EMBEDDING THE KEPLER PROBLEM AS A SURFACE OF REVOLUTION

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Abstract. Solutions of the planar Kepler problem with fixed energy h determine geodesics of the corresponding Jacobi-Maupertuis metric. This is a Riemannian metric on $\mathbb{R}^2$ if $h \geq 0$ or on a disk $\mathcal{D} \subset \mathbb{R}^2$ if $h < 0$. The metric is singular at the origin (the collision singularity) and also on the boundary of the disk when $h < 0$. The Kepler problem and the corresponding metric are invariant under rotations of the plane and it is natural to wonder whether the metric can be realized as a surface of revolution in $\mathbb{R}^3$ or some other simple space. In this note, we use elementary methods to study the geometry of the *Kepler metric* and the embedding problem. Embeddings of the metrics with $h \geq 0$ as surfaces of revolution in $\mathbb{R}^3$ are constructed explicitly but no such embedding exists for $h < 0$ due to a problem near the boundary of the disk. We prove a theorem showing that the same problem occurs for every analytic central force potential. Returning to the Kepler metric, we rule out embeddings in the three-sphere or hyperbolic space, but succeed in constructing an embedding in Minkowski spacetime.

1. The Kepler Problem

The Kepler problem concerns the motion of point mass in the plane under an inverse square central force law. Let $q = (x, y)$ denote the position and $m > 0$ its mass. Newton's equations are

$$m\ddot{q} = -\frac{Gmq}{|q|^3} \tag{1}$$

where $G > 0$ is a constant. Since the mass m drops out, we may as well assume $m = 1$. Furthermore, if $q(t)$ is a solution of (1) then $Q(t) = kq(t), k > 0$ solves a similar equation with G replaced by G/k. So we may also assume that $G = 1$.

The Kepler problem with $m = G = 1$ can be viewed as a Hamiltonian system with Hamiltonian

$$H(q, p) = K(p) - U(q) = \frac{|p|^2}{2} - \frac{1}{|q|}.$$

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Hamilton's equations are

$$\dot{q} = p \qquad \dot{p} = -\frac{q}{|q|^3} \tag{2}$$

where $p = \dot{q}$. The total energy $H(q,p) = h$ is a constant of motion. Another scaling argument shows that we can restrict attention to energy levels $h = 0, 1, -1$. If $(q(t), p(t))$ is a solution of (2) with energy h then consider $(Q(t), P(t)) = (kq(lt), klp(lt))$ with $k > 0, l \neq 0$. It is easy to check that if $k^3l^2 = 1$, $(Q(t), P(t))$ is another solution of (2) but with energy k^2l^2h. Thus, by choosing k, l we can scale the energy by any positive constant to achieve $h = 0, 1, -1$.

Let $H(q,p) = \frac{1}{2}|p|^2 - U(q)$ be any planar Hamiltonian system and fix an energy level $H(q,p) = h$. The corresponding *Jacobi-Mauptertuis metric* is a Riemannian metric on the *Hill's region* $\mathcal{H}(h) = \{q = (x,y) : U(x,y) + h \geq 0\}$ given by

$$g = 2(U(x,y) + h)(dx^2 + dy^2).$$

If $(q(t), p(t))$ is a solution of the Kepler problem with energy h, then according to the Maupertuis variational principle, $q(t) \in \mathcal{H}(h)$ is a geodesic of the metric g [1, 2]. Conversely, such a geodesic can always be parametrized to give a solution $(q(t), p(t)) = (q(t), \dot{q}(t))$ of (2) with energy h.

For the Kepler problem we have

$$g = 2\left(\frac{1}{\sqrt{x^2+y^2}} + h\right)(dx^2 + dy^2) = 2\left(\frac{1}{r} + h\right)(dx^2 + dy^2) \tag{3}$$

where $r = \sqrt{x^2+y^2}$. For $h \geq 0$ the Hill's region is the entire plane, $\mathcal{H}(h) = \mathbb{R}^2$. For $h < 0$ we have a disk, $\mathcal{H}(h) = \{r \leq \frac{1}{|h|}\}$. The metric is singular at the collision singularity, $r = 0$, and also on boundary of the Hill's region when $h < 0$.

