

CHARACTERIZATION OF CYCLE OBSTRUCTION SETS FOR
IMPROPER COLORING PLANAR GRAPHS*ILKYOO CHOI[†], CHUN-HUNG LIU[‡], AND SANG-IL OUM[§]

Abstract. For nonnegative integers k, d_1, \dots, d_k , a graph is (d_1, \dots, d_k) -colorable if its vertex set can be partitioned into k parts so that the i th part induces a graph with maximum degree at most d_i for all $i \in \{1, \dots, k\}$. A class \mathcal{C} of graphs is *balanced k -partitionable* and *unbalanced k -partitionable* if there exists a nonnegative integer D such that all graphs in \mathcal{C} are (D, \dots, D) -colorable and $(0, \dots, 0, D)$ -colorable, respectively, where the tuple has length k . A set X of cycles is a *cycle obstruction set* of a class \mathcal{C} of planar graphs if every planar graph containing none of the cycles in X as a subgraph belongs to \mathcal{C} . This paper characterizes all cycle obstruction sets of planar graphs to be balanced k -partitionable and unbalanced k -partitionable for all k ; namely, we identify all inclusionwise minimal cycle obstruction sets for all k .

Key words. graph coloring, improper coloring, defective coloring, planar graphs, obstruction sets

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1. Introduction. All graphs in this paper are finite and simple, which means no loops and no parallel edges. Let C_k denote a k -cycle, which is a cycle of length k . A set X of cycles is a *cycle obstruction set* of a class \mathcal{C} of planar graphs if every planar graph containing none of the cycles in X as a subgraph belongs to \mathcal{C} .

A graph is *k -colorable* if its vertex set can be partitioned into k color classes so that each color class is an independent set. The celebrated four color theorem [1, 2] (later reproved in [18]) states that every planar graph is 4-colorable. Since there are planar graphs that are not 3-colorable, finding sufficient conditions for a planar graph to be 3-colorable has been an active area of research; many of these conditions can be translated into the language of obstruction sets. Perhaps the most well-known result is the following theorem, known as Grötzsch's theorem [13]:

THEOREM 1.1 (Grötzsch [13]). *Planar graphs with no 3-cycles are 3-colorable.*

In the language of obstruction sets, Grötzsch's theorem states that $\{C_3\}$ is an obstruction set of 3-colorable planar graphs. There is also a vast literature regarding forbidding various cycle lengths to guarantee a planar graph to be 3-colorable; see Table 1 for a summary of some of these results.

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TABLE 1
Forbidding various cycle lengths to guarantee 3-colorability of planar graphs.

| Year | Reference | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Authors |
|------|-----------|---|---|---|---|---|---|---|-------------------|
| 1959 | [13] | × | | | | | | | Grötzsch |
| 2005 | [25] | | × | × | × | | | × | Zhang–Wu |
| 2006 | [24] | | × | × | | × | | | Xu |
| 2010 | [22] | | × | × | | | × | × | Wang–Lu–Chen |
| 2007 | [6] | | × | | × | × | | × | Chen–Raspaud–Wang |
| 2007 | [21] | | × | | × | | × | | Wang–Chen |
| 2011 | [23] | | × | | | × | × | × | Wang–Wu–Shen |

Each result in the aforementioned theorem reveals a new obstruction set of 3-colorable planar graphs. The interest in forbidding various cycle lengths stems from Steinberg's conjecture [20], which states that planar graphs with neither 4-cycles nor 5-cycles are 3-colorable. There was almost no progress after the conjecture was first proposed in 1976, but many partial results were produced after 1991, which is when Erdős [20] proposed the following approach towards Steinberg's conjecture: find the minimum k such that planar graphs with no cycle lengths in $\{4, \dots, k\}$ are 3-colorable. After 40 years of effort by the coloring community to try to prove Steinberg's conjecture, only recently it was disproved via a clever construction by Cohen-Addad et al. [10]. Yet, the question of whether planar graphs with no cycle lengths in $\{4, 5, 6\}$ are 3-colorable or not remains open.

Recently, the following relaxation of proper coloring, also known as *improper coloring*, has attracted much attention: for nonnegative integers k, d_1, \dots, d_k , a graph is (d_1, \dots, d_k) -colorable if its vertex set can be partitioned into k color classes V_1, \dots, V_k so that V_i induces a graph with maximum degree at most d_i for all $i \in \{1, \dots, k\}$. This relaxation allows some prescribed defects in each color class, where defects are measured in terms of the maximum degree of the graph induced by the vertices of a color class. We say a class \mathcal{C} of graphs is *balanced k -partitionable* and *unbalanced k -partitionable* if there exists a nonnegative integer D such that all graphs in \mathcal{C} are (D, \dots, D) -colorable and $(0, \dots, 0, D)$ -colorable, respectively, where the tuple has length k .

There is a vast literature in improper coloring planar graphs. By the four color theorem, planar graphs are 4-colorable, which is equivalent to $(0, 0, 0, 0)$ -colorable, and Cowen, Cowen, and Woodall [11] proved that planar graphs are $(2, 2, 2)$ -colorable. This is best possible in the sense that for any given nonnegative integers d_1 and d_2 , there exists a planar graph that is not $(1, d_1, d_2)$ -colorable; for one such construction see [8]. Therefore, the question of partitioning planar graphs with no extra conditions into at least three subgraphs of bounded maximum degrees is completely solved.

It is often useful to consider girth conditions along with planarity to obtain positive results. Regarding partitioning planar graphs into two parts, for any given nonnegative integers d_1 and d_2 , a planar graph with girth 4 that is not (d_1, d_2) -colorable is constructed in [17]. Yet, Choi et al. [7], Borodin and Kostochka [5], Choi and Raspaud [9], and Škrekovski [19] proved that planar graphs with girth at least 5 are $(1, 10)$ -, $(2, 6)$ -, $(3, 5)$ -, and $(4, 4)$ -colorable, respectively. Also, given a nonnegative integer d , a planar graph with girth 6 that is not $(0, d)$ -colorable is constructed in [3]. On the other hand, it is known that every planar graph with girth at least 7 is $(0, 4)$ -colorable [5]. For other papers regarding improper coloring sparse (not necessarily planar) graphs, see [4, 12, 14, 15, 16].

The previous paragraph concerns girth conditions enforced on planar graphs to obtain positive results. Instead of forbidding all short cycles, we are interested in finding the minimal sets of obstacles in partitioning planar graphs into parts with bounded maximum degrees. We succeed in identifying which cycle lengths are essential obstructions when it comes to partitioning planar graphs in a balanced and unbalanced way. In other words, this paper characterizes all cycle obstruction sets of balanced k -partitionable and unbalanced k -partitionable planar graphs for all k ; namely, we identify all the inclusionwise minimal cycle obstruction sets.

By the four color theorem, the empty set is the (only) inclusionwise minimal cycle obstruction set of both balanced k -partitionable and unbalanced k -partitionable planar graphs when $k \geq 4$. The empty set is also the (only) inclusionwise minimal cycle obstruction set of balanced 3-partitionable planar graphs, since Cowen, Cowen, and Woodall [11] proved that planar graphs are $(2, 2, 2)$ -colorable. For the remaining cases, we characterize the inclusionwise minimal obstruction sets, and for each case there are exactly two. Our main results are the following three theorems:

THEOREM 2.1. *A set S of cycles is an inclusionwise minimal cycle obstruction set of balanced 2-partitionable planar graphs if and only if $S = \{C_4\}$ or S is the set of all odd cycles.*

THEOREM 3.1. *A set S of cycles is an inclusionwise minimal cycle obstruction set of unbalanced 2-partitionable planar graphs if and only if $S = \{C_3, C_4, C_6\}$ or S is the set of all odd cycles.*

THEOREM 4.1. *A set S of cycles is an inclusionwise minimal cycle obstruction set of unbalanced 3-partitionable planar graphs if and only if $S = \{C_3\}$ or $S = \{C_4\}$.*

Theorems 2.1 and 3.1 state that for planar graphs to be balanced 2-partitionable and unbalanced 2-partitionable, respectively, there is only one inclusionwise minimal cycle obstruction set other than the set of all odd cycles. Since forbidding all odd cycles makes the graph bipartite, and thus 2-colorable, which is equivalent to $(0, 0)$ -colorable, the minimal cycle obstructions for planar graphs to be balanced 2-partitionable and unbalanced 2-partitionable is a 4-cycle and all of 3-, 4-, 6-cycles, respectively. Note that previous results by Škrekovski [19] and Borodin and Kostochka [5] imply that planar graphs are balanced 2-partitionable and unbalanced 2-partitionable when the forbidden cycle lengths are 3, 4 and 3, 4, 5, 6, respectively.

Theorem 4.1 states that other than a 3-cycle, there is only one other inclusionwise minimal cycle obstruction set of unbalanced 3-partitionable planar graphs. Since Grötzsch's theorem says that forbidding a 3-cycle in planar graphs guarantees that it is 3-colorable, which is equivalent to $(0, 0, 0)$ -colorable, the minimal cycle obstruction for non-3-colorable planar graphs to be unbalanced 3-partitionable is a 4-cycle.

Note that for both balanced 1-partitioning and unbalanced 1-partitioning, cycle obstruction sets simply do not exist because of planar graphs with arbitrarily large maximum degree. See Table 2 for a complete list of cycle obstruction sets of both balanced k -partitionable and unbalanced k -partitionable planar graphs.

