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Weakly holomorphic modular forms on $\Gamma_0(4)$ and Borcherds products on unitary group $U(2, 1)$

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Abstract

In this note, we construct canonical bases for the spaces of weakly holomorphic modular forms with poles supported at the cusp ∞ for $\Gamma_0(4)$ of integral weight k for $k \leq -1$, and we make use of the basis elements for the case $k = -1$ to construct explicit Borcherds products on unitary group $U(2, 1)$.

Keywords: Borcherds product, Unitary modular form, Heegner divisor, Unitary modular variety

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1 Introduction

In 1998, Borcherds developed a new method to produce meromorphic modular forms on an orthogonal Shimura variety from weakly holomorphic classical modular forms via regularized theta liftings. These meromorphic modular forms have two distinct properties. The first one is the so-called Borcherds product expansion at a cusp of the Shimura variety—his original motivation to prove the Moonshine conjecture. The second is that the divisor of these modular forms are known to be a linear combination of special divisors dictated by the principal part of the input weakly holomorphic forms. The second feature has been extended to produce so-called automorphic green functions for special divisors using harmonic Maass forms via regularized theta lifting by Bruinier [1] and Bruinier-Funke [2], which turned out to be very useful to generalization of the well-known Gross–Zagier formula [3] and the beautiful Gross–Zagier factorization formula of singular moduli [4] to Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$ (see for example [5–11]). On the other hand, the Borcherds product expansion and in particular its integral structure is essential to prove modularity of some generating functions of arithmetic divisors on these Shimura varieties [12, 13]. Borcherds products are also closely related to Mock theta functions (see for example [14] and references there).

We should mention that the analogue of the Borcherds product to unitary Shimura varieties of type $(n, 1)$ has been worked out by Hofmann [15] (see also [12]). The Borcherds product expansion in the unitary case is a little more complicated as it is a Fourier–Jacobi expansion rather than Fourier expansion; the coefficients are theta functions rather than numbers. The purpose of this note is to give some explicit examples of these Borcherds

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product expansion in concrete term. For this reason, we focus on the Picard modular surface $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ associated to the Hermitian lattice $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$ with Hermitian form

$$\langle x, y \rangle = x_1 \bar{y}_3 + x_3 \bar{y}_1 + x_2 \bar{y}_2.$$

Here

$$\mathcal{H} = \{(\tau, \sigma) \in \mathbb{H} \times \mathbb{C} \mid 4\operatorname{Im}(\tau) > |\sigma|^2\},$$

and Γ_L is a subgroup of $U(L)$ defined by (3.4). Our inputs are weakly holomorphic modular forms for $\Gamma_0(4)$ of weight -1 , character $\chi_{-4} := \left(\frac{-4}{\cdot}\right)$ which have poles only at the cusp ∞ , which we denote by $M_{-k}^{l,\infty}(\Gamma_0(4), \chi_{-4}^k)$ with $k = 1$. Our first result (Theorem 2.1) is to give a canonical basis $F_{k,m}$ ($m \geq 1$) for the infinitely dimensional vector space for every $k \geq 1$. The even k case was given by Haddock and Jenkins in [16] in a slightly different fashion. Similar method can be applied to yield a canonical basis for the space of weakly holomorphic forms of $\Gamma_0(4)$ with weight $-k$, character χ_{-4}^k , and having poles only at the cusp 0 (resp. $\frac{1}{2}$).

Next, we use a standard induction procedure to produce vector-valued weakly holomorphic modular forms for $SL_2(\mathbb{Z})$ using our lattice L which will be used to construct Picard modular forms on $U(2, 1)$ (described above). Although the resulting vector-valued modular forms for $SL_2(\mathbb{Z})$ from the three different scalar valued spaces $M_{-k}^{l,P}(\Gamma_0(4), \chi_{-4}^k)$, $P = \infty, 0, \frac{1}{2}$ are linearly independent, they don't generate the whole space. This concludes Part I of our note, which should be of independent interest.

In Part II, we focus on the unitary group $U(2, 1)$ associated to the above Hermitian form and give explicit Borcherds product expansion of the Picard modular forms constructed from $F_m = F_{1,m}$. The delicate part is to choose a proper Weyl chamber, which is a dimensional 3 real manifold and described it explicitly and carefully. Our main formula is Theorem 3.5. We remark that the same method also applies to high dimensional unitary Shimura varieties of unitary type $(n, 1)$ using forms in $M_{1-n}^{l,P}(\Gamma_0(4), \chi_{-4}^k)$ where P is a cusp for $\Gamma_0(4)$. We restrict to $U(2, 1)$ for being as explicit as possible.

2 Part I: vector-valued modular forms

In this part, we derive a canonical basis for the space $M_{-k}^{l,\infty}(\Gamma_0(4), \chi_{-4}^k)$ for any integer $k \geq 0$, and investigate the properties of the vector-valued modular forms arising from $M_{-k}^{l,\infty}(\Gamma_0(4), \chi_{-4}^k)$. For completeness, we will also give canonical bases for $M_{-k}^{l,0}(\Gamma_0(4), \chi_{-4}^k)$ and $M_{-k}^{l,\frac{1}{2}}(\Gamma_0(4), \chi_{-4}^k)$.

2.1 A canonical basis for $M_{-k}^{l,\infty}(\Gamma_0(4), \chi_{-4}^k)$

Let $\chi_{-4}(\cdot) := \left(\frac{-4}{\cdot}\right)$ be the Kronecker symbol modulo 4. Recall that $X_0(4)$ has 3 cusps, represented by ∞ , 0, and $\frac{1}{2}$. For each cusp P , let $M_{-k}^{l,P}(\Gamma_0(4), \chi_{-4}^k)$ denote the space of weakly holomorphic modular forms, which are holomorphic everywhere except at the cusp P , of weight $-k$ on $\Gamma_0(4)$ with character χ_{-4}^k . We will focus mainly on the cusp ∞ and will remark on other cusps (very similar) in the end. We will also denote $M_{-k}^l(\Gamma_0(4), \chi_{-4}^k)$ for the space of weakly holomorphic modular forms for $\Gamma_0(4)$ of weight $-k$ and character χ_{-4}^k .

Let τ be a complex number with positive imaginary part, and set $q = e(\tau) = e^{2\pi i\tau}$, and $q_r = e^{2\pi i\tau/r}$. The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Throughout this paper, we write η_m for $\eta(m\tau)$. The well known Jacobi theta functions are defined by

$$\vartheta_{00}(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \vartheta_{01}(\tau) = \sum_{n=-\infty}^{\infty} (-q)^{n^2}, \quad \vartheta_{10}(\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}.$$

Now we define three functions as follows.

$$\theta_1 = \theta_1(\tau) := \frac{1}{16} \vartheta_{10}^4(\tau) = \frac{\eta_4^8}{\eta_2^4} = q + O(q^2), \quad (2.1)$$

$$\theta_2 = \theta_2(\tau) := \vartheta_{00}^2(\tau) = \frac{\eta_2^{10}}{\eta_1^4 \eta_4^4} = 1 + O(q), \quad (2.2)$$

$$\varphi_{\infty} = \varphi_{\infty}(\tau) := \left(\frac{\eta_1}{\eta_4} \right)^8 = q^{-1} + O(1). \quad (2.3)$$

Here are some basic facts [16] about the functions θ_1 , θ_2 and φ_{∞} .

- (1) $\theta_1(\tau)$ is a holomorphic modular form of weight 2 on $\Gamma_0(4)$ with trivial character, has a simple zero at the cusp ∞ , and vanishes nowhere else.
- (2) $\theta_2(\tau)$ is a holomorphic modular form of weight 1 on $\Gamma_0(4)$ with character χ_{-4} , has a zero of order $\frac{1}{2}$ at the irregular cusp $\frac{1}{2}$, and vanishes nowhere else.
- (3) $\varphi_{\infty}(\tau)$ is a modular form of weight 0 on $\Gamma_0(4)$ with trivial character, has exactly one simple pole at the cusp ∞ and a simple zero at the cusp 0.

The following is a variant of [16] where the case even k has been treated by Haddock and Jenkins. We should mention that similar results for the space of weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ were first obtained in [17] by Duke and Jenkins.

Theorem 2.1 (1) For $k \geq 1$ odd, there is a (canonical) basis $F_{k,m}$ ($m \geq 1$) of $M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4})$ whose Fourier expansion has the following form:

$$F_{k,m} = q^{-\frac{k+1}{2}-m+1} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n.$$

- (2) For $k > 1$ even, there is a (canonical) basis $F_{k,m}$ ($m \geq 1$) of $M_{-k}^{1,\infty}(\Gamma_0(4))$ whose Fourier expansion has the following form:

$$F_{k,m} = q^{-\frac{k}{2}-m+1} + \sum_{n \geq -\frac{k}{2}+1} c(n)q^n,$$

Proof of Theorem 2.1 The proof is similar to those given in [17] and [16], and we include it for completeness. We prove (1) first. Notice that $X_0(4)$ has no elliptic points [18, Section 3.9]. For $F \in M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4})$, the valence formula for $\Gamma_0(4)$ asserts that

$$\sum_{z \in \Gamma_0(4) \backslash \mathbb{H}} \mathrm{ord}_z(F) + \mathrm{ord}_{\infty}(F) + \mathrm{ord}_0(F) + \mathrm{ord}_{1/2}(F) = -\frac{k}{2}.$$

This implies $\text{ord}_{1/2} F \geq \frac{1}{2}$ ($1/2$ is the unique irregular cusp), $\text{ord}_{\infty}(F) \leq -\frac{k+1}{2}$. This implies the uniqueness of the basis $\{F_{k,m}\}$ if it exists. We prove the existence by inductively constructing a sequence of monic polynomials $P_{k,m}(x)$ of degree m ($m \geq 0$) such that $F_{k,m+1} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_{\infty})$ give the basis we seek, i.e., with the following property

$$F_{k,m+1} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_{\infty}) = q^{-\frac{k+1}{2}-m} + \sum_{n \geq -\frac{k-1}{2}} c(n) q^n. \quad (2.4)$$

(1) Notice that $\theta_2 \theta_1^{-\frac{k+1}{2}} \in M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4})$ with

$$\theta_2 \theta_1^{-\frac{k+1}{2}} = q^{-\frac{k+1}{2}} + \sum_{n \geq -\frac{k-1}{2}} c(n) q^n.$$

So we can and will first define $P_{k,0} = 1$.

