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Quantum Groups, Property (T), and Weak Mixing

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Abstract: For second countable discrete quantum groups, and more generally second countable locally compact quantum groups with trivial scaling group, we show that property (T) is equivalent to every weakly mixing unitary representation not having almost invariant vectors. This is a generalization of a theorem of Bekka and Valette from the group setting and was previously established in the case of low dual by Daws, Skalski, and Viselter. Our approach uses spectral techniques and is completely different from those of Bekka–Valette and Daws–Skalski–Viselter. By a separate argument we furthermore extend the result to second countable nonunimodular locally compact quantum groups, which are shown in particular not to have property (T), generalizing a theorem of Fima from the discrete setting. We also obtain quantum group versions of characterizations of property (T) of Kerr and Pichot in terms of the Baire category theory of weak mixing representations and of Connes and Weiss in terms of the prevalence of strongly ergodic actions.

1. Introduction

Introduced by Kazhdan in the 1960s for the purpose of showing that many lattices are finitely generated, property (T) has come to play a foundational role in the study of rigidity in Lie groups, ergodic theory, and von Neumann algebras through the work of Margulis, Zimmer, Connes, Popa, and others [5,30,36]. Over the last twenty-five years it has been extended in stages to the realm of quantum groups, first via Kac algebras [17], then in the algebraic [3] and discrete [14,26] settings, and finally in the general framework of locally compact quantum groups, as defined by Kusterman and Vaes [11]. In one notable recent application, Arano showed in [1,2] that the Drinfeld double of a q-deformation of compact simple Lie group has property (T) and that this implies that the duals of these q-deformations have a central version of property (T), a fact which has inspired progress in the theory of C*-tensor categories and underpins Popa and Vaes's

construction of subfactors with property (T) standard invariant that do not come from groups [28,31].

By definition, a locally compact group G does not have property (T) if it admits a unitary representation that does not have a nonzero invariant vector (*ergodicity*) but does have a net of unit vectors that is asymptotically invariant on each group element (*having almost invariant vectors*). Because ergodicity has poor permanence properties, it can be hard to leverage this definition so as to obtain global information about the representation theory of a group without property (T), and in particular to determine to what extent the kind of flexible behaviour exhibited by amenable groups persists in this more general setting. Bekka and Valette provided a remedy for this in the separable case by showing that one can equivalently replace *ergodicity* above with *weak mixing*, which is characterized by the absence of nonzero finite-dimensional subrepresentions, or alternatively by the ergodicity of the tensor product of the representation with its conjugate [4]. This leads, for example, to a short proof of a theorem of Wang that characterizes property (T) in terms of the isolation of finite-dimensional representations in the spectrum ([4], Sect. 4) and a streamlined proof of the Connes–Weiss characterization of property (T) in terms of strongly ergodic probability-measure-preserving actions ([5], Sect. 6.3).

Using the fact that weak mixing is preserved under tensor products with arbitrary representations, Kerr and Pichot applied the Bekka–Valette theorem to show that if a second countable locally compact group does not have property (T) then within the set of all unitary representations of the group on a fixed separable infinite-dimensional Hilbert space the weakly mixing ones form a dense G_δ in the weak topology [22]. The idea is that any representation will approximately absorb a representation with almost invariant vectors under tensoring (since locally it is as if we were tensoring with the trivial representation) and so such a tensor product will be "close" to the original representation while also inheriting any properties of the second one that are preserved under tensoring, such as weak mixing. By a similar principle requiring a more subtle implementation, Kerr and Pichot also established an analogous conclusion for the measure-preserving actions of the group on a fixed standard atomless probability space, strengthening a result of Glasner, Thouvenot, and Weiss that gave the same conclusion for ergodic actions [15].

Using the theory of positive-definite functions as in [18,29], Daws, Skalski, and Viselter demonstrated in [12] that the conclusion of the Bekka–Valette theorem also holds for second countable discrete unimodular quantum groups with low dual, and as an application they derive analogues of the Connes–Weiss theorem and the representation-theoretic Kerr–Pichot theorem. Low dual is the rather restrictive assumption that there is a bound on the dimensions of the irreducible representations of the quantum group, and the authors of [12] wonder, somewhat pessimistically, whether it can be removed. In the present paper we show that the Bekka–Valette and Kerr–Pichot theorems actually hold for all second countable discrete quantum groups, and even more generally for all second countable locally compact quantum groups with trivial scaling group (Theorems 4.8 and 4.9) as well as for all second countable nonunimodular locally compact quantum groups (Theorem 6.3). The methods of Daws, Skalski, and Viselter can then also be applied to extend their version of the Connes–Weiss theorem to all second countable locally compact quantum groups with trivial scaling group (Theorem 5.1).

Our approach is completely different from those of Daws–Skalski–Viselter and Bekka–Valette and consists in applying the quantum group version of Wang's characterization of property (T) mentioned above in order to reduce the problem to a purely spectral question concerning C*-algebras. In Theorems 3.7 and 3.8 we prove that the fol-

lowing hold for a separable unital C*-algebra A and a fixed separable infinite-dimensional Hilbert space $\mathcal{H}$:

(i) if the spectrum of A contains no isolated finite-dimensional representations then the set of weakly mixing unital representations of A on $\mathcal{H}$ is a dense G_δ, and

(ii) if the set of finite-dimensional representations in the spectrum of A is nonempty and contains only isolated points then the set of weakly mixing unital representations of A on $\mathcal{H}$ is closed and nowhere dense.

A version of the argument establishing (i) for unitary representations of countable discrete groups has also been included in the book [21] by Li and the second author. Theorems 4.8 and 4.9 then follow from (i) and (ii) whenever Wang's characterization of property (T) holds in the quantum group context, and this is known to be the case when the scaling group is trivial (see Sect. 4). By a completely different argument we also prove in Theorem 6.3 that the conclusions Theorems 4.8 and 4.9 are valid for second countable nonunimodular locally compact quantum groups, which we show in particular not to have property (T), generalizing a result of Fima from the discrete case [14].

We begin in Sect. 2 by reviewing some of the basic theory of locally compact quantum groups and their unitary representations as developed by Kustermans and Vaes [23–25,32]. In Sect. 3, we study weak mixing for C*-algebra representations and establish the two key spectral results (i) and (ii) concerning separable unital C*-algebras. In Sect. 4, we discuss weak mixing and property (T) for quantum groups, record the quantum group incarnation of Wang's theorem, and then establish our versions of the Bekka–Valette and Kerr–Pichot theorems. Section 5 contains the Connes–Weiss-type dynamical characterization of property (T). Finally, the nonunimodular case is treated in Sect. 6.

2. Preliminaries

For a C*-algebra A we write $M(A)$ for its multiplier algebra. A representation of A is understood to mean a *-homomorphism from A into the C*-algebra of bounded linear operators on some Hilbert space. When working with tensor products of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we denote by Σ the tensor flip map from $\mathcal{H} \otimes \mathcal{K}$ to $\mathcal{K} \otimes \mathcal{H}$. For linear operators on multiple tensor products, we use leg notation. For example, if U is a unitary operator on a Hilbert space tensor product $\mathcal{H} \otimes \mathcal{K}$ we write U_{13} for the unitary operator on a Hilbert space tensor product of the form $\mathcal{H} \otimes \mathcal{J} \otimes \mathcal{K}$ which is given by $V(U \otimes \mathrm{id})V^{-1}$ where V is the shuffle map $\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{J} \to \mathcal{H} \otimes \mathcal{J} \otimes \mathcal{K}$ defined on elementary tensors by $\xi \otimes \zeta \otimes \kappa \mapsto \xi \otimes \kappa \otimes \zeta$, i.e., $V = \mathrm{id}_{\mathcal{H}} \otimes \Sigma$.