In sections 2, 3, and 4, we prove Theorems 2.1, 3.1, and 4.1, respectively. The constants in all of our main results are probably improvable with some effort. Yet, we focused on simplifying the proofs and using the minimum number of reducible configurations and basic discharging rules in order to improve the readability of the paper. We end this section by posing three questions and some definitions that will be used in the next sections.

TABLE 2
Characterization of inclusionwise minimal cycle obstruction sets.

| k | Balanced | Unbalanced |
|----------------------|------------------------------------|--|
| 4^+ -partitionable | \emptyset | \emptyset |
| 3-partitionable | \emptyset | $\{C_3\}, \{C_4\}$ |
| 2-partitionable | $\{C_{2i+1} : i \geq 1\}, \{C_4\}$ | $\{C_{2i+1} : i \geq 1\}, \{C_3, C_4, C_6\}$ |
| 1-partitionable | does not exist! | does not exist! |

PROBLEM 1.2. *What is the minimum D such that every planar graph with no 4-cycles is (D, D) -colorable?*

PROBLEM 1.3. *What is the minimum D such that every planar graph with no 3-, 4-, 6-cycles is $(0, D)$ -colorable?*

PROBLEM 1.4. *What is the minimum D such that every planar graph with no 4-cycles is $(0, 0, D)$ -colorable?*

The *degree* of a vertex v , denoted by $d(v)$, is the number of edges incident with it. A k -vertex, k^+ -vertex, and k^- -vertex are a vertex of degree exactly k , at least k , and at most k , respectively. Given any embedding of a connected planar graph G on at least two vertices on the plane, for every face f , we say that a boundary walk W_f of f is *canonical* if it traces the edges incident with f according to one of the two obvious cyclic orderings of those edges. The *degree* of a face f , denoted by $d(f)$, is the length of W_f ; note that cut edges are counted twice. A k -face, k^+ -face, and k^- -face is a face of degree exactly k , at least k , and at most k , respectively. For each face f and each vertex v of G , we define $k_{f,v}$ to be the number of triples (e, v, e') such that $e, e' \in E(G)$ and $e e'$ is a subwalk of W_f . It is well known that the degree of f and $k_{f,v}$ is independent of the choice of W_f . Clearly, the degree of f equals $\sum_{v \in V(G)} k_{f,v}$.

2. Balanced 2-partitions. In this section, we prove the following theorem.

THEOREM 2.1. *A set S of cycles is an inclusionwise minimal cycle obstruction set of balanced 2-partitionable planar graphs if and only if $S = \{C_4\}$ or S is the set of all odd cycles.*

We will first show a necessary condition for cycle obstruction sets, and then show that it is sufficient afterwards.

LEMMA 2.2. *If a set S of cycles is an obstruction set of balanced 2-partitionable planar graphs, then either $C_4 \in S$ or S contains all odd cycles.*

Proof. Given a nonnegative integer D and two vertices x and y , let $H_2(D; x, y)$ be the graph consisting of $2D + 1$ internally disjoint x, y -paths of length 2. For a positive integer l and a vertex v_1 , let $H_1(D, l; v_1)$ be the graph obtained from a cycle with vertices v_1, \dots, v_{l+1} and replacing each edge $v_i v_{i+1}$ with a copy of $H_2(D; v_i, v_{i+1})$ where $i \in \{1, \dots, l\}$. Finally, let $H(D, l)$ be the graph obtained from $D + 1$ pairwise disjoint copies of $H_1(D, l; v_1^j)$ and identifying all of v_1^j for $j \in \{1, \dots, D + 1\}$.

Now in any (D, D) -coloring of $H_2(D; x, y)$, it is easy to see that x and y must receive the same color. This implies that the cut-vertex of $H(D, l)$ has $D + 1$ neighbors of the same color, which shows that $H(D, l)$ is not (D, D) -colorable. It is not hard to see that the cycles in $H(D, l)$ have length either 4 or $2l + 1$. Therefore the obstruction set of balanced 2-partitionable planar graphs contains either C_4 or all odd cycles. See Figure 1 for an illustration of $H_2(D; x, y)$ and $H(D, 2)$. \square

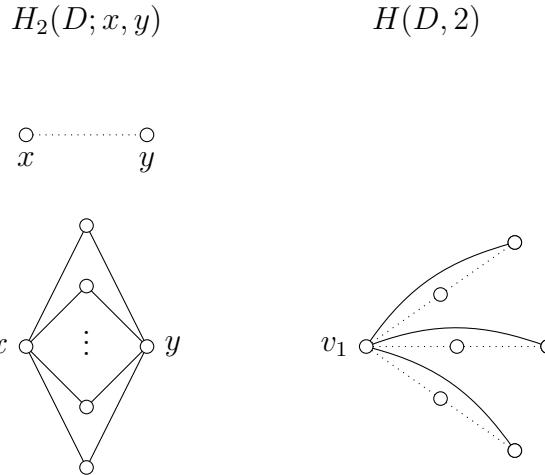


FIG. 1. *Graphs that are not (D, D) -colorable.*

If a planar graph does not contain any odd cycles, then it is bipartite and, thus, it is $(0, 0)$ -colorable and, hence, it is balanced 2-partitionable. The remainder of this section proves that planar graphs with no 4-cycles are balanced 2-partitionable. Note that Lemma 2.2 and Theorem 2.3 imply Theorem 2.1.

THEOREM 2.3. *A planar graph with no 4-cycles is $(5, 5)$ -colorable.*

In the rest of this section, let G be a counterexample to Theorem 2.3 with the minimum number of 3^+ -vertices, and, subject to that, choose one with the minimum number of edges. Also, fix a plane embedding of G . It is easy to see that G is connected and has no 1-vertices. From now on, given a (partially) $(5, 5)$ -colored graph, we will let a and b be the two colors, and we say a vertex with a color is *saturated* if it already has five neighbors of the same color.

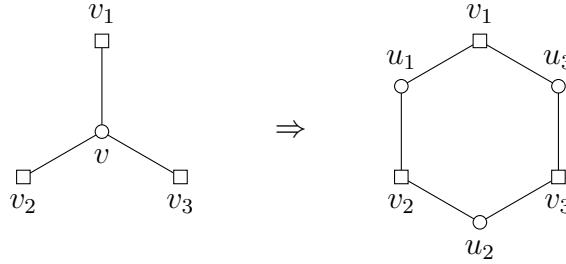
2.1. Structural lemmas.

LEMMA 2.4. *Every edge xy of G has an endpoint with degree at least 7.*

Proof. Suppose to the contrary that x and y are both 6^- -vertices. Since $G \setminus xy$ is a graph with fewer edges than G and the number of 3^+ -vertices did not increase, there is a $(5, 5)$ -coloring $\varphi : V(G) \rightarrow \{a, b\}$ of $G \setminus xy$. If φ is not a $(5, 5)$ -coloring of G , then $\varphi(x) = \varphi(y)$, and either x or y is saturated in $G \setminus xy$. For each saturated vertex z in $\{x, y\}$, we may recolor it with the color in $\{a, b\} \setminus \{\varphi(z)\}$ since all of its neighbors have color $\varphi(z)$ in $G \setminus xy$. We end up with a $(5, 5)$ -coloring of G , which is a contradiction. \square

LEMMA 2.5. *There are no 3-vertices in G .*

Proof. Suppose to the contrary that v is a 3-vertex of G with neighbors v_1, v_2, v_3 . By Lemma 2.4, we know that v_1, v_2, v_3 are 7^+ -vertices. Obtain a graph H from $G - v$ by adding paths $v_1u_1v_2, v_2u_2v_3, v_3u_3v_1$, where u_1, u_2, u_3 are three distinct vertices not in G . Note that H is planar and has no 4-cycles since the pairwise distance between v_1, v_2, v_3 did not change. See Figure 2 for an illustration. Since H has fewer 3^+ -vertices than G , there is a $(5, 5)$ -coloring $\varphi : V(H) \rightarrow \{a, b\}$ of H .

FIG. 2. *Obtaining H from G in Lemma 2.5.*

If $\varphi(v_1) = \varphi(v_2) = \varphi(v_3)$, then we may extend φ to G by using the color in $\{a, b\} \setminus \{\varphi(v_1)\}$ on v . Otherwise, without loss of generality we may assume $\varphi(v_1) = a$ and $\varphi(v_2) = \varphi(v_3) = b$. If $a \in \{\varphi(u_1), \varphi(u_3)\}$, then we may extend φ to G by using a on v . Otherwise, $\varphi(u_1) = \varphi(u_3) = b$, so we may extend φ to G by using b on v . In all cases we end up with a $(5, 5)$ -coloring of G , which is a contradiction. \square

A 3-face is *terrible* if it is incident with a 2-vertex.

LEMMA 2.6. *A 7^+ -vertex v is incident with at most $\min\{\lfloor \frac{d(v)}{2} \rfloor, d(v) - 6\}$ terrible 3-faces.*

Proof. Since G has no 4-cycles, two 3-faces cannot share an edge, and thus v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ terrible 3-faces. Since $\lfloor \frac{d(v)}{2} \rfloor \leq d(v) - 6$ when $d(v) \geq 11$, we may assume $d(v) \leq 10$.