2. Geometry of the Kepler metric

In this section we will study the differential geometry of the metric (3). For simplicity, we will drop the factor of 2 which does not affect the geodesics. The resulting metric will be called the *Kepler metric*. As noted above, we can restrict attention to the cases $h = 0, 1, -1$. For $h = 0, 1$ we have a metric on the plane $\mathbb{R}^2$ while for $h = -1$, the Hill's region is the unit disk $\mathcal{D} = \{r \leq 1\}$. We begin by writing the metric in several different coordinate systems.

Because of the circular symmetry, it is natural to begin with polar coordinates where we have

$$g = \left(\frac{1}{r} + h\right)(dr^2 + r^2\,d\theta^2) = \left(\frac{1}{r} + h\right)dr^2 + r(1 + hr)\,d\theta^2 \tag{4}$$

A standard form for a rotationally symmetric metric is

$$g = ds^2 + a(s)^2 d\theta^2$$

where s is the arclength parameter along a radial line segment. Integrating

$$ds = \sqrt{\frac{1}{r} + h}\, dr$$

gives

$$s = \begin{cases} 2\sqrt{r} & h = 0 \\ \sqrt{r(1+r)} + \sinh^{-1}(\sqrt{r}) & h = 1 \\ \sqrt{r(1-r)} + \sin^{-1}(\sqrt{r}) & h = -1 \end{cases}$$

$$a(r) = \sqrt{r(1+hr)}.$$

To find $a(s)$ one would have to invert the formulas for $s(r)$. This can be done explicitly only for the case $h = 0$ where we find $a(s) = s/2$ and

$$g = ds^2 + \frac{s^2}{4} d\theta^2 \qquad h = 0.$$

It is easy to recognize this as the metric of a circular cone. More details about this will be given later.

Another popular choice of coordinates in the theory of surfaces is isothermal coordinates. If a general, rotation symmetric metric in polar coordinates is written $g = E(u)du^2 + 2F(u)dud\theta + G(u)d\theta^2$ then isothermal coordinate are characterized by requiring that $F(u) = 0$ and $E(u) = G(u)$. In all cases, the isothermal coordinate turns out to be given by

$$u = \ln r \qquad r = e^u$$

and the metric becomes

$$g = e^u(h + e^u)(du^2 + d\theta^2).$$

For $h = 0, 1$ we have $-\infty < u < \infty$ while for $h = -1$ we have $-\infty < u \leq 0$.

Finally, motivated by [5] we can look for symplectic coordinates (τ, θ) such that $F(\tau) = 0$ and $E(\tau)G(\tau) = 1$. When $h = 0$, polar coordinates are already symplectic with $\tau = r, E(\tau) = 1/\tau, G(\tau) = \tau$. For $h = \pm 1$, a short computation shows that

$$\tau = r + \frac{hr^2}{2} = r \pm \frac{r^2}{2} \qquad r = \pm(\sqrt{1 \pm 2\tau} - 1).$$

The metric takes the form

$$g = \frac{1}{\phi(\tau)} d\tau^2 + \phi(\tau) d\theta^2 \qquad \phi(\tau) = \pm\sqrt{1 \pm 2\tau}(\sqrt{1 \pm 2\tau} - 1).$$

The range of τ is $[0, \infty)$ for $h = 0, 1$ and $[0, \frac{1}{2}]$ for $h = -1$.

