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SCATTERING RESONANCES FOR HIGHLY OSCILLATORY POTENTIALS

BY Alexis DROUOT

Abstract. – We study resonances of compactly supported potentials $V_\varepsilon(x) = W(x, x/\varepsilon)$ where $W : \mathbb{R}^d \times \mathbb{R}^d/(2\pi\mathbb{Z})^d \to \mathbb{C}$, d odd. That means that V_ε is a sum of a slowly varying potential, W_0, and one oscillating at frequency $1/\varepsilon$. When $W_0 \equiv 0$ we prove that there are no resonances above the line $\operatorname{Im}\lambda = -A\ln(\varepsilon^{-1})$, except a simple resonance near 0 when $d = 1$. We show that this result is optimal by constructing a one-dimensional example. This settles a conjecture of Duchêne-Vukićević-Weinstein [12]. When $W_0 \neq 0$ and W smooth we prove that resonances in fixed strips admit an expansion in powers of ε. The argument provides a method for computing the coefficients of the expansion. We produce an effective potential converging uniformly to W_0 as $\varepsilon \to 0$ and whose resonances approach resonances of V_ε modulo $O(\varepsilon^4)$. This improves the one-dimensional result of Duchêne, Vukićević and Weinstein and extends it to all odd dimensions.

Résumé. – Nous étudions les résonances de potentiels à support compact $V_\varepsilon(x) = W(x, x/\varepsilon)$, où $W : \mathbb{R}^d \times \mathbb{R}^d/(2\pi\mathbb{Z})^d \to \mathbb{C}$ et d est impair. Ainsi, V_ε est la somme d'un potentiel qui varie lentement W_0 et d'un potentiel qui oscille à fréquence $1/\varepsilon$. Quand $W_0 \equiv 0$ nous prouvons que V_ε n'a pas de résonances dans la zone $\{\operatorname{Im}\lambda \geq -A\ln(\varepsilon^{-1})\}$ mise à part une unique résonance proche de 0 si $d = 1$. Nous montrons par un exemple explicite que ce résultat est optimal. Cela prouve une conjecture de Duchêne-Vukićević-Weinstein [12]. Quand $W_0 \neq 0$ et W est lisse nous montrons que les resonances de V_ε qui restent bornées lorsque ε tend vers 0 admettent une expansion en puissances de ε. Les arguments de la preuve permettent de calculer les coefficients de cette expansion. Nous construisons un potentiel effectif qui converge uniformément vers W_0 lorsque ε tend vers 0 et dont les résonances sont à distance $O(\varepsilon^4)$ de celles de W_0. Cela améliore et étend les résultats de Duchêne, Vukićević et Weinstein à toutes les dimensions impaires.

1. Introduction

In this paper we are interested in the poles of the meromorphic continuation of $(-\Delta + \mathcal{V} - \lambda^2)^{-1}$ where d is odd and $\mathcal{V} : \mathbb{R}^d \to \mathbb{C}$ is a bounded compactly supported potential. These poles called scattering resonances appear in many physical situations, for instance their imaginary parts are the rates of decay of waves scattered by $\mathcal{V}$.

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Let $-\Delta \geq 0$ be the free Laplacian on $\mathbb{R}^d$. The operator $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, well defined as an operator $L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$ for $\operatorname{Im}\lambda > 0$, extends to a meromorphic family of bounded operators $L^2_{\text{comp}}(\mathbb{R}^d) \to H^2_{\text{loc}}(\mathbb{R}^d)$ for $\lambda \in \mathbb{C}$ (see §1.5 for review of notation). This family admits one simple pole at 0 if $d = 1$ and is entire if $d \geq 3$. If $\mathcal{V}$ is a bounded compactly supported function on $\mathbb{R}^d$ then $R_{\mathcal{V}}(\lambda) = (-\Delta + \mathcal{V} - \lambda^2)^{-1}$ is well defined for $\operatorname{Im}\lambda \gg 1$ as an operator $L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$. It extends to a meromorphic family of operators $L^2_{\text{comp}}(\mathbb{R}^d) \to H^2_{\text{loc}}(\mathbb{R}^d)$. In this sense, the resonances of a real-valued potential $\mathcal{V}$—similarly, the poles of the meromorphic continuation of $R_{\mathcal{V}}(\lambda)$—are a generalization of eigenvalues of $-\Delta + \mathcal{V}$: each eigenvalue E of $-\Delta + \mathcal{V}$ is negative and generates a resonance $i\sqrt{-E}$, and conversely every resonance λ of $\mathcal{V}$ in the upper half-plane lies in $i[0,\infty)$ and corresponds to the eigenvalue λ^2. Resonances of $\mathcal{V}$ in the lower half-plane are not related to eigenvalues of $-\Delta + \mathcal{V}$, though they quantize the rate of decay of waves scattered by $\mathcal{V}$. We refer to [15, §2, 3] for a complete introduction to resonances in potential scattering.

Let W be a *bounded* complex valued function with support in $\mathbb{B}^d(0,L) \times \mathbb{T}^d$. We define V_ε as

$$V_\varepsilon(x) = W\left(x, \frac{x}{\varepsilon}\right).$$

If W is formally given by

$$W(x,y) = \sum_{k \in \mathbb{Z}^d} W_k(x) e^{iky}$$

we can write V_ε as a highly oscillatory perturbation of W_0:

$$V_\varepsilon(x) = W_0(x) + V_\sharp(x), \quad V_\sharp(x) = \sum_{k \neq 0} W_k(x) e^{ikx/\varepsilon}. \tag{1.1}$$

In this paper we study resonances of potentials V_ε given by (1.1).

1.1. Main results

The first theorem concerns the case of a vanishing slowly varying part. In the notations of (1.1) we will assume for this result that $W \in L^\infty_0(\mathbb{B}^d(0,L) \times \mathbb{T}^d)$ (i.e., $\operatorname{supp}(W)$ is a compact subset of $\mathbb{B}^d(0,L) \times \mathbb{T}^d$ and W is uniformly bounded) and that moreover,

$$\begin{aligned} &\exists s \in (0,1), \quad \sum_{k \neq 0} \frac{|W_k|_{H^s}}{|k|^s} < \infty \text{ if } d = 1, \\ &\sum_{k \neq 0} \frac{\|W_k\|_1}{|k|} < \infty \text{ if } d \geq 3. \end{aligned} \tag{1.2}$$

Theorem 1. – *Let W be in $L^\infty_0(\mathbb{B}^d(0,L) \times \mathbb{T}^d, \mathbb{C})$ such that $W_0 \equiv 0$ and (1.2) holds. Then there exists C, c, A three positive constants such that*

$$\begin{aligned} &\text{if } d = 1, \ \operatorname{Res}(V_\varepsilon) \setminus \mathbb{D}\left(0, c\varepsilon^{s/2}\right) \subset \left\{\lambda \in \mathbb{C} : \ \operatorname{Im}\lambda \leq C - A\ln(\varepsilon^{-1})\right\}; \\ &\text{if } d \geq 3, \ \operatorname{Res}(V_\varepsilon) \subset \left\{\lambda \in \mathbb{C} : \ \operatorname{Im}\lambda \leq C - A\ln(\varepsilon^{-1})\right\}. \end{aligned}$$

This settles a conjecture of [12]: for odd dimensions $d \geq 3$ and ε small enough the potential V_ε does not have a bound state. In §2.3 we construct a step-like function W such that $V_{\pi/(2n)}$ has a resonance $\lambda_n \sim -i\ln(n)$ as $n \to +\infty$. This shows that one cannot improve the rate of escape of resonances given by Theorem 1 in dimension 1.

In the next statements we always assume that W is smooth. We consider the case $W_0 \neq 0$. If λ_0 is a simple resonance of W_0 we can write

$$R_{W_0}(\lambda) = \frac{iu \otimes v}{\lambda - \lambda_0} + H(\lambda),\ H(\lambda) \text{ holomorphic near } \lambda_0, \tag{1.3}$$

for some functions $u, v \in H^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C})$ called resonant states. As the potential V_ε given by (1.1) converges weakly to W_0, it is natural to expect that resonances of V_ε converge to resonances of W_0. In fact a much stronger statement holds:

THEOREM 2. – *Let W belong to $C_0^\infty(\mathbb{B}^d(0,L)\times\mathbb{T}^d,\mathbb{C})$ and V_ε be given by* (1.1). *Let λ_0 be a simple resonance of W_0. In a neighborhood of λ_0 and for ε small enough the potential V_ε admits a unique resonance λ_ε. Moreover, for any N,*

$$\lambda_\varepsilon = \lambda_0 + c_2\varepsilon^2 + c_3\varepsilon^3 + \cdots + c_{N-1}\varepsilon^{N-1} + O(\varepsilon^N),\ \ c_j \in \mathbb{C}.$$

If u, v are the resonant states of (1.3) *then*

$$\begin{aligned} c_2 &= i\int_{\mathbb{R}^d} \Lambda_0(x)u(x)v(x)dx, & c_3 &= i\int_{\mathbb{R}^d} \Lambda_1(x)u(x)v(x)dx,\\ \Lambda_0 &= \sum_{k\neq 0} \frac{W_k W_{-k}}{|k|^2}, & \Lambda_1 &= -2\sum_{k\neq 0} \frac{W_{-k}((k\cdot D)W_k)}{|k|^4}. \end{aligned} \tag{1.4}$$

If W is real-valued then so are Λ_0 and Λ_1. In §3.1 we will prove a version of Theorem 2 for resonances of higher multiplicity. Theorem 2 implies that perturbations of W_0 by a high frequency potential $V_\sharp$ enjoy some similarities with suitable analytic perturbations of W_0. In fact we have the following

THEOREM 3. – *Assume that W belongs to $C_0^\infty(\mathbb{B}^d(0,L) \times \mathbb{T}^d,\mathbb{C})$ and that V_ε is given by* (1.1). *Let $V_{\text{eff},\varepsilon} = W_0 - \varepsilon^2\Lambda_0 - \varepsilon^3\Lambda_1$ where Λ_0, Λ_1 are given in* (1.4). *For every bounded family $\varepsilon \mapsto \mu_\varepsilon$ of simple resonances of $V_{\text{eff},\varepsilon}$ there exists a family of resonances $\varepsilon \mapsto \lambda_\varepsilon$ of V_ε such that*

$$|\lambda_\varepsilon - \mu_\varepsilon| = O(\varepsilon^4).$$

Conversely for every bounded family $\varepsilon \mapsto \lambda_\varepsilon$ of simple resonances of V_ε there exists a family of resonances $\varepsilon \mapsto \mu_\varepsilon$ of $V_{\text{eff},\varepsilon}$ such that

$$|\lambda_\varepsilon - \mu_\varepsilon| = O(\varepsilon^4).$$

The potential $V_{\text{eff},\varepsilon}$ plays the role of an effective potential. In dimension one Λ_0 was already derived in [12].

We next give a uniform description of the behavior of resonances of V_ε as $\varepsilon \to 0$. For $W_0 \in C_0^\infty(\mathbb{B}^d(0,L),\mathbb{C})$ we define $m_{W_0}(\lambda_0)$ the multiplicity of a resonance λ_0 of W_0. If

ε, B, c, A are given positive constants let $\mathscr{C}_\varepsilon$, $\mathscr{T}_\varepsilon$ and $\mathscr{D}_\varepsilon$ be the sets

$$\mathscr{C}_\varepsilon = \bigcup_{\substack{\lambda \in \mathrm{Res}(W_0),\\ \mathrm{Im}\,\lambda \geq -B}} \mathbb{D}\left(\lambda, c\varepsilon^{2/m_{W_0}(\lambda)}\right), \quad \mathscr{T}_\varepsilon = \bigcup_{\substack{\lambda \in \mathrm{Res}(W_0),\\ \mathrm{Im}\,\lambda \leq -B}} \mathbb{D}\left(\lambda, \langle\lambda\rangle^{-d-1}\right) \tag{1.5}$$

$$\mathscr{D}_\varepsilon = \left\{\lambda \in \mathbb{C} : \ \mathrm{Im}\,\lambda \leq -B, \ |\lambda|^{2d+1} \geq A \ln(\varepsilon^{-1})\right\}.$$

Theorem 4. – *Assume that W belongs to $C_0^\infty(\mathbb{B}^d(0,L) \times \mathbb{T}^d, \mathbb{C})$ and that V_ε is given by* (1.1). *There exists $A > 0$ with the following. For any $B > 0$, there exists $c > 0$ such that for all ε small enough if $\mathscr{C}_\varepsilon$, $\mathscr{T}_\varepsilon$ and $\mathscr{D}_\varepsilon$ are given by* (1.5) *then*

$$\mathrm{Res}(V_\varepsilon) \subset \mathscr{C}_\varepsilon \cup \mathscr{T}_\varepsilon \cup \mathscr{D}_\varepsilon.$$

A different version of Theorem 4 is stated as follow. Let $\varepsilon \mapsto \lambda_\varepsilon$ be a family of resonances of V_ε. Then after passing to a subsequence $\varepsilon_j \to 0$, one of the three following scenarios occurs:

(i) λ_ε converges to a resonance λ_0 of W_0 and $\lambda_\varepsilon = \lambda_0 + O(\varepsilon^{2/m_{W_0}(\lambda_0)})$.
(ii) $\mathrm{Im}\,\lambda_\varepsilon \to -\infty$ and $|\lambda_\varepsilon|$ grows at least like $\ln(\varepsilon^{-1})^{1/(2d+1)}$.
(iii) $\mathrm{Im}\,\lambda_\varepsilon \to -\infty$ and $d(\lambda_\varepsilon, \mathrm{Res}(W_0)) = O(|\lambda_\varepsilon|^{-d-1})$.

In the above we suppressed the subsequence notation. We illustrated these results on Figure 1.1.

Theorems 2, 3 and 4 are consequences of a stronger result. For $\mathscr{V} \in L_0^\infty(\mathbb{B}^d(0,L), \mathbb{C})$ and $\rho \in C_0^\infty(\mathbb{R}^d)$ that is 1 on $\mathrm{supp}(\mathscr{V})$, we define $K_\mathscr{V}(\lambda) = \rho R_0(\lambda)\mathscr{V}$. If $p \geq d+4$ and Ψ is the entire function defined by

$$\Psi(z) = (1+z)\exp\left(-z + \frac{z^2}{2} - \cdots + \frac{(-z)^{p-1}}{p-1}\right) - 1, \tag{1.6}$$

the operator $\Psi(K_\mathscr{V}(\lambda))$ is trace class. This allows us to define the Fredholm determinant

$$D_\mathscr{V}(\lambda) = \mathrm{Det}\left(\mathrm{Id} + \Psi(K_\mathscr{V}(\lambda))\right). \tag{1.7}$$

Apart from the special case of 0 in dimension one, resonances of $\mathscr{V}$ are exactly zeros of $D_\mathscr{V}$— see [16, Theorem 5.4]. To deal with the particular case of the zero resonance in dimension one we define $X_d = \mathbb{C}$ if $d \geq 3$ and $X_1 = \mathbb{C} \setminus \{0\}$. The following result shows that D_V admits an expansion in powers of ε.

Theorem 5. – *Let W in $C_0^\infty(\mathbb{B}^d(0,L) \times \mathbb{T}^d, \mathbb{C})$ and V_ε be the potential given by* (1.1). *Fix $N \geq d+4$ and $p = 4(d+N)N$. If $D_{V_\varepsilon}(\lambda)$ is the Fredholm determinant defined in* (1.7) *then there exists $a_0, \dots, a_{N-1}$ holomorphic functions of $\lambda \in X_d$ such that uniformly on compact subsets of X_d,*

$$D_{V_\varepsilon}(\lambda) = a_0(\lambda) + \varepsilon^2 a_2(\lambda) + \varepsilon^3 a_3(\lambda) + \cdots + \varepsilon^{N-1} a_{N-1}(\lambda) + O(\varepsilon^N).$$

Moreover if Λ_0 and Λ_1 are the potentials defined in Theorem 2 then $a_0(\lambda) = D_{W_0}(\lambda)$,

$$a_2(\lambda) = -D_{W_0}(\lambda) \cdot \mathrm{Tr}\left((\mathrm{Id} + K_{W_0})^{-1}(-K_{W_0})^{p-2} K_{\Lambda_0}\right),$$
$$a_3(\lambda) = -D_{W_0}(\lambda) \cdot \mathrm{Tr}\left((\mathrm{Id} + K_{W_0})^{-1}(-K_{W_0})^{p-2} K_{\Lambda_1}\right).$$

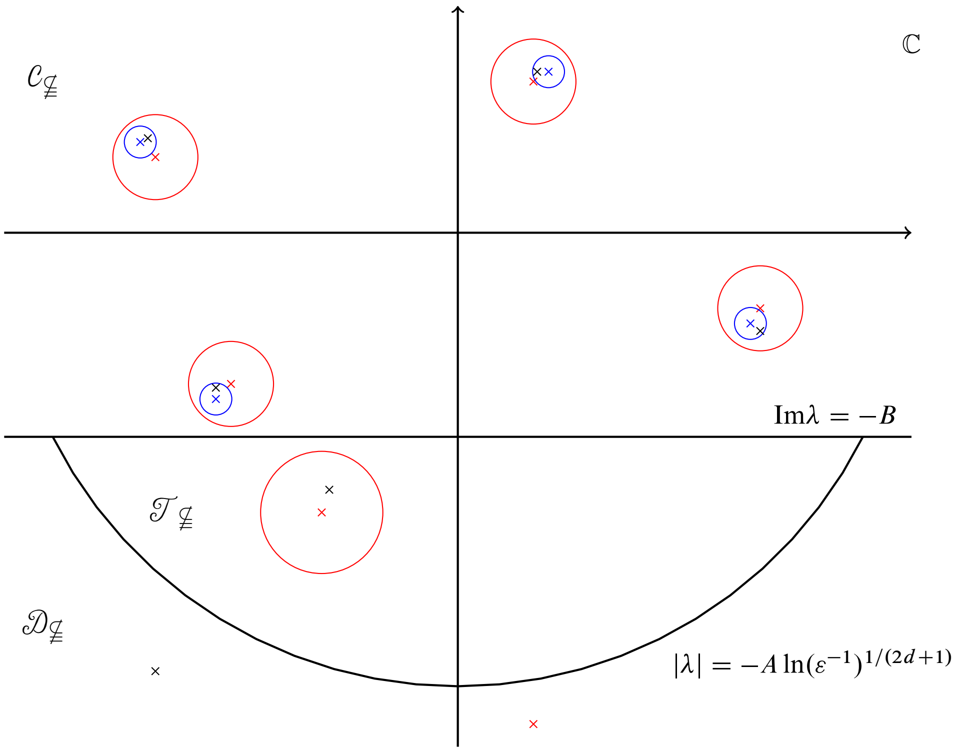

FIGURE 1. The red (resp. black, blue) crosses denote resonances of W_0 (resp. V_ε, $V_{\mathrm{eff},\varepsilon}$). Above the line $\operatorname{Im}\lambda = B$ resonances of $V_{\mathrm{eff},\varepsilon}$ and V_ε lie within red disks of radius $\sim \varepsilon^2$ centered at resonances of W_0. Resonances of $V_{\mathrm{eff},\varepsilon}$ and V_ε in these disks lie within a distance $\sim \varepsilon^4$ from each other. In the middle zone resonances of V_ε lie within disks of radius ~ 1 centered at resonances of W_0. Below both curves $\operatorname{Im}\lambda = -B$ and $|\lambda| = -A\ln(\varepsilon^{-1})^{1/(2d+1)}$ resonances of V_ε, $V_{\mathrm{eff},\varepsilon}$ and W_0 are no longer correlated.

Here again, we note that a perturbation of a potential W_0 by a highly oscillatory potential enjoys similarities with a suitable analytic perturbation of W_0. We will make this observation more precise in §3.2 below.

1.2. Relation with existing work

Our original motivation for investigating highly oscillatory potentials came from Christiansen [5] where it was shown that certain complex-valued oscillatory potentials have no resonances at all. The proof there is based on a priori estimates on solutions of $(\mathrm{Id} + K_{\mathcal{V}}(\lambda))u = 0$. Although real valued potentials have infinitely many resonances—see [22], [24] and references given there—ideas similar to [5] led us to the absence of resonance in strips depending logarithmically on the frequency of oscillations (Theorem 1).

In dimension one scattering resonances of potentials of the form (1.1) have been extensively studied. For W with $W_0 \equiv 0$ and V_ε given by (1.1), Borisov and Gadyl'shin investigate in [4] the behavior of eigenvalues of the Schrödinger operator $D_x^2 + V_\varepsilon$. They give a sufficient condition for an eigenvalue to exist for small ε. Under this condition they derive an expansion of the eigenvalue as $\varepsilon \to 0$. In [3] Borisov refines this result by including potentials that are less regular. These two papers focus on the spectrum and on the eigenvalues rather than on scattering resonances. Scattering theory for operators of the form $D_x^2 + V_\varepsilon$ was systematically presented by Duchêne-Weinstein [14]. In that paper the authors study the behavior of the transmission coefficient of such potentials. They prove that away from possible poles, the transmission coefficient of V_ε converges to that of W_0. They give estimates on the remainder that depend on the regularity of W. The study is later continued in [12]. In that paper Duchêne, Vukićević and Weinstein generalize the result of [4] to general potentials V_ε given by (1.1). They give conditions for the existence of a bound state of V_ε for small ε, whose energy is expressed in terms of an effective potential which is an analytic perturbation of W_0.

Also in dimension one, [2] studies in detail the spectrum of Schrödinger operators with a potential that is the sum of a compactly supported potential and a periodic potential oscillating at frequency $\sim\ \varepsilon^{-1}$. The paper [13] deals with potentials that are a sum of a periodic potential Q_{per} perturbed by a term Q_ε oscillating at frequency ε^{-1}. As $\varepsilon \to 0$ they observe the bifurcation of eigenvalues of $D_x^2 + Q_{\mathrm{av}} + Q_\varepsilon$ at distance ε^4 from the edges of the continuous spectrum of $D_x^2 + Q_{\mathrm{av}}$.

In higher dimension the work [17] deals with general perturbations of operators $-\Delta + W_0$. The perturbation $V_\sharp$ needs to be small when measured in a suitable space. They show that simple resonances of perturbed operators depend analytically on $V_\sharp$. Although such a result applies to potentials given by (1.1) it does not yield an expansion of resonances in powers of ε because $V_\sharp$ does not depend smoothly on ε.

Let us discuss in more detail the relation between our work specialized to dimension one and [12]. By a fine analysis of the scattering coefficients, they show that the transmission coefficient of V_ε is equal to the transmission coefficient of the effective potential

$$V_{\mathrm{eff}}(x) = W_0(x) - \varepsilon^2 \Lambda_0(x), \quad \Lambda_0(x) = \sum_{k \neq 0} \frac{|W_k(x)|^2}{|k|^2}$$

modulo an error of order ε^3. This remarkable result provided further motivation for our investigation. One of the main consequences is [12, Corollary 3.7]: in the case $d = 1$, $W_0 \equiv 0$ and for ε small enough a ground state emerges from the edge of the continuous spectrum of D_x^2, with energy λ_ε given by

$$\lambda_\varepsilon = -\frac{\varepsilon^4}{4} \left(\int_{\mathbb{R}} \Lambda_0(x) dx \right)^2 + O(\varepsilon^5). \tag{1.8}$$

Theorem 2 refines (1.8). Since the functions u, v of (1.3) are given by $u = v = 1/\sqrt{2}$ the energy of the bound state admits the expansion

$$\lambda_\varepsilon = -\frac{\varepsilon^4}{4} \left(\int_{\mathbb{R}} \Lambda_0(x) dx \right)^2 - \frac{\varepsilon^5}{4} \int_{\mathbb{R}} \Lambda_0(x) dx \int_{\mathbb{R}} \Lambda_1(x) dx + O(\varepsilon^6),$$

and in fact λ_ε is even a smooth function of ε. In §1.3 we compare numerically the efficiency of the effective potential $V_{\mathrm{eff},\varepsilon}$ derived here compared to the efficiency of the effective potential derived in [12].

Interest in Schrödinger operators with highly oscillatory potentials has grown since the original version of this work. In [9], we showed the second conjecture of [12]: in dimension two, if $W \in C_0^\infty(\mathbb{R}^2 \times \mathbb{T}^2, \mathbb{R})$ satisfies $\int_{\mathbb{R}^2} W(x,y)dy = 0$ and V_ε is given by (1.1), then $-\Delta + V_\varepsilon$ admits a unique eigenvalue for ε sufficiently small. In addition, this eigenvalue is exponentially close to 0: it is equal to

$$-\exp\left(-\frac{4\pi}{\varepsilon^2\int_{\mathbb{R}^2}\Lambda_0(x)dx + o(\varepsilon^2)}\right), \quad \Lambda_0 = \sum_{k\in\mathbb{Z}^2\setminus 0}\frac{W_k W_{-k}}{|k|^2}.$$

Again, this echoes Simon's result [23, Theorem 3.4] for eigenvalues of weakly coupled Schrödinger operators on the plane.

Dimassi [7] and Dimassi-Duong [8] showed trace formulae and Weyl laws for the operator $-\Delta + \varepsilon^{-2}V_\varepsilon$ for any value of d and $W_0 \equiv 0$. The scaling ε^{-2} enables them to use semiclassical methods to analyze the spectral properties of $-\Delta + V_\varepsilon$. In dimension one, Duchêne-Raymond [11] studied effective potentials, eigenvalues and eigenstates of $-\partial_x^2 + \varepsilon^{-\beta}V_\varepsilon$, for certain values of β and for V_ε real-valued with average zero. The homogenization results are classified in three regimes: weak coupling (corresponding to $\beta \in (2/3, 1)$), critical (corresponding to $\beta = 1$) and semiclassical (corresponding to $\beta \in (1, 3/2)$). As of now, it is not clear how to relate their results to those of [7, 8]. In the discrete 1D ergodic setting (i.e., random or periodic), Klopp [18, 19] and Phong [20, 21] related eigenvalues of Schrödinger operators on large bounded subsets $[-L, L] \subset \mathbb{Z}$ to resonances of the same operator considered on the whole $\mathbb{Z}$, in the regime $L \to \infty$. After rescaling, this is a viscosity limit result for discrete versions of potentials $\varepsilon^{-2}V_\varepsilon$, where V_ε satisfies $W_0 \equiv 0$.

In a very recent paper [10], we prove stability results for resonances of random versions of V_ε (with W_0 non necessarily vanishing), in odd dimensions. We show almost sure convergence of resonances of V_ε to the resonances of W_0. We identify a stochastic and a deterministic regime for the speed of convergence. The type of regime depends whether the (stochastic) low frequency effects due to large deviations overcome the (deterministic) constructive interference produced by highly oscillatory terms.

In a forthcoming work, we will apply the theory developed here to the derivation of edge states in dimer and honeycomb structures.

1.3. Numerical results

Let W be the smooth function on $\mathbb{R} \times \mathbb{T}^1$ defined by

$$W(x,y) = \exp\left(-\frac{x^2}{1-x^2}\right)\mathbb{1}_{[-1,1]}(x)\,(1 + 2\cos(x/2 + y)).$$

Let V_ε be given by (1.1) and Λ_0, Λ_1 the potentials defined in Theorem 2. Thanks to a Matlab simulation whose code was transferred to us by Duchêne, Vukićević and Weinstein we computed numerically the transmission coefficients t_ε of V_ε, t_ε^1 of $V^1_{\mathrm{eff},\varepsilon} = W_0 - \varepsilon^2\Lambda_0$ (the effective potential as derived in [12]) and t_ε^2 of $V^2_{\mathrm{eff},\varepsilon} = W_0 - \varepsilon^2\Lambda_0 - \varepsilon^3\Lambda_1$ (the improved effective potential derived here). In Figure 2 we plotted the graphs of $|t_\varepsilon - t_\varepsilon^j|$ for different values of ε and $j = 1, 2$. For $\varepsilon > 0.1$ neither the approximation of t_ε by t_ε^1 nor t_ε^2 give satisfying results. For $\varepsilon \in [0.01, 0.1]$ it is much better but we still cannot see the improvements

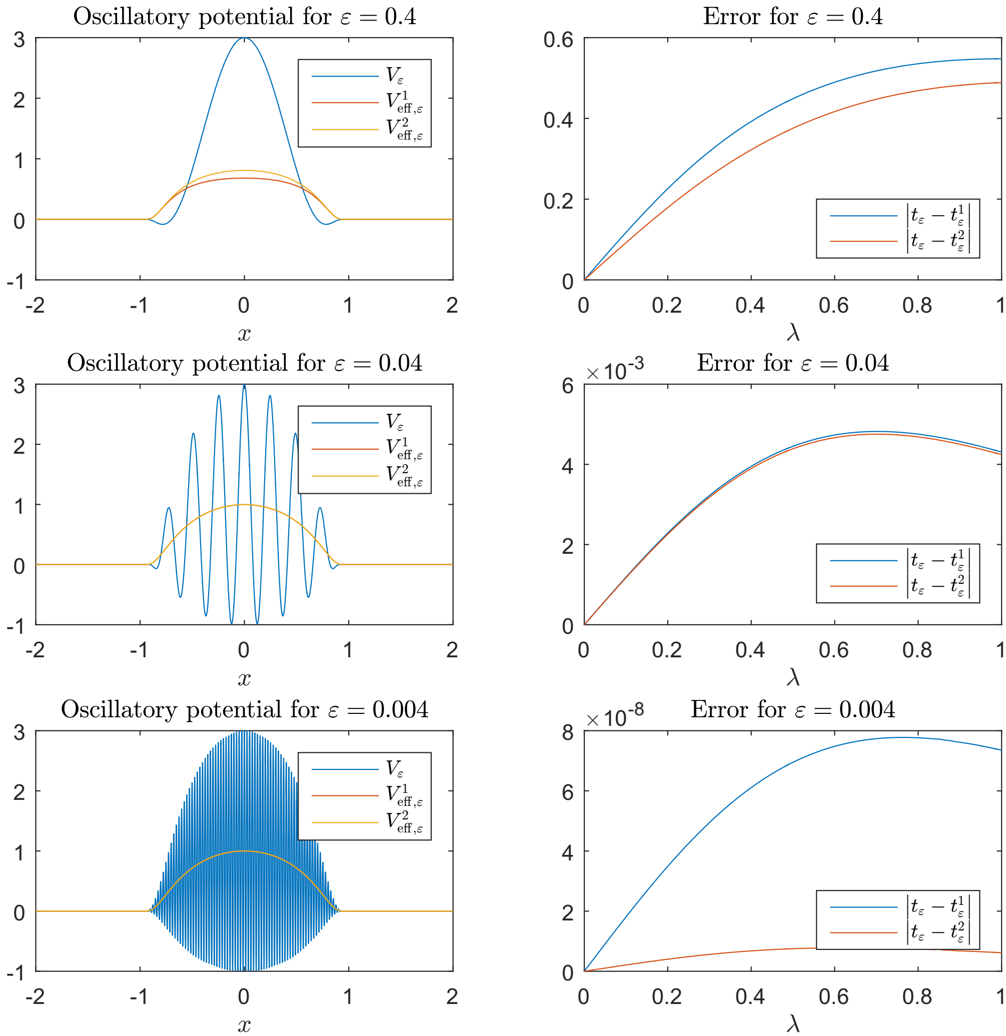

FIGURE 2. Oscillatory potential and errors in approximating the transmission coefficient of V_ε by the transmission coefficient of $V^j_{\mathrm{eff},\varepsilon}$ for different values of ε and $j = 1, 2$.

induced by choosing $V^2_{\mathrm{eff},\varepsilon}$ instead of $V^1_{\mathrm{eff},\varepsilon}$. For $\varepsilon < 0.01$ the approximation of t_ε by t^2_ε instead of t^1_ε gives better results.