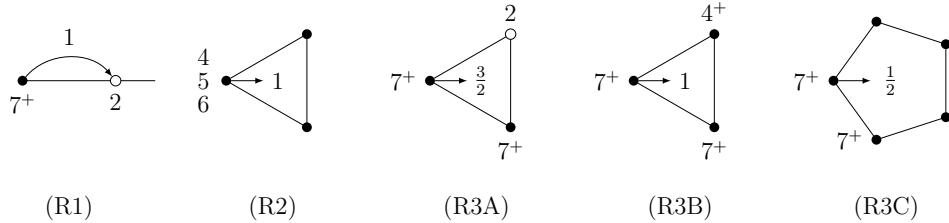
Suppose to the contrary that v is incident with t terrible 3-faces, where $t \geq d(v) - 5$. Let w be a 2-vertex of a terrible 3-face wvu ; note that u is also a 7^+ -vertex by Lemma 2.4. Since $G - w$ is a graph with fewer edges than G and the number of 3^+ -vertices did not increase, there is a $(5, 5)$ -coloring $\varphi : V(G) \setminus \{w\} \rightarrow \{a, b\}$ of $G - w$. If $\varphi(u) = \varphi(v)$, then we may extend φ to G by using the color in $\{a, b\} \setminus \{\varphi(u)\}$ on w . Thus, we may assume $\varphi(u) = a$ and $\varphi(v) = b$. Since using the color b on w should not extend φ to G , we know that v must be saturated by φ .

There are $d(v) - 2t$ neighbors of v in $G - w$ that are not in terrible 3-faces incident with v . Since v has five neighbors with the color b , at least $5 - (d(v) - 2t) = 5 + 2t - d(v)$ neighbors of v in $G - w$ with the color b are incident with a terrible 3-face incident with v . Since neither w nor u is colored with b , there are $t - 1$ terrible 3-faces incident with v that might have a vertex colored with b . Since $t \geq d(v) - 5$ implies $5 + 2t - d(v) > t - 1$, there exists a terrible 3-face xyv where x is a 2-vertex, other than wuv with $\varphi(x) = \varphi(y) = b$. Now, we can extend φ to G by coloring w with b and recoloring x with a , which contradicts the assumption that G has no $(5, 5)$ -coloring. \square

2.2. Discharging. We now define the initial charge at each vertex and each face. For every $v \in V(G)$, let $\mu(v) = 2d(v) - 6$ and for every face $f \in F(G)$, let $\mu(f) = d(f) - 6$. The total initial charge is negative since

$$\begin{aligned} \sum_{z \in V(G) \cup F(G)} \mu(z) &= \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) \\ &= -6|V(G)| + 6|E(G)| - 6|F(G)| = -12 < 0. \end{aligned}$$

The last equality holds by Euler's formula. Recall that a 3-face is terrible if it is incident with a 2-vertex.

FIG. 3. *Discharging rules.*

Here are the discharging rules:

- (R1) Each 7^+ -vertex sends charge 1 to each adjacent 2-vertex.
- (R2) Each 4-, 5-, 6-vertex sends charge 1 to each incident 3-face.
- (R3) Let v be a 7^+ -vertex.
 - (R3A) v sends charge $\frac{3}{2}$ to each incident terrible 3-face.
 - (R3B) v sends charge 1 to each incident 3-face that is not terrible.
 - (R3C) v sends charge $\frac{1}{2}$ to each 5-face f that is incident with v and incident with a neighbor of v with degree at least 7.

See Figure 3 for an illustration of the discharging rules.

We denote the final charge of z by $\mu^*(z)$ for each $z \in V(G) \cup F(G)$. The rest of this section will prove that $\mu^*(z)$ is nonnegative for each $z \in V(G) \cup F(G)$.

CLAIM 2.7. *Every face has nonnegative final charge.*

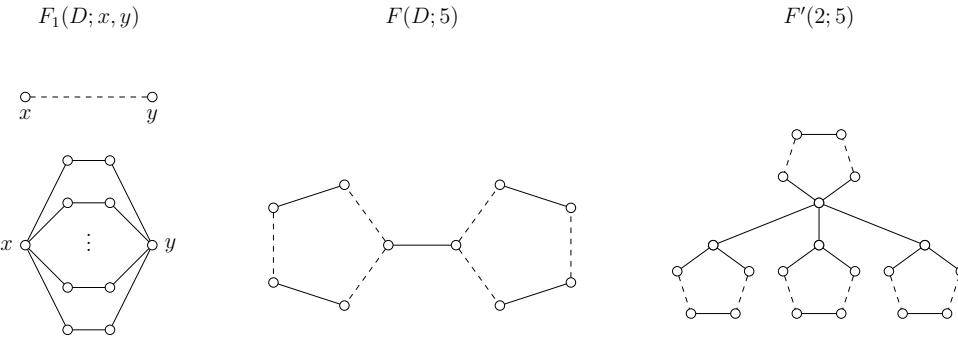
Proof. Let f be a face. It only receives charge and does not give out any charge. Thus if f is a 6^+ -face, then f has nonnegative final charge since $\mu^*(f) = \mu(f) = d(f) - 6 \geq 0$. By Lemma 2.4, every 5-face f is incident with at least three 7^+ -vertices, and at least two of these are adjacent to each other. Thus, by rule (R3C), f receives charge $\frac{1}{2}$ at least twice. Thus, $\mu^*(f) \geq 5 - 6 + 2 \cdot \frac{1}{2} = 0$. Note that f cannot be a 4-face since G has no 4-cycles.

Now assume f is a 3-face. By Lemma 2.4, f is incident with at least two 7^+ -vertices, which must be pairwise adjacent to each other. If f is incident with two 7^+ -vertices and the third vertex is a 4^+ -vertex, then f is not a terrible face. Now f receives either charge 1 twice by rule (R3B) and charge 1 once by rule (R2) or charge 1 three times by rule (R3B). In either case, $\mu^*(f) = 3 - 6 + 3 \cdot 1 = 0$. Note that there are no 3-vertices by Lemma 2.5. If f is incident with exactly two 7^+ -vertices, then the third vertex is a 2-vertex, and f is a terrible 3-face. Thus it receives charge $\frac{3}{2}$ twice by rule (R3A). Thus, $\mu^*(f) = 3 - 6 + 2 \cdot \frac{3}{2} = 0$. \square

CLAIM 2.8. *Each vertex has nonnegative final charge.*

Proof. Each neighbor of a 2-vertex v must be a 7^+ -vertex by Lemma 2.4. Therefore v receives charge 1 twice by (R1). Thus, $\mu^*(v) = 2 \cdot 2 - 6 + 2 \cdot 1 = 0$. Note that there are no 3-vertices by Lemma 2.5, and every vertex is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces since there are no 4-cycles in G . If v is a vertex with $d(v) \in \{4, 5, 6\}$, then v sends charge 1 at most $\lfloor \frac{d(v)}{2} \rfloor$ times by rule (R2). Thus, $\mu^*(v) \geq 2d(v) - 6 - \lfloor \frac{d(v)}{2} \rfloor \geq 0$.

Now assume v is a 7^+ -vertex. We will show that v has nonnegative final charge by a weighting argument. Let $u_1, \dots, u_{d(v)}$ be the neighbors of v in some cyclic order. First give all neighbors of v a weight of 1. If u_i is not a 2-vertex, then split the weight of 1 it received from v , and transfer weight $\frac{1}{2}$ to each of the two faces that are incident with vu_i ; if vu_i is incident with only one face, then transfer the entire weight of 1 to

FIG. 4. Graphs that are not $(0, D)$ -colorable.

this face. Now, every neighbor of v that is a 2-vertex and every face incident with v that is not a terrible 3-face have weight at least the charge that they should receive from v by the discharging rules. Every terrible 3-face has weight at most 1 short of the charge it should receive from v by the discharging rules. Now give weight 1 to each terrible 3-face incident with v . Since v is incident with at most $d(v) - 6$ terrible 3-faces by Lemma 2.6, and each neighbor of v received weight 1 initially, the total weight spent is at most $2d(v) - 6$, which is exactly the initial charge of v . Thus, the total weight sent is no more than the initial charge of v , which proves that the final charge of v is nonnegative. \square

3. Unbalanced 2-partitions. In this section, we prove the following theorem.

THEOREM 3.1. *A set S of cycles is an inclusionwise minimal cycle obstruction set of unbalanced 2-partitionable planar graphs if and only if $S = \{C_3, C_4, C_6\}$ or S is the set of all odd cycles.*

We will first show a necessary condition for cycle obstruction sets, and then show that it is sufficient afterwards.

LEMMA 3.2. *If a set S of cycles is an obstruction set of unbalanced 2-partitionable planar graphs, then either $\{C_3, C_4, C_6\} \subseteq S$ or S contains all odd cycles.*

Proof. For a nonnegative integer D , a positive integer l , and a vertex v , recall that $H(D, l)$ from section 2 is not (D, D) -colorable and the only cycles in $H_1(D, l; v)$ have length either 4 or $2l + 1$. Therefore $H(D, l)$ is not $(0, D)$ -colorable as well. Therefore S contains either C_4 or all odd cycles.