2.1. Locally compact quantum groups. Our main references for generalities on locally compact quantum groups are [24,25,32]. Formally speaking, a (von Neumann algebraic) locally compact quantum group is a von Neumann algebra with coassociative coproduct and left and right Haar weights, but as usual we use the simple notation G so that we can conveniently and suggestively refer to the various objects that are canonically attached to it just as one does for locally compact groups, although there is no longer anything like an underlying group. The von Neumann algebra itself is thus written $L^\infty(G)$, and the coproduct is a unital normal *-homomorphism $\Delta : L^\infty(G) \to L^\infty(G)\overline{\otimes}L^\infty(G)$ satisfying the coassociativity condition

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

The left and right Haar weights are normal semifinite weights φ and ψ on $L^\infty(G)$ such that for every $\omega \in L^\infty(G)_*^+$ one has

$$\varphi((\omega \otimes \mathrm{id})\Delta(a)) = \varphi(a)\omega(1)$$

for all $a \in L^\infty(G)^+$ with $\varphi(a) < \infty$ and

$$\psi((\mathrm{id} \otimes \omega)\Delta(a)) = \psi(a)\omega(1)$$

for all $a \in L^\infty(G)^+$ with $\psi(a) < \infty$. The predual of $L^\infty(G)$ is written as $L^1(G)$, and becomes a completely contractive Banach algebra with respect to the *convolution product*

$$\omega_1 \star \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta, \qquad \omega_1, \omega_2 \in L^1(G).$$

Associated to G is a canonical weakly dense sub-C*-algebra of $L^\infty(G)$, written $C_0(G)$, which plays the role of the C*-algebra of continuous functions vanishing at infinity in the case of ordinary groups. We say that G is *second countable* if $C_0(G)$ is separable. The coproduct restricts to a unital *-homomorphism $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G))$. The algebras $C_0(G)$ and $L^\infty(G)$ are standardly represented on the GNS Hilbert space $L^2(G)$ associated to the left Haar weight. In the case of a locally compact group, the notations $L^\infty(G)$, $L^1(G)$, $C_0(G)$, and $L^2(G)$ have their ordinary meaning.

There is a (left) *fundamental unitary operator* W on $L^2(G) \otimes L^2(G)$ which satisfies the *pentagonal relation* $W_{12}W_{13}W_{23} = W_{23}W_{12}$ and unitarily implements the coproduct Δ on $L^\infty(G)$ via the formula $\Delta(x) = W^*(1 \otimes x)W$. Using W one has $C_0(G) = \overline{\{(\mathrm{id} \otimes \omega)W : \omega \in \mathscr{B}(L^2(G))_*\}}^{\|\cdot\|}$, and one can define the *antipode* of G as the (generally only densely defined) linear operator S on $C_0(G)$ (or $L^\infty(G)$) satisfying the identity $(S \otimes \mathrm{id})W = W^*$. The antipode admits a polar decomposition $S = R \circ \tau_{-i/2}$ where R is an antiautomorphism of $L^\infty(G)$ (the *unitary antipode*) and $\{\tau_t\}_{t \in \mathbb{R}}$ is a one-parameter group of automorphisms (the *scaling group*). In the case of a locally compact group, the scaling group is trivial and the antipode is the antiautomorphism sending a function $f \in C_0(G)$ to the function $s \mapsto f(s^{-1})$. Using the antipode S one can endow the convolution algebra $L^1(G)$ with a densely defined involution by considering the norm-dense subalgebra $L^1_\sharp(G)$ of $L^1(G)$ consisting of all $\omega \in L^1(G)$ for which there exists an $\omega^\sharp \in L^1(G)$ with $\langle \omega^\sharp, x \rangle = \overline{\langle \omega, S(x)^* \rangle}$ for each $x \in \mathcal{D}(S)$. It is known from [23] and Sect. 2 of [25] that $L^1_\sharp(G)$ is an involutive Banach algebra with involution $\omega \mapsto \omega^\sharp$ and norm $\|\omega\|_\sharp = \max\{\|\omega\|, \|\omega^\sharp\|\}$.

Associated to any locally compact quantum group G is its *dual locally compact quantum group* $\widehat{G}$, whose associated algebras, coproduct, and fundamental unitary are given by $C_0(\widehat{G}) = \overline{\{(\omega \otimes \mathrm{id})W : \omega \in \mathscr{B}(L^2(G))_*\}}^{\|\cdot\|} \subseteq \mathscr{B}(L^2(G))$, $L^\infty(\widehat{G}) = C_0(\widehat{G})''$, $\widehat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}$, and $\widehat{W} = \Sigma W^* \Sigma$. Then in fact $W \in M(C_0(G) \otimes C_0(\widehat{G}))$, and the *Pontryagin duality theorem* asserts that the bidual quantum group $\widehat{\widehat{G}}$ is canonically identified with the original quantum group G. One says that a locally compact quantum group G is *compact* if $C_0(G)$ is unital, and *discrete* if $\widehat{G}$ is compact, which is equivalent to $C_0(G)$ being a direct sum of matrix algebras.

For a locally compact quantum group G, we can always assume that the left and right Haar weights are related by $\psi = \varphi \circ R$, where R is the unitary antipode. If the left and right Haar weights φ and ψ of G coincide then we say that G is *unimodular*. In general, the failure of ψ to be left-invariant is measured by the *modular element*, which is a

strictly positive element δ affiliated with $L^\infty(G)$ satisfying the identities $\Delta(\delta) = \delta \otimes \delta$ and $\psi(\cdot) = \varphi(\delta^{1/2} \cdot \delta^{1/2})$. Compact quantum groups are always unimodular, and the corresponding Haar weight can always be chosen to be a state. Although discrete groups are always unimodular, discrete quantum groups need not be. We recall that a discrete quantum group G is said to be of *Kac type* (or a *Kac algebra*) if it is unimodular, which is equivalent to the Haar state on $\widehat{G}$ being a trace.

2.2. Unitary representations.

Definition 2.1. A *unitary representation* of a locally compact quantum group G on a Hilbert space $\mathscr{H}$ is a unitary $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H})) \subseteq \mathscr{B}(L^2(G) \otimes \mathscr{H})$ such that $(\Delta \otimes \mathrm{id})(U) = U_{13}U_{23}$.

In the above definition one can replace $M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ with the larger algebra $L^\infty(G)\overline{\otimes}\mathscr{B}(\mathscr{H})$, for if U is a unitary in the latter which satisfies $(\Delta \otimes \mathrm{id})(U) = U_{13}U_{23}$ then U automatically belongs to the former (see for example Theorem 4.12 of [7]).

Associated to a unitary representation $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ is an adjointable operator on the Hilbert module $C_0(G) \otimes \mathscr{H}$ which we write using the boldface version $\mathbf{U}$ of the symbol in question. The relation between U and $\mathbf{U}$ is given by

$$\langle \mathbf{U}(a \otimes \xi), b \otimes \zeta \rangle = b^*(\mathrm{id} \otimes \omega_{\xi,\zeta})(U)a$$

for all $a, b \in C_0(G)$ and $\xi, \zeta \in \mathscr{H}$, where $\omega_{\xi,\zeta}$ is the vector functional $x \mapsto \langle x\xi, \zeta \rangle$.

Associated to G are two distinguished unitary representations, the one-dimensional *trivial representation* $1_G \in M(C_0(G))$ given by the unit of $L^\infty(G)$, and the *left regular representation* given by the fundamental unitary $W \in M(C_0(G) \otimes C_0(\widehat{G}))$ viewed as an element of $M(C_0(G) \otimes \mathscr{K}(L^2(G)))$.

Two unitary representations $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}_1))$ and $V \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}_2))$ of G are *(unitarily) equivalent* if there is a unitary isomorphism $u : \mathscr{H}_1 \to \mathscr{H}_2$ such that $V = (\mathrm{id} \otimes \mathrm{Ad}\, u)(U)$. A *subrepresentation* of a unitary representation $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ is a unitary representation of the form $Q = (1 \otimes P)U(1 \otimes P) \in L^\infty(G)\overline{\otimes}\mathscr{B}(\mathscr{H}_0)$ where $\mathscr{H}_0$ is a closed subspace of $\mathscr{H}$, P is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_0$, and $1 \otimes P$ commutes with U. In this case, we write $Q \leq U$. A unitary representation $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ is said to be *finite-dimensional* if $\mathscr{H}$ is finite-dimensional.