We close this section with some of the geometric invariants of the Kepler surface. The arclength of the radial rays is $s = \infty$ for $h = 0, 1$ and $s = \frac{\pi}{2}$ for $h = -1$. The area of the disk of radius r centered at the origin is given by

$$A(r) = 2\pi\tau(r) = 2\pi(r + \frac{hr^2}{2}).$$

The total surface area is $S = \infty$ for $h = 0, 1$ and $S = \pi$ for $h = -1$. Finally, the Gauss curvature can be calculated using standard formulas in any of the coordinate systems. The easiest is symplectic coordinates where we have

$$K = -\frac{1}{2}\phi''(\tau) = \begin{cases} 0 & h = 0 \\ \frac{1}{2(1+2\tau)^{\frac{3}{2}}} = \frac{-1}{2(1+r)^{\frac{3}{2}}} & h = 1 \\ \frac{-1}{2(1-2\tau)^{\frac{3}{2}}} = \frac{1}{2(1-r)^{\frac{3}{2}}} & h = -1 \end{cases}$$

Note that in the case $h = 1$ we have negative curvature while for $h = -1$ we have positive curvature with $K(r) \to \infty$ as $r \to 1$.

3. Embedding as a surface of revolution in $\mathbb{R}^3$

Since the Kepler metric is rotationally symmetric, it is natural to wonder if it can be realized as a surface of revolution in Euclidean space $\mathbb{R}^3$. For example, this is given as an exercise in [9, p.254] and is discussed in [8, sec 2.3] but perhaps with insufficient attention to what is under certain square root signs.

Let (ρ, θ, z) denote cylindrical coordinates in $\mathbb{R}^3$, the Euclidean metric is

$$g_{Euc} = d\rho^2 + dz^2 + \rho^2\, d\theta^2.$$

We look for an embedding of the form $\rho = f(r), z = g(r)$ which gives the metric

$$g = (f'(r)^2 + g'(r)^2)\, dr^2 + f(r)^2\, d\theta^2. \tag{5}$$

Comparing with (4) shows that we must have

$$f(r) = \sqrt{r(1+hr)} \qquad f'(r)^2 + g'(r)^2 = \frac{1}{r} + h.$$

When $h = 0$ we find

$$\rho = f(r) = \sqrt{r} \qquad z = g(r) = \sqrt{3r}$$

which describes a line in the (ρ, z) plane with slope $\sqrt{3}$. Rotating this line around the z axis gives a circular cone isometric to the corresponding Kepler surface. The angle between the z axis and the cone is $\pi/6$. Cutting the cone along a radial curve and flattening gives a perfect half-plane, that is, after flattening, the angle between the two cut edges is exactly π. The geodesics, which represent parabolic orbits of the Kepler problem, can be depicted by drawing straight lines on a sheet of paper and gluing the edges of the paper to form a cone. For more about this case see [7].

Next suppose $h = 1$. We find

$$f(r) = \sqrt{r(1+r)} \qquad g'(r)^2 = \frac{3+4r}{4r(1+r)}.$$

Integrating $g'(r)$ gives a solution

$$g(r) = \int_0^r \sqrt{\frac{3+4r}{4r(1+r)}}dr = \int_{\sqrt{3}}^{t(r)} \frac{t^2\,dt}{\sqrt{(t^2-3)(t^2+1)}}$$

where $t(r) = \sqrt{3+4r}$. This is an elliptic integral which can be evaluated with some effort to give

$$g(r) = \frac{3}{2}u - 2E(u,k) + 2\,\mathrm{dn}(u,k)\,\mathrm{sc}(u,k) \quad \text{where} \quad \mathrm{nc}(u,k) = \sqrt{1+\frac{4r}{3}}$$

where $\mathrm{dn}(u,k), \mathrm{sc}(u,k), \mathrm{nc}(u,k)$ are Jacobi elliptic function and $E(u,k)$ is the Jacobi elliptic integral of the second kind with modulus $k = \frac{1}{2}$ [6, p. 60–63]. The generating curve for the surface is given parametrically as $(\rho, z) = (\sqrt{r(1+r)}, g(r))$ and the surface is shown in Figure 1 along with some geodesics. The latter correspond to hyperbolic orbits of the Kepler problem.

