

GENERAL CLARK MODEL FOR FINITE-RANK PERTURBATIONS

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All unitary (contractive) perturbations of a given unitary operator U by finite-rank- d operators with fixed range can be parametrized by $d \times d$ unitary (contractive) matrices Γ ; this generalizes unitary rank-one ($d = 1$) perturbations, where the Aleksandrov–Clark family of unitary perturbations is parametrized by the scalars on the unit circle $\mathbb{T} \subset \mathbb{C}$.

For a strict contraction Γ the resulting perturbed operator T_Γ is (under the natural assumption about star cyclicity of the range) a completely nonunitary contraction, so it admits the functional model.

We investigate the Clark operator, i.e., a unitary operator that intertwines T_Γ (written in the spectral representation of the nonperturbed operator U) and its model. We make no assumptions on the spectral type of the unitary operator U ; an absolutely continuous spectrum may be present.

We first find a universal representation of the adjoint Clark operator in the coordinate-free Nikolski–Vasyunin functional model; the word “universal” means that it is valid in any transcription of the model. This representation can be considered to be a special version of the vector-valued Cauchy integral operator.

Combining the theory of singular integral operators with the theory of functional models, we derive from this abstract representation a concrete formula for the adjoint of the Clark operator in the Sz.-Nagy–Foiaş transcription. As in the scalar case, the adjoint Clark operator is given by a sum of two terms: one is given by the boundary values of the vector-valued Cauchy transform (postmultiplied by a matrix-valued function) and the second one is just the multiplication operator by a matrix-valued function.

Finally, we present formulas for the direct Clark operator in the Sz.-Nagy–Foiaş transcription.

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0. Introduction

The contractive (or unitary) perturbations $U + K$ of a unitary operator U on a Hilbert space H by operators K of finite rank d with fixed range are parametrized by the $d \times d$ contractive (resp. unitary) matrices Γ . Namely, if $\text{Ran } K \subset \mathfrak{R}$, where $\mathfrak{R} \subset H$, $\dim \mathfrak{R} = d$, is fixed, and $\mathbf{B} : \mathbb{C}^d \rightarrow \mathfrak{R}$ is a fixed unitary operator (which we call the coordinate operator), then K is represented as $K = \mathbf{B}(\Gamma - \mathbf{I}_{\mathbb{C}^d})\mathbf{B}^*U$, where Γ is a contraction (resp. a unitary operator) on $\mathbb{C}^d$. Therefore, all such perturbations with $\text{Ran } K \subset \mathfrak{R}$ are represented as $T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I}_{\mathbb{C}^d})\mathbf{B}^*U$, where Γ runs over all $d \times d$ contractive (resp. unitary) matrices.

Recall that T being a *contraction* (contractive) means that $\|T\| \leq 1$.

Focusing on the nontrivial part of the perturbation, we can assume that $\text{Ran } \mathbf{B} = \mathfrak{R}$ is a star-cyclic subspace for U , i.e., $H = \overline{\text{span}}\{U^k\mathfrak{R}, (U^*)^k\mathfrak{R} : k \in \mathbb{Z}_+\}$. Below we will show that star-cyclicity together with the assumption that Γ is a pure contraction ensures that the operator T_Γ is what is called a completely nonunitary contraction, meaning that T_Γ does not have a nontrivial unitary part. Model theory informs us that such T_Γ is unitarily equivalent to its functional model $\mathcal{M}_\theta$, $\theta = \theta_\Gamma$, that is, the compression of the shift operator on the model space $\mathcal{K}_\theta$ with the characteristic function $\theta = \theta_\Gamma$ of T_Γ .

In this paper we investigate the so-called Clark operator, i.e., a unitary operator Φ that intertwines the contraction T_Γ (in the spectral representation of the unperturbed operator U) with its model: $\mathcal{M}_\theta\Phi = \Phi T_\Gamma$, $\theta = \theta_\Gamma$. The case of rank-one perturbations ($d = 1$) was treated by D. Clark [1972] when θ is inner, and later by D. Sarason [1994] under the assumption that θ is an extreme point of the unit ball of H^∞ . For finite-rank perturbations with inner characteristic matrix-valued functions θ , V. Kapustin and A. Poltoratski [2006] studied boundary convergence of functions in the model space $\mathcal{K}_\theta$. The setting of inner characteristic functions corresponds to the operators U that have purely singular spectrum (no a.c. component); see, e.g., [Douglas and Liaw 2013].

In [Liaw and Treil 2016] we completely described the general case of rank-one perturbations (when the measure can have absolutely continuous part, or equivalently, the characteristic function is not necessarily inner).

In the present paper we extend the results from [Liaw and Treil 2016] to finite-rank perturbations with general matrix-valued characteristic functions. We first find a universal representation of the adjoint Clark operator, which features a special case of a matrix-valued Cauchy integral operator. By “universal” we mean that our formula is valid in any transcription of the functional model. This representation is a pretty straightforward, albeit more algebraically involved, generalization of the corresponding result from [Liaw and Treil 2016]; it might look like “abstract nonsense”, since it is proved under the assumption that we pick a model operator that “agrees” with the Clark model (more precisely that the corresponding coordinate/parametrizing operators agree).

However, by careful investigation of the construction of the functional model, using the coordinate-free Nikolski–Vasyunin model we were able to present a formula giving the parametrizing operators for the model that agree with given coordinate operators for a general contraction T ; see Lemma 3.2. Moreover, for the Sz.-Nagy–Foiş transcription of the model we get explicit formulas for the parametrizing operators

in terms of the characteristic function; see [Lemma 3.3](#). Similar formulas can be obtained for other transcriptions of the model.

We also compute the characteristic function of the perturbed operator T_Γ ; the formula involves the Cauchy integral of the matrix-valued measure.

For the Sz.-Nagy–Foiaş transcription of the model we give a more concrete representation of the adjoint Clark operator in terms of the vector-valued Cauchy transform; see [Theorem 8.1](#). This representation looks more natural when one considers spectral representations of the nonperturbed operator U defined with the help of matrix-valued measures; see [Theorem 8.7](#).

0A. Plan of the paper. In [Section 1](#) we set the stage by introducing finite-rank perturbations and studying some of their basic properties. In particular, we discuss the concept of a star-cyclic subspace and find a measure-theoretic characterization for it.

The main result of [Section 2](#) is the universal representation formula for the adjoint Clark operator; see [Theorem 2.4](#). In this section we also introduce the notion of agreement of the coordinate/parametrizing operators and make some preliminary observations about such an agreement.

[Section 3](#) is devoted to the detailed investigation of the agreement of the coordinate/parametrizing operators. Careful analysis of the construction of the model from the coordinate-free point of view of Nikolski–Vasyunin allows us to get for a general contraction T -formulas for the parametrizing operators for the model that agree with the coordinate operators; see [Lemma 3.2](#). Explicit formulas (in terms of the characteristic function) are presented for the case of Sz.-Nagy–Foiaş transcription; see [Lemma 3.3](#).

The characteristic function θ_Γ of the perturbed operator T_Γ is the topic of [Sections 4](#) and [5](#). [Theorem 4.2](#) gives a formula for θ_Γ in terms of a Cauchy integral of a matrix-valued measure. In [Section 5](#) we show that, similarly to the rank-one case, the characteristic functions θ_Γ and θ_0 are related via a special linear fractional transformation. Relations between defect functions Δ_0 and Δ_Γ are also described.

[Section 6](#) contains a brief heuristic overview of what subtle techniques are to come in [Sections 7](#) and [8](#).

In [Section 7](#) we present results about regularizations of the Cauchy transform, and about uniform boundedness of such generalizations, which we need to get the representation formulas in [Section 8](#).

In [Section 8](#) we give a formula for the adjoint Clark operator in the Sz.-Nagy–Foiaş transcription of the model. As in the scalar case, the adjoint Clark operator is given by the sum of two terms: one is in essence a vector-valued Cauchy transform (postmultiplied by a matrix-valued function), and the second one is just a multiplication operator by a matrix-valued function; see [Theorem 8.1](#). In the case of inner characteristic functions (purely singular spectral measure of U) the second term disappears, and the adjoint Clark operator is given by what can be considered a matrix-valued analogue of the scalar *normalized Cauchy transform*; see [Section 8E](#).

[Section 9](#) is devoted to a description of the Clark operator Φ ; see [Theorem 9.2](#).

1. Preliminaries

Consider the family of rank- d perturbations $U + K$ of a unitary operator U on a separable Hilbert space H . If we fix a subspace $\mathfrak{R} \subset H$, $\dim \mathfrak{R} = d$, such that $\text{Ran } K \subset \mathfrak{R}$, then all unitary perturbations of $U + K$

of U can be parametrized as

$$T = U + (X - I_{\mathfrak{R}})P_{\mathfrak{R}}U, \quad (1-1)$$

where X runs over all possible unitary operators in $\mathfrak{R}$.

It is more convenient to factorize the representation of X through the fixed space $\mathfrak{D} := \mathbb{C}^d$ by picking an isometric operator $B : \mathfrak{D} \rightarrow H$, $\text{Ran } B = \mathfrak{R}$. Then any X in (1-1) can be represented as $X = B\Gamma B^*$, where $\Gamma : \mathfrak{D} \rightarrow \mathfrak{D}$ (i.e., Γ is a $d \times d$ matrix). The perturbed operator $T = T_\Gamma$ can be rewritten as

$$T = U + B(\Gamma - I_{\mathfrak{D}})B^*U. \quad (1-2)$$

If we decompose the space H treated as the domain as $H = U^*\mathfrak{R} \oplus (U^*\mathfrak{R})^\perp$, and the same space treated as the target space as $H = \mathfrak{R} \oplus \mathfrak{R}^\perp$, then the operator T can be represented with respect to this decomposition as

$$T = \begin{pmatrix} B\Gamma B^*U & 0 \\ 0 & T_1 \end{pmatrix}, \quad (1-3)$$

where block T_1 is unitary.