Let G be a locally compact quantum group. Let $U \in M(C_0(G)\otimes\mathscr{K}(\mathscr{H}))$ be a unitary representation of G. Write $\overline{\mathscr{H}}$ for the conjugate of $\mathscr{H}$, i.e., the Hilbert space which is the same as $\mathscr{H}$ as an additive group but with the scalar multiplication $(c, \xi) \mapsto \bar{c}\xi$ for $c \in \mathbb{C}$ and inner product $\langle \xi, \zeta \rangle_{\overline{\mathscr{H}}} = \langle \zeta, \xi \rangle_{\mathscr{H}}$. Letting $T : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\overline{\mathscr{H}})$ be the transpose map $T(a)(\bar{\xi}) = \overline{a^*(\xi)}$, we define the *conjugate* of U, written $\overline{U}$, to be the unitary representation

$$(R \otimes T)(U) \in M(C_0(G) \otimes \mathscr{K}(\overline{\mathscr{H}})).$$

The *tensor product* of two unitary representations $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ and $V \in M(C_0(G) \otimes \mathscr{K}(\mathscr{K}))$ is the unitary representation

$$U \odot V := U_{12}V_{13} \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H} \otimes \mathscr{K}))$$
$$\subseteq L^\infty(G)\overline{\otimes}\mathscr{B}(\mathscr{H})\overline{\otimes}\mathscr{B}(\mathscr{K}).$$

There is a bijective correspondence between unitary representations $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ and nondegenerate *-representations $\pi_U : L^1_\sharp(G) \to \mathscr{B}(H)$ ([23], Corollary 2.13). This correspondence is given by

$$\pi_U(\omega) = (\omega \otimes \mathrm{id})U \in \mathscr{B}(H), \qquad \omega \in L^1_\sharp(G).$$

At the level of *-representations of $L^1_\sharp(G)$, the trivial representation 1_G corresponds to the *-character $\omega \mapsto \omega(1)$, and the left regular representation is written as $\omega \mapsto \lambda(\omega) = (\omega \otimes \mathrm{id})W \in C_0(\widehat{G}) \subseteq \mathscr{B}(L^2(G))$. As expected, we have the dual relations

$$C_0(\widehat{G}) = \overline{\lambda(L^1_\sharp(G))}^{\|\cdot\|} \quad \text{and} \quad C_0(G) = \overline{\hat{\lambda}(L^1_\sharp(\widehat{G}))}^{\|\cdot\|}$$

where $\hat{\lambda}$ is the left regular representation of $\widehat{G}$.

Let $C^u_0(\widehat{G})$ denote the universal enveloping C*-algebra of $L^1_\sharp(G)$. This is a universal version of $C_0(\widehat{G})$ which encodes the unitary representation theory of G (since its nondegenerate representations are in bijective correspondence with the nondegenerate *-representations of $L^1_\sharp(G)$ as bounded Hilbert space operators). In particular, the left regular representation W gives rise to a surjective representation $\lambda : C^u_0(\widehat{G}) \to C_0(\widehat{G}) \subseteq \mathscr{B}(L^2(G))$, and the trivial representation gives rise to the (dual) *counit* $\hat{\varepsilon}_u : C^u_0(\widehat{G}) \to \mathbb{C}$. We will generally use the same symbols to denote *-representations of $L^1_\sharp(G)$ and their unique extensions to $C^u_0(\widehat{G})$.

As was shown in [23], $C^u_0(\widehat{G})$ admits a coproduct $\hat{\Delta}_u : C^u_0(\widehat{G}) \to M(C^u_0(\widehat{G}) \otimes C^u_0(\widehat{G}))$ which can be used to turn $(C^u_0(\widehat{G}), \hat{\Delta}_u)$ into a *universal C*-algebraic locally compact quantum group*. For our purposes, we only need the fact that $\hat{\Delta}_u$ allows one to express the tensor product $U \odot V$ of two unitary representations $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ and $V \in M(C_0(G) \otimes \mathscr{K}(\mathscr{K}))$ in terms of the representation $(\pi_U \otimes \pi_V) \circ \sigma \hat{\Delta}_u : C^u_0(\widehat{G}) \to \mathscr{B}(\mathscr{H} \otimes \mathscr{K})$, where σ denotes the tensor flip map on $C^u_0(\widehat{G}) \otimes C^u_0(\widehat{G})$.

Finally, note that at the level of representations of $C^u_0(\widehat{G})$ (or of *-representations of $L^1_\sharp(G)$) the notions of subrepresentation and unitary equivalence of unitary representations reduce to their standard meanings. Indeed, given a unitary representation $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ and a projection P in $\mathscr{B}(\mathscr{H})$, the projection $1 \otimes P$ commutes with U if and only if P commutes with $\pi_U(C^u_0(\widehat{G}))$, in which case the representation $\pi_Q : C^u_0(\widehat{G}) \to \mathscr{B}(P\mathscr{H})$ associated to $Q = (1 \otimes P)U(1 \otimes P)$ is given by $a \mapsto P\pi_U(a)P$. Similarly, if $V \in M(C_0(G) \otimes \mathscr{K}(\mathscr{K}))$ is another unitary representation, then a unitary isomorphism $u : \mathscr{H} \to \mathscr{K}$ implements an equivalence between U and V if and only if it implements a unitary equivalence between π_U and π_V in the sense that $\pi_V = \mathrm{Ad}\, u \circ \pi_U$.

3. Weak Mixing and Representations of C*-Algebras

This section is purely C*-algebraic and aims to establish two results concerning the prevalence of weak mixing among unital representations of a separable unital C*-algebra on a fixed Hilbert space (Theorems 3.7 and 3.8).

Throughout this section A will denote a separable unital C*-algebra. For a fixed Hilbert space $\mathscr{H}$, the set of all unital representations of A on $\mathscr{H}$ will be written $\mathrm{Rep}(A, \mathscr{H})$. We equip $\mathrm{Rep}(A, \mathscr{H})$ with the point-strong operator topology, which is equivalent to the point-weak operator topology, and also to the point-*-strong operator

topology since the strong and *-strong operator topologies agree on the unitary group of $\mathscr{B}(\mathscr{H})$ and A is linearly spanned by its unitaries. Under this topology $\mathrm{Rep}(A, \mathscr{H})$ is a Polish space ([13], Proposition 3.7.1).

Points in the spectrum $\widehat{A}$, while formally defined as equivalence classes of irreducible representations, will be thought of as actual representations via their representatives, following convention. The set of finite-dimensional representations in $\widehat{A}$ will be written $\widehat{A}_{\mathrm{fin}}$.

We begin with a discussion of weak mixing for unital representations of unital C*-algebras.

Definition 3.1. We say that a unital representation of A on a Hilbert space is *weakly mixing* if it has no nonzero finite-dimensional subrepresentations.

Recall that weak mixing for a unitary representation $\pi : G \to \mathscr{U}(\mathscr{H})$ of a group can be expressed in either of the following equivalent ways (see Theorem 2.23 in [21], and note that the countability assumption there is not needed):

(i)   for every finite set $\Omega \subseteq \mathscr{H}$ and $\varepsilon > 0$, there exists an $s \in G$ such that $|\langle \pi(s)\xi, \zeta \rangle| < \varepsilon$ for all $\xi, \zeta \in \Omega$,

(ii) π has no nonzero finite-dimensional subrepresentations.

Since a unital C*-algebra is linearly spanned by its unitaries, from (ii) we immediately obtain the following, justifying the terminology of Definition 3.1.

Proposition 3.2. *A unital representation of A is weakly mixing if and only if its restriction to the unitary group of A is weakly mixing.*

Next we consider a C*-algebra version of Zimmer's notion [36] of weak containment for unitary representations of groups.

Definition 3.3. Let $\pi : A \to \mathscr{B}(\mathscr{H})$ and $\rho : A \to \mathscr{B}(\mathscr{K})$ be unital representations. We write $\pi \prec \rho$ if for every finite set $\Omega \subseteq A$, orthonormal set $\{\xi_1, \ldots, \xi_n\} \subseteq \mathscr{H}$, and $\varepsilon > 0$ there is an orthonormal set $\{\zeta_1, \ldots, \zeta_n\} \subseteq \mathscr{K}$ such that $|\langle \pi(a)\xi_i, \xi_i \rangle - \langle \rho(a)\zeta_i, \zeta_i \rangle| < \varepsilon$ for all $a \in \Omega$ and $i = 1, \ldots, n$.

This is the same as the usual notion of weak containment when the representation π is irreducible, but is different in general (the usual weak containment requires that for every finite set $\Omega \subseteq A$, $\xi \in \mathscr{H}$, and $\varepsilon > 0$ there exist $\zeta_1, \ldots, \zeta_n \in \mathscr{H}$ such that $|\langle \pi(a)\xi, \xi \rangle - \sum_{i=1}^{n} \langle \rho(a)\zeta_i, \zeta_i \rangle| < \varepsilon$ for all $a \in \Omega$ [13]). In fact π is weakly contained in ρ if and only if $\pi \prec \rho^{\oplus \mathbb{N}}$.

The following is a straightforward consequence of Definition 3.3. The version for unitary group representations was noted in the remark after Proposition H.2 in [20].