Given a nonnegative integer D and two vertices x and y , let $F_1(D; x, y)$ be the graph that consists of $2D + 1$ internally disjoint x, y -paths of length 3. See Figure 4 for an illustration of $F_1(D; x, y)$. For an odd integer $l \geq 3$ and a vertex v_1 , let $F_o(D, l; v_1)$ be the graph obtained from an odd cycle with vertices v_1, \dots, v_l by replacing each edge $v_i v_{i+1}$ with $F_1(D; v_i, v_{i+1})$, where i is an odd integer at most l (where v_{l+1} is treated as v_1). Finally, obtain $F(D; l)$ from two disjoint copies of $F_o(D, l; v_1)$ and adding an edge between the two vertices that correspond to v_1 . See Figure 4 for an illustration of $F(D; 5)$.

Now in any $(0, D)$ -coloring of $F_1(D; x, y)$, it is easy to see that x and y cannot both receive the color 2. The two cut-vertices of $F(D; l)$ cannot both receive the color 1 in any $(0, D)$ -coloring, thus at least one cut-vertex v receives the color 2. In the copy that corresponds to $F_o(D, l; v_1)$, either there is an edge with both endpoints colored

with the color 1 or there is a copy of $F_1(D; x, y)$ where both x and y receive the color 2. This shows that $F(D; l)$ is not $(0, D)$ -colorable. It is not hard to see that the only cycles in $F(D; l)$ have length either 6 or $2l + 1$, where l is an odd integer at least 3. Therefore S contains either C_6 or all cycles of lengths $4k + 3$, where k is a positive integer.

For an odd integer $l \geq 3$ and a vertex v_1 , let $F_e(D, l; v_1)$ be the graph obtained from an odd cycle with vertices v_1, \dots, v_l by replacing each edge $v_i v_{i+1}$ with $F_1(D; v_i, v_{i+1})$, where i is an even integer at most l . Finally, let $F'(D; l)$ be the graph obtained from a star with $D + 2$ vertices by attaching a copy of $F_e(D, l; v)$ to each vertex v of the star. See Figure 4 for an illustration of $F'(2; 5)$.

As above, x and y cannot both receive the color 2 in any $(0, D)$ -coloring of $F_1(D; x, y)$. This implies that every cut vertex of $F'(D; l)$ must be colored with color 2 in a $(0, D)$ -coloring. Yet, now there exists a cut vertex of that has $D + 1$ neighbors colored with the color 2, which implies that $F'(D; l)$ is not $(0, D)$ -colorable. It is not hard to see that the cycles in $F'(D; l)$ have length either 6 or $2l - 1$, where l is an odd integer at least 3. Therefore S contains either C_6 or all cycles of lengths $4k + 1$, where k is a positive integer.

Let $T_0(D; x)$ be the graph obtained from $D + 1$ pairwise disjoint 3-cycles by identifying one vertex in each cycle into x . Now let $T(D)$ be the graph obtained from two copies of $T_0(D; x)$ and adding an edge between the two vertices corresponding to x . In any $(0, D)$ -coloring of $T_0(D; x)$, the vertex x must not receive color 2 since it will have $D + 1$ neighbors colored with 2. Yet, in $T(D)$, one of the two cut vertices, which corresponds to x in a copy of $T_0(D; x)$, will receive color 2. This shows that $T(D)$ is not $(0, D)$ -colorable, and it is easy to see that $T(D)$ contains only 3-cycles. Hence S contains C_3 .

To sum up, the obstruction set of unbalanced 2-partitionable planar graphs must contain C_3 , and contains either $\{C_4, C_6\}$ or all odd cycles of length at least five. In other words, S contains either $\{C_3, C_4, C_6\}$ or all odd cycles. \square

If either C_4 or C_6 is not in an obstruction set S of unbalanced 2-partitionable planar graphs, then all odd cycles must be in S . This implies that the graph is bipartite and $(0, 0)$ -colorable, and hence it is unbalanced 2-partitionable. The following theorem shows that $\{C_3, C_4, C_6\}$ is an obstruction set of unbalanced 2-partitionable planar graphs. Note that Lemma 3.2 and Theorem 3.3 imply Theorem 3.1.

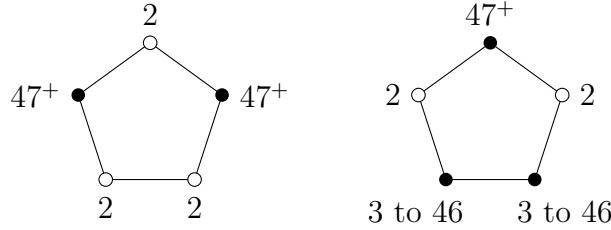
THEOREM 3.3. *A planar graph with no 3-, 4-, 6-cycles is $(0, 45)$ -colorable.*

In this section, let G be a counterexample to Theorem 3.3 with the minimum number of vertices. Also, fix a plane embedding of G . It is easy to see that G is connected and has no 1-vertices. From now on, given a (partially) $(0, 45)$ -colored graph, we will let a and b be the two colors where b is the color class allowed to have maximum degree at most 45, and we say a vertex colored with b is *saturated* if it already has 45 neighbors colored with b .

3.1. Structural lemmas.

LEMMA 3.4. *Any 46^- -vertex is adjacent to a 47^+ -vertex.*

Proof. Suppose to the contrary that a 46^- -vertex v is adjacent to only 46^- vertices. Since $G - v$ is a graph with fewer vertices than G , there is a $(0, 45)$ -coloring φ of $G - v$; choose φ that maximizes the number of neighbors of v with the color a . At least one neighbor of v has color a , since otherwise we can extend φ to all of G by coloring v with color a . Also, every neighbor of v colored b has a neighbor in $G - v$

FIG. 5. *Bad faces.*

with the color a , otherwise it can be recolored by a and violates the choice of φ . Since each neighbor u of v has at most 45 neighbors in $G - v$, u has at most 44 neighbors with the color b in $G - v$. So no neighbor of v is saturated. Hence we can extend φ to G by coloring v with color b . This contradicts that G is a counterexample, and thus proves the claim. \square

Since G has no 3-cycles and no 4-cycles, every 5-face is bounded by a cycle. A *bad face* is a 5-face f where the degrees of the vertices on a boundary walk is as in Figure 5.

LEMMA 3.5. *Any 2-vertex cannot be incident with two bad faces.*

Proof. Suppose to the contrary that a 2-vertex v is incident with two bad faces, where x, v, y, v_1, v_2 and x, v, y, u_1, u_2 are vertices, in this order, of boundary walks of the two bad faces. If $v_1 = u_2$ (or $v_2 = u_1$), then G contains a 3-cycle xv_1v_2 (or yxv_2v_1), which is a contradiction. If $v_1 = u_1$ (or $v_2 = u_2$), then G has a 4-cycle $v_1v_2xu_2$ (or $yxv_1v_2u_1$), which is again a contradiction. Therefore, $\{v_1, v_2\} \cap \{u_1, u_2\} = \emptyset$, and this implies that G contains a 6-cycle with vertices x, v_2, v_1, y, u_1, u_2 , which is a contradiction. \square

LEMMA 3.6. *Any 47^+ -vertex v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ bad faces.*

Proof. Suppose to the contrary that some 47^+ -vertex v is incident with at least $\lfloor \frac{d(v)}{2} \rfloor + 1$ bad faces. Then some edge e incident with v is contained in two different bad faces. By the definition of bad faces, the end of e other than v has degree 2. So this 2-vertex is incident with two different bad faces, contradicting Lemma 3.5. \square

3.2. Discharging. We now define the initial charge at each vertex and each face. For every $v \in V(G)$, let $\mu(v) = 2d(v) - 6$ and for every face $f \in F(G)$, let $\mu(f) = d(f) - 6$. The total initial charge is negative since

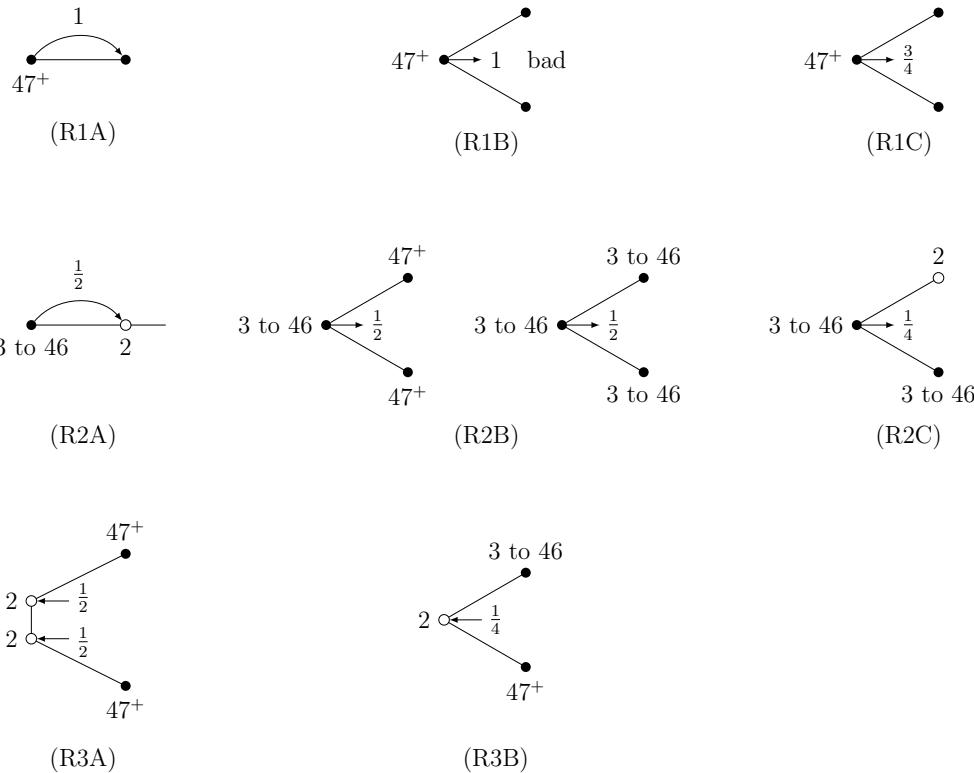
$$\begin{aligned} \sum_{z \in V(G) \cup F(G)} \mu(z) &= \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) \\ &= -6|V(G)| + 6|E(G)| - 6|F(G)| = -12 < 0. \end{aligned}$$

The last equality holds by Euler's formula.

