Quasiconformal surgery and linear differential equations

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Abstract

We describe a new method of constructing transcendental entire functions A such that the differential equation $w'' + Aw = 0$ has two linearly independent solutions with relatively few zeros. In particular, we solve a problem of Bank and Laine by showing that there exist entire functions A of any prescribed order greater than $1/2$ such that the differential equation has two linearly independent solutions whose zeros have finite exponent of convergence. We show that partial results by Bank, Laine, Langley, Rossi and Shen related to this problem are in fact best possible. We also improve a result of Toda and show that the resulting estimate is best possible. Our method is based on gluing solutions of the Schwarzian differential equation $S(F) = 2A$ for infinitely many coefficients A.

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1 Introduction and main result

We consider ordinary differential equations of the form

$$w'' + Aw = 0, \tag{1.1}$$

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where A is an entire function. All solutions w are entire functions. The Wronskian determinant $W(w_1, w_2) = w_1 w_2' - w_1' w_2$ of any two linearly independent solutions w_1 and w_2 is a non-zero constant, and a pair (w_1, w_2) of solutions will be called *normalized* if $W(w_1, w_2) = 1$.

We recall that the *order* of an entire function f is defined by

$$\rho(f) = \limsup_{z\to\infty} \frac{\log\log|f(z)|}{\log|z|}, \tag{1.2}$$

and the *exponent of convergence* of the zeros of f by

$$\lambda(f) = \inf\left\{\lambda > 0\colon \sum_{\{z\neq 0\colon f(z)=0\}} m(z)|z|^{-\lambda} < \infty\right\} = \limsup_{r\to\infty} \frac{\log n(r,0,f)}{\log r},$$

where $m(z)$ is the multiplicity of the zero z, and

$$n(r,0,f) = \sum_{|z|\le r} m(z)$$

denotes the number of zeros of f in the disc $\{z \in \mathbb{C}\colon |z| \le r\}$. It is well-known [34, Chapter I, §5] that

$$\lambda(f) \le \rho(f)$$

for every entire function f.

When A is a polynomial of degree d, then all non-trivial solutions w have order $(d+2)/2$. Some solutions can be free of zeros, but when $d > 0$, the exponent of convergence of zeros of the product

$$E = w_1 w_2$$

of any two linearly independent solutions is equal to $(d+2)/2$.

We refer to [3, Theorem 1] for a proof of these results, most of which are classical. In fact, much more precise estimates on the location of the zeros and the asymptotic behavior of $n(r,0,w)$ can be obtained from the asymptotic integration method; cf. [23, Section 4.6], [24, Section 5.6] and [42, Theorem 6.1].

Some special equations of the form (1.1) with transcendental coefficient A, such as the Mathieu equation, have been intensively studied since the 19-th

century, but one of the first systematic studies of the general case of transcendental entire A is due to Bank and Laine [2, 3]. When A is transcendental, then every solution has infinite order. It is possible that two linearly independent solutions are free of zeros; for example, if p is a polynomial, then

$$w'' - \frac{1}{4}\big(e^{2p} + (p')^2 - 2p''\big)\, w = 0 \tag{1.3}$$

has solutions

$$w_{1,2}(z) = \exp\left(-\frac{1}{2}\left(p(z) \pm \int_0^z e^{p(t)}dt\right)\right). \tag{1.4}$$

However, two linearly independent solutions without zeros can occur only when A is constant, or $\rho(A)$ is a positive integer or ∞. More generally, we have the following result.

Theorem A. *Let E be the product of two linearly independent solutions of the differential equation* (1.1)*, where A is a transcendental entire function of finite order. Then:*

(*i*) *If $\rho(A)$ is not an integer, then*

$$\lambda(E) \geq \rho(A). \tag{1.5}$$

(*ii*) *If $\rho(A) \leq 1/2$, then $\lambda(E) = \infty$.*

These results were proved by Bank and Laine [2, 3], except for the case $\rho(A) = 1/2$ in (*ii*) which is due to Rossi [37] and Shen [40].

Based on these results, Bank and Laine conjectured that whenever $\rho(A)$ is not an integer, then $\lambda(E) = \infty$. This conjecture raised considerable interest; see the surveys [21, 32] and references there. Counterexamples were recently constructed by the present authors [7] who proved the following result.

Theorem B. *There is a dense set of $\rho \geq 1$, such that there exist A with $\rho(A) = \lambda(E) = \rho$. Moreover, one solution for this A is free of zeros.*

This shows that the inequality (1.5) is best possible when $\rho(A) \geq 1$. Rossi [37] showed that if $1/2 \leq \rho(A) < 1$, then (1.5) can be improved to the inequality

$$\frac{1}{\rho(A)} + \frac{1}{\lambda(E)} \leq 2. \tag{1.6}$$

Here and in the following $1/\lambda(E) = 0$ is understood to mean that $\lambda(E) = \infty$. Solving the inequality (1.6) for $\lambda(E)$ we obtain $\lambda(E) \geq \rho(A)/(2\rho(A) - 1)$ for $1/2 < \rho(A) < 1$.

The main result of this paper shows that (1.6) is also best possible.

Theorem 1.1. *For every $\rho \in (1/2, 1)$ there exists an entire function A of order ρ such that the differential equation* (1.1) *has two linearly independent solutions whose product E satisfies*

$$\frac{1}{\rho(A)} + \frac{1}{\lambda(E)} = 2. \tag{1.7}$$

Moreover, $\lambda(E) = \rho(E)$ and one of the solutions of (1.1) *is free of zeros.*

We note that our method also allows to construct coefficients A of preassigned order $\rho(A) \in [1, \infty)$ such that (1.1) has two linearly independent solutions whose product has finite order, with one solution having no zeros; see Corollary 1.2 below.

The main idea in the proofs of this and subsequent results is to glue functions associated to the differential equation (1.1) for different coefficients A. Unlike in [7], where two functions are glued, it is now required to glue infinitely many functions, which creates substantial additional difficulties; see section 2 for a detailed description of the method.

An entire function E is called a *Bank–Laine function* if $E(z) = 0$ implies that $E'(z) \in \{-1, 1\}$. We call a Bank–Laine function *special* if $E(z) = 0$ implies that $E'(z) = 1$. It is known that the product of two normalized solutions of (1.1) is a Bank–Laine function and all Bank–Laine functions arise in this way. If w_1 has no zeros, the corresponding Bank–Laine function is special. It follows from (1.6) and equation (2.2) below that $\rho(E) \geq 1$ for every transcendental Bank–Laine function E; see [41, Theorem 1].

Many authors studied Bank–Laine functions (see the surveys [21, 32]), but it was open until [7] whether there exist Bank–Laine functions of non-integer order – except for those corresponding to a polynomial coefficient A of odd degree, whose order is half an integer. Theorem 1.1 gives a complete answer to the question asked in [2] what the possible orders of Bank–Laine functions are. In fact, prescribing $\rho(A) \in (1/2, 1)$ in (1.7) is equivalent to prescribing $\rho(E) \in (1, \infty)$ in (1.7). Since the exponential function is a special Bank–Laine function of order 1, we obtain the following result.

Corollary 1.1. *For every $\rho \in [1, \infty)$ there exists a special Bank–Laine function of order ρ.*

A major difference between the functions A and E constructed in the proof of Theorem B in [7] and the functions constructed in the proof of Theorem 1.1 is that the functions in [7] have "spiraling" behavior; that is, we have $E(\gamma_1(t)) \to 0$ and $E(\gamma_2(t)) \to \infty$ as $t \to \infty$ on certain logarithmic spirals γ_1 and γ_2.

In contrast, one can show that the functions A and E constructed in the proof of Theorem 1.1 have completely regular growth in the sense of Levin and Pfluger; see [34, Chapter 3]. More precisely, we have the following result.

Theorem 1.2. *The functions A and E in Theorem 1.1 can be chosen such that*

$$\frac{1}{2}\log|A(re^{i\theta})| \sim \log\frac{1}{|E(re^{i\theta})|} \sim r^{\rho}\cos(\rho\theta) \quad \text{for } |\theta| \leq (1-\varepsilon)\frac{\pi}{2\rho} \tag{1.8}$$

while, with $\sigma = \rho/(2\rho-1)$,

$$\log|E(-re^{i\theta})| \sim r^{\sigma}\cos(\sigma\theta) \quad \text{for } |\theta| \leq (1-\varepsilon)\frac{\pi}{2\sigma} \tag{1.9}$$

and

$$|A(-re^{i\theta})| \sim \frac{\sigma^2}{4}r^{2\sigma-2} \quad \text{for } |\theta| \leq (1-\varepsilon)\frac{\pi}{2\sigma} \tag{1.10}$$

uniformly as $r \to \infty$, for any $\varepsilon > 0$.

Since the set of rays of completely regular growth is closed [34, Chapter 3, Theorem 1], it follows that functions A and E satisfying the conclusions of Theorem 1.2 have indeed completely regular growth.

In [5] and various subsequent papers the differential equation (1.1) is studied under suitable hypotheses on the asymptotic behavior of the coefficient A. In particular, the following result was obtained in [5, Theorem 1].

Theorem C. *Let A be a transcendental entire function of finite order with the following property: there exists a subset H of $\mathbb{R}$ of measure zero such that for each $\theta \in \mathbb{R}\backslash H$, either*

(*i*) $r^{-N}|A(re^{i\theta})| \to \infty$ *as* $r \to \infty$, *for each* $N > 0$, *or*

(*ii*) $\int_0^{\infty} r|A(re^{i\theta})|dr < \infty$, *or*

(*iii*) *there exists positive real numbers K and b, and a nonnegative real number n, all possibly depending on θ, such that $(n+2)/2 < \rho(A)$ and $|A(re^{i\theta})| \leq Kr^n$ for all $r \geq b$.*

Let E be the product of two linearly independent solutions of (1.1). *Then* $\lambda(E) = \infty$.

We note that the condition $(n+2)/2 < \rho(A)$ in (*iii*) is sharp by the example (1.3) with the solutions (1.4).

Since $n \geq 0$ we see that (*iii*) can be satisfied only if $\rho(A) > 1$. However, the proof in [5] shows that in (*iii*) one may replace $\rho(A) > (n+2)/2$ by $\rho(E) > (n+2)/2$. Using (1.6) and noting that $\rho(E) \geq \lambda(E)$ we thus also obtain a result for $1/2 < \rho(A) < 1$. A short computation yields the following.

Proposition 1.1. *Let A be a entire function satisfying $1/2 < \rho(A) < 1$. Then the conclusion of Theorem C holds when the condition $(n+2)/2 < \rho(A)$ in (iii) is replaced by $n < \rho(A)/(2\rho(A)-1)$.*

Note that the latter condition is equivalent to

$$\frac{n+2}{2} < 1 + \frac{\rho(A)}{4\rho(A)-2},$$

so for $\rho(A) < 1$ this is indeed a weaker condition. It follows from (1.10) that this modified condition is best possible.

We denote the lower order of an entire function f, which is defined by taking the lower limit in (1.2), by $\mu(f)$. Huang [25] showed that Theorem A, part (ii), and (1.6) hold with $\rho(A)$ replaced by $\mu(A)$. We note that for the function A constructed in the proof of Theorem 1.1 we have $\mu(A) = \rho(A)$.

Toda [44] showed that Theorem A and (1.6), and in fact their refinements obtained by Huang, can be strengthened if the set $\{z \in \mathbb{C}\colon |A(z)| > K\}$ has more than one component for some $K > 0$. Let N be the number of such components. Toda [44, Theorem 3] proved that if $\mu(A) \leq N/2$, then $\lambda(E) = \infty$. Moreover, for $N/2 < \mu(A) < N$ the inequality (1.6) can be improved to

$$\frac{N}{\mu(A)} + \frac{1}{\lambda(E)} \leq 2.$$

Toda actually showed that these results hold for the number N of unbounded components of the set $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$ if $p, K_p > 0$. Note that for $f(z) = z^N + \exp(z^N)$ the set $\{z \in \mathbb{C}\colon |f(z)| > K\}$ is connected for all $K > 0$ while $\{z \in \mathbb{C}\colon |f(z)| > 4|z|^{N+1}\}$ has N unbounded components.

We sharpen Toda's inequality and show that the result obtained is best possible.

Theorem 1.3. *Let A be an entire function, $N \geq 2$ and $p, K_p > 0$. Suppose that the set $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$ has N unbounded components and let E be the product of two linearly independent solutions of the differential equation* (1.1)*. Then $\mu(A) \geq N/2$ and if $\mu(A) < N$, then*

$$\frac{N}{\mu(A)} + \frac{N}{\lambda(E)} \leq 2. \tag{1.11}$$

We note that if $\mu(A) = N$, then we may have $\lambda(E) = 0$ by the examples given in (1.4).

Theorem 1.4. *Let $N \in \mathbb{N}$ and $\rho \in (N/2, N)$. Then there exists an entire function A satisfying $\mu(A) = \rho(A) = \rho$ such that $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$ has N unbounded components for $p = 2N\rho/(2\rho - N) - 2$ and large K_p and such that the differential equation* (1.1) *has two linearly independent solutions whose product E satisfies*

$$\frac{N}{\mu(A)} + \frac{N}{\lambda(E)} = 2. \tag{1.12}$$

Moreover, $\lambda(E) = \rho(E)$ and one of the solutions of (1.1) *is free of zeros.*

One immediate corollary is the following result.

Corollary 1.2. *For every $\rho \in (1/2, \infty)$ there exists an entire function A satisfying $\mu(A) = \rho(A) = \rho$ for which the equation* (1.1) *has two linearly independent solutions w_1 and w_2 such that w_1 has no zeros and $\lambda(w_2) < \infty$.*

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2 Background and underlying ideas

2.1 The Schwarzian derivative and linear differential equations

To every differential equation (1.1) the associated Schwarzian differential equation is given by

$$S(F) := \left(\frac{F''}{F'}\right)' - \frac{1}{2}\left(\frac{F''}{F'}\right)^2 = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = 2A. \tag{2.1}$$

The expression $S(F)$ is called the *Schwarzian derivative* of F. The general solution of (2.1) is $F = w_2/w_1$ where w_1 and w_2 are linearly independent solutions of (1.1). A pair (w_1, w_2) can be recovered from F by the formulas

$$w_1^2 = \frac{1}{F'}, \quad w_2^2 = \frac{F^2}{F'}, \quad \text{and} \quad E = w_1 w_2 = \frac{F}{F'}.$$

It follows that $F' = W(w_1, w_2)/w_1^2$ is free of zeros, and evidently all poles of F are simple. So F is a *locally univalent* meromorphic function. If L is a fractional-linear transformation, then $S(L \circ F) = S(F)$. On the set of all locally univalent meromorphic functions we introduce an equivalence relation by saying that $F_1 \sim F_2$ if $F_1 = L \circ F_2$ for some fractional-linear transformation L. Then the map $F \mapsto A = S(F)/2$ gives a bijection between the set of equivalence classes of locally univalent meromorphic functions and entire functions A.

All these facts were known in the 19-th century; see, e.g., Schwarz's collected papers [39, pp. 351–355], where he also discusses the work of Lagrange, Cayley, Riemann, Klein and others in this context. For a modern exposition, see [38, IV.2.2] or [31, Chapter 6].

We will need some facts about Bank–Laine functions. First of all, we note that the Schwarzian $S(F)$ can be factored as $S(F) = B(F/F')/2$, where

$$B(E) := -2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2}.$$

So the general solution of the differential equation $B(E) = 4A$, that is, the equation

$$4A = -2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2}. \tag{2.2}$$

is a product of a normalized pair of solutions of (1.1). In particular, the general solution E is entire when A is entire, which implies that the general solution is free from movable singularities. All such second order equations linear with respect to the second derivative were classified by Painlevé, Fuchs and Gambier; see, e.g., Ince's book [28, Chapter XIV] for this classification.

The fact that the general solution of (2.2) is the product of a normalized pair of solutions of (1.1) seems to be due to Hermite (see [22] or [48, p. 572]) and can be found already in Julia's collection of exercises [29, Problème no. 33, pp. 193–201] and Kamke's reference book [27, Entry 6.139]. However,

the importance of this equation for the asymptotic study of A and E was shown for the first time by Bank and Laine in [2, 3].

We have seen that for every locally univalent function F, the quotient F/F' is a Bank-Laine function, and all Bank–Laine functions arise in this way. Zeros of a Bank–Laine function E are zeros and poles of F, and E is special if and only if F is entire.

We keep the following permanent notation: (w_1, w_2) is a normalized pair of solutions of (1.1), the quotient $F = w_2/w_1$ is a solution of $S(F) = 2A$, and $E = w_1 w_2 = F/F'$ is a solution of (2.2).

2.2 Description of the method

In this section we describe the underlying ideas of our construction. The formal proofs of our results in sections 3–5 are independent of it. On the other hand, an expert in conformal gluing and the inverse problem of Nevanlinna theory may want to read only this section.

Our approach consists of constructing F by a geometric method (gluing), and then recovering the needed asymptotic properties of A and E. This idea was used for the first time by Nevanlinna [35] and Ahlfors [1]. Nevanlinna solved a special case of the inverse problem of value distribution theory by using the asymptotic theory of the differential equation (1.1) with polynomial coefficients. Ahlfors showed that the same results can be obtained by geometric methods, without appealing to the differential equation (1.1). These two papers initiated a long line of research which culminated in the solution of the inverse problem for functions of finite order with finitely many deficiencies by Goldberg [18, 19]. Further development of these ideas led to the complete solution of the inverse problem by Drasin [14]. The connection with differential equations was not used in this research after [35].

Here we use this connection in the opposite direction to [35]: we construct a locally univalent map geometrically, use a form of the uniformization theorem to obtain F, and then derive asymptotic properties of A and E. There is a subclass of locally univalent maps with especially simple properties: there exists a finite set X such that

$$F\colon \mathbb{C}\backslash F^{-1}(X) \to \overline{\mathbb{C}}\backslash X$$

is a covering. The set of meromorphic functions with this property (not necessarily locally univalent) is called the *Speiser class* and denoted by $\mathcal{S}$;

it plays an important role in holomorphic dynamics [6, 17, 20] and in the general theory of entire and meromorphic functions [13, 19, 15]. We only use the case when $X = \{0, 1, \infty\}$. To visualize functions of class $\mathcal{S}$ one employs a classical tool, line complexes; see [19, Section 7.4] and [36, §11.2]. Consider a graph Γ_0 embedded in the sphere $\overline{\mathbb{C}}$ with two vertices, $\times = i$ and $\circ = -i$, and three edges connecting the two vertices and intersecting the real line exactly once, in the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$, respectively. It is convenient to make this graph symmetric with respect to the real line.

