta}(0) + \int_0^t e^{A(t-s)}\big(0, F(s) + \mathcal{A}G_2(\widetilde{Q}\cdot\nu), 0\big)^\top \mathrm{d}s, \tag{2.12}$$

where the above representation is meaningful in $H_{w,\theta}$, as seen below. In fact, by applying estimates (2.9)–(2.11) to (2.12), we obtain

$$\begin{aligned} &\|A^{1/2} \int_0^t e^{A(t-s)}(0, F(s) + \mathcal{A}G_2(\widetilde{Q}\cdot\nu), 0)^\top \mathrm{d}s\|_{L^2(H_{w,\theta})} \\ &\qquad \leqslant C\|A^{-1/2}(0, F(s) + \mathcal{A}G_2(\widetilde{Q}\cdot\nu), 0)\|_{L^2(H_{w,\theta})}. \end{aligned}$$

Since $D(A) \subset H^4(\Omega) \times H^2(\Omega) \times H^2(\Omega)$, by interpolation $D(A^{1/2}) \subset H^3(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.

This maximal regularity estimate applied to (2.7) translates into the following inequality:

(2.13)
$$\begin{aligned} &\|\widetilde{y}_{w,\theta}(t)\|^2_{H_{w,\theta}} + \int_0^t (\|\widetilde{w}\|^2_{3,\Omega} + \|\widetilde{w}_t\|^2_{1,\Omega} + \|\widetilde{\theta}\|^2_{1,\Omega})\mathrm{d}t \\ &\qquad \leqslant C\|\widetilde{y}_{w,\theta}(0)\|^2_{H_{w,\theta}} + C\int_0^t \|A^{-1/2}(0, F(s) + \mathcal{A}G_2(\widetilde{Q}\cdot\nu), 0)^T\|^2_{L^2(H_{w,\theta})}\mathrm{d}s. \end{aligned}$$

We calculate the norm of the distribution on the right-hand side of (2.13). This leads [49] to the estimate for the term

$$\begin{aligned}\|A^{-1/2}[0,F(s)+\mathcal{A}G_2(\widetilde{Q}\cdot\nu),0]^T\|_{H_{w,\theta}} &\leqslant \|F(s)+\mathcal{A}G_2(\widetilde{Q}\cdot\nu)\|_{[H^1(\Omega)]'}\\ &= \sup_{\|v\|_{1,\Omega}=1} |(F(s)+\mathcal{A}G_2H(s),v)_\Omega|.\end{aligned}$$

Since $G_2^*\mathcal{A}v=-v|_\Gamma$, we obtain

$$\begin{aligned}(\operatorname{div}\{\widetilde{Q}\}+\mathcal{A}G_2(\widetilde{Q}\cdot\nu),v)_\Omega &= -(\widetilde{Q},\nabla v)_\Omega+\langle\widetilde{Q}\cdot\nu,v\rangle_\Gamma+\langle\widetilde{Q}\cdot\nu,G_2^*\mathcal{A}v\rangle_\Gamma\\ &= -(\widetilde{Q},\nabla v)_\Omega+\langle\widetilde{Q}\cdot\nu,v\rangle_\Gamma-\langle\widetilde{Q}\cdot\nu,v\rangle_\Gamma=-(\widetilde{Q},\nabla v)_\Omega.\end{aligned}$$

It remains to estimate the product $(\widetilde{Q},\nabla v)_\Omega$. Here, the more critical is the term $\widetilde{Q}$,

$$\begin{aligned}\|\widetilde{Q}\|_\Omega \leqslant{}& C\big(\|\widetilde{\mathbf{u}}\|_{1,\Omega}\|w^i\|_{2+\epsilon,\Omega}+\|\mathbf{u}^i\|_{1,\Omega}\|\widetilde{w}\|_{2+\epsilon,\Omega}+\|\widetilde{w}\|_{2,\Omega}\|w^i\|_{2,\Omega}^2+\|\widetilde{w}\|_{2,\Omega}\|w^i\|_{2,\Omega}\big)\\ &+C\big(\|\widetilde{\phi}\|_\Omega\|w^i\|_{2+\epsilon,\Omega}+\|\widetilde{w}\|_{2+\epsilon,\Omega}\|\phi^i\|_\Omega\big);\end{aligned}$$

hence,

$$\begin{aligned}|(\widetilde{Q},\nabla v)_\Omega| &\leqslant \|\widetilde{Q}\|_\Omega\|v\|_{1,\Omega}\\ &\leqslant C\big(\|\widetilde{\mathbf{u}}\|_{1,\Omega}+\|\widetilde{w}\|_{2+\epsilon,\Omega}+\|\widetilde{\phi}\|_\Omega\big)\big(\|\mathbf{u}^i\|_{1,\Omega}+\|w^i\|_{2,\Omega}^2+\|w^i\|_{2+\epsilon,\Omega}+\|\phi^i\|_\Omega\big)\|v\|_{1,\Omega}.\end{aligned}$$

For the second term in the definition of F, we have

$$\begin{aligned}(K(\sigma(A)_1-\sigma(A)_2),v)_\Omega &\leqslant C|(\epsilon(\widetilde{\mathbf{u}}),v)_\Omega+(\nabla\widetilde{w}\oplus[\nabla w_1+\nabla w_2],v)_\Omega|\\ &\leqslant C\|\widetilde{y}\|_H(1+\|w^i\|_{2,\Omega})\|v\|_\Omega.\end{aligned}$$

Summing up and also estimating terms corresponding to $\beta(w_1^3-w_2^3)$ and $\widetilde{p}_0$, we find

$$\begin{aligned}&\int_0^T(\|\widetilde{w}\|_{3,\Omega}^2+\|\widetilde{w}_t\|_{1,\Omega}^2+\|\widetilde{\theta}\|_{1,\Omega}^2)\mathrm{d}t\\ &\quad\leqslant C\|\widetilde{y}_{w,\theta}(0)\|_{H_{w,\theta}}^2+C_{\|y_i\|_{X_T}}\int_0^T(\|\widetilde{w}\|_{2+\epsilon}^2+\|\widetilde{\mathbf{u}}\|_{1,\Omega}^2+\|\widetilde{\phi}\|_\Omega^2)\mathrm{d}t\\ &\quad\leqslant C\|\widetilde{y}_{w,\theta}(0)\|_{H_{w,\theta}}^2+C_{\|y_i\|_{X_T}}\int_0^T(\epsilon\|\widetilde{w}\|_{3,\Omega}^2+C_\epsilon\|\widetilde{w}\|_\Omega^2+\|\widetilde{\mathbf{u}}\|_{1,\Omega}^2+\|\widetilde{\phi}\|_\Omega^2)\mathrm{d}t.\end{aligned}$$

Taking ϵ small, we obtain a first smoothing estimate

$$\begin{aligned}&\int_0^T(\|\widetilde{w}\|_{3,\Omega}^2+\|\widetilde{w}_t\|_{1,\Omega}^2+\|\widetilde{\theta}\|_{1,\Omega}^2)\mathrm{d}t\\ &\qquad\leqslant C\|\widetilde{y}_{w,\theta}(0)\|_{H_{w,\theta}}^2+C_{\|y_i\|_{X_T}}\int_0^T(\|\widetilde{w}\|_\Omega^2+\|\widetilde{\mathbf{u}}\|_{1,\Omega}^2+\|\widetilde{\phi}\|_\Omega^2)\mathrm{d}t,\end{aligned}\tag{2.14}$$

and, after applying (2.10),

$$\begin{aligned}&\|\widetilde{y}_{w,\theta}\|_{H_{w,\theta}}^2+\int_0^T(\|\widetilde{w}\|_{3,\Omega}^2+\|\widetilde{w}_t\|_{1,\Omega}^2+\|\widetilde{\theta}\|_{1,\Omega}^2)\,\mathrm{d}t\\ &\qquad\leqslant C\|\widetilde{y}_{w,\theta}(0)\|_{H_{w,\theta}}^2+C_{\|y_i\|_{X_T}}\int_0^T(\|\widetilde{w}\|_\Omega^2+\|\widetilde{\mathbf{u}}\|_{1,\Omega}^2+\|\widetilde{\phi}\|_\Omega^2)\mathrm{d}t.\end{aligned}\tag{2.15}$$

Inserting (2.15) into (2.6) gives

$$\begin{aligned}
&\|\widetilde{y}(t)\|_H^2 + C\int_0^t \Big(\|\widetilde{\mathbf{u}}_t(s)\|_{\Gamma_1}^2 \ + \ \|\nabla\widetilde{\phi}(s)\|_\Omega^2 + \lambda_1\|\widetilde{\phi}(s)\|_\Omega^2\ \Big)\mathrm{d}s\\
&\quad\leqslant C\|\widetilde{y}(0)\|_H^2 + C_{\|y_i\|_{X_T}} T^{r_1}\|\widetilde{w}\|^2_{C([0,T];H^2(\Omega))}\\
&\qquad + C_{\|y_i\|_{X_T}}(T^{r_1}+T^{r_2})\int_0^T \big(\|\widetilde{w}\|_\Omega^2 + \|\widetilde{\mathbf{u}}\|_{1,\Omega}^2 + \|\widetilde{\phi}\|_\Omega^2\big)\mathrm{d}t.
\end{aligned}$$

Taking T sufficiently small, we obtain $\|\widetilde{y}(t)\|_H \equiv 0$ for $t \in [0,T_0]$. The above arguments can be repeated on $[T_0, 2T_0]$, yielding the desired uniqueness of solutions along with continuous dependence on the initial data in the phase space H for as long as a local solution with regularity in X_T exists. □

We shall now proceed with the second step, where we show that local solutions with the desired regularity can, indeed, be constructed.

Proposition 2.2 (Existence). *Assume part* (1.10) *of Assumption* 1*. For every* $y_0 \in H$ *there exists a* $T>0$ *such that there exists a solution* $y \in X_T$ *which satisfies* (1.2) *with prescribed boundary conditions. If, in addition, we also assume* (1.11) *in Assumption* 1*, then said solution is global.*

Proof.

Step 1. In the first step we prove local existence for $t < T_0$ with some $T_0 > 0$. With the prepared background this task is reasonably straightforward. We shall construct a fixed point for the map $\hat{y} \mapsto y$, where $y = (\mathbf{u}, \mathbf{u}_t, w, w_t, \phi, \theta)$ satisfies

$$\begin{aligned}
\mathbf{u}_{tt} - \operatorname{div}\{\sigma[\epsilon(\mathbf{u})]\} + \nabla\phi &= \operatorname{div}\{P(\hat{w})\},\\
\phi_t - \Delta\phi + \operatorname{div}\{\mathbf{u}_t\} &= R(\hat{w}),\\
w_{tt} + \Delta^2 w + \Delta\theta + \beta = \operatorname{div}\{Q(\hat{\mathbf{u}}, \hat{w}, \hat{\phi})\} - K\sigma[A(\hat{\mathbf{u}}, \hat{w})] &+ p_0(w),\\
\theta_t - \Delta\theta - \Delta w_t &= 0,
\end{aligned}$$

with the boundary conditions

$$\begin{aligned}
\sigma[\epsilon(\mathbf{u})] - \phi\nu + \kappa\mathbf{u} + \mathbf{u}_t &= -P(\hat{w})\nu,\\
\Delta w + (1-\nu)B_1 w + \theta &= 0,\\
\partial_\nu(\Delta w) + (1-\mu)B_2 w + \partial_\nu\theta &= Q(\hat{\mathbf{u}}, \hat{w}, \hat{\phi})\nu,
\end{aligned}$$

where

$$\begin{aligned}
\hat{P} &= P(\hat{w}) \equiv \sigma[f(\nabla\hat{w}) + J(\hat{w})],\\
\hat{Q} &= Q(\hat{\mathbf{u}}, \hat{w}, \hat{\phi}) \equiv \sigma[A(\hat{\mathbf{u}}, \hat{w})]\nabla\hat{w} + \hat{\phi}\nabla\hat{w},\\
\hat{R} &= R(\hat{w}) \equiv \nabla\hat{w}\nabla\hat{w}_t.
\end{aligned}$$

For every initial data $y(0) \in H$ consider nonlinear map $\mathcal{T} : B_{X_{T_0}}(R) \to B_{X_{T_0}}(R)$ such that $y = \mathcal{T}(\hat{y})$. Such a map is well defined—owning a priori regularity (in X_T) of the terms P, Q, and R. This was argued already before. For any $\|y(0)\|_H \leqslant R_0$ we will find a suitable (small) $T_0 > 0$ and $R > R_0$ such that the map $\mathcal{T}$ is a contraction on $B_{X_{T_0}}(R)$. By the calculations performed in the proof of Proposition 2.1, one easily shows that for sufficiently small T_0 and suitable R (depending on R_0) the map $\mathcal{T}$ leaves the ball $B_{X_{T_0}}(R)$ invariant. In order to assert the existence of a fixed point, one needs to establish contractivity. To this end we let $\hat{y}_1, \hat{y}_2 \in B_{X_{T_0}}(R)$, and we consider the following notation: $\mathcal{T}(\hat{y}_1) = y_1, \mathcal{T}(\hat{y}_2) = y_2, \mathcal{T}(\hat{y}_1) - \mathcal{T}(\hat{y}_2) = \widetilde{y}$.

By using energy estimate corresponding to the variable $\mathbf{u}$, we obtain

$$\|\widetilde{y}_{u,\phi}(t)\|^2_{H_{u,\phi}} + \int_0^t \big(\|\widetilde{\mathbf{u}}_t(t)\|^2_{\Gamma_1} + \|\nabla\widetilde{\phi}(t)\|^2_\Omega + \lambda_1\|\widetilde{\phi}(t)\|^2_\Gamma\big)\mathrm{d}t$$
$$\leqslant C\|\widetilde{y}_{u,\phi}(0)\|^2_{H_{u,\phi}} + C\int_0^t \Big|-(\hat{P}_1 - \hat{P}_2, \widetilde{\mathbf{u}})_\Omega|_0^t + (\hat{P}_{1,t} - \hat{P}_{2,t}, \epsilon(\widetilde{\mathbf{u}}))_\Omega + (\widetilde{R}, \widetilde{\phi})_\Omega\Big|\mathrm{d}t.$$

We obtain, for some $r_1, r_2 > 0$, the estimate

$$\begin{aligned}
&\|\widetilde{y}_{u,\phi}(t)\|^2_{H_{u,\phi}} + \int_0^t \big(\|\widetilde{\mathbf{u}}_t(t)\|^2_{\Gamma_1} + \|\nabla\widetilde{\phi}(t)\|^2_\Omega + \lambda_1\|\widetilde{\phi}(t)\|^2_\Gamma\big)\mathrm{d}t \\
&\quad\leqslant C_{\|\hat{y}\|_{B_{X_{T_0}}}} T^{r_1}\|\hat{w}^1 - \hat{w}^2\|^2_{C(H^2)} \\
&\qquad + C_{\|\hat{y}\|_{B_{X_{T_0}}}}(T^{r_1} + T^{r_2})\int_0^T (\|\hat{w}^1 - \hat{w}^2\|^2_{3,\Omega} + \|\hat{w}^1_t - \hat{w}^2_t\|^2_{1,\Omega})\mathrm{d}t.
\end{aligned} \tag{2.16}$$

We can see from (2.16) that a finite energy estimate for $\mathbf{u}$ requires a supercritical level of energy for w. In fact, the situation is even more subtle. Obtaining a finite energy estimate for the w equation requires the term $\mathrm{div}\{Q\}$ to be an element of $L^2(\Omega)$. And this is certainly not the case, as we have a loss of $1 + \varepsilon$ derivatives. The reason this loss does not appear in a variational approach which leads at most to an existence of solutions is that there is a cancellation of singularities. However, at the level of construction of solutions and uniqueness, this cancellation does not occur.