From the above decomposition we can immediately see that if Γ is a contraction then T is a contraction (and if Γ is unitary then T is unitary).

In this formula we slightly abuse notation, since formally the operator $B\Gamma B^*U$ is defined on the whole space H . However, this operator clearly annihilates $(U^*\mathfrak{R})^\perp$, and its range belongs to $\mathfrak{R}$, so we can restrict its domain and target space to $U^*\mathfrak{R}$ and $\mathfrak{R}$ respectively. So when such operators appear in the block decomposition we will assume that its domain and target space are restricted.

In this paper we assume that the isometry B is fixed and that all the perturbations are parametrized by the $d \times d$ matrix Γ .

1A. Spectral representation of U . By the spectral theorem the operator U is unitarily equivalent to the multiplication M_ξ by the independent variable ξ in the von Neumann direct integral

$$\mathcal{H} = \int_{\mathbb{T}}^{\oplus} E(\xi) d\mu(\xi), \quad (1-4)$$

where μ is a finite Borel measure on $\mathbb{T}$ (without loss of generality we can assume that μ is a probability measure, $\mu(\mathbb{T}) = 1$).

Let us recall the construction of the direct integral; we present not the most general one, but one that is sufficient for our purposes. Let E be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^\infty$, and let $N : \mathbb{T} \rightarrow \mathbb{N} \cup \{\infty\}$ be a measurable function (the so-called *dimension function*). Define

$$E(\xi) = \overline{\text{span}}\{e_n \in E : 1 \leq n \leq N(\xi)\}.$$

Then the direct integral $\mathcal{H}$ is the subspace of the E -valued space $L^2(\mu; E) = L^2(\mathbb{T}, \mu; E)$ consisting of the functions f such that $f(\xi) \in E(\xi)$ for μ -a.e. ξ .

Note, that the dimension function N and the spectral type $[\mu]$ of μ (i.e., the collection of all measures that are mutually absolutely continuous with μ) are spectral invariants of U , meaning that they define the operator U up to unitary equivalence.

So, without loss of generality, we assume that U is the multiplication M_ξ by the independent variable ξ in the direct integral (1-4).

An important particular case is the case when U is star-cyclic, meaning that there exists a vector $h \in H$ such that $\overline{\text{span}}\{U^n h : n \in \mathbb{Z}\} = H$. In this case $N(\xi) \equiv 1$, and the operator U is unitarily equivalent to the multiplication operator M_ξ in the scalar space $L^2(\mu) = L^2(\mathbb{T}, \mu)$.

In the representation of U in the direct integral it is convenient to give a “matrix” representation of the isometry $\mathbf{B}$. Namely, for $k = 1, 2, \dots, d$ define functions $b_k \in \mathcal{H} \subset L^2(\mu; E)$ by $b_k := \mathbf{B}e_k$; here $\{e_k\}_{k=1}^d$ is the standard orthonormal basis in $\mathbb{C}^d$.

In this notation the operator $\mathbf{B}$, if we follow the standard rules of the linear algebra, is the multiplication by a row B of vector-valued functions,

$$B(\xi) = (b_1(\xi), b_2(\xi), \dots, b_d(\xi)).$$

If we represent $b_k(\xi)$ in the standard basis in E that we used to construct the direct integral (1-4), then $\mathbf{B}$ is just the multiplication by the matrix-valued function of size $(\dim E) \times d$.

1B. Star-cyclic subspaces and completely nonunitary contractions.

Definition 1.1. As was previously mentioned, a subspace $\mathfrak{R}$ is said to be *star-cyclic* for an operator T on H if

$$H = \overline{\text{span}}\{T^k \mathfrak{R}, (T^*)^k \mathfrak{R} : k \in \mathbb{Z}_+\}.$$

For a perturbation (not necessarily unitary) $T = T_\Gamma$ of the unitary operator U given by (1-2) the subspace

$$\mathcal{E} = \overline{\text{span}}\{U^k \mathfrak{R}, (U^*)^k \mathfrak{R} : k \in \mathbb{Z}_+\} = \overline{\text{span}}\{U^k \mathfrak{R} : k \in \mathbb{Z}\} \quad (1-5)$$

is a reducing subspace for both U and T_Γ (i.e., $\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant for both U and T_Γ).

Since $T_\Gamma|_{\mathcal{E}^\perp} = U|_{\mathcal{E}^\perp}$, the perturbation does not influence the action of T_Γ on $\mathcal{E}^\perp$, so nothing interesting for perturbation theory happens on $\mathcal{E}^\perp$; all action happens on $\mathcal{E}$. Therefore, we can restrict our attention to $T_\Gamma|_{\mathcal{E}}$, i.e., assume without loss of generality that $\mathfrak{R} = \text{Ran } \mathbf{B}$ is a star-cyclic subspace for U .

We note the following result.

Lemma 1.2. *Let $\mathfrak{R} = \text{Ran } \mathbf{B}$ be a star-cyclic subspace for U and let Γ be unitary. Then $\mathfrak{R}$ is also a star-cyclic subspace for all perturbed unitary operators $U_\Gamma = T_\Gamma$ given by (1-2).*

We postpone for a moment a proof of this well-known fact.

Definition 1.3. A contraction T in a Hilbert space H is called *completely nonunitary* (c.n.u. for short) if there is no nonzero reducing subspace on which T acts unitarily.

Recall that a contraction is called *strict* if $\|Tx\| < \|x\|$ for all $x \neq 0$.

Lemma 1.4. *If $\mathfrak{R} = \text{Ran } \mathbf{B}$ is a star-cyclic subspace for U and Γ is a strict contraction, then T defined by (1-2) is a c.n.u. contraction.*

Proof. Since Γ is a strict contraction, we get that $\mathbf{B}\Gamma\mathbf{B}^*U|_{U^*\mathfrak{R}}$ is also a strict contraction. Therefore (1-3) implies

$$\begin{aligned}\|Tx\| &= \|x\| &\iff x &\perp U^{-1}\mathfrak{R}, \\ \|T^*x\| &= \|x\| &\iff x &\perp \mathfrak{R}.\end{aligned}$$

Moreover, we can see from (1-3) that if $x \perp U^{-1}\mathfrak{R}$ then $Tx = Uf$ and if $x \perp \mathfrak{R}$ then $T^*x = U^{-1}x$.

Consider a reducing subspace G for T such that $T|_G$ is unitary. Then the above observations imply $G \perp \mathfrak{R}$ and $G \perp U^{-1}\mathfrak{R}$, and that for any $x \in G$

$$T^n x = U^n x \quad \text{as well as} \quad (T^*)^n x = U^{-n} x.$$

Since G is a reducing subspace for T it follows that $U^k x \in G$ for all integers k . But this implies that $U^n x \perp \mathfrak{R}$, or equivalently $x \perp U^n \mathfrak{R}$ for all $n \in \mathbb{Z}$. But $\mathfrak{R}$ is a star-cyclic subspace for U , so we get a contradiction. $\square$

Proof of Lemma 1.2. Assume now that for unitary Γ , the subspace $\text{Ran } \mathbf{B}$ is not a star-cyclic subspace for $U_\Gamma = T_\Gamma$ (but is a star-cyclic subspace for U). Consider the perturbation T_0 ,

$$T_0 = U + \mathbf{B}(\mathbf{0} - I_{\mathfrak{D}})\mathbf{B}^*U.$$

We will show that

$$T_0 = U_\Gamma + \mathbf{B}(\mathbf{0} - I_{\mathfrak{D}})\mathbf{B}^*U_\Gamma. \quad (1-6)$$

By Lemma 1.4 the operator T_0 is a c.n.u. contraction.

But, as we discussed in the beginning of this subsection, if $\text{Ran } \mathbf{B}$ is not star-cyclic for U , then for $\mathcal{E}$ defined by (1-5) the subspace $\mathcal{E}^\perp$ is a reducing subspace for T_Γ (with any Γ) on which T_Γ acts unitarily.

Since by (1-6) the operator T_0 is a perturbation of form (1-2) of the unitary operator T_Γ , we conclude that the operator T_0 has a nontrivial unitary part, and arrive at a contradiction.

To prove (1-6) we notice that

$$T_0 = U - \mathbf{B}\mathbf{B}^*U = U_\Gamma - \mathbf{B}\Gamma\mathbf{B}^*U. \quad (1-7)$$

Direct computations show that

$$U_\Gamma U^* \mathbf{B} = U U^* \mathbf{B} + \mathbf{B}(\Gamma - I_{\mathfrak{D}})\mathbf{B}^* U U^* \mathbf{B} = \mathbf{B} + \mathbf{B}(\Gamma - I_{\mathfrak{D}}) = \mathbf{B}\Gamma.$$

Taking the adjoint of this identity we get $\mathbf{B}^* U U_\Gamma^* = \Gamma^* \mathbf{B}^*$, and so $\Gamma \mathbf{B}^* U = \mathbf{B}^* U_\Gamma$. Substituting $\mathbf{B}^* U_\Gamma$ instead of $\Gamma \mathbf{B}^* U$ in (1-7) we get (1-6). $\square$

1C. Characterization of star-cyclic subspaces. Recall that for an isometry $\mathbf{B} : \mathcal{D} \rightarrow \mathcal{H}$, where $\mathcal{H}$ is the direct integral (1-4), we denoted by $b_k \in \mathcal{H}$ the “columns” of $\mathbf{B}$,

$$b_k = \mathbf{B}e_k,$$

where $e_1, e_2, \dots, e_d$ is the standard basis in $\mathbb{C}^d$.