The preimage $\Gamma = F^{-1}(\Gamma_0)$ is called the *line complex*. Its vertices are labeled by $\times$ and $\circ$, and faces are labeled by $0, 1, \infty$, according to their images. Two line complexes are equivalent if there is a homeomorphism of the plane sending one to another, respecting the labels of vertices and faces. In figures like Figure 1 we draw one representative of the equivalence class, usually not the true preimage of Γ_0 under F. The function F is defined by the equivalence class of its line complex up to an affine change of the independent variable.

It is clear that Γ is a bi-partite connected embedded graph, in which every vertex has degree 3. If F is locally univalent, then the faces (that is, the components of $\mathbb{C}\backslash\Gamma$) can be either 2-gons or ∞-gons. Here 2-gons correspond to a-points of F with $a \in \{0, 1, \infty\}$ and ∞-gons to logarithmic singularities of F^{-1}. Every bipartite connected graph with vertices of degree 3 embedded to the plane is a line complex of some function F of class $\mathcal{S}$ with $X = \{0, 1, \infty\}$, meromorphic either in the unit disk or in the plane.

Our functions F will correspond to the class of line complexes shown in Figure 1. They are parametrized by doubly infinite sequences of non-negative integers $(\ell_k)_{k\in\mathbb{Z}}$, showing the numbers of $-\times=\circ-$ links on the vertical pieces between the infinite horizontal branches. The faces are labeled by their images. Zeros of F correspond to 2-gons on the vertical part of the boundaries of faces labeled ∞ and poles to the 2-gons on the vertical parts of the boundaries of faces labeled 0. So our function f is entire if and only if $\ell_k = 0$ for all odd k.

Example 2.1. $F(z) = e^{e^z}$ corresponds to $\ell_k = 0$, $-\infty < k < \infty$.

Example 2.2. $F(z) = P(e^z)e^{e^z}$, where P is a polynomial of degree d, such that F is locally univalent, corresponds to the line complex with $\ell_k = d$ when k is even and $\ell_k = 0$ when k is odd. To see what this polynomial P might be, we differentiate to obtain $F'(z) = (P'(e^z) + P(e^z))e^z e^{e^z}$. This can be free of zeros when $P'(w) + P(w) = cw^d$, which easily implies that, up to a constant

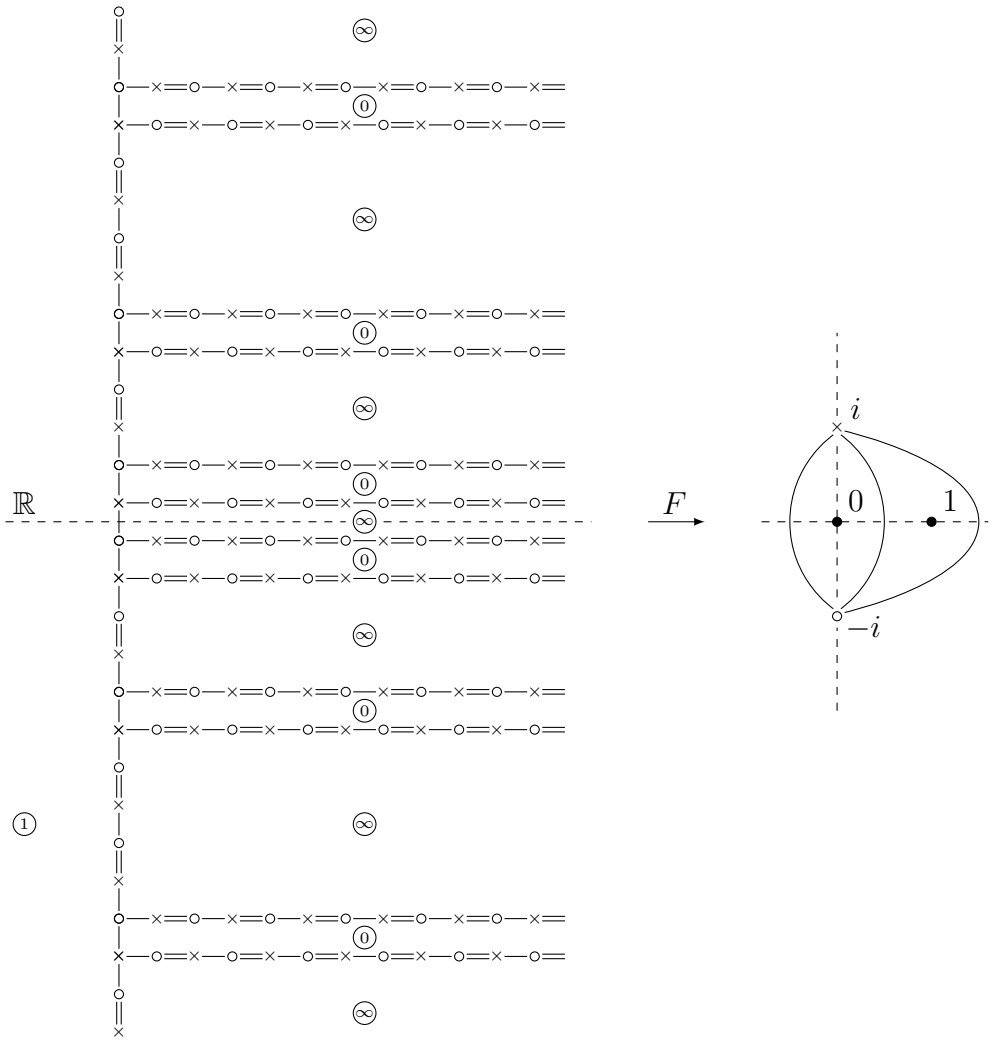

Figure 1: Sketch of the line complex corresponding to $\ell_0 = \ell_{\pm 1} = 0$, $\ell_{\pm 2} = 1$, $\ell_{\pm 3} = 0$, $\ell_{\pm 4} = 2$, and $\ell_{\pm 5} = 0$. The encircled labels indicate to which logarithmic singularities the faces correspond. The graph Γ_0 is shown on the right.

factor,

$$P(w) = \sum_{j=0}^{d} (-1)^j \frac{w^j}{j!},$$

a partial sum of e^{-w}.

We remark that a simple computation using (2.1) shows that the coefficient A corresponding to F is given by

$$A(z) = -\frac{1}{4}e^{2z} - \frac{d}{2}e^{z} - \frac{(d+1)^2}{4}.$$

The case that A in (1.1) has the form $A(z) = R(e^z)$ with a rational function R has been thoroughly studied (see, e.g., [4, 12]) but we shall not use these results.

Example 2.3. $F(z) = R(e^z)e^{e^z}$, where R is a rational function. For F to be locally univalent we need $R'(w)+R(w) = w^p/Q^2(w)$ where Q is a polynomial with distinct roots. Then $R = P/Q$ where $P'Q-PQ'+PQ = cw^p$. Assuming $\deg P = m$, $\deg Q = n$ and $P(0) = Q(0) = 1$ we conclude that $p = m + n$, and P/Q is the (m, n)-Padé approximation of e^{-z}. For this case $\ell_k = n$ when k is even and $\ell_k = m$ when k is odd.

In all these examples F is periodic and the Bank–Laine function $E = F/F'$ is of order 1. In Examples 2.1 and 2.2, the Bank–Laine function is special.

To obtain different orders of E, we consider functions F whose line complex has a non-periodic sequence (ℓ_k). We restrict ourselves to entire functions F and special Bank–Laine functions E, with $\ell_{2k+1} = 0$,

$$\ell_0 = m_1 + m_{-1}, \quad \ell_{2k} = m_k + m_{k+1}, \quad \ell_{-2k} = m_{-k} + m_{-k-1}, \quad k > 0,$$

where $(m_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a sequence of non-negative integers.

The construction of a function F corresponding to such a line complex can be visualized as follows.

Consider the strips

$$\Pi_k = \{x + iy\colon 2\pi(k-1) < y < 2\pi k\}.$$

Let $\mathfrak{F}_k$ be the bordered Riemann surface spread over the plane, which is the image of this strip under the function

$$g_{m_k}(z) = P_{m_k}(e^z)e^{e^z},$$

where $P_{m_k}(w)$ is the partial sum of the Taylor series of e^{-w} of degree $2m_k$, as in Example 2.

All these Riemann surfaces have two boundary components which project onto the ray $(1, +\infty) \in \mathbb{R}$. We glue them together along these rays, in the same order as the strips Π_k are glued together in the plane. The resulting Riemann surface $\mathfrak{F}$ is open and simply connected. Our function F in Theorems 1.1 and 1.2 is the conformal map from $\mathbb{C}$ to $\mathfrak{F}$.

Of course, the uniformization theorem by itself is not sufficient: one has to know that $\mathfrak{F}$ is conformally equivalent to the plane. Moreover, we need to know something about the asymptotic behavior of F to make conclusions about the order of $E = F/F'$ and $A = B(E)$.

So we do the following. Consider the piecewise analytic function g defined by $g(z) = g_{m_k}(z)$ for $z \in \Pi_k$. Each boundary component of the strip Π_k is mapped by g homeomorphically onto the ray $(1, +\infty)$. We shall study the homeomorphisms $\phi_k \colon \mathbb{R} \to \mathbb{R}$ defined by $g_{m_k+1}(\phi_k(x)) = g_{m_k}(x)$. We will see that these homeomorphism are close to the identity on the positive real axis and thus it is easy to glue the restrictions of these functions to the half-strips $\{z \in \Pi_k \colon \operatorname{Re} z > 0\}$ to obtain a quasiregular map U defined in the right half-plane. (For technical reasons we will actually use the functions $u_{m_k}(z) = g_{m_k}(z + s_{m_k})$ instead of $g_{m_k}(z)$, for certain constants s_{m_k}.) The homeomorphisms ϕ_k are close to the linear map $x \mapsto (2m_{k+1}+1)x/(2m_k+1)$ on the negative real axis. In the left half-plane we therefore consider $v_{m_k}(z) = u_{m_k}(z/(2m_k+1))$ instead of u_{m_k} and find that it is easy to glue restrictions of these maps to horizontal half-strips of width $2\pi(m_k+1)$. This way we obtain a quasiregular map V defined in the left half-plane.

We then precompose the maps U and V with appropriate powers and glue the resulting functions to obtain a quasiregular map G. The quasiconformal dilatation K_G of G will satisfy the condition

$$\int_{|z|>1} \frac{K_G(z) - 1}{x^2 + y^2} dx\, dy < \infty. \tag{2.3}$$

Then the theorem of Teichmüller–Wittich–Belinskii [33, §V.6] will guarantee the existence of a homeomorphism $\tau \colon \mathbb{C} \to \mathbb{C}$ with

$$\tau(z) \sim z \quad \text{as } z \to \infty, \tag{2.4}$$

such that $G = F \circ \tau$ for some entire function F. The property (2.4) and explicit estimates for the ϕ_k will give sufficient control of the asymptotic behavior of F to prove the conclusions of Theorems 1.1 and 1.2.

To achieve (2.3) one needs a good control of the homeomorphisms ϕ_k as $x \to \infty$ and as $k \to \infty$. This is achieved by using Szegő-type asymptotics of the partial sums of the exponential which we prove in section 3.

To prove Theorem 1.4, we need a locally univalent function F whose asymptotic behavior is similar to $F_0(z^N)$, where F_0 is the locally univalent function constructed in Theorem 1.1. Of course, $F_0(z^N)$ has a critical point at 0 and thus is not locally univalent. So the idea is to proceed as follows. First we prepare a function F_1 which is similar to F_0, but maps the real line onto $(0, 1)$ reversing the orientation (so it is decreasing on the real line). In fact, the construction would work with $F_1 = 1/F_0$, but then the resulting function F would have poles so that the Bank–Laine function $E = F/F'$ would not be special. Then we glue $F_0(z^N)$ and $F_1(z^N)$ in a suitable way.

To carry out the construction, we will actually work with the quasiregular maps G_0 and G_1 arising in the construction of F_0 and F_1, instead of F_0 and F_1 themselves. Then we consider regions C_j, for $j = 1, \dots, 2N$, which are contained in the sectors $\{z \in \mathbb{C} \colon \pi(j-1)/N < \arg z < \pi j/N\}$ and which are asymptotically close to these sectors; cf. Figure 2.

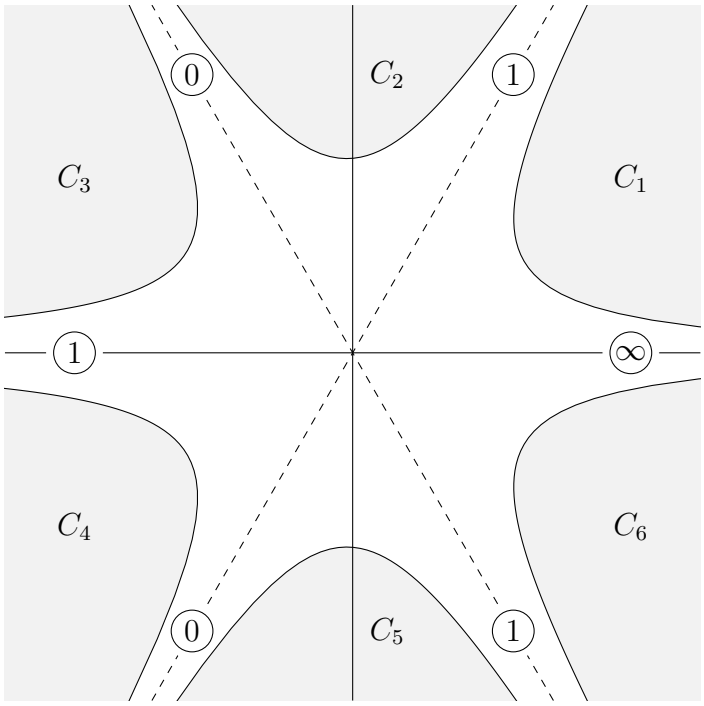

Figure 2: Sketch of the domains C_j for $N = 3$.

For $z \in C_j$ we define $G(z) = G_k(\varphi_j(z^N))$ with some quasiconformal map φ, and $k \in \{0, 1\}$ depending on j. In fact, we will have $G(z) = G_k(z^N)$ in a large subdomain D_j of C_j. Actually, we will take $k = 1$ for $j = 1$ and $j = 2N$ and $k = 0$ otherwise.

By construction, G will map ∂C_j homeomorphically onto one of the intervals $(0, 1)$ and $(1, +\infty)$ of the real line. It remains to define a locally univalent quasiregular function in the complement of $\bigcup_{j=1}^{2N} C_j$ which has these boundary values. In particular, this map will tend to one of the values 0, 1 and ∞ in the strips between the C_j. These values are encircled in Figure 2.

The question whether a locally homeomorphic extension of F to the complement of $\bigcup_{j=1}^{2N} C_j$ with these boundary values exists is a purely topological problem. This is solved in section 5.3. However, we also need the extension to be quasiregular, with dilatation satisfying (2.3). This is achieved by choosing an appropriate shape of the C_j near infinity in section 5.2. Composing the quasiregular map G obtained with a quasiconformal map τ satisfying (2.4) will give our entire function F.

We conclude this section with some general remarks and references. Functions $f \in \mathcal{S}$ are determined by their line complexes labeled by the singular values up to an affine change of the independent variable. It is an important problem to draw conclusions about asymptotic properties of f from the line complex. First of all one has to be able to determine the conformal type of the Riemann surface defined by the line complex [47]. But once the type is determined, one wants to know the asymptotic characteristics like the order of growth, deficiencies, etc. Teichmüller [43] stated the general problem as follows:

Gegeben sei eine einfach zusammenhängende Riemannsche Fläche $\mathfrak{W}$ über der w-Kugel. Man kann sie bekanntlich eineindeutig und konform auf den Einheitskreis $|z| < 1$, auf die punktierte Ebene $z \neq \infty$ oder auf die volle z-Kugel abbilden, so daß w eine eindeutige Funktion von z wird: $w = f(z)$. Die Wertverteilung dieser eindeutigen Funktion ist zu untersuchen.

Dies ist ein Hauptproblem der modernen Funktionentheorie.

This problem has been intensively studied in connection with the inverse problem of value distribution theory [49, 26, 19].

In recent times there is a revival of interest to these questions, which is mainly stimulated by questions of holomorphic dynamics. In this connection, we mention recent remarkable contributions by Bishop [9, 10, 11].

3 Proof of Theorem 1.1

3.1 Preliminary results

Lemma 3.1. *Let $\gamma > 1$. Then there exists a sequence (n_k) of odd positive integers such that the function $h\colon [0,\infty) \to [0,\infty)$ which satisfies $h(0) = 0$ and which is linear on the intervals $[2\pi(k-1), 2\pi k]$ and has slope n_k there satisfies*

$$h(x) = x^\gamma + O(x^{\gamma-2}) + O(1) \tag{3.1}$$

as $x \to \infty$.

Moreover, the function $g\colon [0,\infty) \to [0,\infty)$ defined by $h(g(x)) = x^\gamma$ satisfies

$$g(x) = x + O(x^{-1}) + O(x^{1-\gamma}) \tag{3.2}$$

and

$$g'(x) = 1 + O(x^{-1}) + O(x^{1-\gamma}) \tag{3.3}$$

as $x \to \infty$, where g' denotes either the left or right derivative of g.

Finally,

$$n_k = \gamma(2\pi k)^{\gamma-1} + O(k^{\gamma-2}) + O(1) \tag{3.4}$$

as $k \to \infty$.

Proof. We set $h(0) = 0$ and choose k_0 so that

$$(2\pi k)^\gamma - (2\pi(k-1))^\gamma \geq 4\pi \quad \text{for } k > k_0. \tag{3.5}$$

Such a k_0 exists because $\gamma > 1$.