We shall resolve this issue by appealing, again, to the analyticity and maximal regularity of semigroups associated with thermal plates. To wit, we denote $D(\mathbf{u}, w) \equiv K\sigma[A(\mathbf{u}, w)] + \beta w^3 - p_0(w)$ and write

$$\begin{aligned}
y_{w,\theta}(t) &= e^{At}y_{w,\theta}(0) + \int_0^t e^{A(t-s)}(0, \mathrm{div}\{Q\} + \mathcal{A}G_2(Q\cdot\nu) - D, 0)^\top \mathrm{d}s \\
&= e^{At}y_{w,\theta}(0) + \int_0^t A^{1/2}e^{A(t-s)}A^{-1/2}(0, \mathrm{div}\{Q\} + \mathcal{A}G_2(Q\cdot\nu) - D, 0)^\top \mathrm{d}s.
\end{aligned}$$

Then we find the following estimate on account of the analyticity of the thermal semigroup:

$$\begin{aligned}
\|y_{w,\theta}(t)\|_{H_{w,\theta}} &\leqslant C\|y_{w,\theta}(0)\|_{H_{w,\theta}} \\
&\quad + \int_0^t \frac{e^{-\omega(t-s)}}{\sqrt{t-s}}\|A^{-1/2}(0, \mathrm{div}\{Q\} + \mathcal{A}G_2(Q\cdot\nu) - D, 0)^\top\|_{H_{w,\theta}}\mathrm{d}s.
\end{aligned}$$

Accounting for singularity in the integral, for $p > 2$ we find

$$\begin{aligned}
\|y_{w,\theta}(t)\|_{H_{w,\theta}} &\leqslant C\|y_{w,\theta}(0)\|_{H_{w,\theta}} + \int_0^t \frac{e^{-\omega(t-s)}}{\sqrt{t-s}}\|\mathrm{div}\{Q\} + \mathcal{A}G_2(Q\cdot\nu) - D\|_{[H^1]'}\mathrm{d}s \\
&\leqslant C\|y_{w,\theta}(0)\|_{H_{w,\theta}} + C\|\mathrm{div}\{Q\} + \mathcal{A}G_2(Q\cdot\nu) - D\|_{L^p([H^1]')}.
\end{aligned}$$

On the other hand, as proved earlier,

$$\|\mathrm{div}\{Q\} + \mathcal{A}G_2(Q\cdot\nu)\|_{L^p([H^1]')} \leqslant C\|Q\|_{L^p(L^2)}.$$

Collecting the estimates yields

$$\|y_{w,\theta}(t)\|_{H_{w,\theta}} \leqslant C(\|y_{w,\theta}(0)\|_{H_{w,\theta}} + \|Q\|_{L^p(L^2)} + \|D\|_{L^2(L^2)}).$$

Moreover, by the maximal regularity exploited earlier, we also have

$$\|A^{1/2}y_{w,\theta}\|_{L^2(H_{w,\theta})} \leqslant C(\|y_{w,\theta}(0)\|_{H_{w,\theta}} + \|Q\|_{L^2(L^2)} + C\|D\|_{L^2(L^2)}). \tag{2.17}$$

Taking into consideration the differences of the solutions, we have

$$\|\widetilde{y}_{w,\theta}(t)\|_{H_{w,\theta}} + \|A^{1/2}\widetilde{y}_{w,\theta}\|_{L_2(H_{w,\theta})} \leqslant C(\|\hat{Q}_1 - \hat{Q}_2\|_{L^p(L^2)} + \|\hat{D}_1 - \hat{D}_2\|_{L^2(L^2)}),$$

where $\hat{D}_1 - \hat{D}_2 = D(\hat{\mathbf{u}}^1, \hat{w}^1) - D(\hat{\mathbf{u}}^2, \hat{w}^2)$.

By exploiting the interpolation inclusion

$$L^2(H^3) \cap H^1(H^1) \subset H^{1-\vartheta}(H^{1+2\vartheta}),$$

followed by $H^{1-\vartheta}(0,T) \subset L^{\frac{2}{\epsilon}}(0,T)$, where $2\vartheta = 1+\epsilon$, we obtain with some $r>0$

$$\|\hat{Q}_1 - \hat{Q}_2\|_{L^p(L^2)} \leqslant CT^r\|\hat{y}_1 - \hat{y}_2\|_{X_{T_0}} \sup_i\{\|\hat{y}_i\|_H + \|\hat{y}_i\|_H^2\},$$

$$\|\hat{D}_1 - \hat{D}_2\|_{L^2(L^2)} \leqslant CT^r\|\hat{y}_1 - \hat{y}_2\|_{X_{T_0}} \sup_i\{\|\hat{y}_i\|_H + \|\hat{y}_i\|_H^2\}.$$

Higher norms of w are controlled by smoothing estimate (2.14),

$$\int_0^T \|\widetilde{w}\|_{3,\Omega}^2 + \|\widetilde{w}_t\|_{1,\Omega}^2 \mathrm{d}t \leqslant C(\|\hat{Q}_1 - \hat{Q}_2\|_{L^2(L^2)}^2 + \|\hat{D}_1 - \hat{D}_2\|_{L^2(L^2)}^2),$$

which gives for $p>2$

$$\|\widetilde{y}_{w,\theta}(t)\|_{H_{w,\theta}}^2 + \int_0^T (\|\widetilde{w}\|_{3,\Omega}^2 + \|\widetilde{w}_t\|_{1,\Omega}^2)\mathrm{d}t \leqslant C(\|\hat{Q}_1 - \hat{Q}_2\|_{L^p(L^2)}^2 + \|\hat{D}_1 - \hat{D}_2\|_{L^2(L^2)}^2). \tag{2.18}$$

Going back to $\mathbf{u}$ estimate (2.16) along with (2.18) leads to

$$\|\widetilde{y}(t)\|_H^2 + \int_0^T \left(\|\widetilde{w}\|_{3,\Omega}^2 + \|\widetilde{w}_t\|_{1,\Omega}^2\right)\mathrm{d}t \leqslant C_{\|\hat{y}^i\|_{B_{X_{T_0}}}} T^r\|\hat{y}_1 - \hat{y}_2\|_{X_T}^2 \qquad \text{for some } r>0,$$

which completes the proof of contractivity after taking T sufficiently small.

Step 2. Global bounds. Here, the idea is to explore cancellations occurring in the finite energy estimates and to rely on a smoothing mechanism which leads to linear control of the additional finite energy bounds on H. The energy bounds on H are standard and follow from the application of variational formulation—after accounting for cancellations. This means that we obtain from a standard energy estimate with $p'(s) = p_0(s)$

$$\begin{aligned}\|y(t)\|_H^2 &+ \int_\Omega \big(\frac{\beta}{4}|w(t)|^4 - p(w(t))\big)\mathrm{d}\Omega \\ &+ \int_0^t \big(\|\mathbf{u}_t\|_{\Gamma_1}^2 + \|\nabla\phi\|_\Omega^2 + \|\nabla\theta\|_\Omega^2 + \|\phi\|_\Gamma^2 + \|\theta\|_\Gamma^2\big)\mathrm{d}s \\ &= \|y(0)\|_H^2 + \int_\Omega \big(\frac{\beta}{4}|w(0)|^4 - p(w(0))\big)\mathrm{d}\Omega.\end{aligned}$$

By the dissipativity part of Assumption (1.11) and Gronwall's inequality,

$$\|y(t)\|_H^2 + \int_0^t \left(\|\mathbf{u}_t\|_{\Gamma_1}^2 + \|\nabla\phi\|_\Omega^2 + \|\nabla\theta\|_\Omega^2 + \|\phi\|_\Gamma^2 + \|\theta\|_\Gamma^2\right)\mathrm{d}s \leqslant C_t\|y(0)\|_H^2.$$

On the other hand, from (2.17)

$$\int_0^t \left(\|w_t\|_{1,\Omega}^2 + \|w\|_{3,\Omega}^2\right) \mathrm{d}s \leqslant C\|y(0)\|_H^2 + \int_0^t \left(\|y\|_H + \|y\|_H^2\right)\|w\|_{2+\epsilon,\Omega}\mathrm{d}s$$
$$\leqslant C\|y(0)\|_H^2 + \epsilon\int_0^t \|w\|_{3,\Omega}^2\,\mathrm{d}s + C_\epsilon\int_0^t (\|y\|_H + \|y\|_H^2)^2\mathrm{d}s.$$

Selecting small ϵ and a priori bounds for $\|y\|_H$ gives, for every $T>0$,

$$\sup_{t\in[0,T]} \|y(t)\|_H^2 + \int_0^t \left(\|w_t\|_{1,\Omega}^2 + \|w\|_{3,\Omega}^2\right) \mathrm{d}s \leqslant C_T\|y(0)\|_H^2,$$

which is the desired a priori bound in X_T.

Local solutions with a priori bounds in X_T yield the result stated in Proposition 2.2. □

The proof of Lemma 2.1 follows now by combining the results of Propositions 2.1 and 2.2 □

2.3. **Completion of the proofs of Theorems 1.1 and 1.2.** First, note that Proposition 2.2 provides the actual construction of weak solutions with the regularity characterized by X_T space. Thus, the statement in Theorem 1.1 follows directly from Proposition 2.2 and Lemma 2.1. Existence of weak (finite energy) solutions can also be shown by Faedo–Galerkin's method [70]. However, this method would not exhibit the additional regularity of the vertical displacement, a piece of information critical for the uniqueness and continuous dependence with respect to the data.

As to Theorem 1.2, which provides boosted regularity for more regular and compatible boundary initial data, the arguments from this point on are routine and relay on application of the estimates in Lemma 2.1 to time-differentiated version of the system. This part is now straightforward and hence is omitted. The reader may consult [49] for some technical details.

3. Global attractors: Proof of Theorem 1.3

3.1. **Attractors for quasi-stable systems: Abstract results.** In this subsection we provide several abstract results pertaining to existence and characterization of the attractors for quasi-stable dynamical systems [14, 19].

To facilitate the reading, we shall introduce several concepts from the area of dynamical systems. A dynamical system is a pair $(H, S(t))$, where H is a Banach space and $S(t)$ is a continuous semigroup defined on H. We recall that a set $\mathbf{A}\subset H$ is a global attractor for $(H, S(t))$ if it is compact, fully invariant, and uniformly attracting; that is,

$$S(t)\mathbf{A} = \mathbf{A}$$

and

$$\lim_{t\to\infty} \mathrm{d}_H\{S(t)D, \mathbf{A}\} = 0$$

for any bounded set $D\subset H$, where d_H denotes the Hausdorff semidistance.

As is well known, the existence of a global attractor is granted under suitable dissipativeness and compactness conditions. A dynamical system is called dissipative if it admits a bounded absorbing set, that is, a bounded set $\mathcal{B}\subset H$ such that, for any bounded set $D \subset H$, there exists a time $T_D > 0$ satisfying

$$S(t)D \subset \mathcal{B} \qquad \forall t \geqslant T_D.$$

On the other hand, a dynamical system is called asymptotically smooth if, for any bounded set $D \subset H$ forward invariant ($S(t)D \subset D$, $t \geqslant 0$), there exists a compact set $K \subset \overline{D}$ that uniformly attracts D. Then we have the following classical result; see [5, 18, 19, 32, 42, 71].

Theorem 3.1. *Let $(H, S(t))$ be a dynamical system, dissipative and asymptotically smooth. Then it possesses a unique compact global attractor.*

In the present paper we have exploited the dissipative property characterized by gradient systems, that is, systems possessing a strict Lyapunov function. We recall that $\Phi : H \to \mathbb{R}$ is a strict Lyapunov function if

(i) the map $t \mapsto \Phi(S(t)y)$ is nonincreasing for any $y \in H$;
(ii) if $\Phi(S(t)y) = \Phi(y)$ for all t, then y is a stationary point of $S(t)$.

In the case of a gradient system, the global attractor $\mathbf{A}$ is the unstable manifold of the set of equilibrium points.

The following result is well known. See, for instance, [19, Corollary 7.5.7].

Theorem 3.2. *Let $(H, S(t))$ be an asymptotically smooth gradient system with the corresponding Lyapunov functional denoted by Φ. Suppose that*

$$\Phi(y) \to \infty \quad \textit{if and only if} \quad \|y\|_H \to \infty, \tag{3.19}$$

and that the set of stationary points $\mathcal{N}$ is bounded. Then $(H, S(t))$ has a compact global attractor which coincides with the unstable manifold $\mathbb{M}_+(\mathcal{N})$.

Concerning the asymptotically smooth property, we shall introduce the concept of quasi-stability [19, Chapter 7, Definition 7.9.2].

Let X, Y, Z be three reflexive Banach spaces with X compactly embedded into Y, and define $H = X \times Y \times Z$. Suppose that $(H, S(t))$ is a dynamical system of the form

$$S(t)y = (u(t), u_t(t), \xi(t)), \qquad y = (u(0), u_t(0), \xi(0)) \in H, \tag{3.20}$$

where the functions u and ξ have regularity

$$u \in C([0,\infty); X) \cap C^1([0,\infty); Y), \quad \xi \in C([0,\infty); Z). \tag{3.21}$$

Then we say that $(H, S(t))$ is quasi-stable on a set $B \subset H$ if there exists a compact seminorm n_X on X and nonnegative scalar functions a and c, locally bounded in $[0,\infty)$, and $b \in L^1(0,\infty)$, with $\lim\limits_{t\to\infty} b(t) = 0$, such that

$$\|S(t)y_1 - S(t)y_2\|_H^2 \leqslant a(t)\|y_1 - y_2\|_H^2, \tag{3.22}$$

and, for $S(t)y_i = (u^i(t), u_t^i(t), \xi^i(t))$, $i = 1, 2$,

$$\|S(t)y_1 - S(t)y_2\|_H^2 \leqslant b(t)\|y_1 - y_2\|_H^2 + c(t) \sup_{0\leqslant s\leqslant t} \left[n_X(u^1(s) - u^2(s))\right]^2 \tag{3.23}$$

for any $y_1, y_2 \in B$. In this case the following result holds.

Theorem 3.3 ([19, Proposition 7.9.4])**.** *Let $(H, S(t))$ be a dynamical system given by* (3.20) *and satisfying* 3.21*. Then $(H, S(t))$ is asymptotically smooth if it is quasi-stable on every bounded positively invariant set of H.*

For quasi-stable systems we automatically have both smoothness and finite dimensionality of the attractor. This is guaranteed by the following result.

Theorem 3.4 ([19, Theorems 7.9.6 and 7.9.8])**.** *Let $(H, S(t))$ be a dynamical system given by* (3.20) *and satisfying* (3.21). *Suppose that it has a global attractor* $\mathbf{A}$. *Then if $(H, S(t))$ is quasi-stable on* $\mathbf{A}$*, this global attractor has finite fractal dimension. Moreover, its complete trajectories have additional* (*time*) *regularity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(S(t)y\big)\in L^\infty(\mathbb{R}, H), \qquad y \in \mathbf{A},$$

with the estimate $\|\frac{\mathrm{d}}{\mathrm{d}t}(S(t)y)\|_H \leqslant M, t \in R$, *where* M *depends on* $\sup_{t\geq 0} c(t)$.

Our main result is to establish the existence of a smooth and finite-dimensional global attractor. This will be achieved with the aid of Theorem 3.2, which guarantees the existence of a global attractor, and Theorem 3.4, which provides finite fractal dimension and smoothness. Next we summarize these results into the following corollary.

Corollary 3.1. *Let $(H, S(t))$ be given by* (3.20) *satisfying* (3.21)*, and let it be a quasi-stable gradient system with Lyapunov function satisfying* (3.19) *and a bounded set of stationary points. Then $(H, S(t))$ admits a finite-dimensional global attractor* $\mathbf{A}$ *which is also "smooth":* $\frac{\mathrm{d}}{\mathrm{d}t}\big(S(t)y\big)\in L^\infty(\mathbb{R}, H)$ *for* $y \in \mathbf{A}$. *If, in addition,* $c(t)$ *in* (3.23) *is bounded for* $t > 0$, *there exists an* $M < \infty$ *such that* $\|\frac{\mathrm{d}}{\mathrm{d}t}(S(t)y)\|_H \leqslant M, t \in R$.

3.2. **Proof of Theorem 1.3.** In this subsection we prove Theorem 1.3. This happens in the steps that follows: The corresponding dynamical system $(H, S(t))$ is (1) a gradient system with a Lyapunov function satisfying (3.19), and (2) a quasi-stable system with the appropriate bounds for $c(t)$. The first is a consequence of a new unique continuation property shown for the system under consideration. The latter property is based on results related to hidden regularity of the boundary traces corresponding to vectorial systems—$\mathbf{u}$ displacement—with free boundary conditions and also relies on a use of the analyticity of the semigroup corresponding to the linear part of the thermoelastic model (w, θ).

3.3. **Energy functional.** In what follows we make some remarks on the energy of the system. Along a solution $y = (\mathbf{u}, \mathbf{u}_t, w, w_t, \phi, \theta)$, the energy of the system, with $p'(s) = p_0(s)$, is defined by

$$\mathcal{E}_y(t) \equiv E_k(t) + E_p(t) - \int_\Omega p^+(w)\mathrm{d}\Omega,$$

with kinetic energy $E_k(\cdot)$ and potential energy $E_p(\cdot)$ defined by

$$E_k(t) \equiv \frac{1}{2}\int_\Omega \big(|\mathbf{u}_t|^2 + |w_t|^2\big)\mathrm{d}\Omega$$

and

$$E_p(t) \equiv \frac{1}{2}\int_\Omega \Big(\sigma[A]A + |\phi|^2 + |\theta|^2\Big)\mathrm{d}\Omega + \frac{1}{2}a(w, w) + \frac{\beta}{4}\|w\|^4_{L_4(\Omega)} - \int_\Omega p^-(w)d\Omega + \frac{\kappa}{2}\|\mathbf{u}\|^2_{\Gamma_1},$$

where $p(s) = p^+(s) + p^-(s)$, with $p^-(s) \leqslant 0$. The bilinear form $a(w,z)$ is defined via

$$a(w,z) \equiv \int_\Omega \big(w_{xx}z_{xx} + w_{yy}z_{yy} + \mu w_{xx}z_{yy} + \mu w_{yy}z_{xx} + 2(1-\mu)w_{xy}z_{xy}\big)\mathrm{d}\Omega$$
$$+ l\int_{\Gamma_1} wz\mathrm{d}\Gamma_1.$$

We shall derive formal energy relations, which are easily justified for smooth solutions and later obtained, via density, for weak solutions.