1.4. Plan of the paper

We organize the paper as follows. In §2 we focus on the case $W_0 \equiv 0$ and we prove Theorem 1. The proof relies mainly on an application of the Lippman-Schwinger principle combined with integration by parts. In §2.3 we construct a step-like potential V_ε whose resonances are zeros of a 2×2 explicit determinant. Uniform estimates on this determinant and arguments from complex analysis show that V_ε admits a resonance $\lambda_\varepsilon \sim i\ln(\varepsilon)$.

In §3 we apply Theorem 5 to prove that resonances of potentials of the form (1.1) admit an expansion in powers of ε. We compute the first terms in the expansion using a trace estimate. Then we show that resonances of V_ε are comparable to the one of the effective potential $V_{\text{eff},\varepsilon}$ by comparing two Fredholm determinants. We then prove Theorem 4 using complex analysis arguments.

The Section 4 consists in the proof of Theorem 5. It is by far the hardest part of the paper. We first describe how an expansion of the determinant $D_{V_\varepsilon}(\lambda)$ in powers of ε can be reduced to an expansion on the trace of an operator that takes a complicated form. We split this operator into two parts in a natural way. By arguments of combinatorial nature we will prove that the first part is negligible as $\varepsilon \to 0$ and therefore produces no term in the expansion of D_{V_ε}. We will deal with the second part essentially by deriving an operator-valued expansion of $e^{ik\bullet/\varepsilon} R_0(\lambda) e^{-ik\bullet/\varepsilon}$ in powers of ε. The operators in this expansion will produce all the terms in the expansion of D_V. The expression of the coefficients in the expansion is theoretically traceable directly from the proof. We compute the first few terms. In dimension one the pole of $R_0(\lambda)$ at $\lambda = 0$ will cause some trouble. We will overcome these difficulties by arguments specific to the one-dimensional case but that still rely on trace and determinant computations rather than on ODE techniques.

1.5. Notation

From now on we drop the subscript ε and we fix $L > 0$.

Given a function $W \in L_0^\infty(\mathbb{B}^d(0, L) \times \mathbb{T}^d, \mathbb{C})$, V is the function associated to W by (1.1). We will use the following notation:

- X^d is the set equal to $\mathbb{C} \setminus \{0\}$ when $d = 1$ and equal to $\mathbb{C}$ when $d \geq 3$.
- Any time $\pm$ or $\mp$ appears in an equation, this equation has two meanings: one for the upper subscripts, one for the lower one. For instance, $f(x) = \mp 1$ for $\pm x \geq 1$ means $f(x) = -1$ for $x \geq 1$ and $f(x) = 1$ for $-x \geq 1$.
- If $x \in \mathbb{R}$, $x_- = \max(0, -x)$.
- For $x \in \mathbb{R}^n$, $\langle x \rangle = (1 + |x|^2)^{1/2}$.
- If $z \in \mathbb{C}$ and $r > 0$, $\mathbb{D}(z, r)$ denotes the set of $w \in \mathbb{C}$ with $|z - w| < r$.
- If $x \in \mathbb{R}^d$ and $L > 0$, $\mathbb{B}^d(x, L)$ denotes the set of $y \in \mathbb{R}^d$ with $|x - y| < L$. $\mathbb{T}^d$ is the d-dimensional torus $\mathbb{R}^d/(2\pi\mathbb{Z})^d$.
- Let $\mathcal{H}$ be a space of functions on an open set $\mathcal{U} \subset \mathbb{R}^d$. We write $f \in \mathcal{H}_0$ if f belongs to $\mathcal{H}$ and has compact support in $\mathcal{U}$ and $f \in \mathcal{H}_{\text{loc}}$ if for every $\rho \in C_0^\infty(\mathbb{R}^d)$, $\rho f \in \mathcal{H}$.
- For a potential $\mathcal{V}$, $\text{Res}(\mathcal{V})$ is the set of resonances of $\mathcal{V}$. If $\lambda \in \text{Res}(\mathcal{V})$, $m_{\mathcal{V}}(\lambda)$ is the geometric multiplicity of λ defined by
$$m_{\mathcal{V}}(\lambda) = \text{rank} \oint_\lambda R_{\mathcal{V}}(\mu) d\mu.$$
- If $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert space, we denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ (resp. $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$) the space of bounded (resp. trace class) operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and by $\mathcal{B}(\mathcal{H}_1)$ (resp. $\mathcal{L}(\mathcal{H}_1)$) the space of bounded (resp. trace class) operators from $\mathcal{H}_1$ to itself. If $\mathcal{H}_1 = L^2(\mathbb{R}^d, \mathbb{C})$ we simply write $\mathcal{B} = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{L} = \mathcal{L}(\mathcal{H}_1)$.

– If f is a function on $\mathbb{R}^d$, $\hat{f}$ and $\mathcal{F} f$ both denote the Fourier transform of f:

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx.$$

– We define $H^s(\mathbb{R}^d)$ the space of complex-valued functions f with $\langle \xi \rangle^s \hat{f}(\xi) \in L^2(\mathbb{R}^d)$. If s is an integer we define $W^s(\mathbb{R}^d)$ the space of functions with s derivatives in $L^\infty(\mathbb{R}^d)$ and we write $|\cdot|_{W^s} = \|\cdot\|_s$. Similarly $W_0^s(\mathbb{B}^d(0, L))$ is the space of functions in $W^s(\mathbb{R}^d)$ with support contained in $\mathbb{B}^d(0, L)$.
– For $k \in \mathbb{Z}^d$, $e^{ik\bullet/\varepsilon}$ denotes the multiplication operator by the function $e^{ikx/\varepsilon}$.
– ρ denotes a smooth function that is 1 on $\mathbb{B}^d(0, L)$ and 0 outside $\mathbb{B}^d(0, L + 1)$.
– The operator D is $-i\partial_x$. It is a vector-valued operator in dimension $d > 1$. For $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $k \cdot D$ is the operator $k_1 D_{x_1} + \dots + k_d D_{x_d}$.
– In general if $A(\lambda)$ is a family of operators depending on λ we will write A for $A(\lambda)$ unless there is a possible confusion.

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2. Resonance escaping in the case $W_0 \equiv 0$

In this part we start with preliminary estimates that will be used all along the paper. Then we prove Theorem 1 and construct in §2.3 an example of potential that proves that this theorem is optimal.

2.1. Preliminaries

For $\mathcal{V} \in L_0^\infty(\mathbb{B}^d(0, L), \mathbb{C})$ we define $K_{\mathcal{V}}$ the operator $\rho R_0(\lambda)\mathcal{V}$. We start by the following preliminary:

LEMMA 2.1. – *For all $\alpha, \beta \in \{0, 1, 2\}^d$ with $|\alpha|+|\beta| \leq 2$ and for all $\mathcal{V} \in W_0^{|\beta|}(\mathbb{B}^d(0, L), \mathbb{C})$,*

$$\left| D^\alpha K_{\mathcal{V}} D^\beta \right|_{\mathcal{B}} \leq \begin{cases} C \, \langle \lambda \rangle^{\alpha+\beta} \, |\lambda|^{-1} e^{2L(\operatorname{Im}\lambda)_-} \|\mathcal{V}\|_{|\beta|} & \text{if } d = 1, \\ C \, \langle \lambda \rangle^{|\alpha|+|\beta|-1} \, e^{2L(\operatorname{Im}\lambda)_-} \|\mathcal{V}\|_{|\beta|} & \text{if } d \geq 3. \end{cases}$$

The constant C depends on $d(\operatorname{supp}(\mathcal{V}), \partial\mathbb{B}^d(0, L))$ only.

Such estimates are proved in [15, Theorem 2.1] and follow from Schur's test. We recall that $X_d = \mathbb{C}$ if $d \geq 3$ and $X_1 = \mathbb{C} \setminus \{0\}$. The following lemma characterizes resonances of a potential $\mathcal{V}$ via a Lippman-Schwinger equation.

LEMMA 2.2. – *Let $\mathcal{V} \in L_0^\infty(\mathbb{B}^d(0, L), \mathbb{C})$. $\lambda \in X_d$ is a resonance of $\mathcal{V}$ if and only if there exists $0 \neq u \in L^2$ such that $u = -K_{\mathcal{V}} u$.*

Proof. – For $\lambda \in \mathbb{C}$ if $d \geq 3$ and $\lambda \in \mathbb{C}\backslash\{0\}$ the operator $K_{\mathcal{V}}$ is compact. Thus $\mathrm{Id} + K_{\mathcal{V}}$ is injective if and only if $\mathrm{Id} + K_{\mathcal{V}}$ is invertible. For $\operatorname{Im}\lambda \gg 1$ we can invert $\mathrm{Id} + R_0(\lambda)\mathcal{V}$ via Neumann series. Moreover,

$$
\begin{aligned}
R_{\mathcal{V}}(\lambda) &= (\mathrm{Id} + R_0(\lambda)\mathcal{V})^{-1} R_0(\lambda) = \left(\sum_{n=0}^{\infty}(-R_0(\lambda)\mathcal{V})^n\right) R_0(\lambda) \\
&= \left(\sum_{n=0}^{\infty}(-K_{\mathcal{V}})^n + (1-\rho)\sum_{n=1}^{\infty}(-R_0(\lambda)\mathcal{V})^n\right) R_0(\lambda) \\
&= \left(\mathrm{Id} + (1-\rho)\sum_{n=1}^{\infty}(-R_0(\lambda)\mathcal{V})^n(\mathrm{Id} + K_{\rho})\right)(\mathrm{Id} + K_{\mathcal{V}})^{-1} R_0(\lambda) \\
&= \left(\mathrm{Id} + (1-\rho)\sum_{n=1}^{\infty}(-R_0(\lambda)\mathcal{V})^n - (-R_0(\lambda)\mathcal{V})^{n+1}\right)(\mathrm{Id} + K_{\mathcal{V}})^{-1} R_0(\lambda) \\
&= (\mathrm{Id} - (1-\rho)R_0(\lambda)\mathcal{V})\left(\mathrm{Id} - (\mathrm{Id} + K_{\mathcal{V}})^{-1} K_{\mathcal{V}}\right) R_0(\lambda).
\end{aligned}
$$

The operator $R_0(\lambda)$ meromorphically continues to $\mathbb{C}$ as an operator L^2_{comp} to H^2_{loc} while the operator $(\mathrm{Id} + K_{\mathcal{V}})^{-1}$ meromorphically continues to $\mathbb{C}$ as an operator L^2 to L^2. Thus the identity

$$
R_{\mathcal{V}}(\lambda) = (\mathrm{Id} - (1-\rho)R_0(\lambda)\mathcal{V})\left(\mathrm{Id} - (\mathrm{Id} + K_{\mathcal{V}})^{-1} K_{\mathcal{V}}\right) R_0(\lambda) \tag{2.1}
$$

initially valid for $\operatorname{Im}\lambda \gg 1$ meromorphically continues to all of $\mathbb{C}$. The poles of the RHS are precisely the set of λ such that $\mathrm{Id} + K_{\mathcal{V}}$ is not invertible (apart from $\lambda = 0$ in dimension one) while the poles of the LHS are the resonances of $\mathcal{V}$. This proves the lemma. □

2.2. Escaping of resonances.

We prove here Theorem 1 in the case $d = 1$. Assume that (1.2) holds. If $\lambda \neq 0$ is a resonance of V then by Lemma 2.2 there exists u such that $u = -K_V u$ and $|u|_2 = 1$. It satisfies the a priori estimate

$$
|u|_{H^1} = |K_V u|_{H^1} \leq |K_V|_{\mathcal{B}(H^1,L^2)}|u|_2 \leq C\frac{\langle\lambda\rangle\, e^{2L(\operatorname{Im}\lambda)_-}}{|\lambda|}|W|_\infty|u|_2, \tag{2.2}
$$

in particular it belongs to H^1. The well-known estimate $|fg|_{H^1} \leq |f|_{H^1}|g|_{H^1}$ (valid in dimension one) implies by duality that $|fg|_{H^{-1}} \leq |f|_{H^1}|g|_{H^{-1}}$. The bound (2.2) yields

(2.3)
$$
\begin{aligned}
|u|_2 = |K_V u|_2 &\leq |K_\rho|_{\mathcal{B}(H^{-1},L^2)}|Vu|_{H^{-1}} \\
&\leq C\frac{\langle\lambda\rangle\, e^{2L(\operatorname{Im}\lambda)_-}}{|\lambda|}|V|_{H^{-1}}|u|_{H^1} \leq C\frac{\langle\lambda\rangle^2\, e^{4L(\operatorname{Im}\lambda)_-}}{|\lambda|^2}|V|_{H^{-1}}|W|_\infty|u|_2.
\end{aligned}
$$

To estimate $|K_\rho|_{\mathcal{B}(H^{-1},L^2)}$ we used the adjoint bound i.e., we estimated $|K_\rho(-\overline{\lambda})|_{\mathcal{B}(L^2,H^1)}$ thanks to Lemma 2.1. We claim that $|V|_{H^{-1}} \leq \varepsilon^s|W|_{X^s}$, where $|W|_{X^s} = \sum_{k\neq 0}|k|^{-s}|W_k|_{H^s}$.

Indeed using that $|\langle\xi\rangle^s \widehat{W_k}|_2 = |W_k|_{H^s}$ and $|V|_{H^{-1}} = |\langle\xi\rangle^{-1}\hat{V}|_2$ we have

$$\begin{aligned}
|V|_{H^{-1}} &= \left|\langle\xi\rangle^{-1}\sum_{k\neq 0}\widehat{W_k}(\xi - k/\varepsilon)\right|_2 \\
&\leq \sum_{k\neq 0}\left|\langle\xi\rangle^{-1}\langle\xi - k/\varepsilon\rangle^{-s}\langle\xi - k/\varepsilon\rangle^{s}\widehat{W_k}(\xi - k/\varepsilon)\right|_2 \\
&\leq \sum_{k\neq 0}|\langle\xi\rangle^{-1}\langle\xi - k/\varepsilon\rangle^{-s}|_\infty |W_k|_{H^s} \\
&\leq \sum_{k\neq 0}|\langle\xi\rangle^{-s}\langle\xi - k/\varepsilon\rangle^{-s}|_\infty |W_k|_{H^s} \leq C\sum_{k\neq 0}\langle k/\varepsilon\rangle^{-s}|W_k|_{H^s} \leq \varepsilon^s |W|_{X^s}.
\end{aligned}$$

In the last line we used Peetre's inequality: for every $t \geq 0$ there exists $C > 0$ with

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \quad\Rightarrow\quad \langle x\rangle^{-t}\langle y\rangle^{-t} \leq C\langle x - y\rangle^{-t}. \tag{2.4}$$

Now combining $u = -K_V u$ and $|u|_2 = 1$ with the estimate (2.3) we get

$$1 \leq C\varepsilon^s \frac{\langle\lambda\rangle^2 e^{4L(\operatorname{Im}\lambda)_-}}{|\lambda|^2}|W|_{X^s}^2.$$

Hence either $|\lambda| \leq 1$ and then $|\lambda| \leq c\varepsilon^{s/2}$ for some constant c; or $|\lambda| \geq 1$ and

$$\operatorname{Im}\lambda \leq \frac{1}{4L}\ln\left(C|W|_{X^s}^2\right) - \frac{s}{4L}\ln(\varepsilon^{-1}).$$

This proves Theorem 1 for $d = 1$.

We next prove the theorem in dimension $d \geq 3$. In this case the inequality $|fg|_{H^1} \leq |f|_{H^1}|g|_{H^1}$ no longer holds and we must find another way around. Let W such that $W_0 \equiv 0$ and (1.2) holds and $u \neq 0$ with $|u|_2 = 1$ and

$$u = -K_V u = -\sum_{k\neq 0} K_{W_k} e^{ik\bullet/\varepsilon} u. \tag{2.5}$$

As in the case $d = 1$ u satisfies the a priori estimate $|u|_{H^1} \leq Ce^{C(\operatorname{Im}\lambda)_-}|W|_\infty |u|_2$. Noting that

$$e^{ik\bullet/\varepsilon} = \frac{\varepsilon}{|k|}[k\cdot D, e^{ik\bullet/\varepsilon}] \ \text{ where } \ k\cdot D = \frac{k_1 D_{x_1} + \cdots + k_d D_{x_d}}{|k|},$$

we obtain the commutator identity

$$\varepsilon^{-1}|k|K_{W_k}e^{ik\bullet/\varepsilon} = K_{W_k}(k\cdot D)e^{ik\bullet/\varepsilon} - K_{W_k}e^{ik\bullet/\varepsilon}(k\cdot D).$$

Consequently,

$$\begin{aligned}
\varepsilon^{-1}|k|\left|K_{W_k}e^{ik\bullet/\varepsilon}u\right|_2 &\leq |K_{W_k}(k\cdot D)e^{ik\bullet/\varepsilon}u|_2 + |K_{W_k}e^{ik\bullet/\varepsilon}(k\cdot D)u|_2 \\
&\leq |K_{W_k}(k\cdot D)|_{\mathscr{B}}|u|_2 + |K_{W_k}|_{\mathscr{B}}|(k\cdot D)u|_2 \\
&\leq Ce^{2L(\operatorname{Im}\lambda)_-}\|W_k\|_1|u|_2 + Ce^{C(\operatorname{Im}\lambda)_-}|W_k|_\infty|u|_{H^1} \\
&\leq Ce^{4L(\operatorname{Im}\lambda)_-}\|W_k\|_1(1 + |W|_\infty)|u|_2.
\end{aligned} \tag{2.6}$$

From the second to the third line we used the estimates of Lemma 2.1. From the third to the fourth line we used (2.2). Sum (2.6) over $k \in \mathbb{Z}^d \setminus \{0\}$ to obtain

$$|u|_2 = |K_V u|_2 \leq C\varepsilon e^{4L(\operatorname{Im}\lambda)_-}(1+|W|_\infty)\left(\sum_{k\neq 0}\frac{\|W_k\|_1}{|k|}\right)|u|_2.$$

It follows that

$$1 \leq C\varepsilon e^{4L(\operatorname{Im}\lambda)_-}(1+|W|_\infty)\left(\sum_{k\neq 0}\frac{\|W_k\|_1}{|k|}\right),$$

which implies an upper bound on $\operatorname{Im}\lambda$ of the required form. This ends the proof of Theorem 1.

2.3. Construction of an optimal potential

Here we show that the rate of decay of imaginary parts of resonances of V_ε provided by Theorem 1 is optimal in dimension 1. We construct a function W with $W_0 \equiv 0$ satisfying (1.2) such that the potential V defined by (1.1) has a resonance $\lambda_\varepsilon \sim -i\ln(\varepsilon^{-1})$ with $\varepsilon = \pi/(2n)$. Define W by

$$W(x,y) = \mathbb{1}_{[-1/2,1/2]}(x)\left(\mathbb{1}_{[0,\pi]}(y) - \mathbb{1}_{[-\pi,0]}(y)\right).$$

The k-th Fourier coefficient of W is given by

$$W_k(x) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dfrac{2}{i\pi k}\mathbb{1}_{[-1/2,1/2]}(x) & \text{if } k \text{ is odd.} \end{cases}$$

The function $\mathbb{1}_{[-1/2,1/2]}$ belongs to $H^{1/2-\delta}$ for all $1/2 > \delta > 0$ and

$$\sum_{k\neq 0}|k|^{-1/2+\delta}|W_k|_{H^{1/2-\delta}} \leq c_\delta \sum_{k\neq 0}|k|^{-3/2+\delta} < \infty.$$

Therefore W satisfies (1.2) for every $s \in (0,1/2)$. The potential V associated to W by (1.1) is plotted on Figure 3.

We next characterize resonances of V as zeros of a certain 2×2 determinant.

LEMMA 2.3. – *Let $A_\pm$ be the matrix*

$$A_\pm = \begin{pmatrix} 0 & 1 \\ \pm 1 - \lambda^2 & 0 \end{pmatrix}. \tag{2.7}$$

Then $\lambda \neq 0$ is a resonance of V for $\varepsilon = \pi/(2n)$ if and only if $D(\lambda) = 0$ where

$$D(\lambda) = \operatorname{Det}\left(\left(e^{A_+/2n}e^{A_-/2n}\right)^n\begin{pmatrix}1\\-i\lambda\end{pmatrix}, \begin{pmatrix}1\\ i\lambda\end{pmatrix}\right).$$

Here $\operatorname{Det}(a,b)$ *denotes the determinant of two vectors a, b of $\mathbb{C}^2$.*

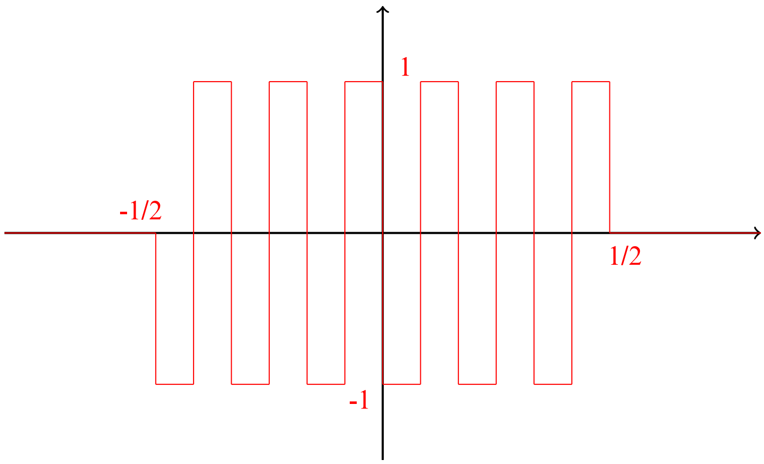

FIGURE 3. The potential V for $\varepsilon = \pi/12$.

Proof. – We recall that since $d = 1$, $\lambda \neq 0$ is a resonance of V if and only if there exists a non zero function $u \in H^2_{\mathrm{loc}}$ with

$$\begin{cases} -u'' + Vu - \lambda^2 u = 0 \\ \quad u(x) = a_\pm e^{\pm i\lambda x}, \quad \pm x \gg 1 \end{cases},$$

see [15, Theorem 2.4]. Using standard uniqueness results for ODEs $\lambda \neq 0$ is a resonance of V if and only if there exists $a \in \mathbb{C}$ such that the boundary problem

$$\begin{cases} \qquad -u'' + Vu - \lambda^2 u = 0, \\ u(-1/2) = 1, \ u'(-1/2) = -i\lambda, \\ \quad u(1/2) = a, \ u'(1/2) = ia\lambda \end{cases} \tag{2.8}$$

admits a non-zero solution u in H^2_{loc}. The ODE

$$\begin{cases} \qquad -u'' + Vu - \lambda^2 u = 0, \\ u(-1/2) = 1, \ u'(-1/2) = -i\lambda \end{cases}$$

admits a unique solution $u \in H^2_{\mathrm{loc}}$. The coefficients of the ODE are constant equal to ± 1 on intervals of length $\pi/(2n)$. Hence u can be explicitly computed using a matrix exponential. A direct calculation shows that

$$\begin{pmatrix} u(1/2) \\ u'(1/2) \end{pmatrix} = \left(e^{A_+/2n} e^{A_-/2n} \right)^n \begin{pmatrix} 1 \\ -i\lambda \end{pmatrix}, \tag{2.9}$$

where $A_\pm$ are the matrices given by (2.7). Putting together (2.8) and (2.9) $\lambda \neq 0$ is a resonance if and only if there exists a such that

$$a \begin{pmatrix} 1 \\ i\lambda \end{pmatrix} = \left(e^{A_+/2n} e^{A_-/2n} \right)^n \begin{pmatrix} 1 \\ -i\lambda \end{pmatrix},$$

that is, if and only if $D(\lambda) = 0$. This ends the proof. □

In order to prove that $V_{\pi/(2n)}$ has a resonance $\lambda_n \sim -i\ln(n)$ we study asymptotics of $D(\lambda)$ uniform in the region $\{(\lambda,n)\ :\ |\lambda| = O(\ln(n))\}$. By the Baker-Hausdorff-Campbell formula, there exists a matrix $Z_n \in M_2(\mathbb{C})$ such that $e^{Z_n} = e^{A_+/n}e^{A_-/n}$. Its asymptotic development is

$$Z_n = \frac{A_+ + A_-}{2n} + \frac{1}{8n^2}[A_+, A_-] + \sum_{m\geq 3} \frac{1}{(2n)^m} p_m(A_+, A_-).$$

The terms $p_m(X,Y)$ are homogeneous polynomial of degree m in the non-commuting variables X, Y. The expansion converges as long as $|A_+| < 2n, |A_-| < 2n$—see [1]. This is realized as long as $|\lambda| = o(\sqrt{n})$, hence when $\lambda = O(\ln(n))$. It yields

$$Z_n = \frac{A_+ + A_-}{2n} + \frac{1}{8n^2}[A_+, A_-] + O\left(n^{-3}\lambda^6\right) \text{ when } \lambda = O(\ln(n)).$$

Therefore

$$\begin{aligned} e^{nZ_n} &= \exp\left(\frac{A_+ + A_-}{2} + \frac{1}{8n}[A_+, A_-] + O\left(n^{-2}\lambda^6\right)\right) \\ &= \exp\left(\frac{A_+ + A_-}{2} + \frac{1}{8n}[A_+, A_-]\right)\left(1 + O\left(n^{-2}\lambda^6\right)\right). \end{aligned}$$

A direct computation leads to

$$\frac{A_+ + A_-}{2} + \frac{1}{8n}[A_+, A_-] = \begin{pmatrix} -1/4n & 1 \\ -\lambda^2 & 1/4n \end{pmatrix}.$$

The eigenvalues are $\pm\nu,\ \ \nu = i\sqrt{\lambda^2 - (4n)^{-2}}$ and therefore

$$\frac{A_+ + A_-}{2} + \frac{1}{8n}[A_+, A_-] = \Omega\Delta\Omega^{-1} \text{ with } \Delta = \begin{pmatrix} -\nu & 0 \\ 0 & \nu \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} 1 & 1 \\ -\nu + (4n)^{-1} & \nu + (4n)^{-1} \end{pmatrix}.$$

Another direct computation gives

$$\begin{aligned} D(\lambda) &= \mathrm{Det}(\Omega)\mathrm{Det}\left(e^{\Delta}\Omega^{-1}\begin{pmatrix} 1 \\ -i\lambda \end{pmatrix}, \Omega^{-1}\begin{pmatrix} 1 \\ i\lambda \end{pmatrix}\right)\left(1 + O\left(n^{-2}\lambda^6\right)\right) \\ &= -\frac{\lambda^2 e^{-\nu}}{2\nu}\left(\left(\frac{\nu}{i\lambda} + 1\right)^2 + (4n\lambda)^{-2} - e^{2\nu}\left(\left(\frac{\nu}{i\lambda} - 1\right)^2 + (4n\lambda)^{-2}\right)\right)\left(1 + O\left(n^{-2}\lambda^6\right)\right) \\ &= -\frac{\lambda^2 e^{-\nu}}{2\nu}\left(4 + O\left((n\lambda)^{-2}\right) - \frac{e^{2i\lambda}}{(4n\lambda)^2}\left(1 + O\left(n^{-2}\lambda^{-1}\right)\right)\right)\left(1 + O\left(n^{-2}\lambda^6\right)\right) \end{aligned}$$

as long as $\lambda = O(\ln(n))$. In order to investigate the behavior of zeros of $D(\lambda)$ we investigate first the behavior of zeros of the function f given by

$$f(\lambda) = 4 - \frac{e^{2i\lambda}}{(4n\lambda)^2}.$$

LEMMA 2.4. – *The zeros of f are given by $\lambda_\nu^{\pm} = i\,\mathcal{W}_\nu(\pm i/8n)$, $\nu \in \mathbb{Z}$ where $\mathcal{W}_\nu$ is the ν-th branch of the Lambert function—see* [6]. *In particular as n goes to infinity $\lambda_1^+ \sim -i\ln(n)$. Moreover, there exists r_0 (independent on n) such that for all n large enough and $\theta \in \mathbb{S}^1$,*

$$|f(\lambda_1^+ + r_0e^{i\theta})| \geq 3r_0. \tag{2.10}$$

Proof. – The equation $f(\lambda)=0$ is equivalent to

$$-i\lambda e^{-i\lambda} = \pm\frac{i}{8n}.$$

Therefore zeros of f are given by $-i\,\mathcal{W}_\nu(\pm i/8n)$. From [6, equation (4.20)] we obtain the asymptotic $\lambda_1^+ \sim -i\ln(n)$. In order to show the lower bound (2.10) we consider $r\in(0,1)$. We prove some estimates that are uniform in n and $\theta\in\mathbb{S}^1$ as $r\to 0$. The identity $f(\lambda_1^+)=0$ yields

$$f(\lambda_1^+ + re^{i\theta}) = 4 - 4\left(\frac{e^{re^{i\theta}}}{1+re^{i\theta}/\lambda_1^+}\right)^2.$$

As $r\to 0$, $e^{re^{i\theta}} = 1 + re^{i\theta} + o(r)$, therefore

$$1-\frac{e^{re^{i\theta}}}{1+re^{i\theta}/\lambda_1^+} = \frac{re^{i\theta}(1-\lambda_1^+)+o(r)}{1+re^{i\theta}/\lambda_1^+}.$$

For n large enough we have $\lambda_1^+\sim -i\ln(n)$ and thus a fortiori $|\lambda_1^+|\geq 2$. This implies

$$\left|1-\frac{e^{re^{i\theta}}}{1+re^{i\theta}/\lambda_1^+}\right| \geq \frac{r/2+o(r)}{1+r/2} = r/2+o(r).$$

Similarly,

$$\left|1+\frac{e^{re^{i\theta}}}{1+re^{i\theta}/\lambda_1^+}\right| \geq 2+O(r).$$

Therefore for r small enough

$$|f(\lambda_1^+ + re^{i\theta})| \geq 4r + o(r) \geq 3r.$$

This completes the proof of the lemma. □

For $\lambda\in\partial\mathbb{D}(\lambda_1^+, r_0)$, $f(\lambda)$ is bounded from below uniformly as $n\to\infty$. Hence, for $\lambda\in\partial\mathbb{D}(\lambda_1^+, r_0)$,

$$\begin{aligned}4+O\left((n\lambda)^{-2}\right) - \frac{e^{2i\lambda}}{(4n\lambda)^2}\left(1+O\left(n^{-2}\lambda^{-1}\right)\right) &= f(\lambda)\left(1+O\left(n^{-2}\lambda^{-1}\right)\right)\\ &= f(\lambda)\left(1+O\left(n^{-2}\ln(n)^{-1}\right)\right).\end{aligned}$$

This implies that for $\lambda\in\partial\mathbb{D}(\lambda_1^+, r_0)$,

$$\begin{aligned}D(\lambda) &= -\frac{\lambda^2 e^{-\nu}}{2\nu} f(\lambda)\left(1+O\left(n^{-2}\ln(n)^{-1}\right)\right)\left(1+O\left(n^{-2}\ln(n)^{6}\right)\right)\\ &= -\frac{\lambda^2 e^{-\nu}}{2\nu} f(\lambda)\left(1+O\left(n^{-2}\ln(n)^{6}\right)\right).\end{aligned}$$

By Rouché's theorem this is enough to ensure that for n large enough, $D(\lambda)$ has exactly one zero on $\mathcal{C}(\lambda_1^+, r_0)$. This proves that there exists a resonance behaving like $-i\ln(n)$.