For $k \leq k_0$ we set $n_k = 1$. Suppose that $k > k_0$, and $h(2\pi(k-1))$ is already defined. Then we define $h(2\pi k) := 2\pi p_k$, where p_k is a positive integer of opposite parity to $h(2\pi(k-1))/(2\pi)$ minimizing $|(2\pi k)^\gamma - 2\pi p_k|$. There are at most two such p, and when there are two, we choose the larger one. Then we interpolate h linearly between $2\pi(k-1)$ and $2\pi k$. Evidently, with this definition,

$$|h(2\pi k) - (2\pi k)^\gamma| \leq 2\pi. \tag{3.6}$$

Next we show that h is strictly increasing. Using (3.5) and (3.6) we have

$$h(2\pi k) \geq (2\pi k)^\gamma - 2\pi > (2\pi(k-1))^\gamma + 2\pi \geq h(2\pi(k-1)).$$

So h is strictly increasing, and its slopes n_k are positive odd integers.

To prove that the function h satisfies (3.1), we note first that $h(2\pi k) = (2\pi k)^\gamma + O(1)$ by construction. For $0 \le t \le 1$ we have

$$\begin{aligned}&(2\pi(k+t))^\gamma - t(2\pi(k+1))^\gamma - (1-t)(2\pi k)^\gamma \\ &= (2\pi k)^\gamma \left(1 + \gamma t/k - t(1+\gamma/k) - 1 + t + O\big(k^{-2}\big)\right) = O(k^{\gamma-2})\end{aligned}$$

as $k \to \infty$, so the straight line connecting the points $(2\pi k, (2\pi k)^\gamma)$ and $(2\pi(k+1), (2\pi(k+1))^\gamma)$ deviates from the graph of the function $x \mapsto x^\gamma$ between the points k and $k+1$ by a term which is $O(k^{\gamma-2})$. This yields (3.1).

To prove (3.2) we note that (3.1) implies that $g(x) = x(1+\delta(x))$ where $\delta(x) \to 0$. Using (3.1) again we see that

$$\begin{aligned}x^\gamma = h(g(x)) &= x^\gamma(1+\delta(x))^\gamma + O(x^{\gamma-2}) + O(1) \\ &= x^\gamma \left(1 + \gamma\delta(x) + O(\delta(x)^2)\right) + O(x^{\gamma-2}) + O(1)\end{aligned}$$

as $x \to \infty$. This yields

$$x^\gamma \delta(x) = O(x^{\gamma-2}) + O(1),$$

from which (3.2) follows.

Similarly we see that

$$h'(x) = \gamma x^{\gamma-1} + O(x^{\gamma-2}) + O(1)$$

and thus

$$h'(g(x)) = \gamma x^{\gamma-1}\,(1+\delta(x))^{\gamma-1} + O(x^{\gamma-2}) + O(1) = \gamma x^{\gamma-1} + O(x^{\gamma-2}) + O(1).$$

Hence

$$g'(x) = \frac{\gamma x^{\gamma-1}}{h'(g(x))} = \frac{1}{1 + O(x^{-1}) + O(x^{1-\gamma})} = 1 + O(x^{-1}) + O(x^{1-\gamma}),$$

which is (3.3).

Finally, by construction we have

$$n_k = \frac{h(2\pi k) - h(2\pi(k-1))}{2\pi} = \frac{(2\pi k)^\gamma - (2\pi(k-1))^{\gamma-1}}{2\pi} + O(1),$$

from which (3.4) easily follows. □

For $m \geq 0$ we will consider the Taylor polynomial

$$P_m(z) = \sum_{k=0}^{2m} (-1)^k \frac{z^k}{k!}$$

of e^{-z} and the functions

$$h_m(z) = P_m(z)e^z$$

and

$$g_m(z) = h_m(e^z) = P_m(e^z) \exp e^z.$$

We shall need some information about the asymptotic behavior of h_m and g_m.

Lemma 3.2. *Let $m \in \mathbb{N}$ and put $n = 2m + 1$. Let $y \geq 0$. Then*

$$\log(h_m(y) - 1) = -\log n! + y + n \log y - \log\Big(1 + \frac{y}{n}\Big) + R(y, n)$$

where

$$|R(y, n)| \leq \frac{24y}{n(n + y)}$$

for $n \geq 24$.

The slightly weaker result that $R(y, n) = O(1/n)$, uniformly in y, can be obtained from the work of Kriecherbauer, Kuijlaars, McLaughlin and Miller [30]. The above estimate is better for small y, which is advantageous for our purposes.

In terms of g_m the estimate of Lemma 3.2 takes the form

$$\log(g_m(x) - 1) = -\log n! + e^x + nx - \log\left(1 + \frac{e^x}{n}\right) + R(e^x, n). \qquad (3.7)$$

Proof of Lemma 3.2. By the formula for the error term of a Taylor series we have

$$e^{-y} - P_m(y) = -\frac{1}{(2m)!} \int_0^y e^{-t}(y - t)^{2m} dt$$

and thus

$$\begin{aligned} h_m(y) - 1 &= \frac{1}{(2m)!}\int_0^y e^{y-t}(y-t)^{2m}dt = \frac{1}{(2m)!}\int_0^y e^u u^{2m}du \\ &= \frac{1}{(2m+1)!}\left(e^y y^{2m+1} - \int_0^y e^u u^{2m+1}du\right) \\ &= \frac{1}{(2m+1)!}e^y y^{2m+1}\left(1 - \int_0^y e^{u-y}\left(\frac{u}{y}\right)^{2m+1}du\right) \\ &= \frac{1}{(2m+1)!}e^y y^{2m+1}\left(1 - y\int_0^1 e^{y(s-1)}s^{2m+1}ds\right). \end{aligned}$$

Since $n = 2m+1$ we thus have

$$\log(h_m(y) - 1) = -\log n! + y + n\log y + \log(1 - yI)\,, \tag{3.8}$$

where

$$I = I(y,n) = \int_0^1 e^{y(s-1)}s^n ds.$$

Since $\log s \le s - 1$ for $s > 0$ we have

$$I \ge \int_0^1 e^{y\log s}s^n ds = \int_0^1 s^{y+n}ds = \frac{1}{y+n+1}. \tag{3.9}$$

For an estimate in the opposite direction we use that

$$\log s \ge s - 1 - (s-1)^2 \quad \text{for } s \ge 1/2. \tag{3.10}$$

With $\delta = \min\{1/2, 1/\sqrt{y}\}$ and $\eta = 1 - \delta$ we write

$$\begin{aligned} I - \frac{1}{y+n+1} &= \int_0^1 e^{y(s-1)}s^n ds - \int_0^1 e^{y\log s}s^n ds \\ &= \int_0^1 \left(e^{y(s-1)}s^n - e^{y\log s}s^n\right)ds \\ &= \int_0^\eta \left(e^{y(s-1)}s^n - e^{y\log s}s^n\right)ds + \int_\eta^1 \left(e^{y(s-1)}s^n - e^{y\log s}s^n\right)ds \\ &= I_1 + I_2. \end{aligned}$$

Now (3.10) yields

$$I_2 = \int_\eta^1 e^{y\log s} s^n \left(e^{y(s-1)-y\log s} s^n - 1\right) ds \le \int_\eta^1 s^{y+n} \left(e^{y(s-1)^2} - 1\right) ds$$

since $\delta \le 1/2$ and thus $s \ge 1/2$ if $s \ge \eta = 1-\delta$. Since $\delta \le 1/\sqrt{y}$ we have $y(s-1)^2 \le 1$ for $s \ge 1-\delta$. Noting that $e^t - 1 \le 2t$ for $0 \le t \le 1$ we obtain

$$\begin{aligned} I_2 &\le 2\int_\eta^1 s^{y+n} y(s-1)^2 ds \le 2\int_0^1 s^{y+n} y(s-1)^2 ds \\ &= \frac{4y}{(y+n+1)(y+n+2)(y+n+3)} \le \frac{4}{(y+n)^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} I_1 &\le \int_0^{1-\delta} e^{y(s-1)} s^n ds = \int_0^{1-\delta} e^{y(s-1)+n\log s} ds \\ &\le \int_0^{1-\delta} e^{(y+n)(s-1)} ds = \frac{1}{y+n}\left(e^{-\delta(y+n)} - e^{-(y+n)}\right) \le \frac{1}{y+n} e^{-\delta(y+n)}. \end{aligned}$$

Since $\delta = \min\{1/2, 1/\sqrt{y}\} \ge 1/(2\sqrt{y+n})$ and since $e^t \ge t^2/2$ and hence $e^{-t} \le 2/t^2$ for $t \ge 0$ this yields

$$I_1 \le \frac{1}{y+n} e^{-\sqrt{y+n}/2} \le \frac{8}{(y+n)^2}.$$

Combining the bounds for I_1 and I_2 with (3.9) we obtain

$$\frac{1}{y+n+1} \le I_1 \le \frac{1}{y+n+1} + \frac{12}{(y+n)^2}.$$

Since

$$\frac{1}{y+n} - \frac{1}{y+n+1} = \frac{1}{(y+n)(y+n+1)} \le \frac{1}{(y+n)^2}$$

we obtain

$$\left| I - \frac{1}{y+n} \right| \le \frac{12}{(y+n)^2}.$$

Combining this with (3.8) we find that

$$\log(h_m(y) - 1) = -\log n! + y + n\log y + \log\left(\frac{n}{y+n} - r(y,n)\right),$$

where

$$|r(y,n)| \le \frac{12y}{(y+n)^2}.$$

Since

$$\log\left(\frac{n}{y+n} - r(y,n)\right) = -\log\left(1+\frac{y}{n}\right) + \log\left(1+\frac{(y+n)r(y,n)}{n}\right)$$

we see that $\log(h_m(x)-1)$ has the form given with

$$R(y,n) = \log\left(1+\frac{(y+n)r(y,n)}{n}\right).$$

To prove the estimate for $R(y,n)$ that was stated in the conclusion we note that $|\log(1+t)| \le 2t$ for $|t| \le 1/2$ and

$$\left|\frac{(y+n)r(y,n)}{n}\right| \le \frac{12y}{n(y+n)} \le \frac{12}{n} \le \frac{1}{2}$$

for $n \ge 24$. It follows that

$$|R(y,n)| = \left|\log\left(1+\frac{(y+n)r(y,n)}{n}\right)\right| \le 2\left|\frac{(y+n)r(y,n)}{n}\right| \le \frac{24y}{n(y+n)}$$

for $n \ge 24$ as claimed. □

Note that

$$g_m'(z) = \frac{1}{(2m)!}\exp(e^z + (2m+1)z) \ne 0 \tag{3.11}$$

so that $g_m\colon \mathbb{R} \to (1,\infty)$ is an increasing homeomorphism. Thus there exists $s_m \in \mathbb{R}$ such that

$$g_m(s_m) = 2. \tag{3.12}$$

Lemma 3.3. *Let $r_0 = -1.27846454\ldots$ be the unique real solution of the equation $e^{r_0} + r_0 + 1 = 0$. Then*

$$s_m = \log n + r_0 - \frac{1}{2r_0}\frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

as $m \to \infty$, with $n = 2m+1$.

Proof. We write $s_m = \log n + r$. Then (3.7) and Stirling's formula yield

$$\begin{aligned} 0 &= -\log n! + ne^r + n\log n + nr - \log(1+e^r) + O\left(\frac{1}{n}\right) \\ &= n(e^r + r + 1) - \log(1+e^r) - \frac{1}{2}\log n - \frac{1}{2}\log 2\pi + O\left(\frac{1}{n}\right). \end{aligned}$$

This implies that $r = r_0 + o(1)$. We write $r = r_0 + t$ so that $t = o(1)$ as $m \to \infty$. We obtain

$$0 = -nr_0 t - \frac{1}{2}\log n + O(1),$$

from which the conclusion easily follows. □

Let now $m, M \in \mathbb{N}$ with $M > m$. We consider the function $\phi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g_M(x) = g_m(\phi(x)).$$

We will consider the functions ϕ for the case that $m = m_{k-1}$ and $M = m_k$ for the sequence (m_k) such that $n_k = 2m_k + 1$, where (n_k) was constructed in Lemma 3.1.

Thus we will consider the behavior of ϕ as $m \to \infty$, but in order to simplify the formulas we suppress the dependence of ϕ from m and M from the notation. We shall assume that there exists a constant $C > 1$ such that $M \le Cm$. We put $n = 2m + 1$ and $N = 2M + 1$. Then clearly

$$N \le Cn. \tag{3.13}$$

In the following, the constants appearing in the Landau notation $O(1/n)$ and $O(1)$ will depend on C, but not on other variables.

Lemma 3.4. *The function ϕ has a unique fixed point p which satisfies*

$$p = \log n + \frac{N \log \frac{N}{n}}{N - n} - 1 + O\left(\frac{1}{n}\right). \tag{3.14}$$

as $m \to \infty$. Moreover, $\phi(x) < x$ for $x < p$ and $\phi(x) > x$ for $x > p$.

Proof. Let $d\colon \mathbb{R} \to \mathbb{R}$, $d(x) = g_M(x) - g_m(x)$. We have to show that d has a unique zero. Now

$$d'(x) = \left(\frac{1}{(2M)!}e^{2(M-m)x} - \frac{1}{(2m)!}\right) e^{(2m+1)x} e^{e^x},$$

from which we see that the derivative has one sign change, namely at the point

$$q = \frac{1}{2(M-m)} \log \frac{(2M)!}{(2m)!}.$$

Moreover, it follows from (3.7) that $d(x) < 0$ if x is negative and of sufficiently large modulus. So d decreases and stays negative on the left of q, and then increases to $+\infty$ on the right of q. We conclude that h has exactly one zero $p > q$, which is the fixed point of ϕ.

To determine the asymptotic behavior of p as $m \to \infty$ we note that (3.7) implies that

$$-\log N! + Np - \log\left(1 + \frac{e^p}{N}\right) = -\log n! + np - \log\left(1 + \frac{e^p}{n}\right) + O\left(\frac{1}{n}\right).$$

We write $p = \log n + r$. It follows that

$$\begin{aligned} & (N-n)r \\ = \; & \log \frac{N!}{n!} - (N-n)\log n + \log\left(1 + \frac{n}{N}e^r\right) - \log(1+e^r) + O\left(\frac{1}{n}\right) \\ = \; & \log \frac{N!}{n!} - (N-n)\log n + \log\left(1 - \frac{N-n}{N}\frac{e^r}{1+e^r}\right) + O\left(\frac{1}{n}\right). \end{aligned}$$

Now

$$\left|\log\left(1 - \frac{N-n}{N}\frac{e^r}{1+e^r}\right)\right| \le \left|\log\left(1 - \frac{N-n}{N}\right)\right| \le c_0 \frac{N-n}{N},$$

with a constant c_0 depending only on the constant C appearing in (3.13). Hence

$$r = \frac{1}{N-n} \log \frac{N!}{n!} - \log n + O\left(\frac{1}{n}\right).$$

An application of Stirling's formula now yields

$$r = \frac{N \log \frac{N}{n}}{N-n} - 1 + \frac{\log \frac{N}{n}}{N-n} + O\left(\frac{1}{n}\right).$$

Since

$$\frac{\log \frac{N}{n}}{N-n} = \frac{\log \frac{N}{n}}{n\left(\frac{N}{n} - 1\right)} \le \frac{1}{n}$$

we actually have

$$r = \frac{N \log \frac{N}{n}}{N-n} - 1 + O\left(\frac{1}{n}\right),$$

so that (3.14) follows. □

Remark. We can write the asymptotic formula for r also in the form

$$r = \frac{\frac{N}{n} \log \frac{N}{n}}{\frac{N}{n} - 1} - 1 + O\left(\frac{1}{n}\right).$$

Since the function $x \mapsto (x \log x)/(x-1)$ is increasing on the interval $(1, C]$ and since $\lim_{x\to 1}(x \log x)/(x-1) = 1$ we deduce that $p = \log n + O(1)$. If $N/n \to 1$ as it will be the case in our application, we even have $p = \log n + o(1)$.

Lemma 3.5. *For $m, M, n, N \in \mathbb{N}$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ with the unique fixed point p as before there exist positive constants $c_1, \ldots, c_8$, depending only on the constant C in* (3.13)*, such that*

$$0 < \phi(x) - x \le c_1 e^{-x/2} \le \frac{c_1}{n} \quad \text{for } x > 8 \log n \tag{3.15}$$

and

$$\left|\phi(x) - \frac{N}{n}x + \frac{1}{n}\log\frac{N!}{n!}\right| \le c_2 e^x \quad \text{for } x < p. \tag{3.16}$$

Moreover, with s_m defined by (3.12) *we have*

$$|\phi(x) - x| \le c_3 \quad \text{for } x > s_m \tag{3.17}$$

and

$$\left|\phi(x) - \frac{N}{n}x - \frac{1}{n}\log\frac{N!}{n!}\right| \le c_4 \quad \text{for } x < p. \tag{3.18}$$

Finally,

$$|\phi'(x) - 1| \le c_5\left(e^{-x/2} + \frac{1}{n}\right) \le \frac{2c_5}{n} \quad \text{for } x > 8 \log n \tag{3.19}$$

and

$$\left|\phi'(x) - \frac{N}{n}\right| \le \frac{c_6 e^x}{n} \quad \text{for } x < p \tag{3.20}$$

as well as

$$c_7 \le |\phi'(x)| \le c_8 \quad \text{for all } x \in \mathbb{R}. \tag{3.21}$$

Remark. Since $s_m < p$ for large m we may assume that the constants c_2, c_4, c_6 are chosen such that (3.16), (3.18) and (3.20) also hold for $x < s_m$.

Proof of Lemma 3.5. Let $y = \phi(x)$ so that $g_M(x) = g_m(y)$. By (3.7) we have

$$\begin{aligned} &- \log N! + e^x + Nx - \log\left(1 + \frac{e^x}{N}\right) + R(e^x, N) \\ = \; &- \log n! + e^y + ny - \log\left(1 + \frac{e^y}{n}\right) + R(e^y, n). \end{aligned} \tag{3.22}$$

Suppose first that $x < p$. Then $y < x$ and thus

$$\begin{aligned} &\left| y - \frac{N}{n}x + \frac{1}{n}\log\frac{N!}{n!} \right| \\ &= \frac{1}{n}\left| e^x - e^y - \log\left(1 + \frac{e^x}{N}\right) + R(e^x, N) + \log\left(1 + \frac{e^y}{n}\right) - R(e^y, n) \right| \\ &\le \frac{1}{n}\left(e^x + e^y + \frac{e^x}{N} + \frac{20e^x}{N^2} + \frac{e^y}{n} + \frac{20e^y}{n^2} \right) \le \frac{44e^x}{n}. \end{aligned}$$

This yields (3.16) and, since $p = \log n + O(1)$, also (3.18).

