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Tat-Leung Yee and Kwok-Pun Ho*

Zygmund inequality of the conjugate function on Morrey-Zygmund spaces

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Abstract: We generalize the Zygmund inequality for the conjugate function to the Morrey type spaces defined on the unit circle $\mathbb{T}$. We obtain this extended Zygmund inequality by introducing the Morrey-Zygmund space on $\mathbb{T}$.

Keywords: Zygmund inequality, Zygmund spaces, Morrey spaces, conjugate function

MSC: 42B20, 42B35, 46E30

1 Introduction

This paper aims to extend the celebrated Zygmund inequality to the Morrey-Zygmund space on the unit circle $\mathbb{T} = \{e^{i\theta} : -\pi < \theta \leqslant \pi\}$. The classical Zygmund inequality gives the borderline behavior of the conjugate function operator on L^1. It shows that the conjugate function operator is a bounded linear mapping from the Zygmund space $L \log L$ to L^1 [1]. The importance of the conjugate function operator stems from its role in the study of Fourier series. Roughly speaking, the boundedness of the conjugate function operator on a rearrangement-invariant Banach function space X on $\mathbb{T}$ yields the convergence of the Fourier series on a subspace of X. The reader is referred to [2, Chapter 3, Theorem 6.10] for the details and the precise statement of this result.

Since the introduction of the classical Morrey spaces on $\mathbb{R}^n$ in [3], several important results in Lebesgue spaces have been extended to Morrey spaces. These include results on the boundedness of the Hardy-Littlewood maximal operator, the singular integral operators, the fractional integral operators [4–8] and the two-weight norm inequalities [9, 10] had been extended to Morrey spaces. Inspired by the recent developments of the studies of Morrey spaces, we investigate the extension of the Zygmund inequality on Morrey spaces. Since we study the conjugate function operator, we consider the Morrey type spaces defined on the unit circle [11].

The main result of this paper establishes the boundedness of the conjugate function operator as a mapping from the Morrey spaces built on Zygmund space to Morrey spaces. The main result of this paper is also related with the results from [12]. The results in [12] consider the Hardy-Littlewood maximal function on the case $q = 1$, while we consider the Hilbert transform for Morrey spaces on $\mathbb{T}$.

This paper is organized as follows. Section 2 contains the definitions and some basic properties of the Zygmund space. The Zygmund inequalities on Morrey-Zygmund spaces are established in Section 3.

Tat-Leung Yee: Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong, China; E-mail: tlyee@eduhk.hk
***Corresponding Author: Kwok-Pun Ho:** Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong, China; E-mail: vkpho@eduhk.hk

2 Definitions and preliminaries

In this section, we present the definitions and some basic properties of the Zygmund spaces. Let $\mathbb{T}$ be the unit circle $\{e^{i\theta} : -\pi < \theta \leqslant \pi\}$ endowed with the measure $\frac{1}{2\pi}d\theta$, where $d\theta$ is the Lebesgue measure on $\mathbb{T}$. We write $f \in L^1$ if

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta < \infty.$$

Definition 2.1. *The Zygmund space $L \log L$ consists of all Lebesgue measurable functions f on $\mathbb{T}$ satisfying*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})|\, d\theta < \infty,$$

where $\log^+ x = \max(\log x, 0)$. *We endow the Zygmund space with the norm*

$$\|f\|_{L \log L} = \int_0^1 f^*(t) \log(1/t)\, dt = \int_0^1 f^{**}(t)\, dt,$$

where $f^*(t)$ is the decreasing rearrangement of f and f^{**} is the maximal function of f^* [2, Chapter 2, Definitions 1.5 and 3.1].

Let M be the Hardy-Littlewood maximal function. A celebrated result from Stein shows that if $f, M(f) \in L^1$, then $f \in L \log L$, see [2, Chapter 4, Theorem 6.7] and [13, Chapter IV, Theorem 5.4]. For the interpolation of operators of joint weak type to Zygmund spaces, the reader may consult [2, Chapter 4, Corollary 6.15] and [13, Chapter IV, Theorem 5.3].

In view of [2, Chapter 4, Theorems 6.4 and 6.5], $L \log L$ is a rearrangement-invariant Banach function space, and for any $1 < p < \infty$,

$$L^p \hookrightarrow L \log L \hookrightarrow L^1. \tag{2.1}$$

The reader is referred to [2, Chapter 2] for the definition and basic properties of rearrangement-invariant Banach function space. In particular, the reader is referred to [2, Chapter 1, Definitions 2.1 and 2.3] for the definition of the associate space of rearrangement-invariant Banach function spaces.