Lemma 1.5. *Let U be the multiplication M_ξ by the independent variable ξ in the direct integral $\mathcal{H}$ given by (1-4), and let $\mathbf{B} : \mathbb{C}^d \rightarrow \mathcal{H}$ be as above. The space $\text{Ran } \mathbf{B} = \text{span}\{b_k : 1 \leq k \leq d\}$ is star-cyclic for U if and only if $\overline{\text{span}}\{b_k(\xi) : 1 \leq k \leq d\} = E(\xi)$ for μ -a.e. ξ .*

Proof. First assume that $\text{Ran } \mathbf{B}$ is not a star-cyclic subspace for U . Then there exists $f \in \mathcal{H} \subset L^2(\mu; E)$, $f \neq 0$ μ -a.e., such that

$$U^l f \perp b_k \quad \text{for all } l \in \mathbb{Z} \text{ and } k = 1, \dots, d,$$

or, equivalently

$$\int_{\mathbb{T}} (f(\xi), b_k(\xi))_E \xi^l d\mu(\xi) = 0 \quad \text{for all } l \in \mathbb{Z} \text{ and } k = 1, \dots, d.$$

But that means for all $k = 1, 2, \dots, d$ we have

$$(f(\xi), b_k(\xi))_E = 0 \quad \mu\text{-a.e.},$$

so on some set of positive μ -measure (where $f(\xi) \neq 0$) we have

$$\overline{\text{span}}\{b_k(\xi) : 1 \leq k \leq d\} \subsetneq E(\xi). \quad (1-8)$$

Vice versa, assume that (1-8) holds on some Borel subset $A \subset \mathbb{T}$ with $\mu(A) > 0$. For $n = 1, 2, \dots, \infty$ define sets $A_n := \{\xi \in A : \dim E(\xi) = n\}$. Then $\mu(A_n) > 0$ for some n . Fix this n and denote the corresponding space $E(\xi)$, $\xi \in A_n$, by E_n .

We know that $\overline{\text{span}}\{b_k(\xi) : 1 \leq k \leq d\} \subsetneq E_n$ on A_n , so there exists $e \in E_n$ such that

$$e \notin \overline{\text{span}}\{b_k(\xi) : 1 \leq k \leq d\}$$

on a set of positive measure in A_n .

Trivially, if $f \in \overline{\text{span}}\{U^k \text{Ran } \mathbf{B} : k \in \mathbb{Z}\}$ then

$$f(\xi) \in \overline{\text{span}}\{b_k(\xi) : 1 \leq k \leq d\} \quad \mu\text{-a.e.},$$

and therefore $f = \mathbf{1}_{A_n} e$ is not in $\overline{\text{span}}\{U^k \text{Ran } \mathbf{B} : k \in \mathbb{Z}\}$. □

1D. The case of star-cyclic U . If U is star-cyclic (i.e., it has a one-dimensional star-cyclic subspace/vector), U is unitarily equivalent to the multiplication operator M_ξ in the scalar space $L^2(\mu)$; of course the scalar space $L^2(\mu)$ is a particular case of the direct integral, where all spaces $E(\xi)$ are one-dimensional.

Lemma 1.5 says that $\text{Ran } \mathbf{B}$ is star-cyclic for U if and only if there is no measurable set A , $\mu(A) > 0$, on which all the functions b_k vanish. If we consider the case when U is star-cyclic, i.e., when it has a star-cyclic vector, we can ask the question:

Does a star-cyclic operator U have a star-cyclic vector that belongs to a prescribed (finite-dimensional) star-cyclic subspace?

The following lemma answers “yes” to that question. Moreover, it implies that if $\text{Ran } \mathbf{B}$ is star-cyclic for $U = M_\xi$ on the scalar-valued space $L^2(\mu)$, then almost all vectors $b \in \text{Ran } \mathbf{B}$ are star-cyclic for U . As the result is measure-theoretic in nature, we formulate it in a general context.

Lemma 1.6. Consider a σ -finite scalar-valued measure τ on a measure space $\mathcal{X}$. Let $b_1, b_2, \dots, b_d \in L^2(\tau)$ be such that

$$\sum_{k=1}^d |b_k| \neq 0 \quad \tau\text{-a.e.}$$

Then for almost all (with respect to the Lebesgue measure) $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{C}^d$ we have

$$\sum_{k=1}^d \alpha_k b_k \neq 0 \quad \tau\text{-a.e. on } \mathcal{X}.$$

Remark. The above lemma also holds for almost all $\alpha \in \mathbb{R}^d$.

Proof of Lemma 1.6. Consider first the case $\tau(\mathcal{X}) < \infty$.

We proceed by induction in d . Clearly, if $|b_1| \neq 0$ τ -a.e. on $\mathcal{X}$, then $\alpha b_1 \neq 0$ τ -a.e. on $\mathcal{X}$ for all $\alpha \in \mathbb{C} \setminus \{0\}$.

Now assume the statement of the lemma for $d = n$ for some $n \in \mathbb{N}$. Deleting a set of τ -measure 0, we can assume that $\sum_{k=1}^{n+1} |b_k| \neq 0$ on $\mathcal{X}$.

Let $\mathcal{Y} := \{x \in \mathcal{X} : \sum_{k=1}^n |b_k(x)| > 0\}$. By the induction assumption, for almost all $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_n)$

$$b(\alpha', x) := \sum_{k=1}^n \alpha_k b_k(x) \neq 0 \quad \text{on } \mathcal{Y}.$$

Fix $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $b(\alpha', x) \neq 0$ on $\mathcal{Y}$. We will show that for any such fixed α'

$$\tau\left(\left\{x \in \mathcal{X} : \sum_{k=1}^{n+1} \alpha_k b_k(x) = 0\right\}\right) > 0 \quad (1-9)$$

only for countably many values of α_{n+1} .

To show this, define for $\beta = \alpha_{n+1} \in \mathbb{C}$ the set

$$\mathcal{X}_\beta := \{x \in \mathcal{X} : b(\alpha', x) + \beta b_{n+1}(x) = 0\}.$$

Let $\tilde{\beta} \in \mathbb{C} \setminus \{0\}$, $\tilde{\beta} \neq \beta$. We claim that the sets $\mathcal{X}_\beta$ and $\mathcal{X}_{\tilde{\beta}}$ are disjoint.

Indeed, the assumption that $\sum_{k=1}^{n+1} |b_k| > 0$ implies $b_{n+1} \neq 0$ on $\mathcal{X} \setminus \mathcal{Y}$, so $\mathcal{X}_\beta, \mathcal{X}_{\tilde{\beta}} \in \mathcal{Y}$. Moreover, solving for b_{n+1} we get that if $\beta \neq 0$, then

$$\mathcal{X}_\beta = \{x \in \mathcal{Y} : b_{n+1}(x) = -b(\alpha', x)/\beta\},$$

and similarly for $\mathcal{X}_{\tilde{\beta}}$. Since $b(\alpha', x) \neq 0$ on $\mathcal{Y}$, we get

$$b(\alpha', x)/\beta \neq b(\alpha', x)/\tilde{\beta} \quad \text{for all } x \in \mathcal{Y},$$

so if $\beta \neq 0$, then $\mathcal{X}_\beta$ and $\mathcal{X}_{\tilde{\beta}}$ are disjoint as preimages of disjoint sets (points).

If $\beta = 0$, then $\mathcal{X}_0 = \mathcal{X} \setminus \mathcal{Y}$, so the sets $\mathcal{X}_{\tilde{\beta}}$ and $\mathcal{X}_0$ are disjoint.

The set $\mathcal{X}$ has finite measure, and $\mathcal{X}$ is the union of disjoint sets $\mathcal{X}_\beta$, $\beta \in \mathbb{C}$. So, only countably many sets $\mathcal{X}_\beta$ can satisfy $\tau(\mathcal{X}_\beta) > 0$. We have proved the lemma for $\tau(\mathcal{X}) < \infty$.

The rest can be obtained by Tonelli's theorem. Namely, define

$$\mathcal{A} := \left\{ (x, \alpha) : x \in \mathcal{X}, \alpha \in \mathbb{C}^{n+1}, \sum_{k=1}^{n+1} \alpha_k b_k(x) = 0 \right\}$$

and let $F = \mathbf{1}_{\mathcal{A}}$. From Tonelli's theorem we can see that

$$\int \mathbf{1}_{\mathcal{A}}(x, \alpha) dm(\alpha) d\tau(x) > 0 \quad (1-10)$$

if and only if for the set of $\alpha \in \mathbb{C}^{n+1}$ of positive Lebesgue measure

$$\tau \left(\left\{ x \in \mathcal{X} : \sum_{k=1}^{n+1} \alpha_k b_k(x) = 0 \right\} \right) > 0.$$

It follows from (1-9) that for almost all $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$

$$\int \mathbf{1}_{\mathcal{A}}(x, \alpha', \alpha_{n+1}) dm(\alpha_{n+1}) d\tau(x) = 0,$$

so, by Tonelli, the integral in (1-10) equals 0. □

2. Abstract formula for the adjoint Clark operator

We now introduce necessary known facts about functional models and then give a general abstract formula for the adjoint Clark operator. To do this we need a new notion of coordinate/parametrizing operators for the model and their agreement: the abstract representation formula (Theorem 2.4) holds under the assumption that the coordinate operators C and C_* agree with the Clark model.