Suppose now that $x > p$. Using

$$\log\left(1 + \frac{e^x}{N}\right) = x - \log N + \log\left(1 + \frac{N}{e^x}\right)$$

and the corresponding formula for $\log(1 + e^y/n)$ we may write (3.17) in the form

$$\begin{aligned} &- \log N! + e^x + (N-1)x + \log N - \log\left(1 + \frac{N}{e^x}\right) + R(e^x, N) \\ = \; &- \log n! + e^y + (n-1)y + \log n - \log\left(1 + \frac{n}{e^y}\right) + R(e^y, n). \end{aligned} \tag{3.23}$$

We write $x = \log n + s$ and $y = \log n + t$ and note that, since $x > p$, we have $t > s \ge -O(1)$ by Lemma 3.4 and the remark following it. We obtain

$$\begin{aligned} &- \log N! + ne^s + (N-1)(\log n + s) + \log N - \log\left(1 + \frac{N}{ne^s}\right) \\ &= - \log n! + ne^t + (n-1)(\log n + t) + \log n - \log\left(1 + \frac{1}{e^t}\right) + O\left(\frac{1}{n}\right) \end{aligned}$$

and hence, using Stirling's formula,

$$\begin{aligned} e^t - e^s &= \frac{1}{n}\left(-\log\frac{N!}{n!} + (N-n)\log n + \log\frac{N}{n}\right) \\ &\quad + \left(\frac{N}{n} - \frac{1}{n}\right)s - \left(1 - \frac{1}{n}\right)t + O\left(\frac{1}{n}\right) \\ &\le \frac{N}{n}s + O(1) \le Cs + O(1). \end{aligned}$$

It follows that

$$0 < \phi(x) - x = t - s \le e^{t-s} - 1 \le Cse^{-s} + O(e^{-s}) = O(1) \quad \text{for } x > p. \tag{3.24}$$

For $x > 8\log n$ we have $s = x - \log n > 3x/4$ and thus $2s/3 > x/2$. It follows that $se^{-s} = O(e^{-2s/3}) = O(e^{-x/2})$ and thus (3.24) yields (3.15). Noting that $p - s_m = O(1)$ by Lemma 3.3 we can also deduce (3.17) from (3.24).

Since $g'_M(x) = g_m(y)\phi'(x)$ by the chain rule, (3.11) yields

$$\phi'(x) = \frac{(n-1)!}{(N-1)!}\exp(e^x + Nx - e^y - ny)$$

and thus

$$\log\phi'(x) = \log n! - \log N! - \log n + \log N + e^x + Nx - e^y - ny.$$

Using (3.23) and (3.24) we obtain

$$\log\phi'(x) = y - x + \log\left(1 + \frac{N}{e^x}\right) - \log\left(1 + \frac{n}{e^y}\right) + O\left(\frac{1}{n}\right)$$

for $x > p$. Together with (3.15) and (3.17), and since for all $A > 0$ there exists $B > 0$ such that $|t-1| \le B|\log t|$ whenever $|\log t| \le A$, the same arguments as the ones used before now yield (3.19), as well as

$$|\log\phi'(x)| = O(1) \quad \text{for } x > p. \tag{3.25}$$

Similarly (3.22) yields

$$\log\phi'(x) = \log\frac{N}{n} + \log\left(1 + \frac{e^x}{N}\right) - \log\left(1 + \frac{e^y}{n}\right) - R(e^x, N) + R(e^y, n)$$

and hence

$$\log\left(\frac{n}{N}\phi'(x)\right) = O\left(\frac{e^x}{n}\right) \quad \text{for } x < p.$$

This yields (3.20) as well as $|\log\phi'(x)| = O(1)$ for $x < p$ which together with (3.25) yields (3.21). □

3.2 Definition of a quasiregular map

The idea is to construct an entire function by gluing functions g_m with different values of m appropriately. Actually, we will first modify the functions g_m slightly to obtain closely related functions u_m and v_m. We then glue restrictions of these maps to half-strips along horizontal lines to obtain quasiregular maps U and V which are defined in the right and left half-plane. Then we will glue these functions along the imaginary axis to obtain a quasiregular map G in the whole plane.

In the next section we will show that the map constructed satisfies the hypothesis of the Teichmüller–Wittich–Belinskii theorem.

The maps U, V and G will commute with complex conjugation, so it will be enough to define them in the upper half-plane. We begin by constructing the map U.

Instead of g_m we consider the map

$$u_m\colon \{z \in \mathbb{C}\colon \operatorname{Re} z \geq 0\} \to \mathbb{C}, \quad u_m(z) = g_m(z + s_m).$$

Note that u_m is increasing on the real line, and maps $[0, \infty)$ onto $[2, \infty)$.

Let (n_k) be the sequence from Lemma 3.1 and write $n_k = 2m_k + 1$. Basically, we would like to put $U(z) = u_{m_k}(z)$ in the half-strip

$$\Pi_k^+ = \{x + iy : x > 0, 2\pi(k-1) < y < 2\pi k\}.$$

However, this function U will be discontinuous. In order to obtain a continuous function we consider the function $\psi_k\colon [0, \infty) \to [0, \infty)$ defined by $u_{m_{k+1}}(x) = u_{m_k}(\psi_k(x))$. This function ψ_k is closely related to the functions ϕ considered in Lemmas 3.4 and Lemma 3.5. In fact, denote by ϕ_k the function ϕ corresponding to $m = m_k$ and $M = m_{k+1}$. Then

$$\psi_k(x) = \phi_k(x + s_{m_{k+1}}) - s_{m_k}. \tag{3.26}$$

We then define $U\colon \{z \in \mathbb{C}\colon \operatorname{Re} z \geq 0\} \to \mathbb{C}$ by interpolating between $u_{m_{k+1}}$ and u_{m_k} as follows: if $2\pi(k-1) \leq y < 2\pi k$, say $y = 2\pi(k-1) + 2\pi t$ where $0 \leq t < 1$, then we put

$$U(x + iy) = u_{m_k}((1-t)x + t\psi_k(x) + iy) = u_{m_k}(x + iy + t(\psi_k(x) - x)).$$

Actually, by $2\pi i$-periodicity we have

$$U(x + iy) = u_{m_k}((1-t)x + t\psi_k(x) + 2\pi i t).$$

The function U defined this way is continuous in the right half-plane.

We now define a function V in the left half-plane. In order to do so, we define

$$v_m\colon \{z \in \mathbb{C}\colon \operatorname{Re} z \leq 0\} \to \mathbb{C}, \quad v_m(z) = g_m\left(\frac{z}{2m+1} + s_m\right).$$

Note that v_m maps $(-\infty, 0]$ monotonically onto $(1, 2]$.

Let (n_k) and (m_k) be as before and put $N_k = \sum_{j=1}^{k} n_j$, with $N_0 = 0$. This time we would like to define $V(z) = v_{m_k}(z)$ in the half-strip

$$\{x + iy : x \leq 0,\ 2\pi N_{k-1} \leq y < 2\pi N_k\},$$

but again this function would be discontinuous, so in order to obtain a continuous function we again interpolate between $v_{m_{k+1}}$ and v_{m_k}. Similarly as before we consider the map $\psi_k\colon (-\infty, 0] \to (-\infty, 0]$ defined by $v_{m_{k+1}}(x) = v_{m_k}(\psi_k(x))$. Then we define $V\colon \{z \in \mathbb{C}\colon \operatorname{Re} z \leq 0\} \to \mathbb{C}$ by interpolating between $v_{m_{k+1}}$ and v_{m_k} as follows: if $2\pi N_{k-1} \leq y < 2\pi N_k$, say $y = 2\pi N_{k-1} + 2\pi n_k t$ where $0 \leq t < 1$, then we put

$$V(x + iy) = v_{m_k}((1-t)x + t\psi_k(x) + iy) = v_{m_k}(x + iy + t(\psi_k(x) - x)).$$

This map V is continuous in the left half-plane.

Now we define our map G by gluing U and V along the imaginary axis. In order to do this we note that by construction we have $U(iy) = V(ih(y))$ and thus $U(ig(y)) = V(iy^\gamma)$ for $y \geq 0$, with the maps h and g from Lemma 3.1. Let now Q be the homeomorphism of the right half-plane

$$H^+ = \{z \in \mathbb{C}\colon \operatorname{Re} z \geq 0\}$$

onto itself which satisfies $Q(\pm iy) = ig(y))$ for $y \geq 0$, which is the identity for $\operatorname{Re} z \geq 1$, and which is defined by interpolation in between; that is,

$$Q(x + iy) = \begin{cases} x + iy & \text{if } x \geq 1, \\ x + i((1-x)g(y) + xy) & \text{if } 0 \leq x \leq 1. \end{cases} \tag{3.27}$$

Then the map $W = U \circ Q$ satisfies $W(iy) = U(ig(y)) = V(iy^\gamma)$.

Let now $\rho \in (1/2, 1)$ as in the hypothesis of Theorem 1.1. We choose $\gamma = 1/(2\rho - 1)$ in the above construction and put $\sigma = \rho\gamma = \rho/(2\rho - 1)$. The

hypothesis that $1/2 < \rho < 1$ corresponds to $\gamma > 1$ as well as $\sigma > 1$. The map

$$G(z) = \begin{cases} W(z^\rho) & \text{if } |\arg z| \le \dfrac{\pi}{2\rho}, \\ V(-(-z)^\sigma) & \text{if } |\arg(-z)| \le \dfrac{\pi}{2\sigma}, \end{cases}$$

is continuous in $\mathbb{C}$. Here, for $\eta > 0$, we denote by z^η the principal branch of the power which is defined in $\mathbb{C}\backslash(-\infty, 0]$.

3.3 Estimation of the dilatation

We will use the Teichmüller–Wittich–Belinskii theorem stated in section 2.2 to show that the map G defined in the previous section has the form $G(z) = F(\tau(z))$ with an entire function F and a homeomorphism τ satisfying $\tau(z) \sim z$ as $z \to \infty$.

For a quasiregular map f, let

$$\mu_f(z) = \frac{f_{\overline{z}}(z)}{f_z(z)} \quad \text{and} \quad K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

In order to apply the Teichmüller–Wittich–Belinskii theorem, we have to estimate $K_G(z) - 1$. We note that

$$K_G(z) - 1 = \frac{2|\mu_G(z)|}{1 - |\mu_G(z)|} = \frac{2|\mu_G(z)|(1 + |\mu_G(z)|)}{1 - |\mu_G(z)|^2} \le \frac{4|\mu_G(z)|)}{1 - |\mu_G(z)|^2}.$$

We begin by estimating $K_U(z) - 1$. Let $2\pi(k-1) \le y < 2\pi k$ so that $y = 2\pi(k-1) + 2\pi t$ where

$$0 \le t = \frac{y}{2\pi} - (k-1) < 1.$$

Then $U(z) = u_{m_k}(q(z))$ where

$$\begin{aligned} q(x+iy) &= x + iy + t(\psi_k(x) - x) \\ &= x + iy + \left(\frac{y}{2\pi} - (k-1)\right)(\psi_k(x) - x). \end{aligned} \tag{3.28}$$

Thus

$$q_z(z) = 1 + a(z) - ib(z) \quad \text{and} \quad q_{\overline{z}}(z) = a(z) + ib(z)$$

with

$$a(x+iy) = \frac{t}{2}(\psi_k'(x)-1) \quad \text{and} \quad b(x+iy) = \frac{1}{4\pi}(\psi_k(x)-x).$$

This yields

$$|\mu_U(z)|^2 = |\mu_q(z)|^2 = \frac{a(z)^2+b(z)^2}{(1+a(z))^2+b(z)^2}$$

and thus

$$\begin{aligned} K_U(z)-1 &\le \frac{4|\mu_q(z)|)}{1-|\mu_q(z)|^2} \\ &= \frac{4\sqrt{(1+a(z))^2+b(z)^2}\sqrt{a(z)^2+b(z)^2}}{1+2a(z)} \\ &= \frac{4(1+a(z)+|b(z)|)(|a(z)|+|b(z)|)}{1+2a(z)}. \end{aligned}$$

Note that if $a(z) < 0$, then

$$a(z) = \frac{t}{2}(\psi_k'(x)-1) \ge \frac{1}{2}(\psi_k'(x)-1)$$

and thus

$$1+2a(z) \ge \psi_k'(x) > 0.$$

Thus we have

$$1+2a(z) \ge \min\{1, \psi_k'(x)\} > 0$$

in any case. Altogether we find that

$$K_U(z)-1 \le \frac{4(1+|\psi_k'(x)-1|+|\psi_k(x)-x|)(|\psi_k'(x)-1|+|\psi_k(x)-x|)}{\min\{1,\psi_k'(x)\}}.$$

With

$$r(x) = |\psi_k'(x)-1|+|\psi_k(x)-x| \tag{3.29}$$

we thus have

$$K_U(z)-1 \le \frac{4(1+r(x))r(x)}{\min\{1,\psi_k'(x)\}}. \tag{3.30}$$

We shall use Lemma 3.5 to estimate the terms occurring here.

Let now

$$S_k = \{x+iy\colon 2\pi(k-1) < y < 2\pi k,\ x > 8\log n_k\}.$$

In order to estimate $K_U(z) - 1$ for $z \in S_k$ we note that Lemma 3.3 yields that that if $x \in S_k$, then $x + s_{m_{k+1}} > 8 \log n_k$ for large k. For such k we deduce from (3.4), (3.15), (3.26) and Lemma 3.3 that

$$\begin{aligned} |\psi_k(x) - x| &= |\phi_k(x + s_{m_{k+1}}) - (x + s_{m_{k+1}}) + s_{m_{k+1}} - s_{m_k}| \\ &\leq \frac{c_1}{n_k} + |\log n_{k+1} - \log n_k| + \frac{1}{2|r_0|} \left| \frac{\log n_{k+1}}{n_{k+1}} - \frac{\log n_k}{n_k} \right| \\ &= O\left(\frac{1}{k^\delta}\right), \end{aligned}$$

with $\delta = \min\{1, \gamma - 1\}$. By (3.4) and (3.19) we also have

$$|\psi_k'(x) - 1| = |\phi_k'(x + s_{m_{k+1}}) - 1| = O\left(\frac{1}{n_k}\right) = O\left(\frac{1}{k^\delta}\right).$$

The last two inequalities imply that $|r(x)| = O(1/k^\delta)$ and hence

$$K_U(z) - 1 = O\left(\frac{1}{k^\delta}\right) \quad \text{for } z \in S_k.$$

by (3.21) and (3.30).

Now

$$\begin{aligned} \int_{S_k} \frac{dx\, dy}{x^2 + y^2} &\leq \int_{S_k} \frac{dx\, dy}{x^2 + 4\pi^2(k-1)^2} \\ &\leq 2\pi \int_0^\infty \frac{dx}{x^2 + 4\pi^2(k-1)^2} = \frac{\pi}{2(k-1)} \leq \frac{\pi}{k} \end{aligned}$$

for $k \geq 2$ and

$$\int_{S_1} \frac{dx\, dy}{x^2 + y^2} \leq 2\pi \int_1^\infty \frac{dx}{x^2} = 2\pi.$$

Combining the last three inequalities and noting that $W(z) = U(z)$ for $z \in S_k$ we deduce that

$$\int_{S_k} \frac{K_W(z) - 1}{x^2 + y^2} dx\, dy = \int_{S_k} \frac{K_U(z) - 1}{x^2 + y^2} dx\, dy \leq \frac{A_1}{k^{1+\delta}} \tag{3.31}$$

for some constant A_1.

Combining (3.17) and (3.21) with (3.30) we also see that U is quasiregular in the right half-plane H^+.

Next we show that Q defined in (3.27) is quasiconformal. Similarly as in the computation of μ_U we find that

$$Q_z(z) = 1 + a(z) - ib(z) \quad \text{and} \quad Q_{\overline{z}}(z) = a(z) + ib(z),$$

where, for $y > 0$,

$$a(x+iy) = \frac{1-x}{2}(g'(y) - 1) \quad \text{and} \quad b(x+iy) = \frac{1}{2}(g(y) - y)$$

and thus

$$|\mu_Q(z)|^2 = \frac{a(z)^2 + b(z)^2}{(1+a(z))^2 + b(z)^2} = \frac{a(z)^2 + b(z)^2}{1 + 2a(z) + a(z)^2 + b(z)^2}.$$

Since $g(y) - y = o(1)$ and $g'(y) - 1 = o(1)$ as $y \to \infty$ by Lemma 3.1, we conclude a and b are bounded. Moreover, $\inf_{y \geq 0} g'(y) > 0$, which implies that $\inf_{z \in H^+} |1 + 2a(z)| > 0$. We deduce that Q is indeed quasiconformal in H^+. Hence $W = U \circ Q$ is quasiregular in H^+.