Lemma 3.1. *The following energy balance is satisfied for all weak solutions:*

$$\mathcal{E}_y(t) + \int_s^t \big(\|\mathbf{u}_t(\tau)\|^2_{\Gamma_1} + \|\nabla\phi(\tau)\|^2_\Omega + \|\nabla\theta(\tau)\|^2_\Omega + \lambda_1\|\phi(\tau)\|^2_\Gamma + \lambda_2\|\theta(\tau)\|^2_\Gamma\big)\mathrm{d}\tau = \mathcal{E}_y(s).$$

Proof. We first take the $L^2(\Omega)$ inner product of (1.2) with $(\mathbf{u}_t, w_t)^\top$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{u}_t\|^2_\Omega + \kappa\|\mathbf{u}_t\|^2_{\Gamma_1}\big) + (\sigma[A], \epsilon(\mathbf{u}_t))_\Omega + (\nabla\phi, \mathbf{u}_t)_\Omega = 0$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|w_t\|^2_\Omega + a(w,w) + \frac{\beta}{2}\|w\|^4_{L^4(\Omega)}\big) + (\sigma[A], (\nabla w \otimes \nabla w_t))_\Omega$$
$$+ (\phi\nabla w, \nabla w_t)_\Omega + (\sigma[A], Jw_t)_\Omega + (\Delta\theta - p_0, w_t)_\Omega = 0.$$

Accounting for the heat transfer,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{u}_t\|^2_\Omega + \|w_t\|^2_\Omega + a(w,w) + \frac{\beta}{2}\|w\|^4_{L^4(\Omega)} + \kappa\|\mathbf{u}_t\|^2_{\Gamma_1} + \|\phi\|^2_\Omega + \|\theta\|^2_\Omega) + (\sigma[A], \frac{\mathrm{d}}{\mathrm{d}t}A)_\Omega$$
$$+ \|\mathbf{u}_t\|^2_{\Gamma_1} + \|\nabla\phi\|^2_\Omega + \|\nabla\theta\|^2_\Omega + \lambda_1\|\phi\|^2_\Gamma + \lambda_2\|\theta\|^2_\Gamma = (p_0, w_t)_\Omega.$$

Hence, by identity $(\frac{\mathrm{d}}{\mathrm{d}t}\sigma[A], A)_\Omega = (\sigma[A], \frac{\mathrm{d}}{\mathrm{d}t}A)_\Omega$ one can show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{u}_t\|^2_\Omega + \|w_t\|^2_\Omega + a(w,w) + \frac{\beta}{2}\|w\|^4_{L^4(\Omega)} + \kappa\|\mathbf{u}_t\|^2_{\Gamma_1} + \|\phi\|^2_\Omega + \|\theta\|^2_\Omega + (\sigma[A], A)_\Omega\big)$$
$$+ \|\mathbf{u}_t\|^2_{\Gamma_1} + \|\nabla\phi\|^2_\Omega + \|\nabla\theta\|^2_\Omega + \lambda_1\|\phi\|^2_\Gamma + \lambda_2\|\theta\|^2_\Gamma = (p_0, w_t)_\Omega.$$

This yields the conclusion. □

Note that $E_y(\cdot) \equiv E_k(\cdot) + E_p(\cdot)$ is topologically equivalent to the phase space H. In fact, due to Korn's inequality [25] and simple algebraic manipulations, we have

$$\|\mathbf{u}\|_{1,\Omega} \leqslant C\big(\|N(\mathbf{u}, w)\|_\Omega + \|\nabla w\|^2_{L_4(\Omega)}\big).$$

Thus,

$$\|\mathbf{u}\|_{1,\Omega} \leqslant C\big(\|A(\mathbf{u}, w)\|_\Omega + \|w\|_\Omega + \|\nabla w\|^2_{L_4(\Omega)}\big).$$

On the other hand, due to positive definitiveness of the bilinear form $a(w,w)$, one obtains $\|w\|^2_{2,\Omega} \leqslant Ca(w,w) \leqslant CE_p$; hence,

$$\|\mathbf{u}\|^2_{1,\Omega} + \|w\|^2_{2,\Omega} + \|\theta\|^2_\Omega + \|\phi\|^2_\Omega \leqslant CE_p^2.$$

We also notice that the energy $\mathcal{E}_y(\cdot)$ is topologically equivalent to $E_y(\cdot)$, giving rise to a topology on the phase space for the variable $y = (\mathbf{u}, \mathbf{u}_t, w, w_t, \theta, \phi) \in H$.

Lemma 3.2. *Let the constant M in Assumption 1 be suitably small or $\alpha < 1$ (resp., $\alpha < 3$) when $\beta = 0$ (resp., $\beta > 0$). Then there exist positive constants M_1, M_2, m_E such that*

$$M_1 E_y(t) - m_E \leqslant \mathcal{E}_y(t) \leqslant M_2 E_y(t) + m_E \qquad \forall t \geqslant 0.$$

Proof. We note first that the following inequality holds:

$$|\mathcal{E}_y(t) - E_y(t)| \leqslant \int_\Omega |p^+(w)| \,\mathrm{d}\Omega.$$

Now, on the strength of the generalized dissipativity assumption imposed on p_0, one obtains for the decomposition $p = p^+ + p^-$

$$|p^+(s)| \leqslant M|s|^{\alpha+1} + m|s|.$$

Since $|s|^{\alpha+1} \leqslant \delta|s|^4 + C_\delta|s|^2$ when $\beta > 0$ (and hence $\alpha < 3$), this implies that

$$|s|^{\alpha+1} \leqslant \delta\frac{\beta}{4}|s|^4 + C_{\delta,\beta}.$$

When $\alpha < 1$ and $\beta = 0$, we then have for an arbitrary small $\delta > 0$

$$|s|^{\alpha+1} \leqslant \delta|s|^2 + C_\delta.$$

Putting these together yields

$$\int_\Omega |p^+(w)| \mathrm{d}\Omega \leqslant \delta\Big(\|w\|_{2,\Omega}^2 + \frac{\beta}{4}\|w\|_{L_4(\Omega)}^4\Big) + C_{\delta,\beta,M,m,\Omega} \qquad \text{if } \beta > 0,$$

$$\int_\Omega |p^+(w)| \mathrm{d}\Omega \leqslant \delta\|w\|_{2,\Omega}^2 + C_{\delta,M,m,\Omega} \qquad \text{if } \beta = 0,$$

where $\delta > 0$ can be taken as arbitrarily small. Thus, the conclusion holds.

Now, for the cases $\alpha = 1$ (when $\beta = 0$) and $\alpha = 3$ (when $\beta > 0$), we obtain from Assumption (1.11) the following estimates:

$$\int_\Omega |p^+(w)| \,\mathrm{d}\Omega \leqslant M\|w\|_{L_4(\Omega)}^4 + C_{M,m,\Omega} \leqslant \frac{4M}{\beta} E_y(t) + C_{M,m,\Omega} \qquad \text{if } \beta > 0,$$

$$\int_\Omega |p^+(w)| \,\mathrm{d}\Omega \leqslant M\|w\|_\Omega^2 + C_{M,m,\Omega} \leqslant 2MC_a E_y(t) + C_{M,m,\Omega} \qquad \text{if } \beta = 0.$$

Thus, assuming the following choice of M, Lemma 3.2 is proved

$$M < \frac{\beta}{4} \qquad \text{if } \beta > 0$$

and

$$M < \frac{1}{2C_a} \qquad \text{if } \beta > 0,$$

where C_a is the embedding constant $\|w\|_\Omega^2 \leqslant C_a a(w,w)$. □

3.4. **Gradient system.** In the present section we will we seek a Lyapunov function for $(H, S(t))$. This is done by taking the energy functional $\mathcal{E}_y(\cdot)$ as a Lyapunov function $\Phi(y)$. In fact, from Lemma 3.1 it follows that $t \mapsto \Phi(S(t)y)$ is a decreasing function for any $y \in H$. The strictness property is a consequence of the unique continuation property stated in the lemma below.

Lemma 3.3. *Assume that $\Gamma_0 \neq \emptyset$. Then the following property holds:*

$$\mathcal{E}_y(t) = \mathcal{E}_y(0) \ \forall t > 0 \Rightarrow S(t)y = y \ \forall t > 0.$$

Proof. The assumption and the energy identity given in Lemma 3.1 yield

$$\mathbf{u}_t = 0 \text{ in } \Gamma_1$$

and

$$\phi = \theta = \operatorname{div}\{\mathbf{u}_t\} = 0 \text{ in } \Omega.$$

These imply that, distributionally, $\Delta w_t = 0$ and $w_t = 0$, $\nabla w_t = 0$ on Γ_0. Taking into account that $\operatorname{meas}(\Gamma_0) > 0$, via the elliptic unique continuation property we conclude that $w_t = 0$. With this information we go back to (1.2), and denoting $\bar{\mathbf{u}} \equiv \mathbf{u}_t$, we find

$$\bar{\mathbf{u}}_{tt} - \operatorname{div}\{\sigma[\epsilon(\bar{\mathbf{u}})]\} = 0 \text{ in } \Omega \times (0, \infty)$$

with the overdetermined boundary conditions

$$\begin{aligned} \bar{\mathbf{u}} &= 0 \text{ on } \Gamma, \\ \sigma[\epsilon(\bar{\mathbf{u}})]\nu &= 0 \text{ on } \Gamma_1. \end{aligned} \tag{3.24}$$

This combined with $\operatorname{div}\{\bar{\mathbf{u}}\} = 0$ in Ω gives

$$\begin{aligned} \bar{\mathbf{u}}_{tt} - \operatorname{div}\{\sigma[\epsilon(\bar{\mathbf{u}})]\} = \bar{\mathbf{u}}_{tt} - \eta(\Delta\bar{\mathbf{u}}_1, \Delta\bar{\mathbf{u}}_2) &= 0 \text{ in } \Omega \times (0, \infty), \\ \bar{\mathbf{u}} &= 0 \text{ on } \Gamma \times (0, \infty), \\ \sigma[\epsilon(\bar{\mathbf{u}})]\nu &= 0 \text{ on } \Gamma_1 \times (0, \infty). \end{aligned}$$

Therefore, we are dealing with two wave equations and overdetermined boundary conditions. The overdetermination allows us to conclude that the corresponding solution enjoys one unit higher regularity than the finite energy solution. This is now accomplished by multipliers analysis applied to the overdetermined problem. In fact, one shows in the same way as in [51] that $\mathbf{u}_t \in C(H^1)$, which gives the result that $\bar{\mathbf{u}} \in C(H^1)$ is of finite energy. In the next step we shall show that

$$\partial_\nu \bar{\mathbf{u}} = 0 \text{ on } \Gamma_1 \times (0, \infty). \tag{3.25}$$

Indeed, in order to prove (3.25), let $\nu = (\nu_1, \nu_2)$ with $\nu_1^2 + \nu_2^2 = 1$. We also have $\bar{\mathbf{u}}_\tau = 0$ on Γ_1, where τ represents the tangential direction to the boundary. Via formulas in [19, p. 299] we obtain

$$\bar{\mathbf{u}}_x = \nu_1 \bar{\mathbf{u}}_\nu, \qquad \bar{\mathbf{u}}_y = \nu_2 \bar{u}_\nu.$$

Then we find the following representation of the stress tensor:

$$\epsilon(\bar{\mathbf{u}}) = \begin{bmatrix} \nu_1 \bar{u}_{1\nu} & 1/2(\nu_1 \bar{u}_{2\nu} + \nu_2 \bar{u}_{1\nu}) \\ 1/2(\nu_1 \bar{u}_{2\nu} + \nu_2 \bar{u}_{1\nu}) & \nu_2 \bar{u}_{2\nu} \end{bmatrix}.$$

Hence,

$$\epsilon(\bar{\mathbf{u}})\nu = \begin{bmatrix} \nu_1 \bar{u}_{1\nu} & 1/2(\nu_1 \bar{u}_{2\nu} + \nu_2 \bar{u}_{1\nu}) \\ 1/2(\nu_1 \bar{u}_{2\nu} + \nu_2 \bar{u}_{1\nu}) & \nu_2 \bar{u}_{2\nu} \end{bmatrix} \nu = M \cdot \bar{\mathbf{u}}_\nu,$$

where after the calculations

$$M = \begin{bmatrix} \nu_1^2 + 1/2\nu_2^2 & 1/2\nu_2\nu_1 \\ 1/2\nu_1\nu_2 & \nu_2^2 + 1/2\nu_1^2 \end{bmatrix}.$$

Since the determinant of M is equal to $1/2(\nu_1^2 + \nu_2^2)^2 = 1/2$, the boundary conditions in (3.24) imply (3.25). In addition, the overdetermined solution is of finite energy. Now we are in position to invoke the unique continuation principle in [65] (or [36, Theorem 1.2], [30]) to claim that $\bar{\mathbf{u}} = \mathbf{u}_t \equiv 0$ in Ω. Thus, the dynamics has been reduced to a stationary problem. ☐

3.5. **Stationary solutions.**

Lemma 3.4. *Let M be as in Lemma 3.2. Assuming that either $\beta > 0$ or $\sup_{i=1,2}\{\|K_i\|_{C(\Omega)}\}$ are sufficiently small, then the set of stationary solutions is bounded.*

Proof. Stationary solutions are given by

$$-\operatorname{div}\{\sigma[A]\} + \nabla\phi = 0 \ \text{ in } \ \Omega\times(0,\infty),$$
$$\Delta^2 w - \operatorname{div}\{\sigma[A]\nabla w + \phi\nabla w\} + K\cdot\sigma[A] + \Delta\theta + \beta w^3 - p_0(w) = 0 \ \text{ in } \ \Omega\times(0,\infty),$$

with boundary conditions

$$\mathbf{u} = 0, w = 0, \nabla w = 0 \ \text{ on } \ \Gamma_0\times(0,\infty),$$
$$\sigma[A]\nu + \kappa\mathbf{u} - \phi\nu = 0 \ \text{ on } \ \Gamma_1\times(0,\infty),$$
$$\Delta w + (1-\mu)B_1 w + \theta = 0 \ \text{ on } \ \Gamma_1\times(0,\infty),$$
$$\partial_\nu(\Delta w) + (1-\mu)B_2 w - \sigma[A]\nu\cdot\nabla w - \phi\partial_\nu w + \partial_\nu\theta = 0 \ \text{ on } \ \Gamma_1\times(0,\infty),$$

with thermal components satisfying

$$\Delta\phi = 0 \ \text{ in } \ \Omega\times(0,\infty)$$

and

$$\Delta\theta = 0 \ \text{ in } \ \Omega\times(0,\infty),$$

subject to $\partial_\nu\phi + \lambda_1\phi = \partial_\nu\theta + \lambda_2\theta = 0$ on $\Gamma\times(0,\infty)$. Due to the positivity of λ_1, λ_2, we have $\theta\equiv 0$, $\phi\equiv 0$. This leads to

$$-\operatorname{div}\{\sigma[A]\} = 0 \ \text{ in } \ \Omega\times(0,\infty),$$
$$\Delta^2 w - \operatorname{div}\{\sigma[A]\nabla w\} + K\cdot\sigma[A] + \beta w^3 - p_0(w) = 0 \ \text{ in } \ \Omega\times(0,\infty),$$

with

$$\mathbf{u} = 0, w = 0, \nabla w = 0 \ \text{ on } \ \Gamma_0\times(0,\infty),$$
$$\sigma[A]\nu + \kappa\mathbf{u} = 0 \ \text{ on } \ \Gamma_1\times(0,\infty),$$
$$\Delta w + (1-\mu)B_1 w = 0 \ \text{ on } \ \Gamma_1\times(0,\infty),$$
$$\partial_\nu(\Delta w) + (1-\mu)B_2 w - \sigma[A]\nu\cdot\nabla w = 0 \ \text{ on } \ \Gamma_1\times(0,\infty).$$

By variational methods one easily shows that the stationary problem admits solutions (u, w) in the phase space $[H^1(\Omega)]^2\times H^2(\Omega)$. We next ask questions about whether these solutions are bounded. Taking the inner product of each equation with $\mathbf{u}$ and w and accounting for boundary conditions yields

$$(\sigma[A], \epsilon(\mathbf{u}))_\Omega + \kappa\|\mathbf{u}\|^2_{\Gamma_1} = 0,$$
$$a(w,w) + \beta\|w\|^4_{L^4(\Omega)} + 2(\sigma[A], f(\nabla w))_\Omega + (\sigma[A], Jw)_\Omega - (p_0(w), w) = 0.$$

This gives, after rescaling the second equation by $\frac{1}{2}$,

$$C_\sigma\|\sigma[A(\mathbf{u},w)]\|^2_\Omega + \frac{1}{2}a(w,w) + \frac{1}{2}\beta\|w\|^4_{L_4} + \kappa\|\mathbf{u}\|^2_{\Gamma_1}$$
$$\leqslant \frac{1}{2}|(p_0(w), w)_\Omega| + \frac{1}{2}|(\sigma[A], Jw)_\Omega|,$$

where C_σ is a positive constant such that $C_\sigma\|\sigma[A(\mathbf{u},w)]\|^2_\Omega \leqslant (\sigma[A], A)_\Omega$.

