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DIFFERENCE OF MODULAR FUNCTIONS AND THEIR CM VALUE FACTORIZATION

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Abstract. In this paper, we use Borcherds lifting and the big CM value formula of Bruinier, Kudla, and Yang to give an explicit factorization formula for the norm of $\Psi(\frac{d_1+\sqrt{d_1}}{2}) - \Psi(\frac{d_2+\sqrt{d_2}}{2})$, where Ψ is the j-invariant or the Weber invariant ω_2. The j-invariant case gives another proof of the well-known Gross-Zagier factorization formula of singular moduli, while the Weber invariant case gives a proof of the Yui-Zagier conjecture for ω_2. The method used here could be extended to deal with other modular functions on a genus zero modular curve.

Contents

1. Introduction

In the 1980s, Gross and Zagier discovered a beautiful factorization formula for the singular moduli [GZ85] in preparation of their well-known Gross-Zagier formula. It was extended slightly by Dorman [Dor88], which can be stated as follows (see Remark 4.1).

Theorem 1.1 (Gross-Zagier, Dorman). *Let $E_i = \mathbb{Q}(\sqrt{d_i})$ be two imaginary quadratic fields of fundamental discriminants d_i with $(d_1, d_2) = 1$, let $F = \mathbb{Q}(\sqrt{D})$ with*

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$D = d_1 d_2$, and let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Let $j(\tau)$ be the well-known j-invariant. Then

$$\sum_{[\mathfrak{a}_i]\in \mathrm{Cl}(E_i)} \log |j(\tau_{\mathfrak{a}_1}) - j(\tau_{\mathfrak{a}_2})|^{\frac{8}{w_1 w_2}}$$
$$= \sum_{\substack{t=\frac{m+\sqrt{D}}{2}\in \mathcal{O}_F \\ |m|<\sqrt{D}}} \sum_{\mathfrak{p} \text{ inert in } E/F} \frac{1+\mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \rho(t\mathfrak{p}^{-1}) \log(\mathrm{N}(\mathfrak{p})).$$

Here w_i is the number of roots of unity in E_i, and for an integral ideal $\mathfrak{a}$ of F,

$$\rho(\mathfrak{a}) = |\{\mathfrak{A} \subset \mathcal{O}_E : \mathrm{N}_{E/F}(\mathfrak{A}) = \mathfrak{a}\}|.$$

Finally, for an integral ideal $\mathfrak{a}_i$ of E_i with

$$\mathfrak{a}_i = \mathbb{Z}a_i + \mathbb{Z}\frac{b_i + \sqrt{d_i}}{2}, \quad a_i = \mathrm{N}(\mathfrak{a}_i),$$

its associated CM point is $\tau_{\mathfrak{a}_i} = \frac{b_i+\sqrt{d_i}}{2a_i}$.

This gives a beautiful factorization formula for $\mathrm{N}(j(\frac{d_1+\sqrt{d_1}}{2}) - j(\frac{d_2+\sqrt{d_2}}{2}))$ (up to sign). In particular, the biggest prime factor of this norm is less than or equal to $D/4$, extremely small compared to the norm. The first few examples of this phenomenon were discovered by Berwick in the 1920s [Ber28]. For example, one has

$$j(\frac{1+\sqrt{-163}}{2}) - j(\frac{1+\sqrt{-3}}{2}) = -2^{18}3^3 5^3 23^3 29^3 = -262537412640768000,$$
$$j(\frac{1+\sqrt{-163}}{2}) - j(i) = -2^6 3^6 7^2 11^2 19^2 127^2 163 = -262537412640769728.$$

In 1997, Yui and Zagier [YZ97] defined a mysterious CM value $\mathbf{f}(\frac{d+\sqrt{d}}{2})$ via the three Weber functions of level 48 (when $d \equiv 1 \pmod 8$ and $3 \nmid d$) and proved that it is defined over the Hilbert class field of $\mathbb{Q}(\sqrt{d})$. They claimed that its Galois conjugates are the CM values at other CM points of the same discriminant d with some modifications, which was later proved by Alice Gee using Shimura's reciprocity law. In addition, Yui and Zagier gave a conjectural factorization formula for the norm of $\mathbf{f}(\frac{d_1+\sqrt{d_1}}{2})^a - \mathbf{f}(\frac{d_2+\sqrt{d_2}}{2})^a$ similar to the Gross-Zagier factorization formula for any positive integer $a|24$. For example, when $a = 24$, the conjecture can be restated as follows.

Conjecture 1.2. *Let the notation be as in Theorem 1.1, and assume further that $d_1 \equiv d_2 \equiv 1 \pmod 8$. Let*

$$\omega_2(\tau) = 2^{12} q \prod_{n>0} (1+q^n)^{24} = 2^{12} \cdot \frac{\Delta(2\tau)}{\Delta(\tau)}$$

be the Weber modular function for $\Gamma_0(2)$. Then

$$\sum_{[\mathfrak{a}_i|\in \mathrm{Cl}(E_i)} \log |\omega_2(\tau_{\mathfrak{a}_1}) - \omega_2(\tau_{\mathfrak{a}_2})|^2$$
$$= \sum_{\substack{t=\frac{m+\sqrt{D}}{2} \\ |m|<\sqrt{D}, \text{odd} \\ m^2\equiv D \pmod{16}}} \sum_{\mathfrak{p} \text{ inert in } E/F} \frac{1+\mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \rho(t\mathfrak{p}^{-1}\mathfrak{p}_t^{-2}) \log(\mathrm{N}(\mathfrak{p})).$$

Here $\mathfrak{p}_t$ is the unique prime ideal of F above 2 such that $\mathrm{ord}_{\mathfrak{p}_t}(t\mathcal{O}_F) \geq 1$, and for each ideal class $[\mathfrak{a}_i] \in \mathrm{Cl}(E_i)$, we choose a representative $\mathfrak{a}_i$ integral with norm prime to 2, i.e.,

$$\mathfrak{a}_i = \mathbb{Z}a_i + \mathbb{Z}\frac{b_i + \sqrt{d_i}}{2}, \quad \text{with } 2 \nmid a_i, \quad a_i > 0, \qquad \tau_{\mathfrak{a}_i} = \frac{b_i + \sqrt{d_i}}{2a_i}.$$

They provided some numerical evidence in their paper. Notice that the biggest prime factor of this norm is less than or equal to $D/16$. In this paper, we will prove this conjecture.

Theorem 1.3. *Conjecture* 1.2 *is true.*

In his 2006 thesis [Sch09], Schofer used regularized theta lifting to generalize the Gross-Zagier factorization formula to small CM values of the so-called Borcherds products on the orthogonal Shimura varieties of type $(n, 2)$. Bruinier and Yang generalized it to big CM values of Hilbert modular forms (which are Borcherds products) over a real quadratic field [BY06]. More recently, Bruinier, Kudla, and Yang ([BKY12]) generalized it to big CM values of Borcherds products on Shimura varieties of orthogonal type $(n, 2)$, following Schofer and [BY09]. On a different track, Lauter, Goren, and Viray have used geometric methods to generalize the Gross-Zagier formula to Igusa's j-invariants for genus two curves, which have important applications to the genus two curve cryptosystem ([GL07], [GL12], [LV15]). Yang also proved Lauter's conjecture on Igusa's j-invariants by combining the result in [BY06] with his work on arithmetic intersection [Yan13]. The big CM value formula in [BY06], [BKY12] has also been used to prove certain cases of the Colmez conjecture ([Yan10a], [Yan10b], [Yan13], [BHK]) and the average Colmez conjecture ([AGHMP15]). Dongxi Ye is extending the result to other modular curves of genus zero [Ye17].

This paper is the first part of our effort to prove Yui and Zagier's conjectural formula using the big CM value formula.

The general idea is as follows. Let Γ be a congruence subgroup such that the compactification of $X_\Gamma = \Gamma\backslash\mathbb{H}$ has genus zero, and let Ψ be a generator of the function field of X_Γ, which is a modular function for Γ. Then the difference function $\Psi(z_1) - \Psi(z_2)$ is a two-variable modular function on $X_\Gamma \times X_\Gamma$ with divisor being the diagonal divisor. We view $X_\Gamma \times X_\Gamma$ as an orthogonal Shimura variety of type $(2, 2)$ associated to $(V = M_2(\mathbb{Q}), Q = N\det)$ for some positive integer N. One can show that the diagonal divisor is a special divisor on the product $X_\Gamma \times X_\Gamma$ so that $\Psi(z_1) - \Psi(z_2)$ has a chance to be a Borcherds lifting (product) of some weakly holomorphic modular forms ([Bor98], [Bru02]). The first task is to find a weakly holomorphic modular form, if any, whose Borcherds lifting is the difference $\Psi(z_1) - \Psi(z_2)$ ([Bor98], [Bru02]; see Section 2). There are two complications even with Bruinier's converse results ([Bru02], [Bru14]). First, when $N > 1$, Bruinier's converse theorem does not apply. Second, there are two variable modular functions whose divisors are only supported on the boundary, so it is not enough to compare the divisors of the Borcherds product with our function only in the open Shimura variety. We also need to understand their boundary behavior. The Borcherds product expansion is important in this aspect. In this paper, we are only successful in this step for the Weber functions ω_i (Section 5) but not for the more interesting Weber functions $\mathbf{f}_i$ of level 48.

The second task is to identify a pair of Heegner points $(\tau_{\mathfrak{a}_1}, \tau_{\mathfrak{a}_2})$ with a big CM point on $X_\Gamma \times X_\Gamma$ associated to the CM number field $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the sense of Bruinier, Kudla, and Yang in [BKY12]. This is done in Section 3. The third task is to apply the big CM value formula in [BKY12] (assuming that Ψ is a Borcherds lifting) to provide the expected formula. One serious problem (for the Yui-Zagier conjecture) is that the big CM cycle in [BKY12] is likely bigger in size than the ideal class groups used in Yui-Zagier's conjectural formula in general. One might need to use Shimura's reciprocity law to analyze the Galois action on the values as in [Gee99] to solve the problem. In the case of ω_2, the condition $d_i \equiv 1 \pmod 8$ allows us to choose an embedding from E_i to $\mathrm{GL}_2(\mathbb{Q})$ so that the ideal class group maps into $X_0(2)$ nicely. Another minor complication (interesting feature) is the explicit computation of the Fourier coefficient of the derivatives of some incoherent Eisenstein series since Schwartz functions are not factorizable in the ω_2 case (Section 5).

Here is the organization of this paper. In Section 2, we review Borcherds lifting, Borcherds product expansion ([Bor98], [Bru02]), and the big CM value formula ([BKY12]). In Section 3, we identify the product $X_\Gamma \times X_\Gamma$ of two copies of a modular curve with a Shimura variety of orthogonal type $(2, 2)$ and identify its big CM points with pairs of the CM points on the modular curve X_Γ. In Section 4, we re-prove Theorem 1.1 using the big CM value formula. In Section 5, we first identify $\omega_2(z_1) - \omega_2(z_2)$ with a Borcherds lifting of some explicit weakly holomorphic modular forms and then use the big CM value formula to prove Theorem 1.3.

2. Borcherds lifting and the big CM value formula

2.1. **Borcherds lifting and Borcherds product expansion.** In this subsection, we review the beautiful work of Borcherds in detail using slightly different conventions and notation for our purpose. Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and let L be an even integral lattice; i.e., $Q(x) = \frac{1}{2}(x, x) \in \mathbb{Z}$ for $x \in L$. Let

$$L' = \{y \in V : (x, y) \in \mathbb{Z}, \text{ for } x \in L\} \supset L$$

be its dual. We assume in this paper that n is even for simplicity. Let $H = \mathrm{GSpin}(V)$, and let $\mathbb{D}$ be the oriented negative 2-planes in $V_\mathbb{R}$. Then for a compact open subgroup K of $H(\mathbb{A}_f)$, there is a Shimura variety X_K defined over $\mathbb{Q}$ such that

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathbb{D} \times H(\mathbb{A}_f)/K).$$

We will identify X_K with $X_K(\mathbb{C})$ in this section. We assume that K fixes L and acts on L'/L trivially. The Hermitian symmetric domain $\mathbb{D}$ has two other useful forms. Let

$$\mathcal{L} = \{w \in V_\mathbb{C} : (w, w) = 0, \quad (w, \bar{w}) < 0\}. \tag{2.1}$$

Then one has an isomorphism

$$\mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}, \qquad w = u + iv \mapsto \mathbb{R}u + \mathbb{R}(-v).$$

This isomorphism gives a complex structure on $\mathbb{D}$, and we can view $\mathcal{L}$ as a line bundle over $\mathbb{D}$—the tautological line bundle. It descends to a line bundle $\mathcal{L}_K$ over X_K—the line bundle of modular forms of weight 1 on X_K. Finally, given an isotropic element $\ell \in V$, choose another element $\ell' \in V$ such that $(\ell, \ell') = 1$, and let $V_0 = (\mathbb{Q}\ell + \mathbb{Q}\ell')^\perp$. Then we have a tube domain (associated to (ℓ, ℓ')):

$$\mathcal{H} = H_{\ell,\ell'} = \{z = x + iy \in V_{0,\mathbb{C}} : Q(y) < 0\}.$$

The map

$$w = w_{\ell,\ell'} : \mathcal{H} \to \mathcal{L}, \qquad w(z) = z + \ell' - (Q(z) + Q(\ell'))\ell$$

gives an isomorphism $\mathcal{H}_{\ell,\ell'} \cong \mathcal{L}/\mathbb{C}^\times$ and actually a nowhere vanishing section of the line bundle $\mathcal{L}$. We emphasize that w depends on the choice of the primitive isotropic vector ℓ and the subspace $\mathbb{Q}\ell + \mathbb{Q}\ell'$ but not ℓ'. Furthermore, this map w induces an action of $\Gamma = K \cap H(\mathbb{Q})^+$ on $\mathcal{H}$ and an automorphy factor $j(\gamma, z)$ characterized by the following identity:

$$\gamma w(z) = \nu(\gamma) j(\gamma, z) w(\gamma z). \tag{2.2}$$

Here $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$, $H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+$, and $\nu(\gamma)$ is the spinor norm of γ. This action preserves the two connected components of $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$. A (meromorphic) function Ψ on $\mathcal{H}^+$ is called a (meromorphic) modular form for Γ of weight k if

$$\Psi(\gamma z) = j(\gamma, z)^k \Psi(z). \tag{2.3}$$

Alternatively, it is a section of the line bundle $\mathcal{L}_K^k$ over X_K.