Recall that a bad face is a 5-face and there are two nonadjacent 2-vertices on that face. For each face f , let W_f be a canonical boundary walk of f . Recall that for any face f and vertex v , $k_{f,v}$ is the number of triples (e, v, e') such that $e, e' \in E(G)$ and eve' is a subwalk of W_f .

Here are the discharging rules:

- (R1) Let v be a 47^+ -vertex.

FIG. 6. *Discharging rules.*

- (R1A) v sends charge 1 to each adjacent vertex.
- (R1B) v sends charge 1 to each incident bad face.
- (R1C) v sends charge $\frac{3}{4}k_{f,v}$ to each incident face f that is not bad.
- (R2) Let v be a vertex where $d(v) \in \{3, \dots, 46\}$.
 - (R2A) v sends charge $\frac{1}{2}$ to each adjacent 2-vertex.
 - (R2B) v sends charge $\frac{t}{2}$ to each incident face f , where t is the number of triples (x, v, y) such that $x, y \in V(G)$, xvy is a subpath in W_f , and either both $d(x), d(y)$ are at least 47, or both $d(x), d(y) \in \{3, \dots, 46\}$.
 - (R2C) v sends charge $\frac{t}{4}$ to each incident face f , where t is the number of triples (x, v, y) such that $x, y \in V(G)$, xvy is a subpath in W_f , $d(x) = 2$, and $d(y) \in \{3, \dots, 46\}$.
- (R3) Let f be a face.
 - (R3A) f sends charge $\frac{1}{2}k_{f,v}$ to each incident 2-vertex v that is adjacent to another 2-vertex.
 - (R3B) f sends charge $\frac{1}{4}k_{f,v}$ to each incident 2-vertex v that is adjacent to a vertex y with $d(y) \in \{3, \dots, 46\}$.

The discharging rule (R1) shows how a 47^+ -vertex distributes its initial charge, (R2) shows how a vertex with degree in $\{3, \dots, 46\}$ sends charge, and (R3) shows how a face sends its charge. Note that by Lemma 3.4, a face does not send charge to a 2-vertex via both (R3A) and (R3B). See Figure 6 for an illustration of the discharging rules.

The rest of this section will prove that the final charge $\mu^*(z)$ is nonnegative for each $z \in V(G) \cup F(G)$.

CLAIM 3.7. *Every vertex has nonnegative final charge.*

Proof. Assume v is a 2-vertex. If v is adjacent to two 47^+ -vertices, then v receives charge 1 from each of its neighbors by (R1A). Thus, $\mu^*(v) = -2 + 2 \cdot 1 = 0$. Note that v is adjacent to at least one 47^+ -vertex by Lemma 3.4. If v is adjacent to another 2-vertex, then v receives charge $2 \cdot \frac{1}{2}$ from the faces incident with v by (R3A). Thus, $\mu^*(v) = -2 + 2 \cdot \frac{1}{2} + 1 = 0$. Otherwise, v is adjacent to a vertex of degree from 3 to 46, which sends charge $\frac{1}{2}$ to v by (R2A). Also, v receives charge $2 \cdot \frac{1}{4}$ from the faces incident with v by (R3B). Thus, $\mu^*(v) = -2 + 2 \cdot \frac{1}{4} + \frac{1}{2} + 1 = 0$.

Assume v is a 47^+ -vertex. By (R1A), v sends charge at most $d(v)$ to its adjacent vertices in total. By Lemma 3.6, v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ bad faces. Since v sends charge 1 to each of its incident bad faces by (R1B) and sends charge $\frac{3}{4}$ to each of its incident faces that are not bad by (R1C), the final charge $\mu^*(v)$ is at least $2d(v) - 6 - d(v) - \lfloor \frac{d(v)}{2} \rfloor - \frac{3}{4} \cdot \lceil \frac{d(v)}{2} \rceil$, which is nonnegative since $d(v) \geq 47$.

Assume $d(v) \in \{4, \dots, 46\}$. We will show that v has nonnegative final charge by using a weighting argument. Let $u_1, \dots, u_{d(v)}$ be the neighbors of v in some cyclic order. First give all neighbors of v a weight of $\frac{1}{2}$. If u_i is not a 2-vertex, then split the weight of $\frac{1}{2}$ it received from v , and transfer weight $\frac{1}{4}$ to each of the two faces that are incident with vu_i (if vu_i is incident with only one face, then transfer weight $\frac{1}{2}$ to this face). Now, every 2-vertex adjacent to v and every face that is incident with v have weight equal to the charge sent from v in the discharging rules. So the total charge sent from v is at most the weight sent from v . Since v has charge $2d(v) - 6 \geq \frac{d(v)}{2}$ when $d(v) \geq 4$, v has nonnegative final charge.

Assume v is a 3-vertex. If v is adjacent to at least two 47^+ -vertices, which each sends charge 1 to v by (R1A), then v is adjacent to at most one 2-vertex. Thus, $\mu^*(v) \geq 0 + 2 - 4 \cdot \frac{1}{2} = 0$. If v is adjacent to exactly one 47^+ -vertex, then v sends charge at most $\frac{1}{2}$ at most twice according to the discharging rules. In either case, $\mu^*(v) \geq 0 + 1 - 2 \cdot \frac{1}{2} = 0$. \square

CLAIM 3.8. *Each 7^+ -face f has nonnegative final charge.*

Proof. We will show that f has nonnegative final charge by using a weighting argument. Pull weight $\frac{3}{4}k_{f,v}$ from each 47^+ -vertex v on f (note that this corresponds to (R1C)), and transfer weight $\frac{3}{8}k_{f,v}$ to each 2-vertex on f that is adjacent to v . Each 2-vertex on f receives weight at least $\frac{3}{8}k_{f,v}$, since it must be adjacent to a 47^+ -vertex, which is on f , by Lemma 3.4. Now if f sends an additional weight of $\frac{1}{8}k_{f,v}$ to each 2-vertex on f , then (R3) is satisfied. By Lemma 3.4, there cannot be three consecutive 2-vertices on a boundary walk of f , so it follows that $\sum k_{f,v} \leq \lfloor \frac{2}{3}d(f) \rfloor$, where the sum is over all 2-vertices incident with f . Therefore, $\mu^*(f) \geq d(f) - 6 - \frac{1}{8} \sum k_{f,v} \geq d(f) - 6 - \frac{1}{8} \lfloor \frac{2}{3}d(f) \rfloor > 0$ when $d(f) \geq 7$, where the sum is over all 2-vertices incident with f . \square

Note that there is no 6-face since G has no 1-vertex and no 3-, 4-, 6-cycles.

CLAIM 3.9. *Each 5-face f has nonnegative final charge.*

Proof. Since G has no 1-vertex, every 5-face is bounded by a cycle. Let v_1, v_2, v_3, v_4, v_5 be the vertices of f in some cyclic order.

Assume f is incident with at most one 2-vertex, and assume v_1 is the 2-vertex, if any. Note that f sends charge $\frac{1}{4}$ to v_1 by (R3B) if it is a 2-vertex. If at least two of v_2, \dots, v_5 are 47^+ -vertices, then f receives charge $\frac{3}{4}$ from each one by (R1C), thus,

$\mu^*(f) \geq -1 + 2 \cdot \frac{3}{4} - \frac{1}{4} > 0$. If exactly one of v_2, \dots, v_5 is a 47^+ -vertex, then without loss of generality we may assume it is v_2 by Lemma 3.4. Now v_4 sends charge $\frac{1}{2}$ to f by (R2B), thus, $\mu^*(f) \geq -1 + \frac{3}{4} + \frac{1}{2} - \frac{1}{4} = 0$. If none of v_2, \dots, v_5 is a 47^+ -vertex, then v_1 cannot be a 2-vertex. Since both v_3 and v_4 send charge $\frac{1}{2}$ by (R2B), it follows that $\mu^*(f) \geq -1 + 2 \cdot \frac{1}{2} = 0$.

Assume f is incident with at least two 2-vertices, where two of them, say v_2 and v_3 , are adjacent to each other. Note that f sends charge $\frac{1}{2}$ to each of v_2 and v_3 by (R3A). By Lemma 3.4, both v_1 and v_4 must be 47^+ -vertices. If v_5 is not a 2-vertex, then f is not a bad face, and v_1, v_4, v_5 send charge $\frac{3}{4}, \frac{3}{4}$, at least $\frac{1}{2}$, respectively, by (R1C) and (R2B). Thus, $\mu^*(f) \geq -1 + 2 \cdot \frac{3}{4} + \frac{1}{2} - 2 \cdot \frac{1}{2} = 0$. If v_5 is a 2-vertex, then f is a bad face, and both v_1 and v_4 send charge 1 each to f by (R1B). Thus, $\mu^*(f) \geq -1 + 2 \cdot 1 - 2 \cdot \frac{1}{2} = 0$.