We put

$$S'_k = \{x + iy \colon 2\pi(k-1) < y < 2\pi k,\ 0 < x < 8 \log n_k\}.$$

Then

$$\int_{S'_k} \frac{dx\,dy}{x^2 + y^2} \leq \frac{1}{4\pi^2(k-1)^2} \int_{S_k} dx\,dy = \frac{8 \log n_k}{2\pi(k-1)^2} \leq A_2 \frac{\log k}{k^2}$$

for some constant A_2 in view of (3.4). Let K be the dilatation of W in H^+; that is, $K = \sup_{z \in H^+} K_W(z)$. We conclude that

$$\int_{S'_k} \frac{K_W(z) - 1}{x^2 + y^2} dx\,dy \leq (K-1) A_2 \frac{\log k}{k^2}. \tag{3.32}$$

Let now $R > 2\pi + 8 \log n_1$ so that $\{z \in \mathbb{C} \colon |z| > R\} \cap S'_1 = \emptyset$. We deduce from (3.31) and (3.32) that

$$\begin{aligned}
&\int_{|z|>R} \frac{K_W(z) - 1}{x^2 + y^2} dx\,dy \\
&\leq \sum_{k=2}^{\infty} \int_{S'_k} \frac{K_W(z) - 1}{x^2 + y^2} dx\,dy + \sum_{k=1}^{\infty} \int_{S_k} \frac{K_W(z) - 1}{x^2 + y^2} dx\,dy \\
&\leq (K-1) A_2 \sum_{k=2}^{\infty} \frac{\log k}{k^2} + A_1 \sum_{k=1}^{\infty} \frac{1}{k^{1+\delta}} dx\,dy < \infty.
\end{aligned}$$

For $|\arg z| < \pi/(2\rho)$ we have $G(z) = W(z^\rho)$. It follows that

$$\begin{aligned}\int_{\substack{|\arg z|<\frac{\pi}{2\rho}\\ |z|>R^{1/\rho}}} \frac{K_G(z)-1}{x^2+y^2}dx\,dy &= \int_{\substack{|\arg z|<\frac{\pi}{2\rho}\\ |z|>R^{1/\rho}}} \frac{K_W(z^\rho)-1}{x^2+y^2}dx\,dy \\ &= \frac{1}{\rho}\int_{\substack{z\in H^+\\ |z|>R}} \frac{K_W(z)-1}{x^2+y^2}dx\,dy < \infty.\end{aligned} \tag{3.33}$$

The estimate of $K_G(z)$ for $\arg(-z) < \pi/(2\sigma)$ is similar. Instead of (3.29) and (3.30) we obtain

$$K_V(z) - 1 \le \frac{4(1+r(x))r(x)}{\min\{1, \psi_k'(x)\}}. \tag{3.34}$$

with

$$r(x) = |\psi_k'(x) - 1| + \frac{1}{n_k}|\psi_k(x) - x| \tag{3.35}$$

and

$$\psi_k(x) = n_k\phi_k\left(\frac{z}{n_{k+1}} + s_{m_{k+1}}\right) - n_k s_{m_k}.$$

Now

$$\begin{aligned}\frac{\psi_k(x)-x}{n_k} &= \phi_k\left(\frac{x}{n_{k+1}} + s_{m_{k+1}}\right) - \frac{n_{k+1}}{n_k}\left(\frac{x}{n_{k+1}} + s_{m_{k+1}}\right) + \frac{1}{n_k}\log\frac{n_{k+1}!}{n_k!} \\ &\quad + \frac{n_{k+1}}{n_k}s_{m_{k+1}} - s_{m_k} - \frac{1}{n_k}\log\frac{n_{k+1}!}{n_k!}.\end{aligned}$$

It follows from Lemma 3.3 and (3.4) that

$$\left|\frac{n_{k+1}}{n_k}s_{m_{k+1}} - s_{m_k} - \frac{1}{n_k}\log\frac{n_{k+1}!}{n_k!}\right| = O\left(\frac{1}{k^\delta}\right).$$

Since $x/n_{k+1} + s_{m_{k+1}} \le -\log n_k$ for $x \le -2n_k\log n_k$ and sufficiently large k by Lemma 3.3, Lemma 3.4 and (3.4), we deduce from (3.16) that

$$\begin{aligned}&\left|\phi_k\left(\frac{x}{n_{k+1}} + s_{m_{k+1}}\right) - \frac{n_{k+1}}{n_k}\left(\frac{x}{n_{k+1}} + s_{m_{k+1}}\right) + \frac{1}{n_k}\log\frac{n_{k+1}!}{n_k!}\right| \\ &\le c_2\exp\left(\frac{x}{n_{k+1}+s_{m_{k+1}}}\right) \le \frac{c_2}{n_k} \quad \text{for } x \le -2n_k\log n_k\end{aligned}$$

and large k. The last three inequalities yield that

$$\frac{1}{n_k}|\psi_k(x) - x| = O\left(\frac{1}{k^\delta}\right) \quad \text{for } x \le -2n_k \log n_k. \tag{3.36}$$

Moreover, the above arguments in conjunction with (3.18) show that

$$\frac{1}{n_k}|\psi_k(x) - x| = O(1) \quad \text{for } -2n_k \log n_k < x < 0. \tag{3.37}$$

Next we note that

$$\psi_k'(x) = \frac{n_k}{n_{k+1}}\phi_k'\left(\frac{z}{n_{k+1}} + s_{m_{k+1}}\right)$$

From (3.20) we can now deduce that

$$|\psi_k'(x) - 1| = O\left(\frac{1}{k}\right) \quad \text{for } x \le -2n_k \log n_k.$$

and

$$|\psi_k'(x) - 1| = O(1) \quad \text{for } -2n_k \log n_k < x < 0.$$

Combining the last two inequalities with (3.34), (3.35), (3.36) and (3.37) we conclude that V is quasiregular in the left half-plane

$$H^- = \{z \in \mathbb{C}\colon \operatorname{Re} z < 0\}$$

and that

$$K_V(z) - 1 = O\left(\frac{1}{k^\delta}\right) \quad \text{for } x \le -2n_k \log n_k. \tag{3.38}$$

In analogy with S_k and S_k' we put

$$T_k = \{x + iy\colon 2\pi N_{k-1} < y < 2\pi N_k,\ x < -2n_k \log n_k\}$$

and

$$T_k' = \{x + iy\colon 2\pi N_{k-1} < y < 2\pi N_k,\ 2n_k \log n_k < x < 0\}.$$

For $k \ge 2$ we have

$$\int_{T_k} \frac{dx\,dy}{x^2 + y^2} \le \int_{T_k} \frac{dx\,dy}{x^2 + 4\pi^2 N_{k-1}^2} \le 2\pi m_k \int_0^\infty \frac{dx}{x^2 + 4\pi^2 N_{k-1}^2} = \frac{\pi m_k}{2N_{k-1}}.$$

Since also

$$\int_{T_1} \frac{dx\,dy}{x^2+y^2} < \infty$$

and since

$$N_k \sim \frac{1}{2\pi} k^\gamma \tag{3.39}$$

as $k \to \infty$ by (3.4) this yields

$$\int_{T_k} \frac{dx\,dy}{x^2+y^2} = O\left(\frac{1}{k}\right).$$

In analogy with (3.31) we can use these estimates and (3.38) to deduce that

$$\int_{T_k} \frac{K_V(z)-1}{x^2+y^2} dx\,dy \leq \frac{A_3}{k^{1+\delta}}$$

for some constant A_3. Similarly, if $k \geq 2$, then

$$\int_{T_k'} \frac{dx\,dy}{x^2+y^2} \leq \frac{1}{4\pi^2 N_{k-1}^2} \int_{T_k'} dx\,dy = \frac{n_k \log n_k}{2\pi N_{k-1}^2} \leq A_4 \frac{\log k}{k^{1+\gamma}}$$

for some constant A_4 by (3.4). The last two inequalities now imply that if $R > 0$, then

$$\int_{|z|>R} \frac{K_V(z)-1}{x^2+y^2} dx\,dy < \infty.$$

For $|\arg(-z)| < \pi/(2\sigma)$ we have $G(z) = V(-(-z)^\sigma)$. Similarly as in (3.33) it follows that

$$\int_{\substack{|\arg(-z)|<\frac{\pi}{2\sigma} \\ |z|>R^{1/\sigma}}} \frac{K_G(z)-1}{x^2+y^2} dx\,dy = \frac{1}{\sigma} \int_{\substack{z\in H^- \\ |z|>R}} \frac{K_V(z)-1}{x^2+y^2} dx\,dy < \infty.$$

Combining this with (3.33) we see that if $r > 0$, then

$$\int_{|z|>r} \frac{K_G(z)-1}{x^2+y^2} dx\,dy < \infty.$$

Thus G satisfies the hypothesis of the Teichmüller–Wittich–Belinskii theorem [33, §V.6]. This theorem yields that there exists a quasiconformal homeomorphism $\tau\colon \mathbb{C} \to \mathbb{C}$ and an entire function F such that

$$G(z) = F(\tau(z)) \quad \text{and} \quad \tau(z) \sim z \quad \text{as } z \to \infty. \tag{3.40}$$

3.4 Asymptotic behavior of F, E and A: proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We begin by estimating the counting function of the zeros of F. Let $r > 0$ and choose $k \in \mathbb{N}$ such that $2\pi(k-1) < r \leq 2\pi k$. It follows from the construction and (3.39) that

$$n(r, 0, U) \leq 4 \sum_{j=1}^{k} m_j \sim 2N_k \sim \frac{1}{\pi} k^\gamma = O(r^\gamma).$$

Similarly, if $2\pi N_{k-1} < r \leq 2\pi N_k$, then

$$n(r, 0, V) \leq 4 \sum_{j=1}^{k} m_j \sim 2N_k \sim \frac{1}{\pi} k^\gamma = O(r).$$

This implies that

$$n(r, 0, G) = O(r^{\rho\gamma}) + O(r^\sigma) = O(r^\sigma) \tag{3.41}$$

since $\rho\gamma = \sigma$. Now (3.40) yields $n(r, 0, F) = O(r^\sigma)$ and hence

$$N(r, 0, F) = O(r^\sigma) \tag{3.42}$$

as $r \to \infty$.

Next we note that the coefficients in the Taylor series expansion of h_m are all non-negative. This implies that $|h_m(z)| \leq h_m(|z|)$ and hence

$$|g_m(z)| \leq g_m(\operatorname{Re} z)$$

for all $z \in \mathbb{C}$. Clearly this implies that

$$|v_m(z)| \leq v_m(\operatorname{Re} z) \quad \text{for } z \in H^- \quad \text{and} \quad |u_m(z)| \leq u_m(\operatorname{Re} z) \quad \text{for } z \in H^+.$$

Let $z = x + iy \in H^-$ and $k \in \mathbb{N}$ with $2\pi N_{k-1} \leq \operatorname{Im} z < 2\pi N_k$. With $t = (y - 2\pi N_{k-1})/(2\pi n_k t)$ we have $0 \leq t < 1$ and

$$\begin{aligned} |V(x+iy)| &= |v_{m_k}((1-t)x + t\psi_k(x) + iy)| \\ &\leq v_{m_k}((1-t)x + t\psi_k(x)) \leq v_{m_k}(0) = g(s_{m_k}) = 2. \end{aligned}$$

Thus

$$|V(z)| \leq 2 \tag{3.43}$$

for $z \in H^-$. For $z \in H^+$ we use the estimate

$$\begin{aligned} |g_m(z)| \leq |g_m(\operatorname{Re} z)| &= \sum_{j=0}^{2m} (-1)^j \frac{1}{j!} e^{j \operatorname{Re} z} \exp(e^{\operatorname{Re} z}) \\ &\leq e^{1+2m \operatorname{Re} z} \exp(e^{\operatorname{Re} z}). \end{aligned} \tag{3.44}$$

Again we choose $k \in \mathbb{N}$ such that $2\pi(k-1) < \operatorname{Im} z \leq 2\pi k$. Then

$$|U(z)| = |u_{m_k}(q(z))| = |g_{m_k}(q(z) + s_{m_k})| \leq g_{m_k}(\operatorname{Re} q(z) + s_{m_k}),$$

with $q(z)$ defined by (3.28). Noting that $m_k = O(k^{\gamma-1}) = O(|z|^{\gamma-1})$ by (3.4) and thus

$$s_{m_k} = \log n_k + O(1) = \log m_k + O(1) = O(\log |z|)$$

we deduce from (3.17) and (3.44) that

$$\begin{aligned} \log |U(z)| &\leq 1 + 2m_k(\operatorname{Re} q(z) + s_{m_k}) + \exp(\operatorname{Re} q(z) + s_{m_k}) \\ &\leq \exp((1 + o(1))|z|) + O(|z|^\gamma) \leq \exp((1 + o(1))|z|) \end{aligned}$$

and hence

$$\log \log |U(z)| \leq (1 + o(1))|z| \quad \text{as } z \to \infty \text{ in } H^+.$$

Together with (3.43) we conclude that

$$\log \log |G(z)| \leq (1 + o(1))|z|^\rho \tag{3.45}$$

as $|z| \to \infty$. Hence (3.40) yields that

$$\log \log |F(z)| \leq (1 + o(1))|z|^\rho$$

as $|z| \to \infty$. The lemma on the logarithmic derivative [19, Section 3.1] now implies that $E = F/F'$ satisfies

$$m\left(r, \frac{1}{E}\right) = O(r^\rho). \tag{3.46}$$

By (3.42), and since the zeros of E are simple, we have

$$N(r, 0, E) = O(r^\sigma)$$

and we conclude that

$$T(r, E) = O(r^\sigma).$$

In particular, E has finite order. Since F is entire, E is clearly a special Bank–Laine function.

The lemma on the logarithmic derivative, together with (2.2) and (3.46), also implies that

$$m(r, A) = 2m\left(r, \frac{1}{E}\right) + O(\log r) = O(r^\rho).$$

We thus have $\lambda(E) \leq \rho(E) \leq \sigma$ and $\rho(A) \leq \rho$. Since

$$\frac{1}{\rho} + \frac{1}{\sigma} = 2$$

by the definition of σ, we deduce from (1.6) that actually $\lambda(E) = \rho(E) = \sigma$ and $\rho(A) = \rho$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We estimate the asymptotics of F, E and A more accurately than in the previous proof. First we note that for $|z| > 4m$ we have

$$\begin{aligned}\left|P_m(z) - \frac{z^{2m}}{(2m)!}\right| &= \left|\sum_{k=0}^{2m-1} (-1)^k \frac{z^k}{k!}\right| \leq \sum_{k=0}^{2m-1} \frac{|z|^k}{k!} = \sum_{k=0}^{2m-1} \frac{|z|^{2m-1}}{|z|^{2m-1-k} k!} \\ &\leq \frac{|z|^{2m-1}}{(2m-1)!} \sum_{k=0}^{2m-1} \frac{1}{2^{2m-1-k}} = \frac{4m}{|z|} \frac{|z|^{2m}}{(2m)!}\end{aligned} \tag{3.47}$$

and thus in particular $P_m(z) \neq 0$.

Let $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon < 1$ and put $H^+_{\varepsilon_1} = \{z \in \mathbb{C} \colon |\arg z| < (1 - \varepsilon_1)\pi/2\}$. Given $z \in H^+$, we choose $k \in \mathbb{N}$ with $2\pi(k-1) < \operatorname{Im} z \leq 2\pi k$. We then have $k = O(\operatorname{Re} z)$ and hence, by (3.4), $\log m_k = o(\operatorname{Re} z)$ and $m_k = o(e^z)$ as $z \to \infty$ in $H^+_{\varepsilon_1}$. We deduce from (3.47) that

$$P_{m_k}(e^z) \sim \frac{e^{2m_k z}}{(2m_k)!}$$

and thus, by Stirling's formula,

$$\log P_{m_k}(e^z) = 2m_k z + \log((2m_k)!) + o(1) \sim 2m_k z \quad \text{as } z \to \infty \text{ in } H^+_{\varepsilon_1}.$$

It follows that

$$\log g_{m_k}(e^z) = (1+o(1))2m_k z + e^z \sim e^z$$

and hence

$$\log U(z) = \log u_{m_k}(z) = \log g_{m_k}(z+s_{m_k}) = e^{(1+o(1))z} \quad \text{as } z \to \infty \text{ in } H^+_{\varepsilon_1}.$$

This implies that

$$\log F(z) = \log U(z^\rho) = \exp((1+o(1))z^\rho) \quad \text{for } |\arg z| < (1-\varepsilon_1)\frac{\pi}{2\rho}$$

and

$$\log\log F(z) = \log\log U(z^\rho) \sim z^\rho \quad \text{for } |\arg z| < (1-\varepsilon_1)\frac{\pi}{2\rho}$$

as $|z| \to \infty$. This implies that

$$\frac{F'(z)}{F(z)\log F(z)} \sim \rho z^{\rho-1} \quad \text{for } |\arg z| < (1-\varepsilon_2)\frac{\pi}{2\rho}$$

and hence

$$\begin{aligned} E(z) = \frac{F(z)}{F'(z)} &\sim \frac{z^{1-\rho}}{\rho \log F(z)} \\ &= \frac{z^{1-\rho}}{\rho}\exp(-(1+o(1))z^\rho) \quad \text{for } |\arg z| < (1-\varepsilon_2)\frac{\pi}{2\rho} \end{aligned}$$

as $|z| \to \infty$. Actually, this yields

$$E(z) = \exp(-(1+o(1))z^\rho) \quad \text{for } |\arg z| < (1-\varepsilon_2)\frac{\pi}{2\rho} \tag{3.48}$$

as $z \to \infty$. This implies that $\log E(z) \sim -z^\rho$ for $|\arg z| < (1-\varepsilon_2)\pi/(2\rho)$ so that

$$\frac{E'(z)}{E(z)} = \frac{d\log E(z)}{dz} \sim -\rho z^{\rho-1} \quad \text{for } |\arg z| < (1-\varepsilon)\frac{\pi}{2\rho}$$

and

$$\frac{E''(z)}{E(z)} - \left(\frac{E'(z)}{E(z)}\right)^2 = \frac{d^2\log E(z)}{dz^2} \sim -\rho(\rho-1)z^{\rho-2} \quad \text{for } |\arg z| < (1-\varepsilon)\frac{\pi}{2\rho}.$$

Together with (2.2) and (3.48) the last two inequalities yields

$$A(z) \sim \frac{1}{E(z)^2} \sim \exp((2+o(1))z^\rho) \quad \text{for } |\arg z| < (1-\varepsilon)\frac{\pi}{2\rho}. \tag{3.49}$$

Now (1.8) follows from (3.48) and (3.49).

The proof of (1.9) and (1.10) is similar. Here we use that

$$\begin{aligned} h_m(z) - 1 &= \frac{1}{(2m)!}\int_0^z \zeta^{2m} e^\zeta d\zeta \\ &= \frac{1}{(2m+1)!} z^{2m+1} + \frac{1}{(2m)!}\int_0^z \zeta^{2m}(e^\zeta - 1)d\zeta \\ &= (1+\eta_m(z))\frac{1}{(2m+1)!} z^{2m+1} \end{aligned}$$

where

$$|\eta_m(z)| \le \frac{2m+1}{|z|^{2m+1}}\left|\int_0^z \zeta^{2m}(e^\zeta - 1)d\zeta\right| \le 2\frac{2m+1}{|z|^{2m+1}}\int_0^{|z|} u^{2m+1}du \le 2|z|$$

for $|z| \le 1$. Thus, with $n = 2m+1$ as before, we have

$$\begin{aligned} \log(v_m(z)-1) &= \log\Big(g_m\Big(\frac{z}{n}+s_m\Big) - 1\Big) = \log\big(h_m\big(e^{z/(n)+s_m}\big) - 1\big) \\ &= -\log(n!) + z + ns_m + \log\left(1 + \eta_m\left(e^{z/n+s_m}\right)\right) \\ &= z + \log\left(1 + \eta_m\left(e^{z/n+s_m}\right)\right) + O(n) \end{aligned}$$

by Lemma 3.3.