3. Applications of Theorem 5

Here we consider $W \in C_0^\infty(\mathbb{B}^d(0,L) \times \mathbb{T}^d, \mathbb{C})$ and V_ε given by (1.1). We assume that Theorem 5 holds and we get directly to the applications. We prove that resonances of V_ε in compact sets admit a full expansion as $\varepsilon \to 0$ (Theorem 2); that they can be well approximated by a small perturbation $V_{\mathrm{eff},\varepsilon}$ of W_0 (Theorem 3); and we give a description of the localization of resonances of V_ε (Theorem 4).

3.1. Expansion of resonances in powers of ε

In this paragraph we prove Theorem 2. We start with the case $d \geq 3$ or $\lambda_0 \neq 0$.

Proof of Theorem 2 assuming $d \geq 3$ or $\lambda_0 \neq 0$. – Let λ_0 be a simple resonance of W_0 with $\lambda_0 \neq 0$ if $d = 1$. For $N \geq d+4$ and $p = 4N(d+N)$ consider $D_V(\lambda)$ given in (1.7). This is a holomorphic function of λ near λ_0. By Theorem 5 it converges to D_{W_0} as $\varepsilon \to 0$ uniformly on a neighborhood of λ_0. Thus by Hurwitz's theorem D_V has exactly one zero λ_ε that converges to λ_0. It follows that for ε small enough and r_0 small enough λ_ε is the only resonance of V on $\mathbb{D}(\lambda_0, r_0)$.

Define $f(\lambda,\varepsilon) = D_V(\lambda)$ if $\varepsilon \neq 0$ and $f(\lambda, 0) = D_{W_0}(\lambda)$ otherwise. By Theorem 5 the function f is of class C^{N-1} in a neighborhood of $(\lambda_0, 0)$. In addition since

$$\frac{\partial f}{\partial \lambda}(\lambda_0, 0) = D'_{W_0}(\lambda_0) \neq 0$$

the implicit function theorem implies that the equation $f(\lambda,\varepsilon) = 0$ has exactly one solution in a neighborhood of $(\lambda_0, 0)$. Using uniqueness it must be $(\lambda_\varepsilon, \varepsilon)$. It follows that the function $\varepsilon \to \lambda_\varepsilon$ is C^{N-1}. As N was arbitrary we conclude that $\varepsilon \to \lambda_\varepsilon$ is C^∞ for ε near 0. Thus for all N,

$$\lambda_\varepsilon = \lambda_0 + \varepsilon c_1 + \cdots + \varepsilon^{N-1} c_{N-1} + O(\varepsilon^N), \quad c_j \in \mathbb{C}.$$

We now derive the values of c_1, c_2, c_3. Let $R_{W_0}(\lambda)$ be the meromorphic continuation of the operator $(-\Delta - \lambda^2 + W_0)^{-1}$. Since λ_0 is a simple resonance of W_0 there exists $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$, $v \in \mathscr{D}'(\mathbb{R}^d, \mathbb{C})$ such that

$$R_{W_0}(\lambda) = \frac{iu \otimes v}{\lambda - \lambda_0} + H(\lambda),$$

where $H(\lambda) : L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$ is a family of operators holomorphic near λ_0. Let f be a smooth compactly supported function on $\mathbb{R}^d$. Since $R_{W_0}(\lambda)(-\Delta + V - \lambda^2) f = f$ we have

$$0 = (iu \otimes v)(-\Delta + V - \lambda_0^2) f = iu \left\langle \overline{v}, (-\Delta + V - \lambda_0^2) f \right\rangle_{\mathscr{D}'} = iu \left\langle (-\Delta + V - \lambda_0^2)^* \overline{v}, f \right\rangle_{\mathscr{D}'}.$$

Since this is valid for arbitrary f it yields $(-\Delta + V - \lambda_0^2)^* \overline{v} = 0$. Thus $v \in H^2_{\mathrm{loc}}$ and $(-\Delta + V - \lambda_0^2) v = 0$ which implies $v + R_0(\lambda_0) W_0 v = 0$.

Let Π_0 be the operator $-i\rho(u \otimes v) W_0$. We claim that the family of operators

$$(-K_{W_0})^{p-2} \left(\mathrm{Id} + K_{W_0}\right)^{-1} - \frac{\Pi_0}{\lambda - \lambda_0} \tag{3.1}$$

is holomorphic in a neighborhood of λ_0. Indeed since $(\mathrm{Id} + K_{W_0})^{-1} = \mathrm{Id} - \rho R_{W_0}(\lambda) W_0$ there exists a family of operators $B(\lambda)$ holomorphic near λ_0 such that

$$(\mathrm{Id} + K_{W_0})^{-1} = \frac{\Pi_0}{\lambda - \lambda_0} + B(\lambda).$$

It leads to

$$\begin{aligned}(-K_{W_0})^{p-2}\left(\mathrm{Id}+K_{W_0}\right)^{-1}-\frac{\Pi_0}{\lambda-\lambda_0}&=(-K_{W_0})^{p-2}\left(\mathrm{Id}+K_{W_0}\right)^{-1}-(\mathrm{Id}+K_{W_0})^{-1}+B(\lambda)\\ &=-\left(\mathrm{Id}-(-K_{W_0})^{p-2}\right)\left(\mathrm{Id}+K_{W_0}\right)^{-1}+B(\lambda)=-(\mathrm{Id}+\cdots+(-K_{W_0})^{p-3})+B(\lambda).\end{aligned}$$

This is as claimed holomorphic near λ_0.

Let $\Lambda\in L^\infty(\mathbb{B}^d(0,L),\mathbb{C})$. We now compute the trace $\mathrm{Tr}\left((-K_{W_0})^{p-2}(\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right)$ modulo a holomorphic function. Since the operator given by (3.1) is holomorphic near λ_0 and trace class there exists a function φ holomorphic near λ_0 such that

$$\mathrm{Tr}\left((-K_{W_0})^{p-2}(\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right)=\frac{\mathrm{Tr}(\Pi_0K_\Lambda)}{\lambda-\lambda_0}+\varphi(\lambda).$$

Using $\Pi_0=-i\rho u\otimes vW_0$ and $v+R_0(\lambda_0)W_0v=0$ we get

$$\begin{aligned}\mathrm{Tr}(\Pi_0K_\Lambda)(\lambda_0)&=-i\int_{\mathbb{R}^d}\rho(x)u(x)v(y)W_0(y)R_0(\lambda_0,y,x)\Lambda(x)dxdy\\ &=-i\int_{\mathbb{R}^d}u(x)\Lambda(x)\left(\int_{\mathbb{R}^d}R_0(\lambda_0,x,y)W_0(y)v(y)dy\right)dx\\ &=-i\int_{\mathbb{R}^d}u(x)\Lambda(x)(R_0(\lambda_0)W_0v)(x)dx=i\int_{\mathbb{R}^d}\Lambda(x)u(x)v(x)dx.\end{aligned}$$

It follows that

$$\mathrm{Tr}\left((-K_{W_0})^{p-2}(\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right)=\frac{i}{\lambda-\lambda_0}\left(\int_{\mathbb{R}^d}\Lambda uv\right)+\varphi(\lambda). \tag{3.2}$$

Apply the Formula (3.2) to $\Lambda=\varepsilon^2\Lambda_0$ to obtain

$$\begin{aligned}D_V(\lambda)&=D_{W_0}(\lambda)\left(1-\mathrm{Tr}\left((-K_{W_0})^{p-2}(\mathrm{Id}+K_{W_0})^{-1}K_{\varepsilon^2\Lambda_0}\right)\right)+O(\varepsilon^3)\\ &=D_{W_0}(\lambda)\left(1-\frac{i\varepsilon^2}{\lambda-\lambda_0}\left(\int_{\mathbb{R}^d}\Lambda_0uv\right)-\varepsilon^2\varphi_0(\lambda)\right)+O(\varepsilon^3).\end{aligned}$$

Here the function φ_0 is holomorphic near λ_0 and does not depend on ε. If g is the holomorphic function such that $g(\lambda)(\lambda-\lambda_0)=D_{W_0}(\lambda)$ then

$$D_V(\lambda)=g(\lambda)\left(\lambda-\lambda_0-i\varepsilon^2\left(\int_{\mathbb{R}^d}\Lambda_0uv\right)-\varepsilon^2(\lambda-\lambda_0)\varphi_0(\lambda)\right)+O(\varepsilon^3). \tag{3.3}$$

Note that as $\varepsilon\to0$ we have $g(\lambda_\varepsilon)\to D'_{W_0}(\lambda_0)\neq0$. Thus specializing the identity (3.3) at $\lambda=\lambda_\varepsilon$ leads to

$$0=\lambda_\varepsilon-\lambda_0-i\varepsilon^2\left(\int_{\mathbb{R}^d}\Lambda_0uv\right)-\varepsilon^2(\lambda_\varepsilon-\lambda_0)\varphi_0(\lambda_\varepsilon)+O(\varepsilon^3).$$

Since $\lambda_\varepsilon-\lambda_0=O(\varepsilon)$ and $\varphi_0(\lambda_\varepsilon)\to\varphi_0(\lambda_0)$ as $\varepsilon\to0$ we obtain

$$\lambda_\varepsilon=\lambda_0+i\varepsilon^2\left(\int_{\mathbb{R}^d}\Lambda_0uv\right)+O(\varepsilon^3). \tag{3.4}$$

This recovers the result of [12].

Now to get the second order correction we apply (3.2) successively to $\Lambda = \varepsilon^2\Lambda_0$ and $\Lambda = \varepsilon^3\Lambda_1$. The same operations as in the previous paragraph lead to

$$\begin{aligned} D_V(\lambda) &= D_{W_0}(\lambda)\left(1 - \mathrm{Tr}\left((-K_{W_0})^{p-2}(\mathrm{Id}+K_{W_0})^{-1}K_{\varepsilon^2\Lambda_0+\varepsilon^3\Lambda}\right)\right) + O(\varepsilon^4) \\ &= g(\lambda)\left(\lambda - \lambda_0 - i\left(\int_{\mathbb{R}^d}\left(\varepsilon^2\Lambda_0 + \varepsilon^3\Lambda_1\right)uv\right) - (\lambda-\lambda_0)(\varepsilon^2\varphi_0(\lambda) + \varepsilon^3\varphi_1(\lambda))\right) + O(\varepsilon^4) \end{aligned}$$

for a function φ_1 holomorphic near λ_0. Here again specialize this identity at $\lambda = \lambda_\varepsilon$ and use $g(\lambda_\varepsilon) \to g(\lambda_0) \neq 0$ to obtain

$$0 = \lambda_\varepsilon - \lambda_0 - i\left(\int_{\mathbb{R}^d}\left(\varepsilon^2\Lambda_0 + \varepsilon^3\Lambda_1\right)uv\right) - (\lambda_\varepsilon - \lambda_0)(\varepsilon^2\varphi_0(\lambda_\varepsilon) + \varepsilon^3\varphi_1(\lambda_\varepsilon)) + O(\varepsilon^4).$$

This time by (3.4) we know that $\lambda_\varepsilon - \lambda_0 = O(\varepsilon^2)$. It follows that

$$\lambda_\varepsilon = \lambda_0 + i\varepsilon^2\left(\int_{\mathbb{R}^d}\Lambda_0 uv\right) + i\varepsilon^3\left(\int_{\mathbb{R}^d}\Lambda_1 uv\right) + O(\varepsilon^4).$$

This proves the theorem. □

In the case $\lambda_0 = 0$ and $d = 1$ we use the following refinement of Theorem 5:

LEMMA 3.1. – *Let W belong to $C_0^\infty([-L,L]\times\mathbb{T}^1,\mathbb{C})$ and V be given by* (1.1). *There exists an entire function h_V satisfying the following:*

(i) *λ_0 is a resonance of V of multiplicity m if and only if it is a zero of h_V of multiplicity m.*

(ii) *There exists $h_4,\dots,h_{N-1}$ such that locally uniformly on $\mathbb{C}$*

$$h_V(\lambda) = \lambda d_{W_0}(\lambda)\left(1 - \mathrm{Tr}\left((\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right)\right) + \varepsilon^4 h_4(\lambda) + \cdots + \varepsilon^{N-1}h_{N-1}(\lambda) + O(\varepsilon^N),$$

where $d_{W_0}(\lambda) = \mathrm{Det}(\mathrm{Id}+K_{W_0})$ and Λ is the potential given by

$$\Lambda = \varepsilon^2\Lambda_0 + \varepsilon^3\Lambda_1 = \varepsilon^2\sum_{k\neq 0}\frac{W_k W_{-k}}{k^2} - 2\varepsilon^3\sum_{k\neq 0}\frac{W_k(DW_{-k})}{k^3}.$$

We defer the proof of Lemma 3.1 to §4.6. The proof of Theorem 2 in the case $\lambda_0 = 0$ and $d = 1$ is the same as in the case $d \neq 1$ or $\lambda_0 \neq 0$ using h_V instead of D_V and we skip the details. We end this part with a version of Theorem 2 for resonances λ_0 of W_0 with higher multiplicity.

THEOREM 6. – *Assume that W belongs to $C_0^\infty(\mathbb{B}^d(0,L)\times\mathbb{T}^d,\mathbb{C})$ and that λ_0 is a resonance of W_0 with multiplicity m. Then in a neighborhood of λ_0 the potential V_ε has exactly m resonances $\lambda_{1,\varepsilon},\dots,\lambda_{m,\varepsilon}$ for ε small enough. In addition for every $j \in [1,m]$ and $N \geq d+4$,*

$$\lambda_{j,\varepsilon} = \lambda_0 + c_{j,2}\varepsilon^{2/m} + c_{j,3}\varepsilon^{3/m} + \cdots + c_{j,N-1}\varepsilon^{(N-1)/m} + O(\varepsilon^{N/m}),\quad c_{j,n} \in \mathbb{C}.$$

Proof. – Let $\lambda_0 \in X_d$ be a resonance of W_0 of multiplicity $m > 1$. Fix $N \geq d+4$ and p, D_V given by Theorem 5. Since locally uniformly on $\mathbb{C}$ we have $D_V(\lambda) \to D_{W_0}(\lambda)$, by Hurwitz's theorem the function D_V has exactly m zeros (counted with multiplicity) converging to λ_0. These zeros admit a Puiseux expansion: there exists $c_{1,1},\dots,c_{m,N-1}$ such that the zeros $\lambda_{1,\varepsilon},\dots,\lambda_{m,\varepsilon}$ of D_V near λ_0 are given by

$$\lambda_{j,\varepsilon} = \lambda_0 + \varepsilon^{1/m}c_{j,1} + \cdots + \varepsilon^{(N-1)/m}c_{j,N-1} + O(\varepsilon^{N/m}).$$

Now since $D_V(\lambda) = D_{W_0}(\lambda) + O(\varepsilon^2)$, $c_{j,1} = 0$. In the case $\lambda_0 = 0$ in dimension one the proof can be modified by considering h_V instead of D_V. This proves Theorem 6. □

3.2. Derivation of an effective potential

In this part we prove Theorem 3. We start by giving a few preliminaries concerning trace class operators and Fredholm determinant. The reader can consult [15, Chapter B] for a complete introduction. The singular values of a compact operator $X : \mathscr{H} \to \mathscr{H}$ are defined as the nonincreasing sequence $s_j(X) = \lambda_j((X^*X)^{1/2})$. In particular $s_0(X) = |X|_{\mathscr{B}(\mathscr{H})}$. The singular values satisfy two remarkable inequalities. If Y is another compact operator then for every j, ℓ,

$$\begin{aligned} s_{j+\ell}(X+Y) &\leq s_j(X) + s_\ell(Y), \\ s_{j\ell}(XY) &\leq s_j(X)s_\ell(Y). \end{aligned}$$

We say that a compact operator X is trace class if the sequence $s_j(X)$ is summable. The trace class norm of X denoted by $|X|_{\mathscr{L}}$ is the sum of the series. If X trace class we can define the trace of X and the Fredholm determinant $\operatorname{Det}(\mathrm{Id} + X)$. This determinant vanishes if and only if $\mathrm{Id} + X$ is not invertible. Recall that $X8d = \mathbb{C}$ for $d \geq 3$, $X_1 = \mathbb{C} \setminus \{0\}$ and that $K_{\mathcal{V}} = \rho R_0(\lambda)\mathcal{V}$.

Lemma 3.2. – *Let $\mathcal{V}$ in $L^\infty(\mathbb{B}^d(0,L),\mathbb{C})$. Uniformly on $\{\operatorname{Im}\lambda \geq 1\}$ and locally uniformly on X_d, $s_j(K_{\mathcal{V}}) \leq C|\mathcal{V}|_\infty j^{-2/d}$. Consequently if $p \geq d$ is an integer the operator $K_{\mathcal{V}}^p$ is trace class and locally uniformly in X_d, uniformly in $\{\operatorname{Im}\lambda \geq 1\}$, $|K_{\mathcal{V}}^p|_{\mathscr{L}} \leq C|\mathcal{V}|_\infty^p$.*

Proof. – We combine [15, Equation (B.3.9] with Lemma 2.1. This gives:

$$s_j(K_{\mathcal{V}}) \leq Cj^{-2/d}|\langle D\rangle^2 K_{\mathcal{V}}|_{\mathscr{B}} \leq C|\mathcal{V}|_\infty j^{-2/d}.$$

This estimate works both locally uniformly on X_d and uniformly on $\{\operatorname{Im}\lambda \geq 1\}$. In order to prove that the operator $K_{\mathcal{V}}^p$ belongs to $\mathscr{L}$ for $p \geq d$ it suffices to prove that the sequence of singular values $s_j(K_{\mathcal{V}}^p)$ is summable. Using the properties of the singular values,

$$\sum_{j=0}^\infty s_j(K_{\mathcal{V}}^p) \leq p\sum_{j=0}^\infty s_{pj}(K_{\mathcal{V}}^p) \leq p\sum_{j=0}^\infty s_j(K_{\mathcal{V}})^p \leq C|\mathcal{V}|_\infty^p \sum_{j=0}^\infty j^{-2p/d}.$$

Since $p \geq d$ the series converges and the lemma follows. □

This lemma implies that for $\mathcal{V} \in L^\infty(\mathbb{B}^d(0,L),\mathbb{C})$ the Fredholm determinant

$$D_{\mathcal{V}}(\lambda) = \operatorname{Det}(\mathrm{Id} + \Psi(K_{\mathcal{V}})), \quad \Psi(z) = (1+z)\exp\left(-z + \frac{z^2}{2} - \cdots + \frac{(-z)^{p-1}}{p-1}\right) - 1$$

is well defined when $\lambda \in X_d$—see [23, Lemma 6.1]. It is an entire function of λ for $d \geq 3$ and is a meromorphic function of λ with a pole at $\lambda = 0$ for $d = 1$. We now show the seemingly unknown:

Lemma 3.3. – *Let $W_0, \Lambda \in L^\infty(\mathbb{B}^d(0,L),\mathbb{C})$. If $p \geq d$ and $D_{W_0+\varepsilon\Lambda}$ is the Fredholm determinant given by* (1.7) *then there exists $b_0, b_1, \dots$ holomorphic functions of $\lambda \in X_d$ such that locally uniformly on X_d,*

$$D_{W_0+\varepsilon\Lambda}(\lambda) = \sum_{j=0}^\infty b_j(\lambda)\varepsilon^j.$$

In addition $b_0(\lambda) = D_{W_0}(\lambda)$ *and*

$$b_1(\lambda) = D_{W_0}(\lambda) \cdot \mathrm{Tr}\left((\mathrm{Id} + K_{W_0})^{-1}(-K_{W_0})^{p-1}K_\Lambda\right).$$

Proof. – Let $W_0, \Lambda \in L^\infty(\mathbb{B}^d(0,L), \mathbb{C})$. By [23, Theorem 3.3] if $p \geq d$ and Ψ is given by (1.6) the determinant $(\varepsilon, \lambda) \mapsto D_{W_0+\varepsilon\Lambda}(\lambda) = \mathrm{Det}(\mathrm{Id} + \Psi(K_{W_0+\varepsilon\Lambda}))$ is an entire function of ε (with $\lambda \in X_d$ fixed) and a holomorphic function of λ on X_d (with ε fixed). Thus by Hartogs's theorem it is analytic on $\mathbb{C} \times X_d$. Write a power expansion of $D_{W_0+\varepsilon\Lambda}$ as follows: $D_{W_0+\varepsilon\Lambda}(\lambda) = \sum_{n=0}^\infty b_n(\lambda)\varepsilon^n$. Since

$$b_n(\lambda) = \frac{1}{n!}\frac{\partial^n D_{W_0+\varepsilon\Lambda}}{\partial\varepsilon^n}|_{\varepsilon=0}(\lambda)$$

the function b_n is holomorphic on X_d. We next identify the coefficients $b_0(\lambda)$ and $b_1(\lambda)$.

Fix $m \geq d$ and assume that $\lambda \in \mathbb{D}(\lambda_0, 1)$, $\mathrm{Im}\,\lambda_0 \gg 1$. By Lemma 2.1 and Lemma 3.2,

$$\left|K_{W_0+\varepsilon\Lambda}^m\right|_{\mathcal{L}} \leq \left|K_{W_0+\varepsilon\Lambda}^{m-d}\right|_{\mathcal{B}} |K_{W_0+\varepsilon\Lambda}^d|_{\mathcal{L}} \leq \frac{C^m}{|\lambda|^{m-d}}.$$

It follows that the series

$$\sum_{m=p}^\infty (-1)^m \frac{K_{W_0+\varepsilon\Lambda}^m}{m}$$

converges absolutely in $\mathcal{L}$ for $\mathrm{Im}\,\lambda \gg 1$ and in addition

$$D_{W_0+\varepsilon\Lambda}(\lambda) = \exp\left(-\sum_{m=p}^\infty (-1)^m \frac{\mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right)}{m}\right), \tag{3.5}$$

see [23, Theorem 6.2]. If $d = 1$ then $\mathrm{Tr}(K_{W_0+\varepsilon\Lambda}) = \mathrm{Tr}(K_{W_0}) + \varepsilon\mathrm{Tr}(K_\Lambda)$. We now obtain a first order Taylor expansion of $\mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right)$ for $m \geq d$. Using the binomial expansion, the cyclicity of the trace and the Taylor-Lagrange inequality,

$$\begin{gathered}\mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right) = \mathrm{Tr}\left(K_{W_0}^m\right) + m\varepsilon\mathrm{Tr}\left(K_{W_0}^{m-1}K_\Lambda\right) + r_m(\varepsilon),\\ |r_m(\varepsilon)| \leq \frac{1}{2}\sup_{\varepsilon' \in [0,\varepsilon]} \frac{\partial^2 \mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right)}{\partial\varepsilon^2}(\varepsilon').\end{gathered} \tag{3.6}$$

We claim that $|r_m(\varepsilon)| \leq \varepsilon^2$ for $\mathrm{Im}\,\lambda$ large enough. The function $\varepsilon \mapsto \mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right)$ is holomorphic and satisfies

$$\left|\mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right)\right| \leq |K_{W_0+\varepsilon\Lambda}^m|_{\mathcal{L}}^d |K_{W_0+\varepsilon\Lambda}^m|_{\mathcal{L}}^{m-d} \leq \frac{C^m}{\langle\lambda\rangle^{m-d}}|W_0 + \varepsilon\Lambda|_\infty^m.$$

when $\mathrm{Im}\,\lambda \geq 1$. Therefore the Cauchy estimate for derivatives of holomorphic functions shows that $|r_m(\varepsilon)| \leq C^m\varepsilon^2 \langle\lambda\rangle^{m-d} (|W_0|_\infty + |\Lambda|_\infty)^m$ when $\mathrm{Im}\,\lambda \geq 1$. This proves the claim. (3.6) implies then

$$\begin{aligned}\sum_{m=p}^\infty (-1)^m \frac{\mathrm{Tr}\left(K_{W_0+\varepsilon\Lambda}^m\right)}{m} &= \sum_{m=p}^\infty (-1)^m \frac{\mathrm{Tr}\left(K_{W_0}^m\right)}{m} + \varepsilon\sum_{m=p}^\infty (-1)^m \mathrm{Tr}\left(K_{W_0}^{m-1}K_\Lambda\right) + O(\varepsilon^2)\\ &= \sum_{m=p}^\infty (-1)^m \frac{\mathrm{Tr}\left(K_{W_0}^m\right)}{m} - \varepsilon\mathrm{Tr}\left((-K_{W_0})^{p-1}(\mathrm{Id} + K_{W_0})^{-1}K_\Lambda\right) + O(\varepsilon^2).\end{aligned}$$

when $\operatorname{Im}\lambda \geq 1$. The following determinant asymptotic follows: for $\operatorname{Im}\lambda$ large enough,

$$\begin{aligned} D_{W_0+\varepsilon\Lambda}(\lambda) &= \exp\left(-\sum_{m=p}^{\infty}(-1)^m \frac{\operatorname{Tr}\left(K_{W_0}^m\right)}{m} + \varepsilon \operatorname{Tr}\left((-K_{W_0})^{p-1}(\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right) + O(\varepsilon^2)\right) \\ &= D_{W_0}(\lambda)\left(1+\varepsilon\operatorname{Tr}\left((-K_{W_0})^{p-1}(\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right)\right) + O(\varepsilon^2). \end{aligned}$$

Thus $b_0(\lambda) = D_{W_0}(\lambda)$ and $b_1(\lambda) = D_{W_0}(\lambda)\operatorname{Tr}\left((-K_{W_0})^{p-1}(\mathrm{Id}+K_{W_0})^{-1}K_\Lambda\right)$ for $\operatorname{Im}\lambda \gg 1$. Since the functions b_0, b_1 are holomorphic by the unique continuation principle these identities must also hold on X_d. This ends the proof of the theorem. □

We are now ready to prove Theorem 3. It is the special case $m = 1$ of

THEOREM 7. – *Let $V_{\mathrm{eff}} = W_0 + \varepsilon^2\Lambda_0 + \varepsilon^3\Lambda_1$ were Λ_0, Λ_1 where defined in Theorem 5. Let μ_ε be a family of resonances of $V_{\mathrm{eff},\varepsilon}$ with multiplicity m. For every $\varepsilon > 0$ there exist m resonances counted with multiplicity $\lambda_{1,\varepsilon}, \dots, \lambda_{m,\varepsilon}$ of V_ε such that*

$$|\lambda_{j,\varepsilon} - \mu_\varepsilon| = O(\varepsilon^{4/m}).$$

Conversely let λ_ε be a family of resonances of V_ε with multiplicity m. For every $\varepsilon > 0$ there exist m resonances counted with multiplicity $\mu_{1,\varepsilon}, \dots, \mu_{m,\varepsilon}$ of $V_{\mathrm{eff},\varepsilon}$ such that

$$|\lambda_{j,\varepsilon} - \mu_{j,\varepsilon}| = O(\varepsilon^{4/m}).$$

Proof. – Assume $d \geq 3$. Fix $N = d + 4$, $p = 4N(d+N)$ and D_V given in Theorem 5. Let $V_{\mathrm{eff}} = W_0 - \varepsilon^2\Lambda_0 - \varepsilon^3\Lambda_1$. By Theorem 5,

$$D_V = D_{W_0}(\lambda)\left(1 + \operatorname{Tr}\left((\mathrm{Id}+K_{W_0})^{-1}(-K_{W_0})^{p-2}K_{-\varepsilon^2\Lambda_0-\varepsilon^3\Lambda_1}\right)\right) + O(\varepsilon^4). \tag{3.7}$$

Define $\mathscr{D}_{\mathscr{V}}$ the Fredholm determinant

$$\mathscr{D}_{\mathscr{V}}(\lambda) = \operatorname{Det}(\mathrm{Id} + \psi(K_{\mathscr{V}})), \quad \psi(z) = \exp\left(\frac{(-z)^{p-1}}{p-1}\right)\Psi(z).$$

The Fredholm determinants $D_{\mathscr{V}}$ defined in (1.7) and $\mathscr{D}_{\mathscr{V}}$ are related through

$$D_{\mathscr{V}}(\lambda) = \exp\left(\frac{\operatorname{Tr}((-K_{\mathscr{V}})^{p-1})}{p-1}\right)\mathscr{D}_{\mathscr{V}}(\lambda).$$

Therefore (3.7) implies $D_V(\lambda) =$

$$\exp\left(\frac{\operatorname{Tr}((-K_{W_0})^{p-1})}{p-1}\right)\mathscr{D}_{W_0}(\lambda)\left(1+\operatorname{Tr}\left((\mathrm{Id}+K_{W_0})^{-1}(-K_{W_0})^{p-2}K_{-\varepsilon^2\Lambda_0-\varepsilon^3\Lambda_1}\right)\right)+O(\varepsilon^4).$$

Lemma 3.3 leads to

$$D_V(\lambda) = \exp\left(\frac{\operatorname{Tr}((-K_{W_0})^{p-1})}{p-1}\right)\mathscr{D}_{V_{\mathrm{eff}}}(\lambda) + O(\varepsilon^4),$$

where $V_{\mathrm{eff}} = W_0 - \varepsilon^2\Lambda_0 - \varepsilon^3\Lambda_1$. Consider now μ_ε a bounded family of resonances of V_{eff} of multiplicity m. As μ_ε is bounded there exist C, r such that for every $\lambda \in \mathbb{D}(\mu_\varepsilon, r)$,

$$\left|\exp\left(\frac{\operatorname{Tr}((-K_{W_0})^{p-1})}{p-1}\right)\mathscr{D}_{V_{\mathrm{eff}}}(\lambda)\right| \geq C|\lambda-\mu_\varepsilon|^m. \tag{3.8}$$

Let $\gamma_\varepsilon = \partial\mathbb{D}(\mu_\varepsilon, c\varepsilon^{4/m})$. If c is small enough then by (3.8) for every $\lambda \in \gamma_\varepsilon$,

$$\left| D_V(\lambda) - \exp\left(\frac{\mathrm{Tr}((-K_{W_0})^{p-1})}{p-1}\right) \mathscr{D}_{V_{\mathrm{eff}}}(\lambda) \right| < \left| \exp\left(\frac{\mathrm{Tr}((-K_{W_0})^{p-1})}{p-1}\right) \mathscr{D}_{V_{\mathrm{eff}}}(\lambda) \right|.$$

By Rouché's theorem this implies that V and V_{eff} have the same number of resonances inside the disk $\mathbb{D}(\mu_\varepsilon, c\varepsilon^{4/m})$. The proof of the convert part is similar and we omit it. This proves Theorem 7 away from the resonance 0 in dimension one.

