Advances in Mathematics 345 (2019) 702–755

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Arithmetic Siegel–Weil formula on $X_0(N)$ ☆

Tuoping Du [a], Tonghai Yang [b,*]

[a] *Department of Mathematics, Northwest University, Xi'an, 710127, P.R. China*
[b] *Department of Mathematics, University of Wisconsin-Madison, United States of America*

ARTICLE INFO

*Article history:*
Received 4 February 2018
Received in revised form 23 December 2018
Accepted 30 December 2018
Available online 18 January 2019
Communicated by Zhiwei Yun

*MSC:*
11G15
11F41
14K22

*Keywords:*
Arithmetic Siegel–Weil formula
Arithmetic intersection

ABSTRACT

In this paper, we proved an arithmetic Siegel–Weil formula and the modularity of some arithmetic theta function on the modular curve $X_0(N)$ when $N$ is square free. In the process, we also constructed some generalized Delta function for $\Gamma_0(N)$ and proved some explicit Kronecker limit formula for Eisenstein series on $X_0(N)$.


## 1. Introduction

It is well-known that there is a deep connection between the leading term of some analytic functions and the arithmetics, such as the class number formula, Birch and Swinnerton–Dyer conjecture, Block–Kato conjecture and the Siegel–Weil formula. Little is known or understood about the possible connection between the second term of these

☆ The first author is partially supported by NSFC (No. 11401470), and the second author is partially supported a NSF grant DMS-1762289.
* Corresponding author.
*E-mail addresses:* dtp1982@163.com (T. Du), thyang@math.wisc.edu (T. Yang).

https://doi.org/10.1016/j.aim.2019.01.015

functions and arithmetic although it started to change in this century. The most famous one is the Kronecker limit formula:

$$E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \Im(\gamma z)^s = 1 + \frac{1}{6}(\log |\Delta(\tau)\Im(z)^6|)s + O(s^2).$$

We refer to [22] for its proof and its beautiful application to class numbers. In 2004, Kudla, Rapoport and Yang ([17]) discovered another second term identity of some Eisenstein series of weight 3/2—the so-called arithmetic Siegel–Weil formula. Roughly speaking, they defined an arithmetic function—a generating function $\widehat{\phi}_{KRY}(\tau)$ of a family of arithmetic divisors in a Shimura curve. They proved that its degree is the special value of some Eisenstein series $\mathcal{E}(\tau, s)$ (weight 3/2) at $s = 1$ and that arithmetic intersection with the (normalized) metrized Hodge bundle on the Shimura curve is the derivative of the same Eisenstein series $\mathcal{E}(\tau, s)$ at $s = 1$ (second term). This case is different from the Kronecker formula in two ways. Firstly, the leading term is already connected with arithmetics by the Siegel–Weil formula and is non-trivial. Secondly, the second term (derivative) is found to be deeply related to the Gillet–Soulé height paring on a Shimura curve. Its analogue in $X_0(1)$ was worked out later by Kudla and Yang, and was reported in [26]. In this case, the Eisenstein series is Zagier's famous Eisenstein series [12] of weight 3/2. In [3], Bruinier and Funke gave a different proof of the main result of [26] using theta lifting. Colmez conjecture [6] can also be viewed as an second term of 'CM' Hecke $L$-functions $L'(0, \chi)$ in terms of Faltings' height. We should mention the breakthrough formula of Zhiwei Yun and Wei Zhang which relates the $n$-th central derivative of the $L$-function of an automorphic representation on $\mathrm{GL}_2$ over a function field and height pairing of some cycles in middle dimension on some Drinfeld space [27]. We also mention the beautiful second term identity in the Siegel–Weil formula (see for example [9] and references there), although it has different flavor.

Later in the book [18, Chapter 4], Kudla proved that the arithmetic theta function $\widehat{\phi}_{KRY}$ is modular. In this paper, we will prove both the arithmetic Siegel–Weil formula and the modularity of a similar arithmetic theta function in the case of modular curve $X_0(N)$ when $N$ is square free. The complication comes mainly from the cusps, and we need to understand the behavior of Kudla's Green functions at cusps carefully. We give a complete description of its behavior at cusps—which is totally new. It is an interesting and likely very challenging question to extend the analysis to high dimensional Shimura varieties of orthogonal type $(n, 2)$. The metrized Hodge bundle has log singularity at cusps presents another complication. The method in [17] in computing the arithmetic intersection does not seem to extend to this case easily. Instead, we will use theta lifting method following [3]. In the process, we also obtain some explicit Kronecker limit formula for Eisenstein series of weigh 0 for $\Gamma_0(N)$, which should be of independent interest. In particular, we construct an explicit modular form (denoted by $\Delta_N$), which gives a rational section of the Hodge bundle and plays an essential role in proving the arithmetic

Siegel–Weil formula. After the arithmetic Siegel–Weil formula is proved, the modularity theorem follows the same method of [18, Chapter 4] with a little modification.

Now we set up notations and describe the main results in a little more detail.

Let

$$V = \left\{ w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in M_2(\mathbb{Q}) : \mathrm{tr}(w) = 0 \right\}, \tag{1.1}$$

with quadratic form $Q(w) = N \det w = -N(w_1^2 + w_2 w_3)$, and let

$$L = \left\{ w = \begin{pmatrix} b & \frac{-a}{N} \\ c & -b \end{pmatrix} \in M_2(\mathbb{Z}) \mid a, b, c \in \mathbb{Z} \right\} \tag{1.2}$$

be an even integral lattice with the dual lattice $L^\sharp$. Then $\mathrm{Spin}(V) \cong \mathrm{SL}_2$ acts on $V$ by conjugation, and the associated Hermitian symmetric domain $\mathbb{D}$ is isomorphic to the upper half plane $\mathbb{H}$. Since $\Gamma_0(N)$ preserves $L$ and acts on $L^\sharp/L$ trivially, we can and will identify $X_0(N)$ with the compactification of the open orthogonal Shimura curve $\Gamma_0(N)\backslash\mathbb{D}$ (see Section 2 for detail).

For each $\mu \in L^\sharp/L$, denote $L_\mu = \mu + L$, and

$$L_\mu[n] = \{w \in L_\mu : Q(w) = n\}.$$

For $\mu \in L^\sharp/L$ and a positive rational number $n \in Q(\mu) + \mathbb{Z}$, let $Z(w) = \mathbb{R}w \in \mathbb{D}$ and define the divisor

$$Z(n, \mu) := \sum_{w \in \Gamma_0(N)\backslash L_\mu[n]} Z(w) \in \mathrm{CH}^1(X_0(N)) \tag{1.3}$$

When $\mu = \mu_r = \mathrm{diag}(\frac{r}{2N}, -\frac{r}{2N})$, this divisor is the same as the Heegner divisors $P_{D,r} + P_{D,-r} \in \mathrm{CH}^1(X_0(N))$ in [11], where $D = -4Nn$ is a discriminant. For a positive real number $v > 0$, let $\Xi(n, \mu, v)$ be the Kudla Green function for $Z(n, \mu)$ in the open modular curve $Y_0(N)$ as defined in [15] (see (5.4) for precise definition). The behavior of $\Xi(n, \mu, v)$ at cusps is complicated and has not been studied before. In Sections 5 and 6, we will prove that it is smooth and of exponential decay when $D = -4Nn$ is not a square, and has singularity along the cusps (Section 6) when $D \neq 0$ is a square. Even worse, when $D = 0$ (which forces $\mu = 0$), $\Xi(0, 0, v)$ has log–log singularity in the sense of [5] (see Section 4). This is the most technical part of this paper.

Let $\mathcal{X}_0(N)$ be the canonical integral model over $\mathbb{Z}$ of $X_0(N)$ as defined in [13] (see Section 6). In the arithmetic part of this paper, we assume $N$ is square free so that $\mathcal{X}_0(N)$ is regular and flat over $\mathbb{Z}$ and is smooth over $\mathbb{Z}[\frac{1}{N}]$. For a point $x \in \mathcal{X}_0(N)$ over a field, since $\{\pm 1\} \subseteq \mathrm{Aut}(x)$, we count $x$ with multiplicity $\frac{2}{|\mathrm{Aut}(x)|}$ for convenience. Let $\mathcal{Z}(n, \mu)$ be the Zariski closure of $Z(n, \mu)$ in $\mathcal{X}_0(N)$, and we obtain a family of arithmetic divisors $\widehat{\mathcal{Z}}(n,\mu,v)$ in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$—arithmetic Chow group with real coefficients in the sense of Gillet–Soulé as follows for $n \neq 0$:

$$\widehat{\mathcal{Z}}(n,\mu,v) = \begin{cases} (\mathcal{Z}(n,\mu), \Xi(n,\mu,v)) & \text{if } n > 0, \\ (0, \Xi(n,\mu,v)) & \text{if } n < 0, D \neq \square, \\ (g(n,\mu,v)\sum_{P \text{ cusps}} \mathcal{P}, \Xi(n,\mu,v)) & \text{if } n < 0, D = \square. \end{cases}$$

Here $g(n,\mu,v)$ is some real number defined in Theorem 6.3, and $\mathcal{P}$ is the Zariski closure of the cusp $P$ in $\mathcal{X}_0(N)$. When $n = 0$, the same formula (as $D$ being a square) gives a 'naive' arithmetic Chow cycle $\widehat{\mathcal{Z}}(0,0,v)^{\mathrm{Naive}}$, which has log–log singularity at the cusps and needs to be modified to make the 'generating series' (to be defined below) modular. Let $\widehat{\omega}_N$ be the metrized Hodge bundle on $\mathcal{X}_0(N)$ with the normalized Petersson metric. It has log singularity at cusps in the sense of Kühn (see Section 4). Its associated arithmetic divisor has log–log singularity at the cusps. It turns out magically that the modified arithmetic divisor

$$\widehat{\mathcal{Z}}(0,0,v) = \widehat{\mathcal{Z}}(0,0,v)^{\mathrm{Naive}} - 2\widehat{\omega}_N - \sum_{p|N} \mathcal{X}_p^0 - (0, \log(\frac{v}{N})) \tag{1.4}$$

belongs to $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$ (Proposition 6.6). Here $\mathcal{X}_p^0$ (resp. $\mathcal{X}_p^\infty$) is the irreducible component of $\mathcal{X}_0(N)$ (mod $p$) containing the reduction of the cusp $P_0$ (resp. $P_\infty$). One of the main results of this paper is the following analogue of the modularity theorem in [18, Chapter 4].

**Theorem 1.1.** *The arithmetic theta function (for $\tau = u + iv$, and $q_\tau = e(\tau) = e^{2\pi i\tau}$)*

$$\widehat{\phi}(\tau) = \sum_{\mu \in L^\sharp/L} \sum_{n \in Q(\mu)+\mathbb{Z}} \widehat{\mathcal{Z}}(n,\mu,v) q_\tau^n e_\mu, \tag{1.5}$$

*is a vector valued modular form for $\Gamma'$ of weight $\frac{3}{2}$, valued in $\mathbb{C}[L^\sharp/L]\otimes\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$. Here $\Gamma'$ is the metaplectic cover of $\mathrm{SL}_2(\mathbb{Z})$ which acts on $\mathbb{C}[L^\sharp/L]$ via the Weil representation $\rho_L$ (see (2.2)) and acts on the arithmetic Chow group trivially. Finally $\{e_\mu : \mu \in L^\sharp/L\}$ is the standard basis of $\mathbb{C}[L^\sharp/L]$.*

Here, the modularity of $\widehat{\phi}(\tau)$ is in the sense of [18, Page 78], i.e., we can write $\widehat{\phi}(\tau) = \phi_{AR}(\tau) + \phi_{SM}(\tau,z)$ formally as Laurent series, where $\phi_{AR}(\tau)$ is a vector valued modular form of weight 3/2 valued in a finite dimensional subspace of $\mathbb{C}[L^\sharp/L]\otimes\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$, and $\phi_{SM}(\tau,z)$ is a smooth function on $\mathbb{H}\times X_0(N)$ and is modular as function of $\tau$ of weight 3/2. Intuitively, it asserts that the formal Laurent series satisfies the transformation law of a modular form of $\mathrm{SL}_2(\mathbb{Z})$ of weight 3/2 and representation $\rho_L$. Alternatively, for every linear map $\ell : \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N)) \to \mathbb{C}$, $\ell(\widehat{\phi}(\tau))$ is a (non-holomorphic) modular form of weight

3/2 of $\mathrm{SL}_2(\mathbb{Z})$ with Weil representation $\rho_L$. We refer to Theorem 8.4 and its proof for more precise meaning of the modularity of $\widehat{\phi}(\tau)$.

To prove the theorem, we will need the decomposition theorem of the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$ and some arithmetic intersection formulas as in [18, Chapter 4]. These intersection formulas are important themselves, which we now describe briefly.

Let

$$E_L(\tau, s) = \sum_{\gamma' \in \Gamma'_\infty \backslash \Gamma'} \left(v^{\frac{s-1}{2}} e_{\mu_0}\right) |_{3/2}\, \gamma'$$

be a vector valued Eisenstein series of weight 3/2, where the Petersson slash operator is defined on functions $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]$ by

$$\left(f\ |_{3/2}\, \gamma'\right)(\tau) = \phi(\tau)^{-3} \rho_L^{-1}(\gamma') f(\gamma\tau),$$

where $\gamma' = (\gamma, \phi) \in \Gamma'$. Let

$$\mathcal{E}_L(\tau, s) = -\frac{s}{4}\pi^{-s-1}\Gamma(s)\zeta^{(N)}(2s)N^{\frac{1}{2}+\frac{3}{2}s}E_L(\tau, s) \tag{1.6}$$

be its normalization, where

$$\zeta^{(N)}(s) = \zeta(s)\prod_{p|N}(1 - p^{-s}).$$

**Remark 1.2.** In the work [17] and [26], the critic point of Eisenstein series is $s = \frac{1}{2}$. In our paper, for the convenience of computation, we define $E_L(\tau, s)$ by a shift of s.

The intersection formulas referred above are given by the following theorem. The third formula is usually called an arithmetic Siegel–Weil formula while the first one (degree formula) is a geometric Siegel–Weil formula.

**Theorem 1.3.** *Let the notations be as above, then*

$$\langle \widehat{\phi}(\tau), a(1) \rangle_{GS} = \frac{1}{2}\deg(\widehat{\phi}(\tau)) = \frac{1}{\varphi(N)}\mathcal{E}_L(\tau, 1),$$
$$\langle \widehat{\phi}(\tau), \mathcal{X}_p^0 \rangle_{GS} = \langle \widehat{\phi}(\tau), \mathcal{X}_p^\infty \rangle_{GS} = \frac{1}{\varphi(N)}\mathcal{E}_L(\tau, 1)\log p, \quad p|N$$

*and*

$$\langle \widehat{\phi}(\tau), \widehat{\omega}_N \rangle_{GS} = \frac{1}{\varphi(N)}\left(\mathcal{E}'_L(\tau, 1) - \sum_{p|N}\frac{p}{p-1}\mathcal{E}_L(\tau, 1)\log p\right).$$

*Here* $a(1) = (0, 1) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))$.

There are three main ingredients in the proof of Theorem 1.3. The first is to analyze and understand the behavior of Kudla's Green function $\Xi(n, \mu, v)$ for all pairs $(n, \mu) \in \mathbb{Q} \times L^\sharp/L$ with $Q(\mu) \equiv n \pmod 1$, in particular when $D = -4Nn \geq 0$ is a square. Here $v > 0$ is a constant. This occupies full Section 5 (general case) and the first part of Section 6. The upshot is an honest definition of the arithmetic divisors $\widehat{\mathcal{Z}}(n, \mu, v)^{\mathrm{Naive}}$ in Theorem 6.3, its modification $\widehat{\mathcal{Z}}(n, \mu, v)$ in (6.11), and the generating function $\widehat{\phi}(\tau)$ above.

To understand $\widehat{\omega}_N$, we actually construct an explicit rational section of $\omega_N^k$, which is isomorphic to the line bundle of modular forms of weight $k$ for $k = 12\varphi(N)$ (the Euler $\varphi$-function), i.e., an explicit modular form $\Delta_N$ of weight $k$ for $\Gamma_0(N)$ as follows:

$$\Delta_N(z) = \prod_{t|N} \Delta(tz)^{a(t)} \tag{1.7}$$

with

$$a(t) = \sum_{r|t} \mu(\frac{t}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(\frac{N}{r})},$$

where $\mu(n)$ is the Möbius function. This is inspired by Kühn's early work on self-intersection of $\widehat{\omega}_N$ with $N = 1$ using the well-known Delta function $\Delta$. One complication here is that $\Delta_N$ has vertical components, see Lemma 6.4. This means that we will need to deal with self-intersections of vertical components (see Section 7).

These ingredients are enough for the first two identities of Theorem 1.3. To prove the last identity, we need to compute the infinity part of the arithmetic intersection, which boils down essentially to self-intersection of $\widehat{\omega}_N$, intersection of vertical components, and the following integral, which can be viewed as a theta lifting:

$$I(\tau, \log \|\Delta_N\|) = \int_{X_0(N)} \log \|\Delta_N\| \Theta_L(\tau, z). \tag{1.8}$$

Here $\Theta_L(\tau, z)$ is the two variable geometric theta kernel of Kudla and Millson defined by (2.6), and the Petersson norm is renormalized as

$$\|f(z)\| = |f(z)(4\pi e^{-C} y)^{\frac{k}{2}}| = e^{-\frac{kC}{2}} \parallel f(z) \parallel_{Pet}, \tag{1.9}$$

with $C = \frac{\log 4\pi + \gamma}{2}$. The theta function $\Theta_L(\tau, z)$ is a vector valued modular form for $\tau$ of weight 3/2 and modular function for the variable $z$ valued in $\Omega^{1,1}(X_0(N))$ for $\Gamma_0(N)$.

To connect this integral with $\mathcal{E}_L'(\tau, 1)$, we follow Bruinier and Funke's idea in [3] in two steps, given by the following two theorems, which are of independent interest.

**Theorem 1.4.** *(Theta lifting of Eisenstein series) Let*

$$E(N,z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}(\Im(\gamma z))^s, \tag{1.10}$$

*be the Eisenstein series of weight* $0$ *for* $\Gamma_0(N)$*, and let*

$$\mathcal{E}(N,z,s):=N^{2s}\pi^{-s}\Gamma(s)\zeta^{(N)}(2s)E(N,z,s) \tag{1.11}$$

*be its normalization. Then*

$$I(\tau,\mathcal{E}(N,z,s))=I(\tau,\mathcal{E}(N,w_Nz,s))=\zeta^*(s)\mathcal{E}_L(\tau,s),$$

*where* $w_N=\left(\begin{smallmatrix}0&-1\\N&0\end{smallmatrix}\right)$ *and* $\zeta^*(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$.

**Theorem 1.5.** *(Kronecker Limit formula for* $\Gamma_0(N)$*) Let the notations be as above, then one has*

$$\lim_{s\to1}\Big(\mathcal{E}(N,z,s)-\varphi(N)\zeta^*(2s-1)\Big)=-\frac{1}{12}\log\big(y^{6\varphi(N)}\mid\Delta_N(z)\mid\big),$$

*and*

$$\lim_{s\to1}\Big(\mathcal{E}(N,w_Nz,s)-\varphi(N)\zeta^*(2s-1)\Big)=-\frac{1}{12}\log\big(y^{6\varphi(N)}\mid\Delta_N^0(z)\mid\big),$$

*where* $\Delta_N^0=\Delta_N|w_N$.