According to [2, Chapter 4, Theorem 6.5], the associate space of $L \log L$ is $L_{\exp}$, where $L_{\exp}$ consists of all Lebesgue measurable functions f satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\lambda |f(e^{i\theta})|)\, d\theta < \infty$$

for some $\lambda > 0$ and where $L_{\exp}$ is endowed with the norm

$$\|f\|_{L_{\exp}} = \sup_{0<t<1} \frac{f^{**}(t)}{1 + \log(1/t)}.$$

The associate space of $L_{\exp}$ is $L \log L$ and $L_{\exp}$ is also a rearrangement-invariant Banach function space. Furthermore, $L_{\exp}$ is the dual space of $L \log L$ (up to equivalence of norms). Therefore, according to the definition of associate spaces, for any $f \in L \log L$ and $g \in L_{\exp}$

$$\int |f(x)g(x)|\, dx \leqslant C\|f\|_{L \log L}\|g\|_{L_{\exp}}, \tag{2.2}$$

see [2, Chapter 1, Theorem 2.4].

In view of [2, Chapter 2, Theorem 5.2], for any Lebesgue measurable set $E \subset \mathbb{T}$, we have

$$\|\chi_E\|_{L \log L}\|\chi_E\|_{L_{\exp}} = |E|. \tag{2.3}$$

We have the following results for the norms of characteristic functions of Lebesgue measurable sets $E \subset \mathbb{T}$ in $L \log L$ and $L_{\exp}$.

Lemma 2.1. *Let E be a Lebesgue measurable set on $\mathbb{T}$. We have*

$$\|\chi_E\|_{L\log L} = |E|(1-\log|E|),$$
$$\|\chi_E\|_{L_{\exp}} = \frac{1}{1-\log|E|}.$$

Proof. For any Lebesgue measurable set E on $\mathbb{T}$, we have $(\chi_E)^* = \chi_{[0,|E|)}$. Therefore,

$$\|\chi_E\|_{L\log L} = \int_0^{|E|} \log(1/t)\,dt = -\int_0^{|E|} \log t\,dt = |E|(1-\log|E|).$$

The identity $\|\chi_E\|_{L_{\exp}} = \frac{1}{1-\log|E|}$ follows from the above result and (2.3). □

Next, we present the celebrated Zygmund inequality. We first recall the definition of the conjugate function operator from [2, Chapter 3, (6.11)]. The conjugate function operator $f \to \tilde{f}$ is defined as the principal value integral

$$\begin{aligned}\tilde{f}(e^{i\theta}) &= \frac{1}{2\pi}\lim_{\epsilon\to 0^+}\int_{\epsilon<|\phi|\leqslant\pi} f(e^{i(\theta-\phi)})\cot(\phi/2)\,d\phi\\ &= \frac{1}{2\pi}\lim_{\epsilon\to 0^+}\int_{\epsilon<|\varphi-\theta|\leqslant\pi} f(e^{i\varphi})\cot((\theta-\varphi)/2)\,d\varphi.\end{aligned}$$

The conjugate function operator can be considered as the periodic analogue of the Hilbert transform on $\mathbb{R}$.

The following is the Zygmund inequality for the conjugate function operator.

Theorem 2.2. *There exists a constant $C > 0$ such that for any $f \in L\log L$, we have*

$$\|\tilde{f}\|_{L^1} \leqslant C\|f\|_{L\log L}.$$

For the proof of the above result, the reader is referred to [2, Chapter 4, Corollary 6.8] and [1].

3 Main result

In this section, we obtain the main result of this paper. Namely, the extension of the Zygmund inequality to Morrey-Zygmund spaces.

Let $\beta, t \in [-\pi,\pi]$, and write $I(\beta,t) = \{e^{i\theta} : \beta - t < \theta < \beta + t\}$ and $\mathbb{I} = \{I(\beta,t) : \beta, t \in (-\pi,\pi]\}$. Note that $\mathbb{T} \in \mathbb{I}$. For any $j \in \mathbb{N}$ and $I = I(\beta,t) \in \mathbb{I}\backslash\{\mathbb{T}\}$, write $2^j I = I(\beta, 2^j t)$. Let $N_I \in \mathbb{N}$ be the smallest positive integer such that $2^{N_I} I = \mathbb{T}$. We now give the definition of the Morrey-Zygmund space on $\mathbb{T}$.