For a vector $x \in V$ with $Q(x) > 0$ and $h \in H(\mathbb{A}_f)$, let

$$\begin{aligned} H_x &= \{g \in H : g(x) = x\}, \\ \mathbb{D}_x &= \{z \in \mathbb{D} : (x, z) = 0\}, \\ \text{and} \quad K_{x,h} &= H_x(\mathbb{A}_f) \cap hKh^{-1}. \end{aligned}$$

Then the map

$$H_x(\mathbb{Q})\backslash(\mathbb{D}_x \times H_x(\mathbb{A}_f)/K_{x,h}) \to X_K(\mathbb{C}), \quad [z, h_1] \mapsto [z, h_1 h]$$

gives a divisor $Z(x, h)$ in X_K. It is actually defined over $\mathbb{Q}$. For a rational number $m > 0$ and $\phi \in S(V_f)$, if there is an $x \in V$ with $Q(x) = m$, we define, following Kudla [Kud97a], the weighted special divisor

$$Z(m, \phi) = \sum_{h \in H_x(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \phi(h^{-1}x) Z(x, h).$$

When there is no $x \in V$ with $Q(x) = m$, we simply set $Z(m, \phi) = 0$.

Associated to the quadratic space V is a reductive dual pair $(\mathrm{SL}_2, O(V))$ and a Weil representation $\omega = \omega_{V,\psi}$ of $\mathrm{SL}_2(\mathbb{A})$ on $S(V_\mathbb{A}) = S(V_f) \otimes S(V_\infty)$, where $V_f = V \otimes_\mathbb{Q} \mathbb{A}_f$ and $V_\infty = V \otimes_\mathbb{Q} \mathbb{Q}_\infty = V \otimes_\mathbb{Q} \mathbb{R}$. Embed $\mathrm{SL}_2(\mathbb{Z})$ into $\mathrm{SL}_2(\hat{\mathbb{Z}})$ diagonally, and let $S_L \subset S(\mathbb{A}_f)$ be the subspace of Schwartz functions ϕ which is supported on $\hat{L}' = L' \otimes_\mathbb{Z} \hat{Z}$ and is $\hat{L}$-translation invariant; i.e., $\phi(x)$ depends only on $x \mod \hat{L}$. Then

$$S_L = \bigoplus_{\mu \in L'/L} \mathbb{C}\phi_\mu, \qquad \phi_\mu = \mathrm{Char}(\mu + \hat{L}).$$

It is easy to check that S_L is $\mathrm{SL}_2(\mathbb{Z})$-invariant under the Weil representation ω; we denote this representation ω_L. One has by definition

$$\omega_L(n(b))\phi_\mu = e(-bQ(\mu))\phi_\mu, \qquad b \in \mathbb{Z}, \tag{2.4}$$
$$\omega_L(w)\phi_\mu = e(\frac{n-2}{8})([L':L])^{-\frac{1}{2}} \sum_{\nu \in L'/L} e((\mu,\nu))\phi_\nu.$$

Here

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and we have used the fact that

$$\psi_f(x) = \psi_\infty(-x) = e(-x)$$

when $x \in \mathbb{Q}$. We also write

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

If we identify $S_L \cong \mathbb{C}[L'/L] = \bigoplus_{\mu \in L'/L} \mathbb{C}e_\mu$ via $\phi_\mu \mapsto e_\mu$ and let ρ_{L_-} be the Weil representation in [Bor98] (also [Bru02]) associated to the quadratic lattice L_-, where $L_- = L$ but with quadratic form $Q_-(x) = -Q(x)$, then one sees immediately that

$$\omega_L = \rho_{L_-}. \tag{2.5}$$

Recall that a meromorphic function $f : \mathbb{H} \to S_L$ is called a *weakly holomorphic modular form* of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ and ω_L if it satisfies the following conditions.

(i) One has $f\,|_{k,\omega_L}\,\gamma = f$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where

$$f\,|_{k,\omega_L}\,\gamma(\tau) = (c\tau+d)^{-k}\omega_L(\gamma)^{-1} f(\tau).$$

(ii) There is an S_L-valued Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{n \le 0} c(n,\mu)\, q^n\, \phi_\mu$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$.

The Fourier polynomial P_f is called the *principal part* of f. We denote the vector space of these weakly holomorphic modular forms by $M^!_{k,\omega_L}$. The Fourier expansion of any $f \in M^!_{k,\omega_L}$ is of the form

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n,\mu)\, q^n\, \phi_\mu. \tag{2.6}$$

With this notation, we define

$$Z(f) = \sum_{n>0, \mu \in L'/L} c(-n,\mu) Z(n,\mu). \tag{2.7}$$

Here $Z(m,\mu) = Z(m,\phi_\mu)$. Let $S_L^\vee$ be the dual space of S_L, the space of linear functionals on S_L, and let $\{\phi_\mu^\vee\}$ be the dual basis in $S_L^\vee$ of the basis $\{\phi_\mu\}$ of S_L.

Recall that the Siegel theta function (for $(z,h) \in X_K$)

$$\theta_L(\tau, z, h) = \sum_{\mu} \theta(\tau, z, h, \phi_\mu)\, \phi_\mu^\vee$$

is an $S_L^\vee$-valued holomorphic modular form of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ and $\omega_L^\vee$ defined as follows (see [BY09, Section 2] or [Kud03] for details). For $z \in \mathbb{D}$, consider the orthogonal decomposition

$$V_{\mathbb{R}} = z \oplus z^\perp, \qquad x = x_z + x_{z^\perp}.$$

Then for $\phi \in S(V_f)$ and $(z,h) \in X_K$, one defines

$$\theta(\tau, z, h, \phi) = v \sum_{x \in V} \phi(h^{-1}x) e(\tau Q(x_{z^\perp}) + \bar{\tau} Q(x_z)). \tag{2.8}$$

Here $v = \mathrm{Im}(\tau)$ is the imaginary part of τ. Notice that $\theta(\tau, z, 1, \phi_\mu) = \overline{\theta(\tau, z, \mu)}$ in comparison with Borcherds' Siegel theta functions.

We consider the regularized theta integral

(2.9)

$$\Phi(z, h, f) = \int_{\mathcal{F}}^{reg} \langle f(\tau), \theta_L(\tau, z, h)\rangle\, d\mu(\tau) = \int_{\mathcal{F}}^{reg} \sum_{\mu \in L'/L} f_\mu(\tau)\theta(\tau, z, h, \phi_\mu) d\mu(\tau)$$

for $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$. Here $\mathcal{F}$ is the standard domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, and we write

$$f(\tau) = \sum_{\mu \in L'/L} f_\mu(\tau)\phi_\mu.$$

The integral is regularized as in [Bor98]; that is, $\Phi(z,h,f)$ is defined as the constant term in the Laurent expansion at $s = 0$ of the function

$$\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), \theta_L(\tau, z, h)\rangle\, v^{-s} d\mu(\tau). \tag{2.10}$$

Here $\mathcal{F}_T = \{\tau \in \mathbb{H};\ |u| \le 1/2,\ |\tau| \ge 1, \text{ and } v \le T\}$ denotes the truncated fundamental domain, and the integrand

$$\langle f(\tau), \theta_L(\tau, z, h)\rangle = \sum_{\mu \in L'/L} f_\mu(\tau)\theta(\tau, z, h, \phi_\mu) \tag{2.11}$$

is the pairing of f with the Siegel theta function, viewed as a linear functional on the space S_L. We remark that our regularized theta integral $\Phi(z,h,f)$ is exactly the same as the one in [Bor98] and [Bru02] when $h = 1$.

The following is the first part of [Bor98, Theorem 13.3] (see also [Bru02, Theorem 3.22]) in our setting.

Theorem 2.1. *Let $f(\tau) = \sum c(m,\mu) q^m \phi_\mu \in M^!_{1-\frac{n}{2}, \omega_L}$ be a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ and ω_L, and assume that $c(m,\mu) \in \mathbb{Z}$ for $m < 0$. Then there is a meromorphic modular form $\Psi(z,h,f))$ of weight $k = c(0,0)/2$ on X_K (with some characters) such that:*

(1) *One has*

$$\mathrm{div}(\Psi(z,h,f)^2) = Z(f) = \sum_{m>0, \mu \in L'/L} c(-m,\mu) Z(m,\mu).$$

Here we count $Z(m,\mu)$ with multiplicity 2 or 1 depending on whether $2\mu \in L$ or not.

(2) *One has*

$$-\log \|\Psi(z,h,f)\|_{\mathrm{Pet}}^4 = \Phi(z,h,f).$$

Here $\| \ \|_{\mathrm{Pet}}$ *is a suitably normalized Petterson norm.*

To describe the Borcherds product expansion formula for $\Psi(z,h,f)$, we need some preparation. First, it works in each connected component. By the strong approximation theorem, one has

$$H(\mathbb{A}_f) = \coprod H(\mathbb{Q})^+ h_j K,$$

so

$$X_K = \coprod X_{\Gamma_j} = \coprod \Gamma_j \backslash \mathbb{D}^+,$$

where $\Gamma_j = H(\mathbb{Q})^+ \cap h_j K h_j^{-1}$ and $\mathbb{D}^+$ is one of the two connected components of $\mathbb{D}$. In this decomposition, one has

$$Z(m,\phi_\mu) = \sum_j Z_{L_j}(m,\mu_j),$$

where $L_j = h_j L = h_j \hat{L} \cap V$, and $\mu_j \in L_j'/L_j$ with $\mu_j - h_j\mu \in \hat{L}_j$, and

$$Z_{L_j}(m,\mu_j) = \{z \in \mathbb{D}^+ : (z,x) = 0 \text{ for some } x \in \mu_j + L_j, Q(x) = m\}.$$

In the following, we will stick with the irreducible component $X_\Gamma = \Gamma\backslash\mathbb{D}^+$ and the lattice L. The other components are the same.

Assume that V has an isotropic line $\mathbb{Q}\ell$ (a cusp). We assume that $\ell \in L$ is primitive, i.e., $L \cap \mathbb{Q}\ell = \mathbb{Z}\ell$. Choose $\ell' \in L'$ with $(\ell,\ell') = 1$. Assume further that $(\ell, L) = N_\ell\mathbb{Z}$ and choose $\xi \in L$ with $(\ell,\xi) = N_\ell$. Let $M = (\mathbb{Q}\ell + \mathbb{Q}\ell')^\perp \cap L$, and let

$$L_0' = \{x \in L' : (\ell,x) \equiv 0 \mod (N_\ell)\} \supset L.$$

Then there is a projection

$$p : L_0' \to M', \quad p(x) = x_M + \frac{(x,\ell)}{N_\ell}\xi_M, \tag{2.12}$$

where x_M and ξ_M are the orthogonal projections of $x, \xi \in V$ to $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. The projection p has the nice property $p(L) \subset M$ although it is not an orthogonal projection anymore (see [Bru02, pp. 40-41]). So it induces a projection from L_0'/L to M'/M.

Next, we define the Weyl chamber for

$$f = \sum f_\mu \phi_\mu = \sum c(m,\mu) q^m \phi_\mu \in M^!_{1-\frac{n}{2},\omega_L}.$$

Define

$$f_M(\tau) = \sum_{\lambda \in M'/M} f_\lambda(\tau)\phi_{\lambda,M} = \sum c_M(m,\lambda) q^m \phi_{\lambda,M}, \tag{2.13}$$

where $\phi_{\lambda,M} = \mathrm{Char}(\lambda + \hat{M})$,

$$f_\lambda(\tau) = \sum_{\substack{\mu \in L_0'/L \\ p(\mu)=\lambda}} f_\mu(\tau). \tag{2.14}$$

Then f_M is an S_M-valued modular form by Borcherds [Bor98, Theorem 5.3] with Weil representation ω_M.

Let $\mathrm{Gr}(M)$ be the set of negative lines in $M_{\mathbb{R}}$ (the Grassmannian), which is a real manifold of dimension $n-1$ (as M has signature $(n-1,1)$). For $\lambda \in M'/M$ and $m \in \mathbb{Q}$ with $m \equiv Q(\lambda) \pmod 1$, let

$$Z_M(m,\lambda) = \{z \in \mathrm{Gr}(M) : (z,x) = 0 \text{ for some } x \in \lambda + M, Q(x) = m\},$$

which is either empty or a real divisor of $\mathrm{Gr}(M)$. The *Weyl chamber* associated to a weakly holomorphic form $f \in M^!_{1-\frac{n}{2},\omega_L}$ is the connected components of (see [Bru02, p. 88])

$$\mathrm{Gr}(M) - \bigcup_{\mu \in L_0'/L} \bigcup_{\substack{m \in Q(\mu)+\mathbb{Z} \\ m>0, c(-m,\mu)\neq 0}} Z_M(m, p(\mu)).$$

Given a Weyl chamber W associated to f, we define its *Weyl vector* $\rho(W,f) = \rho(W, f_M) \in M'$ following Borcherds as in ([Bor98, Section 10.4]; see also [Bru02, p. 88]). Let $\bar{W}$ be the closure of W. If $M \cap \bar{W}$ is anisotropic, it was defined in [Bor98, Section 9] with correction and extension given recently in [BS17, Section 5]. We don't need it in this paper and refer to [BS17] for details. When $M \cap \bar{W}$ is isotropic, choose an isotropic $\ell_M \in M \cap \bar{W}$ and $\ell'_M \in M'$ with $(\ell_M, \ell'_M) = 1$. Let $P = M \cap (\mathbb{Q}\ell_M + \mathbb{Q}\ell'_M)^{\perp}$, which is positive definite of rank $n-2$. Similar to the projection p from L_0'/L to M'/M, one has also a projection p from M_0'/M to P'/P defined in the same way. For the same reason, we have the weakly holomorphic modular form f_P (coming from f_M). Define

$$\rho_{\ell'_M} = \text{constant term of } \theta_P(\tau) f_P(\tau) E_2(\tau)/24, \tag{2.15}$$

$$\rho_{\ell_M} = -\rho_{\ell'_M} Q(\ell'_M) - \frac{1}{4} \sum_{\substack{\lambda \in M_0'/M \\ p(\lambda) = 0 + P}} c_M(0,\lambda) \mathbf{B}_2((\lambda, \ell'_M)) - \frac{1}{2} \sum_{\substack{\gamma \in P' \\ (\gamma, W) > 0}} \sum_{\substack{\lambda \in M_0'/M \\ p(\lambda) = \gamma + P}} c_M(-Q(\gamma),\lambda) \mathbf{B}_2((\lambda, \ell'_M)), \tag{2.16}$$

$$\rho_P = -\frac{1}{2} \sum_{\substack{\gamma \in P' \cap M' \\ (\gamma, W) > 0}} c_M(-Q(\gamma), \gamma)\gamma, \tag{2.17}$$

$$\rho(W,f) = \rho_P + \rho_{\ell_M}\ell_M + \rho_{\ell'_M}\ell'_M. \tag{2.18}$$

Here

$$E_2 = 1 - 24 \sum_{n>0} \sigma_1(n) q^n$$

is the weight 2 Eisenstein series, and $\mathbf{B}_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$ is the second Bernouli polynomial of $\{x\}$, where $0 \le \{x\} = x - [x] < 1$ is the fractional part of x.

Now we can state the beautiful product expansion formula of Borcherds as follows in the signature $(n,2)$ case ([Bor98, Theorem 13.3]; see also [Bru02, Theorem 3.22]).