Lemma 3.4. *Let $\pi : A \to \mathscr{B}(\mathscr{H})$ and $\rho : A \to \mathscr{B}(\mathscr{K})$ be unital representations on separable infinite-dimensional Hilbert spaces. Then $\pi \prec \rho$ if and only if $\pi \in \overline{\{\kappa \in \mathrm{Rep}(A, \mathscr{H}) : \kappa \cong \rho\}}$.*

Denote by $\mathrm{WM}(A, \mathscr{H}) \subseteq \mathrm{Rep}(A, \mathscr{H})$ the subcollection of all weakly mixing representations.

Lemma 3.5. *Let $\mathscr{H}$ be a separable Hilbert space. Then $\mathrm{WM}(A, \mathscr{H})$ is a G_δ in $\mathrm{Rep}(A, \mathscr{H})$.*

Proof. Write G for the unitary group of A. Take an increasing sequence $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ of finite subsets of $\mathscr{H}$ with dense union in $\mathscr{H}$. For every $n \in \mathbb{N}$ define Γ_n to be the set of

all $\varphi \in \text{Rep}(A, \mathcal{H})$ such that there exists a $u \in G$ satisfying $|\langle \varphi(u)\xi, \zeta \rangle| < 1/n$ for all $\xi, \zeta \in \Omega_n$. Then Γ_n is open, and so the set $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$ is a G_δ. By the characterization of weak mixing for unitary group representations described before Proposition 3.2, Γ is precisely the set of all representations whose restriction to G is weakly mixing, which is equal to $\text{WM}(A, \mathcal{H})$ by Proposition 3.2. $\square$

Lemma 3.6. *Suppose that no point in $\widehat{A}_{\text{fin}}$ is isolated in $\widehat{A}$. Let $\rho : A \to \mathcal{B}(\mathcal{H})$ be a representation in $\widehat{A}_{\text{fin}}$. Then there exists a weakly mixing representation θ of A on a separable Hilbert space such that $\rho \prec \theta$.*

Proof. We may assume that ρ is not the limit of a sequence $\{\pi_n\}$ of infinite-dimensional representations in $\widehat{A}$, for in that case the representation $\pi = \bigoplus_{n=1}^{\infty} \pi_n$ is weakly mixing and $\rho \prec \pi$. Since $\widehat{A}$ is second countable ([13], Proposition 3.3.4), we can then find a countable neighbourhood basis $\{U_n\}_{n \in \mathbb{N}}$ for ρ in $\widehat{A}$ such that no U_n contains an infinite-dimensional representation.

Let $n \in \mathbb{N}$. We construct a weakly mixing representation θ_n in $\widehat{A}$ as follows. First we argue that U_n is uncountable. Suppose that this is not the case. As $\widehat{A}$ is a Baire space ([13], Theorem 3.4.13) and U_n is open, U_n is itself a Baire space. For every $\omega \in U_n$ the singleton $\{\omega\}$ is closed by finite-dimensionality (see Sect. 3.6 of [13]), and so by the Baire property there exists an $\omega_0 \in U_n$ such that $\{\omega_0\}$ is open, which means that ω_0 is isolated, contradicting our hypothesis. Thus U_n is uncountable. We can consequently find a $d_n \in \mathbb{N}$ such that U_n contains uncountably many d_n-dimensional representations.

Fix a Hilbert space $\mathcal{H}_n$ of dimension d_n. Denote by $\text{Irr}(A, \mathcal{H}_n)$ the set of irreducible representations in $\text{Rep}(A, \mathcal{H}_n)$. We observe the following:

(i) $\text{Irr}(A, \mathcal{H}_n)$ is open in $\text{Rep}(A, \mathcal{H}_n)$,
(ii)   every equivalence class in $\text{Irr}(A, \mathcal{H}_n)$ is closed in $\text{Irr}(A, \mathcal{H}_n)$,
(iii)   for every open set $U \subseteq \text{Irr}(A, \mathcal{H}_n)$ the set of all elements in $\text{Irr}(A, \mathcal{H}_n)$ which are equivalent to some element of U is open.

Assertion (iii) is clear. To verify (i), let $\{\pi_k\}$ be a convergent sequence in $\text{Rep}(A, \mathcal{H}_n)$ whose terms are not irreducible and let us show that its limit π is not irreducible. For every k choose a nonzero projection $P_k \in \pi_k(A)'$ of rank less than d_n. In view of the finite-dimensionality of $\mathcal{H}_n$, we may assume by passing to a subsequence that the sequence $\{P_k\}$ converges in $\mathcal{B}(\mathcal{H}_n)$, in which case its limit P is a nonzero projection of rank less than d_n which commutes with $\pi(A)$. This means that π is not irreducible, yielding (i). Finally, to verify (ii) we let $\{\pi_k\}$ be a convergent sequence in $\text{Irr}(A, \mathcal{H}_n)$ such that for every k there exists a unitary operator Z_k which conjugates π_k to π_1. By the finite-dimensionality of $\mathcal{H}_n$, there is a subsequence $\{Z_{k_j}\}_j$ that converges to a unitary operator Z, which must then conjugate the limit of $\{\pi_k\}$ to π_1, yielding (ii).

Since open subsets of Polish spaces are themselves Polish spaces ([19], Theorem 3.11), we infer from (i) that $\text{Irr}(A, \mathcal{H}_n)$ is a Polish space. Assertions (ii) and (iii) then permit us to apply a standard selection theorem ([19], Theorem 12.16) which provides a Borel set $B_n \subseteq \text{Irr}(G, \mathcal{H}_n)$ of representatives for the relation of unitary equivalence. Write W_n for the set of all $\pi \in \text{Irr}(A, \mathcal{H}_n)$ which, as elements in $\widehat{A}$, belong to U_n. This is clearly an open set in $\text{Irr}(A, \mathcal{H}_n)$, and it is uncountable by our choice of d_n. Thus $B_n \cap W_n$ is an uncountable Borel set, which means that it is isomorphic as a Borel space to the unit interval with Lebesgue measure and hence admits an atomless Borel probability measure of full support. Let μ_n be the push forward of this measure under the inclusion $B_n \cap W_n \hookrightarrow \text{Irr}(A, \mathcal{H}_n)$, and note that $\mu_n(C) = 0$ for every unitary equivalence class C in $\text{Irr}(A, \mathcal{H}_n)$,

Setting $Y_n = \mathrm{Irr}(A, \mathscr{H}_n)$, we next consider the Hilbert space $L^2(Y_n, \mathscr{H}_n)$ of (classes of) $\mathscr{H}_n$-valued functions on Y_n with inner product

$$\langle f, g \rangle = \int_{Y_n} \langle f(\pi), g(\pi) \rangle \, d\mu_n(\pi).$$

Finally, we define the representation $\theta_n : A \to \mathscr{B}(L^2(Y_n, \mathscr{H}_n))$ by setting

$$(\theta_n(a)\zeta)(\pi) = \pi(a)\zeta(\pi)$$

for $a \in A$, $\zeta \in L^2(Y_n, \mathscr{H}_n)$, and $\pi \in Y_n$.

We next verify that θ_n is weakly mixing. Suppose that this is not the case. Then there is a finite-dimensional irreducible representation $\pi : A \to \mathscr{B}(\mathscr{H})$ and an isometric operator $Z : \mathscr{H} \to L^2(Y_n, \mathscr{H}_n)$ such that $Z\pi(a) = \theta_n(a)Z$ for all $a \in A$. Then for a.e. $\rho \in Y_n$ and every $a \in A$ and $\eta \in \mathscr{H}$ we have

$$(Z\pi(a)\eta)(\rho) = (\theta_n(a)Z\eta)(\rho) = \rho(a)((Z\eta)(\rho)).$$

so that the operator $Z_\rho : \mathscr{H} \to \mathscr{H}_n$ given by $Z_\rho \eta = (Z\eta)(\rho)$ satisfies $Z_\rho \pi(a) = \rho(a)Z$ for all $a \in A$. As Z is isometric, the operator Z_ρ must be nonzero for all ρ in a nonnull subset of Y_n. But each such ρ is equivalent to π by irreducibility, contradicting the fact that the measure of every unitary equivalence class is zero. Therefore θ_n is weakly mixing.