In the case $\beta = 0$ and in view of the fact that $(p_0^-(w), w)_\Omega \leqslant 0$, we find the following bound:

$$\begin{aligned}
&|(p_0^+(w), w)_\Omega + (\sigma[A], Jw)_\Omega| \\
&\qquad \leqslant M\|w\|_\Omega^2 + C_{m,M,\Omega} + \frac{C_\sigma}{2}\|\sigma[A]\|_\Omega^2 + \sup_{i=1,2}\left\{\|K_i\|_{C(\Omega)}\right\}\frac{1}{2C_\sigma}\|w\|_\Omega^2.
\end{aligned}$$

Thus, in order to obtain bounded solutions with $\beta = 0$, one needs

$$\sup_{i=1,2}\left\{\|K_i\|_{C(\Omega)}\right\} < \frac{C_\sigma}{C_a}.$$

If $\beta > 0$, we have

$$|(p_0^+(w), w)_\Omega + (\sigma[A], Jw)_\Omega| \leqslant M\|w\|_{L_4}^4 + \frac{C_\sigma}{2}\|\sigma[A(\mathbf{u}, w)]\|_\Omega^2 + C_{m,M,K,C_\sigma,\Omega},$$

and we do not need any restrictions on K_i. $\square$

3.6. **Quasi-stability inequality.** The goal of this section is to show that our dynamical system satisfies inequality (3.23), which will yield the desired quasi-stability property. Following Corollary 3.1, we are interested in the behavior of the difference of two solutions with initial data from a bounded set $B \subset H$. Here, we denote the difference by

$$\widetilde{y}(t) \equiv (\mathbf{u}^1 - \mathbf{u}^2, w^1 - w^2, \phi^1 - \phi^2, \theta^1 - \theta^2) = (\widetilde{\mathbf{u}}, \widetilde{w}, \widetilde{\phi}, \widetilde{\theta}). \tag{3.26}$$

The flow $\widetilde{y}(\cdot)$ defined in (3.26) satisfies the following problem:

$$\widetilde{\mathbf{u}}_{tt} - \operatorname{div}\left\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\right\} + \nabla\widetilde{\phi} - \operatorname{div}\left\{N_1(\widetilde{w})\right\} = 0, \tag{3.27}$$

$$\widetilde{w}_{tt} + \Delta^2\widetilde{w} + \Delta\widetilde{\theta} - \operatorname{div}\left\{N_2(\widetilde{\mathbf{u}}, \widetilde{w})\right\} + K\cdot\sigma[A(\mathbf{u}^1, w^1) - A(\mathbf{u}^2, w^2)] + P(\widetilde{w}) = 0, \tag{3.28}$$

$$\widetilde{\phi}_t - \Delta\widetilde{\phi} + \operatorname{div}\{\widetilde{\mathbf{u}}_t\} - \left[\nabla w^1\cdot\nabla w_t^1 - \nabla w^2\cdot\nabla w_t^2\right] = 0, \tag{3.29}$$

$$\widetilde{\theta}_t - \Delta\widetilde{\theta} - \Delta\widetilde{w}_t = 0, \tag{3.30}$$

where

$$\begin{aligned}
P &= p_0(w^1) - p_0(w^2) + \beta\widetilde{w}\big((w^1)^2 + (w^2)^2 + w^1w^2\big), \\
N_1 &= \sigma[f(\nabla w^1) - f(\nabla w^2)] + \sigma[J(\widetilde{w})], \\
N_2 &= \sigma[A(\mathbf{u}^1, w^1)]\nabla w^1 - \sigma[A(\mathbf{u}^2, w^2)]\nabla w^2 + \phi^1\nabla w^1 - \phi^2\nabla w^2.
\end{aligned}$$

The displacements $(\widetilde{\mathbf{u}}, \widetilde{w})$ satisfy on $\Gamma_0 \times (0, \infty)$

$$\widetilde{\mathbf{u}} = 0, \quad \widetilde{w} = 0, \quad \nabla\widetilde{w} = 0. \tag{3.31}$$

The boundary conditions on $\Gamma_1 \times (0, \infty)$ are

$$\sigma[\varepsilon(\widetilde{\mathbf{u}})]\nu + N_1(\widetilde{w})\nu + \kappa\widetilde{\mathbf{u}} - \widetilde{\phi}\nu + \widetilde{\mathbf{u}}_t = 0, \tag{3.32}$$

$$\Delta\widetilde{w} + (1 - \mu)B_1\widetilde{w} + \widetilde{\theta} = 0, \tag{3.33}$$

$$\partial_\nu(\Delta\widetilde{w}) + (1 - \mu)B_2\widetilde{w} - N_2(\widetilde{\mathbf{u}}, \widetilde{w})\cdot\nu + \partial_\nu\widetilde{\theta} = 0, \tag{3.34}$$

with the Robin boundary condition on $\Gamma_1 \times (0, \infty)$ for the thermal components.

The corresponding initial data are

$$\begin{aligned} &\widetilde{\mathbf{u}}(\cdot,0)=\mathbf{u}_0^1-\mathbf{u}_0^2, &&\widetilde{\mathbf{u}}_t(\cdot,0)=\mathbf{u}_1^1-\mathbf{u}_1^2,\\ &\widetilde{w}(\cdot,0)=w_0^1-w_0^2, &&\widetilde{w}_t(\cdot,0)=w_1^1-w_1^2,\\ &\widetilde{\phi}(\cdot,0)=\phi_0^1-\phi_0^2, &&\widetilde{\theta}\,(\cdot,0)=\theta_0^1-\theta_0^2. \end{aligned} \tag{3.35}$$

In order to obtain our results, we introduce the energy of the system (3.27)–(3.35),

$$\widetilde{E}(t)=\frac{1}{2}\int_\Omega\Big(|\widetilde{\mathbf{u}}_t|^2+|\widetilde{w}_t|^2+\sigma[\epsilon(\widetilde{\mathbf{u}})]\epsilon(\widetilde{\mathbf{u}})+|\widetilde{\phi}|^2+|\widetilde{\theta}|^2\Big)\mathrm{d}\Omega+\frac{1}{2}a(\widetilde{w},\widetilde{w})+\frac{\kappa}{2}\int_{\Gamma_1}|\widetilde{\mathbf{u}}|^2\mathrm{d}\Gamma_1.$$

We observe that energy $\widetilde{E}(\cdot)$ satisfies

$$\widetilde{E}(t)+D_s^t(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta})=\widetilde{E}(s)+\int_s^t\sum_{i=1}^5 R_i(\tau)\mathrm{d}\tau, \tag{3.36}$$

with

$$\begin{aligned} R_1(t)&=\int_\Omega N_1(\widetilde{w})\widetilde{\mathbf{u}}_t\,\mathrm{d}\Omega+\int_{\Gamma_1}N_1(\widetilde{w})\cdot\nu\widetilde{\mathbf{u}}_t\,\mathrm{d}\Gamma_1,\\ R_2(t)&=\int_\Omega\big(A(\mathbf{u}^1,w^1)\nabla w^1-A(\mathbf{u}^2,w^2)\nabla w^2\big)\nabla\widetilde{w}_t\,\mathrm{d}\Omega,\\ R_3(t)&=-\int_\Omega K\cdot\sigma[A(\mathbf{u}^1,w^1)-A(\mathbf{u}^2,w^2)]\widetilde{w}_t\,\mathrm{d}\Omega,\\ R_4(t)&=-\int_\Omega\big(\phi^1\nabla w^1-\phi^2\nabla w^2\big)\nabla\widetilde{w}_t+\big(\nabla w^1\cdot\nabla w_t^1-\nabla w^2\cdot\nabla w_t^2\big)\widetilde{\phi}\,\mathrm{d}\Omega,\\ R_5(t)&=-\int_\Omega P(\widetilde{w})\widetilde{w}_t\,\mathrm{d}\Omega, \end{aligned} \tag{3.37}$$

and

$$D_s^t(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta})=\int_s^t\big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2+\|\nabla\widetilde{\phi}\|_\Omega^2+\|\nabla\widetilde{\theta}\|_\Omega^2,+\lambda_1\|\widetilde{\phi}\|_\Gamma^2+\lambda_2\|\widetilde{\theta}\|_\Gamma^2\big)\mathrm{d}\tau.$$

The main result of this section is stated below.

Lemma 3.5 (Quasi-stability inequality)**.** *Under the hypotheses of Theorem* 1.3*, let* (3.26) *denote the solution to* (3.27)–(3.35)*. Then, for* $\varepsilon\in(0,\frac{1}{4})$*, there exist constants* $b>0$ *and* $C_1,C_2>0$ *such that*

$$\widetilde{E}(t)\leqslant C_1\widetilde{E}(0)e^{-bt}+C_2\sup_{\tau\in[0,t]}\|\widetilde{\mathbf{u}}(\tau)\|_{1-\varepsilon,\Omega}^2+C_2\sup_{\tau\in[0,t]}\|\widetilde{w}(\tau)\|_{2-\varepsilon,\Omega}^2,$$

with constants depending only on B*.*

Since the proof of Lemma 3.5 is lengthly and needs several energy-type estimates, we outline here a guiding strategy for establishing them:

(i) "Hidden" regularity both for vertical and in-plane displacements are established—Lemmas 3.8 and 3.7—by invoking maximal regularity for thermal shells with free boundary conditions and microlocal estimates applied to a hyperbolic component represented by $\mathbf{u}$.

(ii) Observability inequalities are obtained from multiplier analysis—Section 3.6.2.

(iii) Estimates for R_i—defined in (3.37)—in terms of lower-order terms (l.o.t.) are stated and given in Lemma 3.11.

(iv) Finally, in Section 3.6.4 we complete the stabilizability estimate proof.

We end this section with the following result.

Lemma 3.6. *For every $\varepsilon \in (0,1)$ the following estimates hold,*

(i) $\displaystyle\int_0^T \|f(\nabla w^1) - f(\nabla w^2) + J(\widetilde{w})\|_\Omega^2 \mathrm{d}t \leqslant C_{B,T} \text{ l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}),$

(ii) $\displaystyle\int_0^T \|f(\nabla w^1) - f(\nabla w^2) + J(\widetilde{w})\|_{1,\Omega}^2 \mathrm{d}t \leqslant C_B \int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2 \mathrm{d}t,$

(iii) $\displaystyle\int_0^T \|f(\nabla w^1) - f(\nabla w^2) + J(\widetilde{w})\|_{\Sigma_1}^2 \mathrm{d}t \leqslant C_B \int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2 \mathrm{d}t,$

where the l.o.t. are given by

$$\text{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}) = \sup_{t\in[0,T]} \|\widetilde{\mathbf{u}}(t)\|_{1-\varepsilon,\Omega}^2 + \sup_{t\in[0,T]} \|\widetilde{w}(t)\|_{2-\varepsilon,\Omega}^2.$$

Proof. To prove these estimates, it will suffice to show the desired inequality for the difference of nonlinear terms $f(\nabla w_i)$. We begin with the identity

$$f(\nabla w^1) - f(\nabla w^2) = f(\nabla\widetilde{w}) + \nabla\widetilde{w} \otimes \nabla w^2 + \nabla w^2 \otimes \nabla\widetilde{w}.$$

To show (i), we use $\|u \otimes v\|_\Omega \leqslant C\|u\|_{\varepsilon,\Omega}\|v\|_{1-\varepsilon,\Omega}$:

$$\begin{aligned}\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_\Omega^2 \mathrm{d}t &\leqslant C\int_0^T \Big(\|\widetilde{w}\|_{1+\varepsilon,\Omega}^2 \|\widetilde{w}\|_{2-\varepsilon,\Omega}^2 + \|w^2\|_{1+\varepsilon,\Omega}^2 \|\widetilde{w}\|_{2-\varepsilon,\Omega}^2\Big)\mathrm{d}t \\ &\leqslant C_{B,T} \text{ l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).\end{aligned}$$

To prove (ii), inequality $\|u \otimes v\|_{1,\Omega} \leqslant C\|u\|_{1,\Omega}\|v\|_{1+\varepsilon,\Omega}$ yields

$$\begin{aligned}\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{1,\Omega}^2 \mathrm{d}t &\leqslant C\int_0^T \Big(\|\widetilde{w}\|_{2,\Omega}^2 \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2 + \|w^2\|_{2,\Omega}^2 \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\Big)\mathrm{d}t \\ &\leqslant C_B \int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2 \mathrm{d}t.\end{aligned}$$

The last one is a consequence of the trace theorem and estimate (ii). □

3.6.1. *Higher-order norms of w and boundary trace estimates.* This section is devoted to control higher-order norms of w displacement and also boundary trace terms. These bounds play an essential role in establish the quasi-stability property. The main ingredients here are

(i) analyticity of the corresponding semigroup associated with the linear thermoelastic plate [53],
(ii) trace regularity valid for the linear model of dynamic elasticity [34] (other related analyticity and trace results for elastic coupled systems can be found in [72, 73]).

Lemma 3.7. *Let $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}_t, \widetilde{w}, \widetilde{w}_t, \widetilde{\phi}, \widetilde{\theta})$ be a regular solution of the system* (3.27)–(3.35)*. Then, for any $\varepsilon \in (0, \frac{1}{2})$,*

$$\begin{aligned}&\int_0^T \big(\|\widetilde{w}\|_{3-\varepsilon,\Omega}^2 + \|\widetilde{w}_t\|_{1-\varepsilon,\Omega}^2 + \|\widetilde{\theta}\|_{1-\varepsilon,\Omega}^2\big)\mathrm{d}t \\ &\qquad\leqslant C\widetilde{E}(0) + C_B \int_0^T \|\widetilde{\phi}\|_{1,\Omega}^2 \mathrm{d}t + C_{B,T}\text{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).\end{aligned}$$

Proof. The proof is divided into three parts.

Step 1. Abstract framework. As a starting point for the proof of this result, we will consider the system for $(\widetilde{w},\widetilde{\theta})$ as an abstract evolution equation and, following the proof of Proposition 2.1, we find that the solution $\tilde{y}_{w,\theta}=(\widetilde{w},\widetilde{w}_t,\widetilde{\theta})$ is given by

$$\widetilde{y}_{w,\theta}(t)=e^{At}\widetilde{y}_{w,\theta}(0)+\int_0^t e^{A(t-s)}\big(0,F(s)+\mathcal{A}G_2(N_2(\widetilde{\mathbf{u}},\widetilde{w},\widetilde{\phi})\cdot\nu),0\big)^\top \mathrm{d}s,$$

with $F(s)=\operatorname{div}\{N_2(\widetilde{\mathbf{u}},\widetilde{w},\widetilde{\phi})\}-P(\widetilde{w})-K\cdot\sigma[A(\mathbf{u}^1,w^1)-A(\mathbf{u}^2,w^2)]$.

