Fast Source Reconstruction via ADMM with Elastic Net Regularization

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***Abstract*—Norm-1 regularized optimization algorithms are commonly used for Compressive Sensing applications. In this paper, an optimization algorithm based on the Alternating Direction Method of Multipliers (ADMM) together with the Elastic Net regularization is presented. This type of regularization is a linear combination of the norm-1 and norm-2 regularizations, allowing a solution between the sparsest and the minimum energy solutions, but still enforcing some sparsivity. The combination of these two regularizations and the distributive capabilities of the ADMM algorithm enables a fast sparse signal recovering with minimum error.**

I. Introduction

Compressive Sensing (CS) techniques have been successfully used to recover sparse signals from a much smaller number of measurements than those required by the Nyquist sampling criterion, [1]–[3]. The signal recovery can be represented by solving the following matrix equation:

$$\mathbf{g} = \mathbf{H}\mathbf{u} + \mathbf{w}, \tag{1}$$

where $\mathbf{g} \in \mathbb{C}^{N_m}$ is the vector of measurements, $\mathbf{H} \in \mathbb{C}^{N_m \times N_p}$ is the sensing matrix, $\mathbf{u} \in \mathbb{C}^{N_p}$ is the unknown sparse signal to recover, and $\mathbf{w} \in \mathbb{C}^{N_m}$ is the noise collected. N_p is the number of points in the signal to recover and N_m the number of measurements. Finding the sparsest solution does not ensure that the smallest error is achieved. This paper proposes an algorithm for fast signal reconstruction based on the Alternating Direction Method of Multipliers (ADMM) with Elastic Net regularization [4]. Since this regularization is a linear combination of the norm-1 and norm-2 regularizations, it possesses the property of recovering sparse signals with minimum energy. The ADMM algorithm has shown quasi real time signal reconstruction for imaging application in a distributive scenario [3], [5]. The use of the ADMM with Elastic Net regularization enables a fast sparse signal reconstruction while minimizing the error.

II. Elastic Net regularized ADMM

The ADMM [6] is a method for optimizing convex functions. Its general representation takes the following form:

$$\text{minimize } f(\mathbf{u}) + g(\mathbf{v}) \quad \text{s.t.} \qquad \mathbf{P}\mathbf{u} + \mathbf{Q}\mathbf{v} = \mathbf{c} \tag{2}$$

where f and g are convex, closed, and proper functions over the unknown vectors $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{v} \in \mathbb{C}^m$. A constraint, which relates the variables $\mathbf{u}$ and $\mathbf{v}$, is determined by the known matrices $\mathbf{P} \in \mathbb{C}^{p\times n}$ and $\mathbf{Q} \in \mathbb{C}^{p\times m}$, and vector $\mathbf{c} \in \mathbb{C}^p$.

Equation (1) may be solved by minimizing the convex function $f(\mathbf{u}) = \|\mathbf{H}\mathbf{u} - \mathbf{g}\|_2^2$ together with the Elastic Net regularization $g_\alpha(\mathbf{v}) = \lambda\left((1-\alpha)\|\mathbf{v}\|_1 + \alpha\frac{1}{2}\|\mathbf{v}\|_2^2\right)$, with $0 \leq \alpha \leq 1$. The proposed approach finds the optimal solution via the *consensus*-based ADMM algorithm [3], dividing the sensing matrix and the vector of measurements in M submatrices by rows, and solving the problem in a distributed fashion by minimizing the following expression:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\sum\limits_{i=1}^{M} \|\mathbf{H}_i\mathbf{u}_i - \mathbf{g}\|_2^2 + \lambda\left((1-\alpha)\|\mathbf{v}\|_1 + \alpha\frac{1}{2}\|\mathbf{v}\|_2^2\right) \\ \text{s.t.} & \mathbf{u}_i = \mathbf{v}, \ \ \forall i = 1, ..., M \end{array} \tag{3}$$

where the new variable $\mathbf{v}$ is the *consensus* variable that imposes the agreement of all the partial solutions $\mathbf{u}_i$.

This problem is solved by the following iterative scheme:

$$\mathbf{u}_i^{k+1} = \left(\mathbf{H}_i^*\mathbf{H}_i + \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^*\mathbf{g}_i + \rho\left(\mathbf{v}^k - \mathbf{s}_i^k\right)\right), \tag{4}$$

$$\mathbf{v}^{k+1} = \frac{1}{1+\frac{\alpha\lambda}{\rho M}}\mathbf{S}_{(1-\alpha)\frac{\lambda}{\rho M}}\left(\bar{\mathbf{u}}^{k+1} + \bar{\mathbf{s}}^k\right), \tag{5}$$

$$\mathbf{s}_i^{k+1} = \mathbf{s}_i^k + \mathbf{u}_i^{k+1} - \mathbf{v}^{k+1}. \tag{6}$$

where $\mathbf{s}_i$ is the dual variable for each constraint i and ρ is the augmented parameter. $\mathbf{S}_\kappa(a)$ is the soft thresholding operator, interpreted element-wise [7]. For the case of the Elastic Net regularization, this operator can be interpreted as two step shrinkage, as shown in Fig. 1: first, forcing small input values to be zero (due to the norm-1 regularization); and second, decreasing the amplitude of the non-zero output values (due to the norm-2 regularization). The *matrix inversion lemma* [8] may be applied once to the term $\left(\mathbf{H}_i^*\mathbf{H}_i + \rho\mathbf{I}\right)^{-1}$, since only M matrices of reduced size $\frac{N_m}{M} \times \frac{N_m}{M}$ have to be inverted.