We now concentrate on $d = 1$. In this case by Theorem 5 and Lemma 3.3 the function h_V of Lemma 3.1 satisfies $h_V(\lambda) = \lambda d_{V_{\mathrm{eff}}}(\lambda) + O(\varepsilon^4)$ locally uniformly on X_d. The functions h_V and $d_{V_{\mathrm{eff}}}$ are both entire. By a Cauchy formula, if $\lambda \in \mathbb{D}(0,1)$ then

$$h_V(\lambda) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,2)} \frac{\lambda d_{V_{\mathrm{eff}}}(\mu) d\mu}{\mu - \lambda} + O(\varepsilon^4)$$

and this holds uniformly on $\mathbb{D}(0,1)$. Thus the estimate $h_V(\lambda) = \lambda d_{V_{\mathrm{eff}}}(\lambda) + O(\varepsilon^4)$ holds locally uniformly on $\mathbb{C}$. The end of the proof is the same as in the case $d \geq 3$. □

3.3. Uniform description of the resonant set

Here we prove Theorem 4. Let $W \in C_0^\infty(\mathbb{B}^d(0,L)\times\mathbb{T}^d, \mathbb{C})$ and V associated to W by (1.1). Fix $B > 0$. We first localize resonances of V that are above the line $\mathrm{Im}\,\lambda = -B$. According to (2.1) the set of resonances of V in X_d is the set of λ such that the operator $\mathrm{Id} + K_V(\lambda)$ is not invertible on L^2. Thus if $\lambda \in X_d$ is a resonance then $|K_V|_{\mathscr{B}} \geq 1$. Since for $\mathrm{Im}\,\lambda \geq -B$, $|K_V|_{\mathscr{B}} \leq C|V|_\infty e^{2LB}/|\lambda|$, for ε small enough resonances of V and W_0 in the half plane $\mathrm{Im}\,\lambda \geq -B$ all belong to a same disk $\mathbb{D}(0,\rho)$. By Theorem 5,

$$D_V(\lambda) = D_{W_0}(\lambda) + O(\varepsilon^2) \text{ uniformly on } \mathbb{D}(0,\rho).$$

As D_{W_0} has no zero on $\partial\mathbb{D}(0,\rho)$ we have

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,\rho)} \frac{D_V'(\lambda)}{D_V(\lambda)} d\lambda \to \frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,\rho)} \frac{D_{W_0}'(\lambda)}{D_{W_0}(\lambda)} d\lambda.$$

Therefore W_0 and V have the same (finite) number of resonances on $\mathbb{D}(0,\rho)$ for ε small enough. By Theorem 6 there exists $c > 0$ such that these resonances belong to

$$\mathscr{C}_\varepsilon = \bigcup_{\substack{\lambda_0 \in \mathrm{Res}(W_0),\\ \mathrm{Im}\,\lambda_0 \geq -B}} \mathbb{D}\left(\lambda_0, c\varepsilon^{2/m_{W_0}(\lambda_0)}\right).$$

Now assume that $\lambda \in \mathrm{Res}(V)$ satisfies $\mathrm{Im}\,\lambda \leq -B$ and that λ does not belong to the set $\mathscr{T}_\varepsilon$ defined in (1.5). This means

$$\lambda \notin \bigcup_{\substack{\lambda_0 \in \mathrm{Res}(W_0),\\ \mathrm{Im}\,\lambda_0 \leq -B}} \mathbb{D}\left(\lambda_0, \langle\lambda_0\rangle^{-d-1}\right).$$

Then (see the proof of [15, Theorem 3.49]):

$$\left|(\mathrm{Id} + K_{W_0})^{-1}\right|_{\mathscr{B}} \leq e^{C\langle\lambda\rangle^{2d+1}}.$$

We now reproduce the proof of Theorem 1 for $d \geq 3$. Since $\lambda \in \mathrm{Res}(V)$ there must exist $u \in L^2$ with $u = -K_V u$. In particular u belongs to H^1 with $|u|_{H^1} \leq Ce^{C(\operatorname{Im}\lambda)_-}|W|_\infty |u|_2$. The equation $u = -K_V u$ is equivalent to

$$u = -\left(\mathrm{Id} + K_{W_0}\right)^{-1} K_{V_\sharp} u = -\left(\mathrm{Id} + K_{W_0}\right)^{-1} \sum_{k \neq 0} K_{W_k} e^{ik\bullet/\varepsilon} u,$$

where $V_\sharp(x) = \sum_{k\neq 0} W_k(x) e^{ikx/\varepsilon}$. As in the proof of Theorem 1, we perform an integration by parts on the term $K_{W_k} e^{ik\bullet/\varepsilon} u$:

$$\frac{|k|}{\varepsilon} K_{W_k} e^{ik\bullet/\varepsilon} u = K_{W_k}(k\cdot D) e^{ik\bullet/\varepsilon} u - K_{W_k} e^{ik\bullet/\varepsilon}(k\cdot D)u.$$

This yields

$$\frac{|k|}{\varepsilon} |K_{W_k} e^{ik\bullet/\varepsilon} u|_2 \leq Ce^{2L(\operatorname{Im}\lambda)_-} \|W_k\|_1 |u|_2 + Ce^{2L(\operatorname{Im}\lambda)_-} |W_k|_\infty |u|_{H^1}.$$

Using the a priori bound on $|u|_{H^1}$ and summing over $k \neq 0$ we obtain

$$|K_{V_\sharp} u|_2 \leq C\varepsilon e^{2L(\operatorname{Im}\lambda)_-} |W|_\infty \left(\sum_{k\neq 0} \frac{\|W_k\|_1}{|k|} \right) |u|_2.$$

It follows that

$$|u|_2 = \left|(\mathrm{Id} + K_{W_0})^{-1} K_{V_\sharp} u\right|_2 \leq C\varepsilon e^{2L(\operatorname{Im}\lambda)_-} e^{C\langle\lambda\rangle^{2d+1}} |W|_\infty \left(\sum_{k\neq 0} \frac{\|W_k\|_1}{|k|} \right) |u|_2.$$

Since $u \neq 0$, this implies a lower bound on $|\lambda|$ of the form $|\lambda| \geq A - C\ln(\varepsilon^{-1})^{1/(2d+1)}$. Thus λ belongs to the set $\mathscr{D}_\varepsilon$ defined in (1.5). This ends the proof of Theorem 4.

4. Proof of Theorem 5

We now get to the core of the paper: the proof of Theorem 5. We first explain the ideas. If D_V is the determinant given by (1.7) we can write formally

$$D_V(\lambda) = \exp\left(-\sum_{m=p}^{\infty} \frac{(-1)^m}{m} \mathrm{Tr}\left(K_V^m\right) \right).$$

It order to prove Theorem 5 it seems necessary to obtain an expansion in powers of ε of $\mathrm{Tr}(K_V^m)$. For a potential V given by $V(x) = \sum_{k\in\mathbb{Z}^d} W_k(x) e^{ikx/\varepsilon}$ then $\mathrm{Tr}(K_V^m)$ can be decomposed as a sum of terms of the form

$$T[k_1,\dots,k_m] = \mathrm{Tr}\left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j\bullet/\varepsilon} \right),$$

where $k_1,\dots,k_m \in \mathbb{Z}^d$. We now explain how to obtain an expansion for $T[k_1,\dots,k_m]$. We say that the sequence $k_1,\dots,k_m$ is constructive if $k_1+\cdots+k_m = 0$ and destructive otherwise. We use this terminology for the following reason. In the case of a destructive sequence the behavior of the oscillatory terms $e^{ik_jx/\varepsilon}$ imply $\prod_{j=1}^m e^{ik_jx/\varepsilon} \rightarrow 0$ weakly as $\varepsilon \rightarrow 0$—one sometimes say that the interference between oscillatory terms is destructive, which explains the above terminology. We will prove that in this case $T[k_1,\dots,k_m]$ is of order $O(\varepsilon^N)$ and thus produces no term in the expansion provided by Theorem 5. Now if $k_1, \dots, k_m$ is a constructive sequence let $R(\xi) = (\xi^2 - \lambda^2)^{-1}$ so that formally $R_0(\lambda) = R(D)$. Using the commutation relation $e^{-ik\bullet/\varepsilon} D e^{ik\bullet/\varepsilon} = D + k/\varepsilon$, we have

$$T[k_1, \dots, k_m] = \operatorname{Tr}\left(\prod_{j=1}^{m} \rho R(D) W_{k_j} e^{ik_j\bullet/\varepsilon}\right) = \operatorname{Tr}\left(\prod_{j=1}^{m} \rho R(D + \sigma_j/\varepsilon) W_{k_j}\right), \tag{4.1}$$

where $\sigma_j = k_j + \cdots + k_m$. We note that there are no more oscillatory terms in the second line of (4.1). An expansion of $T[k_1, \dots, k_m]$ follows then from an operator-valued expansion of the operator $R(D + \sigma_j/\varepsilon)$, which in turn follows from an expansion of the function $R(\xi + \sigma_j/\varepsilon)$. The terms in this expansion are specifically created by the constructive interference between oscillatory factors $e^{ik_\ell\bullet/\varepsilon}$.

4.1. Preliminaries on Fredholm determinants

We start by giving a formula for general Fredholm determinants as infinite series. Consider X, Y two trace class operators on L^2 and assume that $\mathrm{Id} + X$ is invertible. Define the Fredholm determinant

$$D(\mu) = \operatorname{Det}(\mathrm{Id} + X + \mu Y).$$

This is a holomorphic function of the variable μ, satisfying the bound $|D(\mu)| \le e^{|X|_{\mathcal{L}} + \mu|Y|_{\mathcal{L}}}$. Expand it in power series: there exists a sequence $\omega_n(X, Y)$ such that

$$D(\mu) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \omega_n(X, Y). \tag{4.2}$$

The terms $\omega_n(X, Y)$ are given by the $n \times n$ determinant

$$\omega_n(X, Y) = \operatorname{Det}(\mathrm{Id} + X) \begin{vmatrix} \tau_1 & n-1 & 0 & \dots & 0 \\ \tau_2 & \tau_1 & n-2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \tau_{n-1} & \ddots & \ddots & \ddots & 1 \\ \tau_n & \tau_{n-1} & \dots & \tau_2 & \tau_1 \end{vmatrix}, \tag{4.3}$$

where $\tau_j = \operatorname{Tr}\left(((\mathrm{Id} + X)^{-1} Y)^j\right)$—see [23, Theorem 6.8].

LEMMA 4.1. – *Let $s \ge 0$ and assume that $\langle D\rangle^s X$ and $\langle D\rangle^s Y$, initially defined as operators from L^2 to H^{-s}, are trace class operator on L^2. Then*

$$|\omega_n(X, Y)| \le |\langle D\rangle^s Y \langle D\rangle^{-s}|_{\mathcal{L}}^n e^{|\langle D\rangle^s X \langle D\rangle^{-s}|_{\mathcal{L}}}. \tag{4.4}$$

Proof. – First note that since $\langle D\rangle^{-s} \in \mathcal{B}$ and $\langle D\rangle^s (X + \mu Y) \in \mathcal{L}$ we can use the cyclicity of the determinant to get

$$\operatorname{Det}(\mathrm{Id} + X + \mu Y) = \operatorname{Det}(\mathrm{Id} + \langle D\rangle^s (X + \mu Y) \langle D\rangle^{-s}).$$

Therefore

$$|\operatorname{Det}(\mathrm{Id} + X + \mu Y)| \le \exp(|\langle D\rangle^s X \langle D\rangle^{-s}|_{\mathcal{L}} + |\mu| |\langle D\rangle^s Y \langle D\rangle^{-s}|_{\mathcal{L}}).$$

This proves that $\mathrm{Det}(\mathrm{Id} + X + \mu Y)$ is an entire function of order 1. Therefore by Cauchy estimates the coefficients $\omega_n(X, Y)$ must satisfy (4.4). This completes the proof. $\square$

4.2. Reduction to a trace expansion

We now start the proof of Theorem 5. Fix $N \geq d + 4$ and $p = 4(d + N)N$. Let W in $C_0^\infty(\mathbb{B}^d(0, L) \times \mathbb{T}^d, \mathbb{C})$, $V, V_\sharp \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ be given by

$$W(x, y) = \sum_{k \in \mathbb{Z}^d} W_k(x) e^{iky}, \quad V(x) = W_0(x) + V_\sharp(x), \quad V_\sharp(x) = \sum_{k \neq 0} W_k(x) e^{ikx/\varepsilon}.$$

We define $|W|_{Z^s} = \sum_{k \in \mathbb{Z}^d} \|W_k\|_s$. This quantity is finite for every $s \geq 0$.

Let X and Y be the trace class operators given by

$$X = \Psi(K_{W_0}), \quad Y = \Psi(K_V) - \Psi(K_{W_0}),$$
$$\Psi(z) = (1 + z) \exp\left(-z + \frac{z^2}{2} - \cdots + \frac{(-z)^{p-1}}{p-1}\right) - 1. \tag{4.5}$$

The expansion (4.2) yields

$$D_V(\lambda) = \mathrm{Det}(\mathrm{Id} + X + Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \omega_n(X, Y).$$

We now reduce this exact infinite expansion to a finite expansion modulo a term of order $O(\varepsilon^{2N})$. We recall that $X_d = \mathbb{C}$ if $d \geq 3$ and $X_1 = \mathbb{C} \setminus \{0\}$.

LEMMA 4.2. – *Locally uniformly on X_d, we have*

$$D_V(\lambda) = \sum_{n=0}^{N} \frac{1}{n!} \omega_n(X, Y) + O(\varepsilon^{2N}).$$

Proof. – It is enough to show that the coefficients $\omega_n(X, Y)$ satisfy the inequality

$$|\omega_n(X, Y)| \leq (C\varepsilon^2)^n \tag{4.6}$$

for all $n \geq 0$. Because of (4.4) it suffices then to estimate $|Y|_{\mathcal{L}}$. Recall that the first $p - 1$ derivatives of Ψ vanish at 0 and write a power series expansion of Ψ as

$$\Psi(z) = \sum_{m=p}^{\infty} \alpha_m z^m, \quad \alpha_m = \frac{1}{m!} \frac{d^m \Psi}{dz^m}(0).$$

Since the function Ψ is entire of order $p - 1$ and type $(p - 1)^{-1}$ the coefficients α_m satisfy the estimate

$$|\alpha_m| \leq C \left(m^{-m} e^m\right)^{1/(p-1)} \leq C(m^{1/2}/m!)^{1/(p-1)} \tag{4.7}$$

—see for instance [25]. Next write

$$Y = \Psi(K_V) - \Psi(K_{W_0}) = \int_0^1 \frac{d}{dt} \Psi(K_{W_0 + tV_\sharp}) dt = \int_0^1 \sum_{m=p}^{\infty} \alpha_m \sum_{\ell=0}^{m-1} K_{W_0+tV_\sharp}^{\ell} K_{V_\sharp} K_{W_0+tV_\sharp}^{m-\ell-1} dt.$$

This yields

$$
\begin{aligned}
&\langle D\rangle^2 \left(\Psi(K_V)-\Psi(K_{W_0})\right)\langle D\rangle^{-2} \\
&=\int_0^1 \sum_{m=p}^{\infty}\alpha_m \sum_{\ell=0}^{m-1}\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)^{\ell-1}\langle D\rangle^2 K_{W_0+tV_\sharp}K_{V_\sharp}\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)^{m-\ell-1} dt.
\end{aligned}
$$

The singular values of $\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}$ are bounded as follows:

$$
s_j\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)\le \left|\langle D\rangle^2 K_{W_0+tV_\sharp}\right|_{\mathscr{B}} s_j\left(\rho\langle D\rangle^{-2}\right)\le C|W|_\infty s_j\left(\rho\langle D\rangle^{-2}\right).
$$

To estimate $s_j\left(\rho\langle D\rangle^{-2}\right)$ we note that as the singular values of an operator X are the square roots of the eigenvalues of XX^*,

$$
s_j\left(\rho\langle D\rangle^{-2}\right)=\lambda_j\left(\rho\langle D\rangle^{-4}\rho\right)^{1/2}\le s_j\left(\rho\langle D\rangle^{-4}\rho\right)^{1/2}\le Cj^{-2/d}. \tag{4.8}
$$

In the last line we used [15, (B.3.9)]. It follows that $s_j\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)\le C|W|_\infty j^{-2/d}$. In addition using the commutation relation

$$
e^{ik\bullet/\varepsilon}=\frac{\varepsilon}{|k|}[(k\cdot D),e^{ik\bullet/\varepsilon}],\quad (k\cdot D)=\frac{k_1D_{x_1}+\cdots+k_dD_{x_d}}{|k|},
$$

we obtain

$$
\begin{aligned}
|K_{V_\sharp}\langle D\rangle^{-2}|_{\mathscr{B}} &\le |K_\rho\langle D\rangle^2|_{\mathscr{B}}|\langle D\rangle^{-2}V_\sharp\langle D\rangle^{-2}|_{\mathscr{B}}\\
&\le \sum_{k\neq 0}|K_\rho\langle D\rangle^2|_{\mathscr{B}}|\langle D\rangle^{-2}W_k e^{ik\bullet/\varepsilon}\langle D\rangle^{-2}|_{\mathscr{B}}\\
&\le \sum_{k\neq 0}\frac{\varepsilon^2}{|k|^2}|K_\rho\langle D\rangle^2|_{\mathscr{B}}|\langle D\rangle^{-2}W_k[(k\cdot D),[(k\cdot D),e^{ik\bullet/\varepsilon}]]\langle D\rangle^{-2}|_{\mathscr{B}}\\
&\le C\varepsilon^2|W|_{Z^2}.
\end{aligned} \tag{4.9}
$$

Consequently,

$$
\begin{aligned}
&s_{(m-2)j}\left(\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)^{\ell-1}\langle D\rangle^2 K_{W_0+tV_\sharp}K_{V_\sharp}\langle D\rangle^{-2}\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)^{m-\ell-1}\right)\\
&\qquad\le s_j\left(\langle D\rangle^2 K_{W_0+tV_\sharp}\langle D\rangle^{-2}\right)^{m-2}|\langle D\rangle^2 K_{W_0+tV_\sharp}|_{\mathscr{B}}|K_{V_\sharp}\langle D\rangle^{-2}|_{\mathscr{B}}\\
&\qquad\le C^m\varepsilon^2|W|_\infty^{m-1}|W|_{Z^2}j^{-2(m-2)/d}.
\end{aligned}
$$

Sum over $\ell\in[0,m-1]$, $j\ge 0$ and note that $m\ge p\ge d+2$ to obtain the bound

$$
\left|\sum_{\ell=0}^{m-1}\langle D\rangle^2 K^{\ell}_{W_0+tV_\sharp}K_{V_\sharp}K^{m-1-\ell}_{W_0+tV_\sharp}\langle D\rangle^{-2}\right|_{\mathscr{L}}\le m^2C^m\varepsilon^2|W|_\infty^{m-1}|W|_{Z^2}.
$$

This yields

$$
\left|\langle D\rangle^2\left(\Psi(K_V)-\Psi(K_{W_0})\right)\langle D\rangle^{-2}\right|_{\mathscr{L}}\le\sum_{m=p}^{\infty}m^2|\alpha_m|C^m\varepsilon^2|W|_\infty^{m-1}|W|_{Z^2}\le C\varepsilon^2,
$$

where the series indeed converges because of the decay of the coefficients α_m proved in (4.7). This ends the proof of the lemma. □

We now show that Theorem 5 can be reduced to the following key result:

LEMMA 4.3. – *Let X, Y be given by* (4.5) *and $\mathscr{T}_X$ be the holomorphic continuation of the operator* $\mathrm{Det}(\mathrm{Id}+X)(\mathrm{Id}+X)^{-1}$ *given in Appendix A. There exist N functions $c_0, c_1, \dots c_{N-1}$ holomorphic on X_d such that for all $1 \leq a \leq N$,*

$$\mathrm{Tr}\left((\mathscr{T}_X Y)^a\right) = c_0(\lambda) + \varepsilon c_1(\lambda) + \cdots + \varepsilon^{N-1} c_{N-1}(\lambda) + O(\varepsilon^N).$$

This holds uniformly locally on X_d.

Assuming that this lemma holds Theorem 5 is only a consequence of a complex analysis argument resumed in

LEMMA 4.4. – *Let $E = \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$, S_0 be a discrete subset of E. Also let $(\lambda, \varepsilon) \to f(\lambda, \varepsilon), g(\lambda, \varepsilon)$ two functions such that $f(\cdot, \varepsilon), g(\cdot, \varepsilon)$ are meromorphic with poles in S_0 and such that $h(\cdot, \varepsilon) = f(\cdot, \varepsilon) g(\cdot, \varepsilon)$ is holomorphic on E. Assume moreover that locally uniformly on $E \setminus S_0$ we have*

$$\begin{aligned} f(\lambda, \varepsilon) &= f_0(\lambda) + \varepsilon f_1(\lambda) + \cdots + \varepsilon^{N-1} f_{N-1}(\lambda) + O(\varepsilon^N) \\ g(\lambda, \varepsilon) &= g_0(\lambda) + \varepsilon g_1(\lambda) + \cdots + \varepsilon^{N-1} g_{N-1}(\lambda) + O(\varepsilon^N), \end{aligned} \tag{4.10}$$

where $f_0, g_0, \dots, f_{N-1}, g_{N-1}$ are meromorphic functions of $\lambda \in \mathbb{C}$. Then there exist holomorphic functions $h_0, \dots, h_{N-1}$ on E such that uniformly locally on E,

$$h(\lambda, \varepsilon) = h_0(\lambda) + \varepsilon h_1(\lambda) + \cdots + \varepsilon^{N-1} h_{N-1}(\lambda) + O(\varepsilon^N). \tag{4.11}$$

Proof. – First note that (4.10) and the fact that $h = fg$ imply that the expansion (4.11) holds for λ away from S_0. It remains to show that the functions h_j are holomorphic on E and that the expansion holds locally uniformly on E. We first note that locally uniformly on $E \setminus S_0$,

$$f_j(\lambda) = \lim_{\varepsilon \to 0} \frac{f(\lambda) - f_0(\lambda) - \cdots - \varepsilon^{j-1} f_{j-1}(\lambda)}{\varepsilon^j},$$

where by convention $f_{-1} = 0$. A uniform limit of holomorphic functions is holomorphic; thus by an immediate recursion $f_0, \dots, f_{N-1}$ must be holomorphic on E. The poles of the f_n are then a subset of the poles of f and thus they all belong to S_0. The same holds for the poles of g_n. Consequently the poles of the h_n belong to S_0. Let n minimal so that h_n has a singularity at a point $\lambda_0 \in S_0$. For r small enough λ_0 is the unique singularity of h_n on $\mathbb{D}(\lambda_0, 2r)$. For every $\varepsilon > 0$, the function

$$H_n(\cdot, \varepsilon) = \frac{h(\cdot, \varepsilon) - \varepsilon h_1 - \cdots - \varepsilon^{n-1} h_{n-1}}{\varepsilon^n}$$

is holomorphic on $\mathbb{D}(\lambda_0, 2r)$. As $\varepsilon \to 0$, $H_n(\lambda, \varepsilon) = O(1)$ and $H_n(\lambda, \varepsilon) \to h_n(\lambda)$, both holding uniformly locally in $\mathbb{D}(\lambda_0, 2r) \setminus \{\lambda_0\}$. By the maximum principle there exists $M > 0$ such that for every $\lambda \in \mathbb{D}(\lambda_0, r) \setminus \{\lambda_0\}$,

$$|h_n(\lambda)| = \lim_{\varepsilon \to 0} |H_n(\lambda, \varepsilon)| \leq \limsup_{\varepsilon \to 0} \sup_{\mu \in \partial \mathbb{D}(\lambda_0, r)} |H_n(\mu, \varepsilon)| \leq M.$$

Therefore h_n is uniformly bounded in a neighborhood of λ_0 and its singularity is removable. It follows that all the h_j are holomorphic on E. Now to prove that (4.11) holds uniformly locally on E we recall that it already holds uniformly locally on $E \setminus S_0$. Now if $\lambda_0 \in S_0$ and $r > 0$ is such that $\overline{\mathbb{D}(\lambda_0, r)} \subset E$ and $\partial\mathbb{D}(\lambda_0, r) \subset E \setminus S_0$ then Cauchy's formula shows

$$h(\lambda) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}(\lambda,r)} \frac{h(\mu)}{\mu-\lambda}d\mu = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}(\lambda,r)} \frac{h_0(\mu) + \cdots \varepsilon^{N-1}h_{N-1}(\mu) + O(\varepsilon^N)}{\mu-\lambda}d\mu$$
$$= h_0(\lambda) + \cdots \varepsilon^{N-1}h_{N-1}(\lambda) + O(\varepsilon^N),$$

with convergence realized uniformly in $\mathbb{D}(\lambda, r)$. This ends the proof. □

Proof of Theorem 5 assuming Lemma 4.3.. – By Lemma 4.2 it suffices to prove that for every $n \in [0, N]$, $\omega_n(X, Y)$ admits an expansion in powers of ε at order N. By (4.3), $\omega_n(X, \mathrm{Det}(\mathrm{Id}+X)Y)$ is a finite combination of terms of the form $\mathrm{Tr}((\mathcal{T}_X Y)^\alpha)$, $1 \le a \le N$. Thus by Lemma 4.3, $\omega_n(X, \mathrm{Det}(\mathrm{Id}+X)Y)$ has an expansion of the form

$$\omega_n(X, \mathrm{Det}(\mathrm{Id}+X)Y) = f_0(\lambda) + \varepsilon f_1(\lambda) + \cdots + \varepsilon^{N-1} f_{N-1}(\lambda) + O(\varepsilon^N). \tag{4.12}$$

Here the convergence holds locally uniformly on X_d. In addition,

$$\omega_n(X, Y) = \frac{1}{\det(\mathrm{Id}+X)^n}\omega_n(X, \mathrm{Det}(\mathrm{Id}+X)Y).$$

Now apply Lemma 4.4 to the case $E = X_d$, $S_0 = \mathrm{Res}(W_0)$, $f = \det(\mathrm{Id}+X)^{-n}$ and $g = \omega_n(X, \mathrm{Det}(\mathrm{Id}+X)Y)$. The meromorphic function f does not depend on ε and its poles in E are exactly the resonances of W_0. The function g is holomorphic on E, depends on ε and admits an expansion given by (4.12). The product $h = fg$ is then meromorphic; by (4.6) it is locally uniformly bounded on E and consequently it is holomorphic on E. Thus $\omega_n(X, Y)$ admits an expansion in powers of ε at order N and Theorem 5 follows. We will compute the first few terms in §4.5 below. □

The next sections are devoted to the proof of Lemma 4.3. We first simplify the expression $\mathrm{Tr}\left((\mathcal{T}_X Y)^a\right)$.

LEMMA 4.5. – *For $a \in [1, N]$, $\mathrm{Tr}((\mathcal{T}_X Y)^a)$ can be written modulo $O(\varepsilon^N)$ as a finite sum of expressions of the form $\mathrm{Tr}(\mathcal{T}_X F_{n_1} \cdots \mathcal{T}_X F_{n_a})$ where $1 \le n_j \le 2N-1$ and*

$$F_n = \sum_{m=p}^{\infty} \alpha_m \sum_{\ell_0+\cdots+\ell_n+n=m} K_{W_0}^{\ell_1} K_{V_\sharp} \cdots K_{W_0}^{\ell_{n-1}} K_{V_\sharp} K_{W_0}^{\ell_n}, \quad \alpha_m = \frac{1}{m!}\frac{d^m\Psi}{dz^m}(0). \tag{4.13}$$

This holds uniformly locally on X_d.