If f is incident with at least two 2-vertices and where no pair is nonadjacent, then f is incident to exactly two 2-vertices by Lemma 3.4. Thus, the only remaining case is when f is incident with exactly two nonadjacent 2-vertices, say v_1 and v_3 . Note that f sends charge $\frac{1}{4}$ to each of v_1 and v_3 by (R3B). If f is incident with at least two 47^+ -vertices, which each sends charge at least $\frac{3}{4}$ to f by (R1), then $\mu^*(f) \geq -1 + 2 \cdot \frac{3}{4} - 2 \cdot \frac{1}{4} = 0$. Now f must be incident with exactly one 47^+ -vertex because f is incident with a 2-vertex, and by Lemma 3.4 we know that v_2 must be the 47^+ -vertex. It follows that $d(v_4), d(v_5) \in \{3, \dots, 46\}$ and, therefore, f is a bad face. Now, v_2, v_4, v_5 send charge $1, \frac{1}{4}, \frac{1}{4}$, respectively, to f by (R1B) and (R2C). Thus, $\mu^*(f) \geq -1 + 1 + 2 \cdot \frac{1}{4} - 2 \cdot \frac{1}{4} = 0$. \square

4. Unbalanced 3-partitions.

In this section, we prove the following theorem.

THEOREM 4.1. *A set S of cycles is an inclusionwise minimal cycle obstruction set of unbalanced 3-partitionable planar graphs if and only if $S = \{C_3\}$ or $S = \{C_4\}$.*

We will first show a necessary condition for cycle obstruction sets, and then show that it is sufficient afterwards.

LEMMA 4.2. *If a set S of cycles is an obstruction set of unbalanced 3-partitionable planar graphs, then either $C_3 \in S$ or $C_4 \in S$.*

Proof. Let $X_0(D; v)$ be the graph that is obtained from starting with $D + 1$ pairwise disjoint copies of K_4 and picking one vertex from each copy of K_4 and identifying them into v . Now let $X(D)$ be the graph obtained from three copies of $X_0(D; v)$ and adding three edges between the three vertices that correspond to v . See Figure 7 for an illustration of $X_0(2; v)$ and $X(2)$. Now in any $(0, 0, D)$ -coloring of $X_0(D; v)$, the vertex v cannot receive the color 3. This is because each copy of $K_4 - v$ must contain a vertex colored with 3, and since there are $D + 1$ copies, v has $D + 1$ neighbors with the same color, which is a contradiction. However, in any $(0, 0, D)$ -coloring of $X(D)$, one vertex v of the three cut vertices must receive the color 3, and this shows that $X(D)$ is not $(0, 0, D)$ -colorable. It is not hard to see that the only cycles in $X(D)$ have length either 3 or 4. \square

If a planar graph does not contain 3-cycles, then it is 3-colorable, which is equivalent to $(0, 0, 0)$ -colorable, by Grötzsch's theorem [13], and thus it is unbalanced 3-partitionable. This means that $\{C_3\}$ is an inclusionwise minimal obstruction set of unbalanced 3-partitionable planar graphs. The remainder of this section proves Theorem 4.3 below, which states that planar graphs with no 4-cycles are unbalanced 3-partitionable. Note that Lemma 4.2 and Theorem 4.3 imply Theorem 4.1.

THEOREM 4.3. *Any planar graph with no 4-cycles is $(0, 0, 117)$ -colorable.*

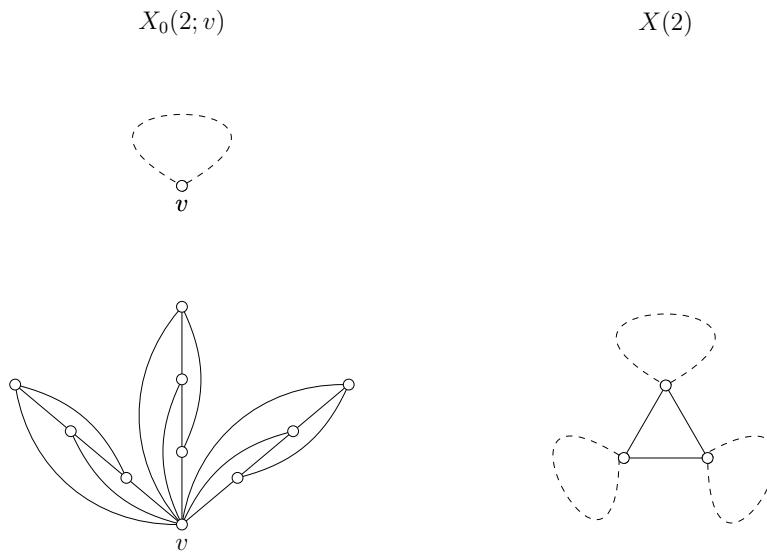


FIG. 7. Graphs that are not $(0, 0, D)$ -colorable.

In this section, let G be a counterexample to Theorem 4.3 with the minimum number of vertices. Also, fix a plane embedding of G . It is easy to see that G is connected and there are no 2^- -vertices in G .

From now on, given a (partially) $(0, 0, 117)$ -colored graph, we will let a, b, c be the color of the color class that is allowed to have maximum degree at most 0, 0, 117, respectively, and we say a vertex colored with c is *saturated* if it already has 117 neighbors colored with c .

4.1. Structural lemmas.

LEMMA 4.4. *A 119^- -vertex is adjacent to a 120^+ -vertex.*

Proof. Suppose to the contrary that a 119^- -vertex v is adjacent to only 119^- -vertices. Since $G - v$ is a graph with fewer vertices than G , there is a $(0, 0, 117)$ -coloring φ of $G - v$. We further assume that φ minimizes the number of neighbors of v colored with c . If there exists a neighbor u of v in G such that $\varphi(u) = c$ and u is saturated, then at most one neighbor of u in $G - v$ has a color in $\{a, b\}$, so we can recolor u to be a color in $\{a, b\}$ that does not appear in its neighborhood in $G - v$, contradicting the minimality of φ . Hence no neighbor u of v with color c is saturated. If no neighbor of v is colored with a color in $\{a, b\}$, then we can extend φ to all of G by coloring v with a color in $\{a, b\}$ that does not appear in the neighborhood of v in G , contradicting that G is a counterexample. So both a and b appear in the neighborhood of v in G , and thus there are at most 117 neighbors of v colored with c . Since no neighbor of v with color c is saturated, we can extend φ to all of G by coloring v with color c , a contradiction. \square

LEMMA 4.5. *Let X be a set of 3-vertices of G such that the subgraph of G induced on X is a path $v_1v_2\dots v_k$, where $k \geq 2$. If x and y are the neighbors of v_k in $G - X$, then $c \in \{\varphi(x), \varphi(y)\}$ and $\varphi(x) \neq \varphi(y)$ for every $(0, 0, 117)$ -coloring φ of $G - X$. Moreover, the vertex in $\{x, y\}$ that receives the color c must be a 116^+ -vertex.*

Proof. Let φ be a $(0, 0, 117)$ -coloring of $G - X$ and let x' and y' be the neighbors of v_1 in $G - X$. For each integer i with $2 \leq i \leq k - 1$, let u_i be the vertex in $G - X$ adjacent to v_i . First we extend φ to a $(0, 0, 117)$ -coloring of $G - v_k$ by defining $\varphi(v_1) \in \{a, b, c\} \setminus \{\varphi(x'), \varphi(y')\}$ and $\varphi(v_i) \in \{a, b, c\} \setminus \{\varphi(v_{i-1}), \varphi(u_i)\}$ for each $i \in \{2, \dots, k-1\}$. If $\varphi(x) = \varphi(y)$, then we can extend φ to be a $(0, 0, 117)$ -coloring of G by further defining $\varphi(v_k)$ to be an element in $\{a, b, c\} \setminus \{\varphi(v_{k-1}), \varphi(x)\}$. This proves $\varphi(x) \neq \varphi(y)$. If either $c \notin \{\varphi(x), \varphi(y)\}$ or the vertex in $\{x, y\}$ with the color c is a 115^- -vertex, then by defining $\varphi(v_k) = c$, we extended φ to a $(0, 0, 117)$ -coloring of G since the degree of v_{k-1} is 3. Therefore, $c \in \{\varphi(x), \varphi(y)\}$. \square

LEMMA 4.6. *Let X be a set of 3-vertices of G such that the subgraph of G induced on X is a path $v_1v_2\dots v_{2k}$ on an even number of vertices. Let u_i be a neighbor of v_i in $G - X$ for each i with $1 \leq i \leq 2k$. Let x and y be the neighbor of v_1 and v_{2k} , respectively, in $G - X$ other than u_1 and u_{2k} . If there exists a $(0, 0, 117)$ -coloring φ of $G - X$ such that $\varphi(u_i) = c$ for every i with $1 \leq i \leq 2k$, then $\varphi(x) = \varphi(y)$.*

Proof. By Lemma 4.5, we may assume $\varphi(x) \neq \varphi(u_1) = c$. Define $\varphi(v_1) = \{a, b\} \setminus \{\varphi(x)\}$ and $\varphi(v_i) = \{a, b\} \setminus \{\varphi(v_{i-1})\}$ for every $2 \leq i \leq 2k$. Since $|X|$ is even, $\varphi(v_1) \neq \varphi(v_{2k})$. That is, $\varphi(v_{2k}) = \varphi(x)$. As this must not extend φ to be a $(0, 0, 117)$ -coloring of G , $\varphi(y) = \varphi(v_{2k})$. Therefore $\varphi(y) = \varphi(x)$. \square

A face f is *annoying* if exactly one vertex incident with f is a 120^+ -vertex and all other vertices incident with f are 3-vertices. We say that two faces are *adjacent* if they share at least one edge.