Theorem 2.2 (Borcherds)**.** *Let the notation be as above. Let W be a Weyl chamber of f whose closure contains ℓ_M. Then the memomorphic automorphic form*

$\Psi(z,f)=\Psi(z,1,f)$ has an infinite product expansion near the cusp $\mathbb{Q}\ell$ (more precisely, when $\mathrm{Im}(z)\in W$ with $-Q(\mathrm{Im}(z))$ sufficiently large):

$$\Psi(z,f)=Ce((z,\rho(W,f)))\prod_{\substack{\lambda\in M'\\(\lambda,W)>0}}\prod_{\substack{\mu\in L_0'/L\\p(\mu)\in\lambda+M}}[1-e((\lambda,z)+(\mu,\ell'))]^{c(-Q(\lambda),\mu)}.$$

Here C is a constant with absolute value

$$\left|\prod_{\delta\in\mathbb{Z}/N_\ell,\delta\neq0}(1-e(\frac{\delta}{N_\ell}))^{\frac{c(0,\frac{\delta}{N_\ell}\ell)}{2}}\right|. \tag{2.19}$$

Sketch of proof. We derive the formula from [Bor98, Theorem 13.3]. Let $L_-=L$ with quadratic form $Q_-(x)=-Q(x)$ so that L_- has signature $(2,n)$, for which we can apply Borcherds' theorem. We use subscript $-$ to indicate the corresponding notation in Borcherds. First notice that the symmetric domain $\mathbb{D}_-=\mathbb{D}$ and the tautological bundle $\mathcal{L}_-=\mathcal{L}$. Since $(\ell,\ell')=1$, one has $(-\ell,\ell')_-=1$. So the tube domains $\mathcal{H}_{\ell,\ell'}$ and $\mathcal{H}_{-\ell,\ell',-}$ are the same too. Furthermore, for $z\in\mathcal{H}_{\ell,\ell'}=\mathcal{H}_{-\ell,\ell',-}$, one has

$$w_-(z)=z+\ell'-(Q_-(z)+Q_-(\ell'))(-\ell)=w(z).$$

Notice that $f|\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)=f$ implies that

$$c(m,\mu)=c(m,-\mu)\quad\text{and}\quad c_M(m,\lambda)=c_M(m,-\lambda). \tag{2.20}$$

Since $(\ell_M,\ell_M')=1$, one has $(\ell_M,-\ell_M')_-=1$. Using [Bor98, Theorem 10.4] (a minor mistake there missing the $\frac14$ summation part), one checks that $\rho_{\ell_M',-}=\rho_{\ell_M'}$ (as $\theta_{P,-}=\overline{\theta_P}$), and

$$\begin{aligned}\rho_{\ell_M,-}&=-\rho_{\ell_M',-}Q_-(\ell_M')+\frac14\sum_{\substack{\lambda\in M_0'/M,\\p(\lambda)=0}}c_M(0,\lambda)\mathbf{B}_2((\lambda,-\ell_M')_-)\\&\quad+\frac12\sum_{\substack{\gamma\in P'\\(\gamma,W)_->0}}\sum_{\substack{\lambda\in M_0'/M,\\p(\lambda)=0+P}}c_M(Q_-(\gamma),\lambda)\mathbf{B}_2((\lambda,-\ell_M')_-)\\&=\rho_{\ell_M'}Q(\ell_M')+\frac14\sum_{\substack{\lambda\in M_0'/M,\\p(\lambda)=\gamma+P}}c_M(0,\lambda)\mathbf{B}_2((\lambda,\ell_M'))\\&\quad+\frac12\sum_{\substack{\gamma\in P'\\(\gamma,W)>0}}\sum_{\substack{\lambda\in M_0'/M,\\p(\lambda)=\gamma+P}}c_M(-Q(\gamma),\lambda)\mathbf{B}_2((\lambda,\ell_M'))\\&=-\rho_{\ell_M}.\end{aligned}$$

In the last identity, we substitute γ by $-\gamma$ and λ by $-\lambda$ and apply (2.20). Similarly, one checks that $\rho_{P,-}=-\rho_P$. So Borcherds' Weyl vector

$$\rho(W,f)_-=\rho_{P,-}+\rho_{\ell_M,-}\ell_M+\rho_{\ell_M'}(-\ell_M')=-\rho(W,f).$$

So [Bor98, Theorem 13.3] gives for $z \in \mathcal{H}_{\ell,\ell'}$,

$$\begin{aligned}\Psi(z,f) &= Ce((z,\rho(W,f)_-)_-) \prod_{\substack{\lambda\in M' \\ (\lambda,W)_->0}} \prod_{\substack{\mu\in L_0'/L \\ p(\mu)=\lambda+M}} [1-e((\lambda,z)_- + (\mu,\ell')_-)]^{c(Q_-(\lambda),\mu)} \\ &= Ce((z,\rho(W,f))) \prod_{\substack{\lambda\in M' \\ (\lambda,W)>0}} \prod_{\substack{\mu\in L_0'/L \\ p(\mu)=\lambda+M}} [1-e((\lambda,z) + (\mu,\ell'))]^{c(-Q(\lambda),\mu)}\end{aligned}$$

as claimed. Here we again replace λ and μ by $-\lambda$ and $-\mu$ and apply (2.20). □

Remark 2.3. It is worthwhile to make a few remarks to clear up some (potentially confusing) differences in different versions.

(1) The sign difference in the formula above and the formula in [Bor98, Theorem 13.3] (and [Bru02, Theorem 3.22]) is due to the fact that they use L_- (signature $(2,n)$) while we use L.
(2) The condition $p(\mu) \in \lambda + M$ here is a more explicit reinterpretation of Borcherds' condition $\mu|M = \lambda$ given by Bruinier ([Bru02, Theorem 3.22]).
(3) The constant C can be taken as the product in (2.19) at a given cusp. However, once it is fixed, the constants at other cusps are determined by this constant (they are in the same connected component).
(4) The conditions that $n \geq 3$ and that M is isotropic in [Bru02] were for convenience and not necessary.
(5) The neighborhood near the cusp $\mathbb{Q}\ell$ where the product formula is valid can be made precise. We refer to [Bru02, Theorem 3.22] for details.
(6) At different cusps, the product formulae look different. This is similar to the different Fourier expansions of a modular form at different cusps.

2.2. **Big CM cycles, incoherent Eisenstein series, and the big CM value formula.** Let E be a CM number field of degree $n+2$ with the maximal totally real subfield F. Let $\sigma_i, 1 \leq i \leq \frac{n}{2}+1$, be distinct real embeddings of F. Choose an element $\alpha \in F$ with $\sigma_{\frac{n}{2}+1}(\alpha) < 0$ and $\sigma_i(\alpha) > 0$ for all $1 \leq i \leq \frac{n}{2}$, and let $W = E$ with the F-quadratic form $Q_F(z) = \alpha z\bar{z}$. Let $W_{\mathbb{Q}} = E$ with the $\mathbb{Q}$-quadratic form

$$Q_{\mathbb{Q}}(z) = \operatorname{tr}_{F/\mathbb{Q}} Q_F(z) = \operatorname{tr}_{F/\mathbb{Q}}(\alpha z\bar{z}).$$

Notice that $(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ is a $\mathbb{Q}$-quadratic space of signature $(n,2)$. Now we assume that $(W_{\mathbb{Q}}, Q_{\mathbb{Q}}) \cong (V,Q)$, where (V,Q) is a given $\mathbb{Q}$-quadratic space of signature $(n,2)$. Write $n_0 = \frac{n}{2}+1$. Then we have

$$V_{\mathbb{R}} \cong \bigoplus_{1\leq i\leq n_0} W_{\sigma_i},$$

where $W_{\sigma_i} = W \otimes_{F,\sigma_i} \mathbb{R}$ has signature $(2,0)$ or $(0,2)$ according to $1 \leq i < n_0$ or $i = n_0$. The negative two plane $W_{\sigma_{n_0}}$ gives rise to two 'big' CM points $z^{\pm}_{\sigma_{n_0}}$, which turn out to be defined over a finite extension of $\sigma_{n_0}(F)$. Define an algebraic torus T over $\mathbb{Q}$ by the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & T & \longrightarrow & \operatorname{Res}_{F/\mathbb{Q}} \mathrm{SO}(W) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & H & \longrightarrow & \mathrm{SO}(V) & \longrightarrow & 1. \end{array} \tag{2.21}$$

Then T is a maximal torus in $H = \mathrm{GSpin}(V)$ (thus the names big CM points and big CM cycles). It is known ([BKY12, Section 2]) that

$$Z(W, z^{\pm}_{\sigma_{n_0}}) = \{z^{\pm}_{\sigma_{n_0}}\} \times (T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T), \qquad K_T = K \cap T(\mathbb{A}_f)$$

is a zero cycle in X_K defined over F, called a big CM cycle of X_K. Let $Z(W)$ be the formal sum of all its Galois conjugates (counting multiplicity), which is a big CM cycle of X_K over $\mathbb{Q}$. We refer to [BKY12, Section 2] for a more precise definition and basic properties of this cycle.

Associated to this quadratic space and the additive adelic character $\psi_F = \psi \circ \mathrm{tr}_{F/\mathbb{Q}}$ is a Weil representation $\omega = \omega_{\psi_F}$ of $\mathrm{SL}_2(\mathbb{A}_F)$ (and thus $T(\mathbb{A}_{\mathbb{Q}})$) on $S(W(\mathbb{A}_F)) = S(V(\mathbb{A}_{\mathbb{Q}}))$. Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of F associated to E/F. Then $\chi = \chi_W$ is also the quadratic Hecke character of F associated to W, and there is an $\mathrm{SL}_2(\mathbb{A}_F)$-equivariant map

$$\lambda = \prod \lambda_v : S(W(\mathbb{A}_F)) \to I(0, \chi), \qquad \lambda(\phi)(g) = \omega(g)\phi(0). \tag{2.22}$$

Here $I(s,\chi) = \mathrm{Ind}_{B_{\mathbb{A}_F}}^{\mathrm{SL}_2(\mathbb{A}_F)} \chi|\cdot|^s$ is the principal series, whose sections (elements) are smooth functions Φ on $\mathrm{SL}_2(\mathbb{A}_F)$ satisfying the condition

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g,s), \qquad b \in \mathbb{A}_F \quad \text{and} \quad a \in \mathbb{A}_F^\times.$$

Here $B = NM$ is the standard Borel subgroup of SL_2. Such a section is called factorizable if $\Phi = \otimes \Phi_v$ with $\Phi_v \in I(s, \chi_v)$. It is called standard if $\Phi|_{\mathrm{SL}_2(\hat{\mathcal{O}}_F)\,\mathrm{SO}_2(\mathbb{R})^{n_0}}$ is independent of s. For a standard section $\Phi \in I(s,\chi)$, its associated Eisenstein series is defined as

$$E(g, s, \Phi) = \sum_{\gamma \in B_F \backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s)$$

for $\Re(s) \gg 0$.

For $\phi \in S(V_f) = S(W_f)$, let Φ_f be the standard section associated to $\lambda_f(\phi) \in I(0, \chi_f)$. For each real embedding $\sigma_i : F \hookrightarrow \mathbb{R}$, let $\Phi_{\sigma_i} \in I(s, \chi_{\mathbb{C}/\mathbb{R}}) = I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}})$ be the unique 'weight one' eigenvector of $\mathrm{SL}_2(\mathbb{R})$ given by

$$\Phi_{\sigma_i}(n(b)m(a)k_\theta) = \chi_{\mathbb{C}/\mathbb{R}}(a)|a|^{s+1}e^{i\theta},$$

for $b \in \mathbb{R}$, $a \in \mathbb{R}^\times$, and $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SO}_2(\mathbb{R})$. We define for $\vec{\tau} = (\tau_1, \dots, \tau_{n_0}) \in \mathbb{H}^{n_0}$,

$$E(\vec{\tau}, s, \phi) = \mathrm{N}(\vec{v})^{-\frac{1}{2}} E(g_{\vec{\tau}}, s, \Phi_f \otimes (\bigotimes_{1 \le i \le n_0} \Phi_{\sigma_i})),$$

where $\vec{v} = \mathrm{Im}(\vec{\tau})$, $\mathrm{N}(\vec{v}) = \prod_i v_i$, and $g_{\vec{\tau}} = (n(u_i)m(\sqrt{v_i}))_{1 \le i \le n_0}$. It is a (non-holomorphic) Hilbert modular form of scalar weight 1 for some congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$. Following [BKY12], we further normalize

$$E^*(\vec{\tau}, s, \phi) = \Lambda(s+1, \chi) E(\vec{\tau}, s, \phi),$$

where ∂_F is the different of F, $d_{E/F}$ is the relative discriminant of E/F, and

$$\Lambda(s, \chi) = A^{\frac{s}{2}} (\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2}))^{n_0} L(s, \chi), \qquad A = \mathrm{N}_{F/\mathbb{Q}}(\partial_F d_{E/F}). \tag{2.23}$$

The Eisenstein series is incoherent in the sense that $\Phi = \bigotimes \Phi_v$ is in the image of λ on $S(\mathcal{C})$, where $\mathcal{C}$ is an incoherent system of quadratic spaces over F_v, given by $\mathcal{C}_v = W_v$ for all places v except the one $v = \sigma_{n_0}$. This incoherence forces $E^*(\vec{\tau}, 0, \phi) = 0$ automatically.

Proposition 2.4 ([BKY12, Proposition 4.6])**.** *Let $\phi \in S(V_f) = S(W_f)$. For a totally positive element $t \in F_+^\times$, let $a(t,\phi)$ be the t-th Fourier coefficient of $E^{*,\prime}(\vec{\tau}, 0, \phi)$ and write the constant term of $E^{*,\prime}(\vec{\tau}, 0, \phi)$ as*

$$\phi(0)\Lambda(0,\chi)\log \mathrm{N}(\vec{v}) + a_0(\phi).$$

Let

$$\mathcal{E}(\tau,\phi) = a_0(\phi) + \sum_{n\in\mathbb{Q}_{>0}} a_n(\phi)\, q^n$$

where (for $n > 0$)

$$a_n(\phi) = \sum_{t\in F_+^\times,\ \mathrm{tr}_{F/\mathbb{Q}}\, t = n} a(t,\phi).$$

Here $F_+^\times$ consists of all totally positive elements in F. Then, writing $\tau^\Delta = (\tau,\dots,\tau)$ for the diagonal image of $\tau \in \mathbb{H}$ in $\mathbb{H}^{n_0}$,

$$E^{*,\prime}(\tau^\Delta, 0, \phi) - \mathcal{E}(\tau,\phi) - \phi(0)\,(\frac{n}{2}+1)\,\Lambda(0,\chi)\,\log v$$

is of exponential decay as v goes to infinity. Moreover, for $n > 0$,

$$a_n(\phi) = \sum_p a_{n,p}(\phi)\log p$$

with $a_{n,p}(\phi) \in \mathbb{Q}(\phi)$, the subfield of $\mathbb{C}$ generated by the values $\phi(x)$, $x \in V(\mathbb{A}_f)$.