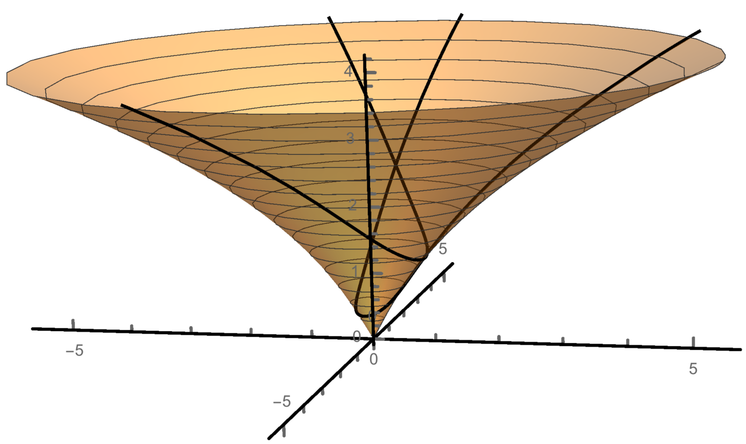

FIGURE 1. Embedding of the $h = 1$ Kepler surface with two geodesics.

Finally, consider the Kepler surface with $h = -1$. Due to the fact that the Kepler metric vanishes on the boundary of the Hill's region, we cannot expect an embedding valid for $0 \le r \le 1$. Instead we might expect an embedding valid for $0 \le r < 1$, perhaps with a cone point or some other singular point at $r = 1$. In fact, we will see that this is not possible.

An embedding with $\rho = f(r), z = g(r)$ must satisfy

$$f(r) = \sqrt{r(1-r)} \qquad f'(r)^2 + g'(r)^2 = \frac{1}{r} - 1.$$

Substituting the first of these equations into the second gives

$$g'(r)^2 = \frac{3-4r}{4r(1-r)}.$$

Clearly this can only be valid for $0 < r \leq \frac{3}{4}$ which indicates that an embedding of the whole surface is impossible.

For $0 < r \leq \frac{3}{4}$ one can solve for $g(r)$ in terms of elliptic functions and elliptic integrals to get

$$g(r) = 2E(u,k) - \frac{1}{2}u \quad \text{where} \quad \mathrm{cn}(u,k) = \sqrt{1 - \frac{4r}{3}}$$

where $\mathrm{cn}(u,k)$ is the Jacobi elliptic function and $E(u,k)$ is the Jacobi elliptic integral of the second kind with modulus $k = \frac{\sqrt{3}}{2}$. The generating curve for the surface is given parametrically as $(\rho, z) = (\sqrt{r(1-r)}, g(r))$ and the surface is shown in Figure 2.

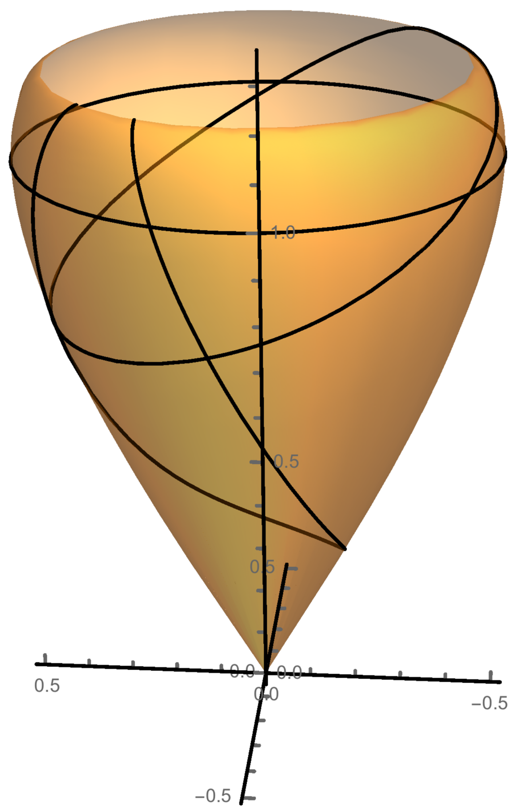

FIGURE 2. Partial embedding of the Kepler surface with $h = -1$ and three geodesics.