Then, for $\varepsilon<\frac{1}{2}$, we obtain

(3.38)
$$\begin{aligned}&A^{\frac{1-\varepsilon}{2}}\widetilde{y}_{w,\theta}(t)\\&\quad=A^{\frac{1-\varepsilon}{2}}e^{At}\widetilde{y}_{w,\theta}(0)+\int_0^t Ae^{A(t-s)}A^{-\frac{1+\varepsilon}{2}}\big(0,F(s)+\mathcal{A}G_2(N_2(\widetilde{\mathbf{u}},\widetilde{w},\widetilde{\phi})\cdot\nu),0\big)^\top \mathrm{d}s.\end{aligned}$$

Step 2. Here, we shall exploit the fact that A generates an analytic semigroup on $H_{w,\theta}$ [53]. In fact, more has been shown recently—A enjoys maximal regularity on any $L^p(L^q)$ with $p,q\in(1,\infty)$ [29]. This, in particular, means that

$$\|\int_0^t Ae^{(t-s)A}f(s)\,\mathrm{d}s\|_{L^p(H_q)}\leqslant C_T\|f\|_{L^p(H_q)}, \tag{3.39}$$

where

$$H_q(\Omega)\equiv W^{2,q}(\Omega)\times L^q(\Omega)\times L^q(\Omega).$$

However, in our case we find it convenient to work with fractional powers of A which have sharp characterization for dissipative and invertible generators, that is,

$$\begin{aligned}&\|\int_0^t A^\alpha e^{(t-s)A}f(s)\mathrm{d}s\|_{H_{w,\theta}}\leqslant C\|f\|_{L^2(H_{w,\theta})}, \qquad \alpha\leqslant\frac{1}{2},\\&\|A^\alpha e^{t\mathcal{A}}x\|_{L^2(H_{w,\theta})}\leqslant C\|x\|_{H_{w,\theta}}, \qquad \alpha\leqslant\frac{1}{2}.\end{aligned} \tag{3.40}$$

Inserting inequalities (3.40) into (3.38), for $\varepsilon\in(0,\frac{1}{2})$, shows that

(3.41)
$$\|A^{\frac{1-\varepsilon}{2}}\widetilde{y}_{w,\theta}(t)\|_{L^2(H_{w,\theta})}\leqslant C\|\widetilde{y}_{w,\theta}(0)\|_{H_{w,\theta}}+C\|A^{-\frac{1+\varepsilon}{2}}\big(0,F(s)+\mathcal{A}G(N_2\cdot\nu),0\big)\|_{L^2(H_{w,\theta})}.$$

We readily find from [7, Proposition 6.1] that

$$D(A^\alpha)\subset H^{2(1+\alpha)}(\Omega)\times H^{2\alpha}(\Omega)\times H^{2\alpha}(\Omega)\qquad \text{for } \alpha\in(0,1). \tag{3.42}$$

Therefore, for $\alpha=\frac{1+\varepsilon}{2}$,

$$C\|A^{-\frac{1+\varepsilon}{2}}\big(0,F(s)+\mathcal{A}G(N_2\cdot\nu),0\big)\|_{H_{w,\theta}}\leqslant C\|F(s)+\mathcal{A}G(N_2\cdot\nu)\|_{-(1+\varepsilon),\Omega}. \tag{3.43}$$

As proved earlier, for every $\psi\in H^{1+\varepsilon}(\Omega)$

(3.44)
$$(F(s)+\mathcal{A}G(N_2\cdot\nu),\psi)_\Omega=-(N_2,\nabla\psi)_\Omega-(P(\widetilde{w})+K\cdot\sigma[A(\mathbf{u}^1,w^1)-A(\mathbf{u}^2,w^2)],\psi)_\Omega.$$

We begin with the most critical term, N_2. Recalling its definition, we obtain

(3.45)
$$\begin{aligned}\big(N_2(\widetilde{\mathbf{u}},\widetilde{w},\widetilde{\phi}),\nabla\psi\big)_\Omega&=\big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2,\nabla\psi\big)_\Omega+\big(\sigma[f(\nabla w^1)-f(\nabla w^2)]\nabla w^2,\nabla\psi\big)_\Omega\\&\quad+\big(\sigma[\epsilon(\mathbf{u}^1)+f(\nabla w^1)]\nabla\widetilde{w},\nabla\psi\big)_\Omega+\big(\widetilde{\phi}\nabla w^2+\phi^1\nabla\widetilde{w},\nabla\psi\big)_\Omega.\end{aligned}$$

The next step is to estimate the inner products in (3.45). Once again, a combination of Hölder, Sobolev, and interpolation inequalities give, for $\varepsilon_1 < \frac{1}{2}$,

$$\begin{aligned}\big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2, \nabla\psi\big)_\Omega &\leqslant C\|\epsilon(\widetilde{\mathbf{u}})\|_{-\varepsilon_1,\Omega}\|\nabla w^2\cdot\nabla\psi\|_{\varepsilon_1,\Omega}\\ &\leqslant C\|\epsilon(\widetilde{\mathbf{u}})\|_{-\varepsilon_1,\Omega}\|\nabla w^2\|_{1,\Omega}\|\nabla\psi\|_{\varepsilon_1+\varepsilon_2,\Omega}.\end{aligned}$$

Take here $\varepsilon_1+\varepsilon_2=\varepsilon$, thereby obtaining

$$\big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2, \nabla\psi\big)_\Omega \leqslant C\|\widetilde{\mathbf{u}}\|_{1-\varepsilon_1,\Omega}\|w^2\|_{2,\Omega}\|\psi\|_{1+\varepsilon,\Omega}.$$

We also find that

$$\big(\sigma[f(\nabla w^1)-f(\nabla w^2)]\nabla w^2, \nabla\psi\big)_\Omega \leqslant C_B\|\widetilde{w}\|_{2-\varepsilon,\Omega}\|w^2\|_{2+\varepsilon,\Omega}\|\psi\|_{1+\varepsilon,\Omega},$$

$$\begin{aligned}\big(\sigma[\epsilon(\mathbf{u}^1)+f(\nabla w^1)]\nabla\widetilde{w}, \nabla\psi\big)_\Omega &\leqslant C_B\|\widetilde{w}\|_\Omega^{\frac{1-2\varepsilon}{3-\varepsilon}}\|\widetilde{w}\|_{3-\varepsilon,\Omega}^{\frac{2+\varepsilon}{3-\varepsilon}}\|\psi\|_{1+\varepsilon,\Omega}\\ &\leqslant \big(\delta\|\widetilde{w}\|_{3-\varepsilon,\Omega}+C_{B,\delta}\|\widetilde{w}\|_\Omega\big)\|\psi\|_{1+\varepsilon,\Omega},\end{aligned}$$

$$\begin{aligned}\big(\widetilde{\phi}\nabla w^2+\phi^1\nabla\widetilde{w}, \nabla\psi\big)_\Omega &\leqslant C\big(\|\widetilde{\phi}\nabla w^2\|_{-\varepsilon,\Omega}+\|\phi^1\nabla\widetilde{w}\|_{-\varepsilon,\Omega}\big)\|\nabla\psi\|_{\varepsilon,\Omega}\\ &\leqslant C\big(\|\widetilde{\phi}\|_\Omega\|w^2\|_{2-\varepsilon,\Omega}+\|\phi^1\|_\Omega\|\widetilde{w}\|_{2-\varepsilon,\Omega}\big)\|\psi\|_{1+\varepsilon,\Omega}.\end{aligned}$$

Similarly, by using forcing assumptions, one can show that

$$(P(\widetilde{w})+K\cdot\sigma[A(\mathbf{u}^1,w^1)-A(\mathbf{u}^2,w^2)],\psi)_\Omega \leqslant C_B\big(\|\widetilde{\mathbf{u}}\|_{1-\varepsilon}+\|\widetilde{w}\|_{2-\varepsilon}\big)\|\psi\|_{1+\varepsilon,\Omega}.$$

Combining these estimates with (3.44), we obtain

$$\|\mathcal{F}(\widetilde{\mathbf{u}},\widetilde{w},\widetilde{\phi})\|^2_{-(1+\varepsilon),\Omega} \leqslant C_B\|\widetilde{\phi}\|^2_\Omega+\delta\|\widetilde{w}\|^2_{3-\varepsilon,\Omega}+C_{B,\delta}\,\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}). \tag{3.46}$$

Step 3. Conclusion. From the estimate (3.46) together with (3.41)–(3.43), and the characterization of $D(A^{\frac{1-\varepsilon}{2}})$, it follows that

$$\begin{aligned}&\int_0^T\big(\|\widetilde{w}\|^2_{3-\varepsilon,\Omega}+\|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega}+\|\widetilde{\theta}\|^2_{1-\varepsilon,\Omega}\big)\mathrm{d}t\\ &\quad\leqslant C\widetilde{E}(0)+C_B\int_0^T\|\widetilde{\phi}\|^2_\Omega\mathrm{d}t+\delta\int_0^T\|\widetilde{w}\|^2_{3-\varepsilon,\Omega}\mathrm{d}t\\ &\qquad+C_{\delta,B,T}\,\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).\end{aligned}$$

Taking $\delta>0$ small enough, we obtain the main result of Lemma 3.7. □

Remark 3.1. The following estimate is an immediate consequence of Lemma 3.7 and the trace theorem,

$$\int_0^T\big(\|\Delta\widetilde{w}\|^2_{\Gamma_0}+\|\widetilde{w}_t\|^2_{\frac{1}{2}-\epsilon,\Gamma_1}\big)\mathrm{d}t \leqslant C\widetilde{E}(0)+C_B\int_0^T\|\widetilde{\phi}\|^2_{1,\Omega}\mathrm{d}t+C_{B,T}\,\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}), \tag{3.47}$$

and it holds for $\epsilon\in(0,\frac{1}{2})$.

Lemma 3.8. *Let $(\widetilde{\mathbf{u}},\widetilde{\mathbf{u}}_t,\widetilde{w},\widetilde{w}_t,\widetilde{\phi},\widetilde{\theta})$ be a regular solution of the system (3.27)–(3.35). Then, for any $\varepsilon\in(0,\frac{1}{4})$ and $\alpha\in(0,\frac{T}{2})$, the following trace regularity is valid:*

$$\begin{aligned}&\int_\alpha^{T-\alpha}\|\nabla\widetilde{\mathbf{u}}\|^2_{\Gamma_1}\mathrm{d}t\\ &\quad\leqslant C_\alpha\int_0^T\big(\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1}+\|\widetilde{\phi}\|^2_{1,\Omega}\big)\mathrm{d}t+C_{\alpha,B}\int_0^T\|\widetilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t+C_{\alpha,B,T}\,\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).\end{aligned}$$

Proof. The proof is divided into three steps.

Step 1. Preliminary estimate. The following inequality holds true:

$$\int_\alpha^{T-\alpha} \|\nabla\widetilde{\mathbf{u}}\|_{\Gamma_1}^2 \,\mathrm{d}t \leqslant C\left(\int_\alpha^{T-\alpha} \|\nabla\widetilde{\mathbf{u}}\,\tau\|_{\Gamma_1}^2 \,\mathrm{d}t + \int_\alpha^{T-\alpha} \|\sigma[\widetilde{\mathbf{u}}]\,\nu\|_{\Gamma_1}^2 \,\mathrm{d}t\right).$$

The conclusion is complete by showing the validation of

$$|\nabla\widetilde{\mathbf{u}}| \leqslant C\big(|\nabla\widetilde{\mathbf{u}}\,\tau| + |\sigma[\widetilde{\mathbf{u}}]\,\nu|\big),$$

where $\nu=(\nu_1,\nu_2)$ and $\tau=(\tau_1,\tau_2)=(-\nu_2,\nu_1)$ denote, respectively, the outward unit normal and the unit tangential vectors, at a point of Γ.

To prove this, let us consider $\nabla\widetilde{\mathbf{u}}$ written as $(\widetilde{\mathbf{u}}_{1,x},\widetilde{\mathbf{u}}_{1,y},\widetilde{\mathbf{u}}_{2,x},\widetilde{\mathbf{u}}_{2,y})$. Then we obtain the following algebraic system:

$$A(\nabla\widetilde{\mathbf{u}})^\top = (\nabla\widetilde{\mathbf{u}}\,\tau, \sigma[\widetilde{\mathbf{u}}]\,\nu)^\top,$$

where

$$A = \begin{bmatrix} \tau_1 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_1 & \tau_2 \\ (\lambda+2\eta)\nu_1 & \eta\nu_2 & \eta\nu_2 & \lambda\nu_1 \\ \lambda\nu_2 & \eta\nu_1 & \eta\nu_1 & (\lambda+2\eta)\nu_2 \end{bmatrix}.$$

Note that $\det(A)= (\lambda+2\eta)\eta$ is constant over Γ. Then we obtain

$$(\nabla\widetilde{\mathbf{u}})^\top = A^{-1}(\nabla\widetilde{\mathbf{u}}\,\tau, \sigma[\widetilde{\mathbf{u}}]\,\nu)^\top,$$

and this implies the required inequality.

Step 2. Tangential derivative of $\mathbf{u}$ estimate. We have the following estimate:

$$\begin{aligned}\int_\alpha^{T-\alpha} \|\nabla\widetilde{\mathbf{u}}\,\tau\|_{\Gamma_1}^2 \mathrm{d}t \leqslant C_\alpha \int_0^T \Big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\phi}\|_{1,\Omega}^2 + \|\sigma[\epsilon(\widetilde{\mathbf{u}})]\nu\|_{\Gamma_1}^2\Big)\mathrm{d}t \\ + C_{\alpha,B}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2 \mathrm{d}t + C_{\alpha,B,T}\ \text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).\end{aligned}\tag{3.48}$$

To establish this, let us consider

$$\widetilde{F} = \operatorname{div}\{\sigma[N_1(\widetilde{w})]\} - \nabla\widetilde{\phi} = \operatorname{div}\{\sigma[f(\nabla w^1) - f(\nabla w^2) + J(\widetilde{w})]\} - \nabla\widetilde{\phi}.\tag{3.49}$$

Then the problem for the $\widetilde{\mathbf{u}}$ displacement is given by

$$\widetilde{\mathbf{u}}_{tt} - \operatorname{div}\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\} = \widetilde{F}\ +\ \text{(boundary conditions)}.$$

After using the established trace regularity for $\mathbf{u}$ stated in [34], we obtain

$$\int_\alpha^{T-\alpha} \|\nabla\widetilde{\mathbf{u}}\,\tau\|_{\Gamma_1}^2 \mathrm{d}t \leqslant C_\alpha \int_0^T \Big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{F}\|_{-\frac12,\Omega}^2 + \|\sigma[\epsilon(\widetilde{\mathbf{u}})]\nu\|_{\Gamma_1}^2 + \|\widetilde{\mathbf{u}}\|_{1-\varepsilon,\Omega}^2\Big)\mathrm{d}t,$$

where we have used the continuous embedding $H^{1-\varepsilon}(\Omega)\subset H^{\frac12+\varepsilon}(\Omega)$.

To conclude this step, we show an estimate for $\widetilde{F}$. For a fixed $\varepsilon\in(0,\frac12)$ we have

$$\|\widetilde{F}(t)\|_{-\frac12,\Omega}^2 \leqslant C_B\|\widetilde{w}\|_{2,\Omega}^2 + C\|\widetilde{\phi}\|_{1,\Omega}^2 + C_B\,\text{l.o.t.}\,(\widetilde{\mathbf{u}},\widetilde{w}) \qquad \forall t\geqslant 0.\tag{3.50}$$

In fact, let $\psi\in H^{\frac12}(\Omega)$. Then Hölder and Sobolev inequalities imply that

$$\begin{aligned}&\big(\operatorname{div}\{\sigma[N_1(\widetilde{w})]\},\psi\big)_\Omega \\ &\quad\leqslant C\big(\|\widetilde{w}\|_{2,\Omega}\|\nabla\widetilde{w}\cdot\psi\|_\Omega + \|\widetilde{w}\|_{2,\Omega}\|\nabla w^2\cdot\psi\|_\Omega + \|w^2\|_{2,\Omega}\|\nabla\widetilde{w}\cdot\psi\|_\Omega + \|\widetilde{w}\|_\Omega\|\psi\|_\Omega\big).\end{aligned}$$

In light of Sobolev embeddings $H^{2-\varepsilon}(\Omega)\subset W^{1,4}(\Omega)$ and $H^{\frac{1}{2}}(\Omega)\subset L^4(\Omega)$, we have

$$\|\widetilde{w}\|_{2,\Omega}\|\nabla\widetilde{w}\cdot\psi\|_{\Omega} \leqslant C_B\|\widetilde{w}\|_{W^{1,4}(\Omega)}\|\psi\|_{L^4(\Omega)} \leqslant C_B\|\widetilde{w}\|_{2-\varepsilon,\Omega}\|\psi\|_{\frac{1}{2},\Omega},$$

$$\|\widetilde{w}\|_{2,\Omega}\|\nabla w^2\cdot\psi\|_{\Omega} \leqslant C\|\widetilde{w}\|_{2,\Omega}\|w^2\|_{W^{1,4}(\Omega)}\|\psi\|_{L^4(\Omega)} \leqslant C_B\|\widetilde{w}\|_{2,\Omega}\|\psi\|_{\frac{1}{2},\Omega},$$

and

$$\|w^2\|_{2,\Omega}\|\nabla\widetilde{w}\cdot\psi\|_{\Omega} \leqslant C_B\|\widetilde{w}\|_{W^{1,4}(\Omega)}\|\psi\|_{L^4(\Omega)} \leqslant C_B\|\widetilde{w}\|_{2-\varepsilon,\Omega}\|\psi\|_{\frac{1}{2},\Omega}.$$

These inequalities combined prove estimate (3.50). Thus, (3.48) is fully proved.

Step 3. Boundary estimate for the stress tensor. For $\varepsilon \in (0,\frac{1}{2})$, we have

$$\int_\alpha^{T-\alpha}\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_{\Gamma_1}\mathrm{d}t$$
$$\leqslant C\int_0^T\big(\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1}+\|\widetilde{\phi}\|^2_{1,\Omega}\big)\mathrm{d}t + C_B\int_0^T\|\widetilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t + C_{\alpha,B,T}\ \text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Indeed, the boundary condition (3.32) readily yields

$$\int_\alpha^{T-\alpha}\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_{\Gamma_1}\mathrm{d}t$$
$$\leqslant C\int_\alpha^{T-\alpha}\big(\|f(\nabla w^1)-f(\nabla w^2)\|^2_{\Gamma_1}+\|\widetilde{w}\|^2_{\Gamma_1}+\|\widetilde{\mathbf{u}}\|^2_{\Gamma_1}+\|\widetilde{\phi}\|^2_{\Gamma_1}+\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1}\big)\mathrm{d}t.$$

This inequality along with trace estimates and Lemma 3.6 promptly imply the desired estimate.