Later in Section 3 we construct the coordinate operators that agree with the Clark model, and in Section 4 we compute the characteristic function, so the abstract Theorem 2.4 will give us concrete, albeit complicated, formulas.

2A. Functional models.

Definition 2.1. Recall that for a contraction T its *defect operators* D_T and D_{T^*} are defined as

$$D_T := (I - T^*T)^{1/2}, \quad D_{T^*} := (I - TT^*)^{1/2}.$$

The *defect spaces* $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$ are defined as

$$\mathfrak{D}_T := \text{clos Ran } D_T, \quad \mathfrak{D}_{T^*} := \text{clos Ran } D_{T^*}.$$

The characteristic function is an (explicitly computed from the contraction T) operator-valued function $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$, where $\mathfrak{D}$ and $\mathfrak{D}_*$ are Hilbert spaces of appropriate dimensions,

$$\dim \mathfrak{D} = \dim \mathfrak{D}_T, \quad \dim \mathfrak{D}_* = \dim \mathfrak{D}_{T^*}.$$

Using the characteristic function θ one can then construct the so-called *model space* $\mathcal{K}_\theta$, which is a subspace of a weighted L^2 space $L^2(\mathbb{T}, W; \mathfrak{D}_* \oplus \mathfrak{D}) = L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$ with an operator-valued

weight W . The model operator $\mathcal{M}_\theta : \mathcal{K}_\theta \rightarrow \mathcal{K}_\theta$ is then defined as the *compression* of the multiplication M_z by the independent variable z ,

$$\mathcal{M}_\theta f = P_{\mathcal{K}_\theta} M_z f, \quad f \in \mathcal{K}_\theta;$$

here $M_z f(z) = zf(z)$.

Let us remind the reader that the norm in the weighted space $L^2(\mathbb{T}, W; H)$ with an operator weight W is given by

$$\|f\|_{L^2(W; H)}^2 = \int_{\mathbb{T}} (W(z)f(z), f(z))_H dm(z);$$

in the case $\dim H = \infty$ there are some technical details, but in the finite-dimensional case considered in this paper everything is pretty straightforward.

The best-known example of a model is the Sz.-Nagy–Foiş (transcription of a) model [Sz.-Nagy et al. 2010]. The Sz.-Nagy–Foiş model space $\mathcal{K}_\theta$ is a subspace of a nonweighted space $L^2(\mathfrak{D}_* \oplus \mathfrak{D})$ (here $W \equiv I$), given by

$$\mathcal{K}_\theta := \begin{pmatrix} H^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2(\mathfrak{D}),$$

where

$$\Delta(z) := (I_{\mathfrak{D}} - \theta^*(z)\theta(z))^{1/2} \quad \text{and} \quad \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2(\mathfrak{D}) = \left\{ \begin{pmatrix} \theta f \\ \Delta f \end{pmatrix} : f \in H^2(\mathfrak{D}) \right\}.$$

In the literature, the case when the vector-valued characteristic function θ is *inner* (i.e., its boundary values are isometries for a.e. $z \in \mathbb{T}$) is often considered. Then $\Delta(z) = \mathbf{0}$ on $\mathbb{T}$, so in that case the second component of $\mathcal{K}_\theta$ collapses completely and the Sz.-Nagy–Foiş model space reduces to the familiar space

$$\mathcal{K}_\theta = H^2(\mathfrak{D}_*) \ominus \theta H^2(\mathfrak{D}).$$

Also, in the literature, see [Sz.-Nagy et al. 2010], the characteristic function is defined up to multiplication by constant unitary factors from the right and from the left. Namely, two functions $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$ and $\tilde{\theta} \in H^\infty(\tilde{\mathfrak{D}} \rightarrow \tilde{\mathfrak{D}}_*)$ are equivalent if there exist unitary operators $U : \mathfrak{D} \rightarrow \tilde{\mathfrak{D}}$ and $U_* : \mathfrak{D}_* \rightarrow \tilde{\mathfrak{D}}_*$ such that $\tilde{\theta} = U_* \theta U^*$.

It is a well-known fact, see [Sz.-Nagy et al. 2010], that two c.n.u. contractions are unitarily equivalent if and only if their characteristic functions are equivalent as described above. So, usually in the literature the characteristic function is understood as the corresponding equivalence class, or an arbitrary representative in this class. However, in this paper, to get correct formulas it is essential to track which representative is chosen.

2B. Coordinate operators, parametrizing operators, and their agreement. Let $T : H \rightarrow H$ be a contraction, and let $\mathfrak{D}$, $\mathfrak{D}_*$ be Hilbert spaces, $\dim \mathfrak{D} = \dim \mathfrak{D}_T$, $\dim \mathfrak{D}_* = \dim \mathfrak{D}_{T^*}$. Unitary operators $V : \mathfrak{D}_T \rightarrow \mathfrak{D}$ and $V_* : \mathfrak{D}_{T^*} \rightarrow \mathfrak{D}_*$ will be called *coordinate operators* for the corresponding defect spaces; the reason for that name is that often spaces $\mathfrak{D}$ and $\mathfrak{D}_*$ are spaces with a fixed orthonormal basis (and one can introduce coordinates there), so the operators introduce coordinates on the defect spaces.

The inverse operators $V^* : \mathfrak{D} \rightarrow \mathfrak{D}_T$ and $V_*^* : \mathfrak{D}_* \rightarrow \mathfrak{D}_{T^*}$ will be called *parametrizing* operators. For a contraction T we will use the symbols V and V_* for the coordinate operators, but for its model $\mathcal{M}_\theta$ the parametrizing operators will be used, and we reserve letters C and C_* for these operators.

Let T be a c.n.u. contraction with characteristic function $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$, and let $\mathcal{M}_\theta : \mathcal{K}_\theta \rightarrow \mathcal{K}_\theta$ be its model. Let also $V : \mathfrak{D}_T \rightarrow \mathfrak{D}$ and $V_* : \mathfrak{D}_{T^*} \rightarrow \mathfrak{D}_*$ be coordinate operators for the defect spaces of T , and $C : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$ and $C_* : \mathfrak{D}_* \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ be the parametrizing operators for the defect spaces of $\mathcal{M}_\theta$ (this simply means that all four operators are unitary).

We say that the operators V, V_* agree with operators C, C_* if there exists a unitary operator $\Phi : \mathcal{K}_\theta \rightarrow \mathcal{H}$ intertwining T and $\mathcal{M}_\theta$,

$$T\Phi = \Phi\mathcal{M}_\theta,$$

and such that

$$C^* = V\Phi|_{\mathfrak{D}_{\mathcal{M}_\theta}}, \quad C_*^* = V_*\Phi|_{\mathfrak{D}_{\mathcal{M}_\theta^*}}. \quad (2-1)$$

The above identities simply mean that the diagrams below are commutative:

$$\begin{array}{ccc} \mathfrak{D}_T & \xrightarrow{V} & \mathfrak{D} \\ & \searrow \Phi & \nearrow C^* \\ & \mathfrak{D}_{\mathcal{M}_\theta} & \end{array} \quad \begin{array}{ccc} \mathfrak{D}_* & \xrightarrow{V_*^*} & \mathfrak{D}_{T^*} \\ & \searrow C_*^* & \nearrow \Phi \\ & \mathfrak{D}_{\mathcal{M}_\theta^*} & \end{array}$$

In this paper, when convenient, we always extend an operator between subspaces to the operator between the whole spaces, by extending it by 0 on the orthogonal complement of the domain; slightly abusing notation we will use the same symbol for both operators. Thus a unitary operator between subspaces E and F can be treated as a partial isometry with initial space E and final space F , and vice versa. With this convention (2-1) can be rewritten as

$$C^* = V\Phi, \quad C_*^* = V_*\Phi.$$

2C. Clark operator. Consider a contraction T given by (1-2) with Γ being a strict contraction. We also assume that $\text{Ran } B$ is a star-cyclic subspace for U , so T is a c.n.u. contraction; see Lemma 1.4.

We assume that U is given in its spectral representation, so U is the multiplication operator M_ξ in the direct integral $\mathcal{H}$.

A Clark operator $\Phi : \mathcal{K}_\theta \rightarrow \mathcal{H}$ is a unitary operator, intertwining this special contraction T and its model $\mathcal{M}_\theta$, $\Phi\mathcal{M}_\theta = T\Phi$, or equivalently

$$\Phi^*T = \mathcal{M}_\theta\Phi^*. \quad (2-2)$$

We name it so after D. Clark, who in [Clark 1972] described it for rank-one perturbations of unitary operators with purely singular spectrum.

We want to describe the operator Φ (more precisely, its adjoint Φ^*) in our situation. In our case, $\dim \mathfrak{D}_T = \dim \mathfrak{D}_{T^*} = d$, and it will be convenient for us to consider models with $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$.