The surface resembles a vase or an amphora. The singularity of the potential at $r = 0$ maps to the cone point on the surface. Figure 2 also shows several geodesics. The circular orbit of the Kepler problem at $r = \frac{1}{2}$ maps to the circular geodesic around the widest part of the vase. Elliptical orbits with apsis exactly at $r = \frac{3}{4}$ just touch the edge of the vase. It turns out the these are the orbits with eccentricity $e = \frac{1}{2}$. Elliptical Kepler orbits with eccentricity $e > \frac{1}{2}$ pass too close to the zero velocity curve ($r > \frac{3}{4}$) and so cannot be fully represented as geodesics on the surface of revolution.

The lack of a global embedding can be explained in a different way using different coordinates. Recall that the Kepler metric for $h = -1$ can be written as

$$g = ds^2 + a(s)^2 d\theta^2$$

where $s = \sqrt{r(1-r)} + \sin^{-1}(\sqrt{r})$ and $a(r) = \sqrt{r(1-r)}$ and where s was found by integrating $ds = \sqrt{\frac{1}{r} - 1}\, dr$. It is a classical result that a metric in this form can be embedded as a surface of revolution if and only if $|a'(s)| \leq 1$ [3, 4]. Here we have

$$a'(s) = \frac{a'(r)}{s'(r)} = \frac{1-2r}{2(1-r)}$$

and $|a'(s)| \leq 1$ if and only if $r \leq \frac{3}{4}$.

In symplectic coordinates, the criterion for existence of an embedding as a surface of revolution is $|\phi'(\tau)| \leq 2$ [5]. For the $h = -1$ Kepler surface we have $\phi(\tau) = \sqrt{1-2\tau}(1 - \sqrt{1-2\tau})$ and

$$\phi'(\tau) = 2 - \frac{1}{\sqrt{1-2\tau}}.$$

We have $|\phi'(\tau)| \leq 2$ if and only if $\tau \leq \frac{15}{32}$. Since $\tau = r - r^2/2$ this again corresponds to $r \leq \frac{3}{4}$.

It turns out that the lack of an embedding near the Hill boundary is not special to the Kepler problem, but is a general feature of planar central force problems. Consider the Jacobi-Maupertuis metric associated to a potential $U(r)$ with $r = \sqrt{x^2+y^2}$. In polar coordinates we have

$$g = (U+h)dr^2 + r^2(U+h)d\theta^2 \tag{6}$$

where we have dropped the factor of 2 as before. A Hill boundary would be a curve where $U(r) + h = 0$ with an open annulus or collar nearby where $U(r) + h > 0$. It turns out that it is generally impossible to embed such an open collar as a surface of revolution.

Theorem 1. *Suppose $U(r)$ is an analytic function such that $U(r) + h$ has a isolated zero at $r = r_0 > 0$. In addition, suppose that there is $\epsilon > 0$ such that $U(r) + h > 0$ for $r_0 - \epsilon < r < r_0$ or for $r_0 < r < r_0 + \epsilon$ or both. Then there is $\delta > 0$ such that the metric (6) on the part of the Hill region $\mathcal{H}(h) = \{U(r) + h \geq 0\}$ with $0 < |r - r_0| < \delta$ does not admit an embedding as a surface of revolution in $\mathbb{R}^3$.*

Proof. As above, a general embedding takes the from $\rho = f(r), z = g(r)$ in cylindrical coordinates in $\mathbb{R}^3$ where $f(r), g(r)$ must satisfy

$$f(r) = r\sqrt{U+h} \qquad f'(r)^2 + g'(r)^2 = \sqrt{U+h}.$$

This leads to the equation

$$g'(r)^2 = -\frac{rU'(r)(rU'(r) + 4(U + h))}{4(U + h)}. \tag{7}$$

It suffices to show that the right hand side of this equation is always negative on an interval of the form $(r_0 - \delta, r_0)$ or $(r_0, r_0 + \delta)$ such that $U(r) + h \geq 0$.

Since $U(r)$ is analytic, r_0 is a finite order zero and we can write

$$U(r) = -h + c(r - r_0)^d/d + O(|r - r_0|^{d+1})$$

for some integer $d \geq 1$ and some $c \neq 0$. If d is even we must have $c > 0$ and we have $U(r) + h \geq 0$ on both sides of r_0. If d is odd then $U(r) + h$ changes sign at r_0 and we want to look on the side with $U(r) + h \geq 0$.