(2) For $m \geq 1$, assume that $P_{k,m-1}(x) \in \mathbb{C}[x]$ is constructed with degree $m-1$, leading coefficient 1, and the property

$$F_{k,m} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m-1}(\varphi_{\infty}) = q^{-\frac{k+1}{2}-m+1} + \sum_{n \geq -\frac{k-1}{2}} c(n) q^n.$$

Then it is easy to see

$$F_{k,m} \varphi_{\infty} = q^{-\frac{k+1}{2}-m} + \sum_{n > -\frac{k+1}{2}-m} d(n) q^n.$$

Let

$$P_{k,m} = x P_{k,m-1} - \sum_{n=-\frac{k+1}{2}-m+1}^{-\frac{k+1}{2}} d(n) P_{k,-n}$$

and

$$F_{k,m+1} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_{\infty}).$$

Then $F_{k,m+1}$ satisfies (2.4). By induction, we prove the existence of the basis $\{F_{k,m}\}$, and (1).

The proof of (2) is similar and is left to the reader. In this case, the basis $\{F_{k,m+1}\}$, $m \geq 0$, has the form

$$F_{k,m+1} = \theta_1^{-\frac{k}{2}} Q_{k,m}(\varphi_{\infty}) = q^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n) q^n \quad (2.5)$$

for a unique monic polynomial $Q_{k,m}$ of degree m . $\square$

The following corollary follows directly from the proof of Theorem 2.1(1).

Corollary 2.2 Every weakly holomorphic modular form $f(\tau) \in M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4}^k)$ with k odd, vanishes at the cusp $1/2$.

2.2 Vector-valued modular form arising from $M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4}^k)$

Let L be an even lattice over $\mathbb{Z}$ with symmetric non-degenerate bilinear form $(\cdot, \cdot)$ and associated quadratic form $Q(x) = \frac{1}{2}(x, x)$. Let L' be the dual lattice of L . Assume that L has rank $2m+2$ and signature $(2m, 2)$. Then the Weil representation of the metaplectic group $\text{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$ factors through $\text{SL}_2(\mathbb{Z})$. Thus we have a unitary representation ρ_L of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, defined by

$$\rho_L(T)\phi_\mu = e(-Q(\mu))\phi_\mu, \quad (2.6)$$

$$\rho_L(S)\phi_\mu = \frac{\sqrt{i}^{2m-2}}{\sqrt{|L'/L|}} \sum_{\beta \in L'/L} e((\mu, \beta))\phi_\beta \quad (2.7)$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, ϕ_μ for $\mu \in L'/L$ are the standard basis elements of $\mathbb{C}[L'/L]$ and $e(z) = e^{2\pi iz}$. We remark that the Weil representation ρ_L depends only on the finite quadratic module $(L'/L, Q)$ (called the discriminant group of L), where $Q(x+L) = Q(x) \pmod{1} \in \mathbb{Q}/\mathbb{Z}$.

Let k be an integer and $\vec{F}$ be a $\mathbb{C}[L'/L]$ valued function on $\mathbb{H}$ and let $\rho = \rho_L$ be a representation of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$. For $\gamma \in \text{SL}_2(\mathbb{Z})$ we define the slash operator by

$$\left(\vec{F}\Big|_{k,\rho}\gamma\right)(\tau) = (c\tau + d)^{-k} \rho(\gamma)^{-1} \vec{F}(\gamma\tau),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{H}$ via $\gamma\tau = \frac{a\tau+b}{c\tau+d}$.

Definition 2.3 Let k be an integer. A function $\vec{F} : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is called a weakly holomorphic vector-valued modular form of weight k with respect to $\rho = \rho_L$ if it satisfies

- (1) $\vec{F}\Big|_{k,\rho}\gamma = F$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$,
- (2) $\vec{F}$ is holomorphic on $\mathbb{H}$,
- (3) $\vec{F}$ is meromorphic at the cusp ∞ .

The space of such forms is denoted by $M_{k,\rho}^1$.

The invariance of T -action implies that $\vec{F} \in M_{k,\rho}^1$ has a Fourier expansion of the form

$$\vec{F} = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n, \phi_\mu) q^n \phi_\mu.$$

Note that $c(n, \phi_\mu) = 0$ unless $n \equiv -Q(\mu) \pmod{1}$.

From now on, we focus on the special case with the discriminant group $L'/L \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with quadratic form $Q(x, y) = \frac{1}{4}(x^2 + y^2) \pmod{1}$. For our purpose (in Sect. 3), it is convenient to identify $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}[i]/2\mathbb{Z}[i]$, where $Q(z) = \frac{1}{4}z\bar{z} \in \mathbb{Q}/\mathbb{Z}$. We write ϕ_0, ϕ_1, ϕ_i and ϕ_{1+i} for the basis elements of $\mathbb{C}[L'/L]$ corresponding to $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ respectively.

Let $F = F(\tau) \in M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4})$ with k odd and positive. Then using $\Gamma_0(4)$ -lifting, we can construct a vector-valued modular form $\vec{F} = \vec{F}(\tau)$ arising from $F(\tau)$ as follows:

$$\vec{F}(\tau) = \sum_{\gamma \in \Gamma_0(4) \backslash \text{SL}_2(\mathbb{Z})} (F|_{-k}\gamma) \rho_L(\gamma)^{-1} \phi_0 = \frac{1}{2} \sum_{\gamma \in \Gamma_1(4) \backslash \text{SL}_2(\mathbb{Z})} (F|_{-k}\gamma) \rho_L(\gamma)^{-1} \phi_0. \quad (2.8)$$

Define modular forms F_0, F_2 and F_3 as follows. Recall that $q_r = e^{2\pi i\tau/r}$. Let

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n.$$

Then for $j \in \{0, 2, 3\}$, we write

$$F_j = \sum_{n=0}^{\infty} a(4n+j)q_4^{4n+j}. \quad (2.9)$$

We also define $F_{1/2}$ to be

$$F_{1/2} = F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} b(n)q_2^n. \quad (2.10)$$

In addition, taking the coset representatives $\{I, S, ST^{-1}, ST, ST^2, ST^2S^{-1}\}$ for $\Gamma_0(4) \backslash \mathrm{SL}_2(\mathbb{Z})$, it is easy to check by (2.6)–(2.7) that

$$\begin{aligned} \rho_L(S)^{-1}\phi_0 &= -\frac{i}{2}(\phi_0 + \phi_1 + \phi_i + \phi_{1+i}), \\ \rho_L(ST^{-1})^{-1}\phi_0 &= -\frac{i}{2}(\phi_0 - i\phi_1 - i\phi_i - \phi_{1+i}), \\ \rho_L(ST)^{-1}\phi_0 &= -\frac{i}{2}(\phi_0 + i\phi_1 + i\phi_i - \phi_{1+i}), \\ \rho_L(ST^2)^{-1}\phi_0 &= -\frac{i}{2}(\phi_0 - \phi_1 - \phi_i + \phi_{1+i}), \\ \rho_L(ST^2S^{-1})^{-1}\phi_0 &= \phi_{1+i}. \end{aligned}$$

Finally, direct calculations yield

$$\vec{F}(\tau) = (-2iF_0 + F)\phi_0 - 2iF_3\phi_1 - 2iF_3\phi_i + (-2iF_2 - F_{1/2})\phi_{1+i}. \quad (2.11)$$

The following theorem gives some basic facts about F_0, F_2, F_3 and $F_{1/2}$.

Theorem 2.4 *With the above definitions, we have*

$$F_0 \in M_{-k}^1(\Gamma_0(4), \chi_{-4}), \quad (2.12)$$

$$F_3 \in M_{-k}^1(\Gamma_0(4), \chi_1) \quad (2.13)$$

where $\chi_1(\gamma) = \chi_{-4}(d)e(-ab/4)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$(2iF_2 + F_{1/2}) \in M_{-k}^1(\Gamma_0(4), \chi_2) \quad (2.14)$$

where $\chi_2(\gamma) = \chi_{-4}(d)e(-ab/2)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

and

$$F_{1/2} \in M_{-k}^1(\delta^{-1}\Gamma_0(4)\delta, \chi_{-4}) \quad (2.15)$$

where $\delta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Proof By (2.11), and [19, Section 3, p.6] or [20, Proposition 4.5], we can show that for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

$$(-2iF_0 + F)|_{-k} \gamma = \chi_{-4}(d)(-2iF_0 + F), \quad (2.16)$$

$$F_3|_{-k} \gamma = \chi_{-4}(d)e(-ab/4)F_3, \quad (2.17)$$

$$(-2iF_2 - F_{1/2})|_{-k} \gamma = \chi_{-4}(d)e(-ab/2)(-2iF_2 - F_{1/2}). \quad (2.18)$$

Since $F \in M_{-k}^1(\Gamma_0(4), \chi_{-4})$, then (2.16) implies (2.12). Relations (2.13) and (2.14) follow directly from (2.17) and (2.18), respectively. The last relation (2.15) follows from the definition of $F_{1/2}$,

$$F_{1/2} = F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

□

Theorem 2.5 *Let k be odd. Let $F = F(\tau) \in M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4})$ with*

$$F(\tau) = \sum_{n=-m}^{\infty} c(n)q^n.$$

Write

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n \quad \text{and} \quad F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} b(n)q_2^n.$$

And let the $\Gamma_0(4)$ -lifting of F be

$$\vec{F}(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n, \phi_\mu) q^n \phi_\mu.$$

Then we have

$$\begin{aligned} (i) \quad & c(n, \phi_0) = -2ia(4n) + c(n), \\ & c(n, \phi_1) = c(n, \phi_i) = -2ia(4n), \\ & c(n, \phi_{1+i}) = -2ia(4n) - b(2n), \end{aligned}$$

(ii) *the principal part of the vector-valued modular form $\vec{F}(\tau)$ is*

$$(c(-m)q^{-m} + \cdots + c(-1)q^{-1}) \phi_0,$$

(iii) *the constant term of the ϕ_0 -component of $\vec{F}(\tau)$ is*

$$c(0, \phi_0) = -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^m c(-n) P_{k, n-\frac{k+1}{2}}(0) + c(0),$$

where $P_{k,n}(x)$ are the polynomials defined as in the proof of Theorem 2.1.