Step 4. Conclusion. Estimates achieved in Steps 2 and 3 applied on the right-hand side of inequality from Step 1 prove the assertion of Lemma 3.8. □

3.6.2. *Observability inequalities.* Here, we obtain a first observability inequality that reconstructs the integral of the energy $\widetilde{E}(\cdot)$ in terms of the dissipation, the l.o.t., and also the boundary trace which, by virtue of Lemmas 3.8 and 3.7, are bounded.

Lemma 3.9. *Let* $(\widetilde{\mathbf{u}},\widetilde{\mathbf{u}}_t,\widetilde{w},\widetilde{w}_t,\widetilde{\phi},\widetilde{\theta})$ *be a solution of the system* (3.27)–(3.35) *with assumption* (1.12) *in force. Then there exists a* $T>0$ *large enough such that for any* $\varepsilon\in(0,\frac{1}{4})$ *the following estimate holds:*

$$\int_0^T\widetilde{E}(t)\mathrm{d}t$$
$$\leqslant C\big(\widetilde{E}(0)+\widetilde{E}(T)\big)+C_B\int_0^T\big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}+\|\widetilde{\phi}\|^2_{1,\Omega}+\|\widetilde{\theta}\|^2_{1,\Omega}\big)\mathrm{d}t+C\int_0^T\|\nabla\widetilde{\mathbf{u}}\|^2_{\Gamma_1}\mathrm{d}t$$
$$+C_B\int_0^T\big(\|\Delta\widetilde{w}\|^2_{\Gamma_0}+\|\widetilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1}+\|\widetilde{w}\|^2_{2+\varepsilon,\Omega}\big)\mathrm{d}t+C_{B,T}\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Proof. The proof of this lemma is divided into several steps.

Step 1. Reconstruction of the kinetic energy of the elastic part. Consider the multiplier $h\nabla\widetilde{\mathbf{u}}$, where $h(x)=x-x_0$. We take here the $[L^2(\Omega)]^2$ inner product with equation (3.27) and, after integration over the $[0,T]$, we obtain the identity

$$\int_0^T\big(\widetilde{\mathbf{u}}_{tt}-\operatorname{div}\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\}+\nabla\widetilde{\phi}-\operatorname{div}\{N_1(\widetilde{w})\},h\nabla\widetilde{\mathbf{u}}\big)_\Omega\mathrm{d}t=0. \tag{3.51}$$

We shall estimate/rewrite each product in (3.51). The first product can be handled by use of divergence formula and integration by parts in time

(3.52)
$$\int_0^T \big(\widetilde{\mathbf{u}}_{tt}, h\nabla\widetilde{\mathbf{u}}\big)_\Omega \mathrm{d}t = \big(\widetilde{\mathbf{u}}_t, h\nabla\widetilde{\mathbf{u}}\big)_\Omega\Big|_0^T + \int_0^T \|\widetilde{\mathbf{u}}_t\|_\Omega^2 \mathrm{d}t - \frac{1}{2}\int_0^T\int_{\Gamma_1} |\widetilde{\mathbf{u}}_t|^2 h\cdot\nu \, d\Gamma_1 \mathrm{d}t.$$

Application of divergence and Gauss theorems allows us to compute the following identity for the second product of (3.51):

(3.53)
$$\int_0^T \big(\mathrm{div}\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\}, h\nabla\widetilde{\mathbf{u}}\big)_\Omega \mathrm{d}t = \int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})]\nu, h\nabla\widetilde{\mathbf{u}}\big\rangle_\Gamma - \big(\sigma[\epsilon(\widetilde{\mathbf{u}})], \nabla(h\nabla\widetilde{\mathbf{u}})\big)_\Omega \mathrm{d}t.$$

Via boundary condition (3.32) and making use of $\big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})]\nu, h\nabla\widetilde{\mathbf{u}}\big\rangle_{\Gamma_0} = \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_0}$, we obtain

$$\int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})]\nu, h\nabla\widetilde{\mathbf{u}}\big\rangle_\Gamma \mathrm{d}t = \int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_0} - \langle N_1(\widetilde{w})\nu + \kappa\widetilde{\mathbf{u}} - \widetilde{\phi}\nu + \widetilde{\mathbf{u}}_t, h\nabla\widetilde{\mathbf{u}}\rangle_{\Gamma_1} \mathrm{d}t.$$

Note that

$$\begin{aligned}\int_0^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})], \nabla(h\nabla\widetilde{\mathbf{u}})\big)_\Omega \mathrm{d}t &= \int_0^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(h\nabla\widetilde{\mathbf{u}})\big)_\Omega \mathrm{d}t \\ &\overset{(4.90)}{=} \int_0^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})\big)_\Omega \mathrm{d}t + \sum_{i,j,k=1}^2 \int_0^T \Big(a_{i,j}, \frac{\partial^2 \widetilde{\mathbf{u}}_i}{\partial x_k \partial x_j} h_k\Big)_\Omega \mathrm{d}t \\ &\overset{(4.89)}{=} \frac{1}{2}\int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_0} + \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_1} \mathrm{d}t.\end{aligned}$$

Taking into account these identities, we can rewrite (3.53) as

(3.54)
$$\begin{aligned}\int_0^T \big(\mathrm{div}\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\}, h\nabla\widetilde{\mathbf{u}}\big)_\Omega \mathrm{d}t = &\frac{1}{2}\int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_0} - \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_1} \mathrm{d}t \\ &- \int_0^T \langle N_1(\widetilde{w})\nu + \kappa\widetilde{\mathbf{u}} - \widetilde{\phi}\nu + \widetilde{\mathbf{u}}_t, h\nabla\widetilde{\mathbf{u}}\rangle_{\Gamma_1} \mathrm{d}t.\end{aligned}$$

The combination of (3.52) and (3.54) with (3.51) yields

$$\begin{aligned}\int_0^T \|\widetilde{\mathbf{u}}_t\|_\Omega^2 \mathrm{d}t = &- \big(\widetilde{\mathbf{u}}_t, h\nabla\widetilde{\mathbf{u}}\big)_\Omega\Big|_0^T + \frac{1}{2}\int_0^T\int_{\Gamma_1} |\widetilde{\mathbf{u}}_t|^2 h\cdot\nu \, \mathrm{d}\Gamma_1 \mathrm{d}t \\ &+ \frac{1}{2}\int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_0} \mathrm{d}t - \frac{1}{2}\int_0^T \big\langle \sigma[\epsilon(\widetilde{\mathbf{u}})], \epsilon(\widetilde{\mathbf{u}})h\cdot\nu\big\rangle_{\Gamma_1} \mathrm{d}t \\ &- \int_0^T \big\langle N_1(\widetilde{w})\nu + \kappa\widetilde{\mathbf{u}} - \widetilde{\phi}\nu + \widetilde{\mathbf{u}}_t, h\nabla\widetilde{\mathbf{u}}\big\rangle_{\Gamma_1} - \big(\nabla\widetilde{\phi} - \mathrm{div}\{N_1(\widetilde{w})\}, h\nabla\widetilde{\mathbf{u}}\big)_\Omega \mathrm{d}t.\end{aligned}$$

Finally, using a combination of geometric condition (1.12), the trace theorem, and Lemma 3.6, we find that

$$
\begin{aligned}
\int_0^T \|\widetilde{\mathbf{u}}_t\|_\Omega^2\,\mathrm{d}t \leqslant C\big(\widetilde{E}(0)+\widetilde{E}(T)\big)+\delta\int_0^T \big(\kappa\|\widetilde{\mathbf{u}}\|_{\Gamma_1}^2+\|\nabla\widetilde{\mathbf{u}}\|_\Omega^2\big)\mathrm{d}t \\
+ C_\delta\int_0^T \big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2+\|\nabla\widetilde{\mathbf{u}}\|_{\Gamma_1}^2+\|\widetilde{\phi}\|_{1,\Omega}^2\big)\mathrm{d}t \\
+ C_{B,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).
\end{aligned}
\tag{3.55}
$$

Here, we used the fact that $\int_0^T \big\langle\sigma[\epsilon(\widetilde{\mathbf{u}})],\epsilon(\widetilde{\mathbf{u}})\big\rangle_{\Gamma_1} h\cdot\nu\,\mathrm{d}t \leqslant C\int_0^T \|\nabla\widetilde{\mathbf{u}}\|_{\Gamma_1}^2\mathrm{d}t$.

Step 2. Reconstruction of the difference of potential and kinetic energies. Now we consider $\mathbf{u}$ as a multiplier. Then we return to equation (3.27) to obtain

$$
\int_0^T \Big(\widetilde{\mathbf{u}}_{tt}-\mathrm{div}\{\sigma[\epsilon(\widetilde{\mathbf{u}})]\}+\nabla\widetilde{\phi}-\mathrm{div}\{N_1(\widetilde{w})\},\widetilde{\mathbf{u}}\Big)_\Omega\mathrm{d}t=0.
\tag{3.56}
$$

After application of the Gauss theorem and recalling boundary conditions (3.31) and (3.32), we have (3.56) turning into

$$
\begin{aligned}
-\int_0^T \|\widetilde{\mathbf{u}}_t\|_\Omega^2\mathrm{d}t+\int_0^T\int_\Omega \sigma[\epsilon(\widetilde{\mathbf{u}})]\epsilon(\widetilde{\mathbf{u}})\,\mathrm{d}\Omega\,\mathrm{d}t+\kappa\int_0^T \|\widetilde{\mathbf{u}}\|_{\Gamma_1}^2\mathrm{d}t \\
= -\big(\widetilde{\mathbf{u}}_t,\widetilde{\mathbf{u}}\big)_\Omega\Big|_0^T-\int_0^T \big\langle N_1(\widetilde{w})\nu+\kappa\widetilde{\mathbf{u}}-\widetilde{\phi}\nu+\widetilde{\mathbf{u}}_t,\widetilde{\mathbf{u}}\big\rangle_{\Gamma_1}\mathrm{d}t \\
-\int_0^T \big(\nabla\widetilde{\phi}-\mathrm{div}\{N_1(\widetilde{w})\},\widetilde{\mathbf{u}}\big)_\Omega\mathrm{d}t.
\end{aligned}
$$

Lemma 3.6 implies that

$$
\begin{aligned}
-\int_0^T \|\widetilde{\mathbf{u}}_t\|_\Omega^2\mathrm{d}t+\int_0^T\int_\Omega \sigma[\epsilon(\widetilde{\mathbf{u}})]\epsilon(\widetilde{\mathbf{u}})\,\mathrm{d}\Omega\,\mathrm{d}t+\kappa\int_0^T \|\widetilde{\mathbf{u}}\|_{\Gamma_1}^2\mathrm{d}t \\
\leqslant C(\widetilde{E}(0)+\widetilde{E}(T))+C\int_0^T \Big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2+\|\widetilde{\phi}\|_{1,\Omega}^2\Big)\mathrm{d}t \\
+ C_B\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t \;+\; C_{B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).
\end{aligned}
\tag{3.57}
$$

Next we consider the multiplier $\widetilde{w}$ for equation (3.28). Then we obtain

(3.58)

$$
-\int_0^T (\|\widetilde{w}_t\|_\Omega^2+a(\widetilde{w},\widetilde{w}))\mathrm{d}t=-\big(\widetilde{w}_t,\widetilde{w}\big)_\Omega\Big|_0^T-\int_0^T\langle\widetilde{\theta},\partial_\nu\widetilde{w}\rangle_{\Gamma_1}-\big(\nabla\widetilde{\theta},\nabla\widetilde{w}\big)_\Omega\mathrm{d}t-R_1,
$$

where

$$
R_1=\int_0^T\big(P(\widetilde{w}),\widetilde{w}\big)_\Omega\mathrm{d}t+\int_0^T\big(N_2,\nabla\widetilde{w}\big)_\Omega\mathrm{d}t+\int_0^T\big(K\cdot\sigma[A(\mathbf{u}^1,w^1)-A(\mathbf{u}^2,w^2)],\widetilde{w}\big)_\Omega\mathrm{d}t.
$$

Here, we have used

$$
\int_0^T\big(\Delta^2\widetilde{w},\widetilde{w}\big)_\Omega\mathrm{d}t=\int_0^T a(\widetilde{w},\widetilde{w})\mathrm{d}t+\int_0^T\langle N_2\cdot\nu-\partial_\nu\widetilde{\theta},\widetilde{w}\rangle_{\Gamma_1}\mathrm{d}t+\int_0^T\langle\widetilde{\theta},\partial_\nu\widetilde{w}\rangle_{\Gamma_1}\mathrm{d}t.
$$

We shall estimate all products on the right-hand side of (3.58). First, trace estimates provide

$$\langle\widetilde{\theta},\partial_\nu\widetilde{w}\rangle_{\Gamma_1}-\big(\nabla\widetilde{\theta},\nabla\widetilde{w}\big)_\Omega\leqslant\delta\|w\|^2_{2,\Omega}+C_\delta\|\theta\|^2_{1,\Omega}. \tag{3.59}$$

Let us estimate R_1. Using the definition of stress $N_2(\cdot,\cdot)$, we find that

$$\begin{aligned}\big(N_2(\widetilde{\mathbf{u}},\widetilde{w}),\nabla\widetilde{w}\big)_\Omega&=\big(\sigma\big[\epsilon(\widetilde{\mathbf{u}})+f(\nabla w^1)-f(\nabla w^2)+J(\widetilde{w})\big],\nabla w^2\otimes\nabla\widetilde{w}\big)_\Omega\\&\quad+\big(\sigma\big[\epsilon(\mathbf{u}^1)+f(\nabla w^1)+J(w^1)\big],\nabla\widetilde{w}\otimes\nabla\widetilde{w}\big)_\Omega\\&\quad+\big(\phi^1\nabla w^1-\phi^2\nabla w^2,\nabla\widetilde{w}\big)_\Omega.\end{aligned}$$

Via inequality $\|u\otimes v\|_\Omega\leqslant C\|u\|_{\varepsilon,\Omega}\|v\|_{1-\varepsilon,\Omega}$, which holds for $\varepsilon\in(0,1)$, we obtain the following estimate:

$$\int_0^T\big(N_2(\widetilde{\mathbf{u}},\widetilde{w}),\nabla\widetilde{w}\big)_\Omega\mathrm{d}t\leqslant\delta\int_0^T\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega\mathrm{d}t+C\int_0^T\|\widetilde{\phi}\|^2_\Omega\mathrm{d}t+C_{B,T,\delta}\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

The last product in R satisfies the estimate

$$\int_0^T\big(K\cdot\sigma[A(\mathbf{u}^1,w^1)-A(\mathbf{u}^2,w^2)],\widetilde{w}\big)_\Omega\mathrm{d}t\leqslant\delta\int_0^T\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega\mathrm{d}t+C_{K,B,T,\delta}\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Forcing term assumptions allows us to show

$$\int_0^T\big(P(\widetilde{w}),\widetilde{w}\big)_\Omega\mathrm{d}t\leqslant C_{B,T}\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Collecting this estimates, we find that

$$|R_1|\leqslant\delta\int_0^T\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega\,\mathrm{d}t+C\int_0^T\|\widetilde{\phi}\|^2_\Omega\,\mathrm{d}t+C_{K,B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}). \tag{3.60}$$

Inserting this and (3.59) into (3.58), we obtain

$$\begin{aligned}-\int_0^T\|\widetilde{w}_t\|^2_\Omega\mathrm{d}t+\int_0^T a(\widetilde{w},\widetilde{w})\mathrm{d}t&\leqslant C(\widetilde{E}(0)+\widetilde{E}(T))+C_B\int_0^T\Big(\|\widetilde{\phi}\|^2_{1,\Omega}+\|\widetilde{\theta}\|^2_{1,\Omega}\Big)\mathrm{d}t\\&\quad+\delta\int_0^T\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega\,\mathrm{d}t+C_{K,B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).\end{aligned}$$

Finally, this estimate combined with (3.57) shows that

(3.61)

$$\begin{aligned}\int_0^T\big(\widetilde{E}_p(t)-\widetilde{E}_k(t)\big)\mathrm{d}t&\leqslant C(\widetilde{E}(0)+\widetilde{E}(T))+C_B\int_0^T\Big(\|\widetilde{\mathbf{u}}\|^2_{1,\Gamma_1}+\|\widetilde{\phi}\|^2_{1,\Omega}+\|\widetilde{\theta}\|^2_{1,\Omega}\Big)\mathrm{d}t\\&\quad+C_B\int_0^T\|\widetilde{w}\|^2_{2+\varepsilon,\Omega}\mathrm{d}t+C_{K,B,T}\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).\end{aligned}$$

Step 3. Reconstruction of kinetic energy of the plate equation. Let $\mathcal{A}_D$ be the Laplace operator acting on $L^2(\Omega)$ with domain $H^2(\Omega)\cap H^1_0(\Omega)$, and let $\mathcal{D}$ be the Dirichlet map

$$\mathcal{D}h=v\Longleftrightarrow\begin{cases}\Delta v=0\text{ in }\Omega,\\ v=h\text{ on }\Gamma.\end{cases}$$