Now set $\theta = \bigoplus_{n=1}^{\infty} \theta_n$. Then θ is weakly mixing since each summand is weakly mixing. It remains to show that $\rho \prec \theta$. Let Ω be a finite subset of A and $\varepsilon > 0$. Then we can find an $n \in \mathbb{N}$ such that for every $\pi \in U_n$ there is an isometry $V : \mathscr{H} \to \mathscr{H}_n$ such that $\|V\rho(a) - \pi(a)V\| < \varepsilon/2$ for all $a \in \Omega$. Since bounded sets in $\mathscr{B}(\mathscr{H}_n)$ are precompact and representations are contractive, we can find an open set $U \subseteq Y_n$ with $\mu(U) > 0$ such that for all $a \in \Omega$ and $\pi, \pi' \in U$ one has $\|\pi(a) - \pi'(a)\| < \varepsilon/2$. Choose a $\pi_0 \in U$ and an isometry $V : \mathscr{H} \to \mathscr{H}_n$ such that for all $a \in \Omega$ we have $\|V\rho(a) - \pi_0(a)V\| < \varepsilon/2$ and hence $\|V\rho(a) - \pi(a)V\| < \varepsilon$ for every $\pi \in U$. Writing $\mathbf{1}_U$ for the indicator function of U, we set $f = \mu_n(U)^{-1/2}\mathbf{1}_U$, which is a unit vector in $L^2(Y_n, \mu_n)$. Define an isometry $\tilde{V} : \mathscr{H} \to L^2(Y_n, \mathscr{H}_n)$ by $\tilde{V}\xi = f \otimes V\xi \in L^2(Y_n, \mu_n) \otimes \mathscr{H}_n \cong L^2(Y_n, \mathscr{H}_n)$. Then for all $a \in \Omega$ and norm-one vectors $\xi \in \mathscr{H}$ we have

$$\begin{aligned}
\|(\tilde{V}\rho(a) - \theta_n(a)\tilde{V})\xi\|^2 &= \|f \otimes V\rho(a)\xi - \theta_n(a)(f \otimes V\xi)\|^2 \\
&= \frac{1}{\mu_n(U)} \int_U \|(V\rho(a) - \pi(a)V)\xi\|^2 \, d\mu_n(\pi) \\
&\leq \frac{1}{\mu_n(U)} \int_U \varepsilon^2 \, d\mu_n(\pi) \\
&= \varepsilon^2,
\end{aligned}$$

so that $\|\tilde{V}\rho(a) - \theta_n(a)\tilde{V}\| < \varepsilon$. We conclude that $\rho \prec \theta$, as desired. $\square$

We now come to the main theorems of this section.

Theorem 3.7. *Suppose that no point in $\widehat{A}_{\mathrm{fin}}$ is isolated in $\widehat{A}$. Let $\mathscr{H}$ be a separable Hilbert space. Then* $\mathrm{WM}(A, \mathscr{H})$ *is a dense G_δ in* $\mathrm{Rep}(A, \mathscr{H})$.

Proof. By Lemma 3.5 it suffices to show the density of WM$(A, \mathcal{H})$. Let $\pi \in \text{Rep}(A, \mathcal{H})$. By a maximality argument involving the collection of direct sums of finite-dimensional subrepresentations of π, we can write $\pi = \pi_0 \oplus \bigoplus_{i \in I} \pi_i$ where π_0 is weakly mixing and π_i is finite-dimensional for every $i \in I$. By decomposing further we may assume that π_i is irreducible for each $i \in I$. By Lemma 3.6, for each $i \in I$ we can find a weakly mixing representation θ_i on a separable Hilbert space such that $\pi_i \prec \theta_i$. Set $\rho = \pi_0 \oplus \bigoplus_{i \in I} \theta_i$. Then $\pi \prec \rho$, and ρ acts on a separable Hilbert space since I is countable by the separability of $\mathcal{H}$. It follows by Lemma 3.4 that π belongs to the closure of the set of all $\kappa \in \text{Rep}(G, \mathcal{H})$ such that $\kappa \cong \rho$, and hence to the closure of WM$(A, \mathcal{H})$, yielding the desired density. $\square$

Theorem 3.8. *Suppose that $\widehat{A}_{\text{fin}} \neq \emptyset$ and each point in $\widehat{A}_{\text{fin}}$ is isolated in $\widehat{A}$. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Then* WM$(A, \mathcal{H})$ *is closed and nowhere dense in* Rep$(A, \mathcal{H})$.

Proof. By assumption there exists a $\rho \in \widehat{A}_{\text{fin}}$. Let $\pi \in \text{Rep}(A, \mathcal{H})$. Then clearly $\pi \prec \pi \oplus \rho$ and so by Lemma 3.4 the representation π belongs to the closure of the set of all $\kappa \in \text{Rep}(G, \mathcal{H})$ such that $\kappa \cong \pi \oplus \rho$, showing that the complement of WM$(A, \mathcal{H})$ is dense in Rep$(A, \mathcal{H})$.

Now let π be a representation in Rep$(A, \mathcal{H})$ which is not weakly mixing. Then we can write $\pi = \pi_0 \oplus \pi_1$ where π_1 is finite-dimensional, and we may assume that π_1 is irreducible. Now suppose that $\{\rho_n\}$ is a sequence in Rep$(A, \mathcal{H})$ converging to π and set $\rho = \bigoplus_{n=1}^{\infty} \rho_n$. Then $\pi \prec \rho$ and hence $\pi_1 \prec \rho$, which implies that π_1 is a subrepresentation of ρ since π_1 is isolated in $\widehat{A}_{\text{fin}}$ ([35], Theorem 1.7). Since π_1 is irreducible, there must exist an $n \in \mathbb{N}$ such that π_1 is a subrepresentation of ρ_n. We deduce from this that π has a neighbourhood in Rep$(A, \mathcal{H})$ which does not intersect WM$(A, \mathcal{H})$. This shows that the complement of WM$(A, \mathcal{H})$ is open and hence completes the proof. $\square$

4. Weak Mixing and Property (T) for Quantum Groups

We now return to the context of quantum groups and discuss the notions of weak mixing and property (T). Let G be a locally compact quantum group.

Definition 4.1. A unitary representation $U \in M(C_0(G) \otimes \mathcal{K}(\mathcal{H}))$ of G is *weakly mixing* if it contains no nonzero finite-dimensional subrepresentation.

Since finite-dimensionality is preserved under the canonical correspondence between unitary representations of G and nondegenerate representations of $C_0^{\text{u}}(\widehat{G})$, by Proposition 3.2 we obtain the following.

Proposition 4.2. *Let $U \in M(C_0(G) \otimes \mathcal{K}(\mathcal{H}))$ be a unitary representation, let π_U be the corresponding representation of $C_0^{\text{u}}(\widehat{G})$, and let π_U^+ be the canonical extension of π_U to a unital representation of the unitization $C_0^{\text{u}}(\widehat{G})^+$. Then U is weakly mixing if and only if the restriction of π_U^+ to the unitary group of $C_0^{\text{u}}(\widehat{G})^+$ is weakly mixing.*

Let $\mathcal{H}$ be a fixed Hilbert space. Write Rep$(G, \mathcal{H})$ for the collection of all unitary representations of G on $\mathcal{H}$, and equip it with the point-strict topology it inherits as a subset of $M(C_0(G) \otimes \mathcal{K}(\mathcal{H}))$. Let WM$(G, \mathcal{H}) \subseteq \text{Rep}(G, \mathcal{H})$ be the subcollection of all weakly mixing representations. Write Rep$(C_0^{\text{u}}(\widehat{G}), \mathcal{H})$ for the collection of nondegenerate representations of $C_0^{\text{u}}(\widehat{G})$ on $\mathcal{H}$ (which is consistent with our notation in Sect. 3 for unital C*-algebras). In Proposition 5.1 of [11] it is shown that the topology

$\mathscr{T}$ induced on $\mathrm{Rep}(C_0^{\mathrm{u}}(\widehat{G}), \mathscr{H})$ under the canonical bijection between $\mathrm{Rep}(G, \mathscr{H})$ and $\mathrm{Rep}(C_0^{\mathrm{u}}(\widehat{G}), \mathscr{H})$ is the point-strict topology. Since the strict topology and the *-strong operator topology coincide on bounded subsets of $\mathscr{B}(\mathscr{H})$, this is the same as the point-*-strong operator topology on $\mathrm{Rep}(C_0^{\mathrm{u}}(\widehat{G}), \mathscr{H})$. Since the *-strong operator topology and the strong operator topology agree on the unitary group of $\mathscr{B}(\mathscr{H})$, and a unital C*-algebra is spanned by its unitaries, we therefore have $\pi_n \to \pi$ in the topology $\mathscr{T}$ on $\mathrm{Rep}(C_0^{\mathrm{u}}(\widehat{G}), \mathscr{H})$ if and only if $\pi_n^+ \to \pi^+$ in the point-strong operator topology on $\mathrm{Rep}(C_0^{\mathrm{u}}(\widehat{G})^+, \mathscr{H})$, where π_n^+ and π^+ are the canonical unital extensions of π_n and π to the unitization $C_0^{\mathrm{u}}(\widehat{G})^+$. Combining these observations with Lemma 3.5, we obtain:

Proposition 4.3. *Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space. Then* WM $(G, \mathscr{H})$ *is a G_δ in* $\mathrm{Rep}(G, \mathscr{H})$.