Definition 3.1. *Let $u : \mathbb{I} \to (0,\infty)$. The Morrey-Zygmund space $M^u_{L\log L}$ consists of all Lebesgue measurable functions f satisfying*

$$\|f\|_{M^u_{L\log L}} = \sup_{I\in\mathbb{I}} \frac{1}{u(I)}\|\chi_I f\|_{L\log L} < \infty.$$

The Morrey space M^u_1 consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M^u_1} = \sup_{I\in\mathbb{I}} \frac{1}{u(I)}\|\chi_I f\|_{L^1} < \infty.$$

The above definition is related to the generalized Morrey spaces of the third kind given in [14]. In view of (2.1), we have $M^u_{L\log L} \hookrightarrow M^u_1$.

When we replace the L^1 norm from the above definition by the L^p norm with $1 < p < \infty$ and take $u(I) = |I|^{\frac{\lambda}{p}}$ with $0 < \lambda < 1$, we have the Morrey spaces on $\mathbb{T}$ studied in [11]. In addition, for the duality theory of Morrey spaces on $\mathbb{T}$, the reader may consult [11].

Since $\mathbb{T} \in \mathbb{I}$, the above definition yields

$$\frac{1}{u(\mathbb{T})}\|f\|_{L\log L} \leqslant \sup_{I\in\mathbb{I}} \frac{1}{u(I)}\|\chi_I f\|_{L\log L} = \|f\|_{M^u_{L\log L}}, \quad f \in M^u_{L\log L}$$
$$\frac{1}{u(\mathbb{T})}\|f\|_{L^1} \leqslant \sup_{I\in\mathbb{I}} \frac{1}{u(I)}\|\chi_I f\|_{L^1} = \|f\|_{M^u_1}, \quad f \in M^u_1.$$

Therefore,

$$\|f\|_{L\log L} \leqslant u(\mathbb{T})\|f\|_{M^u_{L\log L}}, \quad f \in M^u_{L\log L}$$
$$\|f\|_{L^1} \leqslant u(\mathbb{T})\|f\|_{M^u_1}, \quad f \in M^u_1$$

and the embedding constants are $u(\mathbb{T})$. Consequently,

$$M^u_{L\log L} \hookrightarrow L\log L, \quad \text{and} \quad M^u_1 \hookrightarrow L^1. \tag{3.1}$$

The above embeddings are not necessarily valid for the Morrey spaces and the Morrey-Zygmund spaces defined on $\mathbb{R}^n$. They show the major difference between the Morrey spaces and the Morrey-Zygmund spaces on $\mathbb{T}$ and $\mathbb{R}^n$.

The following proposition gives a condition on u which guarantees that $M^u_{L\log L}$ is nontrivial.

Proposition 3.1. *Let $\omega : (0, 1] \to (0, \infty)$ be a Lebesgue measurable function and $u(I) = \omega(|I|)$. If*

$$\sup_{0<t\leqslant 1} \frac{t(1-\log t)}{\omega(t)} < \infty, \tag{3.2}$$

then for any $I \in \mathbb{I}$, $\chi_I \in M^u_{L\log L}$.

Proof. It suffices to verify that $\chi_{\mathbb{T}} \in M^u_{L\log L}$. Then (3.2) yields

$$\|\chi_{\mathbb{T}}\|_{M^u_{L\log L}} = \sup_{J\in\mathbb{I}} \frac{1}{u(J)}\|\chi_J\|_{L\log L} = \sup_{0<t\leqslant 1} \frac{t(1-\log t)}{\omega(t)} < \infty.$$

Therefore, $\chi_{\mathbb{T}} \in M^u_{L\log L}$. □

In particular, if ω is continuous and $\omega(t) > 0$ when $t > 0$, then (3.2) can be relaxed to

$$\lim_{t\to 0^+} \frac{t(1-\log t)}{\omega(t)} < \infty \tag{3.3}$$

because (3.3) guarantees that $\frac{t(1-\log t)}{\omega(t)}$ is a continuous function on $[0, 1]$. Thus, the condition $\sup_{0<t\leqslant 1} \frac{t(1-\log t)}{\omega(t)} < \infty$ is fulfilled. For example, the continuous function $\omega(t) = t^\lambda$, where $0 < \lambda < 1$, satisfies (3.3) and $\omega(t)$ when $t > 0$. Consequently, $M^{\bar u}_{L\log L}$ with $\bar u(I) = |I|^\lambda$ and $0 < \lambda < 1$ is nontrivial. The reader is referred to [15, Proposition 2.6] and [16, Lemma 3.4] for the corresponding results for Morrey type spaces on $\mathbb{R}^n$.