Proof. – Fix $1 \le a \le N$ and define $\mathcal{K}_{\mathcal{V}} = \langle D\rangle K_{\mathcal{V}} \langle D\rangle^{-1}$. Using the cyclicity of the trace,

$$\mathrm{Tr}\left((\mathcal{T}_X Y)^a\right) = \mathrm{Tr}\left((\mathcal{T}_{X'} Y')^a\right)$$
$$X' = \mathrm{Det}(\mathrm{Id} + \Psi(\mathcal{K}_{W_0}))(\mathrm{Id} + \Psi(\mathcal{K}_{W_0}))^{-1}, \quad Y' = \Psi(\mathcal{K}_V) - \Psi(\mathcal{K}_{W_0}).$$

Define

$$\mathcal{E}_{m,n} = \sum_{\ell_0+\cdots+\ell_n+n=m} \mathcal{K}_{W_0}^{\ell_0} \mathcal{K}_{V_\sharp} \cdots \mathcal{K}_{W_0}^{\ell_{n-1}} \mathcal{K}_{V_\sharp} \mathcal{K}_{W_0}^{\ell_n}, \quad \mathcal{F}_n = \sum_{m=p}^{\infty} \alpha_m \mathcal{E}_{m,n}. \tag{4.14}$$

The index n has the following significance: $\mathscr{E}_{m,n}$ is the sum of monomials in $\mathscr{K}_{W_0}$, $\mathscr{K}_{V_\sharp}$ with exactly n factors equal to $\mathscr{K}_{V_\sharp}$. Using the power series expansion of Ψ and $Y' = \Psi(\mathscr{K}_V) - \Psi(\mathscr{K}_{W_0})$ we obtain

$$Y' = \sum_{m=p}^{\infty} \alpha_m (\mathscr{K}_{W_0} + \mathscr{K}_{V_\sharp})^m - \sum_{m=p}^{\infty} \alpha_m \mathscr{K}_{W_0}^m = \sum_{m=p}^{\infty} \alpha_m (\mathscr{E}_{m,1} + \cdots + \mathscr{E}_{m,m}) = \sum_{n=1}^{\infty} \mathscr{F}_n.$$

We claim that

$$\left| \sum_{n=2N}^{\infty} \mathscr{F}_n \right|_{\mathscr{L}} = O(\varepsilon^N). \tag{4.15}$$

In order to prove this start by fixing $\ell_0, \dots, \ell_n$ with $\ell_0 + \cdots + \ell_n + n = m \geq p$. Since $\mathscr{K}_{V_\sharp}$ appears exactly n times in the product $\mathscr{K}_{W_0}^{\ell_0} \mathscr{K}_{V_\sharp} \cdots \mathscr{K}_{W_0}^{\ell_{n-1}} \mathscr{K}_{V_\sharp} \mathscr{K}_{W_0}^{\ell_n}$ we have

$$s_{nj} \left(\mathscr{K}_{W_0}^{\ell_0} \mathscr{K}_{V_\sharp} \cdots \mathscr{K}_{W_0}^{\ell_{n-1}} \mathscr{K}_{V_\sharp} \mathscr{K}_{W_0}^{\ell_n} \right) \leq s_j (\mathscr{K}_{V_\sharp})^n |\mathscr{K}_{W_0}|_{\mathscr{B}}^{m-n}. \tag{4.16}$$

We now prove some estimates on $s_j(\mathscr{K}_{V_\sharp})$. On one hand by the same argument as in (4.9) we have

$$s_j(\mathscr{K}_{V_\sharp}) \leq |\mathscr{K}_{V_\sharp}|_{\mathscr{B}} \leq |\langle D \rangle K_\rho \langle D \rangle|_{\mathscr{B}} \cdot |\langle D \rangle^{-1} V_\sharp \langle D \rangle^{-1}|_{\mathscr{B}} \leq C\varepsilon |W|_{Z^1}.$$

On the other hand by arguments similar to (4.8) we have

$$s_j(\mathscr{K}_{V_\sharp}) \leq |\langle D \rangle K_{V_\sharp}|_{\mathscr{B}} \cdot s_j(\rho \langle D \rangle^{-1}) \leq C|W|_\infty j^{-1/d}.$$

Interpolating both inequalities yields $s_j(\mathscr{K}_{V_\sharp}) \leq C\varepsilon^{1/2}|W|_{Z^1} j^{-1/(2d)}$. Coming back to (4.16) we obtain

$$s_{mj} \left(\mathscr{K}_{W_0}^{\ell_0} \mathscr{K}_{V_\sharp} \cdots \mathscr{K}_{W_0}^{\ell_{n-1}} \mathscr{K}_{V_\sharp} \mathscr{K}_{W_0}^{\ell_n} \right) \leq C^m |W|_{Z^1}^m \varepsilon^{n/2} j^{-n/(2d)}. \tag{4.17}$$

Since $n \geq 2N \geq 2d + 2$ the RHS of (4.17) is summable. Summation over j leads

$$\left| \mathscr{K}_{W_0}^{\ell_0} \mathscr{K}_{V_\sharp} \cdots \mathscr{K}_{W_0}^{\ell_{n-1}} \mathscr{K}_{V_\sharp} \mathscr{K}_{W_0}^{\ell_n} \right|_{\mathscr{L}} \leq m\varepsilon^{n/2} (C|W|_{Z^1})^m$$

Consequently if $\mathscr{E}_{m,n}$ is given by (4.14) then for $n \geq 2N$

$$|\mathscr{E}_{m,n}|_{\mathscr{L}} \leq m \binom{m}{n} \varepsilon^N (C|W|_{Z^1})^m. \tag{4.18}$$

The claim (4.15) follows then from (4.18) and the estimate (4.7) on α_m:

$$\begin{aligned}
\left| \sum_{n=2N}^{\infty} \mathscr{F}_n \right|_{\mathscr{L}} &= \left| \sum_{m=p}^{\infty} \alpha_m (\mathscr{E}_{m,2N} + \cdots + \mathscr{E}_{m,m}) \right|_{\mathscr{L}} \\
&\leq \varepsilon^N \sum_{m=p}^{\infty} m|\alpha_m| (C|W|_{Z^1})^m \left(\binom{m}{2N} + \cdots + \binom{m}{m} \right) \\
&\leq \varepsilon^N \sum_{m=p}^{\infty} m|\alpha_m| (2C|W|_{Z^1})^m = O(\varepsilon^N).
\end{aligned}$$

It follows that we can write Y' as a the sum of a finite combination of the operators $\mathcal{F}_n$ with $1 \leq n \leq m$ and a small error in $\mathcal{L}$:

$$Y' = \sum_{n=1}^{\infty} \mathcal{F}_n = \sum_{n=1}^{2N-1} \mathcal{F}_n + O_{\mathcal{L}}(\varepsilon^N).$$

Therefore $\mathrm{Tr}((\mathcal{T}_{X'} Y')^a)$ is modulo $O(\varepsilon^N)$ a finite sum of expressions of the form

$$\mathrm{Tr}(\mathcal{T}_{X'} \mathcal{F}_{n_1} \cdots \mathcal{T}_{X'} \mathcal{F}_{n_a}),$$

where $1 \leq n_j \leq 2N-1$. Now as $X = \langle D\rangle^{-1} X' \langle D\rangle$, $F_n = \langle D\rangle^{-1} \mathcal{F}_n \langle D\rangle$, and $\mathrm{Tr}((\mathcal{T}_{X'} Y')^a) = \mathrm{Tr}((\mathcal{T}_X Y)^a$, this completes the proof of the lemma. □

To sum up we have proved that Theorem 5 holds if Lemma 4.3 holds, that is if for $a \in [1, N]$, $\mathrm{Tr}((\mathcal{T}_X Y)^a)$ admits an expansion in powers of ε. In addition Lemma 4.3 holds if for all $n_j \in [1, 2N-1]$, $\mathrm{Tr}(\mathcal{T}_X F_{n_1} \cdots \mathcal{T}_X F_{n_a})$ admits an expansion in powers of ε.

We write the operator F_n given in (4.13) in the following form:

$$F_n = \sum_{m=p}^{\infty} \sum_{\{k_\ell\} \in \mathcal{S}_m^n} \alpha_m \left(\prod_{\ell=1}^{m} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon} \right), \tag{4.19}$$

where $\mathcal{S}_m^n$ is the collection of sequences d-tuples $(k_1, \ldots, k_m)$, with exactly n non-vanishing terms. Because of the conclusion of Lemma 4.5 we can restrict our attention to operators F_n with $n \leq 2N-1$. For $n \leq 2N-1$ and $m \geq p$ the sequences of $\mathcal{S}_m^n$ have much more vanishing terms than non vanishing terms. This will allow us to use some arguments of combinatorial nature. The expansion of F_n given by (4.19) leads to

$$\begin{aligned} \prod_{j=1}^{a} \mathcal{T}_X F_{n_j} &= \sum_{m_1,\ldots,m_a=p}^{\infty} \sum_{\{k_\ell^1\} \in \mathcal{S}_{m_1}^{n_1}, \ldots, \{k_\ell^a\} \in \mathcal{S}_{m_a}^{n_a}} \left(\prod_{j=1}^{a} \alpha_{m_j} \mathcal{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right) \\ &= \mathcal{D}_{n_1,\ldots,n_a} + \mathcal{C}_{n_1,\ldots,n_a}, \end{aligned}$$

where

(4.20)

$$\mathcal{D}_{n_1,\ldots,n_a} = \sum_{m_1,\ldots,m_a=p}^{\infty} \sum_{\substack{\{k_\ell^1\} \in \mathcal{S}_{m_1}^{n_1}, \ldots, \{k_\ell^a\} \in \mathcal{S}_{m_a}^{n_a}, \\ k_1^1+\cdots+k_{m_a}^a \neq 0}} \left(\prod_{j=1}^{a} \alpha_{m_j} \mathcal{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right),$$

$$\mathcal{C}_{n_1,\ldots,n_a} = \sum_{m_1,\ldots,m_a=p}^{\infty} \sum_{\substack{\{k_\ell^1\} \in \mathcal{S}_{m_1}^{n_1}, \ldots, \{k_\ell^a\} \in \mathcal{S}_{m_a}^{n_a}, \\ k_1^1+\cdots+k_{m_a}^a = 0}} \left(\prod_{j=1}^{a} \alpha_{m_j} \mathcal{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right).$$

In the next subsection we estimate the trace of the operator $\mathcal{D}_{n_1,\ldots,n_a}$.

4.3. Destructive interaction

The main result of this part is the following:

Lemma 4.6. – *For* $1 \leq a \leq N$ *and* $n_1, \dots, n_a \in [0, 2N-1]$ *let* $\mathscr{D}_{n_1,\dots,n_a}$ *be the trace class operator given by* (4.20). *Then locally uniformly on* X_d*,*

$$\operatorname{Tr}\left(\mathscr{D}_{n_1,\dots,n_a}\right) = O(\varepsilon^N).$$

We start with a few definitions. Let $\{k_\ell\}_{1\leq\ell\leq\nu}$ a sequence of d-tuples in $\mathbb{Z}^d$ of length ν. We say that $\{k_\ell\}_{1\leq\ell\leq\nu}$ is constructive if it satisfies $k_1 + \dots + k_\nu = 0$ and destructive otherwise. Roughly speaking, we will see in Lemma 4.10 below that the terms $\operatorname{Tr}\left(\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon}\right)$ associated with destructive sequences $\{k_\ell\}$ are negligible, i.e., are of order ε^N. This is due to destructive interference between oscillatory terms $e^{ik_\ell\bullet/\varepsilon}$. Similarly, we will see in Lemma 4.14 that if $\{k_j\}$ is constructive, the constructive interference between the oscillatory terms $e^{ik_\ell\bullet/\varepsilon}$ produce an expansion of $\operatorname{Tr}\left(\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right)$ in powers of ε. These are responsible for the terms $a_j\varepsilon^j$ in the expansion of $D_V(\lambda)$.

The treatment of terms associated with destructive sequences is difficult and requires certain preliminaries of combinatorial nature. A sequence of d-tuples $\{k_\ell\}_{1\leq\ell\leq\nu}$ is said to be admissible if

(i) It is destructive.
(ii) It starts and ends with at least N vanishing terms.

A sequence $\{k_\ell\}_{1\leq\ell\leq\nu'}$ with exactly γ non-vanishing terms is said to be good if

(i) It is admissible.
(ii) $\nu' \leq N + N\gamma + 1$.

A subsequence of consecutive d-tuples of an admissible sequence $\{k_\ell\}_{1\leq\ell\leq\nu}$ is said to be good if it takes the form $\{k_\ell\}_{q+1\leq\ell\leq q+\nu'}$ for some q, ν' and if the sequence $\{k_{\ell+q}\}_{1\leq\ell\leq\nu'}$ is good.

A cyclic permutation of $\{k_\ell\}_{1\leq\ell\leq\nu}$ is a sequence equal to

$$(k_{L+1}, \dots, k_\nu, k_1, \dots, k_L)$$

for some $L \geq 0$. We will use below the following version of the pigeonhole principle. Let $\{k_\ell\}_{1\leq\ell\leq\nu}$ a sequence with exactly γ non-vanishing terms. If $\nu \geq N(\gamma+1)$, there exists a subsequence of $\{k_\ell\}_{1\leq\ell\leq\nu}$ made of N consecutive vanishing d-tuples. The next lemma is a combinatorial result allowing us to extract good subsequences of consecutive d-tuples out of admissible subsequences.

Lemma 4.7. – *Every admissible sequence* $\{k_\ell\}_{1\leq\ell\leq\nu}$ *admits a good subsequence of consecutive* d*-tuples.*

Proof. – We prove this lemma by recursion on ν. We can start with $\nu = 2N+1$: there are no admissible sequences of length less or equal than $2N$. Any admissible sequence with length $2N+1$ has at least one non-vanishing term and therefore it is a good sequence. We now fix $\nu \geq 2N+2$ and we assume that all admissible sequences of length strictly less than ν admit a good subsequence of consecutive d-tuples. Let $\{k_\ell\}_{1\leq\ell\leq\nu}$ be an admissible sequence with γ non-vanishing terms.

If $\nu \leq N + \gamma N + 1$ then $\{k_\ell\}_{1\leq\ell\leq\nu}$ is good. Therefore we assume that we have $\nu \geq N + \gamma N + 2$. Consider the subsequence of minimal length of consecutive d-tuples starting at k_1, containing at least one non-zero term and ending with N zeros: $(k_1, \dots, k_{\nu'})$. Let γ' be the number of non-zero terms in this subsequence. Since this sequence is of minimal length the pigeonhole principle implies $\nu' \leq N + \gamma' N + 1$. Hence if $k_1 + \cdots + k_{\nu'} \neq 0$ then this subsequence is good and therefore we are done.

Otherwise the sequence $\{k_\ell\}_{\nu'-N+1\leq\ell\leq\nu}$ is admissible. Indeed it starts and ends with N zeros and it is destructive since $k_1 + \cdots + k_{\nu'} = 0$ and $k_1 + \cdots + k_\nu \neq 0$. Therefore we can apply the induction hypothesis: it admits a good subsequence of consecutive d-tuples. This completes the recursion and the proof. □

LEMMA 4.8. – *Let $\{k_\ell\}_{1\leq\ell\leq\nu}$ be an admissible sequence. Then locally uniformly on X_d,*

$$\left|\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}} \leq C^{\nu^2}\left(\prod_{i=1}^{\nu}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N. \tag{4.21}$$

This lemma is the key to prove Lemma 4.6. Roughly speaking, to prove (4.21), we must realize certain integration by parts at specifically chosen places. Each time we integrate by parts, we win a factor ε but we decrease the order of $\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}$ by one (as a pseudodifferential operator). Starting and ending with N zeroes ensures that after performing N integrations by parts, the resulting operator will still be a pseudodifferential operator of sufficiently small order. We start with a preliminary result:

LEMMA 4.9. – *The operator $I_{\mathcal{V},1}(\lambda) = K_{\mathcal{V}}(\lambda) - K_{\mathcal{V}}(-\lambda)$ is a smoothing operator. In addition there exists a constant C such that uniformly in $\lambda \in \mathbb{C}\setminus\mathbb{D}(0,1)$,*

$$\left|(D^2-\lambda^2)^N I_{\mathcal{V},1}(\lambda)\right|_{\mathcal{B}} \leq C\,\langle\lambda\rangle^{2N+d}\,e^{2L|\operatorname{Im}\lambda|}|\mathcal{V}|_\infty. \tag{4.22}$$

Proof. – The operator $I_{\mathcal{V},1}(\lambda)$ is smoothing as the kernel of the operator $R_0(\lambda) - R_0(-\lambda)$ is given by the smooth function

$$(x,y) \mapsto \frac{i}{2}\frac{\lambda^{d-2}}{(2\pi)^{d-1}}\int_{\mathbb{S}^{d-1}} e^{i\lambda\langle w, x-y\rangle}d\omega,$$

see [15, Theorem 3.4]. In order to prove the estimate (4.22) we note that by the product rule for derivatives the operator $(D^2-\lambda^2)^N I_{\mathcal{V},1}(\lambda)$ is a finite sum of operators of the form

$$\frac{i}{2}\frac{\lambda^{d-2+t}}{(2\pi)^{d-1}}\chi D^\alpha(R_0(\lambda) - R_0(-\lambda))\mathcal{V}, \tag{4.23}$$

where $t \in [0, 2N]$, α is multi-integer with entries in $[1, d]$ and of length $|\alpha| \leq 2N - t$ and $\chi \in \{D^\beta\rho,\ |\beta| \leq 2N\}$. The operators of the form (4.23) have kernel given by

$$(x,y) \mapsto \frac{i}{2}\frac{\lambda^{d-2+t}}{(2\pi)^{d-1}}\chi(x)\left(D_x^\alpha\int_{\mathbb{S}^{d-1}} e^{i\lambda\langle w,\cdot-y\rangle}d\omega\right)(x)\mathcal{V}(y).$$

We have $D_x^\alpha e^{i\lambda\langle w,\cdot-y\rangle}(x) = \omega_\alpha\lambda^{|\alpha|}e^{i\lambda\langle\omega,x-y\rangle}$ where ω_α is by definition $\omega_{\alpha_1}\cdots\omega_{\alpha_{|\alpha|}}$. Hence,

$$\left|\left(D_x^\alpha\int_{\mathbb{S}^{d-1}} e^{i\lambda\langle w,\cdot-y\rangle}d\omega\right)(x)\right| \leq C\,\langle\lambda\rangle^{|\alpha|}\,e^{|\operatorname{Im}\lambda||x-y|} \tag{4.24}$$

uniformly on $\mathbb{C}\setminus\mathbb{D}(0,1)$. Since χ and $\mathcal{V}$ are compactly supported the $\mathcal{B}$-norm of operators of the form (4.23) can be estimated by Schur's lemma and the bound (4.24). Recalling that $t+|\alpha|\leq 2N$ it leads to

$$\left|\frac{i}{2}\frac{\lambda^{d-2+t}}{(2\pi)^{d-1}}\chi D^{\alpha}(R_0(\lambda)-R_0(-\lambda))\mathcal{V}\right|_{\mathcal{B}}\leq C|\chi|_\infty\langle\lambda\rangle^{2N+d}e^{2L|\operatorname{Im}\lambda|}|\mathcal{V}|_\infty.$$

To conclude it suffices to recall that the operator $(D^2-\lambda^2)^N I_{\mathcal{V},1}(\lambda)$ is a finite sum of operators of the form (4.23). This completes the proof of (4.22). $\square$

Proof of Lemma 4.8.. – We divide the proof in three main steps.

1. Fix $M>1$. We first show that

$$\left|\prod_{\ell=1}^{\nu}K_{W_{k_\ell}}e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}}\leq C^{\nu^2}\langle\lambda\rangle^{\nu}\left(\prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N,\quad \operatorname{Im}\lambda\in[1,M],\tag{4.25}$$

uniformly on the set $\{\lambda:\ \operatorname{Im}\lambda\in[1,M]\}$. Let $R(\xi,\lambda)=(\xi^2-\lambda^2)^{-1}$ and

$$A(k,\lambda)=R(D+k/\varepsilon,\lambda)=e^{-ik\bullet/\varepsilon}R_0(\lambda)e^{ik\bullet/\varepsilon}.$$

Define $\sigma_\ell=k_\ell+\cdots+k_\nu$. The commutation relation $e^{-ik\bullet/\varepsilon}De^{ik\bullet/\varepsilon}=D+k/\varepsilon$ shows

$$\begin{aligned}K_{W_{k_1}}e^{ik_1\bullet/\varepsilon}\cdots K_{W_{k_\nu}}e^{ik_\nu\bullet/\varepsilon}&=\rho A(0,\lambda)W_{k_1}e^{ik_1\bullet/\varepsilon}A(0,\lambda)W_{k_2}e^{ik_2\bullet/\varepsilon}\cdots A(0,\lambda)W_{k_\nu}e^{ik_\nu\bullet/\varepsilon}\\&=e^{i\sigma_1\bullet/\varepsilon}\rho A(\sigma_1,\lambda)W_{k_1}A(\sigma_2,\lambda)W_{k_2}\cdots A(\sigma_\nu,\lambda)W_{k_\nu}.\end{aligned}$$

Now define $T_{j-1}=A(\sigma_j,\lambda)\cdots A(\sigma_\nu,\lambda)$ for $j\in[1,\nu]$. Since we are working in the half plane $\{\operatorname{Im}\lambda\geq 1\}$ the operator T_j is well defined and bounded from $H^{-2(\nu-j)}$ to L^2. It admits a bounded inverse T_j^{-1} from L^2 to $H^{-2(\nu-j)}$. Thus, for $j\in[1,\nu-1]$, $A(\sigma_j,\lambda)=T_{j-1}T_j^{-1}$ as an operator on L^2. This yields

$$\begin{aligned}K_{W_{k_1}}e^{ik_1\bullet/\varepsilon}\cdots K_{W_{k_\nu}}e^{ik_\nu\bullet/\varepsilon}&=e^{i\sigma_1\bullet/\varepsilon}\rho T_0\left(T_1^{-1}W_{k_1}T_1\right)\cdots\left(T_{\nu-1}^{-1}W_{k_{\nu-1}}T_{\nu-1}\right)W_{k_\nu}\\&=e^{i\sigma_1\bullet/\varepsilon}\rho T_0\left(\prod_{j=1}^{\nu-1}T_j^{-1}W_{k_j}T_j\right)W_{k_\nu}.\end{aligned}\tag{4.26}$$

The estimate (4.21) for $\operatorname{Im}\lambda\in[1,M]$ follows then from a bound on $|T_j^{-1}W_{k_j}T_j|_{\mathcal{B}}$ and a bound on $|T_0|_{\mathcal{B}}$. We start with the bound on $|T_0|_{\mathcal{B}}$. Since this operator is a Fourier multiplier we have

$$|T_0|_{\mathcal{B}}=\sup_{\xi\in\mathbb{R}^d}\left|\prod_{j=1}^{\nu}R(\xi+\sigma_j/\varepsilon,\lambda)\right|.$$

We reduce this estimate for $\operatorname{Im}\lambda\in[1,M]$ to an estimate for $\lambda=i$. For $\xi\in\mathbb{R}^d$ and $\operatorname{Im}\lambda\in[1,M]$ we have $\left|(\xi^2+1)/(\xi^2-\lambda^2)\right|\leq C\langle\lambda\rangle$. It implies

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^d}\left|\prod_{j=1}^{\nu}R(\xi+\sigma_j/\varepsilon,\lambda)\right|&=\sup_{\xi\in\mathbb{R}^d}\prod_{j=1}^{\nu}\left|R(\xi+\sigma_j/\varepsilon,i)\right|\cdot\left|\frac{(\xi+\sigma_j/\varepsilon)^2+1}{(\xi+\sigma_j/\varepsilon)^2-\lambda^2}\right|\\&\leq(C\langle\lambda\rangle)^{\nu}\sup_{\xi\in\mathbb{R}^d}\left|\prod_{j=1}^{\nu}\left\langle\xi+\sigma_j/\varepsilon\right\rangle^{-2}\right|.\end{aligned}$$

Since the sequence $\{k_\ell\}_{1\le\ell\le\nu}$ is admissible we have $\sigma_1 = \cdots = \sigma_N \neq 0$ and $\sigma_{\nu-N+1} = \cdots = \sigma_\nu = 0$. Thus the sequence $\{\sigma_\ell\}_{1\le\ell\le\nu-1}$ starts with N equal non-vanishing terms and ends with N vanishing terms. Peetre's inequality (see Equation (2.4)) implies

$$\sup_{\xi\in\mathbb{R}^d}\left|\prod_{j=1}^{\nu}\langle\xi+\sigma_j/\varepsilon\rangle^{-2}\right| \le \sup_{\xi\in\mathbb{R}^d}\left|\langle\xi+\sigma_1/\varepsilon\rangle^{-2N}\langle\xi\rangle^{-2N}\right| \le C^\nu\varepsilon^{2N}.$$

It follows that for $\lambda\in[1,M]$, $|T_0|_{\mathcal{B}} \le C^\nu\langle\lambda\rangle^\nu\varepsilon^N$.

We next estimate $|T_j^{-1}W_kT_j|_{\mathcal{B}}$, for any $k\in\mathbb{Z}^d$ and $j\in[0,\nu-1]$. We show that

$$|T_j^{-1}W_kT_j|_{\mathcal{B}} \le C^{\nu-j}\|W_k\|_{2(\nu-j)} \tag{4.27}$$

using a descendent recursion on j. If $j=\nu-1$ then $T_j = A(\sigma)$ for some $\sigma\in\mathbb{Z}^d$. Thus

$$\begin{aligned}A(\sigma)^{-1}W_kA(\sigma) &= W_k + [W_k,(D+\sigma/\varepsilon)^2-\lambda^2]A(\sigma)\\ &= W_k + (D^2W_k)A(\sigma) + 2(DW_k)\cdot(D+\sigma/\varepsilon)A(\sigma).\end{aligned}$$

The operator $A(\sigma) = e^{-i\sigma\bullet/\varepsilon}R_0(\lambda)e^{i\sigma\bullet/\varepsilon}$ is bounded on L^2 with uniform bound when $\operatorname{Im}\lambda\ge 1$. The operator $(D+\sigma/\varepsilon)A(\sigma) = e^{-i\sigma\bullet/\varepsilon}DR_0(\lambda)e^{i\sigma/\varepsilon}$ is also bounded on L^2 with uniform bound when $\operatorname{Im}\lambda\ge1$ as $DR_0(\lambda)$ is bounded on L^2 with uniform bound. Therefore, for a constant C that depends only on d,

$$|A(\sigma)^{-1}W_kA(\sigma)|_{\mathcal{B}} \le C\|W_k\|_2. \tag{4.28}$$

We can assume without loss of generality that

$$C \ge 1 + |A(\sigma)|_{\mathcal{B}} + 2|(D+\sigma/\varepsilon)A(\sigma)|_{\mathcal{B}}. \tag{4.29}$$

The bound (4.28) proves the case $j=\nu-1$ of (4.27). Now assume that (4.27) holds for some $j\in[1,\nu-1]$ and let us prove that it also holds for $j-1$. Write $T_{j-1} = A(\sigma)T_j$ for some σ so that

$$\begin{aligned}T_{j-1}^{-1}W_kT_{j-1} &= T_j^{-1}A(\sigma)^{-1}W_kA(\sigma)T_j\\ &= T_j^{-1}\left(W_k + (D^2W_k)A(\sigma) + 2(DW_k)\cdot(D+\sigma/\varepsilon)A(\sigma)\right)T_j\\ &= \left(T_j^{-1}W_kT_j\right) + 2\left(T_j^{-1}(DW_k)T_j\right)\cdot(D+\sigma/\varepsilon)A(\sigma) + \left(T_j^{-1}(D^2W_k)T_j\right)A(\sigma).\end{aligned}$$

Therefore the bounds follows from the recursion hypothesis applied to the operators $T_j^{-1}W_kT_j$, $T_j^{-1}(D^2W_k)T_j$ and $T_j^{-1}(DW_k)T_j$: we get

$$\begin{aligned}\left|T_{j-1}^{-1}W_kT_{j-1}\right|_{\mathcal{B}} &\le C^{\nu-j}\|W_k\|_{2(\nu-j)} + 2C^{\nu-j}\|DW_k\|_{2(\nu-j)}\cdot|(D+\sigma/\varepsilon)A(\sigma)|_{\mathcal{B}}\\ &\quad + C^{\nu-j}\|D^2W_k\|_{2(\nu-j)}|A(\sigma)|_{\mathcal{B}}\\ &\le C^{\nu-j+1}\|W_k\|_{2(\nu-j+1)}.\end{aligned}$$

In the last line we specifically used (4.29). This ends the recursion and thus the proof of (4.27). The estimate (4.25) follows then from the identity (4.26), and the bounds on $|T_j^{-1}W_kT_j|_{\mathcal{B}}$, $|T_0|_{\mathcal{B}}$.