As discussed above, it can be easily seen from the representation (1-3) that the operators $U^*B : \mathfrak{D} = \mathbb{C}^d \rightarrow \mathfrak{D}_T$ and $B : \mathfrak{D} = \mathbb{C}^d \rightarrow \mathfrak{D}_{T^*}$ are unitary operators canonically (for our setup) identifying $\mathfrak{D}$ with the corresponding defect spaces, i.e., the canonical parametrizing operators for these spaces. The corresponding coordinate operators are given by $V = B^*U$, $V_* = B^*$.

We say that parametrizing operators $C : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$, $C_* : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ agree with the Clark model if the above coordinate operators $V = B^*U$, $V_* = B^*$ agree with the parametrizing operators C , C_* in the sense of Section 2B. In other words, they agree if there exists a Clark operator Φ such that the following diagram commutes:

$$\begin{array}{ccccc}
 \mathfrak{D}_T & \xrightarrow{B^*U} & \mathfrak{D} = \mathbb{C}^d & \xrightarrow{B} & \mathfrak{D}_{T^*} \\
 \searrow \Phi^* & & \swarrow C & & \swarrow C_* \\
 & & \mathfrak{D}_{\mathcal{M}_\theta} & & \mathfrak{D}_{\mathcal{M}_\theta^*} \\
 & & \nwarrow \Phi^* & & \nwarrow \Phi^*
 \end{array} \tag{2-3}$$

Note, that in this diagram one can travel in both directions: to change the direction, one just needs to take the adjoint of the corresponding operator.

Slightly abusing notation, we use C to also denote the extension of C to the model space $\mathcal{K}_\theta$ by the zero operator, and similarly for C_* .

Note that agreement of C and C_* with the Clark model can be rewritten as

$$\Phi^*(B^*U)^* = C, \quad \Phi^*B = C_*. \tag{2-4}$$

And by taking restrictions (where necessary) we find

$$\mathcal{M}_\theta C = C_* \Gamma \quad \text{and} \quad \mathcal{M}_\theta^* C_* = C \Gamma^*. \tag{2-5}$$

We express the action of the model operator and its adjoint in an auxiliary result. The result holds in any transcription of the model. We will need the following simple fact.

Lemma 2.2. *For a contraction T*

$$T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}, \quad T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T.$$

Proof. Since D_T is a strict contraction on $\mathfrak{D}_T$ we get

$$\|Tx\| = \|x\| \iff x \perp \mathfrak{D}_T,$$

and similarly, since T^* is a strict contraction on $\mathfrak{D}_{T^*}$,

$$\|T^*x\| = \|x\| \iff x \perp \mathfrak{D}_{T^*}. \tag{2-6}$$

Thus the operator T is an isometry on $\mathfrak{D}_T^\perp$, so the polarization identity implies $T^*Tx = x$ for all $x \in \mathfrak{D}_T^\perp$. Together with (2-6) this implies $T(\mathfrak{D}_T^\perp) \subset \mathfrak{D}_{T^*}^\perp$, which is equivalent to the inclusion $T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$.

Replacing T by T^* we get $T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$. □

Lemma 2.3. *Let T be as defined in (1-2) with Γ being a strict contraction. Assume also that $\text{Ran } \mathbf{B}$ is star-cyclic (so T is completely nonunitary; see Lemma 1.4).*

Let $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_)$, $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$, be the characteristic function of T , and let $\mathcal{M}_\theta : \mathcal{K}_\theta \rightarrow \mathcal{K}_\theta$ be a model operator. Let $\mathbf{C} : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$ and $\mathbf{C}_* : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ be the parametrizing unitary operators that agree with the Clark model.*

Then

$$\mathcal{M}_\theta = M_z + (\mathbf{C}_* \Gamma - M_z \mathbf{C}) \mathbf{C}^* \quad \text{and} \quad \mathcal{M}_\theta^* = M_{\bar{z}} + (\mathbf{C} \Gamma^* - M_{\bar{z}} \mathbf{C}_*) \mathbf{C}_*^*.$$

Proof. Since the operator $\mathcal{M}_\theta$ acts on $\mathcal{K}_\theta \ominus \mathfrak{D}_{\mathcal{M}_\theta}$ as the multiplication operator M_z , we can trivially write

$$\mathcal{M}_\theta = M_z(\mathbf{I} - P_{\mathfrak{D}_{\mathcal{M}_\theta}}) + \mathcal{M}_\theta P_{\mathfrak{D}_{\mathcal{M}_\theta}}.$$

Recalling that $\mathbf{C} : \mathfrak{D} \rightarrow \mathcal{K}_\theta$ is an isometry with range $\mathfrak{D}_{\mathcal{M}_\theta}$, we can see that $P_{\mathfrak{D}_{\mathcal{M}_\theta}} = \mathbf{C} \mathbf{C}^*$, so

$$M_z(\mathbf{I} - P_{\mathfrak{D}_{\mathcal{M}_\theta}}) = M_z(\mathbf{I} - \mathbf{C} \mathbf{C}^*). \quad (2-7)$$

Using the identity $P_{\mathfrak{D}_{\mathcal{M}_\theta}} = \mathbf{C} \mathbf{C}^*$ and the first equation of (2-5) we get

$$\mathcal{M}_\theta P_{\mathfrak{D}_{\mathcal{M}_\theta}} = \mathcal{M}_\theta \mathbf{C} \mathbf{C}^* = \mathbf{C}_* \Gamma \mathbf{C}^*,$$

which together with (2-7) gives us the desired formula for $\mathcal{M}_\theta$.

To get the formula for $\mathcal{M}_\theta^*$ we represent it as

$$\mathcal{M}_\theta^* = M_{\bar{z}}(\mathbf{I} - P_{\mathfrak{D}_{\mathcal{M}_\theta^*}}) + \mathcal{M}_\theta^* P_{\mathfrak{D}_{\mathcal{M}_\theta^*}}.$$

Using the identities

$$P_{\mathfrak{D}_{\mathcal{M}_\theta^*}} = \mathbf{C}_* \mathbf{C}_*^*, \quad \mathcal{M}_\theta^* P_{\mathfrak{D}_{\mathcal{M}_\theta^*}} = \mathbf{C} \Gamma^* \mathbf{C}_*^*$$

(the first holds because $\mathfrak{D}_{\mathcal{M}_\theta^*}$ is the range of the isometry $\mathbf{C}_*$, and the second one follows from the second equation in (2-5)), we get the desired formula. $\square$

2D. Representation theorem. For a (general) model operator $\mathcal{M}_\theta$, $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$, the parametrizing operators $\mathbf{C} : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$, $\mathbf{C}_* : \mathfrak{D}_* \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ give rise to (uniquely defined) operator-valued functions $\mathbf{C}$ and $\mathbf{C}_*$ on $\mathbb{T}$, where $\mathbf{C}(\xi) : \mathfrak{D} \rightarrow \mathfrak{D} \oplus \mathfrak{D}_*$, $\mathbf{C}_*(\xi) : \mathfrak{D}_* \rightarrow \mathfrak{D} \oplus \mathfrak{D}_*$, $\xi \in \mathbb{T}$, such that for almost all $\xi \in \mathbb{T}$

$$(\mathbf{C}e)(\xi) = \mathbf{C}(\xi)e \quad \text{for all } e \in \mathfrak{D}, \quad (2-8)$$

$$(\mathbf{C}_*e_*)(\xi) = \mathbf{C}_*(\xi)e_* \quad \text{for all } e_* \in \mathfrak{D}_*; \quad (2-9)$$

here $\mathbf{C}e$, $\mathbf{C}_*e_*$ are elements of $\mathcal{K}_\theta$, i.e., functions with values in $\mathfrak{D} \oplus \mathfrak{D}_*$, and $(\mathbf{C}e)(\xi)$, $(\mathbf{C}_*e_*)(\xi)$ are the values of these functions at $\xi \in \mathbb{T}$.

If we fix orthonormal bases in $\mathfrak{D}$ and $\mathfrak{D}_*$, then the k -th column of the matrix of $\mathbf{C}(\xi)$ is defined as $(\mathbf{C}_*e_k)(\xi)$, where e_k is the k -th vector in the orthonormal basis in $\mathfrak{D}$, and similarly for $\mathbf{C}_*$.

If $\mathcal{M}_\theta$ is a model for a contraction $T = T_\Gamma$ with Γ being a strict contraction on $\mathfrak{D} = \mathbb{C}^d$, we can see from (1-3) that $\dim \mathfrak{D}_T = \dim \mathfrak{D}_{T^*} = d$, so we can always pick a characteristic function $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$ (i.e., with $\mathfrak{D}_* = \mathfrak{D} = \mathbb{C}^d$).

The following formula for the adjoint Φ^* of the Clark operator Φ generalizes the “universal” representation theorem [Liaw and Treil 2016, Theorem 3.1] to higher-rank perturbations.