Let $G(r)$ denote the right hand side of (7) and let $\delta r = r - r_0$. The Taylor series for $U(r)$ gives

$$G(r) = -\frac{1}{4} c\, \delta r^{d-2}(d\, r_0^2 + O(\delta r))$$

for $\delta r \neq 0$. Note that the quantity in parentheses is positive for $|\delta r|$ sufficiently small.

If d is even, then $c > 0$ and $\delta r^{d-2} > 0$ and hence $G(r) < 0$ for $|\delta r|$ sufficiently small. So the metric on both sides of r_0 is not embeddable as a surface of revolution. If d is odd, there are two cases. If $c > 0$ then we are interested in the side of r_0 with $\delta r > 0$. Then $\delta r^{d-2} > 0$ and we conclude $G(r) < 0$ for $\delta r > 0$ sufficiently small. If $c < 0$ we are interested in the side of r_0 with $\delta r < 0$. The quantity $c\, \delta r^{d-2} > 0$ and we find $G(r) < 0$ for $\delta r < 0$ sufficiently small. □

4. More attempts to embed the $h = -1$ Kepler surface

Having failed to find a global embedding of the negative energy Kepler surface as a surface of revolution in $\mathbb{R}^3$, we could try to find such and embedding in other familiar spaces such as a three-sphere or a hyperbolic space. First note that an embedding in a higher dimensional Euclidean space is also impossible if we keep the requirement that the surface be represented as a simple kind of surface of revolution. In $\mathbb{R}^{k+2}$, introduce cylindrical coordinates (ρ, θ, z) where now $z \in \mathbb{R}^k$. Consider an embedding of the form $\rho = f(r), z = g(r)$ with $g : [0, 1] \to \mathbb{R}^k$. The reasoning above gives

$$|g'(r)|^2 = \frac{3 - 4r}{4r(1 - r)}$$

and again, we find that embedding the whole surface is impossible. It follows that it is also impossible to embed the Kepler surface into a round sphere or any other rotation invariant submanifold of Euclidean space. We call this kind of surface of revolution *simple* because in higher dimensions there are other, more complicated, types of rotations which could be used to rotate a generating curve.

Next suppose we try to embed the surface in hyperbolic three-space. A hyperbolic metric in the half-space model is

$$g_{hyp} = \frac{d\rho^2 + dz^2 + \rho^2\, d\theta^2}{\alpha^2 z^2}$$

where $\alpha > 0$. With an embedding of the form $\rho = f(r), z = g(r)$ we get

$$g = \frac{(f'(r)^2 + g'(r)^2)\, dr^2 + f(r)^2\, d\theta^2}{\alpha^2 g(r)^2}. \tag{8}$$

Define $h(r) = f(r)/(\alpha g(r))$. Then comparing with (4) gives

$$h(r) = \sqrt{r(1-r)}$$

and a rather complicated differential equation for $g(r)$

$$4r(1-r)(r^2 - r - \alpha^{-2})g'(r)^2 + 4r(1-r)(2r-1)g(r)g'(r) + (3-4r)g(r)^2 = 0.$$

We will see that this cannot be solved by a real differentiable function defined for all $r < 1$. Indeed, it can be viewed as a quadratic equation for the ratio $g'(r)/g(r)$ and the discriminant of this equation is

$$16r(1-r)[4(1-r)^3 + \alpha^{-2}(3-4r)].$$

No matter which $\alpha > 0$ is used, the quantity in square brackets will be negative over some interval $\frac{3}{4} < r_0 < r < 1$ which makes $g'(r)/g(r)$ nonreal.