In particular, when $k = 1$, the constant term of the ϕ_0 -component of $\vec{F}(\tau)$ is

$$c(0, \phi_0) = \sum_{n=1}^m c(-n) \left(\sum_{d|n} (64\chi_{-4}(n/d) + 4\chi_{-4}(d)) d^2 \right). \quad (2.19)$$

Proof Assertion (i) follows directly from (2.11). For the assertion (ii), since F is holomorphic at 0 and $\frac{1}{2}$, then F_j for $j \in \{0, 2, 3\}$ and $F_{1/2}$ will not contribute anything to the principal part of $\vec{F}$. So the principal part of $\vec{F}$ is given by

$$(c(-m)q^{-m} + \cdots + c(-1)q^{-1})\phi_0.$$

For the assertion (iii), we first note by (i) that

$$c(0, \phi_0) = -2ia(0) + c(0).$$

By Theorem 2.1(1), we have

$$F = c(-m)\theta_2\theta_1^{-\frac{k+1}{2}}P_{k,m-\frac{k+1}{2}}(\varphi_\infty) + \cdots + c\left(-\frac{k+1}{2}\right)\theta_2\theta_1^{-\frac{k+1}{2}}P_{k,0}(\varphi_\infty) \quad (2.20)$$

Since θ_1 and θ_2 do not vanish at the cusp 0, and φ_∞ has a simple zero at 0 of width 4, then we have

$$\theta_2\theta_1^{-\frac{k+1}{2}}\varphi_\infty^l \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = O(q^{\frac{l}{4}}),$$

and thus $\theta_2\theta_1^{-\frac{k+1}{2}}\varphi_\infty^l \Big|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ will not contribute anything to the constant term of F_0 when $l \geq 1$. Moreover, simple calculation using the transformation formula for the Dedekind eta function shows that the constant term of the Fourier expansion at the cusp 0 of $\theta_2\theta_1^{-\frac{k+1}{2}}$ is $-(8i)^{k+1}$. Therefore,

$$\begin{aligned} a(0) &= \left(\sum_{n=\frac{k+1}{2}}^m c(-n)P_{k,n-\frac{k+1}{2}}(0)\theta_2\theta_1^{-\frac{k+1}{2}} \right) \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Big|_0 \\ &= -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^m c(-n)P_{k,n-\frac{k+1}{2}}(0) \end{aligned}$$

where $(f)_0$ denotes the constant term of the q -expansion of f . Hence, we have

$$c(0, \phi_0) = -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^m c(-n)P_{k,n-\frac{k+1}{2}}(0) + c(0).$$

For (2.19), according to (iii), we need to show that

$$P_{1,m}(0) = \sum_{d|(m+1)} \chi_{-4}((m+1)/d)d^2 \quad \text{and} \quad c(0) = \sum_{n=1}^m c(-n) \left(4 \sum_{d|n} \chi_{-4}(d)d^2 \right).$$

For the first formula, we first observe that

$$\theta_2\theta_1^{-1}\varphi_\infty^\ell = q^{-\ell-1} + \sum_{j=1}^{\ell} c_\ell(-j)q^{-j} + O(1)$$

for $0 \leq \ell \leq m$. Thus there are $b_1, \dots, b_{m-1}$ such that

$$\begin{aligned} h(\tau) &:= \theta_2\theta_1^{-1}\varphi_\infty^m + b_{m-1}\theta_2\theta_1^{-1}\varphi_\infty^{m-1} + \cdots + b_1\theta_2\theta_1^{-1}\varphi_\infty \\ &= q^{-m-1} + a(-1)q^{-1} + O(1) \end{aligned}$$

for some constant $a(-1)$. Let $g(\tau)$ be defined by

$$g(\tau) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-4}(n/d) d^2 \right) q^n = \sum_{n=1}^{\infty} d_n q^n.$$

It is known [21] that $g(\tau)$ is a weight 3 modular form on $\Gamma_0(4)$ with character χ_{-4} . We note by the basic facts about θ_1 , θ_2 and φ_{∞} that $h(\tau)$ vanishes at the cusps $1/2$ and 0 . Then by [22, Theorem 3.1], we have

$$d_{m+1} + a(-1) = 0, \text{ i.e., } d_{m+1} = -a(-1).$$

Therefore

$$P_{1,m}(0) = d_{m+1} = \sum_{d|(m+1)} \chi_{-4}((m+1)/d) d^2.$$

This proves the first formula. For the second one, the proof is similar by noting that

$$h_1(\tau) := \theta_2 \theta_1^{-1} P_{1,m}(\varphi_{\infty}) = q^{-m-1} + C + O(q)$$

and

$$g_1(\tau) = 1 + 4 \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-4}(d) d^2 \right) q^n$$

is [21] a weight 3 modular form on $\Gamma_0(4)$ with character χ_{-4} . Then again [22, Theorem 3.1] shows that

$$C = 4 \sum_{d|(m+1)} \chi_{-4}(d) d^2.$$

This together with (2.20) proves the second formula. $\square$

Example 2.6 Let $k = 1$ and $F(\tau) = \theta_2 \theta_1^{-1} = \frac{\eta_2^{14}}{\eta_1^4 \eta_4^{12}} \in M_{-1}^{1,\infty}(\Gamma_0(4), \chi_{-4})$. Then we have

$$\vec{F}(\tau) = (-2iF_0 + F) \phi_0 - 2iF_3 \phi_1 - 2iF_3 \phi_i + (-2iF_2 - F_{1/2}) \phi_{1+i} \quad (2.21)$$

where F_0 , F_2 , F_3 and $F_{1/2}$ are defined as in (2.9) and (2.10). We have

$$\begin{aligned} F|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= 32i \frac{\eta(\tau/2)^{14}}{\eta(\tau/4)^4 \eta(\tau)^{12}} \\ &= 32i (1 + 12q^{1/4} + 76q^{2/4} + 352q^{3/4} + 1356q + 4600q^{5/4} \\ &\quad + 14176q^{6/4} + 40512q^{7/4} + \dots) \\ &= 32i (1 + 1356q + O(q^2)) \\ &\quad + 32i (12q^{1/4} + 4600q^{5/4} + O(q^{9/4})) \\ &\quad + 32i (76q^{2/4} + 14176q^{6/4} + O(q^{10/4})) \\ &\quad + 32i (352q^{3/4} + 40512q^{7/4} + O(q^{11/4})), \end{aligned}$$

then

$$\begin{aligned} F_0 &= 32i(1 + 1356q + O(q^2)), \\ F_2 &= 32i(76q^{2/4} + 14176q^{6/4} + O(q^{10/4})) \\ F_3 &= 32i(352q^{3/4} + 40512q^{7/4} + O(q^{11/4})). \end{aligned}$$

And

$$F_{1/2} = F|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 64(q^{1/2} - 8q^{3/2} + 42q^{5/2} + O(q^{7/2})).$$

From (2.21), we note that the principal part of F is $e(-\tau)\phi_0$ and the constant term of the ϕ_0 -component is $c(0, \phi_0) = 68$.

2.3 Canonical bases for $M_{-k}^{l,0}(\Gamma_0(4), \chi_{-4}^k)$ and $M_{-k}^{l,\frac{1}{2}}(\Gamma_0(4), \chi_{-4}^k)$

We complete this section by giving canonical bases for the other two companions of $M_{-k}^{l,\infty}(\Gamma_0(4), \chi_{-4}^k)$.

Let $\theta_3(\tau)$, $\varphi_0(\tau)$ and $\varphi_{1/2}(\tau)$ be defined by

$$\theta_3 = \theta_3(\tau) := \vartheta_{01}^4(\tau) = \frac{\eta_1^8}{\eta_2^4} = 1 + O(q), \quad (2.22)$$

$$\varphi_0 = \varphi_0(\tau) := \left(\frac{\eta_4}{\eta_1}\right)^8 = q + O(q^2), \quad (2.23)$$

$$\varphi_{1/2} = \varphi_{1/2}(\tau) := \frac{\eta_1^8 \eta_4^{16}}{\eta_2^{24}} = q + O(q^2). \quad (2.24)$$

Here are some basic facts about θ_3 , φ_0 and $\varphi_{1/2}$:

- (1) $\theta_3(\tau)$ is a weight 2 modular form on $\Gamma_0(4)$ with trivial character, has a simple zero at the cusp 0, and vanishes nowhere else;
- (2) $\varphi_0(\tau)$ is a weight 0 modular form on $\Gamma_0(4)$ with trivial character, has a simple pole at the cusp 0 and a simple zero at the cusp ∞ , and vanishes nowhere else;
- (3) $\varphi_{1/2}(\tau)$ is a weight 0 modular form on $\Gamma_0(4)$ with trivial character, has a simple pole at the cusp $\frac{1}{2}$ and a simple zero at the cusp ∞ , and vanishes nowhere else.

Theorem 2.7 Let θ_2 , θ_3 and φ_0 be as defined in (2.2), (2.22) and (2.23), respectively.

- (1) For k odd, the set $\{\theta_2\theta_3^{-\frac{k+1}{2}}P_{k,m}(\varphi_0)\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that

$$\theta_2\theta_3^{-\frac{k+1}{2}}P_{k,m}(\varphi_0) \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = q_4^{-\frac{k+1}{2}-m} + \sum_{n=-\frac{k-1}{2}}^{\infty} c(n)q_4^n,$$

is a canonical basis for $M_{-k}^{l,0}(\Gamma_0(4), \chi_{-4})$.

- (2) For k even, the set $\{\theta_3^{-\frac{k}{2}}P_{k,m}(\varphi_0)\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that

$$\theta_3^{-\frac{k}{2}}P_{k,m}(\varphi_0) \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = q_4^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n)q_4^n,$$

is a canonical basis for $M_{-k}^{l,0}(\Gamma_0(4))$.

Theorem 2.8 Let θ_2 and $\varphi_{1/2}$ be as defined in (2.2) and (2.24), respectively. Then the set $\{\theta_2^{-k} P_{k,m}(\varphi_{1/2})\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that

$$\theta_2^{-k} P_{k,m}(\varphi_{1/2}) \Big|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = q^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n)q^n,$$

is a canonical basis for $M_{-k}^{\frac{1}{2}}(\Gamma_0(4), \chi_{-4}^k)$.

Proofs of Theorems 2.7 and 2.8 are similar to that of Theorem 2.1, so we omit the details.