2. We show that an estimate similar to (4.25) holds for $\operatorname{Im}\lambda\in[-M,-1]$. Write $K_{\mathcal{V}}(\lambda) = I_{\mathcal{V},0}(\lambda) + I_{\mathcal{V},1}(\lambda)$ where $I_{\mathcal{V},0}(\lambda) = K_{\mathcal{V}}(-\lambda)$ and $I_{\mathcal{V},1}(\lambda)$ was defined in Lemma 4.9. This

yields

$$\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon} = \sum_{\epsilon_1,\dots,\epsilon_\nu \in \{0,1\}^\nu} \prod_{\ell=1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda) e^{ik_\ell \bullet/\varepsilon}.$$

Fix a sequence $\epsilon_1, \dots, \epsilon_\nu \in \{0,1\}^\nu$. If all the ϵ_j vanish, then

$$\prod_{\ell=1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda) e^{ik_\ell \bullet/\varepsilon} = \prod_{\ell=1}^{\nu} K_{W_{k_\ell}}(-\lambda) e^{ik_\ell \bullet/\varepsilon}.$$

As $\mathrm{Im}(-\lambda) \in [1, M]$ we can bound the norm of this operator by directly applying (4.25). Now assume that at least one of the ϵ_ℓ is equal to 1. The indexes $\ell_1, \dots, \ell_s$ with $\epsilon_{\ell_1} = \dots = \epsilon_{\ell_s} = 1$ split the sequence $k_1, \dots, k_\nu$ in $s+1$ subsequences of consecutive d-tuples, of the form

$$(k_1, \dots, k_{\ell_1 - 1}),\ (k_{\ell_1}, \dots, k_{\ell_2 - 1}),\ \dots, (k_{\ell_s}, \dots, k_\nu). \tag{4.30}$$

At least one of these subsequences is destructive. Let us assume that it is the first one. Then $(k_1, \dots, k_{\ell_1 - 1})$ is destructive and starts with N zeros. It does not necessarily end with N zeros. Write

$$\begin{aligned} &\left| \prod_{\ell=1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda) e^{ik_\ell \bullet/\varepsilon} \right|_{\mathscr{B}} \\ &= \left| \left(\prod_{\ell=1}^{\ell_1 - 1} K_{W_{k_\ell}}(-\lambda) e^{ik_\ell \bullet/\varepsilon} \right) I_{W_{k_{\ell_1}},1}(\lambda) e^{ik_{\ell_1} \bullet/\varepsilon} \right|_{\mathscr{B}} \left| \prod_{\ell=\ell_1+1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda) e^{ik_\ell \bullet/\varepsilon} \right|_{\mathscr{B}}. \end{aligned} \tag{4.31}$$

The second factor of the RHS of (4.31) can be controlled by the estimates of Lemma 2.1:

$$\left| \prod_{\ell=\ell_1+1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda) e^{ik_\ell \bullet/\varepsilon} \right|_{\mathscr{B}} \le \prod_{\ell=\ell_1+1}^{\nu} C_M |W_{k_\ell}|_\infty$$

for a constant C_M depending on M. We deal next with the first factor in the RHS of (4.31). Let $\chi \in \mathbb{C}_0^\infty(\mathbb{B}^d(0,L))$ be equal to 1 on $\mathrm{supp}(\rho)$ and define $\tilde{K}_\rho(\lambda) = \chi R_0(\lambda)\chi$. Since $\mathrm{Im}\,\lambda \le -1$, $\tilde{K}_\rho(-\lambda)^N (D^2 - \lambda^2)^N \rho = \mathrm{Id}$. It follows that

$$\begin{aligned} &\left| \left(\prod_{\ell=1}^{\ell_1 - 1} K_{W_{k_\ell}}(-\lambda) e^{ik_\ell \bullet/\varepsilon} \right) I_{W_{k_{\ell_1}},1}(\lambda) e^{ik_{\ell_1} \bullet/\varepsilon} \right|_{\mathscr{B}} \\ &= \left| \left(\prod_{\ell=1}^{\ell_1 - 1} K_{W_{k_\ell}}(-\lambda) e^{ik_\ell \bullet/\varepsilon} \right) \tilde{K}_\rho(-\lambda)^N (D^2 - \lambda^2)^N \rho I_{W_{k_{\ell_1}},1}(\lambda) \right|_{\mathscr{B}} \\ &\le \left| \left(\prod_{\ell=1}^{\ell_1 - 1} K_{W_{k_\ell}}(-\lambda) e^{ik_\ell \bullet/\varepsilon} \right) \tilde{K}_\rho(-\lambda)^N \right|_{\mathscr{B}} \left| (D^2 - \lambda^2)^N I_{W_{k_{\ell_1}},1}(\lambda) \right|_{\mathscr{B}}. \end{aligned} \tag{4.32}$$

The same arguments used to show (4.25) yield that for $\mathrm{Im}\,\lambda \in [1, M]$,

$$\left| \left(\prod_{\ell=1}^{\ell_1 - 1} K_{W_{k_\ell}}(-\lambda) e^{ik_\ell \bullet/\varepsilon} \right) \tilde{K}_\rho(-\lambda)^N \right|_{\mathscr{B}} \le C^{\ell_1^2} \langle \lambda \rangle^{\ell_1} \left(\prod_{\ell=1}^{\ell_1 - 1} \| W_{k_\ell} \|_{2\nu} \right) \varepsilon^N.$$

By Lemma 4.9, $\left|(D^2-\lambda^2)^N I_{W_{k_{\ell_1}},1}(\lambda)\right|_{\mathcal{B}} \leq \langle\lambda\rangle^{2N+d}\, e^{2L|\operatorname{Im}\lambda|}|W_{k_{\ell_1}}|_\infty$ for $\operatorname{Im}\lambda \in [-1,-M]$. Coming back to (4.32) and putting these bounds together we obtain

$$\left|\left(\prod_{\ell=1}^{\ell_1-1} K_{W_{k_\ell}}(-\lambda)e^{ik_\ell\bullet/\varepsilon}\right) I_{W_{k_{\ell_1}},1}(\lambda)e^{ik_{\ell_1}\bullet/\varepsilon}\right|_{\mathcal{B}} \leq C^{\ell_1^2}\,\langle\lambda\rangle^{\ell_1+2N+d}\, e^{2L|\operatorname{Im}\lambda|}\left(\prod_{\ell=1}^{\ell_1}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N.$$

By (4.31) we conclude that if the first sequence among (4.30) is destructive we have

$$\left|\prod_{\ell=1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda)e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}} \leq C_M^{\nu^2}\,\langle\lambda\rangle^{\nu+2N+d}\, e^{2L|\operatorname{Im}\lambda|}\left(\prod_{\ell=1}^{\ell_1}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N$$

uniformly for λ with $\operatorname{Im}\lambda \in [-1,-M]$. In the case where the first subsequence among (4.30) is not destructive we know that at least one of the subsequence in (4.30) is destructive. This subsequence might not start nor end with N vanishing term. Here again using that the operator $I_{\mathcal{V},1}(\lambda)$ is smoothing we can overcome this difficulty. We skip the additional details. It leads to the bound

$$\left|\prod_{\ell=1}^{\nu} I_{W_{k_\ell},\epsilon_\ell}(\lambda)e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}} \leq C_M^{\nu^2}\,\langle\lambda\rangle^{\nu+4N+2d}\, e^{4L|\operatorname{Im}\lambda|}\left(\prod_{\ell=1}^{\ell_1}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N. \tag{4.33}$$

Sum the bound (4.33) over $\epsilon_1,\dots,\epsilon_\nu \in \{0,1\}^\nu$ to get that when $\operatorname{Im}\lambda \in [-1,-M]$,

$$\left|\prod_{\ell=1}^{\nu} K_{W_{k_\ell}}e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}} \leq C_M^{\nu^2}\,\langle\lambda\rangle^{\nu+4N+2d}\, e^{4L|\operatorname{Im}\lambda|}\left(\prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N. \tag{4.34}$$

3. We conclude the proof by a complex analysis argument. The estimates (4.25) and (4.34) show that (4.21) holds locally for $|\operatorname{Im}\lambda| \geq 1$. Thus it remains to show that it holds locally for $|\operatorname{Im}\lambda| \leq 1$. Fix $u, v \in L^2$ and consider

$$f(\lambda) = \frac{\lambda^\nu}{(\lambda+2i)^{2\nu+4N+2d}}\left\langle \prod_{\ell=1}^{\nu} K_{W_{k_\ell}}e^{ik_\ell\bullet/\varepsilon}u, v\right\rangle.$$

This function is holomorphic and uniformly bounded for $|\operatorname{Im}\lambda| \leq 1$: by Lemma 2.1,

$$|\operatorname{Im}\lambda| \leq 1 \;\Rightarrow\; |f(\lambda)| \leq C^\nu\left(\prod_{\ell=1}^{\nu}|W_{k_\ell}|_\infty\right)|u|_2|v|_2.$$

In addition, (4.25) and (4.34) are uniform estimates on the edge of the strip:

$$|\operatorname{Im}\lambda| = 1 \;\Rightarrow\; |f(\lambda)| \leq C^{\nu^2}\left(\prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_{2\nu}\right)\varepsilon^N|u|_2|v|_2. \tag{4.35}$$

Therefore by the three lines theorem the function f satisfies (4.35) for all λ with $|\operatorname{Im}\lambda| \leq 1$. Taking the supremum over $u, v \in L^2$ shows that (4.21) holds for $|\operatorname{Im}\lambda| \leq 1$. This ends the proof of the lemma. □

Lemma 4.8 is somehow unsatisfying. The bound (4.21) involves a constant C^{ν^2} and the norm $\|W_k\|_{2\nu}$. Both C^{ν^2} and $\|W_k\|_{2\nu}$ grow too fast as $\nu \to \infty$. The proof of the next result, which refines Lemma 4.8, specifically uses the relation between good and admissible sequences given in Lemma 4.7: every admissible sequence admits a good subsequence of consecutive d-tuples.

LEMMA 4.10. – *Let* $\{k_\ell\}_{1\leq\ell\leq\nu}$ *be a destructive sequence with exactly* γ *non-vanishing terms. Let* $s = 2(N+\gamma N+1)$. *If* $\nu \geq (2N+2d)(\gamma+1)$ *then locally uniformly on* X_d

$$\left|\mathrm{Tr}\left(\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right)\right| \leq C^{\nu+s^2}\varepsilon^N \cdot \prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_s.$$

If moreover the sequence $\{k_\ell\}_{1\leq\ell\leq\nu}$ *starts and ends with* $N+d$ *zeros then locally uniformly on* X_d

$$\left|\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{L}} \leq C^{\nu+s^2}\varepsilon^N \prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_s. \tag{4.36}$$

We recall that N is fixed. Because of Lemma 4.5, we will only care about sequences $\{k_j\}$ with at most $2N-1$ non-vanishing term. Hence, we will apply Lemma 4.10 with a parameter s of the lemma at most $2(N+(2N-1)N+1)$. It follows that, in practice, the constant $C^{s^2}\prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_s$ in the LHS of (4.36) will not be growing too fast.

Proof. – First note that since $\nu \geq (2N+2d)(\gamma+1)$ by the pigeonhole principle there exists a cyclic permutation (in the sense described above) of $\{k_\ell\}$ that starts and ends with $N+d$ zeros. Using the cyclicity of the trace we can assume that the sequence $\{k_\ell\}$ starts and ends with $N+d$ zeros. In particular we are reduced to prove (4.36). Since the sequence $\{k_\ell\}$ is now admissible it admits a good subsequence of consecutive d-tuples $\{k_\ell\}_{q+1\leq\ell\leq\nu'+q}$. Without loss of generality $q \geq d$. Write

$$\begin{aligned}\left|\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{L}} \leq &\left|\prod_{\ell=1}^{d} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{L}} \left|\prod_{\ell=d+1}^{q} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}} \\ &\cdot \left|\prod_{\ell=q+1}^{\nu'+q} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}} \left|\prod_{\ell=\nu'+q+1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{B}}.\end{aligned}$$

For λ in compact subsets of X_d the first, second and fourth factor are estimated by Lemma 2.1. The third factor is controlled by (4.21). It leads to

$$\begin{aligned}\left|\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}\right|_{\mathcal{L}} &\leq C^{\nu+\nu'^2}\varepsilon^N\left(\prod_{\ell\leq q,\ \ell\geq\nu'+q+1}|W_{k_\ell}|_\infty\right)\left(\prod_{\ell=q+1}^{\nu'+q}\|W_{k_\ell}\|_{2\nu}\right) \\ &\leq C^{\nu+s^2}\varepsilon^N\prod_{\ell=1}^{\nu}\|W_{k_\ell}\|_s.\end{aligned}$$

This completes the proof of the lemma. □

With this refinement in mind we are now ready for the proof of Lemma 4.6.

Proof of Lemma 4.6. – We divide the proof in 5 main steps.

1. Let $a \in [1, N]$ and $n_1, \dots, n_a \in [1, 2N-1]$. The function $z \mapsto (1+\Psi(z))^{-1}$ is meromorphic with a simple pole at $z=-1$. Write a Taylor expansion of $z \mapsto (1+\Psi(z))^{-1}$ at $z=0$:

$$(\mathrm{Id}+\Psi(z))^{-1} = P_N(z) + z^{2N+2d}\rho_N(z).$$

Here P_N is a polynomial of degree $2N+2d-1$ and ρ_N is a holomorphic function on $\mathbb{C}\backslash\{-1\}$. The pole at -1 is of multiplicity one. Away from resonances of W_0,

$$(\mathrm{Id}+\Psi(K_{W_0}))^{-1} = P_N(K_{W_0}) + K_{W_0}^{N+d}\rho_N(K_{W_0})K_{W_0}^{N+d}.$$

The operator $B_{W_0} = \mathrm{Det}(\mathrm{Id}+\Psi(K_{W_0}))\rho_N(K_{W_0})$, well defined on $\mathbb{C}\setminus \mathrm{Res}(K_{W_0})$, extends to an entire family of operators by Appendix A. Let us write $\mathscr{T}_X = \mathscr{P}_0 + \mathscr{P}_1$, where

$$\mathscr{P}_0 = \mathrm{Det}(\mathrm{Id}+\Psi(K_{W_0}))\cdot P_N(K_{W_0}), \quad \mathscr{P}_1 = K_{W_0}^{N+d}\cdot B_{W_0}\cdot K_{W_0}^{N+d}. \tag{4.37}$$

Fix $m_1, \dots, m_a \geq p$ and for each $1 \leq j \leq a$ a sequence $\{k_\ell^j\} \in \mathscr{S}_{m_j}^{n_j}$, with $k_1^1+\dots+k_{m_a}^a \neq 0$. We define $\gamma = n_1+\dots+n_a$ and $\nu = m_1+\dots+m_a$. Using $\mathscr{T}_X = \mathscr{P}_0+\mathscr{P}_1$ we get

$$\mathrm{Tr}\left(\prod_{j=1}^a \mathscr{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right) = \sum_{\epsilon_1,\dots\epsilon_a\in\{0,1\}^a} \mathrm{Tr}\left(\prod_{j=1}^a C_{\epsilon_j}\prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right).$$

In the following steps we study separately the terms of the RHS sum, depeding on the value of $\epsilon_1, \dots, \epsilon_a \in \{0,1\}^a$.

2. Assume that $\epsilon_1 = \dots = \epsilon_a = 0$. Then

$$\mathrm{Tr}\left(\prod_{j=1}^a C_{\epsilon_j}\prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right) = \mathrm{Tr}\left(\prod_{j=1}^a \mathscr{P}_0\prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right).$$

The sequence $\{k_\ell^j\}$ is destructive, $\nu \geq pa \geq 2(N+d)\cdot 2Na$ and $2Na \geq \gamma+1$. This implies $\nu \geq 2(N+d)(\gamma+1)$. Hence for $s = 2(N+2N^3+1)$ we have $s \geq 2(N+\gamma N+1)$. The assumptions of Lemma 4.10 are satisfied thus

$$\left|\mathrm{Tr}\left(\prod_{j=1}^a \mathscr{P}_0\prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right)\right| \leq C^\nu \varepsilon^N \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_s$$

for a constant C depending only on N, d and $|W_0|_\infty$.

3. Assume that exactly one of the $\epsilon_1, \dots, \epsilon_a \in \{0,1\}^a$ is equal to 1. Using the cyclicity of the trace we can assume without loss of generality that $\epsilon_1 = 1$. Hence

$$\begin{aligned}\mathrm{Tr}&\left(\prod_{j=1}^a C_{\epsilon_j}\prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right)\\ &= \mathrm{Tr}\left(B_{W_0}K_{W_0}^{N+d}\left(\prod_{\ell=1}^{m_1} K_{W_{k_\ell^1}} e^{ik_\ell^1\bullet/\varepsilon}\right)\left(\prod_{j=2}^a \mathscr{P}_0\prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j\bullet/\varepsilon}\right)K_{W_0}^{N+d}\right).\end{aligned}$$

Using (4.36) we obtain again

$$\left|\operatorname{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\right| \\ \leq |B_{W_0}|_{\mathcal{B}} \cdot \left| K_{W_0}^{N+d} \left(\prod_{\ell=1}^{m_1} K_{W_{k_\ell^1}} e^{ik_\ell^1 \bullet/\varepsilon}\right)\left(\prod_{j=2}^{a} \mathscr{P}_0 \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right) K_{W_0}^{N+d}\right|_{\mathcal{L}}.$$

The second factor in the second line is a finite sum of terms studied in Lemma 4.10. Consequently we obtain the bound

$$\left|\operatorname{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\right| \leq C^\nu \varepsilon^N |B_{W_0}|_{\mathcal{B}} \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_s.$$

4. Assume that 2 or more terms among $\epsilon_1, \ldots, \epsilon_a \in \{0,1\}^a$ are equal to 1. Using a circular permutation we can assume without loss of generality that $\epsilon_1 = 1$. Let us prove the following statement: there exists two indexes $j_1, j_2 \in [1, a]$ such that

(i) the sequence $\{k_\ell^j\}_{1\leq \ell \leq m_j}^{j_1 \leq j < j_2}$ is destructive;
(ii) $\epsilon_j = 0$ for all j in the interval (j_1, j_2).

We process by recursion on a. If $a = 2$ this is obvious: either the sequence $\{k_\ell^1\}_{1\leq\ell\leq m_1}$ or the sequence $\{k_\ell^2\}_{1\leq\ell\leq m_2}$ is destructive. Now assume that the statement holds true for all $a' \leq a-1$. Let us prove it for a. Let j_0 be the smallest index with $\epsilon_{j_0} = 1$ and $j_0 > 1$. Then either the sequence $\{k_\ell^j\}_{1\leq\ell\leq m_j}^{1\leq j<j_0}$ is destructive and we are done, or it is constructive. But then the sequence $\{k_\ell^j\}_{1\leq\ell\leq m_j}^{j_0\leq j\leq a}$ is destructive and so we can apply the recursion hypothesis to it. This proves the above claim.

Again using a circular permutation we can assume that $j_1 = 1$. Hence

$$\left|\operatorname{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\right| \\ \leq |B_{W_0}|_{\mathcal{B}} \left| K_{W_0}^{2(N+d)} \left(\prod_{\ell=1}^{m_1} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\left(\prod_{j=2}^{j_2-1} \mathscr{P}_0 \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right) K_{W_0}^{2(N+d)}\right|_{\mathcal{L}} |B_{W_0}|_{\mathcal{B}} \\ \cdot \left|\left(\prod_{\ell=1}^{m_{j_2}} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\left(\prod_{j=j_2+2}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\right|_{\mathcal{B}}.$$

The first line is a finite sum of terms estimated by Lemma 4.10. The second line is controlled by the standards bounds of Lemma 2.1. It leads to

$$\left|\operatorname{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)\right| \leq C^\nu \varepsilon^N \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_s. \tag{4.38}$$

5. Points 2, 3, 4 show that (4.38) holds for all sequences $\epsilon_1, \dots, \epsilon_a \in \{0,1\}^a$. Summing this estimate over all possible $\epsilon_1, \dots, \epsilon_a$ to get

$$\left| \operatorname{Tr} \left(\prod_{j=1}^{a} \mathcal{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right) \right| \leq C^\nu \varepsilon^N \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_s. \tag{4.39}$$

The last step of the proof is to sum the bound (4.39). Recall that

$$\mathcal{D}_{n_1,\dots,n_a} = \sum_{m_1,\dots,m_a=p}^{\infty} \sum_{\substack{\{k_\ell^1\}\in\mathcal{S}_{m_1}^{n_1},\ \dots,\ \{k_\ell^a\}\in\mathcal{S}_{m_a}^{n_a},\\ k_1^1+\dots+k_{m_a}^a\neq 0}} \left(\prod_{j=1}^{a} \alpha_{m_j} \mathcal{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right),$$

where $\mathcal{S}_m^n$ is the set of sequences of length m with n non-vanishing terms. Hence

$$\begin{aligned}
&\left|\operatorname{Tr}\left(\mathcal{D}_{n_1,\dots,n_a}\right)\right| \\
&\leq \sum_{m_1,\dots,m_a=p}^{\infty} \sum_{\substack{\{k_\ell^1\}\in\mathcal{S}_{m_1}^{n_1},\ \dots,\ \{k_\ell^a\}\in\mathcal{S}_{m_a}^{n_a},\\ k_1^1+\dots+k_{m_a}^a\neq 0}} |\alpha_{m_1}\cdots\alpha_{m_a}| \left| \operatorname{Tr} \left(\prod_{j=1}^{a} \mathcal{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right) \right| \\
&\leq \varepsilon^N \sum_{m_1,\dots,m_a=p}^{\infty} |\alpha_{m_1}\cdots\alpha_{m_a}| C^{m_1+\dots+m_a} \sum_{\substack{\{k_\ell^1\}\in\mathcal{S}_{m_1}^{n_1},\ \dots,\ \{k_\ell^a\}\in\mathcal{S}_{m_a}^{n_a},\\ k_1^1+\dots+k_{m_a}^a\neq 0}} \prod_{\ell=1}^{m_1+\dots+m_a} \|W_{k_\ell}\|_s \\
&\leq \varepsilon^N \sum_{m_1,\dots,m_a=p}^{\infty} |\alpha_{m_1}\cdots\alpha_{m_a}| C^{m_1+\dots+m_a} |W|_{Z^s}^{m_1+\dots+m_a} = \varepsilon^N \left(\Phi(C|W|_{Z^s})\right)^a,
\end{aligned}$$

where we recall that $|W|_{Z^s} = \sum_{k\in\mathbb{Z}^d} \|W_k\|_s$ and Φ is defined with $\Phi(z) = \sum_{m=p}^{\infty} |\alpha_m| z^m$. Since Φ is entire, $\Phi(C|W|_{Z^s}) < \infty$. Hence $\operatorname{Tr}(\mathcal{D}_{n_1,\dots,n_a}) = O(\varepsilon^N)$ which completes the proof. □

4.4. Constructive interaction

In this paragraph we prove the following lemma:

Lemma 4.11. – *For $1 \leq a \leq N$ and $n_1, \dots, n_a \in [1, 2N-1]$ let $\mathcal{C}_{n_1,\dots,n_a}$ be the trace class operator given by* (4.20)*. There exist $\varphi_0, \dots, \varphi_{N-1}$ holomorphic functions on X_d such that*

$$\operatorname{Tr}\left(\mathcal{C}_{n_1,\dots,n_a}\right) = \varphi_0(\lambda) + \varepsilon\varphi_1(\lambda) + \dots + \varepsilon^{N-1}\varphi_{N-1}(\lambda) + O(\varepsilon^N)$$

locally uniformly on X_d.

As we will see, the terms $\varepsilon^j \varphi_j(\lambda)$ arise from constructive interference between the terms $e^{ik_j} \bullet /\varepsilon$. The first step in the proof of Lemma 4.11 is an operator valued expansion for $e^{-ik\bullet/\varepsilon} K_{W_k} e^{ik\bullet/\varepsilon}$:

Lemma 4.12. – *For every $n \geq 0$ there exists some operators $A_0, \dots, A_{n-1}, \mathfrak{R}_n$ with*

$$e^{-ik\bullet/\varepsilon} K_{W_k} e^{ik\bullet/\varepsilon} = A_0 + \dots + \varepsilon^{n-1} A_{n-1} + \varepsilon^n \mathfrak{R}_n, \tag{4.40}$$

depending on k, and such that :

(i) A_j *is a pseudodifferential operator of order* $j-2$ *that maps locally supported functions to compactly supported functions. It does not depend on* ε *and there exists* C *such that*

$$s+j \leq N,\ k \in \mathbb{Z}^d \ \Rightarrow\ |A_j|_{\mathcal{B}(H^{s+j}, H^s)} \leq C \|W_k\|_N.$$

(ii) $\mathfrak{R}_n$ *is a pseudodifferential operator of order* $n-1$ *and maps locally supported functions to compactly supported functions. It depends on* ε *and uniformly in* ε *near* 0 *and there exists* C *such that*

$$s+n+1 \leq N,\ k \in \mathbb{Z}^d \ \Rightarrow\ |\mathfrak{R}_n|_{\mathcal{B}(H^{s+n+1}, H^s)} \leq C \|W_k\|_N.$$

Proof. – For $k=0$ there is nothing to prove. Thus we assume $k \neq 0$. In Appendix B we prove that if $R(\xi,\lambda) = (\xi^2-\lambda^2)^{-1}$ then

(4.41)
$$R(\xi+k/\varepsilon,\lambda) = \left(\sum_{j=2}^{n-1} \varepsilon^j p_{j-2}(\xi,\lambda)\right) + \varepsilon^n p_{n-2}(\xi,\lambda) + \varepsilon^{n+1} p_{n-1}(\xi,\lambda) + \varepsilon^n \frac{r_n(\xi,\lambda,\varepsilon)}{(\xi^2-k/\varepsilon)^2-\lambda^2}.$$

Here the $p_j(\xi,\lambda)$ are polynomials in ξ and λ of degree at most j in ξ, depending uniformly on $k/|k|^2$; and $r_n(\xi,\lambda,\varepsilon)$ is a polynomial in ξ and λ of degree at most $n+1$ in ξ and whose coefficients depend smoothly of ε. Since the dependence in k is uniform in $k/|k|^2$, it is uniform for $k \in \mathbb{Z}^d \setminus 0$. It follows that

$$\sup_{k\in\mathbb{Z}^d\setminus 0} \sup_{\xi\in\mathbb{R}^d} \left| \frac{r_n(\xi,\lambda,\varepsilon)}{\langle\xi\rangle^{n+1}} \right| = O(1) \text{ uniformly as } \varepsilon \to 0,\ k \in \mathbb{Z}^d. \tag{4.42}$$

Since $e^{-ik\bullet/\varepsilon} D e^{ik\bullet/\varepsilon} = D + k/\varepsilon$ we have for $\operatorname{Im}\lambda > 0$,

$$e^{-ik\bullet/\varepsilon} R_0(\lambda) e^{ik\bullet/\varepsilon} = \left((D+k/\varepsilon)^2-\lambda^2\right)^{-1}.$$

Therefore the expansion (4.41) implies that for $\operatorname{Im}\lambda > 0$,

$$e^{-ik\bullet/\varepsilon} R_0(\lambda) e^{ik\bullet/\varepsilon} = \left(\sum_{j=2}^{n-1} \frac{\varepsilon^j}{|k|^2} p_{j-2}(D)\right) + \varepsilon^n \left(p_{n-2}(D) + \varepsilon p_{n-1}(D) + R_0(\lambda) r_n(D,\varepsilon)\right).$$

This identity extends analytically to X_d and yields

$$\begin{aligned} e^{-ik\bullet/\varepsilon} K_{W_k} e^{ik\bullet/\varepsilon} &= A_0 + \cdots + \varepsilon^{n-1} A_{n-1} + \varepsilon^n \mathfrak{R}_n, \\ A_0 = A_1 = 0, \quad A_j &= \frac{1}{|k|^2} \rho p_{j-2}(D) W_k \text{ for } j \in [2, n-1], \\ \mathfrak{R}_n &= \rho \left(p_{n-2}(D) + \varepsilon p_{n-1}(D) + R_0(\lambda) r_n(D,\varepsilon)\right) W_k. \end{aligned}$$

The operators A_j are pseudodifferential of order $j-2$ and map locally supported functions to compactly supported functions. For $\operatorname{Im}\lambda > 0$ the operator $K_{W_k}(\lambda)$ is pseudodifferential; the operator $K_{W_k}(\lambda) - K_{W_k}(-\lambda)$ is smoothing. Hence $K_{W_k}(\lambda)$ is pseudodifferential for all $\lambda \in X_d$. As

$$\mathfrak{R}_n = \frac{e^{-ik\bullet/\varepsilon} K_{W_k} e^{ik\bullet/\varepsilon} - A_0 - \cdots - \varepsilon^{n-1} A_{n-1}}{\varepsilon^n}$$

and the RHS is pseudodifferential $\mathfrak{R}_n$ must also be pseudodifferential. To evaluate its order we note that $p_{n-2}(D)$ (resp. $p_{n-1}(D)$) is a differential operator of order $n-4$ (resp. $n-3$) and that $r_n(D)$ is a differential operator of order $n+1$. Thus $R_0(\lambda) r_n(D)$ maps H^{n+1} to H^2 and $\mathfrak{R}_n$ must be of order $n-1$. To prove the required bounds, we note that for $s \leq N$, the multiplication operator $u \mapsto W_k u$ from H^s to itself has norm bounded by $\|W_k\|_N$. Therefore for $s + j \leq N$,

$$|A_j|_{\mathscr{B}(H^{s+j},H^s)} \leq C|p_{j-2}(D)|_{\mathscr{B}(H^{s+j},H^s)}|W_k|_{\mathscr{B}(H^{s+j},H^{s+j})} \leq C\|W_k\|_N.$$

Again, the constant is uniform in $k \in \mathbb{Z}^d \setminus 0$ because $p_j(\xi, \lambda)$ depends uniformly on $k/|k|^2$. This proves (i). Now we prove (ii). For $s + n + 1 \leq N$ the bound (4.42) implies that the operator $r_n(D, \varepsilon)$ (which is a differential operator) satisfies the bound

$$|r_n(D, \varepsilon)|_{\mathscr{B}(H^{s+n+1},H^s)} = O(1) \text{ uniformly as } \varepsilon \to 0.$$

Let $\chi \in C_0^\infty(\mathbb{B}^d(0, L))$ with $\chi = 1$ on $\operatorname{supp}(\chi)$. The operator $\rho R_0(\lambda)\chi$ maps H^s to itself. Consequently, uniformly as $\varepsilon \to 0$

$$\begin{aligned}|\rho R_0(\lambda) r_n(D, \varepsilon) W_k|_{\mathscr{B}(H^{s+n+1},H^s)}& \\ \leq |\rho R_0(\lambda)\chi|_{\mathscr{B}(H^s,H^s)}&|r_n(D, \varepsilon)W_k|_{\mathscr{B}(H^{s+n+1},H^s)} = O(\|W_k\|_N).\end{aligned}$$

The operators $\rho p_{n-2}(D)W_k$ and $\rho p_{n-1}(D)W_k$ do not depend on ε and are bounded from H^{s+n+1} to H^s. This shows (ii) and completes the proof of the lemma. □

Now we prove the same kind of expansion for product of operators of the form (4.40).

LEMMA 4.13. – *Let $\{k_\ell\}_{1\leq\ell\leq\nu}$ be a sequence of d-tuples in $\mathbb{Z}^d$. There exist some operators $\mathscr{A}_0, \dots, \mathscr{A}_{N-1}, \mathscr{R}_N$ with*

$$e^{-i\sigma_1 \bullet/\varepsilon}\left(\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon}\right) = \mathscr{A}_0 + \cdots + \varepsilon^{N-1}\mathscr{A}_{N-1} + \varepsilon^N \mathscr{R}_N, \tag{4.43}$$

where $\sigma_1 = k_1 + \cdots + k_\nu$ and

(i) *$\mathscr{A}_j$ is a pseudodifferential operator of order $j - 2$ and maps locally supported functions to compactly supported functions. It does not depend on ε and*

$$s + j \leq N \ \Rightarrow \ |\mathscr{A}_j|_{\mathscr{B}(H^{s+j},H^s)} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N,$$

(ii) *$\mathscr{R}_N$ is a pseudodifferential operator of order $N - 1$ mapping locally supported functions to compactly supported functions. It depends on ε and uniformly in ε near 0,*

$$s \leq -1 \ \Rightarrow \ |\mathscr{R}_N|_{\mathscr{B}(H^{s+N+1},H^s)} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

This lemma is important when $\sigma_1 = 0$—in this case, the sequence $\{k_\ell\}$ is constructive. It produces an operator-valued expansion of $\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell\bullet/\varepsilon}$. In Lemma 4.14 below, we will explain how to pass to the trace in (4.43). This will generate terms of order ε^j, $j > N$, that will appear later in the expansion of $D_V(\lambda)$ in powers of ε. We refer to §4.5 for the instructive computation of the first few terms.