We consider $H_{\varepsilon_1}^- = \{z \in \mathbb{C}\colon |\arg(-z)| < (1-\varepsilon_1)\pi/2\}$. For $z \in H_{\varepsilon_1}^-$ we choose $k \in \mathbb{N}$ with $2\pi N_{k-1} \le \operatorname{Im} z < 2\pi N_k$. We can deduce from (3.39) and Lemma 3.3 that $\operatorname{Re} z/n_{k-1} + s_{m_k} \to -\infty$. This implies that $e^{z/n_{k-1}+s_{m_k}} \to 0$ and hence $\eta(e^{z/n_{k-1}+s_{m_k}}) \to 0$ as $z \to \infty$ in $H_{\varepsilon_1}^-$. Moreover, $n_k = o(|z|)$ as $z \to \infty$ in $H_{\varepsilon_1}^-$, again by (3.39).

It follows that $\log(v_{m_k}(z)-1) \sim z$ and hence $\log(V(z)-1) \sim z$ as $z \to \infty$ in $H_{\varepsilon_1}^-$. This implies that

$$\log(F(z)-1) \sim -(-z)^\sigma \quad \text{for } |\arg(-z)| < (1-\varepsilon_1)\frac{\pi}{2\sigma}. \tag{3.50}$$

In particular, $F(z) \to 1$ as $z \to \infty$, $|\arg(-z)| < (1-\varepsilon_1)\pi/(2\sigma)$. Similarly as before we conclude that

$$\frac{F'(z)}{F(z)-1} \sim \sigma(-z)^{\sigma-1} \quad \text{for } |\arg(-z)| < (1-\varepsilon_2)\frac{\pi}{2\sigma}. \tag{3.51}$$

and thus, since

$$E(z) = \frac{F(z)}{F'(z)} \sim \frac{1}{F'(z)} = \frac{F(z)-1}{F'(z)} \frac{1}{F(z)-1},$$

we deduce from (3.50) and (3.51) that

$$\begin{aligned} E(z) &= \frac{1}{\sigma(-z)^{\sigma-1}} \exp((1+o(1))(-z)^{\sigma}) \\ &= \exp((1+o(1))(-z)^{\sigma}) \quad \text{for } |\arg(-z)| < (1-\varepsilon_2)\frac{\pi}{2\sigma}. \end{aligned}$$

Thus

$$\log E(z) \sim (-z)^{\sigma} \quad \text{for } |\arg(-z)| < (1-\varepsilon_2)\frac{\pi}{2\sigma},$$

which implies (1.9). As before it follows

$$\frac{E'(z)}{E(z)} \sim -\sigma(-z)^{\sigma-1} \quad \text{for } |\arg(-z)| < (1-\varepsilon)\frac{\pi}{2\sigma}.$$

and

$$\frac{E''(z)}{E(z)} - \left(\frac{E'(z)}{E(z)}\right)^2 \sim -\sigma(\sigma-1)(-z)^{\sigma-2} \quad \text{for } |\arg(-z)| < (1-\varepsilon)\frac{\pi}{2\sigma}.$$

Now (2.2) yields

$$A(z) \sim \left(\frac{E'(z)}{E(z)}\right)^2 \sim \frac{\sigma^2}{4}(-z)^{2\sigma-2} \quad \text{for } |\arg z| < (1-\varepsilon)\frac{\pi}{2\rho},$$

from which (1.10) immediately follows. □

4 Proof of Theorem 1.3

The main idea used in [37, 40, 44] is that (2.2) implies that when A is large, then E is small, except possibly in the set where E''/E or E'/E is large, but the latter set is small by the lemma on the logarithmic derivative. We shall also use this idea, but we will need that every unbounded component of the set $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$ actually contains a path where E tends to zero. In order to prove this we need to show that E is small on certain paths where A is large also on the exceptional set where the logarithmic derivatives are large. The key tool used here is an estimate of harmonic measure.

For $a \in \mathbb{C}$ and $r > 0$ let $D(a,r) = \{z \in \mathbb{C}\colon |z-a| < r\}$ be the disc around a of radius r.

Lemma 4.1. *Let f be a meromorphic function of finite order, $\alpha, \eta > 0$ and $c > \rho(f)$. Let (z_k) be the sequence of zeros and poles of f in $\mathbb{C}\backslash\{0\}$. Then there exists $r_0 > 0$ such that, for $|z| > r_0$,*

$$\left|\frac{f'(z)}{f(z)}\right| \leq \exp((3|z|)^\alpha) \quad \text{if } z \notin \bigcup_k D(z_k, \exp(-|z_k|^\alpha)) \tag{4.1}$$

and

$$\left|\frac{f'(z)}{f(z)}\right| \leq |z|^{c+\eta} \quad \text{if } z \notin \bigcup_k D\big(z_k, |z_k|^{-\eta}\big)\,. \tag{4.2}$$

Remark. The estimate (4.2) is standard [46, p. 74] and (4.1) is proved by the same method.

Proof of Lemma 4.1. We use the estimate [19, Chapter 3, (1.3′)]

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{4RT(R,f)}{(R-|z|)^2} + 2\sum_{|z_k|<R} \frac{1}{|z-z_k|} \quad \text{for } |z| < R, \tag{4.3}$$

which is obtained from the Poisson–Jensen formula and forms the basis for the proof of the lemma on the logarithmic derivative. We choose $R = 2|z|$. Then the first summand on the right side of (4.3) is less than R^{c-1} for large R. The second summand is less than $2n(R)\exp(R^\alpha)$ if $|z - z_k| \geq \exp(-|z_k|^\alpha)$ for all k, where $n(R)$ denotes the number of z_k of modulus at most R. Since $n(R) \leq R^c$ for large R we obtain

$$\left|\frac{f'(z)}{f(z)}\right| \leq R^{c-1} + 2R^c \exp(R^\alpha)$$

for large R, from which (4.1) easily follows. The estimate (4.2) is proved analogously. □

We shall use the following version of the "two constants theorem" which is obtained from a suitable harmonic measure estimate; see [36, p. 113, Satz 4]. There only the case $z = 0$ is stated, but the version below follows immediately.

Lemma 4.2. *Let G be a domain and let $\delta, \Theta \in (0,1)$ and $R > 0$. Let $z \in G$ and suppose that the set of all $r \in (0, R]$ for which the circle $\partial D(z,r)$ intersects the complement of G has measure at least $(1-\Theta)R$.*

Let $w\colon G \to \mathbb{C}$ be an analytic function. Suppose that $|w(\zeta)| < 1$ for all $\zeta \in G$ and that $\limsup_{\zeta\to\xi} |w(\zeta)| \le \delta$ for all $\xi \in \partial G$ satisfying $|\xi - z| < R$. Then

$$|w(z)| \le \delta^M \quad \text{with } M = \frac{2}{\pi} \arcsin \frac{1-\Theta}{1+\Theta}. \tag{4.4}$$

Noting that $\arcsin(1-x) \ge \pi/2 - \sqrt{2x}$ we obtain

$$M \ge \frac{2}{\pi} \arcsin(1-2\Theta) \ge 1 - \frac{4}{\pi}\sqrt{\Theta} \tag{4.5}$$

in (4.4).

The following result was proved in [16].

Lemma 4.3. *Let f be a transcendental entire function, $\varepsilon > 0$, $K_p > 0$ and $p \ge 0$. Suppose that $|f(z)| \le K_p|z|^p$ for z on some curve tending to ∞ and let U be an unbounded component of $\{z \in \mathbb{C}\colon |f(z)| > K_p|z|^p\}$. Then there exists a curve γ tending to ∞ in U such that $|f(z)| > \exp\bigl(|z|^{1/2-\varepsilon}\bigr)$ for z in γ.*

Finally we shall use the following result of Toda [44, Lemma 6].

Lemma 4.4. *Let A and E be entire functions satisfying (2.2). Suppose that $\lambda(E) < \rho(E)$. Then $\mu(E) = \rho(E) = \mu(A) = \rho(A)$, and these numbers are equal to an integer or ∞.*

Proof of Theorem 1.3. Let A and E be as in the statement of the theorem. By the Denjoy–Carleman–Ahlfors Theorem [36, Chapter XI, §4], we have $\mu(A) \ge N/2$. We may also assume that $\lambda(E) < \infty$. By Lemma 4.4 we then have $\rho(E) < \infty$.

Let $U_1, \dots, U_N$ be unbounded components of $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$. Then there are curves $\sigma_1, \dots, \sigma_N$ tending to ∞ "between" these components such that $|A(z)| = K_p|z|^p$ for $z \in \sigma_j$, for $j = 1, \dots, N$. We may assume that the U_j and the σ_j are numbered such that for large r there exist $\varphi_1, \dots, \varphi_N$ and $\theta_1, \dots, \theta_N$ satisfying $\varphi_1 < \theta_1 < \varphi_2 < \cdots < \varphi_N < \theta_N < \varphi_1 + 2\pi$ such that $re^{i\varphi_j} \in U_j$ and $re^{i\theta_j} \in \sigma_j$ for $j = 1, \dots, N$.

Let $\beta \in (0, 1/2)$ and $j \in \{1, \dots, N\}$. By Lemma 4.3 there exists a curve γ_j tending to ∞ in U_j such that

$$|A(z)| \ge \exp\bigl(|z|^\beta\bigr) \quad \text{for } z \in \gamma_j. \tag{4.6}$$

We denote by (z_k) the sequence of zeros of E and E' in $\mathbb{C}\backslash\{0\}$, choose $\alpha \in (0, \beta)$, and put

$$X = \bigcup_k D(z_k, \exp(-|z_k|^\alpha)) .$$

Denote by $n(r)$ be the number of z_k of modulus at most r and let $c > \rho(E)$. Since $\rho(E) = \rho(E')$ we have $n(r) \leq r^c$ for large r. It follows that if r is sufficiently large, then the sum of the diameters of the components of X that intersect the annulus $\{z \in \mathbb{C}\colon r/2 \leq |z| \leq 2r\}$ is at most $(3r)^c \exp(-(r/3)^\alpha)$. Since, noting that $\alpha < 1/2$, we have

$$(3r)^c \exp\Big(-\Big(\frac{r}{3}\Big)^\alpha\Big) \leq \exp\Big(-\frac{1}{2}r^\alpha\Big) \leq r$$

for large r, we see that $\mathbb{C}\backslash X$ has exactly one unbounded component. We denote this component by Ω and put $Y = \mathbb{C}\backslash\Omega$. Then $X \subset Y$ and $\partial Y \subset \partial X$. The above estimate also shows that if $|z|$ is large enough

$$\operatorname{meas}\left\{t \in \left(0, \frac{|z|}{2}\right] : \partial D(|z|, t) \cap Y \neq \emptyset\right\} \leq \exp\Big(-\frac{1}{2}|z|^\alpha\Big) \tag{4.7}$$

for large r.

Next, we deduce from (4.1), applied to $f = E$ and $f = E'$, that

$$\begin{aligned}\left|-2\frac{E''(z)}{E(z)} + \left(\frac{E'(z)}{E(z)}\right)^2\right| &\leq 2\left|\frac{E''(z)}{E'(z)}\right| \cdot \left|\frac{E'(z)}{E(z)}\right| + \left|\frac{E'(z)}{E(z)}\right|^2 \\ &\leq 3\exp(6|z|^\alpha) \quad \text{if } z \in \mathbb{C}\backslash X\end{aligned}$$

and hence, in particular, if $z \in \Omega$, provided $|z|$ is sufficiently large.

It follows from (2.2), (4.6) and the last estimate that if $z \in \gamma_j \cap \Omega$ and $|z|$ is large, then

$$\frac{1}{|E(z)|^2} \geq 4\exp\big(|z|^\beta\big) - 3\exp(6|z|^\alpha) \geq \exp\big(|z|^\beta\big)$$

and hence

$$|E(z)| \leq \exp\Big(-\frac{1}{2}|z|^\beta\Big) \quad \text{for } z \in \gamma_j \cap \Omega, \tag{4.8}$$

provided $|z|$ is sufficiently large. By continuity, this holds for $z \in \operatorname{cl}(\gamma_j \cap \Omega)$. Here and in the following $\operatorname{cl}(\cdot)$ denotes the closure of a set. We use Lemma 4.2

to estimate $E(z)$ for the points z on γ_j which are not in $\operatorname{cl}(\gamma_j \cap \Omega)$. For such a point z, put $R = |z|/2$ and let G be the the component of $D(z,R) \setminus \operatorname{cl}(\gamma_j \cap \Omega)$ which contains z. Since γ_j connects z with $\partial D(z,R)$, we deduce from (4.7) that G satisfies the hypothesis of Lemma 4.2 with $\Theta = \exp(-|z|^\alpha/2)/R$.

We choose $w(\zeta) = E(\zeta)\exp(-|z|^c)$. Since $c > \rho(E)$ we have $|w(\zeta)| < 1$ for $\zeta \in G$, provided $|z|$ is large enough. If $\xi \in \partial G$ and $|\xi - z| < R$, then $\xi \in \partial Y \subset \partial X$ and thus

$$\begin{aligned}\limsup_{\zeta\to\xi} |w(\zeta)| &= |E(\xi)|\exp(-|z|^c)\\ &\le \exp\left(-\frac{1}{2}\left(\frac{|z|}{2}\right)^\beta - |z|^c\right) \le \exp\left(-\frac{1}{4}|z|^\beta - |z|^c\right)\end{aligned}$$

for all $\xi \in \partial G$ satisfying $|\xi - z| < R$. We may thus apply Lemma 4.2 with $\delta = \exp\bigl(-|z|^\beta/4 - |z|^c\bigr)$. Together with (4.5) we thus have

$$\log|E(z)| - |z|^c = \log|w(z)| \le M\log\delta \le -\left(1 - \frac{4}{\pi}\Theta\right)\left(\frac{1}{4}|z|^\beta + |z|^c\right)$$

and hence

$$\log|E(z)| \le -\frac{1}{4}\left(1 - \frac{4}{\pi}\Theta\right)|z|^\beta + \frac{4}{\pi}\Theta|z|^c.$$

Now $\Theta = \exp(-|z|^\alpha/2)/R \le \pi|z|^{\beta-c}/64$ and also $\Theta \le \pi/8$ for large $|z|$. We conclude that $|E(z)| \le \exp\bigl(-|z|^\beta/16\bigr)$ if $z \in \gamma_j$ but $z \notin \operatorname{cl}(\gamma_j \cap \Omega)$. By (4.8) this estimate also holds for all other $z \in \gamma_j$ of sufficiently large modulus; that is, we have

$$|E(z)| \le \exp\left(-\frac{1}{16}|z|^\beta\right) \quad \text{for } z \in \gamma_j. \tag{4.9}$$

We put

$$Z = \bigcup_k D\bigl(z_k, |z_k|^{-c}\bigr).$$

We note that since $c > \rho(E)$, the sum of the radii of the discs forming Z is convergent. By (4.2), applied to E and E',

$$\left|-2\frac{E''(z)}{E(z)} + \left(\frac{E'(z)}{E(z)}\right)^2\right| \le 3|z|^{4c} \quad \text{for } z \notin Z. \tag{4.10}$$

Let $M \in \mathbb{N}$ with $M \geq \max\{2c, p/2\}$. Together with (2.2) the last equation implies that if $z \in \sigma_j \backslash Z$, then

$$\frac{1}{|z|^{2M}|E(z)|^2} \leq K_p|z|^{p-2M} + 3|z|^{4c-2M} = o(1)$$

as $z \to \infty$. Thus $z^M E(z) \to \infty$ as $z \to \infty$ in $\sigma_j \backslash Z$, for $j = 1, \ldots, N$. Since $z^M E(z) \to 0$ as $z \to \infty$ in γ_j for $j = 1, \ldots, N$ by (4.9), we conclude that if $K > 1$ is large, then $\{z \in \mathbb{C}\colon |z^M E(z)| > K\}$ has at least N unbounded components, which we denote by $W_1, \ldots, W_N$.

It follows from (2.2) and (4.10) that if $|z^M E(z)| > K$ and $z \notin Z$, then

$$|A(z)| \leq \frac{3}{4}|z|^{4c} + \frac{1}{4K^2}|z|^{2M} \leq K|z|^{2M}.$$

If K is large, then the component U_j of $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$ contains a component V_j of $\{z \in \mathbb{C}\colon |A(z)| > K|z|^{2M}\}$. With $V = \bigcup_{j=1}^N V_j$ and $W = \bigcup_{j=1}^N W_j$ the above argument shows that

$$V \cap W \subset Z. \tag{4.11}$$

For an unbounded open set D and $r > 0$ we put

$$\theta(r, D) = \operatorname{meas}\{t \in [0, 2\pi]\colon re^{it} \in D\}.$$

Since $\sum_k |z_k|^{-c} < \infty$ we see that $\theta(r, Z) \to 0$ as $r \to \infty$. It thus follows from (4.11) that

$$\theta(r, V) + \theta(r, W) \leq 2\pi + \theta(r, Z) = 2\pi + o(1) \tag{4.12}$$

as $r \to \infty$.