Remark 2.5. There is a minor mistake in [BKY12, Proposition 4.6]) about the constant. The corrected form is

$$E_0^{*,\prime}(\vec{\tau}, 0, \phi) = \phi(0)\Lambda(0,\chi)\log \mathrm{N}(\vec{v}) + a_0(\phi)$$

(i.e., $a_0(\phi)$ might not be a multiple of $\phi(0)$). Direct calculation gives

$$E_0^{*}(\vec{\tau}, s, \phi) = \phi(0)\Lambda(s+1,\chi)(\mathrm{N}(\vec{v}))^{\frac{s}{2}} - (\mathrm{N}(\vec{v}))^{-\frac{s}{2}}\Lambda(s,\chi)\tilde{W}_{0,f}(s,\phi)$$

where (when ϕ is factorizable)

$$\tilde{W}_{0,f}(s,\phi) = \prod_{\mathfrak{p}\nmid\infty} \tilde{W}_{0,\mathfrak{p}}(s,\phi_{\mathfrak{p}}) = \prod_{\mathfrak{p}\nmid\infty} \frac{|A|_{\mathfrak{p}}^{-\frac{1}{2}} L_{\mathfrak{p}}(s+1,\chi)}{\gamma(W_{\mathfrak{p}})L_{\mathfrak{p}}(s,\chi)} W_{0,\mathfrak{p}}(s,\phi_{\mathfrak{p}})$$

is the product of renormalized local Whittaker functions (see (2.25)). With this notation, one has

$$a_0(\phi) = -\Lambda(0,\chi)\tilde{W}_{0,f}'(0,\phi) - 2\phi(0)\Lambda'(0,\chi). \tag{2.24}$$

Notice that $a(t,\phi_\mu) = 0$ automatically unless $\mu + \hat{L}$ represents t, i.e., $t - Q_F(\mu) \in \partial_F^{-1}\mathcal{O}_F$. The following is a special case of the main CM value formula of Bruinier, Kudla, and Yang ([BKY12, Theorem 5.2]).

Theorem 2.6. *Let*

$$f(\tau) = \sum_{\mu\in L'/L} f_\mu(\tau)\phi_\mu = \sum c(m,\mu)q^m\phi_\mu \in M^!_{1-\frac{n}{2},\omega_L}$$

with $c(0,0)=0$*, and let* $\Psi(z,f)$ *be its Borcherds lifting. Then*

$$-\log|\Psi(Z(W),f)|^4 = C(W,K)\left(\sum_{\substack{\mu\in L'/L,\\ m\ge 0\\ m\equiv Q(\mu) \pmod 1}} c(-m,\mu)a_m(\phi_\mu)\right).$$

Here

$$C(W,K)=\frac{\deg(Z(W,z_{\sigma_2}^{\pm}))}{\Lambda(0,\chi)}.$$

To compute the t-th Fourier coefficient $a(t,\phi)$ of $E^{*,\prime}(\vec{\tau},0,\phi)$, one may assume that $\phi=\bigotimes\phi_{\mathfrak{p}}$ is factorizable by linearity. Write for $t\neq 0$

$$E_t^*(\vec{\tau},s,\phi)=\prod_{\mathfrak{p}\nmid\infty}W_{t,\mathfrak{p}}^*(s,\phi)\prod_{j=1}^{n_0}W_{t,\sigma_j}^*(\tau_j,s,\Phi_{\sigma_j}),$$

where

$$W_{t,\mathfrak{p}}^*(s,\phi)=|A|_{\mathfrak{p}}^{-\frac{s+1}{2}}L_{\mathfrak{p}}(s+1,\chi_{\mathfrak{p}})W_{t,\mathfrak{p}}(s,\phi)$$

for finite prime $\mathfrak{p}$ with

$$W_{t,\mathfrak{p}}(s,\phi)=\int_{F_{\mathfrak{p}}}\omega(wn(b))(\phi_{\mathfrak{p}})(0)|a(wn(b))|_{\mathfrak{p}}^s\psi_{\mathfrak{p}}(-tb)\,db, \tag{2.25}$$

and for infinite prime σ_j

$$W_{t,\sigma_j}^*(\tau_j,s,\Phi_{\sigma_j})=v_j^{-1/2}\pi^{-\frac{s+2}{2}}\Gamma(\frac{s+2}{2})\int_{\mathbb{R}}\Phi_{\sigma_j}(wn(b)g_{\tau_j},s)\psi(-bt)db.$$

Here A is defined in (2.23) and $|a(g)|_{\mathfrak{p}}=|a|_{\mathfrak{p}}$ if $g=n(b)m(a)k$ with $k\in \mathrm{SL}_2(\mathcal{O}_{\mathfrak{p}})$.

The following proposition is well-known and is recorded here for reference. Recall that $W=E$ with $Q_F(z)=\alpha z\bar{z}$, $\alpha\in F^\times$.

Proposition 2.7. *For a totally positive number* $t\in F^+$*, let*

$$\mathrm{Diff}(W,t)=\{\mathfrak{p}:\ W_{\mathfrak{p}} \text{ does not represent } t\}$$

be the so-called 'Diff' set of Kudla. Then $|\mathrm{Diff}(W,t)|$ *is finite and odd. Moreover:*

(1) *If* $|\mathrm{Diff}(W,t)|>1$*, then* $a(t,\phi)=0$*.*
(2) *If* $\mathrm{Diff}(W,t)=\{\mathfrak{p}\}$*, then* $W_{t,\mathfrak{p}}^*(0,\phi)=0$*, and*

$$a(t,\phi)=(-2i)^{n_0}W_{t,\mathfrak{p}}^{*,\prime}(0,\phi)\prod_{\mathfrak{q}\nmid\mathfrak{p}\infty}W_{t,\mathfrak{q}}^*(0,\phi).$$

(3) *When* $\mathfrak{p}\nmid\alpha A$ *is unramified in* E/F *and* $\phi_{\mathfrak{p}}=\mathrm{Char}(\mathcal{O}_{E_{\mathfrak{p}}})$*,* $W_{t,\mathfrak{p}}^*(s,\phi)=0$ *unless* $t\in\partial_F^{-1}$*. In this case, one has*

$$\frac{W_{t,\mathfrak{p}}^*(0,\phi)}{\gamma(W_{\mathfrak{p}})}=\begin{cases}1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D}) & \text{if } \mathfrak{p} \text{ is split in } E,\\ \frac{1+(-1)^{\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D})}}{2} & \text{if } \mathfrak{p} \text{ is inert in } E.\end{cases}$$

Here $\gamma(W_{\mathfrak{p}})$ *is the local Weil index (an 8-th root of unity) associated to the Weil representation. Moreover, in this case,* $W_{t,\mathfrak{p}}^*(0,\phi)=0$ *if and only if* $\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D})$ *is odd and* $\mathfrak{p}$ *is inert in* E*. In such a case, one has*

$$\frac{W_{t,\mathfrak{p}}^{*,\prime}(0,\phi)}{\gamma(W_{\mathfrak{p}})}=\frac{1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D})}{2}\log \mathrm{N}(\mathfrak{p}).$$

(4) *One has for* $1 \le j \le n_0$,

$$W^*_{t,\sigma_j}(\tau, 0, \Phi_{\sigma_j}) = -2ie(t\tau), \qquad t > 0,$$

and

$$W^*_{0,\sigma_j}(\tau, s, \Phi_{\sigma_j}) = -i\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})v^{-\frac{s}{2}}.$$

Sketch of proof. The Diff set was first defined by Kudla in [Kud97b]. In our case, the incoherent collection of F_v-quadratic spaces is $\{\mathcal{C}_v\}$ where $\mathcal{C}_v = W_v$ for $v \ne \sigma_{n_0}$ and $\mathcal{C}_{n_0}$ positive definite. The archimedian places are not in the Diff set as t is totally positive. Let $\psi'_F(x) = \psi_F(\frac{x}{\sqrt{D}})$ and $W' = W$ with F-quadratic form $Q'_F(x) = \sqrt{D}Q_F(x) = x\bar{x}$. Then one has as Weil representations on $S(W_f) = S(W'_f)$:

$$\omega_{W,\psi_F} = \omega_{W',\psi'_F},$$

and thus the Whittaker functions have the relation

$$W^{\psi_F}_{t,\mathfrak{p}}(s, \phi) = |\sqrt{D}|^{\frac{1}{2}}_{\mathfrak{p}} W^{\psi'}_{t\sqrt{D},\mathfrak{p}}(s, \phi)$$

for each prime $\mathfrak{p}$ of F. Recall that $W^*_{t,\mathfrak{p}}(0, \phi) = 0$ if $\mathfrak{p} \in \mathrm{Diff}(W, t)$. So (1) is obvious. Claim (3) follows from [Yan05, Proposition 2.1]. Claim (4) is a special case of [KRY99, Proposition 2.6]. Claim (2) follows from

$$E^*_t(\vec{\tau}, s, \phi) = \prod_{\mathfrak{p}\nmid\infty} W^*_{t,\mathfrak{p}}(s, \phi) \prod_{j=1}^{n_0} W^*_{t,\sigma_j}(\tau_j, s, \Phi_{\sigma_j})$$

and (4). □

3. Product of modular curves and its diagonal divisor

3.1. **Product of modular curves as a Shimura variety of orthogonal type (2, 2).** Let N be a positive integer, and let $V = M_2(\mathbb{Q})$ with the quadratic form $Q(X) = N \det X$. Let H be the algebraic group over $\mathbb{Q}$

$$H = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det g_1 = \det g_2\}.$$

Then $H \cong \mathrm{GSpin}(V)$ and acts on V via

$$(g_1, g_2)X = g_1 X g_2^{-1}.$$

One has the exact sequence

$$1 \to \mathbb{G}_m \to H \to \mathrm{SO}(V) \to 1.$$

Recall the Hermitian symmetric domain $\mathbb{D}$ and the tautological line bundle $\mathcal{L}$ in Section 2. For a tube domain, take an isotropic matrix $\ell = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \in L$ and $\ell' = \begin{pmatrix} 0 & 0 \\ \frac{1}{N} & 0 \end{pmatrix} \in V$ with $(\ell, \ell') = 1$. Then the associated tube domain is

$$\mathcal{H}_{\ell,\ell'} = \{\left(\begin{smallmatrix} z_1 & 0 \\ 0 & -z_2 \end{smallmatrix}\right) : y_1 y_2 > 0\}, \qquad y_i = \mathrm{Im}(z_i),$$

together with

$$w : \mathcal{H}_{\ell,\ell'} \to \mathcal{L}, \quad w(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & -z_2 \end{smallmatrix}\right)) = \begin{pmatrix} z_1 & -Nz_1z_2 \\ \frac{1}{N} & -z_2 \end{pmatrix}.$$

Now the following proposition is clear.

Proposition 3.1. *Define*

$$w_N : \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \to \mathcal{L}, \quad w_N(z_1, z_2) = \frac{1}{N}\left(\begin{smallmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{smallmatrix}\right) = w(\begin{pmatrix} \frac{z_1}{N} & 0 \\ 0 & -\frac{z_2}{N} \end{pmatrix}),$$

and

$$pr : \mathcal{L} \to \mathbb{D}, \quad x + iy \mapsto z = \mathbb{R}x + \mathbb{R}(-y).$$

Then pr gives an isomorphism between $\mathcal{L}/\mathbb{C}^\times$ and $\mathbb{D}$, and the composition $pr \circ w$ gives an isomorphism between $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ and $\mathbb{D}$. Moreover, w_N is H-equivariant, where $H \subset \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ via the usual linear fraction transformations

$$(g_1, g_2)(z_1, z_2) = (g_1(z_1), g_2(z_2)),$$

and acts on $\mathcal{L}$ and $\mathbb{D}$ naturally via its action on V. Moreover, one has

$$(g_1, g_2) w_N(z_1, z_2) = \frac{(c_1 z_1 + d_1)(c_2 z_2 + d_2)}{\nu(g_1, g_2)} w_N(g_1(z_1), g_2(z_2)), \tag{3.1}$$

where $\nu(g_1, g_2) = \det g_1 = \det g_2$ is the spin character of $H = \mathrm{GSpin}(V)$. So

$$j(g_1, g_2, z_1, z_2) = (c_1 z_1 + d_1)(c_2 z_2 + d_2).$$

For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, let X_Γ be the associated open modular curve over $\mathbb{Q}$ such that $X_\Gamma(\mathbb{C}) = \Gamma\backslash\mathbb{H}$. Assume $\Gamma \supset \Gamma(M)$ for some integer $M \geq 1$. Let

$$\nu : \mathbb{A}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{A}), \quad \nu(d) = \mathrm{diag}(1, d).$$

Let $K(\Gamma)$ be the product of $\nu(\hat{\mathbb{Z}}^\times)$ and the preimage of $\Gamma/\Gamma(M)$ in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ (under the map $\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/M)$). Let $K = (K(\Gamma) \times K(\Gamma)) \cap H(\mathbb{A}_f)$. Then one has by the strong approximation theorem

$$X_K \cong X_\Gamma \times X_\Gamma.$$

In this way, we have identified the product of two copies of a modular curve X_Γ with a Shimura variety X_K. We will fix this K for a given congruence subgroup Γ in this paper. The tautological line bundle $\mathcal{L}$ descends to a line bundle $\mathcal{L}_K = K\backslash\mathcal{L}$ of modular forms of 2 variables of weight $(1, 1)$ by (3.1).

Let L be an even integral lattice of V, and let L' be its dual with respect to the quadratic form Q. We assume that $\Gamma \times \Gamma$ acts on L'/L trivially. Then for $\mu \in L'/L$ and a rational number $m > 0$ (and $m \equiv Q(\mu) \pmod 1$), the associated special divisor $Z(m, \mu) = Z_K(m, \mu)$ is given in this special case by

$$Z(m, \mu) = (\Gamma \times \Gamma)\backslash\{(z_1, z_2) \in \mathbb{H}^2 : w_N(z_1, z_2) \perp x \text{ for some } x \in \mu + L, Q(x) = m\}. \tag{3.2}$$

Alternatively, $Z(m, \mu)$ is the sum of $Z(x)$, where $x \in \mu + L$ with $Q(x) = m$ modulo the action of $\Gamma \times \Gamma$. Here $Z(x)$ is the subvariety of X_K given by $x^\perp$ (of signature $(1, 2)$):

$$Z(x) = (\Gamma \cap x^{-1}\Gamma x)\backslash\mathbb{H} \cong (\Gamma \times \Gamma)_x\backslash\{(xz, z) : z \in \mathbb{H}\}, \quad [z] \mapsto [xz, z].$$

The linear combinations of these divisors $Z(m, \mu)$ are called the special divisors of X_K.