Proof. – We prove this lemma by recursion. For $\nu = 1$ it is the result of Lemma 4.12. Now assume that Lemma 4.13 holds true for all sequences $\{k_j\}$ of length less or equal to $\nu - 1$. Let $\{k_j\}$ be a sequence of length ν. Define $\sigma_2 = k_2 + \cdots + k_\nu$ so that

$$e^{-i\sigma_1 \bullet/\varepsilon} \prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon} = \left(e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right) \cdot \left(e^{-i\sigma_2 \bullet/\varepsilon} \prod_{\ell=2}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon}\right).$$

Using the recursion hypothesis we have

$$\begin{aligned}\left(e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right) &\cdot \left(e^{-i\sigma_2 \bullet/\varepsilon} \prod_{\ell=2}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon}\right) \\ &= \left(e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right) \mathscr{A}_0 + \cdots + \varepsilon^{N-1} \left(e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right) \mathscr{A}_{N-1} \\ &\quad + \varepsilon^N \left(e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right) \mathscr{R}_N.\end{aligned}$$

We expand below $e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}$ at order $N - j$ as given by Lemma 4.12:

$$e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon} = A_0 + \varepsilon A_1 + \cdots + \varepsilon^{N-j-1} A_{N-j-1} + \varepsilon^{N-j} \mathfrak{R}_{N-j}. \tag{4.44}$$

It leads to

$$\varepsilon^j \left(e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right) \mathscr{A}_j = \varepsilon^j A_0 \mathscr{A}_j + \cdots + \varepsilon^{N-1} A_{N-1-j} \mathscr{A}_j + \varepsilon^N \mathfrak{R}_{N-j} \mathscr{A}_j.$$

The operator $A_{j'} \mathscr{A}_j$ has order $j' - 2 + j - 2 = j' + j - 4 \leq j' + j - 2$ and in the above expression it is weighted with a term $\varepsilon^{j'+j}$. Moreover if $s + j' + j \leq N$ then

$$|A_{j'} \mathscr{A}_j|_{\mathscr{B}(H^{s+j'+j}, H^s)} \leq |A_{j'}|_{\mathscr{B}(H^{s+j'}, H^s)} |\mathscr{A}_j|_{\mathscr{B}(H^{s+j'+j}, H^{s+j'})} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

The remainder $\mathfrak{R}_{N-j} \mathscr{A}_j$ has order $N - j - 1 + j - 2 = N - 3 \leq N - 1$ and satisfies

$$\begin{aligned}|\mathfrak{R}_{N-j} \mathscr{A}_j|_{\mathscr{B}(H^{N+1+s}, H^s)} &\leq |\mathfrak{R}_{N-j}|_{\mathscr{B}(H^{s+N-j+1}, H^s)} |\mathscr{A}_j|_{\mathscr{B}(H^{N+1+s}, H^{N+1+s-j})} \\ &\leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.\end{aligned}$$

The term $e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon} \mathscr{R}_N$ is of order $N - 3 \leq N - 1$ and satisfies

$$\begin{aligned}\left|e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon} \mathscr{R}_N\right|_{\mathscr{B}(H^{s+N+1}, H^s)} &\leq \left|e^{-i\sigma_1 \bullet/\varepsilon} K_{W_{k_1}} e^{i\sigma_1 \bullet/\varepsilon}\right|_{\mathscr{B}(H^s, H^s)} |\mathscr{R}_N|_{\mathscr{B}(H^{s+N+1}, H^s)} \\ &\leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.\end{aligned}$$

This proves that the lemma holds for all sequences of length ν. This completes the recursion and ends the proof. □

The expansion of Lemma 4.13 implies a trace expansion as follows:

LEMMA 4.14. – *Let* $\{k_\ell\}_{1\leq\ell\leq\nu}$ *be a constructive sequence with* γ *non-vanishing terms. Assume that* $\nu \geq N(\gamma+1)$. *Then there exists* $a_0, a_1, \dots, a_{N-1}$ *holomorphic functions on* X_d *such that locally uniformly on* X_d, $|a_j(\lambda)| \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N$ *and*

$$\left|\operatorname{Tr}\left(\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon}\right) - a_0(\lambda) - \varepsilon a_1(\lambda) + \cdots - \varepsilon^{N-1} a_{N-1}(\lambda)\right| \leq \varepsilon^N C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

Proof. – Since $\nu \geq N(\gamma+1)$, there exists a subsequence of $\{k_\ell\}_{1\leq\ell\leq\nu}$ made of N consecutive vanishing d-tuples. Using the cyclicity of the trace we can assume that $k_{\nu-N+1} = \cdots = k_\nu = 0$. The sequence $k_1, \dots, k_{\nu-N}$ is constructive. Therefore we can apply Lemma 4.13 to obtain the expansion

$$\prod_{\ell=1}^{\nu-N} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon} = \mathscr{A}_0 + \cdots + \varepsilon^{N-1} \mathscr{A}_{N-1} + \varepsilon^N \mathscr{R}_N.$$

Here $\mathscr{A}_j$ is pseudodifferential of order $j-2$ and does not depend on ε and $\mathscr{R}_N$ is pseudodifferential of order $N-1$ and satisfies the bound

$$|\mathscr{R}_N|_{\mathscr{B}(H^{N+1},L^2)} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

All these operators map locally supported functions to compactly supported functions. As $k_{\nu-N+1} = \cdots = k_\nu = 0$ we obtain

$$\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon} = \mathscr{A}_0 K_{W_0}^N + \cdots + \varepsilon^{N-1} \mathscr{A}_{N-1} K_{W_0}^N + \varepsilon^N \mathscr{R}_N K_{W_0}^N. \tag{4.45}$$

We recall that $N \geq d$. The operators $\mathscr{A}_j K_{W_0}^N$ have order $j-2-2N \leq -2-N \leq -2-d$ therefore they are trace class. The operator $\mathscr{R}_N K_{W_0}^N$ has order $-N-1 \leq -d$ hence it is also trace class. It satisfies the bound

$$|\mathscr{R}_N K_{W_0}^N|_{\mathscr{B}(H^{1-N},L^2)} \leq |\mathscr{R}_N|_{\mathscr{B}(H^{N+1},L^2)} |K_{W_0}^N|_{\mathscr{B}(H^{1-N},H^{N+1})} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

By [15, Equation (B.3.9)] this implies

$$|\mathscr{R}_N K_{W_0}^N|_{\mathscr{L}} \leq |\mathscr{R}_N K_{W_0}^N|_{\mathscr{B}(H^{1-N},L^2)} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

Taking the trace of both sides of (4.45) yields

$$\left|\operatorname{Tr}\left(\prod_{\ell=1}^{\nu} K_{W_{k_\ell}} e^{ik_\ell \bullet/\varepsilon}\right) - \operatorname{Tr}\left(\mathscr{A}_0 K_{W_0}^N\right) - \cdots - \varepsilon^{N-1} \operatorname{Tr}\left(\mathscr{A}_{N-1} K_{W_0}^N\right)\right| \leq \varepsilon^N C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N.$$

This gives the required expansion. We now need to prove the estimate on the coefficients $a_0, \dots, a_{N-1}$ appearing in the expansion. By [15, Equation (B.3.9)] and the estimate (i) of Lemma 4.13,

$$\begin{aligned} |\operatorname{Tr}(\mathscr{A}_j K_{W_0}^N)| &\leq |\mathscr{A}_j K_{W_0}^N|_{\mathscr{L}} \leq C |\mathscr{A}_j K_{W_0}^N|_{\mathscr{B}(H^{-N},L^2)} \\ &\leq C |\mathscr{A}_j|_{\mathscr{B}(H^N,L^2)} |K_{W_0}^N|_{\mathscr{B}(H^{-N},H^N)} \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_N. \end{aligned}$$

This completes the proof. □

Fix $a \in [1, N]$ and $n_1, \dots, n_a \in [1, 2N-1]$. The operator $\mathscr{C}_{n_1,\dots,n_a}$ defined by (4.20) is a linear combination of operators of the form

$$L[k_\ell^j] = \prod_{j=1}^{a} \mathscr{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}, \tag{4.46}$$

where

(i) For every $j \in [1, a]$, $m_j \geq p$.
(ii) The sequence $\{k_\ell^j\}_{1\leq \ell \leq m_j}^{1\leq j\leq a}$ is constructive.
(iii) For every $j \in [1, a]$, the sequence $\{k_\ell^j\}_{1\leq \ell\leq m_j}$ has n_j non-vanishing terms.

In order to prove Lemma 4.11 we prove an expansion for operators of the form (4.46) where $\{k_\ell^j\}$ satisfies (i), (ii), and (iii). We fix $s = 2(N + 2N^2 + 1)$.

LEMMA 4.15. – *Let $L[k_\ell^j]$ be an operator of the form* (4.46) *where $\{k_\ell^j\}$ satisfies (i), (ii), and (iii) above. Then there exist $b_0[k_\ell^j], \dots, b_{N-1}[k_\ell^j]$ holomorphic functions on X_d such that locally uniformly on X_d we have $|b_i[k_\ell^j]| \leq C^\nu \prod_{\ell=1}^{\nu} \|W_{k_\ell}\|_s$ and*

$$\left|\operatorname{Tr}\left(L[k_\ell^j]\right) - b_0[k_\ell^j] + \dots - b_{N-1}[k_\ell^j]\varepsilon^{N-1}\right| \leq \varepsilon^N C^\nu \prod_{j=1}^{a}\prod_{\ell=1}^{m_j} \|W_{k_\ell^j}\|_s. \tag{4.47}$$

Proof of Lemma 4.11. – Fix $a \in [1, N]$, $n_1, \dots, n_a \in [1, 2N-1]$ and k_ℓ^j satisfying (i), (ii), and (iii) above. Let $\gamma = n_1 + \dots + n_a$ be the number of non-vanishing terms of $\{k_\ell^j\}$. We divide the proof below in 5 main steps.

1. Write $\mathscr{T}_X = \mathscr{P}_0 + \mathscr{P}_1$ where $\mathscr{P}_0$, $\mathscr{P}_1$ were given in (4.37). Then

$$\operatorname{Tr}\left(L[k_\ell^j]\right) = \operatorname{Tr}\left(\prod_{j=1}^{a} \mathscr{T}_X \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right) = \sum_{\epsilon_1,\dots\epsilon_a\in\{0,1\}^a} \operatorname{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right).$$

We recall that since $\{k_\ell^j\}$ has $\gamma \leq (2N-1)a$ non-vanishing terms and length $\nu \geq pa$ we have $\nu \geq N(\gamma+1)$. Fix a sequence $\epsilon_j \in \{0,1\}^a$. In order to prove the lemma it suffices to prove that the term

$$\operatorname{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right) \tag{4.48}$$

admits an expansion in powers of ε at order N.

2. Assume that $\epsilon_1 = \dots = \epsilon_a = 0$. Recall that $\mathscr{P}_0$ is the product of the scalar $\det(\mathrm{Id} + \Psi(K_{W_0})$ with the operator $P_N(K_{W_0})$—which is polynomial in K_{W_0}. Hence, in the case $\epsilon_1 = \dots = \epsilon_a = 0$, (4.48) is a finite sum of terms studied in Lemma 4.14. These all admit an expansion in powers of ε and thus so does (4.48).

3. Assume that ϵ_j has at least one non-zero term. Without loss of generality $\epsilon_1 = 1$. The indexes $j_1, \dots, j_r$ such that $\epsilon_j = 1$ split the sequence $(k_1^1, \dots, k_{m_1}^1, k_1^2, \dots, k_{m_a}^a)$ into $r+1$ subsequences of consecutive d-tuples

$$(k_1^1, \dots, k_{m_{j_1}}^{j^1}), \dots, \ (k_1^{j_r}, \dots, k_{m_a}^a). \tag{4.49}$$

Assume that each of the subsequences in (4.49) is constructive.

Then since $\mathscr{P}_1 = K_{W_0}^{N+d} B_{W_0} K_{W_0}^{N+d}$ we can write (4.48) as the trace of a product of operators of the form

$$B_{W_0} K_{W_0}^{N+d} \prod_{j=j_t}^{j_{t+1}-1} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} K_{W_0}^{N+d}. \tag{4.50}$$

By Lemma 4.13, the operator

$$K_{W_0}^{N+d} \prod_{j=j_t}^{j_{t+1}-1} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} K_{W_0}^{N+d}$$

admits an operator-valued expansion in powers of ε. Thus so does the operator (4.50). Multiplying these expansions over $t = 1, \dots, r$ leads to an operator-valued expansion for the operator

$$\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon},$$

in the spirit of Lemma 4.13. Taking the trace and adapting the proof of Lemma 4.14 shows that (4.48) admits an expansion in powers of ε.

4. Assume that at least one of the sequences in (4.49) is destructive. Without loss of generality $(k_1^1, \dots, k_{j_1}^{m_{j_1}})$ is destructive. Since $\mathscr{P}_1 = K_{W_0}^{N+d} B_{W_0} K_{W_0}^{N+d}$ the operator

$$K_{W_0}^{N+d} \left(\prod_{j=1}^{j_1} \mathscr{P}_0 \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right) K_{W_0}^{N+d} \tag{4.51}$$

appears as one of the factors in the product

$$\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}.$$

In addition since $\mathscr{P}_0$ is the product of the scalar $\det(\mathrm{Id} + \Psi(K_{W_0}))$ with $P_N(K_{W_0})$—which is polynomial on K_{W_0}—it is associated with a destructive sequence, that starts and ends with $N+d$ zeros. Consequently Lemma 4.10 applies and yields

$$\left| K_{W_0}^{N+d} \left(\prod_{j=1}^{j_1} \mathscr{P}_0 \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon} \right) K_{W_0}^{N+d} \right|_{\mathscr{L}} \le C^{\nu+s^2} \varepsilon^N \prod_{j=1}^{j_1} \prod_{\ell=1}^{m_j} \|W_{k_\ell^j}\|_s.$$

This yields the estimate:

$$\mathrm{Tr}\left(\prod_{j=1}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right)$$

$$\leq \left|K_{W_0}^{N+d} B_{W_0}\right|_{\mathscr{B}} \left|K_{W_0}^{N+d}\left(\prod_{j=1}^{j_1} \mathscr{P}_0 \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right) K_{W_0}^{N+d}\right|_{\mathscr{L}}$$

$$\cdot \left|B_{W_0} K_{W_0}^{N+d}\right|_{\mathscr{B}} \left|\prod_{\ell=1}^{m_{j_1}+1} K_{W_{k_\ell^{j_1+1}}} e^{ik_\ell^{j_1+1} \bullet/\varepsilon}\right|_{\mathscr{B}} \left|\prod_{j=j_1+2}^{a} C_{\epsilon_j} \prod_{\ell=1}^{m_j} K_{W_{k_\ell^j}} e^{ik_\ell^j \bullet/\varepsilon}\right|_{\mathscr{B}}$$

$$\leq C^{\nu+s^2} \varepsilon^N \prod_{j=1}^{a} \prod_{\ell=1}^{m_j} \|W_{k_\ell^j}\|_s.$$

This shows that such sequences $\epsilon_1, \ldots, \epsilon_a$ induce negligible contributions.

5. Points 2, 3, 4 include all the possible values of $\epsilon_1, \ldots, \epsilon_a$. The expansion (4.47) follows now from a summation over $\epsilon_1, \ldots, \epsilon_a \in \{0,1\}^a$ of the expansions obtained in Points 2,3. This ends the proof. □

We are now ready to prove Lemma 4.11.

Proof of Lemma 4.11. – Let us recall that for $a \in [1, N]$ and $n_1, \ldots n_a \in [1, 2N-1]$ the operator $\mathcal{C}_{n_1,\ldots,n_a}$ is defined by

$$\mathcal{C}_{n_1,\ldots,n_a} = \sum_{m_1,\ldots,m_a=p}^{\infty} \alpha_{m_1} \cdots \alpha_{m_a} \sum_{\substack{\{k_\ell^1\} \in \mathcal{S}_{m_1}^{n_1}, \ldots, \{k_\ell^a\} \in \mathcal{S}_{m_a}^{n_a}, \\ k_1^1 + \cdots + k_{m_a}^a = 0}} L[k_\ell^j].$$

Here $L[k_\ell^j]$ is given by (4.46), $\mathcal{S}_m^n$ is the set of sequences of length m with n non-vanishing terms and $\alpha_m = \Psi^{(m)}(0)/m!$. The proof consist in showing that the sum of the expansions of $\mathrm{Tr}(L[k_\ell^j])$ provided by Lemma 4.15 is convergent. By Lemma 4.15,

$$\sum_{m_1,\ldots,m_a=p}^{\infty} |\alpha_{m_1} \cdots \alpha_{m_a}| \sum_{\substack{\{k_\ell^1\} \in \mathcal{S}_{m_1}^{n_1}, \ldots, \{k_\ell^a\} \in \mathcal{S}_{m_a}^{n_a}, \\ k_1^1 + \cdots + k_{m_a}^a = 0}} \left|\mathrm{Tr}\left(L[k_\ell^j]\right) - b_0[k_\ell^j] + \cdots - b_{N-1}[k_\ell^j]\varepsilon^{N-1}\right|$$

$$\leq \varepsilon^N \sum_{m_1,\ldots,m_a=p}^{\infty} |\alpha_{m_1} \cdots \alpha_{m_a}| \sum_{\substack{\{k_\ell^1\} \in \mathcal{S}_{m_1}^{n_1}, \ldots, \{k_\ell^a\} \in \mathcal{S}_{m_a}^{n_a}, \\ k_1^1 + \cdots + k_{m_a}^a = 0}} C^{m_1+\cdots+m_a} \prod_{j=1}^{a} \prod_{\ell=1}^{m_j} \|W_{k_\ell}\|_s$$

$$\leq \varepsilon^N \sum_{m_1,\ldots,m_a=p}^{\infty} |\alpha_{m_1} \cdots \alpha_{m_a}| C^{m_1+\cdots+m_a} |W|_{X^s}^{m_1+\cdots+m_a} = \varepsilon^N (\Phi(C|W|_{Z^s}))^a,$$

where we recall that $|W|_{Z^s} = \sum_{k\in\mathbb{Z}^d} \|W_k\|_s$ and $\Phi(z) = \sum_{m=p}^{\infty} |\alpha_m| z^m$. It follows that $\mathrm{Tr}(\mathcal{C}_{n_1,\ldots,n_a})$ has an expansion given by

$$\begin{aligned}\mathrm{Tr}(\mathcal{C}_{n_1,\ldots,n_a}) &= O(\varepsilon^N) + \sum_{m_1,\ldots,m_a=p}^{\infty} \alpha_{m_1}\cdots\alpha_{m_a} \sum_{\substack{\{k_\ell^1\}\in\mathcal{S}_{m_1}^{n_1},\ \ldots,\ \{k_\ell^a\}\in\mathcal{S}_{m_a}^{n_a},\\ k_1^1+\cdots+k_{m_a}^a=0}} b_0[k_\ell^j] + \cdots + b_{N-1}[k_\ell^j]\varepsilon^{N-1}\\ &= \varphi_0 + \cdots + \varepsilon^{N-1}\varphi_{N-1} + O(\varepsilon^N),\end{aligned}$$

where

$$\varphi_i = \sum_{m_1,\ldots,m_a=p}^{\infty} \alpha_{m_1}\cdots\alpha_{m_a} \sum_{\substack{\{k_\ell^1\}\in\mathcal{S}_{m_1}^{n_1},\ \ldots,\ \{k_\ell^a\}\in\mathcal{S}_{m_a}^{n_a},\\ k_1^1+\cdots+k_{m_a}^a=0}} b_i[k_\ell^j].$$

This ends the proof. $\square$

Since

$$\prod_{j=1}^{a} \mathcal{T}_X F_{n_j} = \mathcal{C}_{n_1,\ldots,n_a} + \mathcal{D}_{n_1,\ldots,n_a},$$

the combination of Lemma 4.5, Lemma 4.6 and Lemma 4.11 proves Lemma 4.3. This in turn shows that $D_V(\lambda)$ admits an expansion in powers of ε. In the next section we conclude the proof of Theorem 5 by computing explicitly the first few coefficients in the expansion.

4.5. Computation of coefficients in the expansion

Here we compute the expansion of D_V up to order $O(\varepsilon^4)$. The coefficients that appear are holomorphic functions of λ. Hence it suffices to compute them for $\mathrm{Im}\,\lambda \gg 1$ and to extend the obtained expression to $\mathbb{C}$ by the unique continuation principle. Let $N \geq d+4$ and $p = 4N(d+N)$. If $\mathrm{Im}\,\lambda$ is large enough then $|K_V^p|_{\mathcal{L}} < 1$. In this case the series

$$\ln(1+\Psi(K_V)) = -\sum_{m=p}^{\infty} \frac{(-K_V)^m}{m}$$

converges in $\mathcal{L}$. This implies that for $\mathrm{Im}\,\lambda \gg 1$

$$D_V(\lambda) = \exp\left(-\sum_{m=p}^{\infty} (-1)^m \frac{\mathrm{Tr}\left(K_V^m\right)}{m}\right). \tag{4.52}$$

Hence, to obtain an explicit expansion of $D_V(\lambda)$ at order ε^4, it suffices to obtain an expansion of $\mathrm{Tr}\left(K_V^m\right)$ at order ε^4, for $\mathrm{Im}\,\lambda \gg 1$.

Let us expand $\mathrm{Tr}(K_V^m)$ in the different modes k_j:

$$\mathrm{Tr}(K_V^m) = \sum_{k_1,\ldots,k_m} T[k_1,\ldots,k_m], \quad T[k_1,\ldots,k_m] = \mathrm{Tr}\left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j\bullet/\varepsilon}\right). \tag{4.53}$$

We now fix a sequence $\{k_j\}$ with length $m \geq p$ and we aim to obtain an explicit expansion of $T[k_1,\ldots,k_m]$ at order ε^4. Because of the conclusion of Lemma 4.5, $T[k_1,\ldots,k_m]$ contribute to $O(\varepsilon^4)$ in the sum (4.53) unless the sequence $\{k_j\}$ has $\gamma \leq 2N-1$ non-vanishing terms. We note that $m \geq 2(N + \gamma N + 1)$. Hence, if $k_1 + \cdots + k_m \neq 0$, the sequence $\{k_j\}$ is admissible, and Lemma 4.10 shows that

$$|T[k_1, \ldots, k_m]| = \left| \mathrm{Tr} \left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon} \right) \right|$$
$$\leq C^{m+s^2} \varepsilon^N \prod_{j=1}^{m} \|W_{k_j}\|_s, \quad s = 2(N + \gamma N + 1) \leq 8N^2.$$

Hence, terms $T[k_1, \ldots, k_m]$ with $k_1 + \cdots + k_m \neq 0$ or more than $2N - 1$ non-vanishing k_j contribute to $O(\varepsilon^4)$ in (4.53).

We now focus on constructive sequences $\{k_j\}$ with at most $2N-1$ non-vanishing terms. We follow the construction of the expansion of $\mathrm{Tr}\left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon}\right)$, as explicitly mentioned in the proof of Lemma 4.14. We first perform a cyclic permutation of $\{k_j\}$ so that the resulting sequence ends with N vanishing d-tuples. The next step in the construction of Lemma 4.14 is an expansion of $\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon}$, as realized in Lemma 4.13. We first write

$$\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon} = \prod_{j=1}^{m} \rho R(D + \sigma_j/\varepsilon) W_{k_j}, \tag{4.54}$$

where $R(\xi, \lambda) = (\xi^2 - \lambda^2)^{-1}$. The expansion of $R(\xi + \sigma/\varepsilon, \lambda)$ given in Appendix B induces

$$\rho R(D + \sigma_j/\varepsilon) W_{k_j} = \varepsilon^2 \frac{W_{k_j}}{|\sigma_j|^2} + O_{\mathscr{B}(H^{s+4}, H^s)}(\varepsilon^3) = \varepsilon^2 \mathcal{O}_j \tag{4.55}$$

for an operator $\mathcal{O}_j : H^{s+4} \to H^s$ whose norm is uniformly bounded in ε, λ in compact sets and $\sigma \neq 0$. Assume first that 2 or more of the σ_j are non-zero, say $\sigma_{j_1}, \sigma_{j_2}$ with $j_1 \leq j_2$ maximal—in particular $\sigma_j = 0$ for $j_1 < j < j_2$. Since $\{k_j\}$ ends with N zeroes, we can assume $j_2 \leq m - N$. We perform the expansion (4.55) for the operators $\rho R(D + \sigma_{j_1}/\varepsilon) W_{k_{j_1}}$ and $\rho R(D + \sigma_{j_2}/\varepsilon) W_{k_{j_2}}$ in the product (4.54):

$$\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon} = \varepsilon^4 \left(\prod_{j=1}^{j_1 - 1} \rho R(D + \sigma_j/\varepsilon) W_{k_j} \right) \cdot \mathcal{O}_{j_1} \cdot \left(\prod_{j=j_1+1}^{j_2 - 1} K_{W_{k_j}} \right) \cdot \mathcal{O}_{j_2} \cdot K_{W_0}^{m - j_2}.$$

We can now bound the trace with the trace-class norm, itself controlled by the $H^{-d} \to L^2$ norm:

$$|T[k_1, \ldots, k_m]| = \left| \mathrm{Tr} \left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon} \right) \right| \leq \left| \prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon} \right|_{\mathscr{B}(H^{-d}, L^2)}$$
$$\leq \varepsilon^4 \left| \left(\prod_{j=1}^{j_1 - 1} \rho R(D + \sigma_j/\varepsilon) W_{k_j} \right) \cdot \mathcal{O}_{j_1} \cdot \left(\prod_{j=j_1+1}^{j_2 - 1} K_{W_{k_j}} \right) \cdot \mathcal{O}_{j_2} \right|_{\mathscr{B}(H^8, L^2)}$$
$$\cdot \left| K_{W_0}^{m - j_2} \right|_{\mathscr{B}(H^{-d}, H^8)}.$$

The first factor is uniformly controlled bounded because of the properties of the $\mathcal{O}_j$ and because the $R(D + \sigma_j/\varepsilon)$ are uniformly bounded on L^2; and the second factor is bounded because $K_{W_0}^{m-j_2}$ is a pseudodifferential operator of order $-2(m-j_2) \leq -2N \leq -d-8$. Hence, when two or more of the σ_j are non-zero and $\{k_j\}$ ends with N zeroes,

$$\operatorname{Tr}\left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon}\right) = O(\varepsilon^4).$$

It remains to consider sequences $\{k_j\}$ that ends with N zeroes and that have at most one non-vanishing σ_j. Such sequences must be cyclic perturbations of $(-k, k, 0, \dots, 0)$. Hence, without loss of generalities, we can assume that $\{k_j\}$ is the sequence $(-k, k, 0, \dots, 0)$ for some $k \neq 0$ and get

$$T[k_1, \dots, k_m] = \operatorname{Tr}\left(\prod_{j=1}^{m} K_{W_{k_j}} e^{ik_j \bullet/\varepsilon}\right) = \operatorname{Tr}\left(K_{W_{-k}} R(D + k/\varepsilon) W_k K_{W_0}^{m-2}\right).$$

Because of Appendix B, we know that

$$\rho R(D + k/\varepsilon) W_k = \varepsilon^2 \frac{W_k}{|k|^2} - 2\varepsilon^3 \frac{(k \cdot D) W_k}{|k|^4} + O_{\mathscr{B}(H^{s+5}, H^s)}(\varepsilon^4). \tag{4.56}$$

It suffices to use the same techinque as earlier to obtain

$$T[k_1, \dots, k_m] = \frac{\varepsilon^2}{|k|^2} \operatorname{Tr}\left(K_{W_0}^{m-2} K_{W_{-k}} W_k\right) - 2 \frac{\varepsilon^3}{|k|^4} \operatorname{Tr}\left(K_{W_0}^{m-2} K_{W_{-k}} (k \cdot D) W_k\right) + O(\varepsilon^4).$$

Summing over k and counting the multiplicity m of sequences of the form $(-k, k, 0, \dots, 0)$ due to cyclicity, we conclude that

$$\begin{aligned}\operatorname{Tr}(K_V^m) = \operatorname{Tr}(K_{W_0}^m) + m \sum_{k \neq 0} \frac{\varepsilon^2}{|k|^2} \operatorname{Tr}\left(K_{W_0}^{m-2} K_{W_{-k}} W_k\right) \\ - 2m \sum_{k \neq 0} \frac{\varepsilon^3}{|k|^4} \operatorname{Tr}\left(K_{W_0}^{m-2} K_{W_{-k}} (k \cdot D) W_k\right) + O(\varepsilon^4).\end{aligned}$$

This yields the value of the first four coefficients in the expansion of $D_V(\lambda)$. It is in practice possible to use this method to compute all the other coefficients $a_4, \dots, a_{N-1}$ given by Theorem 5.

4.6. The case $\lambda_0 = 0$ in dimension one

In this part we prove Lemma 3.1. Thus we assume $d = 1$. For $\lambda \neq 0$ the operator $K_{\mathcal{V}}$ is trace class. This allows us to define $d_{\mathcal{V}}(\lambda) = \operatorname{Det}(\operatorname{Id} + K_{\mathcal{V}})$. By [15, Theorem 2.6], the function $\lambda \mapsto \lambda d_{\mathcal{V}}(\lambda)$ is entire. It is related to the modified Fredholm determinant $D_{\mathcal{V}}$ by the identity

$$\lambda \exp\left(-\sum_{m=1}^{p-1} (-1)^m \frac{\operatorname{Tr}(K_{\mathcal{V}}^m)}{m}\right) D_{\mathcal{V}}(\lambda) = \lambda d_{\mathcal{V}}(\lambda). \tag{4.57}$$

If φ is a meromorphic function with a pole at 0 we write $\varphi = \sum_{m \in \mathbb{Z}} \beta_m z^m$ and we define $\operatorname{sing}(\varphi)$ the meromorphic function $\operatorname{sing}(\varphi)(z) = \sum_{m<0} \beta_m z^m$. We recall that Λ is the potential given by

$$\Lambda = \varepsilon^2\Lambda_0 + \varepsilon^3\Lambda_1 = \varepsilon^2 \sum_{k\neq 0} \frac{W_k W_{-k}}{k^2} - 2\varepsilon^3 \sum_{k\neq 0} \frac{W_k(DW_{-k})}{k^3}. \tag{4.58}$$

LEMMA 4.16. – *Let $d = 1$ and $N \geq 4$. For every $m \geq 2$ there exists a holomorphic function $t_m : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with the following:*

(i) $\operatorname{sing}(t_m) = \operatorname{sing}(\operatorname{Tr}(K_V^m))$.
(ii) *Locally uniformly on $\mathbb{C} \setminus \{0\}$,*

$$t_m(\lambda) = \operatorname{Tr}(K_{W_0}^m) + m\operatorname{Tr}(K_{W_0}^{m-2}K_\Lambda) + \cdots + O(\varepsilon^N).$$

Proof of Lemma 3.1 assuming Lemma 4.16. – Let $p = 4N(N+1)$ and set

$$h_V(\lambda) = \lambda \exp\left(-\sum_{m=1}^{p-1}(-1)^m \frac{t_m(\lambda)}{m}\right) D_V(\lambda),$$

where $D_{\mathcal{V}}(\lambda)$ is the determinant defined in (1.7). Equation (4.57) implies that

$$\lambda d_V(\lambda) = h_V(\lambda) \exp\left(\sum_{m=1}^{p-1}(-1)^m \frac{t_m(\lambda) - \operatorname{Tr}(K_V^m)}{m}\right).$$

The function

$$\sum_{m=1}^{p-1}(-1)^m \frac{t_m(\lambda) - \operatorname{Tr}(K_V^m)}{m}$$

is entire thanks to point (i) of Lemma 4.16. Consequently resonances of V (counted with multiplicity) are exactly zeros of h_V (counted with multiplicity).

We next show that the function h_V has an expansion in powers of ε on all of $\mathbb{C}$. For that we use Lemma 4.4 with $S_0 = \{0\}$, $E = \mathbb{C}$,

$$f(\lambda, \varepsilon) = \lambda \exp\left(-\sum_{m=1}^{p-1}(-1)^m \frac{t_m(\lambda)}{m}\right), \quad g(\lambda, \varepsilon) = D_V(\lambda).$$

Both f, g are meromorphic on $\mathbb{C}$ and their only pole is at 0. They both admit an expansion away from $\{0\}$ by Lemma 4.16 for f and by Theorem 5 for g. Their product $h_V = fg$ is entire. Consequently h_V admits an expansion of the form

$$h_V(\lambda) = h_0(\lambda) + \varepsilon h_1(\lambda) + \cdots + \varepsilon^{N-1}h_{N-1}(\lambda) + O(\varepsilon^N)$$

that holds locally uniformly for λ in $\mathbb{C}$. We next compute the first few terms in this expansion. Because of (ii) in Lemma 4.16 and of Theorem 5 we have

$$\begin{aligned} h_V(\lambda) &= \lambda \exp\left(-\sum_{m=1}^{p-1}(-1)^m \frac{t_m(\lambda)}{m}\right) D_V(\lambda) \\ &= \lambda d_{W_0}(\lambda) \exp\left(-\sum_{m=0}^{p-3}(-1)^m \operatorname{Tr}(K_{W_0}^m K_\Lambda)\right)\left(1 - \operatorname{Tr}((\mathrm{Id} + K_{W_0})^{-1}K_{W_0}^{p-2}K_\Lambda)\right) + O(\varepsilon^4) \\ &= \lambda d_{W_0}(\lambda)\left(1 - \operatorname{Tr}((\mathrm{Id} + K_{W_0})^{-1}K_\Lambda)\right) + O(\varepsilon^4). \end{aligned}$$

This ends the proof of Lemma 3.1. □

We next prove Lemma 4.16. We start with a preliminary lemma:

LEMMA 4.17. – *Let $k \in \mathbb{Z}\setminus\{0\}$ and $\varphi : \mathbb{R} \to \mathbb{C}$ be a smooth compactly supported function. Let p_N be the polynomial defined by*

$$p_N(X) = -2\left(1 + 2X + 3X^2 + \cdots + (N+1)X^N\right).$$

Then for every $N \geq 2$,

$$\left|\int_{\mathbb{R}} \varphi(x) e^{ikx/\varepsilon}|x|dx - \varepsilon^2\left(p_{N-3}(\varepsilon D/k)\varphi\right)(0)\right| \leq CN\left(\frac{\varepsilon}{k}\right)^N \|\varphi\|_{N+1}, \tag{4.59}$$

where the constant C depends only on the support of φ.