The proof is now completed by a standard application of the Ahlfors distortion theorem; cf. [36, Chapter XI, §4, no. 267]. Choose $r_0 > 1$ so large that $\partial D(0, r_0)$ intersects all V_j and W_j. Then

$$\log\log M(r, A) \geq \log\log \max_{|z|=r, z\in V_j} \left|\frac{A(z)}{Kz^{2M}}\right| \geq \pi \int_{r_0}^r \frac{dt}{t\theta(t, V_j)} - O(1).$$

By the Cauchy-Schwarz inequality we have

$$N^2 = \left(\sum_{j=1}^N \frac{\sqrt{t\theta(t, V_j)}}{\sqrt{t\theta(t, V_j)}}\right)^2 \leq \sum_{j=1}^N t\theta(t, V_j) \sum_{j=1}^N \frac{1}{t\theta(t, V_j)} = t\theta(t, V) \sum_{j=1}^N \frac{1}{t\theta(t, V_j)}$$

and

$$\left(\log\frac{r}{r_0}\right)^2 = \left(\int_{r_0}^{r}\frac{1}{\sqrt{t\theta(t,V)}}\frac{\sqrt{\theta(t,V)}}{\sqrt{t}}dt\right)^2 \le \int_{r_0}^{r}\frac{dt}{t\theta(t,V)}\cdot\int_{r_0}^{r}\frac{\theta(t,V)}{t}dt$$

so that

$$\begin{aligned}\log\log M(r,A) &\ge \frac{\pi}{N}\int_{r_0}^{r}\sum_{j=1}^{N}\frac{dt}{t\theta(t,V_j)} - O(1)\\ &\ge N\pi\int_{r_0}^{r}\frac{dt}{t\theta(t,V)} - O(1) \ge N\pi\frac{\left(\log\frac{r}{r_0}\right)^2}{\int_{r_0}^{r}\frac{\theta(t,V)}{t}dt} - O(1).\end{aligned}$$

It follows that

$$\frac{N\log r}{\log\log M(r,A)} \le (1+o(1))\frac{1}{\pi\log r}\int_{r_0}^{r}\frac{\theta(t,V)}{t}dt.$$

Noting that $\log\log M(r,E) = \log\log M(r,z^M E(z)) + o(1)$ we obtain

$$\frac{N\log r}{\log\log M(r,E)} \le (1+o(1))\frac{1}{\pi\log r}\int_{r_0}^{r}\frac{\theta(t,W)}{t}dt$$

by the same argument. Adding the last two inequalities and using (4.12) yields

$$\frac{N\log r}{\log\log M(r,A)} + \frac{N\log r}{\log\log M(r,E)} \le 2 + o(1),$$

from which we deduce that

$$\frac{N}{\mu(A)} + \frac{N}{\rho(E)} \le 2. \tag{4.13}$$

Since $\mu(A) < N$, this implies that $N/\rho(E) < 1$ and thus $\rho(E) > N > \mu(A)$. Hence $\rho(E) = \lambda(E)$ by Lemma 4.4, and substituting this in (4.13) yields the conclusion. □

5 Proof of Theorem 1.4

5.1 Preliminaries and outline of the construction

We will use the map constructed in section 3.2, but with $\rho_0 = \rho/N$ and $\sigma_0 = \rho_0/(2\rho_0 - 1)$ instead of ρ and σ. We also assume that $m_1 = m_2 = 0$. We denote the resulting map by G_0. We summarize the properties of G_0 that we need.

Let Q be the homeomorphism of the right half-plane onto itself given by (3.27). The exact definition of Q is irrelevant here but we note that $Q(z) = z$ if $\operatorname{Re} z > 1$. We denote by J^+ the preimage of the half-strip $\{z \in \mathbb{C}\colon \operatorname{Re} z \geq 0,\ |\operatorname{Im} z| \leq 2\pi\}$ under $z \mapsto Q(z^{\rho_0})$ and by J^- the preimage of $\{z \in \mathbb{C}\colon \operatorname{Re} z \leq 0,\ |\operatorname{Im} z| \leq 2\pi\}$ under $z \mapsto -(-z)^{\sigma_0}$. With $s_0 = \log\log 2$ we then have

$$G_0(z) = \begin{cases} \exp\exp(Q(z^{\rho_0}) + s_0) & \text{if } z \in J^+, \\ \exp\exp((-(-z)^{\sigma_0}) + s_0) & \text{if } z \in J^-. \end{cases} \tag{5.1}$$

We note that $J := J^+ \cup J^-$ is bounded by the real axis and a curve in the upper half-plane which, as the real axis, is mapped to $(1, \infty)$ by G_0.

In addition to G_0, we will also consider a modification G_1 of G_0 defined as follows. Let U and V be as in section 3.2. For $\operatorname{Re} z > 0$ we define

$$U_1(z) = \begin{cases} U(z - \pi i) & \text{if } \operatorname{Im} z \geq \pi, \\ \exp(-\exp(z + s_0)) & \text{if } -\pi < \operatorname{Im} z < \pi, \\ U(z + \pi i) & \text{if } \operatorname{Im} z \leq -\pi, \end{cases}$$

and for $\operatorname{Re} z < 0$ we define

$$V_1(z) = \begin{cases} V(z - \pi i) & \text{if } \operatorname{Im} z \geq \pi, \\ \exp(-\exp(z + s_0)) & \text{if } -\pi < \operatorname{Im} z < \pi, \\ V(z + \pi i) & \text{if } \operatorname{Im} z \leq -\pi. \end{cases}$$

The map G_1 is then obtained by gluing U_1 and V_1 in the same way that U and V were glued to obtain G_0. Then

$$G_1(z) = \begin{cases} \exp\exp(-Q(z^{\rho_0}) + s_0) & \text{if } z \in J^+, \\ \exp\exp((-z)^{\sigma_0}) + s_0) & \text{if } z \in J^-. \end{cases} \tag{5.2}$$

We note that for $z \in J$ we have $G_1(z) = 1/G_0(z)$.

For $j = 1, \dots, 2N$ we put

$$\Sigma_j = \left\{ z \in \mathbb{C} \colon (j-1)\frac{\pi}{N} < \arg z < j\frac{\pi}{N} \right\}.$$

Let $H = \{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$ be the upper half-plane and $L = H \setminus J$. Denote by $\overline{L}$ the reflection of L at the real axis; that is, $\overline{L} = \{z \in \mathbb{C} \colon \overline{z} \in L\}$. Denote by D_j the preimage of L or $\overline{L}$ in Σ_j, for $j = 1, \dots, 2N$. We will define a quasiregular map $G \colon \mathbb{C} \to \mathbb{C}$ which satisfies $G(z) = G_0(z^N)$ for $z \in D_j$ if $2 \le j \le 2N-1$ and $G(z) = G_1(z^N)$ for $z \in D_1$ and $z \in D_{2N}$. In the remaining part of the plane we will define G by a suitable interpolation which will require only (5.1) and (5.2).

In order to do so, we will first define G in section 5.2 in a neighborhood of ∞. Thus, for a suitable $r_0 > 0$, we have to define G in certain neighborhoods of the rays $\{z \in \mathbb{C} \colon \arg z = \pi/N, |z| > r_0\}$. This interpolation is comparatively easy if $j \neq 1$ and $j \neq 2N-1$ since in this case G is defined by the same expression in the domains D_j and D_{j+1} adjacent to the ray. (Here we have put $D_{2N+1} = D_1$. In similar expressions the index j will also be taken modulo $2N$.) For $j = 1$ and $j = 2N-1$ the interpolation argument is more elaborate. Next, in section 5.3, we will extend G to the bounded region that remains.

Using that G_0 and G_1 satisfy the hypothesis of the Teichmüller–Wittich–Belinskii Theorem we will then show in section 5.4 that this is also the case for G so that we again have (3.40) with an entire function F and a quasiconformal map $\tau \colon \mathbb{C} \to \mathbb{C}$. As in the proof of Theorem 1.1 we will then define E by $E = F/F'$ and A by (2.2) and show in section 5.5 that these functions have the required properties.

Remark. The function G_1 is introduced only to obtain a special Bank–Laine function E; that is, to obtain that one of the two solutions of (1.1) whose product is E has no zeros. If we use $1/G_0$ instead of G_1, then both solutions have zeros, but (1.11) is still satisfied.

5.2 Interpolation near ∞

Let γ_2 be the curve forming the boundary of L. We may parametrize it as $\gamma_2 \colon \mathbb{R} \to \mathbb{C}$,

$$\gamma_2(t) = \begin{cases} -(-(t+2\pi i))^{1/\sigma_0}) & \text{if } t \le 0, \\ Q^{-1}(t+2\pi i)^{1/\rho_0} & \text{if } t > 0. \end{cases}$$

We shall also need the curve γ_1 in the "middle" between γ_2 and the real axis; that is, $\gamma_1 \colon \mathbb{R} \to \mathbb{C}$,

$$\gamma_1(t) = \begin{cases} -(-(t+i\pi))^{1/\sigma_0}) & \text{if } t \le 0, \\ Q^{-1}(t+i\pi)^{1/\rho_0} & \text{if } t > 0, \end{cases}$$

see Figure 3. Then G_0 maps γ_1 to the interval $(0,1)$ and $G_0(\gamma_1(t)) \to 0$ as $t \to +\infty$ while $G_0(\gamma_1(t)) \to 1$ as $t \to -\infty$.

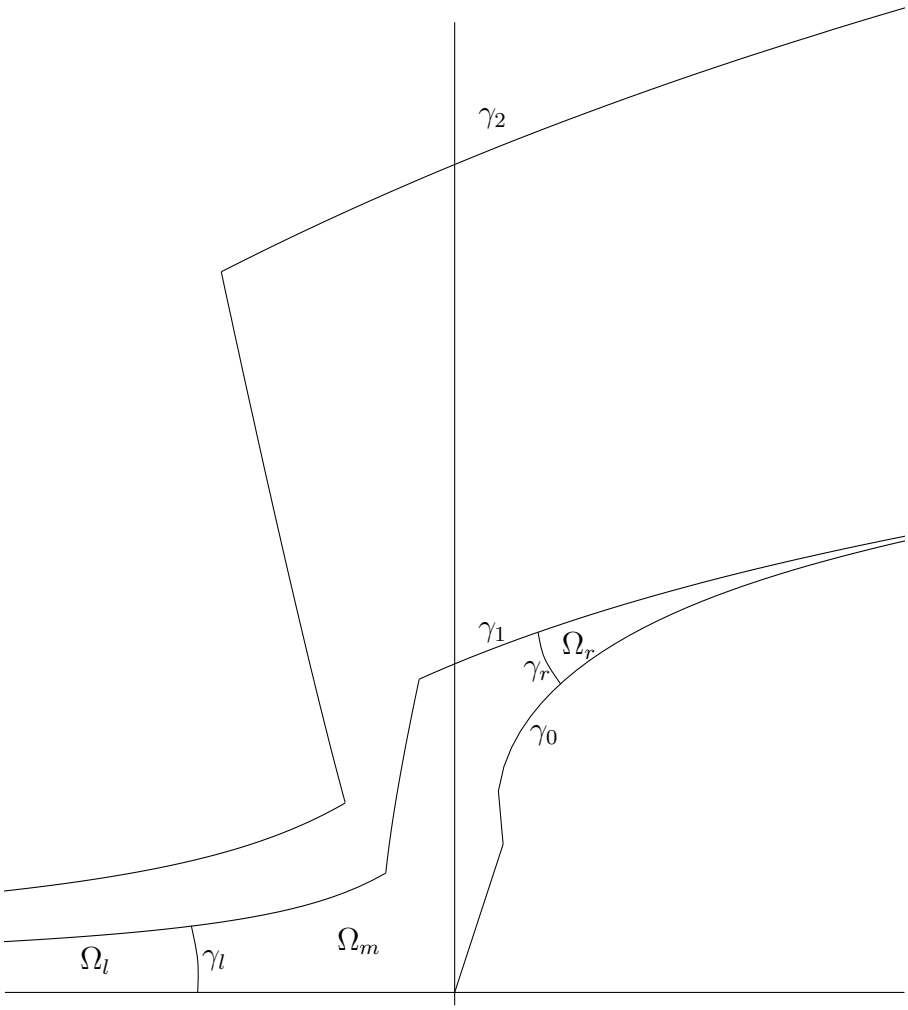

Figure 3: The curves γ_0, γ_1 and γ_2 for $\rho_0 = 3/4$ and $\sigma_0 = 3/2$.

We note that if $x_l < 0$ and $\gamma_l\colon [0,\pi] \to \mathbb{C}$, $\gamma_l(t) = -(-(x_l + it))^{1/\sigma_0}$, then the domain Ω_l which is to the left of γ_l and between γ_1 and the negative real axis is mapped by $z \mapsto -(-z)^{\sigma_0}$ to the half-strip

$$P = \{z \in \mathbb{C}\colon \operatorname{Re} z < x_l \text{ and } 0 < \operatorname{Im} z < \pi\}.$$

We see that if ε small enough, then x_l may be chosen so that Ω_l is mapped univalently onto the half-disc $\{z \in \mathbb{C}\colon |z| < \varepsilon,\ \operatorname{Im} z > 0\}$ by the function $z \mapsto \exp(-(-z)^{\sigma_0} + s_0)$ and hence univalently onto some half-neighborhood of 1 by G_0.

Similarly, we now define a curve γ_0 "below" γ_1 and an arc γ_r connecting γ_0 and γ_1 in the domain between them such that the domain Ω_r between γ_0 and γ_1 and to the right of γ_r is mapped univalently onto some half-neighborhood of 0 by G_0. In order to do so, we define $\gamma_0\colon \mathbb{R} \to \mathbb{C}$ by $\gamma_0(t) = t$ for $t \le 0$ and

$$\gamma_0(t) = \Big(t + i\pi - i \arcsin\Big(\frac{\pi}{e^{t+s_0}}\Big)\Big)^{1/\rho_0} \quad \text{for } t \ge 1.$$

For $0 < t < 1$ we define γ_0 in such a way that the curve γ_0 is below the curve γ_1; see Figure 3.

We find that $G_0(\gamma_0(t))$ is real and negative for $t \ge 1$ and $G_0(\gamma_0(t)) \to 0$ as $t \to \infty$. Moreover, if $x_r > 0$ is large and τ is an arc connecting γ_0 and γ_1 in $\Omega \cap \{z\colon \operatorname{Re} x > x_r\}$, then the image of τ under the function $z \mapsto \exp z^\rho$ is an arc connecting the real axis with the line $\{z \in \mathbb{C}\colon \operatorname{Im} z = \pi\}$. Hence $G_0 \circ \tau$ is an arc which is contained in the intersection of the lower half-plane with a small neighborhood of 0 and which connects a point on the negative real axis with a point on the positive real axis. In fact, for $\varepsilon > 0$ sufficiently small there exists an arc γ_r connecting γ_0 and γ_1 such that Ω_r is mapped univalently onto the half-disc $\{z \in \mathbb{C}\colon |z| < \varepsilon,\ \operatorname{Im} z < 0\}$ by G_0; cf. Figure 3.

Let $\Omega_m = \Omega \backslash (\Omega_r \cup \Omega_l)$ be the "middle piece" of Ω. Let T be the domain above the curve γ_0 and put $S = T \backslash \operatorname{cl}(\Omega_m)$ and $R = T \backslash \operatorname{cl}(\Omega_m \cup \Omega_l) = S \backslash \operatorname{cl}(\Omega_l)$. (Recall that $\operatorname{cl}(\cdot)$ denotes the closure of a set.) Thus we have $L \subset R \subset S \subset T \subset H$.

It is not difficult to see that there exists a quasiconformal map $\varphi\colon H \to T$ which satisfies $\varphi(z) = z$ for $z \in L$ as well as $\varphi(z) = z$ for $\operatorname{Re} z < x_0$ with some $x_0 < 0$. Choosing $\varepsilon > 0$ sufficiently small we may achieve that $x_l < x_0$ and hence that $\varphi(z) = z$ in particular for $z \in \Omega_l$. Moreover, the map φ can be chosen such that $\varphi(z) \sim z$ as $z \to \infty$. In fact, one can show that any quasiconformal map φ with the properties listed before also has the last

property. The map φ extends continuously to the real axis and we may assume that $\varphi(0) = 0$.

We denote by $\overline{L}$, $\overline{R}$, $\overline{S}$ and $\overline{T}$ the reflections of L, R, S and T at the real axis. The function φ extends to the lower half-plane by putting $\varphi(z) = \overline{\varphi(\overline{z})}$ for $\operatorname{Im} z < 0$ and it maps the lower half-plane to $\overline{T}$.

Let A be the closure of $\varphi^{-1}(S) \cup \varphi^{-1}\big(\overline{S}\big)$ and put $B = \{z \in \mathbb{C} \colon z^N \in A\}$; see Figure 4. The boundary of B consists of a Jordan curve $\Gamma \colon [0,1] \to \partial B$ which we may assume to have positive orientation.

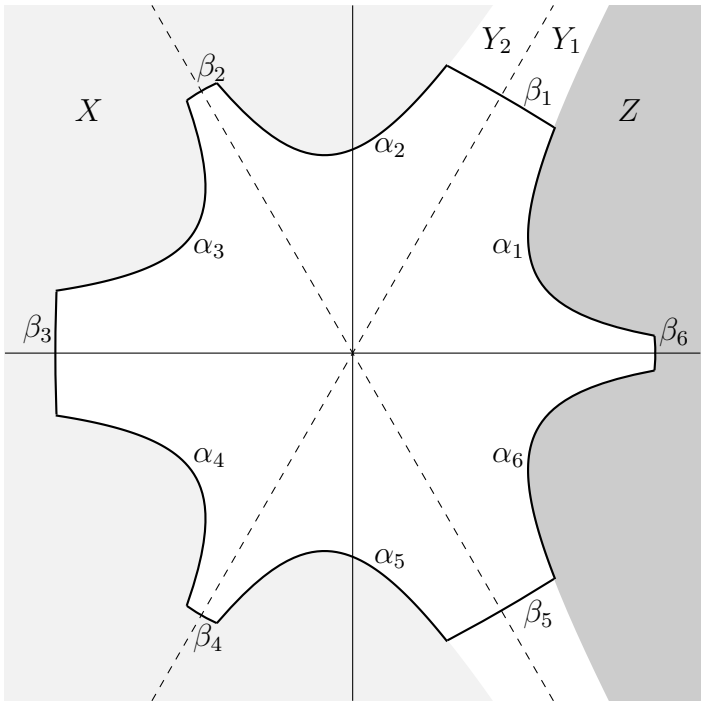

Figure 4: Sketch of the domain B for $N = 3$. The boundary of B consists of the curves $\alpha_1, \ldots, \alpha_6$ and $\beta_1, \ldots, \beta_6$.

The curve Γ splits into curves $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$. Here α_j is the part of Γ which is contained in the sector Σ_j and which by the function $z \mapsto \varphi(z^N)$ is mapped to a subcurve of γ_1 or its reflection in the real axis. For a more precise description of this subcurve, let γ_1^* be the the subcurve of γ_1 that lies between the intersection of γ_1 with γ_l and γ_r, let $-\gamma_1^*$ be the curve γ_1^* with reversed orientation and let $\overline{\gamma_1^*}$ be the reflection of γ_1^* in the real axis. If j is odd, then α_j is mapped bijectively to $-\gamma_1^*$ by $z \mapsto \varphi(z^N)$, and if

j is even, then α_j is mapped bijectively to $\overline{\gamma_1^*}$ by this function. Thus for odd j the function $z \mapsto G_0(\varphi(z^N))$ maps α_j bijectively to the interval $[\varepsilon, e^{-\varepsilon}]$, preserving the orientation. For even j it maps α_j to the same interval, but reversing the orientation. Noting that $G_1(z) = 1/G_0(z)$ for $z \in J \cup \overline{J}$, and thus in particular for z on the curves $-\gamma_1^*$ and $\overline{\gamma_1^*}$, we see that $z \mapsto G_1(\varphi(z^N))$ maps α_j to the interval $[e^{\varepsilon}, 1/\varepsilon]$, reversing the orientation for odd j and preserving the orientation for even j.