Lemma 3.2. *Let $\Gamma = \Gamma(N)$ and $L = M_2(\mathbb{Z})$ with $Q(X) = N \det X$. For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, let*

$$Z_N(\gamma) = \{(\gamma z, z) \in X_{\Gamma(N)} \times X_{\Gamma(N)} : z \in X_{\Gamma(N)}\} \subset X_{\Gamma(N)} \times X_{\Gamma(N)}.$$

Then $Z_N(\gamma) = Z(\frac{1}{N}, \frac{1}{N}\gamma + L)$ *is a special divisor of* X_K.

We denote by $X(N)$ the compactification of $X_{\Gamma(N)}$ (to be compatible with the usual definition of $X(N)$).

Proof. If $x \in \frac{1}{N}\gamma + L$ with $Q(x) = N \det x = 1/N$, then $Nx \in \gamma + NL$ and $\det(Nx) = 1$. So $Nx \in \mathrm{SL}_2(\mathbb{Z})$, and $Nx\gamma^{-1} = \gamma_1 \in \Gamma(N)$, and $x = \gamma_1(\frac{1}{N}\gamma)$. This implies that

$$Z(\frac{1}{N}, \frac{1}{N}\gamma + L) = Z_N(\gamma).$$

□

Corollary 3.3. *Let* X_Γ^Δ *be the diagonal embedding of* X_Γ *into* $X_\Gamma \times X_\Gamma$. *The* X_Γ^Δ *is a special divisor of* $X_\Gamma \times X_\Gamma$ *in the following sense. Assuming* $\Gamma \supset \Gamma(N)$, *we take* $L = M_2(\mathbb{Z})$ *with* $Q(X) = N \det$. *Then the preimage of* X_Γ^Δ *in* $X_{\Gamma(N)} \times X_{\Gamma(N)}$ *is equal to*

$$\sum_{\gamma \in \Gamma/\Gamma(N)} Z_N(\gamma)$$

in the notation of Lemma 3.2.

3.2. **Products of CM cycles as big CM cycles.** For $j = 1, 2$, let $\mathrm{E}_j = \mathbb{Q}(\sqrt{d_j})$ with ring of integers $\mathcal{O}_j = \mathbb{Z}[\frac{d_j + \sqrt{d_j}}{2}]$ of discriminant $d_j < 0$ with $(d_1, d_2) = 1$. In this subsection, we describe how to view a pair of CM points $(\tau_1, \tau_2) \in X_\Gamma \times X_\Gamma$ associated to E_1 and E_2 as a big CM point in X_K in the sense of [BKY12]. For this purpose, let $\mathrm{E} = \mathrm{E}_1 \otimes_{\mathbb{Q}} \mathrm{E}_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with ring of integers $\mathcal{O}_E = \mathcal{O}_1 \otimes_{\mathbb{Z}} \mathcal{O}_2$. Then E is a biquadratic CM number field with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$ and $D = d_1 d_2$.

We define $W = \mathrm{E}$ with the F-quadratic form $Q_F(x) = \frac{Nx\bar{x}}{\sqrt{D}}$. Let $W_{\mathbb{Q}} = W$ with the $\mathbb{Q}$-quadratic form $Q_{\mathbb{Q}}(x) = \mathrm{tr}_{F/\mathbb{Q}} Q_F(x)$. Let $\sigma_1 = 1$ and $\sigma_2 = \sigma$ be two real embeddings of F with $\sigma_j(\sqrt{D}) = (-1)^{j-1}\sqrt{D}$. Then W has signature $(0, 2)$ at σ_2 and $(2, 0)$ at σ_1 respectively, and so $W_{\mathbb{Q}}$ has signature $(2, 2)$. Choose a $\mathbb{Z}$-basis of $\mathcal{O}_E$ as follows:

$$e_1 = 1 \otimes 1, \qquad e_2 = \frac{-d_1 + \sqrt{d_1}}{2} = \frac{-d_1 + \sqrt{d_1}}{2} \otimes 1,$$
$$e_3 = \frac{d_2 + \sqrt{d_2}}{2} = 1 \otimes \frac{d_2 + \sqrt{d_2}}{2}, \qquad e_4 = e_2 e_3.$$

We will drop ⊗ when there is no confusion. Then it is easy to check that

$$(W_{\mathbb{Q}}, Q_{\mathbb{Q}}) \cong (V, Q) = (M_2(\mathbb{Q}), N \det), \qquad \sum x_i e_i \mapsto \left(\begin{smallmatrix} x_3 & x_1 \\ x_4 & x_2 \end{smallmatrix}\right). \tag{3.3}$$

We will identify $(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ with the quadratic space $(V, Q) = (M_2(\mathbb{Q}), N \det)$. Under this identification, the lattice $L = M_2(\mathbb{Z})$ becomes $\mathcal{O}_E$. Then one can check that the maximal torus T in (2.21) can be identified with ([HY12], [BKY12, Section 6])

$$T(R) = \{(t_1, t_2) \in (E_1 \otimes_{\mathbb{Q}} R)^\times \times (E_2 \otimes_{\mathbb{Q}} R)^\times : t_1\bar{t}_1 = t_2\bar{t}_2\},$$

for any $\mathbb{Q}$-algebra R, and the map from T to $\mathrm{SO}(W)$ is given by $(t_1, t_2) \mapsto t_1/\bar{t}_2$. The map from T to H is explicitly given as follows. Define the embedding

$$\iota_j : \mathrm{E}_j \to M_2(\mathbb{Q}), \quad \iota_j(r)(e_{j+1}, e_1)^t = (re_{j+1}, re_1)^t. \tag{3.4}$$

Then $\iota = (\iota_1, \iota_2)$ gives the embedding from T to H.

Extend the two real embeddings of F into a CM type $\Sigma = \{\sigma_1, \sigma_2\}$ of E via

$$\sigma_1(\sqrt{d_i}) = \sqrt{d_i} \in \mathbb{H}, \quad \sigma_2(\sqrt{d_1}) = \sqrt{d_1}, \quad \sigma_2(\sqrt{d_2}) = -\sqrt{d_2}.$$

Since $W_{\sigma_2} = W \otimes_{F,\sigma_2} \mathbb{R} \subset V_{\mathbb{R}}$ has signature $(0, 2)$, it gives two points $z_{\sigma_2}^{\pm}$ in $\mathbb{D}$. In this case, the big CM cycles in Section 2.2 become

$$Z(W, z_{\sigma_2}^{\pm}) = \{z_{\sigma_2}^{\pm}\} \times T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T \in Z^2(X_K) \tag{3.5}$$

and

$$Z(W) = Z(W, z_{\sigma_2}^{\pm}) + \sigma(Z(W, z_{\sigma_2}^{\pm})).$$

For simplicity, we will denote z_{σ_2} for $z_{\sigma_2}^{+}$.

Lemma 3.4. *On $\mathbb{H}^{\pm,2}$, one has $z_{\sigma_2} = (\tau_1, \tau_2) \in \mathbb{H}^2$ and $z_{\sigma_2}^{-} = (\bar\tau_1, \bar\tau_2) \in (\mathbb{H}^-)^2$, where*

$$\tau_j = \frac{d_j + \sqrt{d_j}}{2}.$$

Proof. In the decomposition

$$V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R} = W_{\sigma_1} \oplus W_{\sigma_2}, \qquad W_{\sigma_i} = E \otimes_{F,\sigma_i} \mathbb{R} \cong \mathbb{C}, \ r \mapsto \sigma_i(r),$$

the $\mathbb{R}$-basis $\{e_i, i = 1, 2, 3, 4\}$ becomes

$$e_1 = (1, 1), \qquad e_2 = (-\bar\tau_1, -\bar\tau_1), \qquad e_3 = (\tau_2, \bar\tau_2), \quad \text{and} \quad e_4 = (-\bar\tau_1\tau_2, -\bar\tau_1\bar\tau_2).$$

The negative two plane W_{σ_2} representing $z_{\sigma_2}^{\pm}$ has an $\mathbb{R}$-orthogonal basis

$$u = (0, \sqrt{|d_2|}) \quad \text{and} \quad v = (0, \sqrt{d_2}) \in W_{\sigma_2} \subset V_{\mathbb{R}}.$$

One checks that

$$\begin{aligned} u &= -\frac{D - \sqrt{D}}{2\sqrt{|d_1|}} e_1 - \frac{d_2}{\sqrt{|d_1|}} e_2 + \frac{d_1}{\sqrt{|d_1|}} e_3 + \frac{2}{\sqrt{|d_1|}} e_4 \\ &= \begin{pmatrix} \frac{d_1}{\sqrt{|d_1|}} & -\frac{D-\sqrt{D}}{2\sqrt{|d_1|}} \\ \frac{2}{\sqrt{|d_1|}} & -\frac{d_2}{\sqrt{|d_1|}} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} v &= \frac{\sqrt{d_2}(\sqrt{d_1} + \sqrt{d_2})}{2} e_1 + \frac{\sqrt{d_2}}{\sqrt{d_1}} e_2 - e_3 \\ &= \begin{pmatrix} -1 & \frac{\sqrt{d_2}(\sqrt{d_1}+\sqrt{d_2})}{2} \\ 0 & \frac{\sqrt{d_2}}{\sqrt{d_1}} \end{pmatrix}. \end{aligned}$$

So

$$u - iv = \frac{2}{\sqrt{|d_1|}} \left(\begin{smallmatrix} \tau_1 & -\tau_1\tau_2 \\ 1 & -\tau_2 \end{smallmatrix}\right) = \frac{2N}{\sqrt{|d_1|}} w_N(\tau_1, \tau_2),$$

and

$$u + iv = \frac{2N}{\sqrt{|d_1|}} w_N(\bar\tau_1, \bar\tau_2)$$

as claimed. ∎

Lemma 3.5. *Let $K_j = \iota_j^{-1}(K(\Gamma))$ and let $\mathrm{Cl}(K_j) = E_j^\times \backslash E_{j,f}^\times / K_j$ be the associated class group of E_j. Then there is an injection*

$$p' : T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T \to \mathrm{Cl}(K_1) \times \mathrm{Cl}(K_2)$$

with image

$$\begin{aligned}\mathrm{IM}(p') &= \{(C_1, C_2) \in \mathrm{Cl}(K_1) \times \mathrm{Cl}(K_2) : \exists\ t_j \in E_{j,f}^\times \text{ with } C_j = [t_j], t_1\bar{t}_1 = t_2\bar{t}_2\} \\ &= \{(C_1, C_2) \in \mathrm{Cl}(K_1) \times \mathrm{Cl}(K_2) : \exists \text{ fractional ideals } \mathfrak{a}_i \text{ with } C_j = [\mathfrak{a}_j], \\ &\qquad \mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2)\}.\end{aligned}$$

Proof. Clearly p' is a group homomorphism. We first check that p' is injective. Assume $[t_1, t_2] \in \ker p$, and write $t_j = g_j k_j$ with $g_j \in E_j^\times$ and $k_j \in K_j$. Then $t_1\bar{t}_1 = t_2\bar{t}_2$ implies that

$$\frac{g_1\bar{g}_1}{g_2\bar{g}_2} = \frac{k_2\bar{k}_2}{k_1\bar{k}_1} \in \mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\},$$

so $(g_1, g_2) \in T(\mathbb{Q})$, and $k_2\bar{k}_2 = k_1\bar{k}_1$. This implies that

$$(k_1, k_2) \in K_T = \{(t_1, t_2) \in T(\mathbb{A}_f) : (\iota_1(t_1), \iota_2(t_2)) \in K_\Gamma = K(\Gamma) \times K(\Gamma)\}.$$

So $(t_1, t_2) \in T(\mathbb{Q})K_T$. The first formula for $\mathrm{IM}(p)$ is the definition. To show the second formula, assume $\mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2)$. Let $t_j \in E_{j,f}$ such that its associated ideal is $\mathfrak{a}_j$. Then $t_1\bar{t}_1 = t_2\bar{t}_2 u$ for some $u \in \hat{\mathbb{Z}}^\times$. When $p \nmid d_j$, $u_p = w_p\bar{w}_p$ for some $w_p \in \mathcal{O}_{E_j,p}^\times$. So we can decompose $u = u_1^{-1}u_2$ such that $u_j = w_j\bar{w}_j \in \mathrm{N}_{E_j/\mathbb{Q}}\hat{\mathcal{O}}_{E_j}^\times$. Replacing t_j by $t_j w_j$, we find $t_j \in E_{j,f}^\times$ such that $t_1\bar{t}_1 = t_2\bar{t}_2$ and $[t_j] = [\mathfrak{a}_j]$. ☐

Let H_j be the class field of E_j associated to K_j and let $H = H_1 H_2$ be the composition of H_1 and H_2. By the complex multiplication theory, the point $[z_{\sigma_2}] \in X_K$ is defined over H. Moreover, one has a natural map induced by ι_j in (3.4):
(3.6)

$$\iota_j : \mathrm{Cl}(K_j) \to X_\Gamma = \mathrm{GL}_2(\mathbb{Q})\backslash \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K(\Gamma), \quad \iota_j([t^{-1}]) = [\tau_j, \iota_j(t^{-1})] = \tau_j^{\sigma_t}.$$

Here $\sigma_t \in \mathrm{Gal}(H_j/E_j)$ is associated to $[t]$ by class field theory. The last identity is Shimura's reciprocity law (see for example [Yan16]). We will also write $\tau_j^{\sigma_t} = \tau_j^{\sigma_\mathfrak{a}}$ in ideal language where $[\mathfrak{a}_j] \in \mathrm{Cl}(K_j)$ corresponds to the idele class of t. Now the following two propositions are clear.

Proposition 3.6. *Let $(t_1, t_2) \in T(\mathbb{A}_f)$, and let $\sigma_{t_j} \in \mathrm{Gal}(H_j/E_j)$ be the associated Galois element (to t_j) via the Artin map. Then*

$$[z_{\sigma_2}, (t_1^{-1}, t_2^{-1})] = [\tau_1^{\sigma_{t_1}}, \tau_2^{\sigma_{t_2}}].$$

Proposition 3.7. *Assume $(d_1, d_2) = 1$. Then*

$$\begin{aligned} Z(W, z_{\sigma_2}) &= \sum_{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in \mathrm{IM}(p')} [\tau_1^{\sigma_{\mathfrak{a}_1}}, \tau_2^{\sigma_{\mathfrak{a}_2}}], \\ Z(W, z_{\sigma_2}^-) &= \sum_{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in \mathrm{IM}(p')} [(-\bar{\tau}_1)^{\sigma_{\mathfrak{a}_1}}, (-\bar{\tau}_2)^{\sigma_{\mathfrak{a}_2}}], \\ Z(W) &= \sum_{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in \mathrm{IM}(p')} \left([\tau_1^{\sigma_{\mathfrak{a}_1}}, \tau_2^{\sigma_{\mathfrak{a}_2}}] + [(-\bar{\tau}_1)^{\sigma_{\mathfrak{a}_1}}, \tau_2^{\sigma_{\mathfrak{a}_2}}] + [\tau_1^{\sigma_{\mathfrak{a}_1}}, (-\bar{\tau}_2)^{\sigma_{\mathfrak{a}_2}}] \right. \\ &\qquad \left. + [(-\bar{\tau}_1)^{\sigma_{\mathfrak{a}_1}}, (-\bar{\tau}_2)^{\sigma_{\mathfrak{a}_2}}]\right). \end{aligned}$$

The following lemma will be used later.