Combining the previous theorems, we obtain

**Theorem 1.6.** *One has*

$$I(\tau,1)=\frac{2}{\varphi(N)}\mathcal{E}_L(\tau,1)$$

*and*

$$I(\tau,\log\parallel\Delta_N\parallel)=I(\tau,\log\|\Delta_N^0\|)=-12\mathcal{E}'_L(\tau,1).$$

This paper is organized in two parts as follows. In Part 1, we prove Theorem 1.4 after setting up notation and introduce the theta lifting (2.9) in Section 2. In Section 3, we study some basic properties of $\Delta_N$ and prove the Kronecker limit formula Theorem 1.5 and then Theorem 1.6. We also prove some properties of $\Delta_N$ needed in Part 2.

In Part 2, we first review arithmetic divisors with log–log singularity, metrized line bundles with log singularity, and arithmetic intersection in Section 4 following Kühn [19] and Burgos Gil, Kramer and U. Kühn [5]. In Section 5, we study the behaviors of Kudla's Green functions at cusps in a more general setting (see Theorem 5.1). In Section 6, we focus on the modular curve $X_0(N)$ for square free $N$, and prove Theorem 6.3. We also prove the first two formulas in Theorem 1.3, and reduces the third one to a 'horizontal intersection' theorem Theorem 6.9, which we will prove in Section 7. In Section 8, we will prove the modularity theorem (Theorem 1.1).

Finally, we remark that the technical condition $N$ being square free is only needed in the arithmetic part, mainly to avoid the complication of special fiber of $\mathcal{X}_0(N)$ at $p^2|N$ when $N$ is not square free.

**Acknowledgments.** We thank Haifeng Chu, S. Ehlen, Wanlin Li, S. Marshall, and S. Sankaran for their help during our work in this project. The second author thanks the MPIM-Bonn and the Morningside Center of Mathematics at Beijing (MCM) for their support, part of the work was done when he visited these institutes during summers. The first author thanks Yongju Choie for her help and mentoring and Postech for providing him excellent working environment and financial support when he was a postdoc of Choie in Postech. Part of the work was done there. He also thanks MCM for its support. Part of the work was done when he visited the institute a few times. We thank the referee for his/her careful reading of the paper and valuable suggestion, which makes the exposition clearer.

# Part 1. Theta lifting and Kronecker limit formula

## 2. Basic set-up and theta lifting

Let

$$V = \left\{ w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in M_2(\mathbb{Q}) : \mathrm{tr}(w) = 0 \right\}, \tag{2.1}$$

with the quadratic form $Q(w) = N \det w = -Nw_2w_3 - Nw_1^2$, which has signature $(1, 2)$. Let $L$ be the even integral lattice defined in the introduction with dual lattice $L^\sharp$. We will identify

$$\mathbb{Z}/2N\mathbb{Z} \cong L^\sharp/L, \quad r \mapsto \mu_r = \begin{pmatrix} \frac{r}{2N} & 0 \\ 0 & -\frac{r}{2N} \end{pmatrix}.$$

Let $G = \mathrm{SL}_2 \cong \mathrm{Spin}(V)$ act on $V$ by conjugation, i.e., $g \cdot w = gwg^{-1}$. Notice that $\Gamma_0(N)$ preserves $L$ and acts on $L^\sharp/L$ trivially. Let $\mathbb{D}$ be the Hermitian domain of positive real lines in $V_\mathbb{R}$:

$$\mathbb{D} = \{z \subset V_\mathbb{R} : \dim z = 1 \textit{ and } (\ ,\ )\,|_z > 0\}.$$

The following lemma can be easily checked and is left to the reader.

**Lemma 2.1.** *For $z = x + iy$, define*

$$w(z) = \frac{1}{\sqrt{N}y} \begin{pmatrix} -x & z\overline{z} \\ -1 & x \end{pmatrix}.$$

*Then $z \mapsto [w(z)] = \mathbb{R}w(z)$ gives a $G(\mathbb{R})$-invariant isomorphism between the upper half plane $\mathbb{H}$ and $\mathbb{D}$. It induces an isomorphism between $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ and $\Gamma_0(N)\backslash\mathbb{D}$.*

Let $X_0(N)$ be the usual compactification of $Y_0(N)$. Let $\mathrm{Mp}_{2,\mathbb{R}}$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$, which can be realized as pairs $(g, \phi(g,\tau))$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $\phi(g,\tau)$ is a holomorphic function of $\tau \in \mathbb{H}$ such that $\phi(g,\tau)^2 = j(g,\tau) = c\tau + d$. Let $\Gamma'$ be the preimage of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_{2,\mathbb{R}}$, then $\Gamma'$ is generated by

$$S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right) \quad T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right).$$

We denote the standard basis of $S_L = \mathbb{C}[L^\sharp/L]$ by $\{e_\mu = L_\mu : \mu \in L^\sharp/L\}$. Then there is a Weil representation $\rho_L$ of $\Gamma'$ on $\mathbb{C}[L^\sharp/L]$ given by ([1])

$$\begin{aligned} \rho_L(T)e_\mu &= e(Q(\mu))e_\mu, \\ \rho_L(S)e_\mu &= \frac{e(\frac{1}{8})}{\sqrt{|L^\sharp/L|}} \sum_{\mu' \in L^\sharp/L} e(-(\mu,\mu'))e_{\mu'}. \end{aligned} \tag{2.2}$$

This Weil representation $\rho_L$ is naturally connected to the Weil representation $\omega$ of $\mathrm{Mp}_{2,\mathbb{A}}$ on $S(V_\mathbb{A})$, see [4] for explanation.

Following Kudla and Millson ([16], [3, Section 3]), we decompose for $z = x + iy \in \mathbb{H}$,

$$V_\mathbb{R} = \mathbb{R}w(z) \oplus w(z)^\perp, \quad w = w_z + w_{z^\perp},$$

and define $R(w,z) = -(w_{z^\perp}, w_{z^\perp})$, and the majorant

$$(w,w)_z = (w_z, w_z) + R(w,z).$$

Since $Q(w(z)) = 1$, it is easy to check

$$\begin{aligned} R(w,z) &= \frac{1}{2}(w, w(z))^2 - (w,w), \\ (w,w)_z &= (w, w(z))^2 - (w,w). \end{aligned} \tag{2.3}$$

For $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in V_\mathbb{R}$, we have

$$(w, w(z)) = -\frac{\sqrt{N}}{y}(w_3 z\overline{z} - w_1(z + \overline{z}) - w_2). \tag{2.4}$$

Let $\mu(z) = \frac{dx\,dy}{y^2}$,

$$\varphi^0(w,z) = \left((w, w(z))^2 - \frac{1}{2\pi}\right) e^{-2\pi R(w,z)}\mu(z),$$
$$\varphi(w,\tau,z) = e(Q(w)\tau)\varphi^0(\sqrt{v}w, z), \tag{2.5}$$

which is a Schwartz function on $V_{\mathbb{R}}$ valued in $\Omega^{1,1}(\mathbb{D})$ constructed by Kudla and Millson in [16]. Finally, let

$$\Theta_L(\tau,z) = \sum_{\mu\in L^\sharp/L} \theta_\mu(\tau,z)e_\mu \tag{2.6}$$

be the vector valued Kudla–Millson theta function, where

$$\theta_\mu(\tau,z) = \sum_{n\in\mathbb{Q}, Q(\mu)\equiv n \pmod 1} \omega(n,\mu,v)(z)q^n + \begin{cases} 0 & \text{if } \mu \neq 0, \\ -\frac{1}{2\pi}\mu(z) & \text{if } \mu = 0, \end{cases} \tag{2.7}$$

with ($q = q_\tau = e(\tau)$)

$$\omega(n,\mu,v)(z) = \sum_{0\neq w\in L_\mu[n]} \varphi^0(v^{\frac{1}{2}}w, z) \in \Omega^{1,1}(X_\Gamma), \tag{2.8}$$

where $\tau = u + iv$. It is known that $\Theta_L(\tau,z)$ is a nonholomorphic modular form of weight 3/2 of $(\Gamma', \rho_L)$ valued in $\Omega^{1,1}(X_\Gamma) \otimes \mathbb{C}[L^\sharp/L]$ as a function of $\tau$. It is $\Gamma_0(N)$-invariant as a function of $z$.

The following result of Funke about behavior of $\theta_\mu$ as $z$ goes to the boundary (cusp) is important to our definition of theta lifting.

**Proposition 2.2.** *[3, Proposition 4.1] Fix $\mu \in L^\sharp/L$ and $\tau \in \mathbb{H}$. Let $l = \sigma_l(\infty)$ be a cusp of $X_0(N)$. As a function of $z = x + iy \in \mathbb{H} = \mathbb{D}$, the theta function (recall $z = x + iy$)*

$$\theta_\mu(\tau, \sigma_l z) = O(e^{-Cy^2}), \text{ as } y \longrightarrow \infty$$

*holds uniformly in $x$ for some constant $C > 0$.*

For a (non-holomorphic) modular function $f(z)$ for $\Gamma_0(N)$ (viewed as a subgroup of the Spin group) with moderate growth, the theta lifting

$$I(\tau, f) = \int_{\Gamma_0(N)\backslash\mathbb{D}} f(z)\Theta_L(\tau,z) = \sum_{\mu\in L^\sharp/L} I_\mu(\tau,f)e_\mu \tag{2.9}$$

is absolutely convergent by Proposition 2.2 and is a (non-holomorphic) weight 3/2 modular form of $\Gamma'$ with values in $\mathbb{C}[L^\sharp/L]$.

**Proof of Theorem 1.4.** First, we compute the theta series:

$$\theta_{\mu_r}(\tau, z) = \sum_{w \in L_{\mu_r}} \varphi(w, \tau, z)$$

$$= \sum_{w_1 \in \mathbb{Z}+\frac{r}{2N}, n, w_3 \in \mathbb{Z}} \left( \frac{v}{Ny^2} \big(N(w_3 z\overline{z} - w_1(z+\overline{z})) - n\big)^2 - \frac{1}{2\pi} \right)$$

$$\times e(-N\overline{\tau}w_1^2) e(-\overline{\tau}w_3 n) e\left( \frac{iv}{2Ny^2} \big(N(w_3 z\overline{z} - w_1(z+\overline{z})) - n\big)^2 \right) \mu(z).$$

Let

$$f(X) = (\frac{vX^2}{Ny^2} - \frac{1}{2\pi}) e(-\overline{\tau}w_3 X) e\big(\frac{ivX^2}{2Ny^2}\big),$$

then its Fourier transformation is

$$\widehat{f}(m) = \int_{-\infty}^{\infty} f(X) e(-mX) dX = -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} (\overline{\tau}w_3 + m)^2 e\big(\frac{iNy^2}{2v} (\overline{\tau}w_3 + m)^2\big).$$

Write $t = N(w_3 z\overline{z} - w_1(z+\overline{z}))$. Applying the Poisson summation formula, we obtain

$$\theta_{\mu_r}(\tau, z) = \sum_{w_1 \in \mathbb{Z}+\frac{r}{2N}, m, w_3 \in \mathbb{Z}} e(-N\overline{\tau}w_1^2) e(-\overline{\tau}w_3 t) \widehat{f}(m) e(-mt) \mu(z)$$

$$= -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} \sum_{w_1 \in \mathbb{Z}+\frac{r}{2N}, m, w_3 \in \mathbb{Z}} (\overline{\tau}w_3 + m)^2 e\big(-N\overline{\tau}(w_1 - w_3 x)^2\big)$$

$$\times\, e\big(2N(w_1 - w_3 m/2) m x\big) exp\big(-\frac{\pi N y^2}{v} \mid m + w_3\tau \mid^2 \big) \mu(z).$$

As in [1, Section 4], we define for $\alpha, \beta \in \mathbb{Q}$

$$\Theta_L(\tau, \alpha, \beta) = \sum_{r \in \mathbb{Z}/2N} \sum_{w_1 \in \frac{r}{2N}+\mathbb{Z}} e(-\overline{\tau}(w_1+\beta)^2) e(-\alpha(2w_1+\beta)) e_{\mu_r}. \tag{2.10}$$

For $\gamma' = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \in \Gamma'$, it is easy to check

$$\Theta_L(\tau, ndx, -ncx) = (c\overline{\tau} + d)^{-\frac{1}{2}} \rho_L^{-1}(\gamma') \Theta_L(\gamma'\tau, nx, 0). \tag{2.11}$$

We continue the calculation:

$$
\begin{aligned}
&\Theta_L(\tau, z)\\
&= -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} \sum_{m, w_3 \in \mathbb{Z}} (\overline{\tau} w_3 + m)^2 e^{\left(-\frac{\pi N y^2}{v} |m + w_3 \tau|^2\right)} \Theta_L(\tau, mx, -w_3 x) \mu(z)\\
&= -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^2 \sum_{c,d \in \mathbb{Z}, (c,d)=1} (c\overline{\tau} + d)^2 e^{\left(-\frac{\pi N y^2 n^2}{v} |c\overline{\tau} + d|^2\right)} \Theta_L(\tau, ndx, -ncx) \mu(z)\\
&= -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^2 \sum_{\gamma' \in \Gamma'_\infty \diagdown \Gamma'} (c\overline{\tau} + d)^{\frac{3}{2}} e^{\left(-\frac{\pi N y^2 n^2}{v} |c\overline{\tau} + d|^2\right)} \rho_L^{-1}(\gamma') \Theta_L(\gamma' \tau, nx, 0) \mu(z).
\end{aligned}
$$

Unfolding the integral, we have for $\Re(s) > 1$

$$
\begin{aligned}
I(\tau, E(N, z, s)) &= \int_{\Gamma_\infty \diagdown \mathbb{H}} \Theta_L(\tau, z) y^s\\
&= -v^{-\frac{3}{2}} N^{\frac{3}{2}} \sum_{n=1}^{\infty} n^2 \sum_{\gamma' \in \Gamma'_\infty \diagdown \Gamma'} (c\overline{\tau} + d)^{3/2} \int_0^{\infty} e^{\left(-\frac{\pi N y^2 n^2}{v} |c\overline{\tau} + d|^2\right)} y^{s+1} dy\\
&\quad \times \rho_L^{-1}(\gamma') \int_0^1 \Theta_L(\gamma' \tau, nx, 0) dx.
\end{aligned}
$$

It is easy to check that

$$
\int_0^1 \Theta_L(\gamma' \tau, nx, 0) dx = e_{\mu_0}.
$$

So

$$
\begin{aligned}
&\int_{\Gamma_\infty \diagdown \mathbb{H}} \Theta_L(\tau, z) y^s\\
&= -\frac{1}{2} v^{-\frac{3}{2}} N^{\frac{3}{2}} \sum_{n=1}^{\infty} n^2 \sum_{\gamma' \in \Gamma'_\infty \diagdown \Gamma'} \frac{v^{\frac{s+2}{2}} (c\overline{\tau} + d)^{3/2} \Gamma\left(\frac{s}{2} + 1\right)}{\pi^{\frac{s+2}{2}} \mid c\tau + d \mid^{s+2} N^{\frac{s+2}{2}} n^{s+2}} \rho_L^{-1}(\gamma') e_{\mu_0}\\
&= -\frac{1}{2} N^{\frac{1-s}{2}} \zeta(s) \Gamma\left(\frac{s}{2} + 1\right) \sum_{\gamma' \in \Gamma'_\infty \diagdown \Gamma'} \frac{v^{\frac{s-1}{2}} (c\overline{\tau} + d)^{3/2}}{\pi^{\frac{s}{2}+1} \mid c\tau + d \mid^{s+2}} \rho_L^{-1}(\gamma') e_{\mu_0}\\
&= -N^{\frac{1-s}{2}} \frac{s}{4\pi} \zeta^*(s) \sum_{\gamma' \in \Gamma'_\infty \diagdown \Gamma'} \left(v^{\frac{s-1}{2}} e_{\mu_0}\right) |_{3/2, L} \gamma'.
\end{aligned}
$$

In summary, we have proved

$$
I(\tau, E(N, z, s)) = -N^{\frac{1-s}{2}} \frac{s}{4\pi} \zeta^*(s) E_L(\tau, s),
$$

or equivalently,

$$I(\tau, \mathcal{E}(N, z, s)) = \zeta^*(s)\mathcal{E}_L(\tau, s). \tag{2.12}$$

It is easy to check by definition that

$$\theta_L(\tau, z) = \theta_L(\tau, w_N(z)).$$

This implies that

$$I(\tau, \mathcal{E}(N, w_N(z), s)) = I(\tau, \mathcal{E}(N, z, s)).$$

This proves the theorem.

Taking residue of both sides of the equation (2.12) at $s = 1$, we have the following result.

**Corollary 2.3.**

$$I(\tau, 1) = \frac{2}{\varphi(N)}\mathcal{E}_L(\tau, 1). \tag{2.13}$$

## 3. Kronecker limit formula for the group $\Gamma_0(N)$

We need some preparation before proving Theorem 1.5—the Kronecker Limit formula for $\Gamma_0(N)$. These auxiliary results will also be used in Section 6 and should be of independent interest.

Let

$$C_N(n) = \sum_{a=1,(a,N)=1}^{N} e(\frac{an}{N}) \tag{3.1}$$

be the Ramanujan sum. It has the following properties according to Kluver ([14, p. 411]).