Theorem 2.4 (representation theorem). *Let T be as defined in (1-2) with Γ being a strict contraction and $U = M_\xi$ in $\mathcal{H} \subset L^2(\mu; E)$. Let $\theta = \theta_T$ be a characteristic function of T , and let $\mathcal{K}_\theta$ and $\mathcal{M}_\theta$ be the corresponding model space and model operator.*

Let $C : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$ and $C_ : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ be the parametrizing unitary operators¹ that agree with Clark model, i.e., such that (2-4) is satisfied for some Clark operator Φ . And let $C(z)$ and $C_*(z)$ be given by (2-8) and (2-9), respectively.*

Then the action of the adjoint Clark operator Φ^ is given by*

$$(\Phi^*hb)(z) = h(z)C_*(z)\mathbf{B}^*b + (C_*(z) - zC(z)) \int_{\mathbb{T}} \frac{h(\xi) - h(z)}{1 - z\bar{\xi}} B^*(\xi)b(\xi) d\mu(\xi) \quad (2-10)$$

for any $b \in \text{Ran } \mathbf{B}$ and for all $h \in C^1(\mathbb{T})$; here

$$B^*(\xi) = \begin{pmatrix} b_1(\xi)^* \\ b_2(\xi)^* \\ \vdots \\ b_d(\xi)^* \end{pmatrix}$$

*and $\mathbf{B}^*b = \int_{\mathbb{T}} B^*(\xi)b(\xi) d\mu(\xi)$, as explained more thoroughly in the proof below.*

Remark. The above theorem looks like abstract nonsense because right now it is not clear how to find the parametrizing operators C and C_* that agree with the Clark model. However, Theorem 4.2 below gives an explicit formula for the characteristic function θ (one of the representative in the equivalence class), and Lemma 3.3 gives explicit formulas for C and C_* in the Sz.-Nagy–Foiş transcription that agree with the Clark model for our θ .

When $d = 1$ this formula agrees with the special case of the representation formula derived in [Liaw and Treil 2016]. While some of the ideas of the following proof were originally developed there, the current extension to rank- d perturbations requires several new ideas and a more abstract way of thinking.

Proof of Theorem 2.4. Recall that $U = M_\xi$, so $T = M_\xi + \mathbf{B}(\Gamma - I_{\mathbb{C}^d})\mathbf{B}^*M_\xi$. The intertwining relation $\Phi^*T = \mathcal{M}_\theta\Phi^*$ then can be rewritten as

$$\Phi^*M_\xi + \Phi^*\mathbf{B}(\Gamma - I_{\mathbb{C}^d})\mathbf{B}^*U = \Phi^*T = \mathcal{M}_\theta\Phi^* = [M_z + (C_*\Gamma - M_zC)C^*]\Phi^*; \quad (2-11)$$

here we used Lemma 2.3 to express the model operator in the right-hand side of (2-11).

By the commutation relations in (2-4), the term $\Phi^*\mathbf{B}\Gamma\mathbf{B}^*U$ on the left-hand side of (2-11) cancels with the term $C_*\Gamma C^*\Phi^*$ on the right-hand side of (2-11). Then (2-11) can be rewritten as

$$\begin{aligned} \Phi^*M_\xi &= M_z\Phi^* + \Phi^*\mathbf{B}I_{\mathbb{C}^d}\mathbf{B}^*U - M_zCC^*\Phi^* \\ &= M_z\Phi^* + (C_* - M_zC)\mathbf{B}^*M_\xi; \end{aligned} \quad (2-12)$$

the last identity holds because, by (2-4), we have $\Phi^*\mathbf{B} = C_*$ and $C^*\Phi^* = \mathbf{B}^*U = \mathbf{B}^*M_\xi$.

¹Note that here we set $\mathfrak{D}_* = \mathfrak{D}$, which is possible because the dimensions of the defect spaces are equal.

Right-multiplying (2-12) by M_ξ and using (2-12) we get

$$\begin{aligned}\Phi^* M_\xi^2 &= M_z \Phi^* M_\xi + (C_* - M_z C) \mathbf{B}^* M_\xi^2 \\ &= M_z^2 \Phi^* + M_z (C_* - M_z C) \mathbf{B}^* M_\xi + (C_* - M_z C) \mathbf{B}^* M_\xi^2.\end{aligned}$$

Right-multiplying the above equation by M_ξ and using (2-12) again we get the identity

$$\Phi^* M_\xi^n = M_z^n \Phi^* + \sum_{k=1}^n M_z^{k-1} (C_* - M_z C) \mathbf{B}^* M_\xi^{n-k+1}, \quad (2-13)$$

with $n = 3$. Right-multiplying by M_ξ and applying (2-12) we get by induction that (2-13) holds for all $n \geq 0$. (The case $n = 0$ trivially reads as $\Phi^* = \Phi^*$, and (2-12) is precisely the case $n = 1$.)

We now apply (2-13) to some $b \in \text{Ran } \mathbf{B}$. By commutative diagram (2-3) we get $\Phi^* b = C_* \mathbf{B}^* b$, i.e., $(\Phi^* b)(z) = C_*(z) \mathbf{B}^* b$. Using this identity we get

$$\begin{aligned}(\Phi^* M_\xi^n b)(z) &= z^n (\Phi^* b)(z) + \sum_{k=1}^n z^{k-1} (C_*(z) - zC(z)) \mathbf{B}^* M_\xi^{n-k+1} b \\ &= z^n C_*(z) (\mathbf{B}^* b)(z) + (C_*(z) - zC(z)) \sum_{k=1}^n z^{k-1} \mathbf{B}^* M_\xi^{n-k+1} b.\end{aligned} \quad (2-14)$$

To continue, we recall that $\mathbf{B} : \mathbb{C}^d \rightarrow L^2(\mu; E)$ acts as multiplication by the matrix $B(\xi) = (b_1(\xi), b_2(\xi), \dots, b_d(\xi))$, so its adjoint $\mathbf{B}^* : \mathcal{H} \subset L^2(\mu; E) \rightarrow \mathbb{C}^d$ is given by

$$\mathbf{B}^* f = \int_{\mathbb{T}} B^*(\xi) f(\xi) d\mu(\xi) \quad \text{for } f \in \mathcal{H},$$

where the integral can be expanded as

$$\int_{\mathbb{T}} B^*(\xi) f(\xi) d\mu(\xi) = \begin{pmatrix} \int_{\mathbb{T}} b_1(\xi)^* f(\xi) d\mu(\xi) \\ \int_{\mathbb{T}} b_2(\xi)^* f(\xi) d\mu(\xi) \\ \vdots \\ \int_{\mathbb{T}} b_d(\xi)^* f(\xi) d\mu(\xi) \end{pmatrix}.$$

Using the sum of geometric progression formula we evaluate the sum in (2-14) to

$$\begin{aligned}\sum_{k=1}^n z^{k-1} \mathbf{B}^* M_\xi^{n-k+1} b &= \sum_{k=1}^n z^{k-1} \int_{\mathbb{T}} \xi^{n-k+1} B^*(\xi) b(\xi) d\mu(\xi) \\ &= \int_{\mathbb{T}} \sum_{k=1}^n z^{k-1} \xi^{n-k+1} B^*(\xi) b(\xi) d\mu(\xi) \\ &= \int_{\mathbb{T}} \frac{\xi^n - z^n}{1 - z\bar{z}\xi} B^*(\xi) b(\xi) d\mu(\xi).\end{aligned} \quad (2-15)$$

Thus, we have proved (2-10) for monomials $h(\xi) = \xi^n$, $n \geq 0$. And by the linearity of Φ^* , the representation (2-10) holds for (analytic) polynomials h in ξ .

The argument leading to the determination of the action of Φ^* on polynomials h in $\bar{\xi}$ is similar. But we found that the devil is in the details and therefore decided to include much of the argument.

First observe that the intertwining relation (2-2) is equivalent to $\mathcal{M}_\theta^* \Phi^* = \Phi^* T^*$. Recalling $T^* = U^* + U^* \mathbf{B}(\Gamma^* - \mathbf{I}_{\mathbb{C}^d}) \mathbf{B}^*$ and the resolution of the adjoint model operator $\mathcal{M}_\theta^*$ (see second statement of Lemma 2.3), we obtain

$$M_{\bar{z}} \Phi^* + (C \Gamma^* - \bar{z} C_*) C_*^* \Phi^* = \mathcal{M}_\theta^* \Phi^* = \Phi^* T^* = \Phi^* U^* - \Phi^* U^* \mathbf{B}(\Gamma^* - \mathbf{I}_{\mathbb{C}^d}) \mathbf{B}^*.$$

The terms involving Γ^* on the left-hand side and the right-hand side cancel by the commutation relations in (2-4) (actually by their adjoints). Now, rearrangement and another application of the adjoints of the commutation relations in (2-4) yields

$$\begin{aligned} \Phi^* M_{\bar{\xi}} &= \Phi^* U^* = M_{\bar{z}} \Phi^* + \Phi^* U^* \mathbf{B} \mathbf{I}_{\mathbb{C}^d} \mathbf{B}^* - \bar{z} C_* C_*^* \Phi^* = M_{\bar{z}} \Phi^* + (C - M_{\bar{z}} C_*) \mathbf{B}^* \\ &= M_{\bar{z}} \Phi^* + M_{\bar{z}} (M_z C - C_*) \mathbf{B}^*. \end{aligned} \quad (2-16)$$

In analogy to the above, we right-multiply (2-16) by $M_{\bar{\xi}}$ and apply (2-16) twice to obtain

$$\Phi^* M_{\bar{\xi}}^2 = M_{\bar{z}}^2 \Phi^* + \sum_{k=1}^2 M_{\bar{z}}^k (M_z C - C_*) \mathbf{B}^* M_{\bar{\xi}}^{2-k}.$$

Inductively, we conclude

$$\Phi^* M_{\bar{\xi}}^n = M_{\bar{z}}^n \Phi^* - \sum_{k=1}^n M_{\bar{z}}^k (C_* - M_z C) \mathbf{B}^* M_{\bar{\xi}}^{n-k},$$

which differs in the exponents and in the sign from its counterpart expression in (2-13).