Lemma 3.8. *Assume again that* $(d_1, d_2) = 1$. *Let* $\mathrm{Cl}(E_j)$ *be the ideal class group of* E_j. *Let* $C_j \in \mathrm{Cl}(E_j)$ *be an ideal class for each* $j = 1, 2$. *Then there is an ideal* $\mathfrak{a}_j \in C_j$ *such that* $\mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2)$. *In particular, when* $K_j = \hat{\mathcal{O}}_{E_j}$ *in Lemma* 3.5, *then the map* p' *is an isomorphism.*

Proof. We first show that $H_1 \cap H_2 = \mathbb{Q}$. Let p be a rational prime; then $p \nmid d_1$ or $p \nmid d_2$. When $p \nmid d_j$, p is unramified in H_j and thus in $H_1 \cap H_2$. So every prime p is unramified in $H_1 \cap H_2$, and thus $H_1 \cap H_2 = \mathbb{Q}$. This implies that

$$\mathrm{Gal}(H/\mathbb{Q}) \cong \mathrm{Gal}(H_1/\mathbb{Q}) \times \mathrm{Gal}(H_2/\mathbb{Q}).$$

So there is $\sigma \in \mathrm{Gal}(H/\mathbb{Q})$ such that $\sigma|H_j = \sigma_{C_j}$. In particular, $\sigma \in \mathrm{Gal}(H/E)$, which is abelian. By the class field theory, there is an ideal $\mathfrak{a}$ of E such that $\sigma_{\mathfrak{a}} = \sigma$. Let $\mathfrak{a}_j = \mathrm{N}_{E/E_j}\mathfrak{a}$. Then $\sigma|H_j = \sigma_{\mathfrak{a}_j}$ and $\mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2) = \mathrm{N}(\mathfrak{a})$. Moreover, one has $C_j = [\mathfrak{a}_j]$. ☐

4. Gross and Zagier's singular moduli factorization formula

We will give a different proof of Gross and Zagier's factorization formula (Theorem 1.1) in this section. For this, we take $L = M_2(\mathbb{Z})$ with $Q(X) = \det X$, and $W = E$ with $Q_F(x) = \frac{x\bar{x}}{\sqrt{D}}$, where $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and $F = \mathbb{Q}(\sqrt{D})$ are as in Section 3. In this case, the lattice $L \cong \mathcal{O}_E$ is unimodular.

Proof of Theorem 1.1. Recall the identification at the beginning of Section 3 of the product $X_0(1) \times X_0(1)$ of modular curves with the orthogonal Shimura surface of signature $(2, 2)$ and the isotropic vectors $\ell = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ used for the identification. We also use them as in Theorem 2.2 for Borcherds product expansion. Write

$$j(\tau) - 744 = \sum_{m \geq -1} c(m) q^m.$$

Then Borcherds proved in [Bor95] that

$$j(z_1) - j(z_2) = \Psi(j(\tau) - 744),$$

which can be checked easily by Theorem 2.2. Notice that $\mathrm{Cl}(K_i) = \mathrm{Cl}(E_i)$ is the ideal class group of E_i and $j(-\bar{\tau}_i) = j(\tau_i)$. So the map p' in Lemma 3.5 is an isomorphism, and

$$\begin{aligned} \sum_{(z_1, z_2) \in Z(W)} \log|j(z_1) - j(z_2)| &= 4 \sum_{[\mathfrak{a}_i] \in \mathrm{Cl}(E_i)} \log|j(\tau_1^{\sigma_{\mathfrak{a}_1}}) - j(\tau_2^{\sigma_{\mathfrak{a}_2}})| \\ &= 4 \sum_{[\mathfrak{a}_i] \in \mathrm{Cl}(E_i)} \log|j(\tau_{\mathfrak{a}_1}) - j(\tau_{\mathfrak{a}_2})|. \end{aligned}$$

Here

$$\tau_{\mathfrak{a}_i} = \frac{b_i + \sqrt{d_i}}{2a_i} \quad \text{if } \mathfrak{a}_i = [a_i, \frac{b_i + \sqrt{d_i}}{2}].$$

So one has by Theorem 2.6,

$$-4 \sum_{[\mathfrak{a}_i] \in \mathrm{Cl}(E_i)} \log|j(\tau_{\mathfrak{a}_1}) - j(\tau_{\mathfrak{a}_2})|^4 = C(W, K) a_1(\phi),$$

with $\phi = \mathrm{Char}(\hat{\mathcal{O}}_E)$, and

$$C(W, K) = \frac{|Z(W, \sigma_2^{\pm})|}{\Lambda(0, \chi_{E/F})} = \frac{2h(E_1)h(E_2)}{\Lambda(0, \chi_{E_1/\mathbb{Q}})\Lambda(0, \chi_{E_2/Q})} = \frac{w_1 w_2}{2},$$

where $h(E_i)$ is the class number of E_i. By Proposition 2.7, one has

$$a_1(\phi) = \sum_{\substack{t \in \partial_F^{-1},\ \text{totally positive} \\ \mathrm{tr}_{F/\mathbb{Q}} t = 1}} a(t, \phi).$$

When $|\mathrm{Diff}(W,t)| > 1$, $a(t,\phi) = 0$. When $\mathrm{Diff}(W,t) = \{\mathfrak{p}\}$, $\mathfrak{p}$ is inert in E/F, and $\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D}\mathfrak{p})$ is even, Proposition 2.7 implies that

$$a(t,\phi) = -4\frac{1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D})}{2}\rho(t\sqrt{D}\mathfrak{p}^{-1}) \prod_{\mathfrak{q}<\infty} \gamma(W_{\mathfrak{q}}) \log(\mathrm{N}(\mathfrak{p})),$$

since

$$\prod_{\mathfrak{q}\neq\mathfrak{p}} \frac{W_{t,\mathfrak{q}}^*(0,\phi)}{\gamma(W_{\mathfrak{q}})} = \prod_{\mathfrak{q}\neq\mathfrak{p}} \rho_{\mathfrak{q}}(t\sqrt{D}) = \prod_{\mathfrak{q}} \rho_{\mathfrak{q}}(t\sqrt{D}\mathfrak{p}^{-1}) = \rho(t\sqrt{D}\mathfrak{p}^{-1}).$$

Here we used the fact that $\rho_{\mathfrak{p}}(t\sqrt{D}\mathfrak{p}^{-1}) = 1$ when $\mathfrak{p} \in \mathrm{Diff}(W,t)$. Next, $\gamma(W_{\sigma_1}) = -i = -\gamma(W_{\sigma_2})$ implies that

$$\prod_{\mathfrak{q}<\infty} \gamma(W_{\mathfrak{q}}) = \prod_{\text{all primes } v} \gamma(W_v) = 1.$$

So

$$a(t,\phi) = -2(1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D}))\rho(t\sqrt{D}\mathfrak{p}^{-1})\log(\mathrm{N}(\mathfrak{p})).$$

Notice that the right hand side in the above identity is automatically zero if we replace $\mathfrak{p}$ by other inert primes in E/F since $\rho(t\sqrt{D}\mathfrak{q}^{-1}) = 0$. So we always have

$$a(t,\phi) = -2 \sum_{\mathfrak{p} \text{ inert in } E/F} (1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D}))\rho(t\mathfrak{p}^{-1}\partial_F)\log(\mathrm{N}(\mathfrak{p})).$$

Putting everything together and replacing $t\sqrt{D}$ by t, we obtain the theorem.

Remark 4.1. It is easy to check that our formula coincides with [GZ85, (7.1)] and thus their main formula. Indeed,

$$\sum_{\mathfrak{a}|t\mathcal{O}_F} \chi_{E/F}(\mathfrak{a}) \log \mathrm{N}(\mathfrak{a}) = -\sum_{\mathfrak{p} \text{ inert in } E/F} \frac{(1+\mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F))}{2}\rho(t\mathfrak{p}^{-1})\log(\mathrm{N}(\mathfrak{p})) \tag{4.1}$$

for $t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F$ with $|m| < \sqrt{D}$. To see it, for any fixed integral ideal $\mathfrak{b}$ of F, define

$$f(\mathfrak{b}) = \sum_{\mathfrak{a}|\mathfrak{b}} \chi_{E/F}(\mathfrak{a}) \log \mathrm{N}(\mathfrak{a}).$$

Write $\mathfrak{b} = \prod_{i=1}^n \mathfrak{p}_i^{e_i}$ with $e_i > 0$. Assuming $\mathfrak{p}_1$ is inert in E/F and e_1 is odd, write $\mathfrak{b}_1 = \mathfrak{b}\mathfrak{p}_1^{-e_1}$. Then (recall that $\chi_{E/F}(\mathfrak{p}_1) = -1$)

$$\begin{aligned}
f(\mathfrak{b}) &= \sum_{\mathfrak{a}_1|\mathfrak{b}_1} \sum_{j=0}^{e_1} (-1)^j \chi_{E/F}(\mathfrak{a}_1) \left(j \log \mathrm{N}(\mathfrak{p}_1) + \log \mathrm{N}(\mathfrak{a}_1)\right) \\
&= \left(\sum_{j=0}^{e_1} (-1)^j j \log \mathrm{N}(\mathfrak{p}_1)\right) \left(\sum_{\mathfrak{a}_1|\mathfrak{b}_1} \chi_{E/F}(\mathfrak{a}_1)\right) + \left(\sum_{j=0}^{e_1} (-1)^j\right) f(\mathfrak{b}_1) \\
&= -\frac{1+e_1}{2} \log \mathrm{N}(\mathfrak{p}_1) \sum_{\mathfrak{a}_1|\mathfrak{b}_1} \chi_{E/F}(\mathfrak{a}_1) \\
&= -\frac{1+e_1}{2} \log \mathrm{N}(\mathfrak{p}_1) \prod_{i=2}^{n} \left(\sum_{j=0}^{e_i} \chi_{E/F}(\mathfrak{p}_i)^j\right) \\
&= -\frac{1+e_1}{2} \rho(\mathfrak{b}_1) \log \mathrm{N}(\mathfrak{p}_1) \\
&= -\frac{1+e_1}{2} \rho(\mathfrak{b}\mathfrak{p}_1^{-1}) \log \mathrm{N}(\mathfrak{p}_1).
\end{aligned}$$

In particular, if there is another $\mathfrak{p}_i$ $(i > 1)$ inert in E/F with e_i odd, then $\rho(\mathfrak{b}\mathfrak{p}_1^{-1}) = 0$ and $f(\mathfrak{b}) = 0$. In our case,

$$t\mathcal{O}_F = \prod_{i=1}^n \mathfrak{p}_i^{e_i}.$$

Then $\mathfrak{p}_i \in \mathrm{Diff}(W, t/\sqrt{D})$ if and only if $\mathfrak{p}_i$ is inert in E/F and e_i is odd. When

$$|\,\mathrm{Diff}(W, t/\sqrt{D})| > 1,$$

the above argument shows that $f(t\mathcal{O}_F) = 0$ and (4.1) holds as the right hand side of (4.1) is also zero. When $\mathrm{Diff}(W, t/\sqrt{D}) = \{\mathfrak{p}\}$, say, $\mathfrak{p} = \mathfrak{p}_1$, one has

$$f(t\mathcal{O}_F) = -\frac{1+e_1}{2} \rho(\mathfrak{b}_1)\, \mathrm{N}(\mathfrak{p}_1).$$

The right hand side of (4.1) equals this value too. So (4.1) holds.

5. The Yui-Zagier conjecture for ω_i

5.1. Borcherds product for $\omega_2(z_1) - \omega_2(z_2)$.

In this section, let

$$L = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right) \text{ with } Q(X) = \det X$$

and $\Gamma = \Gamma_0(2)$ in this section. It acts on L'/L trivially, where

$$L'/L = \left\{ \mu_0 = 0, \mu_1 = e_{21}, \mu_2 = \frac{1}{2} e_{12}, \mu_3 = \mu_1 + \mu_2 \right\}.$$

Here e_{ij} is the 2×2 matrix with the (i, j) entry 1 and all other entries 0. It is easy to check $Z(1, \mu_0) = X_{\Gamma_0(2)}^{\Delta}$ in the open variety $X_K = X_{\Gamma_0(2)} \times X_{\Gamma_0(2)}$.

Take the primitive isotropic vector $\ell = -e_{12} \in L$ and the vector $\ell' = e_{21} \in L'$ with $(\ell, \ell') = 1$. Since $(\ell, L) = 2\mathbb{Z}$, we choose $\xi = 2\ell' \in L$ with $(\ell, \xi) = 2$. In this case,

$$L_0' = \{x \in L' : (x, \ell) \equiv 0 \pmod 2\} = \{\left(\begin{smallmatrix} a & b/2 \\ 2c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z}\}, \quad L_0'/L = \{0, \mu_2\}.$$

One also has

$$M = L \cap (\mathbb{Q}\ell + \mathbb{Q}\ell')^{\perp} = \{m(a,b) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : a, b \in \mathbb{Z}\},$$

which is self-dual. So the projection p from L_0'/L to M'/M is zero. We further choose $\ell_M = e_{11}$ and $\ell_M' = e_{22}$ with $(\ell_M, \ell_M') = 1$, so $P = 0$. Finally for a weakly holomorphic modular form $f \in M^!_{0,\omega_L}$ with

$$f(\tau) = \sum_{m,\mu} c(m,\mu) q^m \phi_\mu = \sum_\mu f_\mu \phi_\mu,$$

one has

$$f_M = f_{\mu_0} + f_{\mu_2}.$$

Now Theorem 2.2 gives the following proposition in this special case.

Proposition 5.1. *Let*

$$f(\tau) = \sum_{m,\mu} c(m,\mu) q^m \phi_\mu \in M^!_{0,\omega_L}.$$

Then there is a meromorphic modular form of two variables $\Psi(z_1, z_2, f)$ for $\Gamma_0(2) \times \Gamma_0(2)$ of parallel weight $\frac{c(0,0)}{2}$ with the following product expansion near the cusp $\mathbb{Q}\ell$, with respect to a Weyl chamber W whose closure contains ℓ_M ($z = (z_1, z_2)$):

$$\Psi(z, f) = Ce((\rho(W,f), z)) \prod_{\substack{(m,n)\in\mathbb{Z}^2 \\ (\left(\begin{smallmatrix} -m & 0 \\ 0 & n \end{smallmatrix}\right), W) > 0}} (1 - q_1^n q_2^m)^{c(mn,0)} (1 + q_1^n q_2^m)^{c(mn,\mu_2)}.$$

Here $q_j = e(z_j)$, and $|C| = 2^{\frac{c(0,\mu_2)}{2}}$.