Remark 2.9 For a cusp P , denote by $M_{-k,\rho_L}^{\frac{1}{2}P}$ the space of vector-valued modular forms induced from $M_{-k}^{\frac{1}{2}P}(\Gamma_0(4), \chi_{-4}^k)$ via $\Gamma_0(4)$ -lifting. We have, by (2.11),

$$M_{-k,\rho_L}^{\frac{1}{2}\infty} + M_{-k,\rho_L}^{\frac{1}{2}0} + M_{-k,\rho_L}^{\frac{1}{2}\frac{1}{2}} = M_{-k,\rho_L}^{\frac{1}{2}\infty} \oplus M_{-k,\rho_L}^{\frac{1}{2}0} \oplus M_{-k,\rho_L}^{\frac{1}{2}\frac{1}{2}}.$$

Clearly, $M_{-k,\rho_L}^{\frac{1}{2}\infty} + M_{-k,\rho_L}^{\frac{1}{2}0} + M_{-k,\rho_L}^{\frac{1}{2}\frac{1}{2}}$ is a subspace of $M_{-k,\rho_L}^{\frac{1}{2}}$. In general, the former space may not be equal to the latter one. We first note that every vector-valued modular form in $M_{-k,\rho_L}^{\frac{1}{2}\infty} + M_{-k,\rho_L}^{\frac{1}{2}0} + M_{-k,\rho_L}^{\frac{1}{2}\frac{1}{2}}$ must have the same component functions at ϕ_1 and ϕ_i . We now give an example of functions in $M_{-1,\rho_L}^{\frac{1}{2}}$ that does not have this property. Let $F(\tau) = \theta_2 \theta_1^{-1} \in M_{-1}^{\frac{1}{2}}(\Gamma_0(4), \chi_{-4})$. Then as above we write the $\Gamma_0(4)$ -lifting of $F(\tau)$ as

$$\vec{F}(\tau) = (-2iF_0 + F)\phi_0 - 2iF_3\phi_1 - 2iF_3\phi_i + (-2iF_2 - F_{1/2})\phi_{1+i}$$

where

$$F_j = \sum_{n=0}^{\infty} a(4n+j)q_4^{4n+j},$$

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n$$

and

$$F_{1/2} = F|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

By (2.13), we know that $F_3(\tau) \in M_{-1}^1(\Gamma_1(4), \chi)$ where $\chi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = e(-b/4)$. Now we do $\Gamma_1(4)$ -lifting on $F_3(\tau)$ against ϕ_1 , namely,

$$\vec{F}_3(\tau) = \sum_{\gamma \in \Gamma_1(4) \backslash \text{SL}_2(\mathbb{Z})} (F_3|_{-1} \gamma) \rho_L(\gamma)^{-1} \phi_1,$$

and get

$$\vec{F}_3(\tau) = -4if_0\phi_0 + (2F_3 + 4if_3)\phi_1 + (-4if_3 - 2f_{1/2})\phi_i + 4if_2\phi_{1+i}$$

where

$$f_j = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} \tilde{a}(4n+j)q_4^{4n+j},$$

$$F_3|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} \tilde{a}(n) q_4^n$$

and

$$f_{1/2} = F_3|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Now the component functions at ϕ_1 and ϕ_i are $2F_3 + 4if_3$ and $-4if_3 - 2f_{1/2}$, respectively. We can compute and verify that they are not the same. Therefore, $\tilde{F}_3(\tau)$ is not in the space $M_{-k, \rho_L}^{1, \infty} + M_{-k, \rho_L}^{1, 0} + M_{-k, \rho_L}^{1, \frac{1}{2}}$.

3 Part II: Borcherds products on $U(2, 1)$

It is well-known that the vector-valued weakly modular forms construction in Part I can be used to construct memomorphic modular forms on Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$ with Borcherds product formulas and known divisors. In this part, we focus on one special case to make it very explicitly—the Picard modular surfaces over $\mathbb{k} = \mathbb{Q}(i)$. In particular, we describe a Weyl chamber explicitly and write down the Borcherds product expression concretely.

This part is devoted to deriving Borcherds products lifted from a vector-valued modular form arising from $M_{-1}^{1, \infty}(\Gamma_0(4), \chi_{-4})$.

3.1 Picard modular surfaces over $\mathbb{k} = \mathbb{Q}(i)$

Let $(V, \langle \cdot, \cdot \rangle)$ be a Hermitian vector space over $\mathbb{k}$ of signature $(2, 1)$ and let $H = U(V)$, where $U(V)$ denotes the unitary group associated to V . Let $V_{\mathbb{C}} = V \otimes_{\mathbb{k}} \mathbb{C}$, and

$$\mathcal{L} = \{w \in V_{\mathbb{C}} \mid \langle w, w \rangle < 0\}.$$

Then $\mathcal{K} = \mathcal{L}/\mathbb{C}^{\times}$ is the Hermitian domain for $H(\mathbb{R})$, and $\mathcal{L}$ is the tautological line bundle over $\mathcal{K}$. For a congruence subgroup Γ of $H(\mathbb{Q})$, the associated Picard modular surface $X_{\Gamma} = \Gamma \backslash \mathcal{K}$ is defined over some number field.

Given an isotropic line $\mathbb{k}\ell$ (i.e., a cusp), choose another isotropic element ℓ' with $\langle \ell, \ell' \rangle \neq 0$. Let $V_0 = (\mathbb{k}\ell + \mathbb{k}\ell')^{\perp}$, and let

$$\mathcal{H} = \mathcal{H}_{\ell, \ell'} = \left\{ (\tau, \sigma) \in \mathbb{H} \times V_{0, \mathbb{C}} \mid \operatorname{Im} \tau > \frac{\langle \sigma, \sigma \rangle}{4|\langle \ell', \ell \rangle|^2} \right\}.$$

Then the map

$$\mathcal{H} \rightarrow \mathcal{L}, (\tau, \sigma) \mapsto z(\tau, \sigma) = 2i\langle \ell', \ell \rangle \tau \ell + \sigma + \ell' \quad (3.1)$$

gives rise to an isomorphism $\mathcal{H} \cong \mathcal{K}$. It is also a nowhere vanishing section of the line bundle $\mathcal{L}$. Using this map, we can define action of $H(\mathbb{R})$ on $\mathcal{H}$ and automorphy factor $j(\gamma, \tau, \sigma)$ via the equation

$$\gamma z(\tau, \sigma) = j(\gamma, \tau, \sigma) z(\gamma(\tau, \sigma)). \quad (3.2)$$

Indeed, both $\gamma z(\tau, \sigma)$ and $z(\gamma(\tau, \sigma))$ are in $\mathcal{L}$ and they become the same in $\mathcal{K}$, so they are different by a multiplication constant, namely, the automorphy factor $j(\tau, \sigma)$.

Definition 3.1 Let Γ be a unitary modular group. A holomorphic automorphic form of weight k and with character χ for Γ is a function $g : \mathcal{H} \rightarrow \mathbb{C}$, with the following properties:

- (1) g is holomorphic on $\mathcal{H}$,
- (2) $g(\gamma(\tau, \sigma)) = j(\gamma; \tau, \sigma)^k \chi(\gamma) g(\tau, \sigma)$ for all $\gamma \in \Gamma$.

We remark that a holomorphic modular form g for Γ is automatically holomorphic at the cusps.

Now we make everything concrete and explicit. First choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of V with Gram matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so $V = \bigoplus_{i=1}^3 \mathbf{k} \mathbf{e}_i \cong \mathbf{k}^3$ with Hermitian form

$$\langle x, y \rangle = x_1 \bar{y}_3 + x_2 \bar{y}_2 + x_3 \bar{y}_1 = {}^t x J \bar{y}, \quad (3.3)$$

and

$$H = H(\mathbb{Q}) = \{h \in \mathrm{GL}_3(\mathbf{k}) \mid h J {}^t \bar{h} = J\}.$$

We take the lattice

$$L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2} \mathbb{Z}[i]$$

(instead of the typical $\mathbb{Z}[i]^3$). Its $\mathbb{Z}$ -dual lattice is

$$L' = \{v \in V \mid \mathrm{Tr}_{\mathbf{k}/\mathbb{Q}} \langle v, L \rangle \subset \mathbb{Z}\} = \mathbb{Z}[i] \oplus \frac{1}{2} \mathbb{Z}[i] \oplus \frac{1}{2} \mathbb{Z}[i]$$

So $L'/L \cong \frac{1}{2} \mathbb{Z}[i]/\mathbb{Z}[i]$ with quadratic form $Q(x) = x \bar{x} \in \frac{1}{4} \mathbb{Z}/\mathbb{Z}$, which is the same finite quadratic module considered in Part I. Let

$$\begin{aligned} \mathrm{U}(L) &= \{g \in H \mid gL = L\} \\ &= H \cap \left\{ \begin{pmatrix} \mathbb{Z}[i] & \mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & \mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \frac{1}{2}\mathbb{Z}[i] & \frac{1}{2}\mathbb{Z}[i] & \mathbb{Z}[i] \end{pmatrix} \right\}. \end{aligned}$$

be the stabilizer of L in H , and Γ_L be the subgroup of $\mathrm{U}(L)$ which acts on the discriminant group L'/L trivially:

$$\Gamma_L = \mathrm{U}(L) \cap \left\{ \begin{pmatrix} \mathbb{Z}[i] & 2\mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & 1 + 2\mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & 2\mathbb{Z}[i] & \mathbb{Z}[i] \end{pmatrix} \right\}. \quad (3.4)$$

Take the cusp $\ell = \mathbf{e}_1$ and $\ell' = \mathbf{e}_3$. Then $V_0 \cong \mathbf{k}$ with Hermitian form $\langle x, y \rangle = x \bar{y}$, and

$$\mathcal{H} = \{(\tau, \sigma) \in \mathbb{H} \times \mathbb{C} \mid 4\mathrm{Im}(\tau) > |\sigma|^2\}.$$

Moreover, one has for $\gamma = (a_{ij}) \in H$

$$\gamma(\tau, \sigma) = \left(\frac{a_{11}\tau + (2i)^{-1}a_{12}\sigma + (2i)^{-1}a_{13}}{2ia_{31}\tau + a_{32}\sigma + a_{33}}, \frac{2ia_{21}\tau + a_{22}\sigma + a_{23}}{2ia_{31}\tau + a_{32}\sigma + a_{33}} \right).$$

and

$$j(\gamma, \tau, \sigma) = \frac{\langle \gamma z, \ell \rangle}{\langle \ell', \ell \rangle} = 2i\tau a_{31} + a_{32}\sigma + a_{33}.$$

Our Picard modular surface is the quotient space $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ of $\mathcal{H}$ modulo the action of Γ_L .