Next we recall the definition of property (T).

Definition 4.4. Let G be a locally compact quantum group and let $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ be a unitary representation of G. A vector $\xi \in \mathscr{H}$ is said to be *invariant* for U if $U(\eta \otimes \xi) = \eta \otimes \xi$ for all $\eta \in L^2(G)$. We say that U *has almost invariant vectors* if there is a net $\{\xi_i\}_i$ of unit vectors in $\mathscr{H}$ such that

$$\|U(\eta \otimes \xi_i) - \eta \otimes \xi_i\| \to 0$$

for all $\eta \in L^2(G)$, which by Proposition 3.7 of [11] is equivalent to

$$\|\pi_U(a)\xi_i - \hat{\varepsilon}_{\mathrm{u}}(a)\xi_i\| \to 0$$

for all $a \in C_0^{\mathrm{u}}(\widehat{G})$.

Definition 4.5. A locally compact quantum group G has *property (T)* if every unitary representation of G having almost invariant vectors has a nonzero invariant vector.

In order to establish the two main results of this section, Theorems 4.8 and 4.9, we require a characterization of property (T) in terms of the isolation of finite-dimensional representations in the spectrum. Specifically, we will need the equivalence of (i) and (iv) in Theorem 4.7 below. For locally compact groups the equivalence between (i), (iii), and (iv) in Theorem 4.7 is due to Wang [35], and is known more generally for locally compact quantum groups with trivial scaling group, although it does not seem to be explicitly stated in this generality in the literature (see Remark 5.3 of [26] and Sect. 3 of [8]). For quantum groups the idea is to adapt the argument of Bekka, de la Harpe, and Valette for groups in Sect. 1.2 of [5]. This requires Lemma 4.6, which generalizes a well known fact for locally compact groups. The discrete case of Lemma 4.6 appears in Sect. 2.5 of [26], although the proof there works more generally, as observed in Proposition 7.2 of [10]. See also Sect. 3 of [8] and the first paragraph of the proof of Proposition 3.5 in [34].

Lemma 4.6. *Let U and V be finite-dimensional unitary representations of a second countable locally compact quantum group G with trivial scaling group. Then $1_G \leq U \odot \bar{V}$ if and only if U and V contain a common nonzero subrepresentation.*

Armed with Lemma 4.6, one can now establish the following result by repeating mutatis mutandis the argument in Sect. 1.2 of [5], as was done in Sect. 3 of [8] (the results in [8] were only stated for G of Kac type, but their arguments readily generalize to our present context).

Theorem 4.7. *For a second countable locally compact quantum group G with trivial scaling group the following are equivalent:*

(i) *G has property (T),*
(ii) *$\hat{\varepsilon}_u$ is isolated in the spectrum of $C_0^u(\widehat{G})$,*
(iii) *every finite-dimensional representation in the spectrum of $C_0^u(\widehat{G})$ is isolated,*
(iv) *there exists a finite-dimensional representation in the spectrum of $C_0^u(\widehat{G})$ which is isolated.*

Theorem 4.8. *A second countable locally compact quantum group G with trivial scaling group has property (T) if and only if every weakly mixing unitary representation of G fails to have almost invariant vectors.*

Proof. For the nontrivial direction, if G does not have property (T) then by Theorem 4.7 and Lemma 3.6 there is a weakly mixing unitary representation of G with almost invariant vectors. ☐

Theorem 4.9. *Let G be a second countable locally compact quantum group with trivial scaling group. Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space. If G does not have property (T) then $\mathrm{WM}(G, \mathscr{H})$ is a dense G_δ in $\mathrm{Rep}(G, \mathscr{H})$, while if G has property (T) then $\mathrm{WM}(G, \mathscr{H})$ is closed and nowhere dense in $\mathrm{Rep}(G, \mathscr{H})$.*

Proof. Apply Theorem 4.7 in conjunction with Theorems 3.7 and 3.8. ☐

5. Property (T) and Strongly Ergodic Actions

We establish in Theorem 5.1 a quantum group version of a result of Connes and Weiss [9] for countable discrete groups. It was verified by Daws, Skalski, and Viselter under the additional hypothesis that the quantum group is discrete and has low dual ([12], Theorem 9.3). In fact to obtain the conclusion we can simply apply the argument of Daws, Skalski, and Viselter by replacing their Theorem 7.3 with our Theorem 4.8, or rather a slight strengthening of the latter in line with Remark 7.4 of [12], as we explain below.

An *n.s.p. (normal-state-preserving) action* $G \curvearrowright^\alpha (N, \sigma)$ is a normal injective unital *-homomorphism $\alpha : N \to L^\infty(G) \overline{\otimes} N$, where N is a von Neumann algebra with a faithful normal state σ, such that $(\mathrm{id} \otimes \alpha)\alpha = (\Delta \otimes \mathrm{id})\alpha$ and $(\mathrm{id} \otimes \sigma)\alpha(x) = \sigma(x)1$ for all $x \in N$. We drop the symbol α if we don't need to refer to it explicitly. The action α is *ergodic* if the only elements $x \in N$ satisfying $\alpha(x) = 1 \otimes x$ are the scalar multiples of the identity.

Let $G \curvearrowright^\alpha (N, \sigma)$ be an n.s.p. action. A bounded net $\{x_i\} \subset N$ is said to be *asymptotically invariant* if for every normal state $\omega \in L^1(G)$ we have $(\omega \otimes \mathrm{id})(\alpha(x_i)) - x_i \to 0$ strongly, and *trivial* if $x_i - \sigma(x_i)1 \to 0$ strongly. The action is said to be *strongly ergodic* if every asymptotically invariant net is trivial.

Recall that R denotes the unitary antipode of G, which acts on $\mathscr{B}(L^2(G))$ as $R(x) = J_R x^* J_R$ where J_R is the modular conjugation associated to the left Haar weight on $L^\infty(G)$. We say that a unitary representation $U \in M(C_0(G) \otimes \mathscr{K}(\mathscr{H}))$ of G is *self-conjugate* (referred to as condition $\mathscr{R}$ in [12]) if there exists an anti-unitary operator $J : \mathscr{H} \to \mathscr{H}$ such that the anti-isomorphism $j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ given by $x \mapsto J x^* J^*$ satisfies $(R \otimes j)(U) = U$.

Let U be a unitary representation of G on a Hilbert space $\mathscr{H}$. Recall that the conjugate representation $\overline{U}$ on the conjugate Hilbert space $\overline{\mathscr{H}}$ is defined as $(R \otimes T)(U)$ where

$T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\overline{\mathcal{H}})$ is the map given by $T(b)\overline{\xi} = \overline{b^*\xi}$. Let $V = U \odot \overline{U}$ be the tensor product of U and $\overline{U}$. Write J for the anti-unitary operator on $\mathcal{H} \otimes \overline{\mathcal{H}}$ given on elementary tensors by $J(\xi \otimes \overline{\zeta}) = \zeta \otimes \overline{\xi}$. and let $j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the anti-isomorphism given by $x \mapsto Jx^*J^*$. Let $y = \sum_{i \in I} a_i \otimes b_i$ be a finite sum of elementary tensors in $L^\infty(G) \otimes \mathcal{B}(\mathcal{H})$. Since $R(a_i R(a_j)) = a_j R(a_i)$ and $j(b_i \otimes T(b_j)) = b_j \otimes T(b_i)$ for all $i, j \in I$, it follows that the element

$$x = y_{12}[(R \otimes T)y]_{13} = \sum_{i,j \in I} a_i R(a_j) \otimes b_i \otimes T(b_j) \in L^\infty(G) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\overline{\mathcal{H}})$$

satisfies

$$(R \otimes j)(x) = x.$$

Since the maps $R \otimes j$ and $R \otimes T$ are *-strongly continuous, multiplication is *-strongly continuous on the unit ball of $\mathcal{B}(L^2(G) \otimes \mathcal{H} \otimes \overline{\mathcal{H}})$, and U is a *-strong limit of operators of norm at most one in $L^\infty(G) \otimes \mathcal{B}(\mathcal{H})$ by Kaplansky density, we conclude that $(R \otimes j)(V) = V$, so that V is self-conjugate.