LEMMA 4.7. *If an annoying 5-face f is adjacent to only annoying 3-faces and annoying 5-faces, then f is adjacent to at most two 3-faces.*

Proof. Let $f = wx'xyy'$, where w is the 120^+ -vertex on f and x', x, y, y' are all 3-vertices. For $e \in \{wx', x'x, xy, yy', y'w\}$, let f_e be the face incident with e other than f .

Suppose to the contrary that f is adjacent to three annoying 3-faces. Since G has no 4-cycles, two 3-faces cannot share an edge, and two faces incident with the same 3-vertex must share an edge. Since x', x, y, y' are all 3-vertices, this implies that $f_{xy}, f_{wx'}, f_{y'w}$ must be the annoying 3-faces adjacent to f .

Let $f_{xy} = xyz$ so that z is a 120^+ -vertex and the common neighbor of x and y . Also, let $f_{wx'} = wx'x$ and $f_{y'w} = wy'y_1$ so that x_1 and y_1 is the common neighbor of w, x' and w, y' , respectively, which must be a 3-vertex. Note that z, x_1, y_1 must be all distinct since otherwise that would imply the existence of a 4-cycle.

Let z_1 be the common neighbor of z and x_1 . Since $f_{xx'}$ is an annoying 5-face, z_1 must be a 3-vertex. Also, $z_1 \notin \{z, x, y, x', y', x_1, y_1, w\}$ since there are no 4-cycles. Let z_2 be the common neighbor of z and y_1 . Similarly, z_2 is a 3-vertex and $z_2 \notin \{z, x, y, x', y', x_1, y_1, w\}$. Note that $z_1 \neq z_2$, since z has degree at least 120, $f_{xx'}$ and $f_{yy'}$ are 5-faces, and xyz is a 3-face. Note that the subgraph of G induced on $\{x_1, x', x, y, y', y_1\}$ is a path. See Figure 8 for an illustration.

Suppose that z_1z_2 is not an edge of G . Set $H = (G - \{x, y, x', y'\}) \cup x_1y_1$. Note that H is still a plane graph with no 4-cycles, since z_1z_2 is not an edge. Since H is a graph with fewer vertices than G , there is a $(0, 0, 117)$ -coloring φ of H . Note that φ is a $(0, 0, 117)$ -coloring of $G - \{x, y, x', y'\}$. If $\varphi(z) \neq c$, then let

$$\begin{aligned} \varphi(x) &= c, & \varphi(x') &\in \{a, b, c\} \setminus \{\varphi(x_1), \varphi(w)\}, \\ \varphi(y') &\in \{a, b, c\} \setminus \{\varphi(w), \varphi(y_1)\}, & \varphi(y) &\in \{a, b, c\} \setminus \{\varphi(z), \varphi(y')\}, \end{aligned}$$

to extend φ to all of G . Hence $\varphi(z) = c$. Since w is a 120^+ -vertex, by Lemma 4.5,

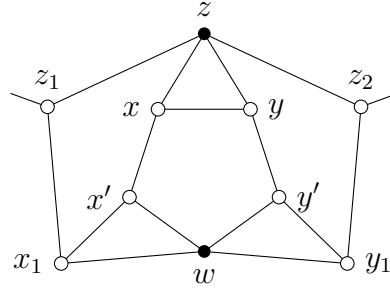


FIG. 8. Figure for Lemma 4.7.

$\varphi(w) = c$ and $\{\varphi(x_1), \varphi(y_1)\} \subseteq \{a, b\}$. Since $\varphi(z) = \varphi(w) = c$, Lemma 4.6 implies that $\varphi(x_1) = \varphi(y_1)$. However, $\{\varphi(x_1), \varphi(y_1)\} \subseteq \{a, b\}$ and x_1y_1 is an edge of H , so $\varphi(x_1) \neq \varphi(y_1)$, a contradiction.

Therefore, z_1z_2 is an edge of G . Since $G - \{x, y, x', y', x_1, y_1\}$ is a graph with fewer vertices than G , there exists a $(0, 0, 117)$ -coloring φ for $G - \{x, y, x', y', x_1, y_1\}$. If $\varphi(z) \neq c$, then let

$$\begin{aligned} \varphi(x) &= c, & \varphi(x_1) &\in \{a, b, c\} \setminus \{\varphi(z_1), \varphi(w)\}, \\ \varphi(x') &\in \{a, b, c\} \setminus \{\varphi(x_1), \varphi(w)\}, & \varphi(y_1) &\in \{a, b, c\} \setminus \{\varphi(z_2), \varphi(w)\}, \\ \varphi(y') &\in \{a, b, c\} \setminus \{\varphi(w), \varphi(y_1)\}, & \varphi(y) &\in \{a, b, c\} \setminus \{\varphi(z), \varphi(y')\}, \end{aligned}$$

to extend φ to all of G , which is a contradiction. Hence $\varphi(z) = c$. Since w is a 120^+ -vertex, by Lemma 4.5, $\varphi(w) = c$ and $\{\varphi(z_1), \varphi(z_2)\} \subseteq \{a, b\}$. Since $\varphi(z) = \varphi(w) = c$, Lemma 4.6 implies that $\varphi(z_1) = \varphi(z_2)$. However, $\{\varphi(z_1), \varphi(z_2)\} \subseteq \{a, b\}$ and z_1z_2 is an edge of $G - \{x, y, x', y', x_1, y_1\}$, so $\varphi(z_1) \neq \varphi(z_2)$, a contradiction. \square

4.2. Discharging. We now define the initial charge at each vertex and each face. For every $z \in V(G) \cup F(G)$, let $\mu(z) = d(z) - 4$. The total initial charge is negative since

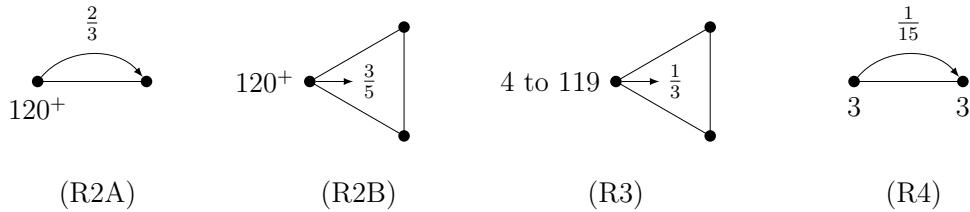
$$\begin{aligned} \sum_{z \in V(G) \cup F(G)} \mu(z) &= \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) \\ &= -4|V(G)| + 4|E(G)| - 4|F(G)| = -8 < 0. \end{aligned}$$

The last equality holds by Euler's formula.

Here are the discharging rules:

- (R1) Each 5^+ -face f sends charge $\frac{k_{f,v}}{r}(d(f) - 4)$ to each incident 3-vertex v , where $r = \sum k_{f,u}$ and the sum is over all 3-vertices u incident with f .
- (R2) Let v be a 120^+ -vertex.
 - (R2A) v sends charge $\frac{2}{3}$ to each neighbor.
 - (R2B) v sends charge $\frac{3}{5}$ to each incident 3-face.
- (R3) Each vertex v , where $d(v) \in \{4, \dots, 119\}$, sends charge $\frac{1}{3}$ to each incident 3-face.
- (R4) Each 3-vertex that is not incident with a 3-face sends charge $\frac{1}{15}$ to each adjacent 3-vertex.

The discharging rule (R1) shows how a face distributes its initial charge, (R2) shows how a 120^+ -vertex sends charge, (R3) shows how a vertex with degree in

FIG. 9. *Discharging rules.*

$\{4, \dots, 119\}$ sends charge, and (R4) shows how a 3-vertex that is not incident with a 3-face sends charge to an adjacent 3-vertex. See Figure 9 for an illustration of the discharging rules.

The rest of this section will prove that the sum of the final charge $\mu^*(z)$ is non-negative for $z \in V(G) \cup F(G)$. Note that every 5^+ -face has nonnegative final charge since it only distributes its initial charge, which is positive. There are no 4-faces since there are no 4-cycles, and each edge is incident with at most one 3-face since there are no 4-cycles. We will first show that each 4^+ -vertex has nonnegative final charge. Then, instead of counting 3-vertices and 3-faces separately, we will compute the final charge of 3-faces and 3-vertices together.

CLAIM 4.8. *Every 4^+ -vertex v has nonnegative final charge.*

Proof. Note that v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces since there are no 4-cycles. If v is a 119^- -vertex, then by Lemma 4.4, v has a neighbor u that is a 120^+ -vertex. By (R2A), u sends charge $\frac{2}{3}$ to v , and by (R3), v sends charge at most $\frac{1}{3} \cdot \lfloor \frac{d(v)}{2} \rfloor$ to its incident 3-faces. Thus, $\mu^*(v) \geq d(v) - 4 + \frac{2}{3} - \frac{1}{3} \cdot \lfloor \frac{d(v)}{2} \rfloor \geq 0$ when $d(v) \geq 4$.