Let P_ℓ be the stabilizer of the cusp $k\ell$ in H . Then $P_\ell = N_\ell M_\ell$ with

$$M_\ell = \{m(a, b) = \text{Diag}(a, b, \bar{a}^{-1}) \mid a \in k^\times, b \in k^1\},$$

$$N_\ell = \left\{ n(b, c) = \begin{pmatrix} 1 & -2\bar{b} & -2b\bar{b} + 2ic \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix} \mid b \in k, c \in \mathbb{Q} \right\},$$

where $k^1 = \{a \in k \mid a\bar{a} = 1\}$ is the norm one group. Notice that N_ℓ is a Heisenberg group action on $\mathcal{H}_{\ell, \ell'}$ via

$$n(b, c)(\tau, \sigma) = (\tau + c + i\bar{b}(\sigma + b), \sigma + b).$$

In particular

$$n(0, c)(\tau, \sigma) = (\tau + c, \sigma).$$

Let

$$\Gamma_{L, \ell} = \Gamma_L \cap N_\ell = \{n(b, c) \mid b \in \mathbb{Z}[i], c \in \mathbb{Z}\}.$$

Then for a holomorphic modular form $f(\tau, \sigma)$ for Γ_L , we have a Fourier–Jacobi expansion at the cusp $k\ell$:

$$f(\tau, \sigma) = \sum_{n \geq 0} f_n(\sigma) q^n. \quad (3.5)$$

3.2 The hermitian space V as a quadratic space

As mentioned in the previous subsection, the hermitian space V can be viewed as a quadratic space $V_{\mathbb{Q}}$ of signature (4,2) associated with bilinear form induced from the hermitian form:

$$(x, y) = \text{Tr}_{k/\mathbb{Q}} \langle x, y \rangle.$$

Then the lattice L can be considered as a quadratic $\mathbb{Z}$ -lattice in $V_{\mathbb{Q}}$. Denote by

$$\text{SO}(V_{\mathbb{Q}}) = \{g \in \text{SL}(V_{\mathbb{Q}}) \mid (gx, gy) = (x, y) \text{ for all } x, y \in V_{\mathbb{Q}}\}$$

the special orthogonal group of $V_{\mathbb{Q}}$ and its set of real points as $\text{SO}(V_{\mathbb{Q}})(\mathbb{R}) \cong \text{SO}(4, 2)$. A model for the symmetric domain of $\text{SO}(V_{\mathbb{Q}})(\mathbb{R})$ is the Grassmannian of two-dimensional negative definite subspaces of $V_{\mathbb{Q}}$, denoted by Gr_O . It can be realized as a tube domain $\mathcal{H}_O$ as follows. Denote by $V_{\mathbb{Q}}(\mathbb{C})$ the complex quadratic space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ with $(\cdot, \cdot)$ extended to a $\mathbb{C}$ -valued bilinear form.

Now we view L as a $\mathbb{Z}$ -lattice. Let $e_1 \in L$ be a primitive isotropic lattice vector and choose an isotropic dual vector $e_2 \in L'$ with $(e_1, e_2) = 1$. Denote by K the Lorentzian $\mathbb{Z}$ -sublattice $K = L \cap e_1^\perp \cap e_2^\perp$ with respect to $(\cdot, \cdot)$. The tube domain model $\mathcal{H}_O$ is one of the two connected components of the following subset of $K \otimes_{\mathbb{Z}} \mathbb{C}$

$$\{Z = X + iY \mid X, Y \in K \otimes_{\mathbb{Z}} \mathbb{R}, Q(Y) < 0\}.$$

Recall that $\ell = \mathbf{e}_1$ and $\ell' = \mathbf{e}_3$. We define

$$e_1 = \ell, e_2 = \frac{1}{2}\ell', e_3 = -i\ell, e_4 = -\frac{i}{2}\ell'$$

where we denote by $\hat{\mu}$ the endomorphism of $V_{\mathbb{Q}}(\mathbb{R})$ induced from the scalar multiplication with μ . Then we can check that $\{e_1, e_2, e_3, e_4\}$ is a basis for $(\mathbb{Z}[i]\ell + \mathbb{Z}[i]\ell') \otimes_{\mathbb{Z}} \mathbb{Q}$ and we can see that $K \otimes_{\mathbb{Z}} \mathbb{R} = ((\mathbb{Q}e_3 + \mathbb{Q}e_4) \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (V_0 \otimes_{\mathbb{Z}} \mathbb{R})$. Thus we can identify Y with $y_1e_3 + y_2e_4 + \sigma \in K \otimes_{\mathbb{Z}} \mathbb{R}$. Now denote by $\mathcal{C}$ the set of $Y = y_1e_3 + y_2e_4 + \sigma$ with $y_1y_2 + Q(\sigma) < 0$, $y_1 < 0$ and $y_2 > 0$. We can fix $\mathcal{H}_O$ as the component for which $Y \in \mathcal{C}$. Therefore, $\mathcal{H}_O = K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathcal{C}$.

In addition, the tube domain $\mathcal{H}_O$ can be mapped biholomorphically to any one of the two connected components of a negative cone of $\mathbb{P}^1(V_{\mathbb{Q}})(\mathbb{C})$ given by

$$\{[Z_L] \in \mathbb{P}^1(V_{\mathbb{Q}})(\mathbb{C}) \mid (Z_L, Z_L) = 0, (Z_L, \tilde{Z}_L) < 0\}.$$

We fix this component and denote it by $\mathcal{K}_O$. For each $[Z_L]$, we can uniquely represent it as

$$Z_L = e_2 - q(Z)e_1 + Z$$

with $Z \in \mathcal{H}_O$.

3.3 Embedding of $\mathcal{H}$ into $\mathcal{H}_O$

As in [15, Section 4], we can embed $\mathcal{H}$ into $\mathcal{H}_O$ via

$$(\tau, \sigma) \rightarrow \iota(\tau, \sigma) = -\tau e_3 + ie_4 + \mathfrak{z}(\sigma) \quad (3.6)$$

where

$$\mathfrak{z}(\sigma) = \frac{\hat{1}}{2}\sigma + i\left(-\frac{\hat{i}}{2}\right)\sigma. \quad (3.7)$$

Similarly, $\mathcal{K}_U$ can be embedded into $\mathcal{K}_O$ through the identifications between $\mathcal{K}_U$ and $\mathcal{H}$, and between $\mathcal{K}_O$ and $\mathcal{H}_O$. Namely,

$$z = \ell' + 2i\tau\ell + \sigma \rightarrow Z_L = -i\tau e_1 + e_2 - \tau e_3 + ie_4 + \mathfrak{z}(\sigma). \quad (3.8)$$

3.4 Weyl chambers of $K \otimes_{\mathbb{Z}} \mathbb{R}$

In Theorem 2.1 (1), we have shown that $F_{1,m} = q^{-m} + O(1)$ for $m \geq 1$, form a canonical basis for $M_{-1}^{\infty}(\Gamma_0(4), \chi_{-4})$. Therefore, to study the Borchers product lifted from M_{-1,ρ_L}^{∞} , it suffices to start with $F_{1,m}$. Since we only deal with weight -1 in the rest of this paper, we will simply write $F_m = F_{1,m}$, and $\tilde{F}_m = \tilde{F}_{1,m}$.

For general definitions of the following, we refer the reader to [1, Chapter 3.1]. For $\kappa \in K$ with $q(\kappa) > 0$, denote by $\kappa^{\perp}$ the orthogonal complement of κ in $K \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\mathcal{D}_K$ the Grassmannian of negative 1-lines of $K \otimes_{\mathbb{Z}} \mathbb{R}$, which can be realized as

$$\begin{aligned} \mathcal{D}_K &= \{\mathbb{R}w \subset K_{\mathbb{R}} \mid q(w) < 0\} \\ &\cong \{w = y_1e_3 + e_4 + (y_3 + iy_4) \mid y_i \in \mathbb{R}, q(w) < 0\}. \end{aligned}$$

Then by considering the Grassmannian of negative 1-lines of $\kappa^{\perp}$, it corresponds to a codimension 1 sub-manifold of the Grassmannian $\mathcal{D}_K$ of $K \otimes_{\mathbb{Z}} \mathbb{R}$.

In our case, a Heegner divisor of index $(m, 0)$, $H_K(m, 0)$, is a locally finite union of codimension 1 sub-manifolds of $\mathcal{D}_K$, namely,

$$H_K(m, 0) = \{z \in \mathcal{D}_K \mid \exists \kappa \in K \text{ with } q(\kappa) = m \text{ and } (z, \kappa) = 0\}$$

Let $\vec{F}_m(\tau)$ be the vector-valued modular form arising from F_m . It is known by Theorem 2.5 that the principal part of $\vec{F}_m(\tau)$ is $q^{-m}\phi_0$. The Weyl chambers attached to $\vec{F}_m(\tau)$ are the connected components W_m of

$$\mathcal{D}_K - H_K(m, 0).$$

Fix a Weyl chamber W_m of $\mathcal{D}_K$, we can also define the corresponding Weyl chambers of $K \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathcal{H}$ by

$$W_{m,K} = \{w \in K \otimes_{\mathbb{Z}} \mathbb{R} \mid \mathbb{R}w \in W_m\},$$

$$W_{m,U} = \left\{ (\tau, \sigma) \in \mathcal{H} \mid \operatorname{Im}(\iota(\tau, \sigma)) = -\operatorname{Im} \tau e_3 + e_4 - \frac{\hat{i}}{2} \sigma \in W_{m,K} \right\},$$

respectively. In the following lemma, we give an explicit description of the Weyl chamber that we use to construct Borcherds product in Theorem 3.5.

Lemma 3.2 (1) Let

$$W_m = \left\{ y_1 e_3 + e_4 + (y_3 + iy_4) \in \mathcal{D}_K \mid \begin{array}{l} y_1 < r^2 + s^2 - m + 2ry_3 + 2sy_4 \quad \forall r, s \in \mathbb{Z}, \\ 1 + 2ty_3 + 2hy_4 > 0 \quad \forall t, h \in \mathbb{Z}, t^2 + h^2 = m, \\ ty_3 + hy_4 > 0, \quad \forall t, h \in \mathbb{Z}, t^2 + h^2 = m, t > 0, \\ y_4 > 0 \text{ if } m \text{ is a square.} \end{array} \right\}$$

$$\subset \left\{ y_1 e_3 + e_4 + (y_3 + iy_4) \in \mathcal{D}_K \mid \begin{array}{l} k_2 y_1 < -k_1 + 2k_3 y_3 + 2k_4 y_4 \quad \forall k_i \in \mathbb{Z}, k_2 > 0, k_1 k_2 + k_3^2 + k_4^2 = m, \\ k + 2ty_3 + 2hy_4 > 0 \quad \forall k, t, h \in \mathbb{Z}, k > 0, t^2 + h^2 = m, \\ ty_3 + hy_4 > 0, \quad \forall t, h \in \mathbb{Z}, t^2 + h^2 = m, t > 0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right\}. \quad (3.9)$$

Then W_m is a Weyl chamber containing e_3 .