III. Numerical Results

The Elastic Net regularized ADMM has been tested in a simple 2D problem of source reconstruction. The geometry is defined as described in Fig. 2. Five point sources of unit value are located in a 5×5 square normalized over the wavelength λ_0. N_{Rx} receivers are placed in a circular position of radius 10, with an angular separation of $\theta = \frac{2\pi}{N_{Rx}}$. The measured field is described by the following equation:

$$g_m = \sum_{n=1}^{N_p} u_n e^{-jkd_{nm}}, \tag{7}$$

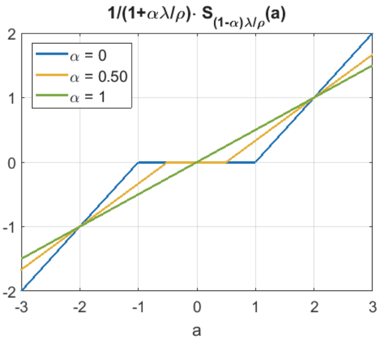

Fig. 1. Graphical representation of the soft-thresholding operator. Small input values are set to zero due to the influence of the norm-1, and non-zero output values are reduced due to the norm-2. For $\alpha = 0$ there is norm-1 regularization only, and for $\alpha = 1$, there is norm-2 regularization only.

where $k = \frac{2\pi}{\lambda_0}$ and d_{nm} is the distance between the $n-th$ pixel and the $m-th$ receiver. This can be expressed in the linear matrix equation proposed in Eq. (1), where $\mathbf{H} = \{e^{-jkd_{nm}}\}_{nm}$.

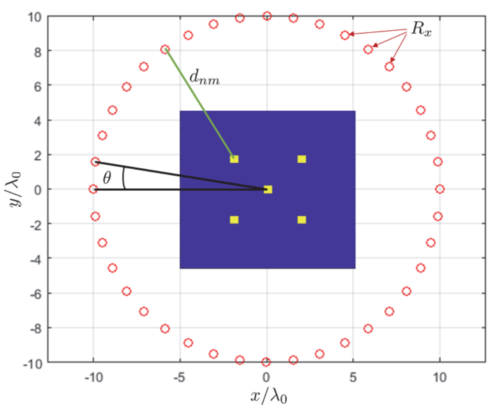

Fig. 2. System configuration. Yellow points represent the sources over the dark blue square region of interest. Red circles represent the receivers. All distances are normalized by the wavelength λ_0.

Figure 3 depicts the reconstruction error when varying the parameter α for different number of measurements for a $SNR = 50dB$. Two cases are analyzed: (a) making use of the computed sensing matrix $\mathbf{H}$ (Fig. 3(a)), and (b) when multiplying the sensing matrix by random values from a Gaussian distribution of zero mean and standard deviation $\sqrt{N_{Rx}}$ (Fig. 3(b)), to enhance the restricted isometry constant. The parameters used for the configuration and for the ADMM algorithm are shown in Table I. The reconstruction error is defined as

$$\epsilon = \frac{\|\mathbf{u}_0 - \mathbf{u}\|_2}{\|\mathbf{u}_0\|_2} \cdot 100\ (\%), \tag{8}$$

where $\mathbf{u}_0$ is the desired source to recover. The results show that a linear combination of the norm-1 and norm-2 regularizations can reduce the reconstruction error when trying to recover a sparse signal with a limited number of measurements. The distributive capabilities of the *consensus*-based ADMM allows to solve the problem in $1.6s$ for 200 iterations in a Matlab 2017 code using a Titan V 5120 cores GPU with double precision.

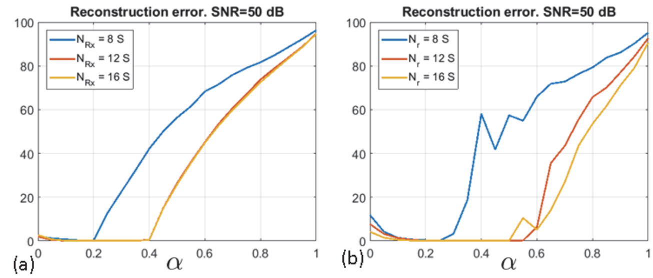

Fig. 3. Reconstruction error varying the parameter α for different numbers of receivers with a SNR of $50dB$, for (a) regular sensing matrix, and (b) sensing matrix multiplied by random gaussian values.

TABLE I
PARAMETERS FOR THE NUMERICAL EXAMPLE.

Parameter	Description	Value
N_p	Number of pixels	441
N_m	Number of measurements	$N_m = N_{Rx}$
S	Sparsity level	5
ρ	Augmented parameter	0.01
λ	Regularization parameter	0.01
M	Number of rows divisions	5

IV. CONCLUSION

This paper has presented the mathematical formulation of a new distributed algorithm based on the Elastic Net regularized ADMM. This technique reduces the reconstruction error of sparse signals due to the combined properties of the norm-1 and norm-2 regularizations. The method has been tested in 2D source reconstruction. The distributive capabilities of the ADMM allows minimum error fast sparse signal recovery.

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