Now if U has almost invariant vectors then so does V, as is easily seen, and if U is weakly mixing and G has trivial scaling group then V is weakly mixing by Theorem 3.11 of [34]. It thus follows from Theorem 4.8 that if G has trivial scaling group and does not have property (T) then it admits a weakly mixing self-conjugate unitary representation with almost invariant vectors.

This extra self-conjugacy condition is needed in the argument of Daws–Skalski–Viselter, who apply it in Lemma 9.2 of [12] so as to permit the use of Vaes's construction of actions on the free Araki–Wood factors from [33]. Now that we also have it in our more general setting, we can apply the argument in Sect. 9 of [12] to deduce the following theorem. Note that the assumption of trivial scaling group is not merely required for the application of Theorem 3.11 of [34] in the previous paragraph, but is also a hypothesis in Lemma 9.2 of [12]. Here $(\mathscr{L}F_\infty, \tau)$ is the von Neumann algebra of the free group on a countably infinite set of generators along with its canonical tracial state.

Theorem 5.1. *For a second countable locally compact quantum group G with trivial scaling group the following are equivalent:*

  (i) *G has property (T),*
 (ii) *every weakly mixing n.s.p. action $G \curvearrowright (N, \sigma)$ is strongly ergodic,*
(iii) *every ergodic n.s.p. action $G \curvearrowright (N, \sigma)$ is strongly ergodic.*

One can also replace (N, σ) with the fixed pair $(\mathscr{L}F_\infty, \tau)$ in (ii) and (iii).

6. The General Nonunimodular Case

In this final section, our aim is to prove Theorem 6.3, which extends Theorems 4.8 and 4.9 so as to cover the general nonunimodular case. It shows in particular that a nonunimodular second countable locally compact quantum group cannot have property (T), which in the discrete situation was established in [14]. As one of the referees has pointed out, this lack of property (T) can also be seen by viewing the modular element δ below as generating a morphism from G to $\mathbb{R}$ (see Sect. 1.4 of [12]) and then applying Theorem 4.7, along with the appendix, in [12].

Let G be a second countable locally compact quantum group. Recall that the modular element of G is strictly positive unbounded operator δ on $L^2(G)$ affiliated with the

von Neumann algebra $L^\infty(G)$. The map $t \mapsto \delta^{it}$ from $\mathbb{R}$ to $L^\infty(G) \subseteq \mathscr{B}(L^2(G))$ is continuous in the strong operator topology by Stone's theorem and so it defines a unitary operator U on $L^2(\mathbb{R}, L^2(G)) \cong L^2(\mathbb{R}) \otimes L^2(G)$ which, when viewing $L^2(\mathbb{R}, L^2(G))$ as the Hilbert space direct integral of copies of $L^2(G)$ over $\mathbb{R}$ with respect to Lebesgue measure, is a decomposable operator and as such is expressed by the direct integral $\int_{\mathbb{R}}^{\oplus} \delta^{it} \, dt$. Thus for all $\eta_1, \eta_2 \in L^2(G)$ and $\xi_1, \xi_2 \in L^2(\mathbb{R})$ we have

$$\langle U(\eta_1 \otimes \xi_1), \eta_2 \otimes \xi_2 \rangle = \int_{\mathbb{R}} \langle \xi_1(t) \delta^{it} \eta_1, \xi_2(t) \eta_2 \rangle \, dt$$

$$= \int_{\mathbb{R}} \xi_1(t) \overline{\xi_2(t)} \langle \delta^{it} \eta_1, \eta_2 \rangle \, dt. \qquad (1)$$

Since $\Delta(\delta^{it}) = \delta^{it} \otimes \delta^{it}$ (see for example the proof of Proposition 1.9.11 in [32]), we see that $(\Delta \otimes \mathrm{id})U$ and $U_{13}U_{12}$ are both decomposable operators on $L^2(\mathbb{R}, L^2(G) \otimes L^2(G)) \cong L^2(G) \otimes L^2(G) \otimes L^2(\mathbb{R})$ which can be expressed as the direct integral $\int_{\mathbb{R}}^{\oplus} \delta^{it} \otimes \delta^{it} \, dt$. Thus U is a unitary representation of G on $L^2(\mathbb{R})$. (In fact, $U \in L^\infty(G) \overline{\otimes} L^\infty(\mathbb{R})$ can be regarded as a unitary representation of both G and $\mathbb{R}$ simultaneously.)

Lemma 6.1. *The unitary representation U of G has almost invariant vectors.*

Proof. For every $n \in \mathbb{N}$, writing $\mathbf{1}_{[0, \frac{1}{n}]}$ for the indicator function of $[0, \frac{1}{n}]$ we set $\xi_n = \sqrt{n} \mathbf{1}_{[0, \frac{1}{n}]}$, which is a unit vector in $L^2(\mathbb{R})$. Let $\eta \in L^2(G)$ be a unit vector. Using the formula (1), for every n we have

$$|\langle U(\eta \otimes \xi_n), \eta \otimes \xi_n \rangle - 1| = n \int_{[0, \frac{1}{n}]} \langle (\delta^{it} - 1)\eta, \eta \rangle \, dt$$

and the expression on the right converges to zero as $n \to \infty$ by the strong operator continuity of the map $t \mapsto \delta^{it}$. Thus U has almost invariant vectors. $\quad\square$

Let V be any unitary representation of G on a separable Hilbert space $\mathscr{H}$, and consider the tensor product representation $Z = V \odot U = V_{12}U_{13}$. Viewing $L^2(\mathbb{R}, L^2(G) \otimes \mathscr{H}) \cong L^2(G) \otimes \mathscr{H} \otimes L^2(\mathbb{R})$ as the Hilbert space direct integral of copies of $L^2(G) \otimes \mathscr{H}$ over $\mathbb{R}$ with respect to Lebesgue measure, the operator V_{12} is decomposable and can expressed by the direct integral $\int_{\mathbb{R}}^{\oplus} V \, dt$, so that for $\eta \in L^2(G)$ and $\zeta \in L^2(\mathbb{R}, \mathscr{H}) \cong \mathscr{H} \otimes L^2(\mathbb{R})$ the vector $V_{12}^*(\eta \otimes \zeta)$, viewed as an element of $L^2(\mathbb{R}, L^2(G) \otimes \mathscr{H})$, is equal to $t \mapsto V^*(\eta \otimes \zeta(t))$. Thus for $\eta_1, \eta_2 \in L^2(G)$ and $\zeta_1, \zeta_2 \in L^2(\mathbb{R}, \mathscr{H}) \cong \mathscr{H} \otimes L^2(\mathbb{R})$ we have the formula

$$\langle Z(\eta_1 \otimes \zeta_1), \eta_2 \otimes \zeta_2 \rangle = \langle U_{13}(\eta_1 \otimes \zeta_1)), V_{12}^*(\eta_2 \otimes \zeta_2) \rangle$$

$$= \int_{\mathbb{R}} \langle \delta^{it} \eta_1 \otimes \zeta_1(t), V^*(\eta_2 \otimes \zeta_2(t)) \rangle \, dt. \qquad (2)$$

Lemma 6.2. *Suppose that G is not unimodular. Let V be any unitary representation of G on a separable Hilbert space $\mathscr{H}$. Then the tensor product representation $Z = V \odot U$ is weakly mixing. Moreover, if $\mathscr{H}$ is infinite-dimensional then the closure of the set of unitary conjugates of Z in $\mathrm{Rep}(G, \mathscr{H})$ contains V.*

Proof. Since G is not unimodular, there is a real number $c \neq 0$ such that e^c belongs to the spectrum of the modular element δ. The equation $\Delta(\delta) = \delta \otimes \delta$ then implies, via elementary spectral theory, that e^{nc} belongs to the spectrum of δ for every $n \in \mathbb{N}$.