(2) Let

$$K_m = \left\{ \lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4) \in K' \mid \begin{array}{l} (\lambda, W_m) > 0, \\ (Q(\lambda) = m \text{ with } \lambda_3, \lambda_4 \in 2\mathbb{Z}) \\ \text{or } (Q(\lambda) \leq 0) \end{array} \right\}$$

where $(\lambda, W_m) > 0$ means that $(\lambda, w) > 0$ for all $w \in W_m$. Then

$$K_m = \left\{ \lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4) \mid \begin{array}{l} \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{Z}, \\ \lambda_2 > 0, \\ \text{or } (\lambda_2 = 0 \text{ and } \lambda_1 > 0), \\ \text{or } (\lambda_2 = \lambda_1 = 0 \text{ and } \lambda_3 > 0), \\ \text{or } (\lambda_2 = \lambda_1 = \lambda_3 = 0 \text{ and } \lambda_4 > 0) \end{array} \right\}.$$

Proof For Assertion (1), it is clear that W_m contains e_3 since the set of (y_3, y_4, y_1) determined by the inequalities in W_m contains $y_1 = -\infty$. We only need to show W_m is actually a Weyl chamber.

Write $\kappa = k_1 e_3 + k_2 e_4 + k_3 + ik_4 \in K$ with $k_i \in \mathbb{Z}$. Since $(-\kappa)^\perp = \kappa^\perp$, we can assume $k_2 \geq 0$. By the definition of Weyl chamber W_m , we can see that a Weyl chamber W_m can be viewed as a connected component of $\mathbb{R}^3$ cut out by the planes

$$k_2 y_1 + k_1 + 2k_3 y_3 + 2k_4 y_4 = 0$$

for all $k_1, \dots, k_4 \in \mathbb{Z}$ with $k_2 \geq 0$ and $k_1 k_2 + k_3^2 + k_4^2 = m$.

When $k_2 = 0$ and m is representable as a sum of two squares, then we have planes

$$k_1 + 2k_3 y_3 + 2k_4 y_4 = 0$$

perpendicularly passing through the (y_3, y_4) -plane. In this case, the connected components are determined by the connected components of the $y_3 - y_4$ plane cut out by the lines

$$k_1 + 2k_3y_3 + 2k_4y_4 = 0,$$

and it is easy to find that one of the connected components C_1 can be identified as

$$\left\{ (y_3, y_4) \in \mathbb{R}^2 \left| \begin{array}{l} 1+2ty_3+2hy_4 > 0 \ \forall t, h \in \mathbb{Z}, t^2+h^2=m, \\ ty_3+hy_4 > 0, \ \forall t, h \in \mathbb{Z}, t^2+h^2=m, t>0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right. \right\}$$

which is a subset of

$$\left\{ (y_3, y_4) \in \mathbb{R}^2 \left| \begin{array}{l} k+2ty_3+2hy_4 > 0 \ \forall k, t, h \in \mathbb{Z}, k>0, t^2+h^2=m, \\ ty_3+hy_4 > 0, \ \forall t, h \in \mathbb{Z}, t^2+h^2=m, t>0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right. \right\}.$$

When $k_2 > 0$, with the aid of MAPLE, we can check that there is a connected component C_2 of $\mathbb{R}^3$ covered by

$$y_1 = r^2 + s^2 - m + 2ry_3 + 2sy_4$$

for $r, s \in \mathbb{Z}$. Such a connected component contains $y_1 < -m$, and all the other planes

$$k_2y_1 = -k_1 + 2k_3y_3 + 2k_4y_4$$

for $k_1, \dots, k_4 \in \mathbb{Z}$ with $k_2 > 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$. In conclusion, $W_m = C_1 \cap C_2$ is a connected component of $\mathbb{R}^3$ cut out by the planes

$$k_2y_1 + k_1 + 2k_3y_3 + 2k_4y_4 = 0$$

for all $k_1, \dots, k_4 \in \mathbb{Z}$ with $k_2 \geq 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$, and thus W_m is a Weyl chamber. For the case $m = 1$, we can visualize it by a 3D-plot. See Fig. 1.

Now let us prove Assertion (2).

- (i) Suppose that $Q(\lambda) = m$ and $\lambda_3, \lambda_4 \in 2\mathbb{Z}$ which imply that $\lambda \in K$. By (3.9), we note that $y_1e_3 + e_4 + (y_3 + iy_4) \in W_m$ implies that

$$k_2y_1 < -k_1 + 2k_3y_3 + 2k_4y_4$$

for all $k_i \in \mathbb{Z}$ with $k_2 > 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$, which is equivalent to

$$k_2y_1 + k_1 + 2k_3y_3 + 2k_4y_4 > 0$$

for all $k_i \in \mathbb{Z}$ with $k_2 < 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$. Therefore, when $\lambda_2 \neq 0$ and $Q(\lambda) = m$, that is, $\lambda_1(-\lambda_2) + \frac{1}{4}(\lambda_3^2 + \lambda_4^2) = m$, $(\lambda, W_m) > 0$ if and only if $-\lambda_2 < 0$, that is, $\lambda_2 > 0$. Similarly, by the other conditions given in (3.9), we can conclude that when $Q(\lambda) = m$, $(\lambda, W_m) > 0$ if and only if $\lambda_2 < 0$, or $(\lambda_2 = 0$ and $\lambda_1 > 0)$, or $(\lambda_2 = \lambda_1 = 0$ and $\lambda_3 > 0)$, or $(\lambda_2 = \lambda_1 = \lambda_3 = 0$ and $\lambda_4 > 0)$.

- (ii) Now suppose that $Q(\lambda) \leq 0$, that is, $\lambda_1\lambda_2 + \frac{1}{4}(\lambda_3^2 + \lambda_4^2) \leq 0$. By (3.9), we know that

$$y_1 < r^2 + s^2 - m + 2ry_3 + 2sy_4$$

for all $r, s \in \mathbb{Z}$. By [1, Lemma 3.2], it is known that if $(\lambda, w_0) > 0$ for a $w_0 \in W_m$, then $(\lambda, W_m) > 0$. Thus $(\lambda, W_m) > 0$ if and only if $\lambda_2 > 0$. When $\lambda_2 = 0$, since $Q(\lambda) \leq 0$, then $\lambda_3 = \lambda_4 = 0$, and thus $(\lambda, w) = \lambda_1$ for $w \in W_m$. This implies that $(\lambda, W_m) > 0$ if and only if $\lambda_1 > 0$ when $\lambda_2 = 0$. $\square$

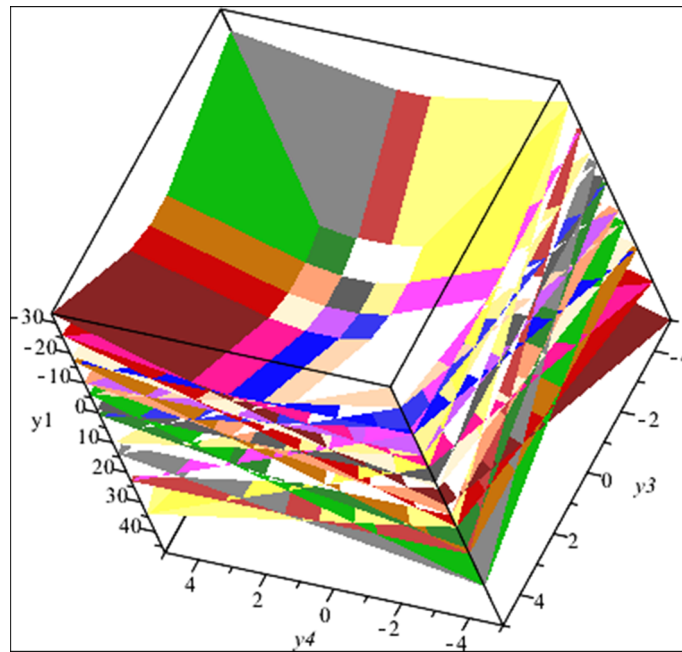


Fig. 1 A 3D-plot for the Weyl chamber W_1

3.5 The Weyl vector for $\vec{F}_m$

In this subsection, we aim to compute the Weyl vector $\rho(W_m, \vec{F}_m)$. We first recall a nice summary of the explicit computations of Weyl vector given in [11, Subsection 2.1] (also see [23, Thm. 10.4] for original definitions).

Let L be a $\mathbb{Z}$ -lattice with quadratic form $Q(\cdot)$ of a quadratic space V of type $(n, 2)$ and L' be its dual lattice. Take $\ell_L \in L$ and $\ell'_L \in L'$ to be such that $Q(\ell_L) = Q(\ell'_L) = 0$ and $(\ell_L, \ell'_L) = 1$. Assume that $(\ell_L, L) = N_L \mathbb{Z}$ for some unique positive integer and choose $\xi \in L$ with $(\ell_L, \xi) = N_L$. Let $K = L \cap (\mathbb{Q}\ell_L + \mathbb{Q}\ell'_L)^\perp$ and let

$$L'_0 = \{x \in L' \mid (\ell_L, x) \equiv 0 \pmod{N_L}\} \subset L'.$$

Then there is a projection

$$p: L'_0 \rightarrow K', \quad p(x) = x_K + \frac{(x, \ell_L)}{N_L} \xi_K,$$

where x_K and ξ_K are the orthogonal projections of $x, \xi \in V$ to $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$. So it induces a projection from L'_0/L to K'/K . Next, for

$$\vec{f} = \sum f_{\mu} \phi_{\mu} = \sum c(m, \phi_{\mu}) q^m \phi_{\mu} \in M_{1-\frac{n}{2}, \rho_L}^1,$$

define

$$\vec{f}_K = \sum_{\lambda \in K'/K} f_{\lambda} \phi_{\lambda, K} = \sum c_K(m, \lambda) q^m \phi_{\lambda, K},$$

where $\phi_{\lambda, K}$ is the basis element associated to λ of $\mathbb{C}[K'/K]$, and

$$f_{\lambda} = \sum_{\substack{\mu \in L'_0/L \\ p(\mu) = \lambda}} f_{\mu}.$$

For a Weyl chamber W , take $\ell_K \in K \cap \overline{W}$, where $\overline{W}$ denotes the closure of the Weyl chamber W , and $\ell'_K \in K'$ with $Q(\ell_K) = Q(\ell'_K) = 0$ and $(\ell_K, \ell'_K) = 1$, and let $P =$