**Lemma 3.1.** *(Kluver) Let $t = (N, n)$ be the greatest common divisor of $N$ and $n$. Then one has*

$$C_N(n) = \frac{\varphi(N)}{\varphi(\frac{N}{t})}C_{\frac{N}{t}}(1) = \sum_{r|t} \mu(\frac{N}{r})r.$$

*Here $\varphi$ is the classical Euler $\varphi$-function, and $\mu(t)$ is the well-known Möbius function. In particular, one has $C_N(1) = \mu(N)$.*

**Lemma 3.2.** *For a positive integer $N$ and a divisor $t$ of $N$, let*

$$a_N(t) = \sum_{r|t} \mu(\frac{t}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(\frac{N}{r})}$$

*be as in the introduction. Then the following are true.*

(1) *If $Q\|N$, i.e., $Q|N$ and $(Q, N/Q) = 1$, write $t = t_1t_2$. Then $a_N(t) = a_Q(t_1)a_{N/Q}(t_2)$.*
(2) *One has*

$$\sum_{t|N} a_N(t) = \varphi(N),$$

$$\sum_{t|N} ta_N(t) = N\varphi(N)\prod_{p|N}(1+p^{-1}),$$

$$\sum_{t|N} t^{-1}a_N(t) = 0 \quad \textit{when } N > 1.$$

**Proof.** (1) is clear. For (2), we check the second identity and leave the others to the reader. We drop the subscript $N$ from now on as $N$ will be fixed. One has

$$\begin{aligned}
\sum_{t|N} ta(t) &= \sum_{t|N} t \sum_{r|t} \mu(\frac{t}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(\frac{N}{r})} \\
&= \varphi(N)\sum_{r|N}\frac{\mu(\frac{N}{r})}{\varphi(\frac{N}{r})}\sum_{t|\frac{N}{r}} rt\mu(t) \quad (\text{ replacing } t \text{ by } rt) \\
&= N\varphi(N) \sum_{\substack{r|N \\ \text{r square free}}} \frac{\mu(r)}{r\varphi(r)}\sum_{t|r} t\mu(t) \quad (\text{ replacing } N/r \text{ by } r) \\
&= N\varphi(N) \sum_{\substack{r|N \\ \text{r square free}}} \frac{1}{r} = N\varphi(N)\prod_{p|N}(1+p^{-1}). \quad \square
\end{aligned}$$

**Proposition 3.3.** *Let $\Delta_N(z)$ be defined as in (1.7). Then ($q_z = e(z)$)*

$$\Delta_N(z) = q_z^{N\varphi(N)\prod_{p|N}(1+p^{-1})}\prod_{n\geq 1}(1-q_z^n)^{24C_N(n)}.$$

**Proof.** Let

$$\tilde{\Delta}_N(z) = \prod_n (1-q_z^n)^{C_N(n)}, \text{ and } \tilde{\Delta}(z) = \prod_{n=1}^{\infty}(1-q_z^n).$$

Suppose that there are numbers $b(t)$ with

$$\tilde{\Delta}_N(z) = \prod_{t|N} \tilde{\Delta}(tz)^{b(t)},$$

which implies by Lemma 3.1

$$\begin{aligned}
\prod_{t|N} \prod_{(n,\frac{N}{t})=1} (1-q_z^{tn})^{\frac{\varphi(N)}{\varphi(N/t)}\mu(N/t)} &= \prod_{t|N}\prod_{n}(1-q_z^{tn})^{b(t)} \\
&= \prod_{t|N}\prod_{t'|\frac{N}{t}} \prod_{(n,\frac{N}{tt'})=1} (1-q_z^{tt'n})^{b(t)} \\
&= \prod_{r|N}\prod_{t|r} \prod_{(n,\frac{N}{r})=1} (1-q_z^{rn})^{b(t)} \\
&= \prod_{r|N} \prod_{(n,\frac{N}{r})=1} (1-q_z^{rn})^{\sum_{t|r} b(t)}.
\end{aligned}$$

So for every $r|N$, one has

$$\sum_{t|r} b(t) = \frac{\varphi(N)}{\varphi(N/r)}\mu(N/r). \tag{3.2}$$

By Möbius inverse formula, one has

$$b(t) = \sum_{r|t} \mu(\frac{t}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(\frac{N}{r})} = a(t).$$

So we have proved that

$$\tilde{\Delta}_N(z) = \prod_{t|N} \tilde{\Delta}(tz)^{a(t)}.$$

Combining this with Lemma 3.2 (2), we obtain the lemma. □

Recall ([21]) that cusps of $X_0(N)$ are given by $P_{\frac{aQ}{N}} = \frac{aQ}{N}$, where $Q|N$ and $a \in (\mathbb{Z}/(Q,N/Q)\mathbb{Z})^\times$. In particular, when $Q\|N$, i.e., $Q|N$ and $(Q,N/Q)=1$, there is a unique cusp $P_{\frac{Q}{N}}$ associated to it. $Q=1$ is associated to $P_\infty = P_{\frac{1}{N}}$, and $Q=N$ is associated to $P_0 = P_1$. Assume $Q\|N$, and let

$$W_Q = \begin{pmatrix} \alpha & \beta \\ \gamma\frac{N}{Q} & Q\delta \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma\frac{N}{Q} & Q\delta \end{pmatrix} \in \Gamma_0(N/Q)$$

be an Atkin–Lehner involution matrix with $w_Q(P_\infty) = P_{\frac{Q}{N}}$. Notice that when $N$ is a square free, $P_{\frac{Q}{N}}$, $Q|N$, give all the cusps of $X_0(N)$. The following proposition gives Fourier expansion of $\Delta_N$ at cusps associated to $Q\|N$.

**Proposition 3.4.** *Assume $Q\|N$. For $t|N$, write $t_0 = (t, Q)$ for their greatest common divisor. Then*

$$\Delta_N|W_Q(z) = C_Q \prod_{t|N} \Delta(\frac{t}{t_0}\frac{Q}{t_0}z)^{a_N(t)}$$

*where*

$$C_Q = Q^{6\varphi(N)} \prod_{t_0|Q} t_0^{-12\varphi(\frac{N}{Q})a_Q(t_0)}.$$

*In particular,* $\mathrm{ord}_p\, C_Q = 0$ *for* $p \nmid Q$*. Moreover,* $\Delta_N(z)$ *does not vanish at cusps associated to* $Q\|N$ *(with* $Q \neq 1$*).*

**Proof.** Write $k = 12\varphi(N)$, $t = t_0 t_1$, and $Q = t_0 Q_1$. Then

$$\begin{aligned}\Delta_N|W_Q(z) &= \frac{Q^{\frac{k}{2}}}{(\gamma N z + Q\delta)^k} \prod_{t|N} \Delta(\frac{\alpha Q t z + t\beta}{\gamma N z + Q\delta})^{a_N(t)} \\ &= \frac{Q^{\frac{k}{2}}}{(\gamma N z + Q\delta)^k} \prod_{t|N} \Delta(\frac{\alpha t_0(t_1 Q_1 z) + t_1\beta}{\gamma \frac{N}{t_1 Q}(t_1 Q_1 z) + Q_1\delta})^{a_N(t)} \\ &= A_Q \prod_{t|N} \Delta(t_1 Q_1 z)^{a_N(t)},\end{aligned}$$

where (recall Lemma 3.2)

$$A_Q = Q^{\frac{k}{2}} \prod_{t|Q} t_0^{-12 a_N(t)} = C_Q.$$

On the other hand, the leading $q$-power exponent of $\Delta_N|W_Q$ is given by the above calculation (recall again Lemma 3.2)

$$\begin{aligned}\sum_{t|Q} t_1 Q_1 a_N(t) &= \sum_{t_0|Q} \frac{Q}{t_0} a_Q(t_0) \sum_{t_1|\frac{N}{Q}} t_1 a_{\frac{N}{Q}}(t_1) \\ &= \begin{cases} 0 & \text{if } Q > 1, \\ N\varphi(N)\prod_{p|N}(1+p^{-1}) & \text{if } Q = 1. \end{cases}\end{aligned}$$

This proves the result. □

**Proof of Theorem 1.5.** Recall the Whittaker function ([25, Chapter 2]) for $y > 0$ and $\alpha, \beta \in \mathbb{C}$:

$$W(y,\alpha,\beta)=\Gamma(\beta)^{-1}\int_0^\infty (1+h)^{\alpha-1}h^{\beta-1}e^{-yh}dh. \tag{3.3}$$

Define

$$\begin{aligned} &t_n(y,\alpha,\beta)\\ &=\begin{cases} i^{\beta-\alpha}(2\pi)^{\alpha+\beta}n^{\alpha+\beta-1}e^{-2\pi ny}\Gamma(\alpha)^{-1}W(4\pi ny,\alpha,\beta), & \text{if } n>0,\\ i^{\beta-\alpha}(2\pi)^{\alpha+\beta}\mid n\mid^{\alpha+\beta-1} e^{-2\pi|n|y}\Gamma(\beta)^{-1}W(4\pi\mid n\mid y,\beta,\alpha), & \text{if } n<0,\\ i^{\beta-\alpha}(2\pi)^{\alpha+\beta}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta}, & \text{if } n=0. \end{cases}\end{aligned}$$

One has by calculation ($z=x+iy\in\mathbb{H}$)

$$\begin{aligned} E(N,z,s)&=\frac{y^s}{2\zeta^{(N)}(2s)}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(N,n)=1}}\frac{1}{|mNz+n|^{2s}}\\ &=y^s+\frac{y^s}{N^{2s}\zeta^{(N)}(2s)}\sum_{m=1}^\infty\sum_{\substack{1\le a<N\\(a,N)=1}}\sum_{j\in\mathbb{Z}}|mz+\frac{a}{N}+j|^{-2s}\\ &=y^s+\frac{y^s}{N^{2s}\zeta^{(N)}(2s)}\sum_{n\in\mathbb{Z}}\sum_{m=1}^\infty t_n(my,s,s)\sum_{a\in(\mathbb{Z}/N)^\times}e(\frac{n(mNx+a)}{N}). \end{aligned}$$

Write

$$\mathcal{E}(N,z,s)=N^{2s}\pi^{-s}\Gamma(s)\zeta^{(N)}(2s)E(N,z,s)=\sum_{k\in\mathbb{Z}}a_k(z,s)e(kx).$$

Then we have

$$a_0(z,s)=N^{2s}\pi^{-s}\Gamma(s)\zeta^{(N)}(2s)y^s+\varphi(N)y^s\pi^{-s}\frac{(2\pi)^{2s}\Gamma(2s-1)(4\pi y)^{1-2s}\zeta(2s-1)}{\Gamma(s)}.$$

Simple calculation gives

$$a_0(z,s)=\varphi(N)\Big(\frac{1}{2(s-1)}-\frac{\log y}{2}-\frac{\log 4\pi-\gamma}{2}+\frac{\pi}{6}yN\prod_{p|N}(1+p^{-1})\Big)+O(s-1). \tag{3.4}$$

On the other hand,

$$\zeta^*(2s-1)=\frac{1}{2(s-1)}-\frac{1}{2}\big(\log 4\pi-\gamma\big)+O(s-1). \tag{3.5}$$

So

$$\lim_{s\to 1}(a_0(z,s)-\varphi(N)\zeta^*(2s-1)) = \varphi(N)(-\frac{\log y}{2}+\frac{\pi}{6}yN\prod_{p|N}(1+p^{-1})). \tag{3.6}$$

For $k > 0$, one has

$$\begin{aligned} a_k(z,s) &= y^s\pi^{-s}\Gamma(s)\sum_{mn=k} t_n(my,s,s) \sum_{a=1,(a,N)=1}^{N} e(nmx+anN^{-1}) \\ &= y^s\pi^{-s}\Gamma(s)(2\pi)^{2s}\frac{W(4\pi ky,s,s)}{\Gamma(s)e^{2\pi ky}}\sum_{mn=k} n^{2s-1}C_N(n). \end{aligned}$$

As

$$W(4k\pi y,1,1)=\frac{1}{4k\pi y},$$

one has

$$a_k(z,1)=\frac{e^{-2\pi ky}}{k}\sum_{n|k} nC_N(n). \tag{3.7}$$

It is easy to see from definition that $a_{-k}(z,1) = a_k(z,1)$. Therefore,

$$\begin{aligned} \mathcal{E}(N,z,s) &= \sum_{k=-\infty}^{\infty} a_k(z,s)e(kx) \\ &= a_0(z,s)+\sum_{k>0}\frac{1}{k}\sum_{n|k} nC_N(n)q_z^k+\sum_{k>0}\frac{1}{k}\sum_{n|k} nC_N(n)\overline{q_z^k}+O(s-1) \\ &= a_0(z,s)+\sum_{n=1}^{\infty} C_N(n)\sum_{m=1}^{\infty}\frac{1}{m}(q_z^{mn}+\bar{q_z}^{mn})+O(s-1). \end{aligned}$$

Combining this with (3.6) and Proposition 3.3, we obtain

$$\begin{aligned} &\lim_{s\to 1}\left(\mathcal{E}(N,z,s)-\varphi(N)\zeta^*(2s-1)\right) \\ &= -\frac{\varphi(N)}{2}\log y+\frac{N\varphi(N)\prod_{p|N}(1+p^{-1})\pi y}{6}-\sum_{n=1}^{\infty}\log|1-q_z^n|^2 \\ &= -\frac{1}{12}\log\left(y^{6\varphi(N)}\mid\Delta_N(z)\mid\right), \end{aligned}$$

as claimed. The second one follows from this identity immediately by applying $w_N$ on both sides.

**Proof of Theorem 1.6.** The first identity is just restatement of Corollary 2.3. For the second identity, we have by Theorems 1.5, 1.4 and Corollary 2.3

$$\begin{aligned}
&-\frac{1}{12}I(\tau, \log \mid \Delta_N(z)y^{6\varphi(N)} \mid) \\
&= \lim_{s\to 1}\Big(I(\tau, \mathcal{E}(N,z,s)) - I(\tau, \varphi(N)\zeta^*(2s-1))\Big) \\
&= \lim_{s\to 1}\Big(\zeta^*(s)\mathcal{E}_L(\tau,s) - 2\zeta^*(2s-1)\mathcal{E}_L(\tau,1)\Big).
\end{aligned}$$

Now the second identity for $\log\|\Delta_N(z)\|$ follows from elementary calculation of the Laurent expansion (just first two terms) of the functions in the above expression. We leave the detail to the reader.

**Proposition 3.5.** *(1) The generalized Delta function $\Delta_N(z)$ of level $N$ vanishes at the cusp $\infty$ with vanishing order $N\varphi(N)\prod_{p|N}(1+p^{-1})$, and does not vanish at other cusps.*

*(2)*

$$\Delta_N^0(z) = \Delta_N(z)|w_N = C_N \prod_{t|N} \Delta(tz)^{a(\frac{N}{t})} \in M_k(N) \tag{3.8}$$

*has vanishing order $\varphi(N)N\prod_{p|N}(1+p^{-1})$ at the cusp $P_0$ and does not vanish at other cusps. Here $C_N$ is the constant given in Proposition 3.4.*

**Proof.** This proposition is clear at cusp $P_{Q/N}$ with $Q\|N$ by Proposition 3.4. In particular, it is true when $N$ is square free, which is all we need in Part 2. The general case follows from the Kronecker limit formula at the cusp $P$. Write

$$N^{2s}\pi^{-s}\Gamma(s)\zeta^{(N)}(2s) = A + B(s-1) + O((s-1)^2),$$

and $\alpha = \frac{\varphi(N)}{A}$. According to [10, (21)], for a cusp $P$, there is $\sigma = \sigma_P \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma(P_\infty) = P$, and

$$\begin{aligned}
&\lim_{s\to 1}\left(E(N,\sigma z,s) - \frac{\alpha}{2(s-1)}\right) \\
&= \beta_P - \frac{\alpha}{2}\log y + y\delta_{P,P_\infty} + \sum_{m>1}(\phi_{P,m}q_z^m + \overline{\phi_{P,m}q_z^m}),
\end{aligned}$$

for some constant $\beta_P$. Here $\delta_{P,P_\infty}$ is the Kronecker $\delta$-symbol. So simple calculation gives for $P \neq P_\infty$

$$\lim_{s\to 1}\left(\mathcal{E}(N,\sigma z,s)-\varphi(N)\zeta^*(2s-1)\right)$$
$$=\gamma_P-\frac{\varphi(N)}{2}\log y+A\sum_{m>1}(\phi_{P,m}q_z^m+\overline{\phi_{P,m}q_z^m}),$$

for some constant $\gamma_P$. One has thus by Theorem 1.5

$$\log\left(y^{6\varphi(N)}\mid\Delta_N(\sigma(z))\mid\right)=-12\gamma_P+6\varphi(N)\log y-12A\sum_{m>1}(\phi_{P,m}q_z^m+\overline{\phi_{P,m}q_z^m}).$$

Equivalently,

$$\log|\Delta_N(\sigma(z))|=-12\gamma_P-12A\sum_{m>1}(\phi_{P,m}q_z^m+\overline{\phi_{P,m}q_z^m}),$$

which goes to $-12\gamma_P$ when $y\to\infty$. So $\Delta_N(z)$ does not vanish at the cusp $P=\sigma(P_\infty)$. $\square$

Recall that the Eisenstein series $E(N,z,s)$ has the Fourier expansion

$$E(N,z,s)=\sum_{n\in\mathbb{Z}}c_n(y,s)e(nx),$$

where the constant term has the form

$$c_0(y,s)=y^s+\Phi(s)y^{1-s},$$

with

$$\Phi(s)=\frac{\varphi(N)\pi^{\frac{1}{2}}\zeta(2s-1)\Gamma(s-\frac{1}{2})}{N^{2s}\zeta^{(N)}(2s)}. \tag{3.9}$$

Simple calculation gives the following lemma, which will be used in the proof of Theorem 6.9.

**Lemma 3.6.** *Write*

$$\Phi(s)=\frac{C_{-1}}{s-1}+C_0+O(s-1).$$

*Then*

$$C_{-1}=Res_{s=1}\Phi(s)=\frac{3}{\pi r},$$
$$C_0=-\frac{6}{\pi r}\left(\log 4\pi-1+12\zeta'(-1)+\sum_{p|N}\frac{p^2}{p^2-1}\log p\right),$$

*where* $r=[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]=N\Pi_{p|N}(1+p^{-1})$.

We remark that $C_0$ is the so-called scattering constant $C_{P_\infty,P_\infty}$ in [20].

## Part 2. Arithmetic intersection and derivative of Eisenstein series

In this part, we will focus on the arithmetic intersection on the modular curve $X_0(N)$ and prove Theorems 1.3 and 1.1. We will assume from now on that $N$ is square free.

### 4. Metrized line bundles with log singularity and arithmetic divisors with log–log singularities

The Gillet–Soulé height pairing (see [24]) has been extended to arithmetic divisors with log–log singularities or equivalently metrized bundles with log singularities ([5], [20], [19]). It is also extended to arithmetic divisors with $L_1^2$-Green functions ([2]). In this paper, we will use Kühn's set-up in [20], which is most convenient in our situation. Actually, for simplicity, we use a stronger condition which is easier to state and enough for our purpose.

Let $\mathcal{X}$ be a regular, proper and flat stack over $\mathbb{Z}$ of dimension 2 (called arithmetic surface), and denote $X = \mathcal{X}(\mathbb{C})$. For a finite subset $S = \{S_1, \cdots, S_r\}$ of $X$, let $Y = X - S$ be its complement. For $\epsilon > 0$, let $B_\epsilon(S_j)$ be the open disc of radius $\epsilon$ centered at $S_j$, and $X_\epsilon = X - \bigcup_j B_\epsilon(S_j)$. Let $t_j$ be a local parameter at $S_j$. A metrized line bundle $\widehat{\mathcal{L}} = (\mathcal{L}, \| \|)$ with log singularity (with respect to $S$) is a line bundle $\mathcal{L}$ over $\mathcal{X}$ together with a metric $\| \|$ on $\mathcal{L}_\infty = \mathcal{L}(\mathbb{C})$ satisfying the following conditions:

(1) $\| \|$ is a smooth Hermitian metric on $\mathcal{L}_\infty$ when restricting to $Y$.
(2) For each $S_j \in S$ and a (non-trivial) section $s$ of $\mathcal{L}$, there is a real number $\alpha_j$ and a positive smooth function $\varphi$ on $B_\epsilon(S_j)$ such that

$$\|s(t_j)\| = (-\log |t_j|^2)^{\alpha_j} |t_j|^{\mathrm{ord}_{S_j}(s)} \varphi(t_j)$$

hold for all $t_j \in B_\epsilon(S_j) - \{0\}$ (here $t_j = 0$ corresponds to $S_j$).

Notice that $\widehat{\mathcal{L}}$ with log singularity is a regular metrized line bundle if and only if all $\alpha_j = 0$. We will denote $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathcal{X}, S)$ for the group of metrized line bundles with log singularity (with respect to $S$) with $\mathbb{R}$-coefficients (i.e. allowing formally $\widehat{\mathcal{L}}^c$ with $c \in \mathbb{R}$).