Proof. – By rescaling ε to ε/k we see that it suffices to prove the lemma in the case $k = 1$. Define

$$I[\varphi] = \int_{\mathbb{R}} e^{ix/\varepsilon}\varphi(x)|x|dx, \quad J[\varphi] = \frac{1}{i}\int_{\mathbb{R}} e^{ix/\varepsilon}\varphi(x)\mathrm{sgn}(x)dx.$$

By an integration by parts

$$I[\varphi] = -\varepsilon\left(J[\varphi] + I[D\varphi]\right),$$
$$J[\varphi] = \varepsilon\left(2\varphi(0) - J[D\varphi]\right).$$

Consequently,

$$I[\varphi] = \varepsilon^2\left(-2\varphi(0) + 2J[D\varphi] + I[D^2\varphi]\right). \tag{4.60}$$

We prove by recursion: for every $n \geq 0$

$$I[\varphi] = \varepsilon^2(p_n(-\varepsilon D)\varphi)(0) + \varepsilon^{n+2}I[(-D)^{n+2}\varphi] - \varepsilon^{n+2}(n+2)J[(-D)^{n+1}\varphi], \tag{4.61}$$

where $p_n = -2(1 + 2X + 3X^2 + \cdots + (n+1)X^n)$. For $n = 2$ this holds by Equation (4.60). Now assume (4.61) holds for some n. Then

$$\begin{aligned} I[\varphi] &= \varepsilon^2[p_n(-\varepsilon D)\varphi](0) + \varepsilon^{n+3}\left(-J[(-D)^{n+2}\varphi] + I[(-D)^{n+3}\varphi]\right) \\ &\quad - \varepsilon^{n+3}(n+2)\left(2[(-D)^{n+1}\varphi](0) + J[(-D)^{n+2}\varphi]\right) \\ &= \varepsilon^2[p_{n+1}(-\varepsilon D)\varphi](0) + \varepsilon^{n+3}I[(-D)^{n+3}\varphi] - \varepsilon^{n+3}(n+3)J[(-D)^{n+2}\varphi], \end{aligned}$$

where $p_{n+1} = p_n - 2(n+2)x^{n+1}$. This ends the recursion. Equation (4.59) follows from (4.61) and the estimate $|I[D^N\varphi]| \leq C\varepsilon\|\varphi\|_{N+1}$, $|J[D^N\varphi]| \leq C\varepsilon\|\varphi\|_{N+1}$. □

Proof of Lemma 4.16. – In dimension one the kernel of the free resolvent $R_0(\lambda)$ is given by $R_0(\lambda, x, y) = ie^{i\lambda|x-y|}/(2\lambda)$. We decompose it as follows:

$$R_0(\lambda, x, y) = \frac{f_0(\lambda, x-y)}{\lambda} + f_1(\lambda, x-y)|x-y|,$$
$$f_0(\lambda, x-y) = \frac{i}{2}\cos(\lambda|x-y|), \quad f_1(\lambda, x-y) = -\frac{\sin(\lambda|x-y|)}{2\lambda|x-y|}.$$

The functions f_0 and f_1 are both smooth on $\mathbb{C}\times\mathbb{R}$. This induces a decomposition of $K_{\mathcal{V}}(\lambda)$ given by

$$K_{\mathcal{V}}(\lambda) = E_{\mathcal{V},0}(\lambda) + E_{\mathcal{V},1}(\lambda),$$
$$E_{\mathcal{V},\epsilon}(\lambda, x, y) = \rho(x)\frac{f_0(\lambda, x-y)}{\lambda}\mathcal{V}(y), \quad E_{\mathcal{V},1}(\lambda, x, y) = \rho(x)f_1(\lambda, x-y)|x-y|\mathcal{V}(y).$$

Thus $K_{\mathcal{V}}(\lambda)$ is the sum of a smoothing operator $E_{\mathcal{V},0}(\lambda)$ with a pole at $\lambda = 0$ and of an operator $E_{\mathcal{V},1}(\lambda)$ which is not smoothing but has no pole. We now define

$$t_m(\lambda) = \begin{cases} \mathrm{Tr}(K_V^m) - \mathrm{Tr}(E_{V,1}^m) + \mathrm{Tr}(E_{W_0,1}^m) + m\mathrm{Tr}(E_{W_0,1}^{m-2}E_{\Lambda,1}) & \text{if } m \geq 3, \\ \mathrm{Tr}(K_V^2) + \mathrm{Tr}(E_{W_0,1}^2) - \mathrm{Tr}(E_{V,1}^2) & \text{if } m = 2, \end{cases} \tag{4.62}$$

where Λ is the potential given by (4.58). Since $\mathrm{Tr}(E_{W_0,1}^m) - \mathrm{Tr}(E_{V,1}^m) + m\mathrm{Tr}(E_{W_0,1}^{m-2}E_{\Lambda,1})$ and $\mathrm{Tr}(E_{W_0,1}^2) - \mathrm{Tr}(E_{V,1}^2)$ are both entire function of λ we have $\mathrm{sing}(t_m) = \mathrm{sing}(\mathrm{Tr}(K_V^m))$. It remains to show that the function t_m satisfies the expansion given by (ii). Write

$$\begin{aligned}\mathrm{Tr}(K_V^m) &= \sum_{\epsilon_1,\dots,\epsilon_m \in \{0,1\}^m} \mathrm{Tr}\left(\prod_{j=1}^m E_{V,\epsilon_j}\right) \\ &= \mathrm{Tr}\left(E_{V,1}^m\right) + \sum_{k_1,\dots,k_m} \sum_{\substack{\epsilon_1 \dots,\epsilon_m \in \{0,1\}^m \\ \epsilon_1 \cdot \dots \cdot \epsilon_m = 0}} \mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j \bullet/\varepsilon}\right).\end{aligned}$$

We first claim that for every N and locally uniformly on $\mathbb{C} \setminus \{0\}$,

$$\sum_{k_1+\dots+k_m \neq 0} \sum_{\substack{\epsilon_1 \dots,\epsilon_m \in \{0,1\}^m \\ \epsilon_1 \cdot \dots \cdot \epsilon_m = 0}} \mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j \bullet/\varepsilon}\right) \leq C\varepsilon^N \|W\|_{Z^N}^m. \tag{4.63}$$

Fix a sequence $\epsilon_1, \dots, \epsilon_m \in \{0,1\}^m$ with $\epsilon_1 \cdot \dots \cdot \epsilon_m = 0$ and $k_1, \dots, k_m \in \mathbb{Z}$ with $k_1 + \dots + k_m \neq 0$. There exists j_0 with $\epsilon_{j_0} = 0$. Using the cyclicity of the trace we can assume without loss of generality that $j_0 = 1$. Let $n = m - \epsilon_1 - \dots - \epsilon_m$. Using the explicit expression of the kernel of the operators $E_{\mathcal{V},\epsilon}$ we have

$$\mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j \bullet/\varepsilon}\right) = \lambda^{-n} \int_{\mathbb{R}^m} \left(\prod_{j=1}^m f_{\epsilon_j}(x_j - x_{j-1})|x_j - x_{j-1}|^{\epsilon_j} W_{k_j}(x_j) e^{ik_j x_j/\varepsilon} dx_j\right) dx_1,$$

where by convention $x_0 = x_m$. The substitution $x_j = y_1 + \dots + y_j$, $j \in [1,m]$ and the explicit expression of the kernels of $E_{\mathcal{V},0}$ and $E_{\mathcal{V},1}$ yield

$$\mathrm{Tr}\left(\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j \bullet/\varepsilon}\right)\right) = \lambda^{-n} \int_{\mathbb{R}} e^{i\sigma_1 y_1/\varepsilon} I(y_1) dy_1,$$

where $\sigma_j = k_j + \dots + k_m$, $z = y_2 + \dots + y_{m-1}$ and

$$I(y_1) = W_{k_1}(y_1) \int_{\mathbb{R}^{m-1}} f_0(z + y_m) \prod_{j=2}^m f_{\epsilon_j}(y_j)|y_j|^{\epsilon_j} W_{k_j}(y_1 + \dots + y_j) e^{i\sigma_j y_j/\varepsilon} dy_j.$$

The function $y_1 \mapsto I(y_1)$ is smooth and compactly supported. Since $\sigma_1 \neq 0$ N integrations by parts give the estimate

$$\left|\int_{\mathbb{R}} e^{i\sigma_1 y_1/\varepsilon} I(y_1) dx_1\right| \leq C\varepsilon^N \|I\|_N \leq C\varepsilon^N \prod_{j=1}^m \|W_{k_j}\|_N.$$

Therefore

$$\left| \sum_{k_1+\cdots+k_m\neq 0} \sum_{\substack{\epsilon_1\dots,\epsilon_m\in\{0,1\}^m \\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j\bullet/\varepsilon}\right)\right|$$
$$\leq C\varepsilon^N \sum_{k_1,\dots,k_m} \prod_{j=1}^m \|W_{k_j}\|_N \leq C\varepsilon^N \|W\|^m_{Z^N},$$

where we recall that $|W|_{Z^N} = \sum_{k\in\mathbb{Z}^d} \|W_k\|_s$. This proves (4.63).

We next show that the function

$$\sum_{k_1+\cdots+k_m=0} \sum_{\substack{\epsilon_1\dots,\epsilon_m\in\{0,1\}^m \\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j\bullet/\varepsilon}\right)$$

admits an expansion in powers of ε. It suffices to prove that for any fixed sequence $\{\epsilon_j\}$ with $\epsilon_1 = 0$ the function

$$\sum_{k_1+\cdots+k_m=0} \mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j\bullet/\varepsilon}\right) \tag{4.64}$$

admits an expansion in powers of ε. Fix $k_1,\dots,k_m$ with $k_1+\cdots+k_m = 0$. We define F_{m-1} and F_s, $s\in[1,m-2]$ recursively as follows:

$$\begin{cases} F_{m-1}(y_1,\dots,y_{m-1}) = \int_{\mathbb{R}} f_0(z+y_m) f_{\epsilon_m}(y_m) W_{k_m}(y_1+\cdots+y_m) e^{i\sigma_m y_m/\varepsilon}|y_m|^{\epsilon_m} dy_m \\ F_{s-1}(y_1,\dots,y_{s-1}) = \int_{\mathbb{R}} f_{\epsilon_s}(y_s) W_{k_s}(y_1+\cdots+y_s) F_s(y_1,\dots,y_s) e^{i\sigma_s y_s/\varepsilon}|y_s|^{\epsilon_s} dy_s, \end{cases}$$

where $z = y_2+\cdots+y_{m-1}$. Let $F_0(\lambda)$ be given by

$$F_0(\lambda) = \mathrm{Tr}\left(\prod_{j=1}^m E_{W_{k_j},\epsilon_j} e^{ik_j\bullet/\varepsilon}\right) = \lambda^{-n}\int_{\mathbb{R}} W_{k_1}(y_1)F_1(y_1)dy_1.$$

We prove recursively that $F_{m-1}, F_{m-2},\dots,F_1,F_0$ admit an expansion in powers of ε. The fact that F_{m-1} admits an expansion in powers of ε is a consequence of Lemma 4.17. The coefficients are smooth functions of $y_1,\dots,y_{m-1}$. The recursive formula defining F_{m-2} shows that F_{m-2} also admits an expansion in powers of ε whose coefficients are smooth functions of $y_1,\dots,y_{m-2}$. The same recursive scheme shows that $F_{m-3},\dots,F_0$ admit an expansion in powers of ε. The sum over $k_1,\dots,k_m$ with $k_1+\cdots+k_m = 0$ of the coefficients converge (we skip the details) and we conclude that (4.64) admits an expansion in powers of ε. Finally we sum over all sequences $\{\epsilon_j\}$ with at least one vanishing term and we use (4.63) to deduce that

$$\begin{aligned} t_m(\lambda) &= \mathrm{Tr}(K_V^m) + \mathrm{Tr}(E^m_{W_0,1}) - \mathrm{Tr}(E^m_{V,1}) + \delta_{m\neq 2} m\mathrm{Tr}(E^{m-2}_{W_0,1}E_{\Lambda,1}) \\ &= \sum_{\substack{\epsilon_1,\dots,\epsilon_m\in\{0,1\}^m, \\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^m E_{V,\epsilon_j}\right) + \delta_{m\neq 2} m\mathrm{Tr}(E^{m-2}_{W_0,1}E_{\Lambda,1}) + \mathrm{Tr}(E^m_{W_0,1}) \end{aligned}$$

admits an expansion in powers of ε.

To end the proof we must compute the first terms in the expansion of the function t_m. We fix $N = 4$ and work modulo $O(\varepsilon^4)$. The only sequence $\{k_j\}$ that can generate non-negligible terms is $(0, \dots, 0, -k, k)$ up to cyclic permutation—see §4.5. We fix $\{\epsilon_j\}$ and we estimate

$$\sum_{k\neq 0} \operatorname{Tr}\left(\left(\prod_{j=1}^{m-2} E_{W_0,\epsilon_j} \right) E_{W_{-k},\epsilon_{m-1}} e^{-ik\bullet/\varepsilon} E_{W_k,\epsilon_m} e^{ik\bullet/\varepsilon} \right).$$

Assume that $m \geq 3$ and define G by

$$\begin{aligned} &G(\lambda, y_1, \dots, y_{m-1}) \\ &\quad = \sum_{k\neq 0} W_{-k}(y_1 + z) \int_{\mathbb{R}} f_0(z + y_m) f_{\epsilon_m}(y_m) W_k(y_1 + z + y_m) e^{iky_m/\varepsilon} |y_m|^{\epsilon_m} dy_m, \end{aligned}$$

where we recall that $z = y_2 + \cdots + y_{m-1}$. We first deal with the case $\epsilon_m = 1$. This implies $f_{\epsilon_m}(0) = f_1(0) = -1/2$. Apply Lemma 4.17 to obtain the asymptotic

$$\begin{aligned} G(\lambda, y_1, \dots, y_{m-1}) = &\sum_{k\neq 0} \left(\frac{\varepsilon}{k}\right)^2 f_0(z) W_{-k}(y_1 + z) W_k(y_1 + z) \\ &- 2\sum_{k\neq 0} \left(\frac{\varepsilon}{k}\right)^3 f_0(z) W_{-k}(y_1 + z)(DW_k)(y_1 + z) \\ &- 2\sum_{k\neq 0} \left(\frac{\varepsilon}{k}\right)^3 (Df_0)(z) W_{-k}(y_1 + z) W_k(y_1 + z) + O(\varepsilon^4). \end{aligned}$$

Since $\sum_{k\neq 0} W_{-k} W_k / k^3 = 0$ we can remove the last term that appears in the expansion of G and write $G(\lambda, y_1, \dots, y_{m-1}) = f_0(z)\Lambda(y_1 + z) + O(\varepsilon^4)$. This expansion combined with the inverse substitution $y \mapsto x$ variables yields

$$\begin{aligned} &\sum_{k\neq 0} \operatorname{Tr}\left(\left(\prod_{j=1}^{m-2} E_{W_0,\epsilon_j} \right) E_{W_{-k},\epsilon_{m-1}} e^{-ik\bullet/\varepsilon} E_{W_k,1} e^{ik\bullet/\varepsilon} \right) + O(\varepsilon^4) \\ &\quad = \sum_{k\neq 0} \lambda^{-n} \int_{\mathbb{R}^{m-1}} f_0(z) W_0(y_1) \left(\prod_{j=2}^{m-2} f_{\epsilon_j}(y_j) |y_j|^{\epsilon_j} W_0(y_1 + \cdots + y_j) dy_j \right) \\ &\qquad \cdot f_{\epsilon_{m-1}}(y_{m-1}) \Lambda(z) dy_1 dy_{m-1} \\ &\quad = \sum_{k\neq 0} \lambda^{-n} \int_{\mathbb{R}^{m-1}} \left(\prod_{j=1}^{m-2} f_{\epsilon_j}(y_j - y_{j-1}) |x_j - x_{j-1}|^{\epsilon_j} W_0(x_j) dx_j \right) \\ &\qquad \cdot f_{\epsilon_{m-1}}(x_{m-1} - x_{m-2}) \Lambda(x_{m-1}) dx_1 dx_{m-1} \\ &\quad = \operatorname{Tr}\left(\left(\prod_{j=1}^{m-2} E_{W_0,\epsilon_j} \right) E_{\Lambda,\epsilon_{m-1}} \right). \end{aligned}$$

This gives an estimate of G in the case $\epsilon_m = 1$. In the case $\epsilon_m = 0$ the kernel of $E_{W_k,0}$ is smooth and we can integrate by parts by parts to obtain $G(\lambda, y_1, \dots, y_{m-1}) = O(\varepsilon^4)$. Summing these estimates of G over all possible values of $\epsilon_1, \dots, \epsilon_{m-1}, \epsilon_m$ and using the cyclicity of the trace yield

$$
\begin{aligned}
&\sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^{m} E_{V,\epsilon_j}\right) - \sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^{m} E_{W_0,\epsilon_j}\right)\\
&\quad = m \sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0}} \sum_{k\neq 0} \mathrm{Tr}\left(\left(\prod_{j=1}^{m-2} E_{W_0,\epsilon_j}\right) E_{W_{-k},\epsilon_{m-1}} e^{-ik\bullet/\varepsilon} E_{W_k,\epsilon_m} e^{ik\bullet/\varepsilon}\right) + O(\varepsilon^4)\\
&\quad = m \sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0,\ \epsilon_m=1}} \mathrm{Tr}\left(\left(\prod_{j=1}^{m-2} E_{W_0,\epsilon_j}\right) E_{\Lambda,\epsilon_{m-1}}\right) + O(\varepsilon^4).
\end{aligned}
$$

Recall that $t_m(\lambda)$ is given by (4.62) to conclude that

$$
\begin{aligned}
t_m(\lambda) &= \sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^{m} E_{W_0,\epsilon_j}\right) + \mathrm{Tr}(E_{W_0,1}^m)\\
&\quad + \sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0}} \mathrm{Tr}\left(\prod_{j=1}^{m} E_{V,\epsilon_j}\right) + m\mathrm{Tr}(E_{W_0,1}^{m-2} E_{\Lambda,1})\\
&= \mathrm{Tr}(K_{W_0}^m) + m \sum_{\substack{\epsilon_1,\ldots,\epsilon_m\in\{0,1\}^m\\ \epsilon_1\cdots\cdots\epsilon_m=0,\ \epsilon_m=1}} \mathrm{Tr}\left(\left(\prod_{j=1}^{m-2} E_{W_0,\epsilon_j}\right) E_{\Lambda,\epsilon_{m-1}}\right) + O(\varepsilon^4)\\
&= \mathrm{Tr}(K_{W_0}^m) + m\mathrm{Tr}\left(K_{W_0}^{m-2} K_\Lambda\right) + O(\varepsilon^4).
\end{aligned}
$$

We finally deal with the case $m = 2$. If $\epsilon_1, \epsilon_2 \in \{0, 1\}$ then

$$
\begin{aligned}
&\mathrm{Tr}\left(E_{W_{-k},\epsilon_{m-1}} e^{-ik\bullet/\varepsilon} E_{W_k,\epsilon_m} e^{ik\bullet/\varepsilon}\right)\\
&\quad = \lambda^{\epsilon_1+\epsilon_2-2} \int_{\mathbb{R}} f_{\epsilon_1}(y_2) f_{\epsilon_2}(y_2) W_{-k}(y_1) W_k(y_1 + y_2) |y_2|^{\epsilon_1+\epsilon_2} e^{iky_2/\varepsilon} dy_1 dy_2.
\end{aligned}
$$

If $\epsilon_1+\epsilon_2$ is even then one can integrate by parts many times in y_2 and obtain $O(\varepsilon^4)$. Otherwise $\epsilon_1 + \epsilon_2 = 1$ and $f_{\epsilon_1} f_{\epsilon_2} = f_0 f_1$. In particular $f_0 f_1(0) = 1/(4i)$ and $(f_0 f_1)'(0) = 0$. This yields

$$
\begin{aligned}
&\sum_{\epsilon_1,\epsilon_2,\ \epsilon_1\epsilon_2=0} \sum_{k\neq 0} \mathrm{Tr}(E_{W_{-k},\epsilon_{m-1}} e^{-ik\bullet/\varepsilon} E_{W_k,\epsilon_m} e^{ik\bullet/\varepsilon})\\
&\quad = 2\lambda^{-1} \sum_{k\neq 0} \int_{\mathbb{R}} W_{-k}(y_1) \int_{\mathbb{R}} f_0(y_2) f_1(y_2) W_k(y_1 + y_2) |y_2| e^{iky_2/\varepsilon} dy_2 dy_1\\
&\quad = 2 \sum_{k\neq 0} \frac{i}{2\lambda} \int_{\mathbb{R}} W_{-k}(y_1) \left(\left(\frac{\varepsilon}{k}\right)^2 W_k(y_1) - 2\left(\frac{\varepsilon}{k}\right)^3 DW_k(y_1)\right) dy_1 + O(\varepsilon^4)\\
&\quad = 2\mathrm{Tr}(K_\Lambda) + O(\varepsilon^4).
\end{aligned}
$$

Together with (4.63) this gives $t_2(\lambda) = \mathrm{Tr}(K_{W_0}^2) + 2\mathrm{Tr}(K_\Lambda) + O(\varepsilon^4)$. This completes the proof of the lemma. □

Appendix A

Analytic continuation of some Fredholm operators

Let $T(\lambda)$ be a holomorphic family of trace-class operators on a Hilbert space. In finite dimension, the operator $\det(\mathrm{Id} + T(\lambda))(\mathrm{Id} + T(\lambda))^{-1}$, defined away from the poles of $(\mathrm{Id} + T(\lambda))^{-1}$, extends to an entire family of operators known as the comatrix of $\mathrm{Id} + T(\lambda)$. In infinite dimension a similar statement holds:

LEMMA A.1. – *Consider $\mathscr{H}$ a Hilbert space, $\mathscr{U}$ an open connected subset of $\mathbb{C}$ and $T(\lambda)$ a holomorphic family of trace class operators for $\lambda \in \mathscr{U}$. Assume that $\mathrm{Id} + T(\lambda_0)$ is invertible for some $\lambda_0 \in \mathscr{U}$. Then the family of operators*

$$\mathscr{T}(\lambda) = \mathrm{Det}(\mathrm{Id} + T(\lambda))(\mathrm{Id} + T(\lambda))^{-1}$$

initially defined for λ away from the poles of $(\mathrm{Id} + T(\lambda))^{-1}$ extends to a holomorphic family of operators on $\mathscr{U}$. Moreover,

$$|\mathscr{T}(\lambda)|_{\mathscr{B}(\mathscr{H})} \leq \mathrm{Det}\left(\mathrm{Id} + \left(T(\lambda)^* T(\lambda)\right)^{1/2}\right) \leq e^{2|T(\lambda)|_{\mathscr{L}}}. \tag{A.1}$$

Proof. – The proof uses the Gohberg-Sigal theory of residues—see [15, Appendix C.4]. By analytic Fredholm theory, $(\mathrm{Id} + T(\lambda))^{-1}$ defines a meromorphic family of operators with poles of finite rank. Fix $\mu \in \mathscr{U}$ a pole of $(\mathrm{Id} + T(\lambda))^{-1}$ and λ in a punctured neighborhood of μ. We can write

$$\mathrm{Id} + T(\lambda) = U_1(\lambda)\left(\Pi_0 + \sum_{m=1}^{N} (\lambda - \mu)^{\kappa_m} \Pi_m\right) U_2(\lambda),$$

where $U_1(\lambda), U_2(\lambda)$ are holomorphic families of invertible operators, $\kappa_m \geq 1$, Π_m has rank 1 for $m > 0$, $\Pi_m \Pi_{m'} = \delta_{mm'} \Pi_m$, $\mathrm{rank}(\mathrm{Id} - \Pi_0) < \infty$. Therefore

$$(\mathrm{Id} + T(\lambda))^{-1} = U_2(\lambda)^{-1}\left(\Pi_0 + \sum_{m=1}^{N} (\lambda - \mu)^{-\kappa_m} \Pi_m\right) U_1(\lambda)^{-1}.$$

The holomorphic function $\lambda \mapsto \mathrm{Det}(\mathrm{Id} + T(\lambda))$ has a zero at μ, of multiplicity $\sum_{m=1}^N \kappa_m$—see [15, equation $(C.4.7)$]. It follows that the operator $\mathscr{T}(\lambda)$ can indeed be analytically continued at $\lambda = \mu$ with

$$\mathscr{T}(\mu) = \begin{cases} 0 & \text{if } N > 1 \\ \dfrac{\mathrm{Det}(\mathrm{Id} + T(\lambda))}{(\lambda - \mu)^{\kappa_1}}\Big|_{\lambda = \mu} U_2(\mu)^{-1} \Pi_1 U_1(\mu)^{-1} & \text{if } N = 1. \end{cases}$$

The first bound in (A.1) follows from [15], (B.4.7). For the second one, note first that

$$\mathrm{Det}\left(\mathrm{Id} + \left(T(\lambda)^* T(\lambda)\right)^{1/2}\right) \leq \exp\left(\left|\left(T(\lambda)^* T(\lambda)\right)^{1/2}\right|_{\mathscr{L}}\right).$$

Finally we note that

$$s_{2j}\left(\left(T(\lambda)^*T(\lambda)\right)^{1/2}\right) \leq s_j\left(T(\lambda)^*\right)^{1/2} s_j\left(T(\lambda)\right)^{1/2} \leq s_j\left(T(\lambda)\right),$$

$$\left|\left(T(\lambda)^*T(\lambda)\right)^{1/2}\right|_{\mathcal{L}} \leq 2\sum_{j=0}^{\infty} s_{2j}\left(\left(T(\lambda)^*T(\lambda)\right)^{1/2}\right) \leq 2\sum_{j=0}^{\infty} s_j\left(T(\lambda)\right) \leq 2|T(\lambda)|_{\mathcal{L}}.$$

This concludes the proof. □

Appendix B

Expansion of $R(\lambda, \xi + k/\varepsilon)$

In this appendix we study the Taylor development of rational functions of the form $F(\varepsilon) = (1 + a\varepsilon + b\varepsilon^2)^{-1}$. Such functions are analytic for small values of ε and therefore there exists $u_k \in \mathbb{C}$ with $F(\varepsilon) = \sum_{j\geq 0} u_j \varepsilon^j$. Since $F(\varepsilon)(1 + a\varepsilon + b\varepsilon^2) = 1$, the u_k must satisfy the recursion relation

$$\begin{cases} u_0 = 1, \\ u_1 = -a, \\ u_j = -au_{j-1} - bu_{j-2}. \end{cases}$$

For ε small enough the Taylor development of F takes the form

$$F(\varepsilon) = \sum_{j=0}^{J-1} u_j\varepsilon^j + r_J(\varepsilon), \quad r_J(\varepsilon) = \sum_{j=J}^{\infty} u_j \varepsilon^j.$$

We have moreover

$$\begin{aligned}(1 + a\varepsilon + b\varepsilon^2)r_J(\varepsilon) &= (1 + a\varepsilon + b\varepsilon^2)\sum_{j=J}^{\infty} u_j\varepsilon^j \\ &= u_J\varepsilon^J + u_{J+1}\varepsilon^{J+1} + au_J\varepsilon^{J+1} + \sum_{j=J+2}^{\infty}(u_j + au_{j-1} + bu_{j-2})\varepsilon^j \\ &= u_J\varepsilon^J + u_{J+1}\varepsilon^{J+1} + au_J\varepsilon^{J+1}.\end{aligned}$$

Consequently for small values of ε,

$$F(\varepsilon) = \left(\sum_{j=0}^{J-1} u_j\varepsilon^j\right) + \frac{u_J + u_{J+1}\varepsilon + au_J\varepsilon}{1 + a\varepsilon + b\varepsilon^2}\varepsilon^J$$

and this identity extends meromorphically to all of $\mathbb{C}$. If a and b are polynomial of respective degree 1 and 2 in a parameter ξ then by an immediate recursion u_j is a polynomial of degree at most j in ξ. In particular, (4.41) holds:

$$\begin{aligned}
R(\xi + k/\varepsilon) &= \frac{\varepsilon^2}{|k|^2} \frac{1}{1 + \varepsilon k \cdot \xi/|k|^2 + \varepsilon^2(\xi^2 - \lambda^2)/|k|^2} \\
&= \frac{\varepsilon^2}{|k|^2} \left(\left(\sum_{j=0}^{J-1} u_j \varepsilon^j \right) + \frac{u_J + u_{J+1}\varepsilon + a u_J \varepsilon}{1 - 2\varepsilon k \cdot \xi/|k|^2 + \varepsilon^2(\xi^2 - \lambda^2)/|k|^2} \varepsilon^J \right) \\
&= \left(\sum_{j=2}^{J-1} \frac{u_{j-2}}{|k|^2} \varepsilon^j \right) + \frac{u_{J-1}}{|k|^2} \varepsilon^J + \frac{u_{J-1}}{|k|^2} \varepsilon^{J+1} + \frac{u_J + u_{J+1}\varepsilon + a u_J \varepsilon}{(\xi - k/\varepsilon)^2 - \lambda^2} \varepsilon^J .
\end{aligned}$$

Because of the recursion formula defining the u_j, their dependence in k depends uniformly on $k/|k|^2$ and $|k|^{-2}$—hence uniformly on $k/|k|^2$ only. The first terms in this expansion are given by

$$R(\xi + k/\varepsilon) = \frac{\varepsilon^2}{|k|^2} - 2\varepsilon^3 \frac{k \cdot \xi}{|k|^4} - \varepsilon^4 \frac{\xi^2 - \lambda^2}{|k|^4} + 4\varepsilon^4 \frac{(k \cdot \xi)^2}{|k|^6} + O(\varepsilon^5).$$

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