Let $\mathcal{E}$ be a nonzero finite-dimensional subspace of $L^2(\mathbb{R}) \otimes \mathcal{H}$. Choose an orthonormal basis $\{\zeta_1, \ldots, \zeta_K\}$ for $\mathcal{E}$. Let $\varepsilon > 0$. Then there are a $b > 0$ and an $N \in \mathbb{N}$ such that for each $k = 1, \ldots, K$ we can find a $\zeta_k' \in L^2([-b, b], \mathcal{H}) \cong L^2([-b, b]) \otimes \mathcal{H}$ which is an $\mathcal{H}$-valued step function on $[-b, b]$ taking at most N different values and satisfying $\|\zeta_k' - \zeta_k\| < \varepsilon/3$. Set $M = \max_{k=1,\ldots,K} \max_{t \in [-b,b]} \|\zeta_k'(t)\|$. By the proof of the Riemann–Lebesgue lemma for step functions, there is an $n \in \mathbb{N}$ depending only on N and M such that, setting $s = nc$, every step function $f : [-b, b] \to \mathbb{C}$ which takes at most N^2 values and is bounded in modulus by M^2 satisfies

$$\left| \int_{[-b,b]} f(t) e^{ist} \, dt \right| < \frac{\varepsilon}{6} \tag{3}$$

Write $\mathcal{K}$ for the range of the spectral projection of δ corresponding to $[0, 2e^s]$. Since e^s belongs to the spectrum of δ, we can find a norm-one vector $\eta \in \mathcal{K}$ such that $\|\delta\eta - e^s\eta\|$ is small enough so that by the continuous functional calculus, applied to δ acting (boundedly) on $\mathcal{K}$, we have $\|\delta^{it}\eta - e^{ist}\eta\| < \varepsilon/(12bM^2)$ for all $t \in [-b, b]$. For every norm-one vector $\theta \in L^2(G)$ and $k = 1, \ldots, n$ the function $t \mapsto \langle \eta \otimes \zeta_1'(t), V^*(\theta \otimes \zeta_k'(t))\rangle$ on $[-b, b]$ is a step function which takes at most N^2 values and is bounded in modulus by M^2, and so using the formula (2) and applying (3) we obtain

$$|\langle Z(\eta \otimes \zeta_1'), \theta \otimes \zeta_k'\rangle| = \left| \int_{[-b,b]} \langle \delta^{it}\eta \otimes \zeta_1'(t), V^*(\theta \otimes \zeta_k'(t))\rangle \, dt \right|$$

$$\leq \left| \int_{[-b,b]} \langle e^{ist}\eta \otimes \zeta_1'(t), V^*(\theta \otimes \zeta_k'(t))\rangle \, dt \right|$$

$$+ \left| \int_{[-b,b]} \langle (\delta^{it}\eta - e^{ist}\eta) \otimes \zeta_1'(t), V^*(\theta \otimes \zeta_k'(t))\rangle \, dt \right|$$

$$\leq \left| \int_{[-b,b]} \langle \eta \otimes \zeta_1'(t), V^*(\theta \otimes \zeta_k'(t))\rangle e^{ist} \, dt \right|$$

$$+ 2b \cdot \sup_{t \in [-b,b]} \|\delta^{it}\eta - e^{ist}\eta\| \|\zeta_1'(t)\| \|\zeta_k'(t)\|$$

$$< \frac{\varepsilon}{6} + 2b \cdot \frac{\varepsilon}{12bM^2} \cdot M^2$$

$$= \frac{\varepsilon}{3}$$

and hence

$$|\langle Z(\eta \otimes \zeta_1), \theta \otimes \zeta_k\rangle| \leq |\langle Z(\eta \otimes \zeta_1'), \theta \otimes \zeta_k'\rangle| + \|\zeta_1 - \zeta_1'\| + \|\zeta_k - \zeta_k'\|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \tag{4}$$

Now given a norm-one vector $\kappa \in L^2(G) \otimes \mathcal{E}$ we can write it as $\sum_{k=1}^{K} \theta_k \otimes \zeta_k$ where $\sum_{k=1}^{K} \|\theta_k\|^2 = 1$, and so if we take $\varepsilon = 1/K$ then from (4) we get

$$|\langle Z(\eta \otimes \zeta_1), \kappa\rangle| \leq \sum_{k=1}^{K} |\langle Z(\eta \otimes \zeta_1), \theta_k \otimes \zeta_k\rangle| < \sum_{k=1}^{K} \frac{1}{n} \|\theta_k\| \leq K\varepsilon = 1.$$

Since the vector $Z(\eta \otimes \zeta_1)$ has norm one by the unitarity of Z, it follows that $Z(\eta \otimes \zeta_1) \notin L^2(G) \otimes \mathscr{E}$, which shows that $L^2(G) \otimes \mathscr{E}$ is not Z-invariant. We conclude that Z has no nonzero finite-dimensional subrepresentations.

Suppose now that $\mathscr{H}$ is infinite-dimensional and let us show that the closure of the set of unitary conjugates of Z in $\text{Rep}(G, \mathscr{H})$ contains V. The argument is similar to that in the proof of Lemma 5.3 in [11], whose antecedents in the group setting can be found in [6,22]. Fix an orthonormal basis $\{\zeta_k\}_{k=1}^{\infty}$ of $\mathscr{H}$. Let Ω be a finite set of norm-one elements in $C_0(G)$. Let $K \in \mathbb{N}$ and $\varepsilon > 0$. Working in the Hilbert module $C_0(G) \otimes \mathscr{H}$, there is an integer $L \geq K$ such that for each $a \in \Omega$ and $k = 1, \ldots, K$ we can find $x_{a,k,1}, \ldots, x_{a,k,L}, y_{a,k,1}, \ldots, y_{a,k,L} \in \mathscr{H}$ satisfying $\|\mathbf{V}(a \otimes \zeta_k) - \sum_{l=1}^{L} x_{a,k,l} \otimes \zeta_l\| < \varepsilon/3$ and $\|\mathbf{V}^*(a \otimes \zeta_k) - \sum_{l=1}^{L} y_{a,k,l} \otimes \zeta_l\| < \varepsilon/3$. Using the characterization of having almost invariant vectors given in Proposition 3.7(v) of [11], Lemma 6.1 yields a unit vector $\xi \in \mathscr{H}$ such that $\|\mathbf{U}(a \otimes \xi) - a \otimes \xi\| < \varepsilon/3$ for all $a \in \Omega$. Since $\mathscr{H}$ is separable and infinite-dimensional we can find a unitary isomorphism $u : \mathscr{H} \otimes L^2(\mathbb{R}) \to \mathscr{H}$ which sends $\zeta_k \otimes \xi$ to ζ_k for every $k = 1, \ldots, L$. Then for every $a \in \Omega$ and $k = 1, \ldots, K$ we have

$$\begin{aligned}
&\|((\text{id} \otimes u)\mathbf{Z}(\text{id} \otimes u)^{-1} - \mathbf{V})(a \otimes \zeta_k)\| \\
&\leq \|(\text{id} \otimes u)\mathbf{V}_{12}(\mathbf{U}_{13}(a \otimes \zeta_k \otimes \xi) - a \otimes \zeta_k \otimes \xi)\| \\
&\quad + \left\|(\text{id} \otimes u)\left(\mathbf{V}_{12}(a \otimes \zeta_k \otimes \xi) - \sum_{l=1}^{L} x_{a,k,l} \otimes \zeta_l \otimes \xi\right)\right\| \\
&\quad + \left\|\sum_{l=1}^{L} x_{a,k,l} \otimes \zeta_l - \mathbf{V}(a \otimes \zeta_k)\right\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

and similarly $\|((\text{id} \otimes u)\mathbf{Z}^*(\text{id} \otimes u)^{-1} - \mathbf{V}^*)(a \otimes \zeta_k)\| < \varepsilon$. We conclude that V belongs to the closure of the set of unitary conjugates of Z in $\text{Rep}(G, \mathscr{H})$. $\quad\square$

We can now conclude with the main result of the section.

Theorem 6.3. *Let G be a second countable nonunimodular locally compact quantum group. Then there exists a weakly mixing unitary representation of G which has almost invariant vectors. Moreover, if $\mathscr{H}$ is a separable infinite-dimensional Hilbert space then $\text{WM}(G, \mathscr{H})$ is a dense G_δ in $\text{Rep}(G, \mathscr{H})$.*

Proof. The representation U has almost invariant vectors by Lemma 6.1, and it is weakly mixing by Lemma 6.2, as we can take V there to be the trivial representation.

Finally, if $\mathscr{H}$ is a separable infinite-dimensional Hilbert space then $\text{WM}(G, \mathscr{H})$ is a dense G_δ in $\text{Rep}(G, \mathscr{H})$ by Lemmas 3.5 and 6.2. $\quad\square$

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