A pair $\widehat{\mathcal{Z}} = (\mathcal{Z}, g)$ is called an arithmetic divisor with log–log-singularity (along $S$) if $\mathcal{Z}$ is a divisor of $\mathcal{X}$, and $g$ is a smooth function away from $Z \cup S$ ($Z = \mathcal{Z}(\mathbb{C})$), and satisfying the following conditions:

$$dd^c g + \delta_Z = [\omega],$$

$$g(t_j) = -2\alpha_j \log\log(\frac{1}{|t_j|^2}) - 2\beta_j \log |t_j| - 2\psi_j(t_j) \quad \text{near } S_j,$$

for some smooth function $\psi_j$ and some $(1,1)$-form $\omega$ which is smooth away from $S$. Let $\widehat{\mathcal{L}}$ be the metrized line bundle associated to $\widehat{\mathcal{Z}}$ with canonical section $s$ with $-\log\|s\|^2 = g$, then $\widehat{\mathcal{Z}}$ is of log–log-singularity if and only if $\widehat{\mathcal{L}}$ has log-growth and

$$\alpha_j(g) = \alpha_j(s), \quad \beta_j(g) = \mathrm{ord}_{S_j}(s), \quad \psi_j(t_j) = \log\varphi(t_j). \tag{4.1}$$

We define $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S)$ be the quotient of the $\mathbb{R}$-linear combination of the arithmetic divisors of $\mathcal{X}$ with log–log growth along $S$ by $\mathbb{R}$-linear combinations of the principal arithmetic divisors with log–log growth along $S$. One has $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathcal{X}, S) \cong \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S)$. The following is a [20, Proposition 1.4].

**Proposition 4.1.** *There is an extension of the Gillet–Soulé height paring to*

$$\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S) \times \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S) \to \mathbb{R}$$

*such that if $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are divisors intersect properly, then*

$$\langle(\mathcal{Z}_1, g_1), (\mathcal{Z}_2, g_2)\rangle = (\mathcal{Z}_1.\mathcal{Z}_2)_{fin} + \frac{1}{2} g_1 * g_2$$

*where the star product is defined to be*

$$g_1 * g_2 = g_1(Z_2 - \sum_j \mathrm{ord}_{S_j}(Z_2) S_j) + 2\sum_j \mathrm{ord}_{S_j} Z_2\,(\alpha_{\mathcal{L}_1,j} - \psi_{1,j}(0))$$
$$- \lim_{\epsilon\to 0}\left(2\sum_j (\mathrm{ord}_{S_j} Z_2)\alpha_{\mathcal{L}_1,j}\log(-2\log\epsilon) - \int_{X_\epsilon} g_2\omega_1\right).$$

*Here $Z_i = \mathcal{Z}_i(\mathbb{C})$, $\widehat{\mathcal{L}}_i$ is the associated metrized line bundle with the canonical section $s_i$. $\alpha_{\mathcal{L}_i,j}$ and $\psi_{i,j}$ are associated to $g_i$ and cusp $S_j$. Finally, $\omega_i$ is the $(1,1)$-form associate to $g_i$ via the following equation*

$$dd^c[g_i] + \delta_{Z_i} = [\omega_i].$$

We remark that the pairing is also symmetric. In particular, one has for any $a(f) = (0, f) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S)$,

$$\langle(\mathcal{Z}, g), a(f)\rangle = \frac{1}{2}\int_X f\omega. \tag{4.2}$$

We define the degree map

$$\deg : \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S) \to \mathbb{R}, \quad \deg(\mathcal{Z}, g) = \int_X \omega = \langle(\mathcal{Z}, g), (0, 2)\rangle. \tag{4.3}$$

It is just $\deg Z$ when $g$ is a Green function of $Z = \mathcal{Z}(\mathbb{C})$ without log–log singularity.

We will denote $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}) = \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, \text{empty})$ for the usual arithmetic Gillet–Soulé Chow group with real coefficients.

## 5. Kudla's Green function

Let $V = \{w \in M_2(\mathbb{Q}) : \mathrm{tr}(w) = 0\}$ be the quadratic space with quadratic form $Q(w) = N \det w$, and let $\mathbb{D}$ be the associated Hermitian symmetric domain of positive lines in $V_{\mathbb{R}}$ as in Section 2. Recall that $\mathrm{SL}_2 = \mathrm{Spin}(V)$ acts on $\mathbb{D}$ by conjugation, and $\mathbb{D}$ can be identified with $\mathbb{H}$ (Lemma 2.1) via

$$w(z) = \frac{1}{\sqrt{N}y}\begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix}, \quad z = x + iy \in \mathbb{H}. \tag{5.1}$$

Let $L$ be an even integral lattice with dual lattice $L^\sharp$ (arbitrary in this section). Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index which fixes $L$ and acts on $L^\sharp/L$ trivially. We denote $\bar{\Gamma} = \Gamma/(\Gamma \cap \{\pm 1\})$. For each pair $(n, \mu) \in \mathbb{Q} \times L^\sharp/L$ with $n > 0$, $Q(\mu) \equiv n \pmod 1$, let $Z(n, \mu)$ be the associated Heegner divisor given by

$$Z(n, \mu) = \Gamma \backslash \{\mathbb{R}w : w \in L_\mu[n]\}.$$

Kudla defined a nice Green function for $Z(n, \mu)$ in his seminal work [15], which we now briefly review. The purpose of this section is to understand its behavior at the cusps.

For $r > 0$ and $s \in \mathbb{R}$, let

$$\beta_s(r) = \int_1^\infty e^{-rt} t^{-s} dt \tag{5.2}$$

and

$$\xi(w, z) = \beta_1(2\pi R(w, z)), \tag{5.3}$$

be Kudla's $\xi$-function. For $\mu \in L^\sharp/L$, $n \in Q(\mu) + \mathbb{Z}$ and $v \in \mathbb{R}_{>0}$, define

$$\Xi(n, \mu, v)(z) = \sum_{0 \neq w \in L_\mu[n]} \xi(v^{\frac{1}{2}} w, z). \tag{5.4}$$

Then Kudla has proved on $Y_0(N)$ ([15]) that $\Xi(n, \mu, v)$ is a Green function for $Z(n, \mu)$ and satisfies the following Green current equation:

$$dd^c[\Xi(n, \mu, v)] + \delta_{Z(n,\mu)} = [\omega(n, \mu, v)]$$

when $n > 0$. When $n \leq 0$, $\Xi(n, \mu, v)$ is still well-defined and actually smooth on $Y_0(N)$ while $Z(n, \mu) = 0$. So $\Xi(n, \mu, v)$ is a Green function for $Z(n, \mu)$ for all $n$. The purpose of this section is to understand its behavior at cusps, which is quite complicated and subtle.

Let $\mathrm{Iso}(V)$ be the set of isotropic non-zero vectors of $V$, i.e., $0 \neq \ell \in V$ with $Q(\ell) = 0$. Given $\ell = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iso}(V)$, let $P_\ell = \frac{a}{c}$ be the associated cusp, which depends only on the isotropic line $\mathbb{Q}\ell$. Two isotropic lines give the same cusp in $\Gamma\backslash\mathbb{H}$ if and only if there is $\gamma \in \Gamma$ such that $\mathbb{Q}\gamma \cdot \ell_1 = \mathbb{Q}\ell_2$.

Let $\ell_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Iso}(V)$ and let $P_\infty = \infty$ be its associated cusp. In general, for an isotropic element $\ell$, there exists $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\mathbb{Q}\sigma_\ell \cdot \ell_\infty = \mathbb{Q}\ell$. Then

$$\sigma_\ell^{-1}\Gamma_\ell\sigma_\ell = \{\pm \left(\begin{smallmatrix} 1 & m\kappa_\ell \\ 0 & 1 \end{smallmatrix}\right), m \in \mathbb{Z}\},$$

where $\Gamma_\ell$ is the stabilizer of $\ell$ and $\kappa_\ell > 0$ is the classical width of the associated cusp $P_\ell$, and $q_\ell$ is a local parameter at the cusp $P_\ell$. On the other hand, there is another positive number $\beta_\ell > 0$, depending on $L$ and the cusp $P_\ell$, such that $\left(\begin{smallmatrix} 0 & \beta_\ell \\ 0 & 0 \end{smallmatrix}\right)$ is a primitive element in $\mathbb{Q}\ell_\infty \bigcap \sigma_\ell^{-1} \cdot L$. We denote $\varepsilon_\ell = \frac{\kappa_\ell}{\beta_\ell}$ and call it Funke constant at cusp $P_\ell$ although Funke called it width at $P_\ell$ in [8, Section 3]. We will simply denote $\kappa = \kappa_\infty$.

The main purpose of this section is to prove the following technical theorem.

**Theorem 5.1.** *Let the notation be as above. Let $0 \neq \ell \in \mathrm{Iso}(V)$ be an isotropic vector and $P_\ell$ be the associated cusp.*

(1) *When $D = -4nN$ is not a square, $\Xi(n, \mu, v)$ is smooth and of exponential decay at the cusp $P_\ell$.*
(2) *When $D = -4nN > 0$ is a square. Then $\Xi(n, \mu, v)$ has log singularity at the cusp $P_\ell$ with*

$$\Xi(n, \mu, v) = -g(n, \mu, v, P_\ell)(\log |q_\ell|^2) - 2\psi_\ell(n, \mu, v; q_\ell).$$

*Here $q_\ell$ is a local parameter at the cusp $P_\ell$,*

$$\alpha_\Gamma(n, \mu, P_\ell) = \sum_{w \in L_\mu[n] \mod \Gamma} \delta_w,$$

*where $0 \le \delta_w \le 2$ is the number of isotropic lines $\mathbb{Q}\ell_w \in \mathrm{Iso}(V)$ which is perpendicular to $w$ and belongs to the same cusp as $\ell$, and*

$$g(n, \mu, v, P_\ell) = \frac{1}{8\pi\sqrt{-nv}}\beta_{3/2}(-4nv\pi)\alpha_\Gamma(n, \mu, P_\ell).$$

*Finally, $\psi_\ell(n, \mu, v; q_\ell)$ is a smooth function of $q_\ell$ (as two real variables $q_\ell$ and $\bar{q_\ell}$) and*

$$\lim_{q_\ell \to 0} \psi_\ell(n, \mu, v; q_\ell) = 0.$$

(3) *When $D = 0$, one has*

$$\Xi(0,\mu,v) = -g(0,\mu,v,P_\ell)(\log|q_\ell|^2) - 2\log(-\log|q_\ell|^2)$$
$$-2\psi_\ell(0,\mu,v;q_\ell),$$

*where $q_\ell$ is the local parameter at $P_\ell$ with respect to the classical width $\kappa_\ell$, $g(0,\mu,v,P_\ell) = \frac{\varepsilon_\ell}{2\pi\sqrt{vN}}$. Here $\varepsilon_\ell$ is the Funke constant of $\ell$. Finally, $\psi_\ell(0,\mu,v;q_\ell)$ is a smooth function of $q_\ell$ (as two real variables $q_\ell$ and $\bar{q}_\ell$) and*

$$\lim_{q_\ell\to 0}\psi_\ell(0,\mu,v;q_\ell) = \begin{cases} \log\frac{\varepsilon_\ell}{4\pi\sqrt{Nv}} - \frac{1}{2}f(0) & \text{if } \mu \in L, \\ \frac{1}{2}\log\frac{\varepsilon_\ell^2}{4Nv\pi^3} + \frac{\gamma_1(0)}{2} - \sum_{n=1}^\infty \frac{\cos(\frac{2\pi n\mu_\ell}{\beta_\ell})}{n} & \text{if } \mu \notin L. \end{cases}$$

*Here $f(0) = \gamma - \log(4\pi)$ is defined in Lemma 5.2,*

$$\gamma_1(0) = \int_1^\infty e^{-y}\frac{dy}{y} + \int_0^1 \frac{e^{-y}-1}{y}dy$$

*and*

$$\sigma_\ell^{-1}\cdot L_\mu \cap \mathbb{Q}\ell = \{\left(\begin{smallmatrix} 0 & \mu_\ell+m\beta_\ell \\ 0 & 0 \end{smallmatrix}\right) : m \in \mathbb{Z}\}.$$

The proof is long and technical and will occupy the next few subsections.

### *5.1. Two lemmas*

**Lemma 5.2.** *Let $a > 0$ and $z = x + iy \in \mathbb{C}$. Then*

(1) *When $z \notin \mathbb{R}$, one has*

$$\sum_{n\in\mathbb{Z}} \beta_1(\pi a^2|z+n|^2) = \frac{1}{a}\sum_{n\in\mathbb{Z}} e(nx)\int_1^\infty e^{-\pi a^2y^2t - \frac{\pi n^2}{a^2t}} t^{-\frac{3}{2}}dt.$$

(2) *When $z = x \in \mathbb{R} - \mathbb{Z}$, one has*

$$\sum_{n\in\mathbb{Z}} \beta_1(\pi a^2(x+n)^2) = 2\sum_{n\in\mathbb{Z}} e(nx)\int_0^{\frac{1}{a}} e^{-\pi n^2t^2}dt.$$

*Moreover, one has near $a = 0$*

$$\sum_{n\in\mathbb{Z}} \beta_1(\pi a^2(x+n)^2) = \frac{2}{a} + f(a,x),$$

*for some smooth function $f(a,x)$ near $a=0$ with*

$$f(0,x)=\lim_{a\to 0} f(a,x)=2\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}.$$

(3) *One has*

$$\sum_{0\neq n\in\mathbb{Z}}\beta_1(\pi a^2n^2)=2\int_0^{\frac{1}{a}}\left(\sum_{n\in\mathbb{Z}}e^{-\pi n^2t^2}-\int_{\mathbb{R}}e^{-\pi x^2t^2}dx\right)dt.$$

*Moreover, one has near $a=0$*

$$\sum_{0\neq n\in\mathbb{Z}}\beta_1(\pi a^2n^2)=\frac{2}{a}+2\log a+f(a),$$

*for some smooth function $f(a)$ with*

$$f(0)=\lim_{a\to 0} f(a)=\gamma-\log(4\pi),$$

*where $\gamma$ is the Euler constant.*

**Proof.** Let

$$f(n)=\beta_1(\pi a^2|z+n|^2)=\beta_1(\pi a^2y^2+\pi a^2(x+n)^2).$$

Then its Fourier transformation is

$$\begin{aligned}\widehat{f}(n)&=\int_{\mathbb{R}}f(\alpha)e(-\alpha n)d\alpha\\&=\frac{e(nx)}{a}\int_1^{\infty}e^{-\pi a^2y^2t-\frac{\pi n^2}{a^2t}}t^{-\frac{3}{2}}\,dt.\end{aligned}$$

Now applying the Poisson summation formula, one obtains the formula in (1). When $y=0$, simple substitution gives part of (2) with $x\notin\mathbb{Z}$. To see the behavior of the sum near $a=0$, notice that the right-hand side is equal to $\frac{2}{a}+f(a,x)$ with

$$f(a,x)=2\sum_{n=1}^{\infty}(e(nx)+e(-nx))\int_0^{\frac{1}{a}}e^{-\pi n^2t^2}dt.$$

It is clearly smooth near $a=0$ if we define

$$f(0,x)=2\sum_{n=1}^{\infty}(e(nx)+e(-nx))\int_0^{\infty}e^{-\pi n^2t^2}dt=2\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}.$$

To prove (3), take $z=i\epsilon$ in (1), and let $\epsilon$ goes to zero, we obtain

$$\sum_{0\neq n\in\mathbb{Z}}\beta_1(\pi a^2n^2)=\lim_{\epsilon\to 0}\left[\frac{1}{a}\sum_{n\in\mathbb{Z}}\int_1^{\infty}e^{-\pi a^2\epsilon^2t-\frac{\pi n^2}{a^2}t^{-1}}t^{-\frac{3}{2}}dt-\beta_1(\pi a^2\epsilon^2)\right].$$

By the Fourier inversion formula, one has

$$\frac{1}{a}\int_1^{\infty}\int_{\mathbb{R}}e^{-\pi a^2\epsilon^2t-\frac{\pi x^2}{a^2}t^{-1}}t^{-\frac{3}{2}}dxdt=\beta_1(\pi a^2\epsilon^2).$$

So

$$\begin{aligned}
&\sum_{0\neq n\in\mathbb{Z}}\beta_1(\pi a^2n^2)\\
&=\frac{1}{a}\lim_{\epsilon\to 0}\int_1^{\infty}e^{-\pi a^2\epsilon^2t}\left[\sum_{n\in\mathbb{Z}}e^{-\frac{\pi n^2}{a^2t}}-\int_{\mathbb{R}}e^{-\frac{\pi n^2}{a^2t}}dx\right]t^{-\frac{3}{2}}dt\\
&=\frac{2}{a}\int_1^{\infty}\left[\sum_{n\in\mathbb{Z}}e^{-\frac{\pi n^2}{a^2t}}-\int_{\mathbb{R}}e^{-\frac{\pi n^2}{a^2t}}dx\right]t^{-\frac{3}{2}}dt\\
&=2\int_0^{\frac{1}{a}}(\sum_{n\in\mathbb{Z}}e^{-\pi n^2t^2}-\int_{\mathbb{R}}e^{-\pi x^2t^2}dx)dt\\
&=\frac{2}{a}-4\int_0^{\frac{1}{a}}\int_0^1e^{-\pi x^2t^2}dxdt+4\sum_{n=1}^{\infty}\left[\int_0^{\frac{1}{a}}e^{-\pi n^2t^2}dt-\int_0^{\frac{1}{a}}\int_n^{n+1}e^{-\pi x^2t^2}dxdt\right]\\
&=\frac{2}{a}-4g_0(a)+4\sum_{n=1}^{\infty}g_n(a),
\end{aligned}$$

with obvious meaning of $g_n(a)$. Here we have used the fact that the integrand in the last integral is negative. The term $\frac{2}{a}$ comes from the term $n=0$ in the sum. We remark that the formula looks formally like ($z=0$)

$$\sum_{n\neq 0}f(n)=\sum_{n\in\mathbb{Z}}\widehat{f}(n)-\int_{\mathbb{R}}\widehat{f}(x)dx.$$

What we did is to regularize the right hand side to make it meaningful.