The curve β_j connects α_j and α_{j+1} and is mapped onto the concatenation of the curves γ_l and $\overline{\gamma_l}$ by $z \mapsto \varphi(z^N)$ if j is odd and onto that of γ_r and $\overline{\gamma_r}$ if j is even. Thus $z \mapsto G_0(\varphi(z^N))$ maps β_j to a loop surrounding the point 1 once for odd j and to a loop surrounding the point 0 once for even j. Using again that $G_1(z) = 1/G_0(z)$ for $z \in J \cup \overline{J}$ we see that $z \mapsto G_1(\varphi(z^N))$ maps β_j to a loop around 1 and ∞, respectively.

We will define a quasiregular map $G\colon \mathbb{C} \to \mathbb{C}$ first in B and later extend it to the (bounded) complement of B. In order to do so, put $B_j = B \cap \Sigma_j$ for $j = 1, \dots, 2N$, let

$$\eta\colon \bigcup_{j=1}^{2N} \Sigma_j \to T \cup \overline{T}, \quad \eta(z) = \varphi(z^N),$$

and denote by η_j the restriction of η to Σ_j. Then η_j maps Σ_j univalently onto T or $\overline{T}$, depending on whether j is odd or even. Since $\varphi(z) \sim z$ as $z \to \infty$, we have

$$\eta(z) \sim z^N \tag{5.3}$$

as $z \to \infty$.

Let

$$X = \eta_2^{-1}\big(\overline{R}\big) \cup \eta_{2N-1}^{-1}(R) \cup \bigcup_{j=3}^{2N-2} B_j \quad \text{and} \quad Z = \eta_1^{-1}(R) \cup \eta_{2N}^{-1}\big(\overline{R}\big)\,.$$

The sets X and Z are shown in light and dark gray in Figure 4.

We put $G(z) = G_0(\eta(z))$ for $z \in X$ and $G(z) = G_1(\eta(z))$ for $z \in Z$. The map G extends continuously to the parts of the rays $\{re^{ik\pi/N}\colon r > 0\}$ that are contained in B, for $k = 0$ and $k = 2, \dots 2N-2$. Thus G extends continuously to the closures of X and Z. Note that since $\varphi(z) = z$ for $z \in L \cup \overline{L}$ we indeed have $G(z) = G_0(z^N)$ for $z \in D_j$ if $2 \le j \le 2N-1$ and $G(z) = G_1(z^N)$ for $z \in D_j$ if $j = 1$ or $j = 2N$, as said in section 5.1.

Next we define G in the remaining part of B; that is, to $B\backslash(\mathrm{cl}(X)\cup\mathrm{cl}(Z))$. As we will define a map which commutes with complex conjugation, it suffices to define G in the remaining part in the upper half-plane. We put (cf. Figure 4)

$$Y_1 = \eta_1^{-1}(\Omega_l), \quad Y_2 = \eta^{-1}\big(\overline{\Omega_l}\big) \quad \text{and} \quad Y_0 = \partial Y_1 \cap \partial Y_2.$$

Hence $Y_0 = \{re^{i\pi/N} \colon r > r_0\}$ for some $r_0 > 0$.

In order to define G in $Y_1 \cup Y_2 \cup Y_0$, we note that since $\varphi(z) = z$ for $z \in \Omega_l$, we have $\eta(z) = z^N$ for $z \in Y_1 \cup Y_2 \cup Y_0$. Also, $z \mapsto -(-\eta(z))^{\sigma_0} = -(-z)^{N\sigma_0}$ maps Y_1 univalently onto the half-strip P and Y_2 univalently onto

$$\overline{P} = \{z \in \mathbb{C} \colon \operatorname{Re} < x_l \text{ and } -\pi < \operatorname{Im} z < 0\}.$$

Moreover, Y_0 is mapped to $(-\infty, x_l]$ by this map.

We consider the quasiconformal maps

$$\tau_1 \colon P \to \mathbb{C}, \quad \tau_1(x+iy) = x + i\frac{y+\pi}{2},$$

and $\tau_2 \colon \overline{P} \to \mathbb{C}$, $\tau_2(z) = \overline{\tau_1(\overline{z})}$. Then we define

$$G(z) = \begin{cases} \exp\big(-\exp\big(\tau_1\big(-(-z)^{N\sigma_0}\big) + s_0\big)\big) & \text{if } z \in Y_1, \\ \exp(-i\exp(-(-z)^{\sigma_0} + s_0)) & \text{if } z \in Y_0, \\ \exp\big(\exp\big(\tau_2\big(-(-z)^{N\sigma_0}\big) + s_0\big)\big) & \text{if } z \in Y_2. \end{cases}$$

Note that G has already defined on $\mathrm{cl}(X)$ and thus in particular on $\partial X \cap \partial Y_2$, and that we have $G(z) = \exp\exp(-(-\eta(z))^{\sigma_0} + s_0))$ for $z \in \partial X \cap \partial Y_2$. Since $\tau_2(x - i\pi) = x - i\pi$ for $x < x_l$ we conclude that G is continuous in $\partial X \cap \partial Y_2$. Similarly, $G(z) = \exp(-\exp(-(-\eta(z))^{\sigma_0} + s_0)))$ for $z \in \partial Z \cap \partial Y_1$ and thus G is also continuous there. Clearly G also extends continuously to the remaining parts of ∂Y_1 and ∂Y_2. Overall we have thus defined a continuous map G in B which is quasiregular in the interior of B.

5.3 Extension of G to the complement of B

To extend G to the complement of B, recall that the boundary of B is given by the Jordan curve $\Gamma \colon [0,1] \to \partial B$ which consists of the curves $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$.

For $2 \le j \le 2N-1$ the function G maps α_j to the interval $[\varepsilon, e^{-\varepsilon}]$, preserving the orientation for odd j and reversing the orientation for even j. Moreover, α_1 and α_{2N} are mapped to $[e^{\varepsilon}, 1/\varepsilon]$, preserving the orientation for α_1 and reversing the orientation for α_{2N}.

Also, for odd $j \ne 1$ the function G maps β_j to a loop surrounding the point 1 once and for even $j \ne 2N$ it maps β_j to a loop surrounding the point 0 once. The curve β_{2N} is mapped to a loop around ∞ and β_1 and β_{2N-1} are mapped to half-loops around 1, connecting the points e^{ε} and $e^{-\varepsilon}$.

To define a quasiregular map in the interior of Γ which has this boundary behavior suppose first that N is odd. We note that there exists a quasiconformal map from the sector $\{z \in \mathbb{C}\colon |\arg z| < \pi/(2N),\ |z| < 1\}$ onto $\mathbb{C}\backslash[0,\infty)$ such that the continuous extension to the boundary maps 0 to ∞, the points $e^{\pm i\pi/(2N)}$ to 1, and 1 to 0. We may assume that this map is symmetric with respect to the real axis; that is, it commutes with complex conjugation. Reflecting $N-1$ times along the sides of sectors we obtain a locally univalent quasiregular map defined in the half-disc $\{z \in \mathbb{C}\colon \operatorname{Re} z \ge 0,\ |z| \le 1\}$. For even N we do the same construction, starting with a quasiconformal homeomorphism of the sector $\{z \in \mathbb{C}\colon 0 < \arg z < \pi/N,\ |z| < 1\}$ onto $\mathbb{C}\backslash[0,\infty)$ such that the continuous extension to the boundary maps 0 to ∞, the points 1 and $e^{i\pi/N}$ to 1, and $e^{i\pi/(2N)}$ to 0. Reflecting $N-1$ times we again obtain a locally univalent quasiregular map, defined in the half-disc $\{z \in \mathbb{C}\colon \operatorname{Re} z \ge 0,\ |z| \le 1\}$ and symmetric with respect to the real axis.

The boundary curve of the half-disc, beginning at the origin, is mapped – in the following order – to the intervals $[\infty, 1]$, $[1, 0]$, $[0, 1]$, $[1, 0]$, $\ldots$, $[1, 0]$, $[0, 1]$, $[1, \infty]$. Removing small neighborhoods of the $2N$-th roots of unity and a small neighborhood of 0 from the half-disc we obtain a domain Δ and a locally univalent quasiregular map $\nu\colon \Delta \to \mathbb{C}$ such that, for a suitable parametrization $\Gamma_0\colon [0,1] \to \partial\Delta$ of the boundary of Δ, we have $\nu(\Gamma_0(t)) = G(\Gamma(t))$.

Furthermore, there exists a quasiconformal map $\phi\colon \Delta \to \mathbb{C}\backslash B$ which has a continuous extension to the boundary of Δ that satisfies $\phi(\Gamma_0(t)) = \Gamma(t)$ for all $t \in [0,1]$. Finally, we put $G(z) = \nu(\phi^{-1}(z))$ for $z \in \mathbb{C}\backslash B$. Then G is a locally quasiregular map defined in $\mathbb{C}$.

Remark. The quasiregular map ν in the above proof was defined by an ad hoc construction. A systematic way to construct such maps was described by Nevanlinna [35, no. 16]; see also [42, no. 44].

A special case of the result proved there yields the following. Let a poly-

gon K_p with p sides and a continuous function $F\colon \partial K_p \to [0,\infty]$ be given. Suppose that F maps each side of K_p homeomorphically onto one of the intervals $[0,1]$ and $[1,\infty]$, with each interval occurring as the image of at least one side. Then there exists a locally homeomorphic extension of F to K_p.

Nevanlinna used this to construct functions in the class $\mathcal{S}$ by gluing logarithmic ends to the sides of this polygon. Instead, we glue restrictions of our maps G_0 and G_1 to half-planes to the sides of this polygon.

5.4 Estimate of the dilatation

Let L' be the preimage of $L \cup \overline{L}$ under $z \mapsto z^N$. Since $\varphi(z) = z$ for $z \in L \cup \overline{L}$ we have $G(z) = G_0(z^N)$ for $z \in L' \cap \Sigma_j$ if $2 \le j \le 2N-1$ and $G(z) = G_1(z^N)$ for $z \in L' \cap \Sigma_j$ if $j = 1$ or $j = 2N$. For odd j satisfying $2 \le j \le 2N-1$ we thus have

$$\int_{L' \cap \Sigma_j} \frac{K_G(z)-1}{x^2+y^2} dx\, dy = \frac{1}{N} \int_L \frac{K_{G_0}(z)-1}{x^2+y^2} dx\, dy,$$

and for even j in that range the same equation holds with L replaced by $\overline{L}$ on the right side. For $j = 1$ and $j = 2N$ these equations hold with G_0 replaced by G_1. Since G_0 and G_1 satisfy the hypothesis of the Teichmüller–Wittich–Belinskii Theorem we conclude that

$$\int_{L'} \frac{K_G(z)-1}{x^2+y^2} dx\, dy < \infty.$$

Next we note that $\gamma_2(t)^{1/N} = (t+2\pi i)^{1/(N\rho_0)}$ for large t. Since $N\rho_0 > N/2 \ge 1$ we thus have $\operatorname{Im} \gamma_2(t) \to 0$ as $t \to +\infty$. So the distance of $\gamma_2(t)^{1/N}$ to the positive real axis tends to 0 as $t \to +\infty$. Similarly we see that the distance of $\gamma_2(t)^{1/N}$ to the ray $\{z \in \mathbb{C}\colon \arg z = \pi/N\}$ tends to 0 as $t \to -\infty$. We conclude that there exists $R > 1$ such that $\mathbb{C} \setminus (D(0,R) \cup L')$ is contained in strips of width 1 around the rays $\{z \in \mathbb{C}\colon \arg z = j\pi/N\}$, for $j = 1, \dots, 2N$. Now

$$\int_{\substack{|z|>R \\ |\operatorname{Im} z| < \frac{1}{2}}} \frac{1}{x^2+y^2} dx\, dy \le 2 \int_{\frac{1}{2}R}^{\infty} \frac{dx}{x^2} = \frac{4}{R}.$$

Denoting by $K = \sup_{z \in \mathbb{C}} K_G(z)$ the dilatation of G we conclude that

$$\int_{\mathbb{C} \setminus (D(0,R) \cup L')} \frac{K_G(z)-1}{x^2+y^2} dx\, dy \le \frac{4NK}{R} < \infty.$$

Overall we see that the quasiregular map G satisfies hypothesis of the Teichmüller–Wittich–Belinskii Theorem. Thus there exist an entire function F and a quasiconformal map $\tau\colon \mathbb{C} \to \mathbb{C}$ satisfying (3.40).

5.5 Completion of the proof

Equation (3.41) says that $n(r, 0, G_0) = O(r^{\sigma_0})$ as $r \to \infty$ and the same argument shows that $n(r, 0, G_0) = O(r^{\sigma_0})$. Noting that G has no zeros for $z \in \mathbb{C}\backslash(X \cup Z)$ while $G(z) = G_0(\eta(z))$ for $z \in X$ and $G(z) = G_1(\eta(z))$ for $z \in Z$, we find that $n(r, 0, G) = O(|\eta(z)|^{\sigma_0})$ and hence, with $\sigma = N\sigma_0$,

$$n(r, 0, G) = O(r^{N\sigma_0}) = O(r^{\sigma})$$

by (5.3).

Similarly, (3.45) says that $\log\log|G_0(z)| \leq (1 + o(1))|z|^{\rho_0}$ as $|z| \to \infty$, and the same argument yields that $\log\log|G_1(z)| \leq (1 + o(1))|z|^{\rho_0}$. Since G is bounded in $\mathbb{C}\backslash(X \cup Z)$, we now conclude from (5.3) that

$$\log\log|G(z)| \leq (1 + o(1))|z|^{N\rho_0} = (1 + o(1))|z|^{\rho}$$

as $|z| \to \infty$. As in the proof of Theorem 1.1, we now deduce from (3.40) that and the last two equations that

$$N(r, 0, F) = O(r^{\sigma})$$

and

$$\log\log|F(z)| \leq (1 + o(1))|z|^{\rho},$$

which together with the lemma on the logarithmic derivative again yields that $E = F/F'$ satisfies $m(r, 1/E) = O(r^{\rho})$ and $N(r, 0, E) = O(r^{\sigma})$ so that $T(r, E) = O(r^{\sigma})$. Hence

$$\log|E(z)| = O(|z|^{\sigma}).$$

As in the proof of Theorem 1.1 the lemma on the logarithmic derivative also implies that $m(r, A) = 2m(r, 1/E) + O(\log r) = O(r^{\rho})$ so that altogether we have $\lambda(E) \leq \rho(E) \leq \sigma$ and $\mu(A) \leq \rho(A) \leq \rho$.

For odd j satisfying $1 \leq j \leq 2N - 1$ and $\varepsilon > 0$ we put

$$S_j(\varepsilon) = \left\{ z \in \mathbb{C}\colon \left| \arg z - \frac{j\pi}{2N} \right| \leq (1 - \varepsilon)\frac{\pi}{4N} \right\}.$$

Let $0 < \varepsilon_1 < \varepsilon_2$. It follows from the construction that

$$|F(z)| \to 1 \quad \text{as } |z| \to \infty \text{ in } S_j(\varepsilon_1).$$

Hence $|F'(z)| = o(1)$ as $z \to \infty$ in $S_j(\varepsilon_2)$ and

$$\log \frac{1}{|F'(z)|} \sim \log |E(z)| = O(|z|^\sigma) \quad \text{as } |z| \to \infty \text{ in } S_j(\varepsilon_1).$$

This implies that

$$\left|\frac{F''(z)}{F'(z)}\right| = O(|z|^{\sigma-1}) \quad \text{and} \quad \left|\frac{d}{dz}\frac{F''(z)}{F'(z)}\right| = O(|z|^{\sigma-2}) \quad \text{as } |z| \to \infty \text{ in } S_j(\varepsilon_2).$$

Using (2.1) we see that

$$|A(z)| = O(|z|^{2\sigma-2}) \quad \text{as } |z| \to \infty \text{ in } S_j(\varepsilon_2). \tag{5.4}$$

For even j we denote by T_j the sector between S_{j-1} and S_{j+1}. It follows from the construction that if $R > 1$ is sufficiently large, then each T_j contains infinitely many components of the set $\{z \in \mathbb{C}\colon |F(z)| > R\}$. Moreover, for each component U of this set we have

$$M_U(r) := \max_{\substack{|z|=r \\ z\in U}} |F(z)| \geq \exp\exp((1-o(1))r^\rho) \tag{5.5}$$

as $r \to \infty$.

For large r we choose $z_r \in U$ such that $|z_r| = r$ and $|F(z_r)| = M_U(r)$. It follows from [8, equation (2.10)] that with

$$a(r) = \frac{d \log M_U(r)}{\log r}$$

we have, for all $k \in \mathbb{N}$,

$$F^{(k)}(z_r) \sim \left(\frac{a(r)}{z_r}\right)^k F(z_r) \quad \text{as } r \to \infty,\ r \notin L,$$

where $L \subset [1, \infty)$ is some exceptional set of finite logarithmic measure. Now (2.1) yields that

$$A(z) = -\frac{1}{4}\left(\frac{a(r)}{z_r}\right)^2 \quad \text{as } r \to \infty,\ r \notin L.$$

Noting that [8, equation (2.6)]

$$a(r) \geq (1 - o(1))\frac{\log M_U(r)}{\log r} \geq \exp((1 - o(1))r^\rho)$$

as $r \to \infty$ we deduce from (5.4) and (5.5) that $\{z \in \mathbb{C}\colon |A(z)| > K_p|z|^p\}$ has at least N unbounded components if $p = 2N\rho/(2\rho - 1) - 2 = 2\sigma - 2$ and K_p is sufficiently large.

Since

$$\frac{N}{\rho} + \frac{N}{\sigma} = \frac{1}{\rho_0} + \frac{1}{\sigma_0} = 2,$$

it now follows from (1.11) that we actually have $\lambda(E) = \rho(E) = \sigma$ and $\mu(A) = \rho(A) = \rho$. Hence (1.12) holds.

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