Elliptic regularity [59] shows that

$$\|\mathcal{A}_D^{-1}v\|_{2,\Omega}\leqslant C\|v\|_\Omega\ v\in L^2(\Omega)$$

and

$$\mathcal{D}\in\mathcal{L}(H^s(\Gamma),H^{s+\frac{1}{2}}(\Omega)),\qquad s\in\mathbb{R}.$$

For $v \in H^2(\Omega)$ we have

$$-v + \mathcal{D}(v|_\Gamma) \in D(\mathcal{A}_D) \quad \text{and} \quad \mathcal{A}_D^{-1}\Delta v = -v + \mathcal{D}(v|_\Gamma). \tag{3.62}$$

Now, taking $\mathcal{A}_D^{-1}\theta$ as a multiplier and back to equation (3.28), we obtain, after integration in time,

$$\int_0^T \big(\widetilde{w}_{tt} + \Delta^2\widetilde{w} + \Delta\widetilde{\theta} + P(\widetilde{w}) - \operatorname{div}\{N_2\} + K\cdot\sigma[A(\mathbf{u}^1, w^1) - A(\mathbf{u}^2, w^2)], \mathcal{A}_D^{-1}\widetilde{\theta}\big)_\Omega \mathrm{d}t = 0.$$

Proceeding as before, we obtain

$$\begin{aligned}
&\int_0^T \big(\widetilde{w}_{tt}, \mathcal{A}_D^{-1}\widetilde{\theta}\big)_\Omega \mathrm{d}t + \int_0^T a(\widetilde{w}, \mathcal{A}_D^{-1}\widetilde{\theta})\mathrm{d}t \\
&\quad = -\int_0^T \Big(\langle\widetilde{\theta}, \partial_\nu(\mathcal{A}_D^{-1}\widetilde{\theta})\rangle_{\Gamma_1} - \langle\Delta\widetilde{w}, \partial_\nu(\mathcal{A}_D^{-1}\widetilde{\theta})\rangle_{\Gamma_0} - \big(\nabla\widetilde{\theta}, \nabla(\mathcal{A}_D^{-1}\widetilde{\theta})\big)_\Omega\Big)\mathrm{d}t - R_2,
\end{aligned} \tag{3.63}$$

with

$$\begin{aligned}
R_2 = \int_0^T &\big(P(\widetilde{w}), \mathcal{A}_D^{-1}\widetilde{\theta}\big)_\Omega + \big(N_2, \nabla(\mathcal{A}_D^{-1})\big)_\Omega \\
&+ \big(K\cdot\sigma[A(\mathbf{u}^1, w^1) - A(\mathbf{u}^2, w^2)], \mathcal{A}_D^{-1}\widetilde{\theta}\big)_\Omega \mathrm{d}t.
\end{aligned}$$

Integration by parts in time variable and identity (3.62) implies that

$$\int_0^T \big(\widetilde{w}_{tt}, \mathcal{A}_D^{-1}\widetilde{\theta}\big)_\Omega \mathrm{d}t = \big(\widetilde{w}_t, \mathcal{A}_D^{-1}\widetilde{\theta}\big)_\Omega\Big|_0^T + \int_0^T \|\widetilde{w}_t\|_\Omega^2 - \big(\widetilde{w}_t, \mathcal{D}(\widetilde{w}_t|_\Gamma) - \widetilde{\theta} + \mathcal{D}(\widetilde{\theta}|_\Gamma)\big)_\Omega \mathrm{d}t.$$

A combination of the Hölder, Young, and Sobolev inequalities allows us to conclude that

$$\int_0^T \big(\widetilde{w}_t, \mathcal{D}(\widetilde{w}_t|_\Gamma)\big)_\Omega \mathrm{d}t \leqslant \delta_0 \int_0^T \|\widetilde{w}_t\|_\Omega^2 \mathrm{d}t + C_{\delta_0}\int_0^T \|\widetilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1}\mathrm{d}t, \tag{3.64}$$

$$\int_0^T \big(\widetilde{w}_t, \widetilde{\theta} - \mathcal{D}(\widetilde{\theta}|_\Gamma)\big)_\Omega \mathrm{d}t \leqslant \delta_0 \int_0^T \|\widetilde{w}_t\|_\Omega^2 \mathrm{d}t + C_{\delta_0}\int_0^T \|\widetilde{\theta}\|^2_{1,\Omega}\mathrm{d}t, \tag{3.65}$$

$$\int_0^T a(\widetilde{w}, \mathcal{A}_D^{-1}\widetilde{\theta})\,\mathrm{d}t \leqslant \delta\int_0^T \|\widetilde{w}\|^2_{2,\Omega}\mathrm{d}t + C_\delta\int_0^T \|\widetilde{\theta}\|^2_\Omega \mathrm{d}t, \tag{3.66}$$

$$\int_0^T \big(\nabla\widetilde{\theta}, \nabla(\mathcal{A}_D^{-1}\widetilde{\theta})\big)_\Omega - \big\langle\widetilde{\theta}, \partial_\nu(\mathcal{A}_D^{-1}\widetilde{\theta})\big\rangle_{\Gamma_1}\mathrm{d}t \leqslant C\int_0^T \|\widetilde{\theta}\|^2_{1,\Omega}\mathrm{d}t, \tag{3.67}$$

$$\int_0^T \big\langle\Delta\widetilde{w}, \partial_\nu(\mathcal{A}_D^{-1}\widetilde{\theta})\big\rangle_{\Gamma_0}\mathrm{d}t \leqslant C\int_0^T \|\Delta\widetilde{w}\|^2_{\Gamma_0}\mathrm{d}t + C\int_0^T \|\widetilde{\theta}\|^2_\Omega \mathrm{d}t. \tag{3.68}$$

It remains to estimate the nonlinear terms R_2. Its estimate can be adapted to the same case as in (3.60), the R_1 estimate. Then we find that

$$|R_2| \leqslant \delta\int_0^T \|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega\,\mathrm{d}t + C_{B,\delta}\int_0^T \big(\|\widetilde{\phi}\|^2_{1,\Omega} + \|\widetilde{\theta}\|^2_{1,\Omega}\big)\,\mathrm{d}t + C_{B,T,\delta}\,\text{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}). \tag{3.69}$$

Therefore, inequalities (3.64)–(3.68) and (3.69) applied in (3.63), for $\delta_0 > 0$ small enough, yield

(3.70)
$$\int_0^T \|\widetilde{w}_t\|_\Omega^2 \,\mathrm{d}t \leqslant C(\widetilde{E}(0)+\widetilde{E}(T)) + \delta\int_0^T \|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|_\Omega^2 \,\mathrm{d}t + C_{B,\delta}\int_0^T \Big(\|\widetilde{\theta}\|_{1,\Omega}^2 + \|\widetilde{\phi}\|_{1,\Omega}^2\Big)\,\mathrm{d}t$$
$$+ C_B\int_0^T \Big(\|\Delta\widetilde{w}\|_{\Gamma_0}^2 + \|\widetilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1} + \|\widetilde{w}\|^2_{2+\varepsilon,\Omega}\Big)\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Step 4. Completion of the proof. The conclusion follows from (3.55), (3.61), (3.70) and selecting suitable $\delta > 0$ small. □

Next we establish a second observability inequality. Here, the integral of the linear energy is bounded by dissipation, l.o.t., and nonlinearities (3.37).

Lemma 3.10. *Let $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}_t, \widetilde{w}, \widetilde{w}_t, \widetilde{\phi}, \widetilde{\theta})$ be a solution of the system (3.27)–(3.35). Then for $\alpha \in (0, \frac{T}{2})$ there exist positive constants $C_\alpha, C_{\alpha,B}, C_{\alpha,B,T}$ such that*
$$T\widetilde{E}(T) + \int_0^T \widetilde{E}(t)\mathrm{d}t + \int_0^T (\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\phi}\|_{1,\Omega}^2)\mathrm{d}t + \Big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\Big)^2$$
$$\leqslant C\widetilde{E}(0) + (C+2\alpha)\widetilde{E}(T) + C_{\alpha,B}\, D_0^T\,(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta})$$
$$+ C\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}) + C_\alpha R(0,T) + \int_0^T R(s,T)\mathrm{d}s,$$
where $R(s_1,s_2) = \int_{s_2}^{s_1} \Big|\sum_{i=1}^5 R_i(t)\Big|\,\mathrm{d}t$.

Proof. The estimate in Lemma 2.2 applied to the interval $[\alpha, T-\alpha]$ and Lemma 3.8 allows us to conclude that
$$\int_\alpha^{T-\alpha} \widetilde{E}(t)\mathrm{d}t \leqslant C\big(\widetilde{E}(\alpha)+\widetilde{E}(T-\alpha)\big) + C_{\alpha,B}\int_\alpha^{T-\alpha} \Big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\theta}\|_{1,\Omega}^2 + \|\widetilde{\phi}\|_{1,\Omega}^2\Big)\mathrm{d}t$$
$$+ C_{\alpha,B}\int_\alpha^{T-\alpha} \Big(\|\Delta\widetilde{w}\|_{\Gamma_0}^2 + \|\widetilde{w}_t\|^2_{\frac{1}{2}-\varepsilon,\Gamma_1} + \|\widetilde{w}\|^2_{2+\varepsilon,\Omega}\Big)\mathrm{d}t$$
$$+ C_{\alpha,B,T}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

By use of the interpolation inequality $\|w\|_{2+\varepsilon,\Omega} \leqslant C\|w\|_\Omega^{\frac{1-2\varepsilon}{3-\epsilon}}\|w\|_{3-\varepsilon,\Omega}^{\frac{2+\varepsilon}{3-\varepsilon}}$ combined with Lemma 3.7 and estimate (3.47), we obtain

(3.71)
$$\int_\alpha^{T-\alpha} \widetilde{E}(t)\mathrm{d}t \leqslant C\big(\widetilde{E}(0)+\widetilde{E}(\alpha)+\widetilde{E}(T-\alpha)\big) + C_{\alpha,B} D_0^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta}) + C_{\alpha,B,T}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

The next step of the proof is to extend this integral to the interval $(0,T)$. To this end, via energy equality (3.36), we find that
$$\int_0^\alpha \widetilde{E}(t)\mathrm{d}t \leqslant \alpha\widetilde{E}(\alpha) + \int_0^\alpha R(0,t)\,\mathrm{d}t,$$
$$\int_{T-\alpha}^T \widetilde{E}(t)\mathrm{d}t \leqslant \alpha\widetilde{E}(T-\alpha) + \int_{T-\alpha}^T R(0,t)\,\mathrm{d}t,$$
$$\widetilde{E}(\alpha) \leqslant \widetilde{E}(T) + D_\alpha^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta}) + R(\alpha,T),$$
$$\widetilde{E}(T-\alpha) \leqslant \widetilde{E}(T) + D_{T-\alpha}^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta}) + R(T-\alpha,T),$$

and thus

$$\int_0^\alpha \widetilde{E}(t)\mathrm{d}t + \int_{T-\alpha}^T \widetilde{E}(T)\mathrm{d}t \leqslant 2\alpha\widetilde{E}(T) + 2D_0^T(\widetilde{\mathbf{u}}, \widetilde{\phi}, \widetilde{\theta}) + C_{\alpha,T}R(0,T). \tag{3.72}$$

Identity (3.36) also shows that the following estimate holds:

$$T\widetilde{E}(T) \leqslant \int_0^T \widetilde{E}(s)\mathrm{d}s + \int_0^T R(s,t)\,\mathrm{d}s. \tag{3.73}$$

A combination of estimates (3.72), (3.73) with (3.71) produces

$$\begin{aligned} T\widetilde{E}(T) + \int_0^T \widetilde{E}(t)\mathrm{d}t \leqslant{}& C\widetilde{E}(0) + \big(C + 2\alpha\big)\widetilde{E}(T) + C_{\alpha,B}\ D_0^T\ (\widetilde{\mathbf{u}}, \widetilde{\phi}, \widetilde{\theta}) \\ & + C_{\alpha,T}R(0,T) + \int_0^T R(s,T)\mathrm{d}s + C_{\alpha,B,T}\mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}). \end{aligned} \tag{3.74}$$

Next we add $\int_0^T \big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\phi}\|_{1,\Omega}^2\big)\mathrm{d}t + \big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\big)^2$ to both sides of (3.74) and, using the following estimate, we complete the proof of Lemma 3.10:

$$\Big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\Big)^2 \leqslant C\big(\widetilde{E}(0) + D_0^T(\widetilde{\mathbf{u}}, \widetilde{\phi}, \widetilde{\theta}) + R(0,T)\big).$$

This last step is needed to absorb some terms produced by the following result. $\square$

3.6.3. *Handling of R_i.* Here, we aim to obtain estimates for $R_i(t)$, $1 \leqslant i \leqslant 5$.

Lemma 3.11. *With reference to the nonlinearities R_i, we have*

$$\begin{aligned} \max\Big\{ \int_0^T \int_s^T \Big| \sum_{i=1}^5 R_i(t)\Big|\mathrm{d}t\mathrm{d}s\,, \int_0^T \Big|\sum_{i=1}^5 R_i(t)\Big|\mathrm{d}t\Big\} & \\ \leqslant \delta \int_0^T \Big(\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|_\Omega^2 + \|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\mathbf{u}}_t\|_\Omega^2 + \|\widetilde{w}_t\|_\Omega^2 + \|\widetilde{\phi}\|_{1,\Omega}^2\Big)\,\mathrm{d}t\ +\ \delta\widetilde{E}(T) & \\ + \delta\big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\big)^2 + \delta\int_0^T \big(\|\widetilde{w}_t\|_{1-\varepsilon,\Omega}^2 + \|\widetilde{w}\|_{3-\varepsilon,\Omega}^2\big)\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}). & \end{aligned}$$

Proof. Combining Hölder and Lemma 3.6, we have

$$\begin{aligned} & \int_0^T\int_s^T R_1(t)\mathrm{d}t\mathrm{d}s \\ & \quad \leqslant \delta\int_0^T \big(\|\widetilde{\mathbf{u}}_t\|_\Omega^2 + \|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2\big)\mathrm{d}t + C_{B,T,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}). \end{aligned} \tag{3.75}$$

Recalling the definition of $A(\cdot,\cdot)$, we compute

(3.76)
$$\begin{aligned}
&\int_0^T\int_s^T R_2(t)\mathrm{d}t\mathrm{d}s\\
&\quad (I_1) \quad = \int_0^T\int_s^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2 + \sigma[J(\widetilde{w})]\nabla w^2, \nabla\widetilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s\\
&\quad (I_2) \quad + \int_0^T\int_s^T \big(\sigma\big[f(\nabla w^1)-f(\nabla w^2)\big]\nabla w^2, \nabla\widetilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s\\
&\quad (I_3) \quad + \int_0^T\int_s^T \big(\sigma\big[\epsilon(\mathbf{u}^1)+f(\nabla w^1)+J(w^1)\big]\nabla\widetilde{w}, \nabla w_t\big)_\Omega \mathrm{d}t\mathrm{d}s.
\end{aligned}$$

The goal now is to estimate the products (I_1)–(I_3) on the right-hand side of (3.76). Integrating by parts in time, we obtain

$$\begin{aligned}
\int_0^T\int_s^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2, \nabla\widetilde{w}_t\big)_\Omega \mathrm{d}t\mathrm{d}s &= \int_0^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2, \nabla\widetilde{w}\big)_\Omega\Big|_s^T \mathrm{d}s\\
&- \int_0^T\int_s^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}}_t)]\nabla w^2, \nabla\widetilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s \qquad (3.77)\\
&- \int_0^T\int_s^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w_t^2, \nabla\widetilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s.
\end{aligned}$$

Using $\|uv\|_\Omega \leqslant C\|u\|_{\varepsilon,\Omega}\|v\|_{1-\varepsilon,\Omega}$ for $\varepsilon<1$, we estimate

$$\int_0^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w^2, \nabla\widetilde{w}\big)_\Omega\Big|_s^T \mathrm{d}s \leqslant \delta\widetilde{E}(T) + \delta\int_0^T \|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|_\Omega^2 \mathrm{d}t + C_{B,T,\delta}\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

After integration by parts in space variable, we make use of trace theorem obtaining

$$\begin{aligned}
&\int_0^T\int_s^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}}_t)]\nabla w^2, \nabla\widetilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s\\
&\qquad \leqslant C_T\int_0^T \Big(\|\nabla w^2\cdot\nabla\widetilde{w}\|_{\Gamma_1}\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1} + \|\nabla w^2\cdot\nabla\widetilde{w}\|_{1,\Omega}\|\widetilde{\mathbf{u}}_t\|_\Omega\Big)\mathrm{d}t\\
&\qquad \leqslant C_{B,T,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + \delta\int_0^T \|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2\mathrm{d}t + \delta\int_0^T \|\widetilde{\mathbf{u}}_t\|_\Omega^2\mathrm{d}t.
\end{aligned}$$