First,

$$\begin{aligned}
g_0(a) &= \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt + \int_1^{\frac{1}{a}} \int_0^1 e^{-\pi x^2 t^2} dx dt \\
&= \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt + \int_1^{\frac{1}{a}} \int_0^\infty e^{-\pi x^2 t^2} dx dt - \int_1^{\frac{1}{a}} \int_1^\infty e^{-\pi x^2 t^2} dx dt \\
&= -\frac{1}{2} \log a + \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt - \int_1^{\frac{1}{a}} \int_1^\infty e^{-\pi x^2 t^2} dx dt.
\end{aligned}$$

We will prove the following identity in Lemma 5.3 below.

$$\int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt - \int_1^\infty \int_1^\infty e^{-\pi x^2 t^2} dx dt = \frac{1}{4}(\gamma + \log 4\pi). \tag{5.5}$$

Then we have

$$\lim_{a \to 0} (g_0(a) + \frac{1}{2} \log a) = \frac{1}{4}(\gamma + \log 4\pi).$$

Next, we have

$$\begin{aligned}
&\lim_{a \to 0} \sum_{n=1}^\infty \left[ \int_0^{\frac{1}{a}} e^{-\pi n^2 t^2} dt - \int_0^{\frac{1}{a}} \int_n^{n+1} e^{-\pi x^2 t^2} dx dt \right] \\
&= \sum_{n=1}^\infty \left[ \int_0^\infty e^{-\pi n^2 t^2} dt - \int_n^{n+1} \int_0^\infty e^{-\pi x^2 t^2} dt dx \right] \\
&= \frac{1}{2} \sum_{n=1}^\infty (\frac{1}{n} - \log \frac{n+1}{n}) = \frac{1}{2} \gamma.
\end{aligned}$$

In summary, we have

$$\sum_{0 \neq n \in \mathbb{Z}} \beta_1(\pi a^2 n^2) = \frac{2}{a} + 2 \log a + f(a),$$

for some smooth function $f(a)$ near $a = 0$ with

$$f(0) = \lim_{a \to 0} f(a) = -(\gamma + \log(4\pi)) + 2\gamma = \gamma - \log(4\pi).$$

This proves the proposition. $\square$

**Lemma 5.3.**

$$\int_0^1\int_0^1 e^{-\pi x^2t^2}dxdt - \int_1^\infty\int_1^\infty e^{-\pi x^2t^2}dxdt = \frac{1}{4}(\gamma+\log 4\pi). \tag{5.6}$$

**Proof.** Recall the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds \tag{5.7}$$

which behaves like $\frac{2}{\sqrt{\pi}}x$ when $x \to 0$. Write the left hand side of (5.6) as $I_1 - I_2$ in an obvious way. A simple substitution and an integration by parts give

$$\begin{aligned} I_1 &= \int_0^1\int_0^1 e^{-\pi x^2t^2}dxdt = \int_0^1 \frac{\operatorname{erf}(\sqrt{\pi}x)}{2x}dx \\ &= \left[\frac{\operatorname{erf}(\sqrt{\pi}x)\log x}{2}\right]_0^1 - \int_0^1 e^{-\pi x^2}\log x dx = -\int_0^1 e^{-\pi x^2}\log x dx, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_1^\infty\int_1^\infty e^{-\pi x^2t^2}dxdt = \int_1^\infty \frac{1-\operatorname{erf}(\sqrt{\pi}x)}{2x}dx \\ &= \int_1^\infty e^{-\pi x^2}\log x dx. \end{aligned}$$

So

$$I_1 - I_2 = -\int_0^\infty e^{-\pi x^2}\log x dx = -\frac{1}{4\sqrt{\pi}}\left[\Gamma'(\frac{1}{2}) - \log\pi\Gamma(\frac{1}{2})\right]$$

by simple substitution $y = \pi x^2$. Now the result follows from the formulas

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \text{ and } \Gamma'(\frac{1}{2}) = -\sqrt{\pi}(\gamma+\log 4). \quad \square$$

**Lemma 5.4.** *Assume that* $D = -4Nn = (2Nm)^2 > 0$ *is a square. For any* $w = w(m, r) = m\begin{pmatrix} 1 & 2r \\ 0 & -1 \end{pmatrix} \in L_\mu[n]$ *with* $(w, \ell_\infty) = 0$*, define*

$$\Xi_\infty(w, z) = \sum_{\gamma\in\bar{\Gamma}_\infty} \xi(w, \gamma z).$$

*Then for any* $v > 0$

$$\Xi_\infty(\sqrt{v}w, z) = -(\log|q_\kappa|)\frac{\sqrt{N}}{2\pi\sqrt{Dv}}\sum_{n\in\mathbb{Z}} e(\frac{n}{\kappa}(x+r))\int_1^\infty e^{-\left(\frac{tDv}{N}+n^2\frac{N}{tDv}\frac{y^2}{\kappa^2}\right)\pi}\frac{dt}{t^{\frac{3}{2}}},$$

*where* $q_\kappa = e(z/\kappa)$ *is a local parameter of* $X_\Gamma$ *at the cusp* $P_\infty$. *Moreover, one has near the cusp* $P_\infty$ *(*$q_\kappa = 0$*)*

$$\Xi_\infty(\sqrt{v}w, z) = -(\log|q_\kappa|^2)\frac{\sqrt{N}}{4\pi\sqrt{Dv}}\beta_{\frac{3}{2}}(\frac{Dv\pi}{N}) + f(\sqrt{v}w, z),$$

*where* $f(\sqrt{v}w, z)$ *is a smooth function of* $x$ *and* $y$ *near* $P_\infty$ *and*

$$\lim_{y\to\infty} f(\sqrt{v}w, z) = 0.$$

**Proof.** One has $\bar{\Gamma}_\infty = \{\left(\begin{smallmatrix}1 & \kappa\mathbb{Z}\\ 0 & 1\end{smallmatrix}\right)\}$ and

$$\begin{aligned} R(\sqrt{v}w, \left(\begin{smallmatrix}1 & n\kappa\\ 0 & 1\end{smallmatrix}\right) z) &= \frac{v}{2}(w, w(z+n\kappa))^2 - v(w,w) \\ &= \frac{Dv}{2Ny^2} \mid z + n\kappa + r \mid^2 . \end{aligned}$$

So one has by Lemma 5.2,

$$\begin{aligned} \Xi_\infty(\sqrt{v}w, z) &= \sum_{n\in\mathbb{Z}} \beta_1(\frac{\pi Dv}{Ny^2}|z+r+n\kappa|^2) \\ &= \frac{y\sqrt{N}}{\kappa\sqrt{Dv}}\sum_{n\in\mathbb{Z}} e(\frac{n}{\kappa}(x+r))\int_1^\infty e^{-\left(\frac{tD}{N}+n^2\frac{N}{tD}\frac{y^2}{\kappa^2}\right)\pi}\frac{dt}{t^{\frac{3}{2}}} \\ &= -(\log|q_\kappa|^2)\frac{\sqrt{N}}{4\pi\sqrt{Dv}}\beta_{\frac{3}{2}}(\frac{\pi D}{N}) + f(\sqrt{v}w, z) \end{aligned}$$

with

$$f(\sqrt{v}w, z) = -(\log|q_\kappa|^2)\frac{\sqrt{N}}{4\pi\sqrt{Dv}}\sum_{0\neq n\in\mathbb{Z}} e(\frac{n}{\kappa}(x+r))\int_1^\infty e^{-\left(\frac{tDv}{N}+n^2\frac{N}{tDv}\frac{y^2}{\kappa^2}\right)\pi}\frac{dt}{t^{\frac{3}{2}}}.$$

Since

$$\frac{tDv}{N} + n^2\frac{N}{tDv}\frac{y^2}{\kappa^2} \geq \frac{2|n|y}{\kappa},$$

one sees for all $n \neq 0$

$$\left| e(\frac{n}{\kappa}(x+r)) \int_1^\infty e^{-\left(\frac{tDv}{N}+n^2\frac{N}{tDv}\frac{y^2}{\kappa^2}\right)\pi} \frac{dt}{t^{\frac{3}{2}}} \right| \le 2e^{-2\frac{|n|y}{\kappa}\pi},$$

and

$$|f(\sqrt{v}w,z)| \le \frac{4\sqrt{N}y}{\kappa\sqrt{Dv}} \sum_{n=1}^\infty e^{-2\frac{n\pi y}{\kappa}}$$

which is of exponential decay as $y \mapsto \infty$. This proves the lemma. □

*5.2. Proof of Theorem 5.1*

**Proof.** Now we are ready to start proof of Theorem 5.1. By linear fractional transformation, we may assume that $\ell = \ell_\infty$ is associated to the cusp $P_\infty$. Then $q_\ell = q_\kappa$ where $\kappa$ is the width of the cusp $P_\infty$ defined at the beginning of Section 5. We divide the proof into three steps: general set-up and the case $D = -4Nn$ is not a square, $D > 0$ being a square, and finally the case $D = 0$.

**Step 1: Set-up and the case that $D$ is not a square.** We write

$$\Xi(n,\mu,v) = \sum_{w\in L_\mu[n] \mod \Gamma} \Xi(\sqrt{v}w,z), \quad \Xi(\sqrt{v}w,z) = \sum_{\gamma\in\bar{\Gamma}_w\backslash\bar{\Gamma}} \xi(\sqrt{v}w,\gamma z). \tag{5.8}$$

For $w = \left(\begin{smallmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{smallmatrix}\right) \in L_\mu[n]$, let $\tilde{w} = \begin{pmatrix} w_3 & -w_1 \\ -w_1 & -w_2 \end{pmatrix} = S^{-1}\cdot w$ with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\tilde{w}$ is symmetric. Simple calculation gives for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$R(w,\gamma z) = \frac{N}{2y^2}\left[h_{\tilde{w}}(\gamma,z)\right]^2 - n, \tag{5.9}$$

where

$$h_{\tilde{w}}(\gamma,z) = (az+b,cz+d)\tilde{w}\overline{(az+b,cz+d)}^t = Q_{\tilde{w}}(a,c)y^2 + Q_{\tilde{w}}(ax+b,cx+d)$$

is the Hermitian form on $(\mathbb{R}z+\mathbb{R})^2$, and $Q_{\tilde{w}}$ is the quadratic form on $\mathbb{R}^2$ associated to $\tilde{w}$. Notice that $\{(az+b,cz+d) : \gamma\in\Gamma\}$ is a subset of a lattice of $(\mathbb{R}z+\mathbb{R})^2$, so for any positive number $M$

$$\#\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty\backslash\Gamma : |h_{\tilde{w}}(\gamma,z)| \le M \quad \text{and} \quad 0 < |Q_{\tilde{w}}(a,c)| \le M\}$$

are finite and of polynomial growth as functions of $M$. Moreover there is a positive number $M_0$ such that if $Q_{\tilde{w}}(a,c) \ne 0$ for some $\gamma\in\Gamma$, then $|Q_{\tilde{w}}(a,c)| \ge M_0$. In such a case, we have

$$R(w, \gamma z) \sim \frac{N}{2} Q_{\tilde{w}}(a, c)^2 y^2$$

as $y \to \infty$. Recall that

$$\beta_1(t) = O(e^{-t}/t)$$

as $t \to \infty$. Therefore the terms with $Q_{\tilde{w}}(a, c) \neq 0$ in the sum $\Xi(\sqrt{v}w, z)$ goes to zero in an exponential decay fashion. So we have proved the following lemma.

**Lemma 5.5.** *Let the notation be as above. If there is no* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *such that* $Q_{\tilde{w}}(a, c) = 0$*, then* $\Xi(\sqrt{v}w, z)$ *is smooth at the cusp* $P_\infty$ *and is of exponential decay as* $y \to \infty$*.*

When $D$ is not a square, the quadratic form $Q_{\tilde{w}}$ does not represent 0. So $\Xi(n, \mu, v)$ is of exponential decay in this case when $y \to \infty$. This proves (1).

**Step 2: Next, we assume $D = -4Nn > 0$ is a square**. In this case, $\bar{\Gamma}_w = 1$

$$0 = Q_{\tilde{w}}(a, c) = w_3 a^2 - 2w_1 ac - w_2 c^2$$

has exactly two integral solutions $(a_i, c_i) \in \mathbb{Z}^2$ such that $\gcd(a_i, c_i) = 1$, $a_i > 0$ or $a_i = 0, c_i = 1$. So $w^\perp \cap \mathrm{Iso}(V)$ consists exactly two cusps $\mathbb{Q}\ell_{a_i,c_i}$ where $\ell_{a,c} = \left(\begin{smallmatrix} ac & -a^2 \\ c^2 & -ac \end{smallmatrix}\right)$.

For a fixed solution $(a, c)$, if there is $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, then the cusp $P_{\frac{a}{c}}$ (corresponding to $\mathbb{Q}\ell_{a,c}$) is $\Gamma$-equivalent to $P_\infty$: $\gamma_0 P_\infty = P_{\frac{a}{c}}$, and all $\gamma = \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \Gamma$ with $Q_{\tilde{w}}(a, c) = 0$ is of the form $\gamma_0\gamma_1$ with $\gamma_1 \in \Gamma_\infty$. Therefore the sum related to this solution $(a, c)$ is

$$\begin{aligned}
\sum_{\substack{\gamma = \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \bar{\Gamma} \\ Q_{\tilde{w}}(a,c)=0}} \xi(\sqrt{v}w, \gamma z) &= \sum_{\gamma_1 \in \bar{\Gamma}_\infty} \xi(\sqrt{v}\gamma_0^{-1} \cdot w, \gamma_1 z) \\
&= \Xi_\infty(\sqrt{v}\gamma_0^{-1} \cdot w, z) \\
&= -(\log |q_\kappa|^2) \frac{\sqrt{N}}{4\pi\sqrt{Dv}} \beta_{\frac{3}{2}}(\frac{Dv\pi}{N}) + f(\sqrt{v}\gamma_0^{-1} \cdot w, z)
\end{aligned}$$

by Lemma 5.4. Recall $\lim_{y\to\infty} f(\sqrt{v}\gamma_0^{-1} \cdot w, z) = 0$ by Lemma 5.4. So we have by Lemma 5.5,

$$\begin{aligned}
\Xi(\sqrt{v}w, z) &= \sum_{\substack{Q_{\tilde{w}}(a,c)=0 \\ \gcd(a,c)=1 \\ a>0 \text{ or } a=0, c=1}} \sum_{\gamma = \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \bar{\Gamma}} \xi(\sqrt{v}w, z) + \sum_{\substack{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma} \\ Q_{\tilde{w}}(a,c) \neq 0}} \xi(\sqrt{v}w, z) \\
&= -\frac{\delta_w \sqrt{N}}{4\pi\sqrt{Dv}} \beta_{\frac{3}{2}}(\frac{Dv\pi}{N})(\log |q_\kappa|^2) + \psi(w, z)
\end{aligned}$$

with $\psi(w,z)$ smooth at the cusp $P_\infty$ and

$$\lim_{y\to\infty} \psi(w,z) = 0.$$

Combining this with Lemma 5.5, we proved (3) of Theorem 5.1.

**Step 3: Finally we assume** $n = 0$. Each vector $0 \neq w \in L_\mu[0]$ corresponds to an isotropic line and thus a cusp. We regroup the sum in $\Xi(0,\mu,v)$ in terms of $\Gamma$-equivalent cusp classes $[P_r]$, where $r \in \mathbb{Q}$ or $\infty$. Let $\ell_r = \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix}$ be an associated isotropic vector for a rational number $r$ and recall $\ell_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$$\Xi(0,\mu,v) = \sum_{[P_r]} \sum_{0\neq w \in L_\mu[0]\cap \mathbb{Q}\ell_r} \Xi(\sqrt{v}w, z). \tag{5.10}$$

First consider the sum $[P_\infty]$ part. Let

$$L_\mu[0] \cap \mathbb{Q}\ell_\infty = \{w_m = \begin{pmatrix} 0 & \mu_\infty + m\beta_\infty \\ 0 & 0 \end{pmatrix} \neq 0 : \, m \in \mathbb{Z}\},$$

where $\beta_\infty = \beta_{\ell_\infty}$ is the constant defined at the beginning of this section and $\mu_\infty \in \mathbb{Q}$. Notice that two different $w_m$s are not $\Gamma$-equivalent, and $\Gamma_{w_m} = \Gamma_\infty$. Simple calculation gives

$$\begin{aligned}\Xi(\sqrt{v}w_m, z) &= \sum_{\gamma\in\bar{\Gamma}_\infty\backslash\bar{\Gamma}} \beta_1(\frac{\pi N v|cz+d|^4(m\beta_\infty+\mu_\infty)^2}{y^2}) \\ &= \beta_1(\frac{\pi N v(m\beta_\infty+\mu_\infty)^2}{y^2}) + \sum_{\gamma\in\bar{\Gamma}_\infty\backslash\bar{\Gamma}, c>0} \beta_1(\frac{\pi N v|cz+d|^4(m\beta_\infty+\mu_\infty)^2}{y^2}).\end{aligned}$$

When $\mu_\infty \notin \beta_\infty\mathbb{Z}$ (i.e., $\mu \notin L$), one has by Lemma 5.2

$$\begin{aligned}&\sum_{0\neq w\in L_\mu[0]\cap\mathbb{Q}\ell_\infty} \Xi(\sqrt{v}w, z) \\ &= \sum_{0\neq m\in\mathbb{Z}} \beta_1(\frac{\pi N v(m\beta_\infty+\mu_\infty)^2}{y^2}) + e(\mu, z) \\ &= -\beta_1(\frac{\pi N v \mu_\infty^2}{y^2}) + \frac{2y}{\beta_\infty\sqrt{Nv}} + f(\frac{\beta_\infty\sqrt{Nv}}{y}, \frac{\mu_\infty}{\beta_\infty}) + e(\mu, z).\end{aligned}$$

Here

$$e(\mu, z) = \sum_{0\neq m\in\mathbb{Z}} \sum_{\gamma\in\bar{\Gamma}_\infty\backslash\bar{\Gamma}, c>0} \beta_1(\frac{\pi N v|cz+d|^4(m\beta_\infty+\mu_\infty)^2}{y^2}).$$

Recall that near $t = 0$

$$\beta_1(t) = -\log t + \gamma_1(t)$$

with

$$\gamma_1(t) = \int_1^\infty e^{-y}\frac{dy}{y} + \int_t^1 \frac{e^{-y}-1}{y}dy.$$

So we have for $\mu \notin L$ (recall $y = -\frac{\kappa}{2\pi}\log|q_\kappa|$)

$$\sum_{0\neq w\in L_\mu[0]\cap\mathbb{Q}\ell_\infty} \Xi(\sqrt{v}w, z) = -\log|q_\kappa|^2\frac{\varepsilon_\infty}{\pi\sqrt{Nv}} - 2\log(-\log|q_\kappa|^2) + \psi(\mu, z), \tag{5.11}$$

where

$$\psi(\mu, z) = -\log\frac{\varepsilon_\infty^2}{4Nv\pi^3} - \gamma_1(\frac{\pi Nv\beta_\infty^2}{y^2}) + f(\frac{\beta_\infty\sqrt{Nv}}{y}, \frac{\mu_\infty}{\beta_\infty}) + e(\mu, z).$$

It is easy to see that $\beta_1(t) = O(e^{-t})$ as $t \to \infty$, and

$$e^{-\frac{\pi Nv|cz+d|^4(m\beta_\infty+\mu_\infty)^2}{y^2}} \le e^{-\pi Nv(c^4y^2+\frac{(cx+d)^4}{y^2})(m\beta_\infty+\mu_\infty)^2}, \tag{5.12}$$

which is uniformly of exponential decay (with respect to $c, d, m \in \mathbb{Z}, c > 0, m \neq 0$) as $y$ goes to infinity. So $e(\mu, z)$ is of exponential decay as $y$ goes to infinity. This implies

$$\lim_{y\to\infty}\psi(\mu, z) = -\log\frac{\varepsilon_\infty^2}{4Nv\pi^3} - \gamma_1(0) + 2\sum_{n=1}^\infty\frac{\cos(\frac{2\pi n\mu_\infty}{\beta_\infty})}{n}. \tag{5.13}$$

For $\mu \in L$ (i.e., $\mu = 0$ in $L^\sharp/L$), one has

$$\begin{aligned}\sum_{0\neq w\in L[0]\cap\mathbb{Q}\ell_\infty} \Xi(\sqrt{v}w, z) &= \frac{2y}{\beta_\infty\sqrt{Nv}} + 2\log\frac{\sqrt{Nv}\beta_\infty}{y} + f(\frac{\sqrt{Nv}\beta_\infty}{y}) + e(0, z)\\ &= -\frac{\varepsilon_\infty}{2\pi\sqrt{Nv}}\log|q_{\ell_\infty}|^2 - 2\log(-\log|q_{\ell_\infty}|^2) + \psi(0, z),\end{aligned}$$

with

$$\psi(0, z) = 2\log\frac{4\pi\sqrt{Nv}}{\varepsilon_\infty} + f(\frac{\sqrt{Nv}\beta_\infty}{y}) + e(0, z).$$

So one has

$$\lim_{y\to\infty} \psi(0,z) = 2\log\frac{4\pi\sqrt{Nv}}{\varepsilon_\infty} + f(0), \tag{5.14}$$

as $e(0,z)$ is of exponential decay as $y$ goes to the infinity.