Now assume v is a 120^+ -vertex. Then v sends charge at most $\frac{2d(v)}{3}$ to its neighbors by (R2A) and v sends charge at most $\frac{3}{5} \cdot \lfloor \frac{d(v)}{2} \rfloor$ to its incident 3-faces by (R2B). Thus, $\mu^*(v) \geq d(v) - 4 - \frac{2d(v)}{3} - \frac{3}{5} \cdot \lfloor \frac{d(v)}{2} \rfloor \geq 0$ when $d(v) \geq 120$. \square

Note that a 6^+ -face and 5^+ -face send charge at least $\frac{1}{3}$ and at least $\frac{1}{5}$, respectively, to each incident 3-vertex. In particular, a 5-face that is incident with at least one 3-vertex and at least two 4^+ -vertices sends charge at least $\frac{1}{4}$ and at least $\frac{1}{3}$, respectively, to each incident 3-vertex.

CLAIM 4.9. *Each 3-vertex v that is not incident with a 3-face has positive final charge.*

Proof. By Lemma 4.4, v has a 120^+ -vertex u as a neighbor. The faces incident with v send charge at least $3 \cdot \frac{1}{5}$ to v by (R1) and u sends charge $\frac{2}{3}$ to v by (R2). Also v loses charge $\frac{1}{15}$ at most twice by (R4). Thus, $\mu^*(v) \geq -1 + \frac{3}{5} + \frac{2}{3} - \frac{2}{15} > 0$. \square

CLAIM 4.10. *If f is a 3-face that is incident with three 3-vertices x, y, z , then the sum of the final charge of f, x, y, z is nonnegative.*

Proof. Let x', y', z' be the neighbor of x, y, z , respectively, that is not on f . Since there are no 4-cycles, x', y', z' are pairwise distinct. By Lemma 4.4, x', y', z' are all 120^+ -vertices. Since x, y, z are 3-vertices, xx', yy', zz' are not contained in 3-faces. Therefore each face that is adjacent to f is incident with at least two 120^+ -vertices. Thus, each of x, y, z receives charge at least $\frac{2}{3}$ from the incident faces by (R1). Now

x', y', z' sends charge $\frac{2}{3}$ to x, y, z , respectively, by (R2A). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) + \mu^*(z) \geq -4 + 3 \cdot \frac{2}{3} + 3 \cdot \frac{2}{3} = 0$. \square

CLAIM 4.11. *If f is a 3-face xyz that is incident with exactly two 3-vertices x and y , then the sum of the final charge of f, x, y is nonnegative.*

Proof. Let x' and y' be the neighbor of x and y , respectively, that is not on f . Note that x' and y' are distinct, and xx' and yy' are not contained in any 3-faces since x and y are 3-vertices and G has no 4-cycles.

Assume z is not a 120^+ -vertex. This implies that x' and y' are both 120^+ -vertices by Lemma 4.4. Therefore each face that is adjacent to f is incident with at least two 4^+ -vertices. Thus, each of x and y receives charge at least $\frac{2}{3}$ from the incident faces by (R1). Now x' and y' sends charge $\frac{2}{3}$ to x and y , respectively, by (R2A). Also z sends charge $\frac{1}{3}$ to f by (R3). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + 2 \cdot \frac{2}{3} + 2 \cdot \frac{2}{3} + \frac{1}{3} = 0$.

Assume z is a 120^+ -vertex. This implies that x, y, f receive charge $\frac{2}{3}, \frac{2}{3}$, at least $\frac{3}{5}$, from z by (R2A), (R2A), (R2B), respectively; note that the sum of these charges is $\frac{29}{15}$. Let f_{xy}, f_{zx}, f_{zy} be the face incident with xy, zx, zy , respectively, that is not f . It is possible that f_{xy}, f_{zy} , and f_{zx} are not pairwise distinct. Assume that one of x', y' is a 4^+ -vertex. Without loss of generality, we may assume that x' is a 4^+ -vertex. By (R1), f_{zx} and f_{xy} give charges at least $\frac{1}{3}$ and at least $\frac{1}{2}$ to x and x, y , respectively. Also, f_{zy} gives charge $\frac{1}{4}$ to y by (R1). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + \frac{29}{15} + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} > 0$. So we may assume both x' and y' are 3-vertices. By Lemma 4.4, x' and y' must have a neighbor x'' and y'' , respectively, that is a 120^+ -vertex. Note that $x'' = y''$ is possible.

If none of x'' and y'' is incident with f_{xy} , then f_{xy} sends charge at least $\frac{2}{5}$ to x and y by (R1), and each f_{zx} and f_{zy} sends charge at least $\frac{1}{3}$ to x and y , respectively, by (R1). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + \frac{29}{15} + \frac{2}{5} + 2 \cdot \frac{1}{3} = 0$. If exactly one of x'' and y'' is incident with f_{xy} , then without loss of generality, we may assume x'' is incident with f_{zx} and y'' is incident with f_{xy} . Now, by (R1), f_{xy} sends charge at least $\frac{1}{4}$ to each of x and y , and f_{zx} and f_{zy} sends charge at least $\frac{1}{3}$ and at least $\frac{1}{4}$ to x and y , respectively. Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + \frac{29}{15} + \frac{1}{3} + 3 \cdot \frac{1}{4} > 0$.

Assume both x'' and y'' are incident with f_{xy} . If $x'' \neq y''$, then $d(f_{xy}) \geq 6$ and f_{xy} sends charge at least $\frac{1}{3}$ to each of x and y by (R1), and f_{zx} and f_{zy} sends charge at least $\frac{1}{4}$ to x and y , respectively, by (R1). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + \frac{29}{15} + \frac{2}{3} + 2 \cdot \frac{1}{4} > 0$. Now consider the case when $x'' = y''$, so f_{xy} sends charge $\frac{1}{4}$ to each of x and y by (R1). If one of f_{zy} and f_{zx} is either a 6^+ -face or a 5-face that is not annoying, then it sends charge at least $\frac{1}{3}$ to y or x by (R1) and the other face still sends charge to x or y at least $\frac{1}{4}$ by (R1). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + \frac{29}{15} + \frac{1}{3} + 3 \cdot \frac{1}{4} > 0$. So assume each of f_{zx} and f_{zy} is an annoying 5-face, which sends charge $\frac{1}{4}$ by (R1). In particular, x' and y' have degree 3. Therefore, x and y receive a total of charge 1 by the surrounding faces.

If f' is a 3-face incident with x' , then it is incident with x', x'' , and a vertex on f_{zx} other than z . Since f_{zx} is an annoying 5-face, f' is an annoying 3-face. So every 3-face incident with x' is annoying. Similarly, every 3-face incident with y' is annoying. Since $xyy'x''x'$ is an annoying 5-face and xyz is an annoying 3-face, either one of x', y' is not incident with any 3-face, or some 3-face is incident with both x', y' , by Lemma 4.7. The later implies that x' is adjacent to y' , which is a contradiction since $x'y'yx$ is now a 4-cycle. Hence one of x', y' is not incident with any 3-face, and that vertex sends charge $\frac{1}{15}$ to either x or y by (R4). Thus, $\mu^*(f) + \mu^*(x) + \mu^*(y) \geq -3 + \frac{29}{15} + 1 + \frac{1}{15} = 0$. \square

CLAIM 4.12. *If f is a 3-face xyz that is incident with exactly one 3-vertex x , then the sum of the final charge of f and x is nonnegative.*

Proof. By Lemma 4.4, x has a neighbor x' that is a 120^+ -vertex.

Assume $x' \notin \{y, z\}$. The sum of charge received from the faces incident with x is at least $2 \cdot \frac{1}{3}$ by (R1). Also, x' sends charge $\frac{2}{3}$ to x by (R2A). Each of y and z sends charge at least $\frac{1}{3}$ to f by either (R2B) or (R3). Thus, $\mu^*(f) + \mu^*(x) \geq -2 + 4 \cdot \frac{1}{3} + \frac{2}{3} = 0$.

So we may assume $x' \in \{y, z\}$. Without loss of generality, assume $x' = y$. The sum of charge received from the faces incident with x is at least $2 \cdot \frac{1}{4}$ by (R1). Now x' sends charges $\frac{2}{3}$ and $\frac{3}{5}$ to x and f by (R2A) and (R2B), respectively. Also, z sends charge at least $\frac{1}{3}$ to f by either (R2B) or (R3). Thus, $\mu^*(f) + \mu^*(x) \geq -2 + 2 \cdot \frac{1}{4} + \frac{2}{3} + \frac{3}{5} + \frac{1}{3} > 0$. \square

CLAIM 4.13. *If f is a 3-face xyz that is incident with no 3-vertices, then the final charge of f is nonnegative.*

Proof. Since each of x, y, z is a 4^+ -vertex, each of x, y, z sends charge at least $\frac{1}{3}$ to f by either (R2B) or (R3). Thus, $\mu^*(f) \geq -1 + 3 \cdot \frac{1}{3} = 0$. \square

Since no 3-vertex is contained in two different 3-faces, the sum of the final charge on all 3-faces and all 3-vertices is nonnegative by Claims 4.9, 4.10, 4.11, 4.12, and 4.13.

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