As $M^u_{L\log L} \subset M^u_1$, the preceding proposition also guarantees that $\mathbb{I} \subset M^u_1$ when u satisfies (3.2). When $u \equiv 1$, L^1 is identical to M^u_1. However, in general, M^u_1 is a proper subspace of L^1. Let $\bar u(I) = |I|^\lambda$ where $\frac{1}{2} < \lambda < 1$. Define $h(e^{i\theta}) = \theta^{-\frac{1}{2}}$ when $0 < \theta < \pi$ and $h(e^{i\theta}) = 0$ otherwise. We have $\log^+ h(\theta) = -\frac{1}{2}\log\theta$ where $0 < \theta < 1$ and $\log^+ h(\theta) = 0$ otherwise. By using the L'Hospital rule, we find a constant $C > 0$ such that

$$0 < -\frac{1}{2}\theta^{-\frac{1}{2}}\log\theta < C\theta^{-\frac{2}{3}}, \quad 0 < \theta < 1.$$

We have

$$\frac{1}{2\pi}\int_{\mathbb{T}} |h(\theta)|\log^+|h(\theta)|\,d\theta = -\frac{1}{4\pi}\int_0^1 \theta^{-\frac{1}{2}}\log\theta\,d\theta \leqslant C\int_0^1 \theta^{-\frac{2}{3}}\,d\theta < \infty.$$

That is, $h \in L\log L$. On the other hand, for any $0 < a < \pi$, we have

$$\|\chi_{(0,a)}h\|_{L^1} = \frac{1}{2\pi}\int_0^a \theta^{-\frac{1}{2}}\,d\theta = \frac{a^{\frac{1}{2}}}{\pi}.$$

Therefore,

$$\sup_{a\in(0,\pi)} \frac{1}{\bar{u}((0,a))}\|\chi_{(0,a)}h\|_{L^1} = \sup_{a\in(0,\pi)} \frac{a^{\frac{1}{2}-\lambda}}{\pi} = \infty.$$

That is, $h \in L\log L\backslash M_1^{\bar{u}}$.

Since $L\log L \subset L^1$ and (2.1) assures that $M^{\bar{u}}_{L\log L} \subset M_1^{\bar{u}}$, $M^{\bar{u}}_{L\log L}$ is a proper nontrivial subset of $L\log L$ and $M_1^{\bar{u}}$ is also a proper nontrivial subset of L^1. The embedding (3.1) also guarantees that the conjugate function operator is well defined on $M^u_{L\log L}$. In view of the results in [17–22], the action of the singular integral operators on Morrey type spaces on $\mathbb{R}^n$ cannot directly be defined by the principal value integral. It shows another difference between the Morrey spaces and the Morrey-Zygmund spaces on $\mathbb{T}$ and $\mathbb{R}^n$.

The reader is referred to [23] for the study of the weak type estimate for maximal commutator and commutator of maximal function on the Morrey-Zygmund spaces defined on $\mathbb{R}^n$.

We are now ready to present and establish the main result of this paper, the Zygmund inequalities on Morrey-Zygmund spaces. These inequalities give the boundedness of the conjugate function operator as a mapping from the Morrey-Zygmund space $M^u_{L\log L}$ to the Morrey space M^u_1.

Theorem 3.2. *Let $u, w : \mathbb{I} \to (0,\infty)$. If there exists a constant $C > 0$ such that for any $I \in \mathbb{I}$, u and w satisfy*

$$\sum_{i=0}^{N_I} \frac{\|\chi_I\|_{L^1}}{\|\chi_{2^iI}\|_{L\log L}} w(2^iI) \leqslant Cu(I), \tag{3.4}$$

then we have

$$\|\tilde{f}\|_{M_1^u} \leqslant C\|f\|_{M^w_{L\log L}}, \quad \text{for all } f \in M^w_{L\log L}$$

for some $C > 0$ independent of $f \in M^w_{L\log L}$.