Now look at the sum of $[P_r]$ part, where $P_r$ is not $\Gamma$-equivalent to $P_\infty$. This implies that there is no $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $\gamma(\infty) = \frac{a}{c} = r$. For $w = m\left(\begin{smallmatrix} r & -r^2 \\ 1 & -r \end{smallmatrix}\right) \in L_\mu[0] \cap \mathbb{Q}\ell_r$ so that $\tilde{w} = S^{-1}\cdot w = m\left(\begin{smallmatrix} 1 & -r \\ -r & r^2 \end{smallmatrix}\right)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, one has

$$R(w,\gamma z) = \frac{1}{2}(w, w(\gamma z))^2 = \frac{Nm^2}{2}\frac{|(a-rc)z+(b-rd)|^4}{y^2} \sim \frac{m^2N}{2}(a-rc)^4y^2$$

as $y\to\infty$ as $a - rc \neq 0$ for all $\gamma\in\Gamma$. So

$$\Xi(\sqrt{v}w, z) = \sum_{\gamma\in\bar{\Gamma}_w\backslash\bar{\Gamma}} \beta_1(2\pi R(w,\gamma z))$$

is smooth and of exponential decay at the cusp $P_\infty$. Putting everything together, we obtain the result for $\Xi(0,\mu,v)$ at the cusp $P_\infty$. This finally proves Theorem 5.1. □

**Corollary 5.6.** *Let the notation and assumption be as in Theorem 5.1 and let $D = -4nN$. Then $\Xi(n,\mu,v)$ is a Green function for $Z(n,\mu,v)^{\mathrm{Naive}}$ in the usual Gillet–Soulé sense for $n\neq 0$ and with (at most) log–log singularity when $n = 0$, and*

$$dd^c[\Xi(n,\mu,v)] + \delta_{Z(n,\mu,v)^{\mathrm{Naive}}} = [\omega(n,\mu,v)].$$

*Here $\omega(n,\mu,v)$ is the differential defined in (2.8)*

$$Z(n,\mu,v)^{\mathrm{Naive}} = \begin{cases} Z(n,\mu) & \text{if } D<0,\\ \sum_{P_\ell \ \text{cusps}} g(n,\mu,v,P_\ell)P_\ell & \text{if } D\geq 0 \text{ is a square},\\ 0 & \text{if otherwise.}\end{cases}$$

**Proof.** Away from the singularity divisor $Z(n,\mu.v)^{\mathrm{Naive}}$, one has by [15, Proposition 11.1]

$$dd^c\Xi(n,\mu,v) = \omega(n,\mu,v).$$

Near the cusps, it is given by Theorem 5.1, and we leave the detail to the reader following the idea in [15, Proposition 11.1]. □

## 6. Modular curve $\mathcal{X}_0(N)$ and the main theorem

From now on, we focus on the specific lattice $L$ given in Section 2 and $\Gamma = \Gamma_0(N)$ with $N$ square free. So our modular curve is $X_0(N) = Y_0(N)\cup S$ the cusp set $S = \{P_{\frac{1}{M}} : M|N\}$ with $P_{\frac{1}{M}}$ is the cusp associated to $\frac{1}{M}$ (as $N$ is square free). Let

$$\ell_{\frac{1}{M}} = \begin{pmatrix} -M & 1 \\ -M^2 & M \end{pmatrix}$$

be an associated isotropic element.

### *6.1. Some numerical results on Kudla Green functions*

**Lemma 6.1.** *The Funke constant for $P_{\frac{1}{M}}$ is $\varepsilon_{\frac{1}{M}} = N$, independent of the choices of the cusps.*

**Proof.** Take $\sigma_M = \left(\begin{smallmatrix} 1 & 0 \\ M & 1 \end{smallmatrix}\right)$. Then $\sigma_M \cdot \ell_\infty = \ell_{\frac{1}{M}}$, and

$$\sigma_M^{-1} \cdot L \bigcap \mathbb{Q}\ell_\infty = \begin{pmatrix} 0 & \frac{1}{M}\mathbb{Z} \\ 0 & 0 \end{pmatrix}.$$

So we have $\beta_{\frac{1}{M}} = \frac{1}{M}$. Next, we know that

$$\sigma_M^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \sigma_M = \begin{pmatrix} 1+Mx & x \\ -M^2x & 1-Mx \end{pmatrix} \in \Gamma_0(N) \tag{6.1}$$

if and only if $x \in \frac{N}{M}\mathbb{Z}$. This implies $\kappa_{\frac{1}{M}} = \frac{N}{M}$. So $\varepsilon_{\frac{1}{M}} = N$. $\square$

**Lemma 6.2.** *When $D = -4nN > 0$ is a square and $L_\mu[n] \neq \phi$, one has for every cusp $P_\ell$*

$$\alpha_{\Gamma_0(N)}(n, \mu, P_\ell) = \begin{cases} \sqrt{D}, & \text{if } 2\mu \notin L, \\ 2\sqrt{D}, & \text{if } 2\mu \in L. \end{cases}$$

**Proof.** We will drop the subscript $\Gamma_0(N)$ in the proof. We first assume $P_\ell = P_\infty$. Recall

$$\alpha(n, \mu, P_\infty) = \sum_{w \in L_\mu[n] \mod \Gamma_0(N)} \delta_w,$$

where $\delta_w$ is the number of the isotropic lines $\mathbb{Q}\ell$ which is perpendicular to $w$ and whose associated cusp is $\Gamma_0(N)$-equivalent to $P_\infty$. Replacing $w$ by its $\Gamma_0(N)$-equivalent element if necessary we may and will assume $(w, \ell_\infty) = 0$ (for $\delta_w \neq 0$). This implies

$$w = w(a,b) = \begin{pmatrix} \frac{a}{2N} & \frac{b}{N} \\ 0 & -\frac{a}{2N} \end{pmatrix}$$

with

$$a^2 = D, \quad a \equiv r \mod (2N). \tag{6.2}$$

So

$$w(a,b)^\perp \cap \mathrm{Iso}(V) = \mathbb{Q}\ell_\infty \cup \mathbb{Q}\ell(a,b), \quad \ell(a,b) = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}. \tag{6.3}$$

On the other hand, it is straightforward to check that $w(a, b_1)$ is $\Gamma_0(N)$-equivalent to $w(a, b_2)$ if and only if $b_1 \equiv b_2 \pmod{a}$. Therefore, we only need to consider these $w(a, b)$ with $a$ satisfying (6.2) and $b \pmod{a}$. There are at most $2|a|$ of them.

Now divide the proof into two cases: $N \nmid r$ (i.e., $2\mu_r \notin L$) and $N|r$ (i.e. $2\mu_r \in L$).

Assume first that $N \nmid r$. Then (6.2) has a unique solution $a$, and for this $a$, the cusp $P_{-\frac{b}{a}} = P_{\ell(a,b)}$ is not $\Gamma_0(N)$-equivalent to $P_\infty$. So $\delta_w = 1$ for each $w(a, b)$. Therefore we have

$$\alpha(n, \mu, P_\infty) = |a| = \sqrt{D}$$

in this case.

Next we assume $N|r$. In this case (6.2) has two solutions $a = \sqrt{D}$ and $-\sqrt{D}$. One has also $N|a$. It is not hard to verify via calculation that $w(a, b)$ and $w(-a, b')$ are $\Gamma_0(N)$-equivalent if and only if $a_2 = \gcd(a, b) = \gcd(a, b')$ has the following properties: $a = Na_2z$ and $b = a_2w$ with $\gcd(Nz, w) = 1$, and $b' = a_2x$ for some $x$ with $xw - Nyz = 1$ for some integer $y$. Moreover, in such a case, $b' \pmod{a}$ is uniquely determined by $b \pmod{a}$.

Write $a = Na_1$ and $(a, b) = a_2$ with $b = a_2w$.

**Subcase 1**: We first assume $a_2|a_1$. In this case, we can write $a_1 = a_2z$ and thus $a = a_2Nz$ with $(w, Nz) = 1$. So $\delta_{w(\pm a,b)} = 2$. On the other hand, $w(\epsilon a, b)$ is $\Gamma_0(N)$-equivalent to $w(-\epsilon a, bx)$ with $xw - Nyz = 1$ for some $x, y \in \mathbb{Z}$. So the four pairs $(\pm a, b)$ and $(\pm a, bx)$ contribute 4 to the sum of $\delta_w$.

**Subcase 2**: Next we assume $a_2 \nmid a_1$. This means $\gcd(a_2, N) > 1$. So the cusp $P_{\frac{b}{\pm a}} = P_{\frac{a_2 z}{\pm N a_1}}$ is not $\Gamma_0(N)$-equivalent to the cusp $P_\infty$. This implies $\delta_{w(\pm a,b)} = 1$. On the other hand, for such a pair $(\epsilon a, b)$, $w(\epsilon a, b)$ is not $\Gamma_0(N)$-equivalent to any other $w(\pm a, b')$.

Combining the two subcases, we see that

$$\alpha(n, \mu, P_\infty) = 2|a|$$

in this case. This proves the lemma for the cusp $P_\infty$.

Next, we show that $\alpha(n, \mu, P_{\frac{1}{M}})$ does not depend on the cusp $P_{\frac{1}{M}}$ in the following sense.

$$\alpha(n, \mu, P_{\frac{1}{M}}) = \alpha(n, W_Q \mu W_Q^{-1}, P_\infty), \tag{6.4}$$

where $Q = \frac{M}{N}$, and $W_Q$ is the associated Atkin–Lehner involution defined as follows. Since $(M, Q) = 1$, there exist $\alpha, \beta \in \mathbb{Z}$ with $\alpha Q - M\beta = 1$, so $\left(\begin{smallmatrix} 1 & \beta \\ M & Q\alpha \end{smallmatrix}\right) \in \Gamma_0(M)$. Let

$$W_Q = \left(\begin{smallmatrix} 1 & \beta \\ M & Q\alpha \end{smallmatrix}\right) \left(\begin{smallmatrix} Q & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} Q & \beta \\ N & Q\alpha \end{smallmatrix}\right)$$

be the associated Atkin–Lehner operator. Then one has

$$W_Q \Gamma_0(N) W_Q^{-1} = \Gamma_0(N).$$

It is easy to verify

$$W_Q L_\mu[n] W_Q^{-1} = L_{W_Q \mu W_Q^{-1}}[n], \quad W_Q \ell_\infty W_Q^{-1} = \begin{pmatrix} Q\alpha & -\beta \\ M\alpha & -M\beta \end{pmatrix} = \ell'.$$

Notice that $P_{\ell'} = P_{\frac{1}{M}}$. So there is a bijective map

$$L_\mu[n] \bigcap \ell'^{,\perp} \longleftrightarrow L_{W_Q \mu W_Q^{-1}}[n] \bigcap \ell_\infty^\perp,$$
$$w \longleftrightarrow W_Q^{-1} w W_Q.$$

This proves (6.4), and thus the lemma. □

Now we can refine Theorem 5.1 and Corollary 5.6 as

**Theorem 6.3.** *Let the notation and assumption be as above and let $D = -4nN$. Then $\Xi(n, \mu, v)$ is a Green function for $Z(n, \mu, v)^{\mathrm{Naive}}$ with (at most) log–log singularity, and*

$$dd^c[\Xi(n, \mu, v)] + \delta_{Z(n,\mu,v)^{\mathrm{Naive}}} = [\omega(n, \mu, v)].$$

*Here $\omega(n, \mu, v)$ is the differential defined in (2.8)*

$$Z(n, \mu, v)^{\mathrm{Naive}} = \begin{cases} Z(n, \mu) & \text{if } D < 0, \\ g(n, \mu, v) \sum_{M|N} \mathcal{P}_{\frac{1}{M}} & \text{if } D \geq 0 \text{ is a square}, \\ 0 & \text{if otherwise}, \end{cases}$$

*and*

$$g(n, \mu, v) = \begin{cases} \frac{\sqrt{N}}{4\pi\sqrt{v}} \beta_{3/2}(-4nv\pi) & \text{if } n \neq 0, \mu \notin \frac{1}{2}L/L, \\ \frac{\sqrt{N}}{2\pi\sqrt{v}} \beta_{3/2}(-4nv\pi) & \text{if } n \neq 0, \mu \in \frac{1}{2}L/L, \\ \frac{\sqrt{N}}{2\pi\sqrt{v}} & \text{if } n = 0, \mu = 0, \\ 0 & \text{if } n = 0, \mu \neq 0. \end{cases}$$

*Moreover, for every $M|N$,*

(1) *when $D$ is not a square, the Green function $\Xi(n, \mu, v)$ is of exponential decay near cusp $P_{\frac{1}{M}}$.*
(2) *When $D = -4Nn > 0$ is a square, one has*

$$\Xi(n, \mu, v) = -g(n, \mu, v)(\log |q_M|^2) - 2\psi_M(n, \mu, v; q_M),$$

*where $q_M$ is a local parameter at $P_{\frac{1}{M}}$, and $\psi_M(n, \mu, v; q_M)$ is of exponential decay near $P_M$. Here $P_{\frac{1}{N}} = P_\infty$, and $\psi_N = \psi_\infty$.*

(3) *When* $D = 0$, $\Xi(0, \mu, v) = 0$ *when* $\mu \notin L$, *and*

$$\Xi(0,0,v) = -g(0,0,v)(\log|q_M|^2) - 2\log(-\log|q_M|^2) \\ -2\psi_M(0,\mu,v;q_M),$$

*and*

$$\lim_{|q_M|\to 0} \psi_M(0,0,v;q_M) = \log \frac{\sqrt{N}}{4\pi\sqrt{v}} - \frac{1}{2} f(0).$$

*Here* $f(0) = \gamma - \log(4\pi)$ *is defined in Lemma 5.2.*

## 6.2. Integral model

Following [13], let $\mathcal{Y}_0(N)$ ($\mathcal{X}_0(N)$) be the moduli stack over $\mathbb{Z}$ of cyclic isogenies of degree $N$ of elliptic curves (generalized elliptic curves) $\pi : E \to E'$ such that $\ker \pi$ meets every irreducible component of each geometric fiber. The stack $\mathcal{X}_0(N)$ is regular, proper, and flat over $\mathbb{Z}$ and smooth over $\mathbb{Z}[\frac{1}{N}]$ such that $\mathcal{X}_0(N)(\mathbb{C}) = X_0(N)$ as $N$ is square free. It is a DM-stack. For convenience, we count each point $x$ with multiplicity $\frac{2}{|\operatorname{Aut}(x)|}$ instead of $\frac{1}{|\operatorname{Aut}(x)|}$. When $p|N$, the special fiber $\mathcal{X}_0(N) \pmod p$ has two irreducible components $\mathcal{X}_p^\infty$ and $\mathcal{X}_p^0$. Both of them are isomorphic to $\mathcal{X}_0(N/p) \mod p$, and they intersect at supersingular points. We require $\mathcal{X}_p^\infty$ to contain the cusp $\mathcal{P}_\infty \pmod p$ and $\mathcal{X}_p^0$ to contain the cusp $\mathcal{P}_0 \pmod p$. Here for each divisor $Q|N$, let $\mathcal{P}_{\frac{Q}{N}}$ be the boundary arithmetic curve associated to the cusp $P_{\frac{Q}{N}}$, which is the Zariski closure of $P_{\frac{Q}{N}}$ in $\mathcal{X}_0(N)$ and has a nice moduli interpretation too. We refer to [7] for detail. It is known that $\mathcal{P}_{\frac{Q}{N}}$ mod $p$ lies in $\mathcal{X}_p^\infty$ (resp. $\mathcal{X}_p^0$) if and only if $p \nmid Q$ (resp. $p|Q$).

For $r \in \mathbb{Z}/2N$, $\mu_r = \operatorname{diag}(r/2N, -r/2N) \in L^\sharp/L$ and a positive rational number $n \in Q(\mu_r) + \mathbb{Z}$, let $D = -4Nn \equiv r^2 \mod 4N$, $k_D = \mathbb{Q}(\sqrt{D})$ and the order $\mathcal{O}_D = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$ of discriminant $D$. When $D < 0$, let $\mathcal{Z}(n, \mu_r)$ be the flat closure of $Z(n, \mu_r)$ in $\mathcal{X}_0(N)$.

## 6.3. The metrized Hodge bundle

Let $\omega_N$ be the Hodge bundle on $\mathcal{X}_0(N)$ (see [13]). Then there is a canonical isomorphism $\omega_N^2 \cong \Omega_{\mathcal{X}_0(N)/\mathbb{Z}}(-S)$, which is also canonically isomorphic to the line bundle of modular forms of weight 2 for $\Gamma_0(N)$. Here $S$ is the set of cusps. For a positive integer $N$, let $\mathcal{M}_k(N)$ be the line bundle of weight $k$ with the normalized Petersson metric

$$\|f(z)\| = |f(z)(4\pi e^{-C} y)^{\frac{k}{2}}|$$

as defined in (1.9). This gives a metrized line bundle $\widehat{\mathcal{M}}_k(N)$ and also induces a metric on $\omega_N$ so that the associated metrized line bundle $\widehat{\omega}_N$ satisfies $\widehat{\omega}_N^k \cong \widehat{\mathcal{M}}_k(N)$. From now on, we denote

$$k = 12\varphi(N), \quad r = N\prod_{p|N}(1+p^{-1}) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = \frac{3}{\pi}\,\mathrm{vol}(X_0(N), \mu(z)). \tag{6.5}$$

Recall that $\Delta_N(z)$ and $\Delta_N^0(z)$ are both rational sections of $\mathcal{M}_k(N)$.

**Lemma 6.4.**

$$\mathrm{Div}\,\Delta_N = \frac{rk}{12}\mathcal{P}_\infty - k\sum_{p|N}\frac{p}{p-1}\mathcal{X}_p^0 \tag{6.6}$$

*and*

$$\mathrm{Div}\,\Delta_N^0 = \frac{rk}{12}\mathcal{P}_0 - \frac{k}{2}\sum_{p|N}\frac{p+1}{p-1}\mathcal{X}_p^\infty - \frac{k}{2}\sum_{p|N}\mathcal{X}_p^0. \tag{6.7}$$

*Here $r$ and $k$ are given by (6.5).*

**Proof.** Since

$$\Delta_N \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(z) = N^{-6\varphi(N)}\Delta_N^0(\frac{z}{N}) = N^{-12\varphi(N)}\Pi_{t|N}t^{12a(\frac{N}{t})}\Delta(\frac{tz}{N})^{a(\frac{N}{t})},$$

we have

$$\mathrm{Div}\Delta_N = \frac{rk}{12}\mathcal{P}_\infty + \sum_{p|N}(-12\varphi(N) + 12\sum_{M|\frac{N}{p}}a(M))\mathcal{X}_p^0.$$

One has by (3.2)

$$\sum_{M|\frac{N}{p}}a(M) = -\varphi(\frac{N}{p}), \tag{6.8}$$

so

$$\mathrm{Div}\,\Delta_N = \frac{rk}{12}\mathcal{P}_\infty + \sum_{p|N}(-12\varphi(N) - 12\varphi(\frac{N}{p}))\mathcal{X}_p^0.$$

Notice $\varphi(\frac{N}{p}) = \frac{1}{p-1}\varphi(N)$, one has

$$\mathrm{Div}\,\Delta_N = \frac{rk}{12}\mathcal{P}_\infty - k\sum_{p|N}\frac{p}{p-1}\mathcal{X}_p^0$$

as claimed. The second identity follows the same way and is left to the reader. □

The following lemma is clear.