Using both the Hölder inequality and the Sobolev embedding $H^{1+\varepsilon}(\Omega)\subset L^\infty(\Omega)$, we find that

$$\begin{aligned}
\int_0^T\int_s^T \big(\sigma[\epsilon(\widetilde{\mathbf{u}})]\nabla w_t^2, \nabla\widetilde{w}\big)_\Omega \mathrm{d}t\mathrm{d}s &\leqslant C_T \sup_{t\in[0,T]}\|\epsilon(\widetilde{\mathbf{u}})\|_\Omega \int_0^T \big(\|\nabla w_t^2\|_\Omega\|\widetilde{w}\|_{2+\varepsilon,\Omega}\big)\mathrm{d}t\\
&\leqslant \delta\left(\sup_{t\in[0,T]}\widetilde{E}^{\frac12}(t)\right)^2 + C_{B,T,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t.
\end{aligned}$$

Choosing $\varepsilon_0 < 1 - 2\varepsilon$, we have $H^{2-\varepsilon}(\Omega) \subset H^{1+\varepsilon+\varepsilon_0}(\Omega)$. Then, making use of this, we get

(3.78)
$$\begin{aligned}\int_0^T \int_s^T \left(\sigma[J(\widetilde{w})]\nabla w^2, \nabla \widetilde{w}_t\right)_\Omega \mathrm{d}t\mathrm{d}s &\leqslant C \int_0^T \|\widetilde{w}\|_{1,\Omega} \|\nabla w^2\|^2_{\varepsilon+\varepsilon_0,\Omega} \|\nabla \widetilde{w}_t\|^2_{-\varepsilon,\Omega} \mathrm{d}t \\ &\leqslant \delta \int_0^T \|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega}\, \mathrm{d}t + C_{B,T,\delta} \mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).\end{aligned}$$

Recalling these estimates in (3.77), we find that

$$\begin{aligned}|I_1| &\leqslant \delta \widetilde{E}(T) + \delta \int_0^T \left(\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega + \|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1} + \|\widetilde{\mathbf{u}}_t\|^2_\Omega \right) \mathrm{d}t \\ &\quad + \delta \int_0^T \|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega} \mathrm{d}t + C_{B,T,\delta} \int_0^T \|\widetilde{w}\|^2_{2+\varepsilon,\Omega} \mathrm{d}t + C_{B,T,\delta} \mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).\end{aligned} \tag{3.79}$$

Let us estimate I_2 (3.76). Taking $\varepsilon_0 > 0$ as before and via Lemma 3.6—estimate (ii)—we obtain

$$|I_2| \leqslant C_{B,T,\delta} \int_0^T \|\widetilde{w}\|^2_{2+\varepsilon,\Omega} \mathrm{d}t + \delta \int_0^T \|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega} \mathrm{d}t. \tag{3.80}$$

It remains to establish an estimate for I_3. Integration by parts in time and space and the use of $\||\nabla \widetilde{w}|^2\|_{1,\Omega} = \|\nabla \widetilde{w} \cdot \nabla \widetilde{w}\|_{1,\Omega} \leqslant C \|\widetilde{w}\|_{1,\Omega} \|\widetilde{w}\|_{2+\varepsilon,\Omega}$ yield

$$\begin{aligned}&\int_0^T \int_s^T \left(\sigma\left[\epsilon(\mathbf{u}^1)\right] \nabla \widetilde{w}, \nabla \widetilde{w}_t\right)_\Omega \mathrm{d}t\mathrm{d}s \\ &\qquad \leqslant C_T \|\epsilon(\mathbf{u}^1(T))\|_\Omega \||\nabla \widetilde{w}(T)|^2\|_\Omega + C_T \int_0^T \|\epsilon(\mathbf{u}^1)\|_\Omega \||\nabla \widetilde{w}|^2\|_\Omega\, \mathrm{d}t \\ &\qquad\quad + C_T \int_0^T \|\mathbf{u}^1_t\|_{\Gamma_1} \||\nabla \widetilde{w}|^2\|_{\Gamma_1} \mathrm{d}t + C_T \int_0^T \|\mathbf{u}^1_t\|_\Omega \||\nabla \widetilde{w}|^2\|_{1,\Omega}\, \mathrm{d}t \\ &\qquad \leqslant C_{B,T} \int_0^T \|\widetilde{w}\|^2_{2+\varepsilon,\Omega} \mathrm{d}t + C_{B,T}\ \mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).\end{aligned}$$

Similar to the proof of (3.78), we see that

$$\int_0^T \int_s^T \left(\sigma\left[f(\nabla w^1) + J(w^1)\right] \nabla \widetilde{w}, \nabla \widetilde{w}_t\right)_\Omega \mathrm{d}t\, \mathrm{d}s \leqslant \delta \int_0^T \|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega} \mathrm{d}t + C_{B,T,\delta}\ \mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).$$

These inequalities lead to the following estimate:

$$|I_3| \leqslant \delta \int_0^T \|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega} \mathrm{d}t + C_{B,T} \int_0^T \|\widetilde{w}\|^2_{2+\varepsilon,\Omega} \mathrm{d}t + C_{B,T,\delta}\ \mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}). \tag{3.81}$$

Inserting the estimates (3.79), (3.80), and (3.81) into (3.76) implies that

$$\begin{aligned}\int_0^T \int_s^T R_2(t) \mathrm{d}t\mathrm{d}s &\leqslant \delta \widetilde{E}(T) + \delta \int_0^T \left(\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|^2_\Omega + \|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1} + \|\widetilde{\mathbf{u}}_t\|^2_\Omega \right) \mathrm{d}t \\ &\quad + \delta \int_0^T \|\widetilde{w}_t\|^2_{1-\varepsilon,\Omega} \mathrm{d}t + C_{B,T,\delta} \int_0^T \|\widetilde{w}\|^2_{2+\varepsilon,\Omega} \mathrm{d}t + C_{B,T,\delta}\ \mathrm{l.o.t.}(\widetilde{\mathbf{u}}, \widetilde{w}).\end{aligned} \tag{3.82}$$

Analogously to the above, we can show that

(3.83)
$$\int_0^T \int_s^T R_3(t)\mathrm{d}t\mathrm{d}s \leqslant \delta\widetilde{E}(T) + \delta\int_0^T \Big(\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|_\Omega^2 + \|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\mathbf{u}}_t\|_\Omega^2 + \|\widetilde{w}_t\|_\Omega^2\Big)\mathrm{d}t$$
$$+ C_{B,T,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

To conclude, we estimate $\int_0^T\int_s^T R_5(t)\mathrm{d}t\mathrm{d}s$. We may proceed as in (3.78) to show that

(3.84)
$$\int_0^T \int_s^T R_4(t)\mathrm{d}t\mathrm{d}s$$
$$= \int_0^T \int_s^T \Big(-\big(\phi^1\nabla\widetilde{w},\nabla\widetilde{w}_t\big)_\Omega + \big(\nabla\widetilde{w}\cdot\nabla w_t^2,\widetilde{\phi}\big)_\Omega + \big(\widetilde{\phi}\nabla\widetilde{w},\nabla\widetilde{w}_t\big)_\Omega\Big)\mathrm{d}t\mathrm{d}s$$
$$\leqslant \delta\int_0^T \Big(\|\widetilde{\phi}\|_{1,\Omega}^2 + \|\widetilde{w}_t\|_{1-\varepsilon,\Omega}^2\Big)\mathrm{d}t + \delta\Big(\sup_{t\in[0,T]}\widetilde{E}^{\frac{1}{2}}(t)\Big)^2$$
$$+ C_{B,T,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Combining (3.75), (3.82), (3.83), (3.84), and the interpolation inequality

$$C_{B,T,\delta}\int_0^T \|\widetilde{w}\|_{2+\varepsilon,\Omega}^2\mathrm{d}t \leqslant \delta\int_0^T \|\widetilde{w}\|_{3-\varepsilon,\Omega}^2\mathrm{d}t + C_{B,T,\delta}\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}),$$

we find that $\int_0^T\int_s^T |\sum R_i(t)|\,\mathrm{d}t\mathrm{d}s$ satisfies the desired estimate, and the same holds for $\int_0^T |\sum R_i(t)|\,\mathrm{d}t$. □

3.6.4. *Quasi-stability inequality: Proof.*

Proof of Lemma 3.5. Combining Lemma 3.11 with the estimate given in Lemma 3.10, we find that

(3.85)
$$T\widetilde{E}(T) + \int_0^T \widetilde{E}(t)\mathrm{d}t + \int_0^T \big(\|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\phi}\|_{1,\Omega}^2\big)\mathrm{d}t + \big(\sup_{t\in[0,T]}\widetilde{E}^{\frac{1}{2}}(t)\big)^2$$
$$\leqslant \delta\int_0^T\Big(\|\sigma[\epsilon(\widetilde{\mathbf{u}})]\|_\Omega^2 + \|\widetilde{\mathbf{u}}_t\|_{\Gamma_1}^2 + \|\widetilde{\mathbf{u}}_t\|_\Omega^2 + \|\widetilde{w}_t\|_\Omega^2 + \|\widetilde{\phi}\|_{1,\Omega}^2\Big)\mathrm{d}t + \delta\big(\sup_{t\in[0,T]}\widetilde{E}^{\frac{1}{2}}(t)\big)^2$$
$$+ \delta\int_0^T\big(\|\widetilde{w}_t\|_{1-\varepsilon,\Omega}^2 + \|\widetilde{w}\|_{3-\varepsilon,\Omega}^2\big)\mathrm{d}t + C\widetilde{E}(0) + (C_\delta + 2\alpha)\widetilde{E}(T) + C_{B,\alpha}D_0^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta})$$
$$+ C_{\alpha,B,T,\delta}\,\mathrm{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).$$

Then this estimate with Lemma 3.7 and (3.47) yields

$$
\begin{aligned}
T\widetilde{E}(T) + \int_0^T \widetilde{E}(t)\mathrm{d}t + \int_0^T \big(\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1} + \|\widetilde{\phi}\|^2_{1,\Omega}\big)\mathrm{d}t + \big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\big)^2 \\
\leqslant \delta \int_0^T \widetilde{E}(t)\mathrm{d}t + \delta \int_0^T \big(\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1} + \|\widetilde{\phi}\|^2_{1,\Omega}\big)\mathrm{d}t + \delta\Big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\Big)^2 \\
+ C\widetilde{E}(0) + (C_\delta + 2\alpha)\widetilde{E}(T) + C_{B,\alpha}\, D_0^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta}) + C_{\alpha,B,T,\delta}\ \text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).
\end{aligned}
$$

Taking $\delta > 0$ small enough, $T > 4C_\delta = T_0$, and $\alpha = C_\delta < \frac{T}{2}$, we have

$$
\begin{aligned}
T\widetilde{E}(T) + \int_0^T \widetilde{E}(t)\mathrm{d}t + \int_0^T \big(\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1} + \|\widetilde{\phi}\|^2_{1,\Omega}\big)\mathrm{d}t + \big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\big)^2 \\
\leqslant\ C\widetilde{E}(0) + C_B D_0^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta}) + C_{B,T}\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).
\end{aligned}
\tag{3.86}
$$

Energy identity (3.36) and estimate (3.11) allow us to conclude the following estimate for $D_0^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta})$:

$$
\begin{aligned}
D_0^T(\widetilde{\mathbf{u}},\widetilde{\phi},\widetilde{\theta}) &\leqslant \widetilde{E}(0) - \widetilde{E}(T) + \int_\alpha^T \left|\sum_{i=1}^5 R_i(t)\right| \mathrm{d}t \\
&\leqslant \widetilde{E}(0) - \widetilde{E}(T) + \delta\widetilde{E}(T) + \delta\int_0^T \widetilde{E}(t)\mathrm{d}t + \delta\int_0^T \big(\|\widetilde{\mathbf{u}}_t\|^2_{\Gamma_1} + \|\widetilde{\phi}\|^2_{1,\Omega}\big)\mathrm{d}t \\
&\quad + \delta\big(\sup_{t\in[0,T]} \widetilde{E}^{\frac{1}{2}}(t)\big)^2 + C_{B,T,\delta}\ \text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).
\end{aligned}
$$

This inequality applied in (3.86) and also taking δ small provide for fixed $T > T_0$

$$
\widetilde{E}(T) \leqslant \frac{C_B}{1+C_B}\widetilde{E}(0) + C_{B,T}\text{l.o.t.}(\widetilde{\mathbf{u}},\widetilde{w}).
$$

Via the stabilization argument we obtain the desired inequality. □

Remark 3.2. We note that the arguments leading to observability estimates can also be used in the study of controllability of thermoelastic systems [3, 4].

3.7. **Completion of the proof of Theorem 1.3.**

Proof of Theorem 1.3. As already noted, our dynamical system $(H, S(t))$ is a gradient one (Lemma 3.3) with a bounded set of stationary points (Lemma 3.4). Next, via Lemma 3.5, we can conclude that $(H, S(t))$ is asymptotically compact. To reach the conclusion, we shall apply Theorem 3.2, and this is achieved showing the validity of (3.19). The last is a consequence of Lemma 3.2, and therefore $(H, S(t))$ has a global attractor $\mathbf{A}$. Theorem 3.4 implies that $\mathbf{A}$ has finite fractal dimension and further "time" regularity,

$$
\left\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)y_0\right\|_H \leqslant C, \qquad \forall t\in\mathbb{R},\ \forall y_0\in\mathbf{A}.
$$

Feeding back this information into the problem (1.2)–(1.9) and applying elliptic regularity, we find the additional regularity on the attractor $\mathbf{A}$. This ends the proof of Theorem 1.3. □

4. Appendix

4.1. **Tensor identities.** For the reader's convenience, we present some elementary tensor identities—see also [43, 48, 49]—used in Sections 3.6.2 and 3.6.1. Let ϵ be the strain tensor, and let $h(x) = x - x_0$ be the vector field given for $x_0 \in \mathbb{R}^2$. Then we have the identity

$$\epsilon(h\nabla\mathbf{u}) = \epsilon(\mathbf{u}) + \mathscr{R}, \tag{4.87}$$

where $\mathbf{u} = (u_1, u_2)$ and $\mathscr{R}$ is the tensor defined by

$$\mathscr{R} = \begin{bmatrix} \sum_{i=1}^{2} \frac{\partial^2 u_1}{\partial x_1 \partial x_i} h_i & \frac{1}{2}\sum_{i=1}^{2} \left[\frac{\partial^2 u_1}{\partial x_i \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_i}\right] h_i \\ \frac{1}{2}\sum_{i=1}^{2} \left[\frac{\partial^2 u_1}{\partial x_i \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_i}\right] h_i & \sum_{i=1}^{2} \frac{\partial^2 u_2}{\partial x_i \partial x_2} h_i \end{bmatrix}.$$

Let $A = [a_{i,j}]$ be a symmetric tensor. Then

$$A \cdot \mathscr{R} = \sum_{i,j,k=1}^{2} a_{k,j} \frac{\partial^2 u_j}{\partial x_k \partial x_i} h_i. \tag{4.88}$$

Let $B = [b_{i,j}]$ be a symmetric tensor given by the relation $a_{j,i} = \sum_{l=1}^{2} c_{j,l} b_{l,i}$, where $c_{j,i}$ are symmetric coefficients. Then

$$\begin{aligned} \operatorname{div}\{(A\cdot B)h\} &= A\cdot B \operatorname{div}\{h\} + \sum_{i,j,k,l=1}^{2} c_{i,l} \frac{\partial}{\partial x_k}\left[b_{l,j} b_{j,i}\right] h_k \\ &= A\cdot B \operatorname{div}\{h\} + 2\sum_{i,j,k=1}^{2} a_{j,i} \frac{\partial b_{j,i}}{\partial x_k} h_k. \end{aligned}$$

In particular, if $A = \sigma[\epsilon(\mathbf{u})]$ and $B = \epsilon(\mathbf{u})$, then

$$\operatorname{div}\{(\sigma[\epsilon(\mathbf{u})], \epsilon(\mathbf{u}))_\Omega h\} = 2\big(\sigma[\epsilon(\mathbf{u})], \epsilon(\mathbf{u})\big)_\Omega + 2\sum_{i,j,k=1}^{2} \Big(a_{i,j}, \frac{\partial^2 u_i}{\partial x_k \partial x_j} h_k\Big)_\Omega. \tag{4.89}$$

Finally, from identities (4.87) and (4.88), with $A = \sigma[\epsilon(\mathbf{u})]$, we obtain

$$\big(\sigma[\epsilon(\mathbf{u})], \epsilon(h\nabla\mathbf{u})\big)_\Omega = \big(\sigma[\epsilon(\mathbf{u})], \epsilon(\mathbf{u})\big)_\Omega + \sum_{i,j,k=1}^{2} \Big(a_{i,j}, \frac{\partial^2 u_i}{\partial x_k \partial x_j} h_k\Big)_\Omega. \tag{4.90}$$

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