Proposition 5.2. (1) *Let $M^{!,0}_{0,\omega_L}$ be the subspace of $M^!_{0,\omega_L}$ consisting of constant vector $f = \sum a_i \phi_{\mu_i}$. Then it is of dimension 2 with a basis $\{\phi_{\mu_0} + \phi_{\mu_1}, \phi_{\mu_0} + \phi_{\mu_2}\}$.*

(2) *One has*

$$\Psi(z, \phi_{\mu_0} + \phi_{\mu_1}) = \eta(z_1)\eta(z_2),$$
$$\Psi(z, \phi_{\mu_0} + \phi_{\mu_2}) = \sqrt{2}\eta(2z_1)\eta(2z_2),$$
$$\Psi(z, \phi_{\mu_2} - \phi_{\mu_1}) = \frac{1}{\sqrt{2}}\mathbf{f}_2(z_1)\mathbf{f}_2(z_2).$$

Here $\mathbf{f}_2(z) = \omega_2(z)^{\frac{1}{24}} = \sqrt{2}\frac{\eta(2z)}{\eta(z)}$ is also a famous Weber function.

Proof. Recalling that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $n(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$,

$$n(1)(f) = a_0 e_0 + a_1 e_1 + a_2 e_2 - a_3 e_3 = f$$

if and only if $a_3 = 0$. Next, assuming $a_3 = 0$, then

$$w(f) = \sum a_i \omega_L(w)(e_i) = \frac{1}{2}\left[(\sum a_i)e_0 + \sum_{i=1}^{3}(a_0 + a_i - \sum_{j\neq i} a_j)e_i\right] = f$$

if and only if $a_0 = a_1 + a_2$. This proves (1). In such a case, $f = a_1(e_0 + e_1) + a_2(e_0 + e_2)$.

To prove (2), notice that

$$\mathrm{Gr}(M) = \{\mathbb{R}\left(\begin{smallmatrix} a & 0 \\ 0 & -1 \end{smallmatrix}\right) : a > 0\} \cong \mathbb{R}_{>0}.$$

Since $f = a_1(e_0+e_1) + a_2(e_0+e_2)$ has no negative term, one sees that $\mathrm{Gr}(M)$ has only one Weyl chamber, i.e., itself with respect to f. A vector $\lambda = \left(\begin{smallmatrix} -m & 0 \\ 0 & n \end{smallmatrix}\right)$ satisfies $(\lambda, W) > 0$ if and only if $m, n \ge 0$ but not both equal to 0. One also has

$$\rho(W, f) = \frac{2a_2 + a_1}{24}(-\ell_M + \ell'_M).$$

Now the proposition is clear from Theorem 2.2 if we just take $C = 2^{c(0,\mu_2)/2}$. □

Proposition 5.3. *Let*

$$f = 12(\phi_{\mu_2} - \phi_{\mu_1}) + \sum_{\gamma \in \Gamma_0(2)\backslash \mathrm{SL}_2(\mathbb{Z})} (2^{12}\omega_2^{-1} + 12)|\gamma \omega_L(\gamma)^{-1}\phi_{\mu_0} \in M^!_{0,\omega_L}.$$

Then

$$c(0,\mu_0) = c(0,\mu_1) = c(0,\mu_3) = 0, \quad c(0,\mu_2) = 24, \tag{5.1}$$

and

$$\Psi(z, f) = \omega_2(z_1) - \omega_2(z_2).$$

Proof. Direct calculation gives

$$\begin{aligned} f = (q^{-1} - 98028q - 10749952q^2 - 432133182q^3 + \cdots)\phi_{\mu_0} \\ +(-98296q - 10747904q^2 - 432144384q^3 + \cdots)\phi_{\mu_1} \\ +(24 - 98296q - 10747904q^2 - 432144384q^3 + \cdots)\phi_{\mu_2} \\ +(4096q^{\frac{1}{2}} + 1228800q^{\frac{3}{2}} + 74244096^{\frac{5}{2}} + \cdots)\phi_{\mu_3}. \end{aligned}$$

In particular, (5.1) holds, and $Z(f) = Z(1, \mu_0) = X^{\Delta}_{\Gamma_0(2)}$ in X_K. This implies that

$$g(z_1, z_2) = \frac{\Psi(z_1, z_2, f)}{\omega_2(z_1) - \omega_2(z_2)}$$

has no zeros or poles in the open Shimura variety X_K; i.e., its divisor is supported on the boundaries $\{P\} \times X_0(2)$ and $X_0(2) \times \{P\}$, where P runs through the cusps 0 and ∞ of $X_0(2)$. We now use Borcherds product expansion to show that $g(z_1, z_2)$ has no zeros or poles on the boundaries and thus has to be a constant.

The weakly holomorphic form f gives rise to two Weyl chambers

$$\mathrm{Gr}(M) - Z_M(1, \mu_0) = W^{\pm},$$

where

$$W^{\pm} = \{\mathbb{R}\left(\begin{smallmatrix} a & 0 \\ 0 & -1 \end{smallmatrix}\right) : a^{\pm 1} > 1\}.$$

We choose the Weyl chamber W^+ whose closure contains ℓ_M. Then for $\lambda = \left(\begin{smallmatrix} -m & 0 \\ 0 & n \end{smallmatrix}\right) \in M'$, $(\lambda, W^+) > 0$ if and only if

$$m + n \ge 0, \qquad n \ge 0, \quad \text{and} \quad m^2 + n^2 > 0.$$

Direct calculation using (2.15)–(2.18) gives the Weyl vector

$$\begin{aligned} \rho(W^+, f) &= -\frac{1}{24}(c(0,\mu_0) + c(0,\mu_2))\ell_M + (-c(-1,\mu_0) + \frac{c(0,\mu_0) + c(0,\mu_2)}{24})\ell'_M \\ &= -e_{11}. \end{aligned}$$

We can take the constant

$$C = -2^{c(0,\mu_2)/2} = -2^{12}.$$

One has by Proposition 5.1 that

$$\begin{aligned}\Psi(z,f) &= -2^{12}q_2(1-q_1q_2^{-1})\prod_{m,n\geq 0, m+n>0}(1-q_1^nq_2^m)^{c(mn,0)}(1+q_1^nq_2^m)^{c(mn,\mu_2)}\\ &= 2^{12}(q_1-q_2)\prod_{m,n>0}(1-q_1^nq_2^m)^{c(mn,0)}(1+q_1^nq_2^m)^{c(mn,\mu_2)}.\end{aligned}$$

This product formula shows that $\Psi(z,f)$ has no zeros or poles along the boundary $\{\infty\}\times X_{\Gamma_0(2)}$ and $X_{\Gamma_0(2)}\times\{\infty\}$. Since $\omega_2(z_1)-\omega_2(z_2)$ has the same property, $g(z_1,z_2)$ has no zeros or poles in these boundaries. Fixing a $z_2\in\mathbb{H}$, the function $g(z_1,z_2)$ of z_1 has then only zeros or poles at the cusp $\{0\}$ in $X_0(2)$ and is thus independent of z_1: $g(z_1,z_2)=g(z_2)$. This implies that $g(z_2)$ has only zeros or poles at the cups 0 and is thus a constant $g(z_2)=A$. Therefore,

$$\Psi(z_1,z_2,f)=A(\omega_2(z_1)-\omega_2(z_2)).$$

Comparing the leading coefficients on both sides, one sees that $A=1$. □

5.2. **Proof of Theorem 1.3.** Now we start to prove Theorem 1.3. Under the isomorphism

$$(M_2(\mathbb{Q}),\det)\cong(E,\mathrm{tr}_{F/\mathbb{Q}}\frac{x\bar{x}}{\sqrt{D}}),\quad \left(\begin{smallmatrix}x_3 & x_1\\ x_4 & x_2\end{smallmatrix}\right)\mapsto\sum x_ie_i,$$

one has

$$L\cong\mathbb{Z}+\mathbb{Z}\frac{D+\sqrt{D}}{2}+\mathbb{Z}\frac{-d_1+\sqrt{d_1}}{2}+\mathbb{Z}\frac{d_2+\sqrt{d_2}}{2},$$

which is of index 2 in $\mathcal{O}_E$, but is not an $\mathcal{O}_F$-lattice unfortunately. By Proposition 5.3, we have

$$\omega_2(z_1)-\omega_2(z_2)=\Psi(z,f).$$

Lemma 5.4. *Assume $d_j\equiv 1 \pmod 8$. Then*

$$\iota_j^{-1}(K(\Gamma_0(2)))=\hat{\mathcal{O}}_{E_j}^\times.$$

Proof. We work the case $j=2$. Then case $j=1$ is the same. For $r=x+y\frac{d_2+\sqrt{d_2}}{2}\in E_{2,f}^\times$, one has

$$\iota_2(r)=\begin{pmatrix}x+dy & y\frac{d-d^2}{4}\\ y & x\end{pmatrix}.$$

So $\iota_2(r)\in K(\Gamma_0(2))$ if and only if $y\in 2\hat{\mathbb{Z}}$. This implies that

$$\iota_2^{-1}(K(\Gamma_0(2)))=(\hat{\mathbb{Z}}+2\hat{\mathcal{O}}_{E_2})^\times.$$

Since $d_2\equiv 1 \pmod 8$, 2 is split in E_2 and

$$\mathcal{O}_{E_2,2}^\times=\mathbb{Z}_2^\times\times\mathbb{Z}_2^\times=(1+2\mathcal{O}_{E_2,2}).$$

So

$$(\hat{\mathbb{Z}}+2\hat{\mathcal{O}}_{E_2})^\times=\hat{\mathcal{O}}_{E_2}^\times.$$

□

This lemma and Lemma 3.8 imply that the class projection p' in Lemma 3.5 is an isomorphism. By Proposition 3.6, one has

$$\tau_j^{\sigma_{\mathfrak{a}_j}}=\tau_{\mathfrak{a}_j}=\frac{b_j+\sqrt{d_j}}{2a_j}\quad\text{if } \mathfrak{a}_j=[a_j,\frac{b_j+\sqrt{d_j}}{2}], 2\nmid a_j.$$

On the other hand,

$$\omega_2(-\bar{\tau}_j) = \omega_2(\tau_j - d_j) = \omega_2(\tau_j).$$

So one has again by Proposition 3.7

$$\sum_{(z_1,z_2)\in Z(W)} \log|\omega_2(z_1) - \omega_2(z_2)| = 4 \sum_{[\mathfrak{a}_j]\in \mathrm{Cl}(E_j)} \log|\omega_2(\tau_{\mathfrak{a}_1}) - \omega_2(\tau_{\mathfrak{a}_2})|.$$

So we have by Theorem 2.6,

$$-4 \sum_{[\mathfrak{a}_j]\in \mathrm{Cl}(E_j)} \log|\omega_2(\tau_{\mathfrak{a}_1}) - \omega_2(\tau_{\mathfrak{a}_2})|^4 = C(W,K)[a_1(\phi)+24a_0(\tilde{\phi})] = 2[a_1(\phi)+24a_0(\tilde{\phi})],$$

with $\phi = \mathrm{Char}(\hat{L})$ and $\tilde{\phi} = \mathrm{Char}(\mu_2 + \hat{L})$. Here

$$C(W,K) = \frac{\deg Z(W, z_{\sigma_2}^{\pm})}{\Lambda(0,\chi)} = \frac{w_1 w_2}{2} = 2.$$

Now Theorem 1.3 follows from the following lemma, which we will prove in the next subsection.

Lemma 5.5. *Let the notation be as above. Then*

(1)

$$a_0(\tilde{\phi}) = 0,$$

(2)

$$a_1(\phi) = -4 \sum_{\substack{t=\frac{m+\sqrt{D}}{2} \\ |m|<\sqrt{D},\mathrm{odd} \\ m^2\equiv D \pmod{16}}} \sum_{\mathfrak{p} \text{ inert in } E/F} \frac{1+\mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \rho(t\mathfrak{p}^{-1}\mathfrak{p}_t^{-2}) \log(\mathrm{N}(\mathfrak{p})).$$

5.3. Whittaker functions and proof of Lemma 5.5.

Lemma 5.6. *Let $W = \mathbb{Q}_2^2$ with the quadratic form $Q(x) = \alpha^{-1}x_1x_2$ with $\alpha \in \mathbb{Z}_2^\times$. For $a = 0, 1$, let*

$$M_a = \{(x_1,x_2) \in \mathbb{Z}_2^2 : x_1 + x_2 \equiv a \pmod 2\}$$

and

$$\varphi_a = \mathrm{Char}(M_a), \qquad \tilde{\varphi}_a = \mathrm{Char}((\frac{1}{2},\frac{1}{2}) + M_a).$$

Let ψ be an unramified additive character of $\mathbb{Q}_2$.

(1) *When $a = 0$, the local Whittaker function $W_{t\alpha}(s,\varphi_a) = 0$ unless $t \in \mathbb{Z}_2$, and*

$$\frac{W_{t\alpha}(s,\varphi_0)}{\gamma(W)} = \begin{cases} \frac{1}{2} & \text{if } t \in \mathbb{Z}_2^\times, \\ \frac{1}{2} - 2^{-s} + (1-2^{-1-s})\sum_{n=1}^{o(t)} 2^{-ns} & \text{if } t \in 2\mathbb{Z}_2, \end{cases}$$

where $o(t) = \mathrm{ord}_2 t$. In particular,

$$\frac{W_{t\alpha}(0,\varphi_0)}{\gamma(W)} = \begin{cases} \frac{1}{2} & \text{if } o(t) = 0, \\ \frac{o(t)-1}{2} & \text{if } o(t) \geq 1. \end{cases}$$

(2) *When $a = 1$, the local Whittaker function $W_{t\alpha}(s, \varphi_a) = 0$ unless $t \in \mathbb{Z}_2$, and*

$$\frac{W_{t\alpha}(s,\varphi_1)}{\gamma(W)} = \begin{cases} \frac{1}{2}(1-2^{-s}) & \text{if } t \in \mathbb{Z}_2^\times, \\ \frac{1}{2}(1+2^{-s}) & \text{if } t \in 2\mathbb{Z}_2. \end{cases}$$

In particular,

$$\frac{W_{t\alpha}(0,\varphi_1)}{\gamma(W)} = \begin{cases} 0 & \text{if } o(t) = 0, \\ 1 & \text{if } o(t) \geq 1. \end{cases}$$

(3) *One has*

$$W_{t\alpha}(s, \tilde{\varphi}_a) = 0$$

unless $t - \frac{1+2a}{4} \in \mathbb{Z}_2$, in which case it is the constant $\frac{1}{2}\gamma(W)$. In particular, $W_0(s, \tilde{\varphi}_a) = 0$.