**Lemma 6.5.** *Let $q_z = e(z)$ be a local parameter of $X_0(N)$ at the cusp $P_\infty$.*

(1) *The metrized line bundle $\widehat{\omega}_N^k = \widehat{\mathcal{M}}_k(N)$ has log singularity along cusps with all $\alpha$-index $\alpha_P = \frac{k}{2}$ at every cusp $P$. At the cusp $P_\infty$, one has*

$$\|\Delta_N(z)\| = (-\log|q_z|^2)^{\frac{k}{2}}|q_z|^{\frac{r}{12}k}\varphi(q_z),$$

*with*

$$\varphi(q_z) = e^{-\frac{kC}{2}}\prod_{n=1}^{\infty}|(1-q_z)^{24C_N(n)}|.$$

(2) *Both $\widehat{\mathrm{Div}}(\Delta_N) = (\mathrm{Div}(\Delta_N), -\log\|\Delta_N(z)\|^2)$ and $\widehat{\mathrm{Div}}(\Delta_N^0) = (\mathrm{Div}(\Delta_N^0), -\log\|\Delta_N^0(z)\|^2)$ are arithmetic divisors (on $\mathcal{X}_0(N)$) associated to $\widehat{\omega}_N^k$ with log–log singularity at cusps.*

We also consider the arithmetic divisor on $\mathcal{X}_0(N)$:

$$\widehat{\Delta}_N = (\frac{rk}{12}\mathcal{P}_\infty, -\log\|\Delta_N(z)\|^2). \tag{6.9}$$

One has

$$\widehat{\mathrm{Div}}(\Delta_N) = \widehat{\Delta}_N - k\sum_{p|N}\frac{p}{p-1}\mathcal{X}_p^0. \tag{6.10}$$

Define

$$\widehat{\mathcal{Z}}(n,\mu,v) = \begin{cases}\widehat{\mathcal{Z}}(n,\mu,v)^{\mathrm{Naive}} - 2\widehat{\omega}_N - \sum_{p|N}\mathcal{X}_p^0 - (0,\log(\frac{v}{N})) & \text{if } n=0, \mu=0,\\ \widehat{\mathcal{Z}}(n,\mu,v)^{\mathrm{Naive}} & \text{otherwise.}\end{cases} \tag{6.11}$$

The arithmetic generating function ($q = e(\tau)$) in the introduction is defined to be

$$\widehat{\phi}(\tau) = \sum_{\substack{n\in\frac{1}{2N}\mathbb{Z}\\ \mu\in L^\sharp/L\\ Q(\mu)\equiv n \pmod 1}} \widehat{\mathcal{Z}}(n,\mu,v)q^n e_\mu \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N), S)\otimes\mathbb{C}[L^\sharp/L][[q,q^{-1}]]. \tag{6.12}$$

At this moment, we simply view it as a Laurent series with coefficients in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N), S)\otimes\mathbb{C}[L^\sharp/L]$.

**Proposition 6.6.** *One has*

$$\widehat{\phi}(\tau) \in (\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N))\otimes\mathbb{C}[L^\sharp/L])[[q,q^{-1}]].$$

**Proof.** By Theorem 6.3, it suffices to check the case for $\widehat{\mathcal{Z}}(0,0,v)$. Notice that $\Delta(\tau)$ is a section of $\omega_N^{12}$. So we have by Theorem 6.3

$$\widehat{\mathcal{Z}}(0,0,v) = (\mathcal{Z}, g)$$

with

$$\mathcal{Z} = -\frac{\sqrt{N}}{2\pi\sqrt{v}} \sum_{M|N} \mathcal{P}_{\frac{1}{M}} - \frac{1}{6} \operatorname{Div} \Delta - \sum_{p|N} \mathcal{X}_p^0,$$
$$g = \Xi(0,0,v) + \frac{1}{6} \log \|\Delta\|^2 - \log \frac{v}{N}.$$

For each $M|N$, choose $\sigma_M \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_M(\infty) = \frac{1}{M}$. Then Theorem 6.3(3) asserts

$$\begin{aligned}\Xi(0,0,v)(\sigma_M(z)) &= -\frac{\sqrt{N}}{2\pi\sqrt{v}}(\log |q_M|^2) - 2\log(-\log |q_M|^2) + \text{smooth} \\ &= -\frac{\sqrt{N}}{2\pi\sqrt{v}}(\log |q|^2) - 2\log(-\log |q|^2) + \text{smooth},\end{aligned}$$

where $q_M = q^{\frac{M}{N}}$, as the width of the cusp $P_{\frac{N}{M}}$ is $\frac{N}{M}$. On the other hand,

$$\log \|\Delta(\sigma_M(z))\|^2 = \log \|\Delta(z)\|^2 = \log(|q|^2) + 12\log(-\log |q|^2) + \text{smooth}.$$

So we know

$$g(\sigma_M(z)) = (-\frac{\sqrt{N}}{2\pi\sqrt{v}} + \frac{1}{6}) \log(|q|^2) + \text{smooth}$$

has just log singularity. □

We first record the following proposition, which is clear by (4.2) and Corollary 2.3.

**Proposition 6.7.** *Let the notation be as above, then*

$$\deg \widehat{\phi}(\tau) = \sum_{n,\mu} \deg(\widehat{\mathcal{Z}}(n,\mu,v)) q^n e_\mu = \langle \widehat{\phi}(\tau), a(2) \rangle = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1).$$

*In general, for* $a(f) = (0, f) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}_0(N), S)$, *we have*

$$\langle \widehat{\phi}(\tau), a(f) \rangle = \frac{1}{2} \int_{X_0(N)} f(z) \Theta_L(\tau, z) = \frac{1}{2} I(\tau, f)$$

*is a vector valued modular form valued in* $S_L$ *for* $\Gamma'$ *of weight* $3/2$ *and representation* $\rho_L$.

**Proposition 6.8.** *For every prime $p|N$, one has*

$$\langle \widehat{\phi}(\tau), \mathcal{X}_p^0 \rangle = \langle \widehat{\phi}(\tau), \mathcal{X}_p^\infty \rangle = \frac{1}{\varphi(N)} \mathcal{E}_L(\tau, 1) \log p.$$

**Proof.** Since

$$R(w, w_N z) = R(w_N^{-1} \cdot w, z),$$

and $w_N \cdot L_\mu = L_{-\mu}$, one has by definition

$$w_N^* \Xi(n, \mu, v) = \Xi(n, -\mu, v) = \Xi(n, \mu, v).$$

This implies

$$w_N^* \hat{Z}(n, \mu, v)^{\mathrm{Naive}} = \hat{Z}(n, \mu, v)^{\mathrm{Naive}}$$

on the generic fiber. Since the divisors $\mathcal{Z}(n, \mu, v)^{\mathrm{Naive}}$ are all horizontal (flat closure of $Z(n, \mu, v)^{\mathrm{Naive}}$), we have

$$w_N^* \hat{\mathcal{Z}}(n, \mu, v)^{\mathrm{Naive}} = \hat{\mathcal{Z}}(n, \mu, v)^{\mathrm{Naive}}.$$

One has also $w_N^* \widehat{\Delta}_N = \widehat{\Delta}_N^0$ and $w_N^* \mathcal{X}_p^0 = \mathcal{X}_p^\infty$. Direct calculation using Lemma 6.4 shows

$$w_N^* \widehat{\mathcal{Z}}(0, 0, v) = \widehat{\mathcal{Z}}(0, 0, v),$$

and so

$$w_N^* (\widehat{\phi}(\tau)) = \widehat{\phi}(\tau).$$

Since $w_N$ is an isomorphism, we have

$$\begin{aligned} &\langle \widehat{\phi}(\tau), \mathcal{X}_p^0 \rangle = \langle \widehat{\phi}(\tau), \mathcal{X}_p^\infty \rangle \\ &= \frac{1}{2} \langle \widehat{\phi}(\tau), \mathcal{X}_p \rangle = \frac{1}{2} \langle \widehat{\phi}(\tau), (0, \log p^2) \rangle \\ &= \frac{1}{2} \deg \widehat{\phi}(\tau) \log p = \frac{1}{\varphi(N)} \mathcal{E}_L(\tau, 1) \log p. \end{aligned}$$

Here we have used the fact that the principal arithmetic divisor $\widehat{\mathrm{Div}}(p) = (\mathcal{X}_p, -\log p^2)$. This proves the proposition. □

**Proof of Theorem 1.3.** Now Theorem 1.3 follows from Propositions 6.7 and 6.8, equation (6.10), and the following theorem, which will be proved in next section.

**Theorem 6.9.** *Let the notation be above. Then*

$$\langle \widehat{\phi}(\tau), \widehat{\Delta}_N \rangle_{GS} = 12\mathcal{E}'_L(\tau, 1).$$

## 7. The proof of Theorem 6.9

### 7.1. Some preparation

**Lemma 7.1.** *Two different cusps of $X_0(N)$ reduce to two different cusps modulo $p$ for every prime number $p$. So $\langle \mathcal{P}_{\frac{1}{M_1}}, \mathcal{P}_{\frac{1}{M_2}} \rangle = 0$ if $M_1 \not\equiv M_2 \pmod N$.*

**Proof.** We only need to consider primes $p|N$. If $p$ divides exactly one of the $M_1$ and $M_2$, the two cusps landed in two different branches of $\mathcal{X}_p$ and thus do not coincide. When $p$ divides both of them, their reductions $\bar{\mathcal{P}}_{\frac{1}{M_j}}$ both landed in $\mathcal{X}_p^0$. On the other hand, $\mathcal{X}_p^0$ is isomorphic to the reduction of $\mathcal{X}_0(N/p)$, under which cusps correspond to cusps. Counting the number of cusps, we see that different cusps which landed in $\mathcal{X}_p^0$ are still different in the reduction. This proves the lemma. □

**Lemma 7.2.** *One has for each $p|N$,*

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle = -\langle \mathcal{X}_p^0, \mathcal{X}_p^0 \rangle = -\langle \mathcal{X}_p^\infty, \mathcal{X}_p^\infty \rangle = \frac{r(p-1)}{12(p+1)} \log p.$$

**Proof.** Recall that $\mathcal{X}_p^\infty$ and $\mathcal{X}_p^0$ are both isomorphic to the special fiber $\mathcal{X}_0(\frac{N}{p})_p = \mathcal{X}_0(\frac{N}{p}) \pmod p$ and that they intersect properly exactly at the supersingular points. So

$$\begin{aligned}
\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle &= \sum_{\substack{x \in \mathcal{X}_0(\frac{N}{p})_p(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} \frac{2}{|Aut(x)|} \\
&= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(\frac{N}{p})] \sum_{\substack{x \in \mathcal{X}_0(1)_p(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} \frac{2}{|Aut(x)|} \\
&= \frac{r}{p+1} \sum_{\substack{x \in \mathcal{X}_0(1)_p(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} \frac{2}{|Aut(x)|}.
\end{aligned}$$

It is well-known (see for example [13, corollary 12.4.6]) that

$$\frac{p-1}{24} = \sum_{j \in \overline{F}_p, E_j \text{supersingular}} \frac{1}{|Aut(E_j)|}, \tag{7.1}$$

where $j$ is the $j$-invariant. So

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle = \frac{r(p-1)}{12(p+1)}.$$

On the other hand

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^\infty \rangle + \langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle = \langle \mathcal{X}_p^\infty, (0, \log p^2) \rangle = 0.$$

So

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^\infty \rangle = -\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle. \quad \square$$

**Lemma 7.3.** *One has*

(1)

$$\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = r(\frac{\zeta(-1)}{2} + \zeta'(-1)) + \frac{r}{12} C,$$

*where* $C = \frac{\log(4\pi)+\gamma}{2}$ *is the normalization constant in (1.9)*

(2)

$$\langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle = k^2 r(\frac{\zeta(-1)}{2} + \zeta'(-1)) + \frac{k^2 r C}{12} + \frac{k^2 r}{12} \sum_{p|N} \frac{p^2}{p^2-1} \log p.$$

**Proof.** Let $\widehat{\omega}_{N,\mathrm{Pet}}^k$ be the Hodge bundle with the Petersson metric (via its isomorphism to $\mathcal{M}_k(N)$)

$$\|f(z)\|_{\mathrm{Pet}} = |f(z)(4\pi y)^{\frac{k}{2}}| = \|f(z)\| e^{\frac{kC}{2}}.$$

According to [19, Theorem 6.1], we have

$$\langle \widehat{\omega}_{\mathrm{Pet}}, \widehat{\omega}_{\mathrm{Pet}} \rangle = r(\frac{\zeta(-1)}{2} + \zeta'(-1)). \tag{7.2}$$

So

$$\begin{aligned} \langle \widehat{\omega}_N, \widehat{\omega}_N \rangle &= \langle \widehat{\omega}_{\mathrm{Pet}}, \widehat{\omega}_{\mathrm{Pet}} \rangle + 2\langle \widehat{\omega}_{\mathrm{Pet}}, (0, C) \rangle \\ &= r(\frac{\zeta(-1)}{2} + \zeta'(-1)) + \deg(\widehat{\omega}_{\mathrm{Pet}}) C \\ &= r(\frac{\zeta(-1)}{2} + \zeta'(-1)) + \frac{r}{12} C \end{aligned}$$

as claimed.

Next, one has

$$\begin{aligned}\langle \widehat{\Delta}_N, \widehat{\Delta}_N\rangle &= \langle \widehat{\Delta}_N, \widehat{\mathrm{Div}}(\Delta_N)\rangle + k\sum_{p|N}\frac{p}{p-1}\langle \widehat{\Delta}_N, \mathcal{X}_p^0\rangle \\ &= \langle \widehat{\mathrm{Div}}(\Delta_N), \widehat{\mathrm{Div}}(\Delta_N)\rangle + k\sum_{p|N}\frac{p}{p-1}\langle \mathcal{X}_p^0, \widehat{\mathrm{Div}}(\Delta_N)\rangle \\ &= k^2\langle \widehat{\omega}_N, \widehat{\omega}_N\rangle - k^2\sum_{p|N}(\frac{p}{p-1})^2\langle \mathcal{X}_p^0, \mathcal{X}_p^0\rangle \\ &= k^2 r(\frac{\zeta(-1)}{2} + \zeta'(-1)) + \frac{k^2 rC}{12} + \frac{k^2 r}{12}\sum_{p|N}\frac{p^2}{p^2-1}\log p,\end{aligned}$$

by Lemma 7.2 □

**Remark 7.4.** We remark that Lemma 7.3 can be proved directly using our explicit description of sections of $\omega_N^k$ without using [19, Theorem 6.1]. Indeed, one has

$$\langle \widehat{\omega}_N^k, \widehat{\omega}_N^k\rangle = \langle \widehat{\mathrm{Div}}(\Delta_N), \widehat{\mathrm{Div}}(\Delta_N^0)\rangle.$$

Now direct calculation gives the lemma. We leave the detail to the reader.

*7.2. Proof of Theorem 6.9*

In this section, we prove Theorem 6.9, which amounts to check term by term on their Fourier coefficients.

By Theorem 1.6 and (2.6), it suffices to prove

$$\langle \widehat{\mathcal{Z}}(n,\mu,v), \widehat{\Delta}_N\rangle = \begin{cases} -\int_{X_0(N)} \log\|\Delta_N(z)\|\omega(n,\mu,v) & \text{if } n\neq 0, \\ -\int_{X_0(N)} \log\|\Delta_N(z)\|(\omega(0,0,v) - \frac{dxdy}{2\pi y^2}) & \text{if } n=0, \mu=0. \end{cases} \tag{7.3}$$

The case $n=0, \mu\neq 0$ is trivial as both sides are zero.

We divide the proof into three cases: $D$ is not a square, $D>0$ is a square, and $D=0$.

**Case 1**: We first assume that $D$ is not a square. In this case, $\mathcal{Z}(n,\mu,v)$ and $\mathcal{P}_\infty$ has no intersection at all. By Proposition 4.1 and Theorem 6.3, one has

$$\langle \widehat{\mathcal{Z}}(n,\mu,v), \widehat{\Delta}_N\rangle = -\int_{X_0(N)} \log\|\Delta_N\|\omega(n,\mu,v).$$

This proves the case that $D$ is not a square.

**Case 2**: Now we assume that $D$ is a square. This case is complicated due to self-intersection at $\mathcal{P}_\infty$. We work out the case $D=0$ and leave the similar (and slightly easier) case $D>0$ to the reader. Let

$$\widehat{\mathcal{Z}}_1(0,0,v) = \widehat{\mathcal{Z}}(0,0,v)^{Naive} - \frac{12g(0,0,v)}{rk}\widehat{\Delta}_N = (\mathcal{Z}_1(0,0,v), \Xi_1(0,0,v)).$$

Then

$$\langle \widehat{\mathcal{Z}}(0,0,v)^{Naive}, \widehat{\Delta}_N \rangle = \langle \widehat{\mathcal{Z}}_1(0,0,v), \widehat{\Delta}_N \rangle + \frac{12g(0,0,v)}{rk}\langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle.$$

We have

$$\begin{aligned} &\langle \widehat{\mathcal{Z}}_1(0,0,v), \widehat{\Delta}_N \rangle \\ &= \sum_{0<M|N, M<N} \frac{rk}{12} g(n,\mu,v) \langle \mathcal{P}_{\frac{1}{M}}, \mathcal{P}_\infty \rangle + \frac{rk}{12}(\alpha_{\mathcal{Z}_1,P_\infty} - \psi_{1,\infty}(0,0,v,0)) \\ &\quad - \lim_{\epsilon\to 0}\left( \frac{rk}{12}\alpha_{\mathcal{Z}_1,P_\infty} \log(-\log\epsilon^2) - \frac{1}{2}\int_{X_0(N)_\epsilon} -\log\|\Delta_N\|^2 \omega_1 \right), \end{aligned}$$

where

$$\omega_1 = \omega(0,0,v) - \frac{12g(0,0,v)}{r}\frac{dxdy}{4\pi y^2}$$

and

$$\alpha_{\mathcal{Z}_1,P_\infty} = 1 - \frac{6}{r}g(0,0,v).$$

So the limit is equal to

$$\begin{aligned} &\frac{rk}{12}\alpha_{\mathcal{Z}_1,P_\infty} \lim_{\epsilon\to 0}\left( \log(-\log\epsilon^2) + \frac{12}{rk}\int_{X_0(N)_\epsilon} \log\|\Delta_N\|^2 \frac{dxdy}{4\pi y^2} \right) \\ &+ \lim_{\epsilon\to 0}\int_{X_0(N)_\epsilon} \log\|\Delta_N\| (\omega(0,0,v) - \frac{dxdy}{2\pi y^2}) \\ &= \frac{rk}{12}\alpha_{\mathcal{Z}_1,P_\infty} \lim_{\epsilon\to 0}\left( \log(-\log\epsilon^2) + \frac{12}{rk}\int_{X_0(N)_\epsilon} \log\|\Delta_N\|^2 \frac{dxdy}{4\pi y^2} \right) \\ &+ \int_{X_0(N)} \log\|\Delta_N\| (\omega(0,0,v) - \frac{dxdy}{2\pi y^2}). \end{aligned}$$

Recall ([20, Lemma 2.8]) that

$$\lim_{\epsilon\to 0}\left(\log(-\log\epsilon^2)+\frac{12}{rk}\int\limits_{X_0(N)_\epsilon}\log\|\Delta_N\|^2\frac{dxdy}{4\pi y^2}\right)=\frac{r\pi}{3}C_0+2\log(4\pi)-C. \tag{7.4}$$