How about an embedding into three-dimensional Minkowski space ? We have

$$g_{Mink} = d\rho^2 - dz^2 + \rho^2\, d\theta^2.$$

and if $\rho = f(r), z = g(r)$ we get the metric

$$g = (f'(r)^2 - g'(r)^2)\, dr^2 + f(r)^2\, d\theta^2. \tag{9}$$

which leads, as above, to $f(r) = \sqrt{r(1-r)}$ but

$$g'(r)^2 = -\frac{3-4r}{4r(1-r)}.$$

This one can only be solved for $\frac{3}{4} \le r < 1$. Comparing this with the discussion for Euclidean space suggests trying an embedding in a three-dimensional semi-Riemannian space whose signature varies from point to point. However, no natural candidates for such a space come to mind so in the next section we turn to four-dimensional Minkowski space.

5. Embedding Kepler in Minkowski spacetime

On the positive side, we will now show that embeddings into four-dimensional Minkowski space are possible. The Minkowski metric in an appropriate version of cylindrical coordinates is

$$g_{Mink} = d\rho^2 + dz^2 - dt^2 + \rho^2\, d\theta^2.$$

An embedding as a simple surface of revolution would take the form $\rho = f(r), z = g(r), t = h(r)$. We have

$$g = (f'(r)^2 + g'(r)^2 - h'(r)^2)\,dr^2 + f(r)^2\,d\theta^2.$$

which leads, to $f(r) = \sqrt{r(1-r)}$ and

$$g'(r)^2 - h'(r)^2 = \frac{3-4r}{4r(1-r)}.$$

This equation has many solutions which can be found by splitting the right-hand side as the difference of two nonnegative functions. For example suppose we choose

$$h(r) = 2 - 2\sqrt{1-r} \qquad h'(r)^2 = \frac{1}{1-r} = \frac{4r}{4r(1-r)}.$$

Then we get

$$g'(r)^2 = \frac{3}{4r(1-r)} \qquad g(r) = \sqrt{3}\,\arcsin(\sqrt{r}).$$

The profile curve

$$(\rho, z, t) = (\sqrt{r(1-r)}, \sqrt{3}\,\arcsin(\sqrt{r}), 2 - 2\sqrt{1-r})$$

along with its projections is shown in Figures 3 and 4. The profile curve meets the plane $\rho = 0$ (shaded in Figure 3) when $r = 0, 1$ and these will be conical singularities on the surface. Note that the boundary of the Hill's region (the unit circle $r = 1$) is collapsed to a point. This was to be expected since its length in the Kepler metric is 0. The point with $r = \frac{3}{4}$ is indicated in the figures but appears unremarkable. However, one can check that the slope of projection of the profile curve to the (z, t) plane reaches 1 at this point. The surface itself is spacelike, but is tangent to the light cone at $r = 1$.

In view of the nonuniqueness of the embedding, the details of these curves should not be taken too seriously. Perhaps there is a way to single out a unique embedding by imposing some further geometrical constraints, but it is not clear how to proceed.

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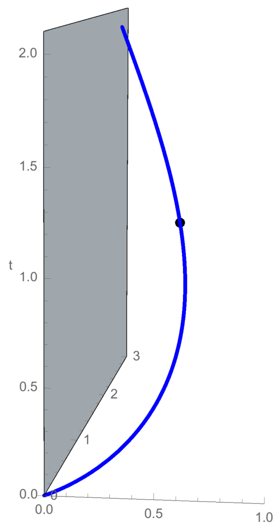

FIGURE 3. Profile curve for an embedding in spacetime

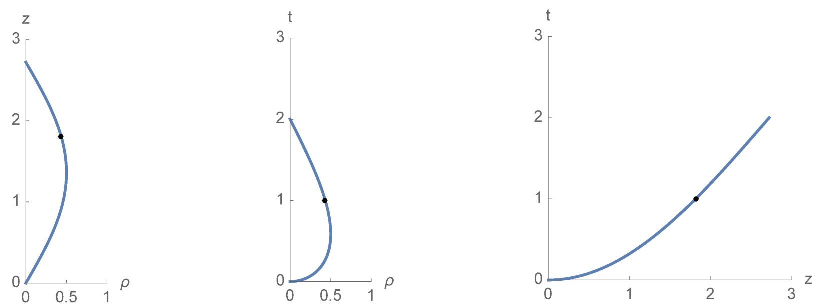

FIGURE 4. Projections of the curve in Figure 3

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