Through an application of this identity to b and by the commutative diagram (2-3), we see

$$\begin{aligned} (\Phi^* M_{\bar{\xi}}^n b)(z) &= \bar{z}^n (\Phi^* b)(z) - \sum_{k=1}^n \bar{z}^k (C_*(z) - z C(z)) \mathbf{B}^* M_{\bar{\xi}}^{n-k} b \\ &= \bar{z}^n C_*(z) (\mathbf{B}^* b)(z) - (C_*(z) - z C(z)) \sum_{k=1}^n \bar{z}^k \mathbf{B}^* M_{\bar{\xi}}^{n-k} b. \end{aligned}$$

As in (2-15), but here with the geometric progression

$$-\sum_{k=1}^n (\bar{z})^k (\bar{\xi})^{n-k} = \frac{(\bar{\xi})^n - (\bar{z})^n}{1 - \bar{\xi} \bar{z}},$$

we can see (2-10) for monomials $\bar{\xi}^n$, $n \in \mathbb{N}$. And by the linearity of Φ^* , we obtain the same formula (2-10) for functions h that are polynomials in $\bar{\xi}$.

We have proved (2-10) for trigonometric polynomials f . The theorem now follows by a standard approximation argument, developed in [Liaw and Treil 2009]. The application of this argument to the current situation is a slight extension of the one used in [Liaw and Treil 2016]. Fix $f \in C^1(\mathbb{T})$ and let $\{p_k\}$ be a sequence of trigonometric polynomials with uniform-on- $\mathbb{T}$ approximations $p_k \rightrightarrows f$ and $p'_k \rightrightarrows f'$. In particular, we have $|p'_k|$ is bounded (with bound independent of k) and $p_k \rightarrow f$ as well as

$p_k b \rightarrow f b$ in $L^2(\mu; E)$. Since Φ^* is a unitary operator, it is bounded and therefore we have convergence on the left-hand side $\Phi^* p_k b \rightarrow \Phi^* f b$ in $\mathcal{K}_\theta$.

To investigate convergence on the right-hand side, first recall that the model space is a subspace of the weighted space $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$.

So convergence of the first term on the right-hand side happens, since $p_k \rightrightarrows f$ and the operator norm $\|C_* B^*\| = 1$ implies $p_k C_*(z) B^* b = p_k C_* B^* b \rightarrow f C_* B^* b = f C_*(z) B^* b$ in $\mathcal{K}_\theta$.

Lastly, to see convergence of the second term on the right-hand side, consider auxiliary functions $f_k := f - p_k$. We have $f_k \rightrightarrows 0$ and $f'_k \rightrightarrows 0$. Let $I_{\xi, z} \subset \mathbb{T}$ denote the shortest arc connecting ξ and z . Then by the intermediate value theorem

$$|f_k(\xi) - f_k(z)| \leq \|f'_k\|_\infty |I_{\xi, z}| \quad \text{for all } \xi, z \in \mathbb{T}.$$

By virtue of the geometric estimate $|I_{\xi, z}| \leq \frac{\pi}{2} |\xi - z|$, we obtain

$$\left| \frac{f_k(\xi) - f_k(z)}{1 - \bar{\xi}z} \right| \leq \frac{\pi}{2} \|f'_k\|_\infty \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

And since B^* is bounded as a partial isometry, we conclude the componentwise uniform convergence

$$\int \frac{p_k(\xi) - p_k(z)}{1 - \bar{\xi}z} B^*(\xi) b(\xi) d\mu(\xi) \rightrightarrows \int \frac{f(\xi) - f(z)}{1 - \bar{\xi}z} B^*(\xi) b(\xi) d\mu(\xi), \quad z \in \mathbb{T}.$$

By Lemma 3.4 below, the functions $W^{1/2}C$ and $W^{1/2}C_*$ are bounded, and so is the function $W^{1/2}C_1$, $C_1(z) := C_*(z) - zC(z)$. That means the multiplication operator $f \mapsto C_1 f$ is a bounded operator from $L^2(\mathfrak{D})$ to $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$ (recall that in our case $\mathfrak{D} = \mathfrak{D}_*$ and we use $\mathfrak{D}_*$ here only for consistency with the general model notation).

The uniform convergence implies the convergence in $L^2(\mathfrak{D})$, so the boundedness of the multiplication by C_1 implies the convergence in norm in the second term in the right-hand side of (2-10) (in the norm of $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$). $\square$

3. Model and agreement of operators

We want to explain how to get operators C and C_* that agree with each other.

To do that we need to understand in more detail how the model is constructed, and what operator gives the unitary equivalence of the function and its model.

Everything starts with the notion of unitary dilation. Recall that for a contraction T in a Hilbert space H its unitary dilation is a unitary operator $\mathcal{U}$ on a bigger space $\mathcal{H}$, $H \subset \mathcal{H}$, such that for all $n \geq 0$

$$T^n = P_H \mathcal{U}^n|_H. \quad (3-1)$$

Taking the adjoint of this identity we immediately get

$$(T^*)^n = P_H \mathcal{U}^{-n}|_H. \quad (3-2)$$

A dilation is called *minimal* if it is impossible to replace $\mathcal{U}$ by its restriction to a reducing subspace and still have the identities (3-1) and (3-2).

The structure of minimal unitary dilations is well known.

Theorem 3.1 [Nikolski and Vasyunin 1998, Theorem 1.4; Nikolski 2002b, Theorem 1.1.16]. *Let $\mathcal{U} : \mathcal{H} \rightarrow \mathcal{H}$ be a minimal unitary dilation of a contraction T . Then $\mathcal{H}$ can be decomposed as $\mathcal{H} = G_* \oplus H \oplus G$, and with respect to this decomposition $\mathcal{U}$ can be represented as*

$$\mathcal{U} = \begin{pmatrix} \mathcal{E}_*^* & 0 & 0 \\ D_{T^*} V_*^* & T & 0 \\ -VT^* V_*^* & VD_T & \mathcal{E} \end{pmatrix}, \quad (3-3)$$

where $\mathcal{E} : G \rightarrow G$ and $\mathcal{E}_* : G_* \rightarrow G_*$ are pure isometries, V is a partial isometry with initial space $\mathfrak{D}_T$ and final space $\ker \mathcal{E}^*$ and V_* is a partial isometry with initial space $\mathfrak{D}_{T^*}$ and final space $\ker \mathcal{E}_*^*$.

Moreover, any minimal unitary dilation of T can be obtained this way. Namely if we pick auxiliary Hilbert spaces G and G_* and isometries $\mathcal{E}$ and $\mathcal{E}_*$ there with $\dim \ker \mathcal{E}^* = \dim \mathfrak{D}_T$, $\dim \ker \mathcal{E}_*^* = \dim \mathfrak{D}_{T^*}$ and then pick arbitrary partial isometries V and V_* with initial and final spaces as above, then (3-3) will give us a minimal unitary dilation of T .

The construction of the model then goes as follows. We take auxiliary Hilbert spaces $\mathfrak{D}$ and $\mathfrak{D}_*$, $\dim \mathfrak{D} = \dim \mathfrak{D}_T$, $\dim \mathfrak{D}_* = \dim \mathfrak{D}_{T^*}$, and construct operators $\mathcal{E}$ and $\mathcal{E}_*$ such that $\ker \mathcal{E}^* = \mathfrak{D}$, $\ker \mathcal{E}_*^* = \mathfrak{D}_*$. We can do that by putting $G = \ell^2(\mathfrak{D}) = \ell^2(\mathbb{Z}_+; \mathfrak{D})$, and defining

$$\mathcal{E}(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots), \quad x_k \in \mathfrak{D},$$

and similarly for $\mathcal{E}_*$.

Picking arbitrary partial isometries V and V_* with initial and final spaces as in (3-3), we get a minimal unitary dilation $\mathcal{U}$ of T given by (3-3).

Remark. Above, we were speaking a bit informally, identifying $x \in \mathfrak{D}$ with the sequence $(x, 0, 0, 0, \dots) \in \ell^2(\mathfrak{D})$, and $x_* \in \mathfrak{D}_*$ with $(x_*, 0, 0, 0, \dots) \in \ell^2(\mathfrak{D})$.

To be absolutely formal, we need to define canonical embeddings $e : \mathfrak{D} \rightarrow G = \ell^2(\mathfrak{D})$, $e_* : \mathfrak{D}_* \rightarrow G_* = \ell^2(\mathfrak{D}_*)$ with

$$e(x) := (x, 0, 0, 0, \dots), \quad x \in \mathfrak{D}, \quad (3-4)$$

$$e_*(x_*) := (x_*, 0, 0, 0, \dots), \quad x_* \in \mathfrak{D}_*. \quad (3-5)$$

Then, picking arbitrary unitary operators $V : \mathfrak{D}_T \rightarrow \mathfrak{D}$, $V_* : \mathfrak{D}_{T^*} \rightarrow \mathfrak{D}_*$, we rewrite (3-3) to define the corresponding unitary dilation as

$$\mathcal{U} = \begin{pmatrix} \mathcal{E}_*^* & 0 & 0 \\ D_{T^*} V_*^* e_*^* & T & 0 \\ -e V T^* V_*^* e_*^* & e V D_T & \mathcal{E} \end{pmatrix}. \quad (3-6)$$

The reason for being so formal is that if $\dim \mathfrak{D}_T = \dim \mathfrak{D}_{T^*}$ it is often convenient to put $\mathfrak{D} = \mathfrak{D}_*$, but we definitely want to be able to distinguish between the cases when $\mathfrak{D}$ is identified with $\ker \mathcal{E}$ and when it is identified with $\ker \mathcal{E}_*$.