Here $C_0$ is the scattering constant given in Lemma 3.6, and $C$ is the normalization constant in Petersson norm. Combining this with Corollary 3.6, we obtain

$$\begin{aligned}&\langle\widehat{\mathcal{Z}}_1(0,0,v),\widehat{\Delta}_N\rangle\\&=\frac{rk}{12}(1-\frac{6}{r}g(0,0,v))\left(12\zeta(-1)+24\zeta'(-1)+C+2\sum_{p|N}\frac{p^2}{p^2-1}\log p\right)\\&-\frac{rk}{12}\psi_{1,\infty}(0,0,v,0)-\int\limits_{X_0(N)}\log\|\Delta_N\|(\omega(0,0,v)-\frac{dxdy}{2\pi y^2}).\end{aligned}$$

Here we recall $\zeta(-1)=-\frac{1}{12}$. On the other hand, Theorem 5.1 implies

$$\begin{aligned}\psi_{1,\infty}(0,0,v,0)&=\lim_{y\to\infty}(\psi_\infty(0,0,v,q_z)-\frac{12}{rk}g(0,0,v)\log\phi(q_z))\\&=-\frac{1}{2}\log(\frac{v}{N})-(1-\frac{6}{r}g(0,0,v))C.\end{aligned}$$

Therefore, one has by Lemma 7.3

$$\begin{aligned}&\langle\widehat{\mathcal{Z}}(0,0,v)^{Naive},\widehat{\Delta}_N\rangle\\&=\langle\widehat{\mathcal{Z}_1}(0,0,v),\widehat{\Delta}_N\rangle+\frac{12g(0,0,v)}{rk}\langle\widehat{\Delta}_N,\widehat{\Delta}_N\rangle\\&=\frac{rk}{24}\log(\frac{v}{N})+\frac{2}{k}\langle\widehat{\Delta}_N,\widehat{\Delta}_N\rangle-\int\limits_{X_0(N)}\log\|\Delta_N\|(\omega(0,0,v)-\frac{dxdy}{2\pi y^2}),\end{aligned}$$

and

$$\begin{aligned}\langle\widehat{\mathcal{Z}}(0,0,v),\widehat{\Delta}_N\rangle&=\langle\widehat{\mathcal{Z}}(0,0,v)^{\mathrm{Naive}},\widehat{\Delta}_N\rangle-\frac{2}{k}\langle\widehat{\Delta}_N,\widehat{\Delta}_N\rangle\\&\quad-\sum_{p|N}\frac{p+1}{p-1}\langle\mathcal{X}_p^0,\widehat{\Delta}_N\rangle-\langle(0,\log(\frac{v}{N})),\widehat{\Delta}_N\rangle\\&=-\int\limits_{X_0(N)}\log\|\Delta_N\|(\omega(0,0,v)-\frac{dxdy}{2\pi y^2}).\end{aligned}$$

This proves the case $D=0$.

## 8. Modularity of the arithmetic theta function

In this section, we will prove the modularity of $\widehat{\phi}(\tau)$. To simplify the notation, we denote in this section $X = X_0(N)$ and $\mathcal{X} = \mathcal{X}_0(N)$, and let $S$ be the set of cusps of $X$. Let $g_{\mathrm{GS}}$ be a Gillet–Soulé Green function for the divisor $\operatorname{Div}\Delta_N$ (without log–log singularity), and let $\widehat{\Delta}_{\mathrm{GS}} = (\operatorname{Div}\Delta_N, g_{\mathrm{GS}}) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$, and $f_N = g_{\mathrm{GS}} + \log \|\Delta_N\|^2$. Then $a(f_N) = (0, f_N) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, S)$ and

$$\widehat{\Delta}_{\mathrm{GS}} = \widehat{\operatorname{Div}}(\Delta_N) + a(f_N).$$

Theorem 6.9 and Proposition 6.7 imply the following proposition immediately.

**Proposition 8.1.** *The Gillet–Soulé height pairing $\langle \widehat{\phi}, \widehat{\Delta}_{\mathrm{GS}}\rangle$ is a vector valued modular form of $\Gamma'$ valued in $\mathbb{C}[L^\sharp/L]$ of weight $3/2$ and representation $\rho_L$.*

Now we are ready to prove Theorem 1.1 following the idea in [18, Chapter 4] with $\widehat{\omega}$ replaced by $\widehat{\Delta}_{\mathrm{GS}}$. Let $\mu_{GS} = c_1(\widehat{\Delta}_{\mathrm{GS}})$, $A(X)$ be the space of smooth functions $f$ on $X$ which are conjugation invariant ($Frob_\infty$-invariant), and let $A^0(X)$ be the subspace of functions $f \in A(X)$ with

$$\int_X f \mu_{GS} = 0.$$

For each $p|N$, let $\mathcal{Y}_p = \mathcal{X}_p^\infty - p\mathcal{X}_p^0$, then $\langle \mathcal{Y}_p, \widehat{\Delta}_{\mathrm{GS}}\rangle = 0$. Let $\mathcal{Y}_p^\vee = \frac{1}{\langle \mathcal{Y}_p, \mathcal{Y}_p\rangle}\mathcal{Y}_p$. Finally let $\widetilde{\mathrm{MW}}$ be the orthogonal complement of $\mathbb{R}\widehat{\Delta}_{\mathrm{GS}} + \sum_{p|N}\mathbb{R}\mathcal{Y}_p^\vee + \mathbb{R}a(1) + a(A^0(X))$ in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$. Then one has

**Proposition 8.2.** *([18, Propositions 4.1.2, 4.1.4])*

$$\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}) = \widetilde{\mathrm{MW}} \oplus (\mathbb{R}\widehat{\Delta}_{\mathrm{GS}} + \sum_{p|N}\mathbb{R}\mathcal{Y}_p^\vee + \mathbb{R}a(1)) \oplus a(A^0(X)).$$

*More precisely, every $\widehat{Z} = (\mathcal{Z}, g_Z)$ decomposes into*

$$\widehat{Z} = \widetilde{Z}_{MW} + \frac{\deg \widehat{Z}}{\deg \widehat{\Delta}_{\mathrm{GS}}}\widehat{\Delta}_{\mathrm{GS}} + \sum_{p|N}\langle \widehat{Z}, \mathcal{Y}_p\rangle \mathcal{Y}_p^\vee + 2\kappa(\widehat{Z})a(1) + a(f_{\widehat{Z}})$$

*for some $f_{\widehat{Z}} \in A^0(X)$, where*

$$\kappa(\widehat{Z}) \deg \widehat{\Delta}_{\mathrm{GS}} = \langle \widehat{Z}, \widehat{\Delta}_{\mathrm{GS}}\rangle - \frac{\deg \widehat{Z}}{\deg \widehat{\Delta}_{\mathrm{GS}}}\langle \widehat{\Delta}_{\mathrm{GS}}, \widehat{\Delta}_{\mathrm{GS}}\rangle.$$

**Proposition 8.3.** *([18, Remark 4.1.3]) Let* $\mathrm{MW} = J_0(N) \otimes_{\mathbb{Z}} \mathbb{R}$. *Then* $\widehat{\mathcal{Z}} = (\mathcal{Z}, g_Z) \mapsto Z \in \mathrm{MW}$ *induces an isomorphism*

$$\widetilde{\mathrm{MW}} \cong \mathrm{MW},$$

*where* $Z$ *is the generic fiber of* $\mathcal{Z}$. *The inverse map is given as follows. Given a rational divisor* $Z \in J_0(N)$, *let* $g_Z$ *be the unique harmonic Green function for* $Z$ *such that*

$$d_z d_z^c g_Z - \delta_Z = \frac{\deg(Z)}{\deg \widehat{\Delta}_{\mathrm{GS}}} \mu_{GS},$$

$$\int_X g_Z \mu_{GS} = 0.$$

*Let* $\mathcal{Z}$ *be a divisor of* $\mathcal{X}$ *with rational coefficients such that its generic fiber is* $Z$, *and it is orthogonal to every irreducible components of the closed fiber* $\mathcal{X}_p$ *for each prime* $p$. *Finally let*

$$\widetilde{\mathcal{Z}} = \widehat{\mathcal{Z}} - 2a(\langle \widehat{\Delta}_{\mathrm{GS}}, \widehat{\mathcal{Z}} \rangle), \quad \widehat{\mathcal{Z}} = (\mathcal{Z}, g_Z).$$

*Then the map* $Z \mapsto \widetilde{\mathcal{Z}}$ *is the inverse isomorphism.*

Finally, let $\Delta_z$ be the Laplacian operator with respect to $\mu_{\mathrm{GS}}$. Then the space $A^0(X)$ has an orthonormal basis $\{f_j\}$ with

$$\Delta_z f_j + \lambda_j f_j = 0, \quad \langle f_i, f_j \rangle = \delta_{ij}, \quad \text{and } 0 < \lambda_1 < \lambda_2 < \cdots,$$

where the inner product is given by

$$\langle f, g \rangle = \int_{X_0(N)} f \bar{g} \mu_{\mathrm{GS}}.$$

In particular, every $f \in A^0(X)$ has the decomposition

$$f(z) = \sum \langle f, f_j \rangle f_j. \tag{8.1}$$

Recall also ([18, (4.1.36)]) that

$$d_z d_z^c f = \frac{1}{2} \Delta_z(f) \mu_{\mathrm{GS}}. \tag{8.2}$$

With the above preparation, we are now ready to restate Theorem 1.1 in a slightly more precise form as follows.

**Theorem 8.4.** *Let the notation be as above. Then*

$$\widehat{\phi}(\tau) = \tilde{\phi}_{\mathrm{MW}}(\tau) + \phi_{\mathrm{GS}}(\tau)\widehat{\Delta}_{\mathrm{GS}} + \sum_{p|N} \phi_p(\tau)\mathcal{Y}_p^{\vee} + \phi_1(\tau)a(1) + a(\phi_{SM})$$

*where $\phi_p$, $\phi_1$, and $\phi_{\mathrm{GS}}$ are real analytic modular forms of $\Gamma'$ of weight $3/2$ and representation $\rho_L$ valued in $\mathbb{C}[L^\sharp/L]$, $\tilde{\phi}_{\mathrm{MW}}(\tau)$ is a modular form of $\Gamma'$ of weight $3/2$ and representation $\rho_L$ valued in finite dimensional vector space $\widetilde{MW} \otimes \mathbb{C}[L^\sharp/L]$. Finally, there is a smooth $\mathbb{C}[L^\sharp/L]$-valued function $\Phi(\tau, z)$ on $\mathbb{H} \times X_0(N)$ which is a modular form of $\Gamma'$ of weight $3/2$ and representation $\rho_L$ on the variable $\tau$ such that its $q$-expansion (with respect to $\tau$) is $\phi_{SM}$.*

**Proof.** In this proof, the intersection, degree and differential for generation functions are computed by coefficients.

Under the isomorphism in Proposition 8.3, $\widetilde{\phi}_{MW}$ becomes (here we use Manin's well-known result that the divisor of degree 0 supported on cusps is torsion and is thus zero in $\mathrm{CH}^1_{\mathbb{R}}(X)$)

$$\phi(\tau)_{\mathbb{Q}} - \frac{\deg \widehat{\phi}}{\deg \widehat{\Delta}_{\mathrm{GS}}} \mathrm{Div}(\Delta_N)_{\mathbb{Q}} = \sum_{n>0,\mu} (Z(n,\mu) - \deg Z(n,\mu) P_\infty) q_\tau^n e_\mu,$$

which is modular by either the main result of Gross–Kohnen–Zagier [11] (note that Jacobi forms there are the same as vector valued modular forms we used here), or Borcherds' modularity result for $\phi(\tau)_{\mathbb{Q}}$ (see [18, Theorem 4.5.1]) and Proposition 6.7. Next,

$$\phi_{\mathrm{GS}}(\tau)\langle \widehat{\Delta}_{\mathrm{GS}}, a(1)\rangle = \langle \widehat{\phi}, a(1)\rangle$$

implies that $\phi_{\mathrm{GS}}(\tau)$ is modular by Proposition 6.7. For a given $p|N$, $\phi_p(\tau) = \langle \widehat{\phi}, \mathcal{Y}_p\rangle$ is modular by Proposition 6.8. The identity

$$\langle \widehat{\phi}, \widehat{\Delta}_{\mathrm{GS}}\rangle = \phi_{\mathrm{GS}}\langle \widehat{\Delta}_{\mathrm{GS}}, \widehat{\Delta}_{\mathrm{GS}}\rangle + \phi_1(\tau)\langle a(1), \widehat{\Delta}_{\mathrm{GS}}\rangle$$

implies that $\phi_1(\tau)$ is modular by Proposition 8.1.

Finally, we have by Proposition 8.2,

$$\phi_{SM}(\tau, z) = \Xi_L(\tau, z) - g_{MW} - \phi_{\mathrm{GS}}(\tau) g_{\mathrm{GS}} - \phi_1(\tau) = \sum \phi_{SM}(n, \mu, v; z) q^n e_\mu, \quad (8.3)$$

with all Fourier coefficients $\phi_{SM}(n, \mu, v; z)$ being smooth functions. Here

$$\Xi_L(\tau, z) = (\Xi(0,0,\mu) + \frac{2}{k}\log\|\Delta_N\|^2 - \log\frac{v}{N})e_0 + \sum_{n\neq 0,\mu} \Xi(n,\mu,v) q^n e_\mu.$$

Recall that

$$d_z d_z^c \Xi(n,\mu,v) + \delta_{Z(n,\mu)} = \omega(n,\mu,v)\mu_{GS} \tag{8.4}$$

and

$$d_z d_z^c (g_{MW} + \phi_{\mathrm{GS}}(\tau) g_{\mathrm{GS}}) + \sum_{n,\mu} \delta_{Z(n,\mu)} q^n e_\mu = \frac{\deg \widehat{\phi}}{\deg \widehat{\Delta}_{GS}} \mu_{GS}. \tag{8.5}$$

So we have

$$d_z d_z^c \phi_{SM}(n,\mu,v) = (\tilde{\omega}(n,\mu,v) - \alpha_{0,\mu} - \frac{\deg \widehat{\mathcal{Z}}(n,\mu,v)}{\deg \widehat{\Delta}_{GS}})\mu_{GS}.$$

Here we write

$$\omega(n,\mu,v) = \tilde{\omega}(n,\mu,v)\mu_{GS}, \quad \Theta_L(\tau,z) = \tilde{\Theta}_L(\tau,z)\mu_{GS},$$

and

$$\alpha_{0,\mu} = \begin{cases} 0 & \text{if } \mu \neq 0, \\ \frac{1}{2\pi}\frac{\mu(z)}{\mu_{GS}} & \text{if } \mu = 0. \end{cases}$$

Since its Fourier coefficients have spectral decomposition as smooth functions of $z$, we have the spectral decomposition

$$\phi_{SM}(n,\mu,v;z) = \sum_{\lambda>0} \langle \phi_{SM}(n,\mu,v), f_\lambda \rangle f_\lambda(z),$$

as $\langle \phi_{SM}(n,\mu,v), 1\rangle = 0$. Then

$$\begin{aligned}
\langle \phi_{SM}(n,\mu,v), f_\lambda \rangle &= -\frac{1}{\lambda} \int_{X_0(N)} \phi_{SM}(n,\mu,v)\Delta_z(\bar{f}_\lambda)\mu_{GS} \\
&= -\frac{2}{\lambda} \int_{X_0(N)} \phi_{SM}(n,\mu,v) d_z d_z^c \bar{f}_\lambda \\
&= -\frac{2}{\lambda} \int_{X_0(N)} d_z d_z^c \phi_{SM}(n,\mu,v) \bar{f}_\lambda \\
&= -\frac{2}{\lambda} \int_{X_0(N)} (\tilde{\omega}(n,\mu,v) - \alpha_{0,\mu} - \frac{\deg \widehat{\mathcal{Z}}(n,\mu,v)}{\deg \widehat{\Delta}_{GS}})\bar{f}_\lambda \mu_{GS} \\
&= -\frac{2}{\lambda} \int_{X_0(N)} (\tilde{\omega}(n,\mu,v) - \alpha_{0,\mu})\bar{f}_\lambda \mu_{GS} \\
&= -\frac{2}{\lambda} \langle (\tilde{\omega}(n,\mu,v) - \alpha_{0,\mu}, f_\lambda(z) \rangle.
\end{aligned}$$

Define

$$\Phi(\tau, z) = -2\sum_{\lambda>0} \lambda^{-1}\langle\tilde{\Theta}_L(\tau, z), f_\lambda(z)\rangle f_\lambda(z)$$
$$= 2\Delta_z^{-1}(\tilde{\Theta}_L(\tau, z) - \tilde{\Theta}_L(\tau)). \tag{8.6}$$

Here

$$\tilde{\Theta}_L(\tau, z) = \tilde{\Theta}_L(\tau) + \sum_{\lambda>0}\langle\tilde{\Theta}_L(\tau, z), f_\lambda(z)\rangle f_\lambda(z)$$

is the spectral decomposition of the smooth two variable theta function $\tilde{\Theta}_L(\tau, z)$, and

$$\tilde{\Theta}_L(\tau) = \frac{\int_{X_0(N)} \tilde{\Theta}_L(\tau, z)\mu_{GS}}{\int_{X_0(N)} 1\mu_{GS}}$$

is the constant term of the theta kernel function $\tilde{\Theta}_L(\tau, z)$. Recall (2.6) and (2.7), we see that the $(n, \mu)$-th Fourier coefficients of $\Phi(\tau, z)$ and $\phi_{SM}(\tau, z)$ coincide. This proves the theorem except the claim that $\Phi(\tau, z)$ is a smooth function on two variables $(\tau, z)$, which we now sketch a proof. Indeed, as $\tilde{\Theta}_L(\tau, z)$ is a fixed smooth function of $\tau$ and $z$, integration by parts in the $z$ variable gives (recall that $f_\lambda$ is an eigenfunction with eigenvalue $\lambda$)

$$\partial_\tau^\alpha\langle\tilde{\Theta}(\tau, z), f_\lambda(z)\rangle \ll_{N,\alpha,K} \lambda^{-N}$$

for any integer $\alpha \geq 0$, $N > 0$, and all $\tau \in K_1$, where $K_1$ is any compact subset of $\mathbb{H}$. On the other hand, for any integer $\beta \geq 0$, Standard Sobolev estimates for $f_\lambda$ and its derivatives give (see for example [23, Chapter 3])

$$\partial_z^\beta f_\lambda(z) \ll \lambda^{a(\beta)}$$

for all $z$ and some number $a(\beta)$ (actually we can take $a(\beta) = \beta/2 + 1$). This shows that

$$\partial_\tau^\alpha\partial_z^\beta\Phi(\tau, z) = \sum_{\lambda>0} \frac{1}{\lambda}\partial_\tau^\alpha\langle\tilde{\Theta}_L(\tau, z), f_\lambda(z)\rangle\partial_z^\beta f_\lambda(z)$$

converges locally uniformly for any $\alpha$ and $\beta$. So $\Phi(\tau, z)$ is smooth in both $\tau$ and $z$ variable. The modularity of $\Phi(\tau, z)$ follows from that of $\theta\Theta(\tau, z)$. Taking $\phi_{SM}(\tau, z) = \Phi(\tau, z)$, we prove the theorem. □

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