Sketch of proof. By the definition and unfolding, one has

$$\begin{aligned} \frac{W_{t\alpha}(s,\varphi_a)}{\gamma(W)} &= \int_{\mathbb{Q}_2} J_a(b)\psi(-tb)|a(wn(b))|^s \, db \\ &= \int_{\mathbb{Z}_2} J_a(b)\psi(-tb)\, db + \sum_{n\geq 1} 2^n \int_{\mathbb{Z}_2^\times} J_a(2^{-n}b)\psi(-2^{-n}tb)|a(wn(2^{-n}b))|^s \, db, \end{aligned}$$

where

$$J_a(b) = \int_{M_a} \psi(bx_1x_2)\, dx_1 dx_2.$$

Then one checks that

$$J_1(b) = \frac{1}{2}\,\mathrm{Char}(\frac{1}{2}\mathbb{Z}_2)(b), \tag{5.2}$$

$$J_0(b) = \begin{cases} \frac{1}{2} & \text{if } b \in \mathbb{Z}_2, \\ 0 & \text{if } b \in \frac{1}{2}\mathbb{Z}_2^\times, \\ |b|^{-1} & \text{if } b \notin \frac{1}{2}\mathbb{Z}_2, \end{cases} \tag{5.3}$$

and

$$|a(wn(b))| = \min(1, |b|^{-1}).$$

Now a direct calculation proves (1) and (2). For (3), one has similarly

$$\frac{W_{t\alpha}(s,\tilde{\varphi}_a)}{\gamma(W)} = \int_{\mathbb{Q}_2} \tilde{J}_a(b)\psi(-tb)|a(wn(b))|^s \, db,$$

where

$$\tilde{J}_a(b) = \int_{(\frac{1}{2},\frac{1}{2})+M_a} \psi(bx_1x_2)\, dx_1dx_2 = \tilde{J}_a^{(0)}(b) + \tilde{J}_a^{(1)}(b).$$

Here (after a simple substitution)

$$\begin{aligned} 4\tilde{J}_a^{(j)}(b) &= \int_{\mathbb{Z}_2}\int_{\mathbb{Z}_2} \psi(b(\frac{1}{2}+j+2y_1)(\frac{1}{2}-j+a+2y_2))dy_1\, dy_2 \\ &= \psi((\frac{1}{2}+j)(\frac{1}{2}-j+a)b)\,\mathrm{Char}(\mathbb{Z}_2)(b). \end{aligned}$$

So

$$\frac{4W_{t\alpha}(s,\tilde{\varphi}_a)}{\gamma(W)} = \sum_{j=0}^{1}\int_{\mathbb{Z}_2}\psi((\frac{1}{2}+j)(\frac{1}{2}-j+a)b)\psi(-tb)\,db$$
$$= 2\int_{\mathbb{Z}_2}\psi((\frac{1+2a}{4}-t)b)\,db$$
$$= 2\,\mathrm{Char}(\frac{1+2a}{4}+\mathbb{Z}_2)(t).$$

□

To compute $a_1(\phi)$ and $a_0(\tilde{\phi})$, we keep the notation in the proof of Theorem 1.3. Recall that

$$L = \mathbb{Z} + \mathbb{Z}\frac{D+\sqrt{D}}{2} + \mathbb{Z}\frac{-d_1+\sqrt{d_1}}{2} + \mathbb{Z}\frac{d_2+\sqrt{d_2}}{2}$$

is not an $\mathcal{O}_F$-lattice as $\frac{D+\sqrt{D}}{2}\frac{-d_1+\sqrt{d_1}}{2} \notin L$. So ϕ and $\tilde{\phi}$ are *not* factorizable over primes of F. Instead one has only

$$\phi = \phi_2\prod_{\mathfrak{p}\nmid 2}\phi_{\mathfrak{p}} \quad \text{and} \quad \tilde{\phi} = \tilde{\phi}_2\prod_{\mathfrak{p}\nmid 2}\phi_{\mathfrak{p}},$$

where $\phi_{\mathfrak{p}} = \mathrm{Char}(\mathcal{O}_{E,\mathfrak{p}})$ for a prime (ideal) $\mathfrak{p}$ of F not dividing 2, $\phi_2 = \mathrm{Char}(L_2)$, and $\tilde{\phi}_2 = \mathrm{Char}(\frac{1}{2}+L_2)$. So we need to take special care at the local calculation at $p=2$. We focus on ϕ and $a_1(\phi)$ first.

Our assumption implies also that 2 splits in E completely. Write

$$2\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2, \qquad \mathfrak{p}_i\mathcal{O}_E = \mathfrak{P}_i\bar{\mathfrak{P}}_i.$$

Let $\sqrt{D}\in\mathbb{Z}_2$ and $\sqrt{d_i}\in\mathbb{Z}_2$ be some prefixed square roots of D and d_i respectively with $\sqrt{d_1}\sqrt{d_2} = -\sqrt{D}$. We identify $F_{\mathfrak{p}_i}$, $E_{\mathfrak{P}_i}$, and $E_{\bar{\mathfrak{P}}_i}$ with $\mathbb{Q}_2$ as follows:

$$F_{\mathfrak{p}_i} \cong \mathbb{Q}_2, \quad \sqrt{D}\mapsto(-1)^{i-1}\sqrt{D},$$
$$E_{\mathfrak{P}_i} \cong \mathbb{Q}_2, \quad \sqrt{D}\mapsto(-1)^{i-1}\sqrt{D}, \sqrt{d_i}\mapsto\sqrt{d_i},$$
$$E_{\bar{\mathfrak{P}}_i} \cong \mathbb{Q}_2, \quad \sqrt{D}\mapsto(-1)^{i-1}\sqrt{D}, \sqrt{d_i}\mapsto-\sqrt{d_i}.$$

With this identification, we can check that $L_2 = L\otimes_{\mathbb{Z}}\mathbb{Z}_2$ is given by

$$L_2 = \{x=(x_1,x_2,x_3,x_4)\in E_{\mathfrak{P}_1}\times E_{\bar{\mathfrak{P}}_1}\times E_{\mathfrak{P}_2}\times E_{\bar{\mathfrak{P}}_2}\cong\mathbb{Q}_2^4 :\ x_i\in\mathbb{Z}_2, \sum x_i\in 2\mathbb{Z}_2\},$$

with quadratic form

$$Q(x) = \frac{x_1x_2}{\sqrt{D}} + \frac{x_3x_4}{-\sqrt{D}} = Q_{\mathfrak{p}_1}(x_1,x_2) + Q_{\mathfrak{p}_2}(x_3,x_4).$$

The embedding from L to L_2 is given by

$$x\mapsto(\sigma_1(x),\sigma_1(\bar{x}),\sigma_2(x),\sigma_2(\bar{x})),$$

where $\sigma_1(\sqrt{d_i}) = \sqrt{d_i}$ and $\sigma_2(\sqrt{d_i}) = (-1)^i\sqrt{d_i}$. So

$$L_2 = (M_0\times M_0)\cup(M_1\times M_1),$$

where M_a is given as in Lemma 5.6. This implies that

$$\phi_2 = \mathrm{Char}(L_2) = \phi_{\mathfrak{p}_1,0}\phi_{\mathfrak{p}_2,0} + \phi_{\mathfrak{p}_1,1}\phi_{\mathfrak{p}_2,1},$$

where $\phi_{\mathfrak{p}_i,a}$ is φ_a in Lemma 5.6. Correspondingly, we have

$$\phi = \phi_0 + \phi_1, \qquad a(t,\phi) = a(t,\phi_0) + a(t,\phi_1),$$

where $\phi_i = \phi_{\mathfrak{p}_1,i}\phi_{\mathfrak{p}_2,i}\prod_{\mathfrak{p}\nmid 2}\phi_{\mathfrak{p}}$. Now Proposition 2.7 and the proof of Theorem 1.1 give

(5.4)

$$\begin{aligned} &a(t,\phi_i)\\ &= -4\sum_{\mathfrak{p}\ \text{inert in E/F}} \frac{1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D})}{2}\rho^{(2)}(t\mathfrak{p}^{-1}\partial_F)\prod_{j=1}^{2}\frac{W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_j}(0,\phi_{\mathfrak{p}_j,i})}{\gamma(W_{\mathfrak{p}_j})}\log(\mathrm{N}(\mathfrak{p})).\end{aligned}$$

Here $\psi'(x) = \psi_F(x/\sqrt{D})$ and

$$\rho^{(2)}(\mathfrak{a}) = \prod_{\mathfrak{p}\nmid 2}\rho_{\mathfrak{p}}(\mathfrak{a})$$

as in the proof of Theorem 1.1.

Lemma 5.7. *Assume again that* $d_1\equiv d_2\equiv 1 \pmod 8$. *Let* $t\in\partial_F^{-1}$ *with* $\mathrm{tr}_{F/\mathbb{Q}}(t)=1$. *Then there is a unique prime ideal* $\mathfrak{p}_t$ *with* $t\sqrt{D}\in\mathfrak{p}_t$. *Moreover,*

$$W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_1}(0,\phi_{\mathfrak{p}_1,1})W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_2}(0,\phi_{\mathfrak{p}_2,1}) = 0,$$

and

$$\frac{W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_1}(0,\phi_{\mathfrak{p}_1,0})W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_2}(0,\phi_{\mathfrak{p}_2,0})}{\gamma(W_{\mathfrak{p}_1})\gamma(W_{\mathfrak{p}_2})} = \mathrm{ord}_{\mathfrak{p}_t}(t\sqrt{D})-1 = \rho_{\mathfrak{p}_t}(t\sqrt{D}\mathfrak{p}_t^{-2}).$$

Proof. Write $t = \frac{m+\sqrt{D}}{2\sqrt{D}} \in \partial_F^{-1}$. Recall the two natural embeddings $\sigma_i : F \hookrightarrow F_{\mathfrak{p}_i}$, $i = 1, 2$. Since

$$\sigma_1(t\sqrt{D})\sigma_2(t\sqrt{D}) = \frac{m^2-D}{4} \equiv 0 \pmod 2, \quad \sigma_1(t\sqrt{D})-\sigma_2(t\sqrt{D}) \equiv 1 \pmod 2,$$

one sees that exactly one of $\mathrm{ord}_{\mathfrak{p}_i}(\sigma_i(t\sqrt{D}))$ is positive while the other one is zero. For simplicity, let $\mathfrak{p}_t = \mathfrak{p}_1$ with $\mathrm{ord}_{\mathfrak{p}_1}(t\sqrt{D}) \ge 1$ and let $\mathrm{ord}_{\mathfrak{p}_2}(t\sqrt{D}) = 0$. Then Lemma 5.6 implies that

$$W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_2}(0,\phi_{\mathfrak{p}_2,1}) = 0.$$

The same lemma also implies (recall $L(1,\chi_{\mathfrak{p}_i}) = 2$) that

$$\begin{aligned}\frac{W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_1}(0,\phi_{\mathfrak{p}_1,0})W^{*,\psi'}_{t\sqrt{D},\mathfrak{p}_2}(0,\phi_{\mathfrak{p}_2,0})}{\gamma(W_{\mathfrak{p}_1})\gamma(W_{\mathfrak{p}_2})} &= 4\cdot\frac{1}{2}\cdot\frac{1}{2}(\mathrm{ord}_{\mathfrak{p}_1}(t\sqrt{D})-1)\\ &= \rho_{\mathfrak{p}_t}(t\sqrt{D}\mathfrak{p}_t^{-2}).\end{aligned}$$

□

Now, one has by Lemma 5.7 and (5.4)

$$\begin{aligned} a(t,\phi_1) &= 0,\\ a(t,\phi_0) &= -4\sum_{\mathfrak{p}\ \text{inert in E/F}}\frac{1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D})}{2}\rho(t\mathfrak{p}\partial_F\mathfrak{p}_t^{-2})\log(\mathrm{N}(\mathfrak{p})).\end{aligned}$$

Here $\mathfrak{p}_t$ is the only prime ideal of F above 2 with $t\sqrt{D} \in \mathfrak{p}_t$. Replacing t by $t/\sqrt{D}$, one obtains for $t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F$ with $|m| < \sqrt{D}$:

$$a(t/\sqrt{D}, \phi) = -4 \sum_{\mathfrak{p} \text{ inert in E/F}} \frac{1 + \mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \rho(t\mathfrak{p}\mathfrak{p}_t^{-2}) \log(\mathrm{N}(\mathfrak{p})).$$

The condition $\rho(t\mathfrak{p}\mathfrak{p}_t^{-2}) \neq 0$ implies that $t \in \mathfrak{p}_t^2$ and so

$$\mathrm{N}(t) = \frac{m^2 - D}{4} \equiv 0 \pmod 4, \qquad \text{i.e., } m^2 \equiv D \pmod{16}.$$

This proves the second identity in Lemma 5.5:

$$a_1(\phi) = \sum_{\substack{t = \frac{m+\sqrt{D}}{2} \\ |m| < \sqrt{D}, \text{odd} \\ m^2 \equiv D \pmod{16}}} \sum_{\mathfrak{p} \text{ inert in } E/F} \frac{1 + \mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \rho(t\mathfrak{p}^{-1}\mathfrak{p}_t^{-2}) \log(\mathrm{N}(\mathfrak{p})).$$

Now we prove $a_0(\tilde{\phi}) = 0$. The same argument as above gives

$$\tilde{\phi} = \tilde{\phi}_2 \prod_{\mathfrak{p} \nmid 2} \phi_{\mathfrak{p}}$$

and

$$\tilde{\phi}_2 = \tilde{\phi}_{\mathfrak{p}_1,0} \tilde{\phi}_{\mathfrak{p}_2,0} + \tilde{\phi}_{\mathfrak{p}_1,1} \tilde{\phi}_{\mathfrak{p}_2,1}$$

with $\tilde{\phi}_{\mathfrak{p}_i,a}$ being $\tilde{\varphi}_a$ in Lemma 5.6. So

$$W_{0,2}(s, \tilde{\phi}_2) = \sum_{a=0}^{1} \prod_{i=0}^{1} W_{0,\mathfrak{p}_i}(s, \tilde{\phi}_{\mathfrak{p}_i,a}) = 0$$

by Lemma 5.6. This implies that

$$W_{0,f}(s, \tilde{\phi}) = 0,$$

and thus $a_0(\tilde{\phi}) = 0$ by Remark 2.5 (and $\tilde{\phi}(0) = 0$). This proves Lemma 5.5 and thus Theorem 1.3.

Remark 5.8. When $d_i \equiv 1 \pmod 8$ are not satisfied, the big CM value formula will still give a factorization formula for the CM values of $\omega_2(z_1) - \omega_2(z_2)$ although the summation will be over the ring class group of E_i with conductor 2 when $d_i \equiv 5 \pmod 8$ (see Lemma 5.4). We leave the details to the reader.

Remark 5.9. The Weber function ω_2 has two companions, $\omega_1(\tau) = w(\omega_2)$ and $\omega_0(\tau) = \omega_1(\tau + 1)$. So the results on ω_2 can easily be transferred to its companions ω_0 and ω_1.

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