Proof. Let $I = I(\beta, t) \in \mathbb{I}$, $\beta, t \in [-\pi,\pi]$. We consider the two cases

1. $|I| \geqslant \frac{1}{2}$,
2. $|I| < \frac{1}{2}$.

For the first case, $|I| \geqslant \frac{1}{2}$, we find that $2I = \mathbb{T}$. In view of Lemma 2.1 and (3.4), we have

$$w(\mathbb{T}) \leqslant C\frac{|\mathbb{T}|(1-\log|\mathbb{T}|)}{|I|}u(I) = C\frac{1}{|I|}u(I) \leqslant Cu(I)$$

for some $C > 0$. Consequently,

$$\frac{1}{u(I)}\|\chi_I\tilde{f}\|_{L^1} \leqslant \frac{1}{u(I)}\|\tilde{f}\|_{L^1} \leq 2\frac{1}{w(\mathbb{T})}\|\tilde{f}\|_{L^1} \leqslant 2\|f\|_{M^w_{L\log L}}.$$

Next, we consider the case $|I| < \frac{1}{2}$. As $|I| < \frac{1}{2}$, we have $N_I \geqslant 2$ and $|2I| = 2|I| < 1$. Define $f_1 = \chi_{2I}f$ and $f_j = \chi_{2^jI\backslash 2^{j-1}I}f$ where $2 \leqslant j \leqslant N_I$. Therefore, $\tilde{f} = \sum_{j=1}^{N_I}\tilde{f}_j$. We find that

$$\|\chi_I\tilde{f}_1\|_{L^1} \leqslant \|\tilde{f}_1\|_{L^1} \leqslant C\|f_1\|_{L\log L} = C\|\chi_{2I}f\|_{L\log L}. \tag{3.5}$$

According to (3.4), we have

$$\frac{w(2I)}{u(I)} \leqslant C\frac{\|\chi_{2I}\|_{L\log L}}{\|\chi_I\|_{L^1}} = C\frac{|2I|(1-\log|2I|)}{|I|} \leqslant C\frac{|2I|}{|I|} = 2C$$

because $|2I| < 1$. That is, $\frac{1}{u(I)} \leqslant C\frac{1}{w(2I)}$. Therefore, by multiplying $\frac{1}{u(I)}$ on both sides of (3.5), we get

$$\frac{1}{u(I)}\|\chi_I \tilde{f}_1\|_{L^1} \leqslant C\frac{1}{w(2I)}\|\chi_{2I}f\|_{L\log L} \leqslant C\|f\|_{M^w_{L\log L}}. \tag{3.6}$$

We consider f_j where $2 \leqslant j \leqslant N_I$. We find that for any $\theta \in I$ and $\varphi \in 2^jI\backslash 2^{j-1}I$, we have $|\theta - \varphi| > 2^{j-1}|I|$. Therefore,

$$\left|\cot\frac{\theta-\varphi}{2}\right| \leqslant \frac{1}{\left|\sin\frac{\theta-\varphi}{2}\right|} \leqslant \frac{2}{|\theta-\varphi|} \leqslant C\frac{1}{2^j|I|}$$

for some constant $C > 0$ independent of j and I.

In view of the above inequalities, we have

$$\begin{aligned}|\chi_I(\theta)\tilde{f}_j(\theta)| &\leqslant \frac{\chi_I(\theta)}{2\pi}\int_{2^jI\backslash 2^{j-1}I}|f(e^{i\varphi})\cot((\theta-\varphi)/2)|\,d\varphi\\ &\leqslant C\frac{\chi_I(\theta)}{|2^jI|}\int_{2^jI\backslash 2^{j-1}I}|f(e^{i\varphi})|\,d\varphi\\ &\leqslant C\frac{\chi_I(\theta)}{|2^jI|}\|\chi_{2^jI}f\|_{L\log L}\|\chi_{2^jI}\|_{L_{\exp}}\end{aligned}$$

where we use (2.2) to establish the last inequality.