$K \cap (\mathbb{Q}\ell_K + \mathbb{Q}\ell'_K)^\perp$, which is positive definite of rank $n - 2$. Similar to the projection p from L'_0/L to K'/K , one also has a projection, also denoted by p , from K'_0/K to P'/P defined in the same way. Similarly, we have a weakly holomorphic modular form $\vec{f}_P$ induced by $\vec{f}_K$. Then we can compute and express the Weyl vector $\rho(W, \vec{f})$ associated to W and $\vec{f}$ as

$$\rho(W, \vec{f}) = \rho_{\ell_K} \ell_K + \rho_{\ell'_K} \ell'_K + \rho_P,$$

where

$$\begin{aligned} \rho_{\ell_K} = & -\frac{1}{4} \sum_{\substack{\lambda \in K'_0/K \\ p(\lambda)=0+P}} c_K(0, \lambda) B_2((\lambda, \ell'_K)) \\ & - \frac{1}{2} \sum_{\substack{\gamma \in P' \\ (\gamma, W) > 0}} \sum_{\substack{\lambda \in K'_0/K \\ p(\lambda)=\gamma+P}} c_K(-Q(\gamma), \lambda) B_2((\lambda, \ell'_K)), \end{aligned} \quad (3.10)$$

$B_2(x) := \{x\}^2 - \{x\} + \frac{1}{6}$ is the second Bernoulli polynomial, $\{x\}$ is the fractional part of x ,

$$\begin{aligned} \rho_{\ell'_K} = & \text{constant term of } \langle \vec{\theta}_P, \vec{f}_P \rangle E_2 / 24, \\ \vec{\theta}_P := & \sum_{\gamma \in P'/P} \sum_{\lambda \in \gamma+P} e(Q(\lambda)\tau) \phi_\gamma, \end{aligned} \quad (3.11)$$

$E_2 := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ is the holomorphic Eisenstein series of weight 2,

$$\rho_P = -\frac{1}{2} \sum_{\substack{\gamma \in P' \cap K' \\ (\gamma, W) > 0}} c_K(-Q(\gamma), \gamma) \gamma. \quad (3.12)$$

Now for our case, we set $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$, $\ell_L = e_1$, $\ell'_L = e_2$, $\vec{f} = \vec{F}_m$, $W = W_m$, $\ell_K = e_3$ and $\ell'_K = e_4$, where e_1, e_2, e_3 and e_4 are defined as in Subsection 3.2. It is easy to check that $K = \mathbb{Z}e_3 \oplus \mathbb{Z}e_4 \oplus P$ and $P = \mathbb{Z}[i]e_2$ where e_2 is defined as in Subsection 3.1. Direct calculations show that $L'_0 = L'$, $K'_0 = K'$ and $L'/L \cong K'/K \cong P'/P$. Write

$$\vec{F}_m = \sum_{\mu \in L'/L} F_{m,\mu} \phi_\mu = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Q}} c(n, \mu) q^n \phi_\mu.$$

Then under $L'/L \cong K'/K \cong P'/P$, direct calculations show that

$$\vec{F}_{m,K} = \vec{F}_{m,P} = \vec{F}_m.$$

Note that by Theorem 2.5, for $n > 0$, $c_K(-n, \lambda) \neq 0$ if and only if $c_K(-n, \lambda) = c_K(-m, 0)$, which equals 1. Also, we can compute and express $\vec{\theta}_P$ as

$$\begin{aligned} \vec{\theta}_P = & \left(\sum_{r,s \in \mathbb{Z}} e((r^2 + s^2)\tau) \right) \phi_0 + \left(\sum_{r,s \in \mathbb{Z}} e\left(\left(\frac{1}{4} + r + r^2 + s^2\right)\tau\right) \right) \phi_1 \\ & + \left(\sum_{r,s \in \mathbb{Z}} e\left(\left(r^2 + s^2 + s + \frac{1}{4}\right)\tau\right) \right) \phi_i + \left(\sum_{r,s \in \mathbb{Z}} e\left(\left(\frac{1}{2} + r + s + r^2 + s^2\right)\tau\right) \right) \phi_{1+i}. \end{aligned} \quad (3.13)$$

Now let us first compute ρ_{e_3} . Since $K'_0/K = K'/K \cong P'/P$, then $\lambda \in K'_0/K$ such that $p(\lambda) = 0 + P$ if and only if $\lambda = 0 + K$. In addition, as we point out above that for

$n > 0$, $c_K(-n, \lambda) \neq 0$ if and only if $c_K(-n, \lambda) = c_K(-m, 0) = 1$, then we can see that $c_K(-Q(\gamma), \lambda) = 1$ if and only if $\gamma \in P$ with $Q(\gamma) = m$, i.e., $\gamma = (r + si)\mathbf{e}_2$ with $r, s \in \mathbb{Z}$ and $r^2 + s^2 = m$. In addition, by Lemma 3.9, to have $(\gamma, W_m) > 0$, we must have $(r > 0)$ or $(r = 0 \text{ and } s > 0)$. Now by the definition of ρ_{e_3} and the above analysis, we have

$$\begin{aligned}\rho_{e_3} &= -\frac{1}{24}c_K(0, 0) - \frac{1}{2} \sum_{\substack{r^2+s^2=m \\ r>0 \\ \text{or } (r=0 \text{ and } s>0)}} \frac{1}{6} \\ &= -\frac{1}{6} \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 - \frac{1}{6}\sigma_{\chi_{-4}}(m),\end{aligned}$$

where $\sigma_{\chi_{-4}}(m) = \sum_{d|m} \chi_{-4}(d)$ follows from the well known fact (see, e.g., [24, Thm. 3.2.1]) that the number of integral solutions of $r^2 + s^2 = m$ is given by $4\sigma_{\chi_{-4}}(m)$.

For the e_4 -component ρ_{e_4} , we first note that the non- ϕ_0 -component functions of θ_P have no constant terms, and the non- ϕ_0 -component functions of $\vec{F}_P$ have no negative power terms. In addition, the ϕ_0 -component function of θ_P is

$$\sum_{r,s \in \mathbb{Z}} e((r^2 + s^2)\tau) = 1 + 4 \sum_{n=1}^{\infty} \sigma_{\chi_{-4}}(n)q^n,$$

and the ϕ_0 -component function of $\vec{F}_P$ is $q^{-m} + c(0, 0) + O(q)$. Therefore, the constant term of $\langle \vec{\theta}_P, \vec{f}_P \rangle_{E_2}$ is the constant term of

$$\left(1 + 4 \sum_{n=1}^{\infty} \sigma_{\chi_{-4}}(n)q^n\right) (q^{-m} + c(0, 0) + O(q)) \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n\right),$$

which is

$$4\sigma_{\chi_{-4}}(m) - 24\sigma_1(m) - 96 \left(\sum_{\substack{k+l=m \\ k,l \geq 1}} \sigma_{\chi_{-4}}(k)\sigma_1(l) \right) + c(0, 0).$$

Thus by Theorem 2.5, we have

$$\begin{aligned}\rho_{e_4} &= \frac{1}{6} \left[\sigma_{\chi_{-4}}(m) - 6\sigma_1(m) - 24 \left(\sum_{\substack{k+l=m \\ k,l \geq 1}} \sigma_{\chi_{-4}}(k)\sigma_1(l) \right) \right. \\ &\quad \left. + \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 \right].\end{aligned}$$

For ρ , notice that for $n > 0$, $c_K(-n, \lambda) \neq 0$ if and only if $\gamma \in P$ with $Q(\gamma) = m$, i.e., $\gamma = (r + si)\mathbf{e}_2$ with $r, s \in \mathbb{Z}$ and $r^2 + s^2 = m$. Notice also that $(\gamma, W_m) > 0$ implies $(r > 0)$ or $(r = 0 \text{ and } s > 0)$. So by similar calculations, we have

$$\rho_P = -\frac{1}{2} \sum_{\substack{r^2+s^2=m \\ r>0 \\ \text{or } (r=0 \text{ and } s>0)}} (r + si)\mathbf{e}_2.$$

Summing up, we conclude with the following proposition.

Proposition 3.3 *Let $\vec{F}_m$ be defined as in Subsection 3.4, and let W_m be the Weyl chamber given by (3.9). Then the Weyl vector associated to the Weyl chamber W_m and the weakly holomorphic modular form $\vec{F}_m$ is*

$$\rho(W_m, \vec{F}_m) = \rho_{e_3} e_3 + \rho_{e_4} e_4 + \rho_P,$$

where

$$\rho_{e_3} = -\frac{1}{6} \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 - \frac{1}{24} \sigma_{\chi_{-4}}(m),$$

$$\rho_{e_4} = \frac{1}{6} \left[\sigma_{\chi_{-4}}(m) - 6\sigma_1(m) - 24 \left(\sum_{\substack{k+l=m \\ k,l \geq 1}} \sigma_{\chi_{-4}}(k) \sigma_1(l) \right) + \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 \right],$$

$$\rho_P = -\frac{1}{2} \sum_{\substack{r^2+s^2=m \\ r>0 \\ \text{or } (r=0 \text{ and } s>0)}} (r+si) e_2,$$

$$\text{and } \sigma_{\chi_{-4}}(m) = \sum_{d|m} \chi_{-4}(d).$$

3.6 Heegner divisors for Γ_L

Let $\lambda \in L'$ be a lattice vector with positive norm, i.e., $\langle \lambda, \lambda \rangle > 0$. The orthogonal complement of λ in $\mathcal{K}_U$ is a closed analytic subset of codimension 1, which we denote as follows

$$\mathbf{H}(\lambda) = \{[z] \in \mathcal{K}_U \mid \langle z, \lambda \rangle = 0\}.$$

By identification between $\mathcal{K}_U$ and $\mathcal{H}$, $\mathbf{H}(\lambda)$ can also be considered as a closed analytic subset of $\mathcal{H}$, and we call such set a prime Heegner divisor on $\mathcal{H}$. Given $\beta \in L'/L$ and $m \in \mathbb{Z}_{>0}$, a Heegner divisor of index (m, β) in $\mathcal{H}$ is defined as the locally finite sum

$$\mathbf{H}(m, \beta) = \sum_{\substack{\lambda \in \beta + L \\ Q(\lambda) = m}} \mathbf{H}(\lambda).$$

The associated Heegner divisor in $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is $\mathbf{Z}(m, \beta) = \Gamma_L \backslash \mathbf{H}(m, \beta)$.

3.7 Borchers products

In this section, we give a family of new Borchers products explicitly by using the results of Hofmann [15, Thms. 4, 5 and Cor. 1]. We first summarize Hofmann's results as follows.