Thus, (2.3) guarantees that

$$|\tilde{f}_j(\theta)| \leqslant C\frac{\chi_I}{\|\chi_{2^jI}\|_{L\log L}}\|\chi_{2^jI}f\|_{L\log L}.$$

By applying the norm $\|\cdot\|_{L^1}$ and multiplying $\frac{1}{u(I)}$ on both sides of the above inequality, we obtain

$$\begin{aligned}\frac{1}{u(I)}\|\chi_I\tilde{f}_j\|_{L^1} &\leqslant C\frac{\|\chi_I\|_{L^1}}{\|\chi_{2^jI}\|_{L\log L}}\frac{w(2^jI)}{u(I)}\frac{1}{w(2^jI)}\|\chi_{2^jI}f\|_{L\log L}\\ &\leqslant C\frac{\|\chi_I\|_{L^1}}{\|\chi_{2^jI}\|_{L\log L}}\frac{w(2^jI)}{u(I)}\|f\|_{M^w_{L\log L}}\end{aligned} \tag{3.7}$$

for some $C > 0$ independent of j and I.

As a result of (3.4), (3.6) and (3.7), we have

$$\begin{aligned}\frac{1}{u(I)}\|\chi_I\tilde{f}\|_{L^1} &\leqslant \sum_{j=1}^{N_I}\frac{1}{u(I)}\|\chi_I\tilde{f}_j\|_{L^1}\\ &\leqslant C\sum_{j=1}^{N_I}\frac{\|\chi_I\|_{L^1}}{\|\chi_{2^jI}\|_{L\log L}}\frac{w(2^jI)}{u(I)}\|f\|_{M^w_{L\log L}} \leqslant C\|f\|_{M^w_{L\log L}}\end{aligned}$$

for some $C > 0$ independent of $I \in \mathbb{I}$.

Finally, by taking the supremum over $I \in \mathbb{I}$, we establish

$$\|\tilde{f}\|_{M^u_1} = \sup_{I\in\mathbb{I}}\frac{1}{u(I)}\|\chi_I\tilde{f}\|_{L^1} \leqslant C\|f\|_{M^w_{L\log L}}.$$

□

The result in Theorem 3.2 sharpens the classical Zygmund inequality in the sense that when we consider the function in the subspace $M^w_{L\log L}$ of $L\log L$, the image of the conjugate function belongs to M^u_1 which is a proper subspace of L^1. The assumption (3.4) is related with [24, (1.3)].

We give some examples on u and w for which (3.4) are fulfilled. Let $0 < \lambda < 1$ and $w(I) = \|\chi_I\|^{\lambda}_{L\log L}$ and $u(I) = \|\chi_I\|^{\lambda}_{L^1}$. We have

$$\begin{aligned}\sum_{i=0}^{N_I} \frac{\|\chi_I\|_{L^1}}{\|\chi_{2^iI}\|_{L\log L}} \frac{w(2^iI)}{u(I)} &= \sum_{i=0}^{N_I} \left(\frac{\|\chi_I\|_{L^1}}{\|\chi_{2^iI}\|_{L\log L}}\right)^{1-\lambda} \\ &\leqslant \sum_{i=0}^{N_I} \left(\frac{|I|}{|2^iI|(1-\log|2^iI|)}\right)^{1-\lambda} \\ &\leqslant \sum_{i=0}^{N_I} \left(\frac{|I|}{|2^iI|}\right)^{1-\lambda} \leqslant C\end{aligned}$$

and, hence, (3.4) is satisfied.

Observe $\log|2^iI| < 0$. As such, whenever $u(I) = w(I) = |I|^{\lambda}$ (where $0 < \lambda < 1$), we get $\|\chi_{2^iI}\|_{L^1} = |2^iI| \leqslant |2^iI|(1-\log|2^iI|) = \|\chi_{2^iI}\|_{L\log L}$. Consequently,

$$\sum_{i=0}^{N_I} \frac{\|\chi_I\|_{L^1}}{\|\chi_{2^iI}\|_{L\log L}} \frac{w(2^iI)}{u(I)} = \sum_{i=0}^{N_I} \frac{|I|}{|2^iI|} \left(\frac{|2^iI)|}{|I|}\right)^{\lambda} \leqslant \sum_{i=0}^{N_I} \left(\frac{|I|}{|2^iI|}\right)^{1-\lambda} < C.$$

Therefore, (3.4) is fulfilled and Theorem 3.2 is valid for the Morrey spaces on $\mathbb{T}$ studied in [11] when the L^p norm is replaced by the L^1 norm.

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