Theorem 3.4 (Hofmann) *Let $\mathbb{F}$ be an imaginary quadratic field. Let L be an even hermitian lattice of signature $(m, 1)$ with $m \geq 1$, and $\ell \in L$ a primitive isotropic vector. Let $\ell' \in L'$ an isotropic vector with $\langle \ell, \ell' \rangle \neq 0$. Further assume that L is the direct sum of a hyperbolic plane $H \cong \mathcal{O}_{\mathbb{F}} \oplus \partial_{\mathbb{F}}^{-1}$ and a definite part D with $\langle D, H \rangle = 0$.*

Given a weakly holomorphic modular form $f \in M_{1-m, \rho_L}^!$ with Fourier coefficients $c(n, \beta)$ satisfying $c(n, \beta) \in \mathbb{Z}$ for $n < 0$, there is a meromorphic function $\Psi(\tau, \sigma; f)$ on $\mathcal{H}$ with the following properties:

- (1) $\Psi(\tau, \sigma; f)$ is an automorphic form of weight $c(0, \phi_0)/2$ for Γ_L with some multiplier system χ of finite order.
- (2) The zeros and poles of $\Psi(\tau, \sigma; f)$ lie on Heegner divisors. The divisor of $\Psi(\tau, \sigma; f)$ on $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is given by

$$\operatorname{div}(\Psi(\tau, \sigma; f)) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(\beta) \\ n > 0}} c(-n, \phi_\beta) \mathbf{H}(n, \beta).$$

The multiplicities of $\mathbf{H}(n, \beta)$ are 2 if $2\beta = 0$ in L'/L , and 1 otherwise.

- (3) For a Weyl chamber W whose closure contains the cusp $\mathbb{Q}e_3$, $\Psi(\tau, \sigma; f)$ has an infinite product expansion of the form

$$\Psi(\tau, \sigma; f) = Ce\left(\frac{\langle z, \rho(W, f) \rangle}{\langle \ell, \ell' \rangle}\right) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \left[1 - e\left(\frac{\langle z, \lambda \rangle}{\langle \ell, \ell' \rangle}\right)\right]^{c(-Q(\lambda), \lambda)},$$

where $z = z(\tau, \sigma) = \ell' + \delta \langle \ell, \ell' \rangle \tau \ell + \sigma$, δ is the square root of the discriminant of $\mathbb{F}$, the constant C has absolute value 1 and $\rho(W, f)$ is the Weyl vector attached to W and f .

- (4) The lifting is multiplicative: $\Psi(\tau, \sigma; f + g) = \Psi(\tau, \sigma; f) \Psi(\tau, \sigma; g)$.
- (5) Let W be a Weyl chamber such that the cusp corresponding to ℓ is contained in the closure of W . If this cusp is neither a pole nor a zero of $\Psi(\tau, \sigma; f)$, then we have

$$\lim_{\tau \rightarrow \infty} \Psi(\tau, \sigma; f) = Ce\left(\overline{\rho(W, f)_\ell}\right) \prod_{\substack{\lambda \in K' \\ \lambda = \frac{1}{2} \kappa \delta \ell \\ \kappa \in \mathbb{Q}_{>0}}} \left(1 - e\left(-\frac{1}{2} \kappa \delta\right)\right)^{c(0, \lambda)}$$

where $\overline{\rho(W, f)_\ell}$ denotes the complex conjugate of the ℓ -component of the Weyl vector $\rho(W, f)$.

By specializing Theorem 3.4 in our case, we obtain the main result of this note.

Theorem 3.5 Let $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$ with respect to the standard basis over $\mathbb{Z}[i]$ with hermitian form defined in (3.3). We set $\ell = (1, 0, 0)$ and $\ell' = (0, 0, 1)$. Let $\vec{F}_m$ be the vector-valued modular form arising from $F_m = \theta_2 \theta_1^{-1} P_{1, m-1}(\varphi_\infty)$ and denote by $c(n, \phi_\mu)$ the Fourier coefficient of index (n, ϕ_μ) of $\vec{F}_m$. Then there is a meromorphic function $\Psi(\tau, \sigma; F_m) = \Psi(\tau, \sigma; \vec{F}_m)$ on $\mathcal{H}$ with the following properties:

- (1) $\Psi(\tau, \sigma; \vec{F}_m)$ is an automorphic form of weight

$$32 \sum_{d|m} \chi_{-4}(n/d) d^2 + 2 \sum_{d|m} \chi_{-4}(d) d^2$$

for Γ_L , with some multiplier system χ of finite order.

- (2) The zeros and poles of $\Psi(\tau, \sigma; \vec{F}_m)$ lie on Heegner divisors. The divisor of $\Psi(\tau, \sigma; \vec{F}_m)$ on $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is given by

$$\operatorname{div}(\Psi(\tau, \sigma; \vec{F}_m)) = \mathbf{Z}(m, 0) = \Gamma_L \backslash \mathbf{H}(m, 0),$$

where

$$H(m, 0) = \sum_{\substack{(r_1, s_1, r_2, s_2, r_3, s_3) \in \mathbb{Z}^6 \\ r_1 r_3 + s_1 s_3 + r_2^2 + s_2^2 = m}} \left\{ (\tau, \sigma) \in \mathcal{H} \left| \begin{array}{l} r_1 + 2r_2 \operatorname{Re} \sigma + 2s_2 \operatorname{Im} \sigma + s_3 \operatorname{Re} \tau - r_3 \operatorname{Im} \tau = 0, \\ s_1 + 2r_2 \operatorname{Im} \sigma - 2s_2 \operatorname{Re} \sigma + s_3 \operatorname{Im} \tau + r_3 \operatorname{Re} \tau = 0 \end{array} \right. \right\}.$$

(3) For the Weyl chamber W_m described in (3.9), $\Psi(\tau, \sigma; \vec{F}_m)$ has an infinite product expansion near the cusp $\mathbb{Q}e_3$ (precisely, when $(\tau, \sigma) \in W_{m,U}$ with $\operatorname{Im} \tau$ sufficiently large):

$$\Psi(\tau, \sigma; F_m) = A_1(\tau, \sigma) A_2(\sigma) A_3(\sigma) A_4(\sigma) A_5(\tau, \sigma), \quad (3.14)$$

where

$$(i) \quad A_1(\tau, \sigma) = e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma)$$

where ρ_{e_3} , ρ_{e_4} and ρ are defined as in Proposition 3.3,

$$(ii) \quad A_2(\sigma) = \begin{cases} [1 - e(-i\sigma\sqrt{m})] & \text{if } m \text{ is a square,} \\ 1 & \text{otherwise,} \end{cases}$$

$$(iii) \quad A_3(\sigma) = \prod_{\substack{(k_3, k_4) \in \mathbb{Z}_{>0}^2 \\ k_3^2 + k_4^2 = m}} [1 - e(\sigma(k_3 + ik_4))] [1 - e(\sigma(k_3 - ik_4))],$$

$$(iv) \quad A_4(\sigma) = \prod_{\substack{n_3, n_4 \in \mathbb{Z} \\ n_3^2 + n_4^2 = m}} \prod_{n_2 \in \mathbb{Z}_{>0}} [1 - e(in_2)e(\sigma(n_3 - in_4))] \\ \times \prod_{n_2 \in \mathbb{Z}_{>0}} (1 - e(in_2))^{c(0,0)}$$

with

$$c(0, 0) = c(0, \phi_0) = \sum_{d|m} (64\chi_{-4}(m/d) + 4\chi_{-4}(d)) d^2,$$

$$(v) \quad A_5(\tau, \sigma) = \prod_{\substack{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \\ n_1 > 0}} \left[1 - e\left(n_1\tau + \sigma\left(\frac{n_3}{2} - i\frac{n_4}{2}\right) + in_2\right) \right]^{c(n_1n_2 - \frac{1}{4}(n_3^2 + n_4^2), \phi_{\vec{n}})}$$

with $\vec{n} = n_2e_3 - n_1e_4 + \frac{1}{2}(n_3 + in_4)$.

(4) If the cusp corresponding to ℓ is neither a pole nor a zero of $\Psi(\tau, \sigma; \vec{F}_m)$, then we have

$$\lim_{\tau \rightarrow i\infty} \Psi(\tau, \sigma; \vec{F}_m) = e(i\rho_{e_3}) \prod_{k=1}^{\infty} (1 - e(ki))^{c(0, \phi_0)}$$

where

$$\rho_{e_3} = -\frac{1}{6} \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 - \frac{1}{24} \sigma_{\chi_4}(m)$$

is defined as in Proposition 3.3, and

$$c(0, \phi_0) = \sum_{d|m} (64\chi_{-4}(m/d) + 4\chi_{-4}(d)) d^2$$

is defined as in Theorem 2.5.

Proof Assertion (1) follows from Theorem 2.5 and Theorem 3.4 (1). Assertion (2) follows directly from Theorem 3.4 (2).

Then by Theorem 3.4 (3) together with Lemma 3.2 and Proposition 3.3, we have that $\Psi(\tau, \sigma; \vec{F}_m)$ has the following infinite product expansion near the cusp $\mathbb{Q}e_3$

$$\begin{aligned} & \psi(\tau, \sigma; \vec{F}_m) \\ &= e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma) \\ & \times \prod_{\substack{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{Z}^4 \\ \lambda_2 > 0, \\ \text{or } \lambda_2=0 \text{ and } \lambda_1 > 0, \\ \text{or } \lambda_2=\lambda_1=0 \text{ and } \lambda_3 > 0, \\ \text{or } \lambda_2=\lambda_1=\lambda_3=0 \text{ and } \lambda_4 > 0.}} \left[1 - e\left(\lambda_2\tau + \sigma\left(\frac{\lambda_3}{2} - i\frac{\lambda_4}{2}\right) + i\lambda_1\right) \right]^{c\left(\lambda_1\lambda_2 - \frac{1}{4}(\lambda_3^2 + \lambda_4^2), \phi_\lambda\right)} \end{aligned}$$

where $\lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4)$, and ρ_{e_3} , ρ_{e_4} and ρ are given as in Proposition 3.3. We first set $A_1(\tau, \sigma) = e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma)$. Then by decomposing the infinite product according to the four cases in its product index set, we can easily rewrite it as (3.14).

Finally, for Assertion (4), we first note that in our case, $K' = \mathbb{Z}i \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}i$ and $\delta = 2i$, then $\lambda \in K'$ and $\lambda = \frac{1}{2}\kappa\delta\ell = \kappa i\ell$ with $\kappa \in \mathbb{Q}_{>0}$ imply that $\kappa \in \mathbb{Z}_{>0}$ and $c(0, \lambda) = c(0, \phi_0)$. Together with the Weyl vector attached to W_m and $\vec{F}_m$ shown in Subsection 3.5, Theorem 3.4 (5) proves Assertion (4). $\square$

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