We then define *functional embeddings* $\pi : L^2(\mathfrak{D}) \rightarrow \mathcal{H}$ and $\pi_* : L^2(\mathfrak{D}_*) \rightarrow \mathcal{H}$ by

$$\begin{aligned}\pi\left(\sum_{k \in \mathbb{Z}} z^k e_k\right) &= \sum_{k \in \mathbb{Z}} \mathcal{U}^k e(e_k), \quad e_k \in \mathfrak{D}, \\ \pi_*\left(\sum_{k \in \mathbb{Z}} z^k e_k\right) &= \sum_{k \in \mathbb{Z}} \mathcal{U}^{k+1} e_*(e_k), \quad e_k \in \mathfrak{D}_*.\end{aligned}$$

We refer the reader to [Nikolski and Vasyunin 1998, Section 1.6] or to [Nikolski 2002b, Section 1.2] for the details. Note that there $\mathfrak{D}$ and $\mathfrak{D}_*$ were abstract spaces, $\dim \mathfrak{D} = \dim \ker \mathcal{E}^*$ and $\dim \mathfrak{D}_* = \dim \ker \mathcal{E}_*^*$, and the unitary operators $v : \mathfrak{D} \rightarrow \ker \mathcal{E}^*$, $v_* : \mathfrak{D}_* \rightarrow \ker \mathcal{E}_*^*$ used in the formulas there are just the canonical embeddings e and e_* in our case.

Note that π and π_* are isometries.

Note also that for $k \geq 0$

$$\begin{aligned}\mathcal{U}^k e(e) &= \mathcal{E}^k e, \quad e \in \mathfrak{D}, \\ \mathcal{U}^{-k} e_*(e_*) &= \mathcal{E}_*^k e_*, \quad e_* \in \mathfrak{D}_*,\end{aligned}$$

so

$$\pi(H^2(\mathfrak{D})) = G, \quad \pi_*(H_-^2(\mathfrak{D}_*)) = G_*.$$

The characteristic function is then defined as follows. We consider the operator $\theta = \pi_*^* \pi : L^2(\mathfrak{D}) \rightarrow L^2(\mathfrak{D}_*)$. It is easy to check that $M_z \theta = \theta M_z$, so θ is multiplication by a function $\theta \in L^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$. It is not hard to check that θ is a contraction, so $\|\theta\|_\infty \leq 1$. Since

$$\pi(H^2(\mathfrak{D})) = G \perp G_* = \pi_*(H_-^2(\mathfrak{D}_*)),$$

we can conclude that $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$.

The characteristic function $\theta = \theta_T$ can be explicitly computed, see [Nikolski 2002b, Theorem 1.2.10],

$$\theta_T(z) = V_*(-T + z D T^* (I_{\mathcal{H}} - z T^*)^{-1} D_T) V^*|_{\mathfrak{D}}, \quad z \in \mathbb{D}. \quad (3-7)$$

Note that the particular representation of θ depends on the coordinate operators V and V_* identifying defect spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$ with the abstract spaces $\mathfrak{D}$ and $\mathfrak{D}_*$.

To construct a model (more precisely its particular transcription), we need to construct a unitary map Ψ between the space $\mathcal{H}$ of the minimal unitary dilation $\mathcal{U}$ and its spectral representation.

Namely, we represent $\mathcal{U}$ as a multiplication operator in some subspace $\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_\theta$ of $L^2(\mathfrak{D}_* \oplus \mathfrak{D})$ or its weighted version.

We need to construct a unitary operator $\Psi : \mathcal{H} \rightarrow \tilde{\mathcal{K}}$ intertwining $\mathcal{U}$ and M_z on $\tilde{\mathcal{K}}$, i.e., such that

$$\Psi \mathcal{U} = M_z \Psi. \quad (3-8)$$

Note that if T is a completely nonunitary contraction, then $\pi(L^2(\mathfrak{D})) + \pi_*(L^2(\mathfrak{D}_*))$ is dense in $\mathcal{H}$.

So, for Ψ to be unitary it is necessary and sufficient that Ψ^* acts isometrically on $\pi(L^2(\mathfrak{D}))$ and on $\pi_*(L^2(\mathfrak{D}_*))$, and that for all $f \in L^2(\mathfrak{D})$, $g \in L^2(\mathfrak{D}_*)$

$$(\Psi^* \pi f, \Psi^* \pi g)_{\tilde{\mathcal{K}}} = (\pi f, \pi_* g)_{\mathcal{H}} = (\theta f, g)_{L^2(\mathfrak{D}_*)}; \quad (3-9)$$

the last equality here is just the definition of θ .

Of course, we need Ψ^* to be onto, but that can be easily accomplished by restricting the target space $\tilde{\mathcal{K}}$ to $\text{Ran } \Psi^*$.

Summing up, we have

$$\begin{array}{ccccccc} \mathcal{H} & = & G & \oplus & H & \oplus & G_* \\ \downarrow \Psi^* & & \downarrow \Psi^*|_G & & \downarrow \Psi^*|_H & & \downarrow \Psi^*|_{G_*} \\ \tilde{\mathcal{K}} & = & \mathcal{G} & \oplus & \mathcal{K}_\theta & \oplus & \mathcal{G}_* \end{array}$$

3A. Pavlov transcription. Probably the easiest way to construct the model is to take $\tilde{\mathcal{K}}$ to be the weighted space $L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W)$, where the weight W is picked to make the simplest operator Ψ^* an isometry, and is given by

$$W(z) = \begin{pmatrix} I_{\mathfrak{D}_*} & \theta(z) \\ \theta(z)^* & I_{\mathfrak{D}} \end{pmatrix}. \quad (3-10)$$

Now the operator Ψ^* is defined on $\pi(L^2(\mathfrak{D}))$ and on $\pi_*(L^2(\mathfrak{D}_*))$ as

$$\begin{aligned} \Psi^* \left(\sum_{k \in \mathbb{Z}} \mathcal{U}^k e(e_k) \right) &= \sum_{k \in \mathbb{Z}} z^k \begin{pmatrix} 0 \\ e_k \end{pmatrix}, \quad e_k \in \mathfrak{D}, \\ \Psi^* \left(\sum_{k \in \mathbb{Z}} \mathcal{U}^k e_*(e_k) \right) &= \sum_{k \in \mathbb{Z}} z^{k-1} \begin{pmatrix} e_k \\ 0 \end{pmatrix}, \quad e_k \in \mathfrak{D}_*, \end{aligned} \quad (3-11)$$

or equivalently

$$\begin{aligned} \Psi^*(\pi f) &= \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad f \in L^2(\mathfrak{D}), \\ \Psi^*(\pi_* f) &= \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad f \in L^2(\mathfrak{D}_*). \end{aligned}$$

The incoming and outgoing spaces $\mathcal{G}_* = \Psi^* G_*$, $\mathcal{G} = \Psi^* G$ are given by

$$\mathcal{G}_* := \text{clos}_{\tilde{\mathcal{K}}} \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix} : f \in H_-^2(\mathfrak{D}_*) \right\}, \quad \mathcal{G} := \text{clos}_{\tilde{\mathcal{K}}} \left\{ \begin{pmatrix} 0 \\ f \end{pmatrix} : f \in H^2(\mathfrak{D}) \right\},$$

and the model space $\mathcal{K} = \mathcal{K}_\theta$ is defined as

$$\mathcal{K}_\theta = \tilde{\mathcal{K}} \ominus (\mathcal{G}_* \oplus \mathcal{G}).$$

3B. Sz.-Nagy–Foiaş transcription. This transcription appears when one tries to make the operator Ψ^* act into a nonweighted space $L^2(\mathfrak{D}_* \oplus \mathfrak{D})$. We make the action of the operator Ψ^* on $\pi_*(L^2(\mathfrak{D}_*))$ as simple as possible,

$$\Psi^* \left(\sum_{k \in \mathbb{Z}} \mathcal{U}^k e_*(e_k) \right) = \sum_{k \in \mathbb{Z}} z^{k-1} \begin{pmatrix} e_k \\ 0 \end{pmatrix}, \quad e_k \in \mathfrak{D}_*; \quad (3-12)$$

this is exactly as in (3-11). The action of Ψ^* on $\pi(L^2(\mathfrak{D}))$ is defined as

$$\Psi^* \left(\sum_{k \in \mathbb{Z}} \mathcal{U}^k e(e_k) \right) = \sum_{k \in \mathbb{Z}} z^k \begin{pmatrix} \theta e_k \\ \Delta e_k \end{pmatrix}, \quad e_k \in \mathfrak{D}, \quad (3-13)$$

where $\Delta(z) = (\mathbf{I} - \theta(z)^* \theta(z))^{1/2}$. Then (3-12) and (3-13) can clearly be rewritten as

$$\Psi^*(\pi f) = \begin{pmatrix} \theta f \\ \Delta f \end{pmatrix}, \quad f \in L^2(\mathfrak{D}), \quad (3-14)$$

$$\Psi^*(\pi_* f) = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad f \in L^2(\mathfrak{D}_*). \quad (3-15)$$

Note, that θ in the top entries in (3-13) and (3-14) is necessary to get (3-9); after (3-12), equivalently (3-15), is chosen, one does not have any choice here. The term Δ in the bottom entries of (3-13) and (3-14) is there to make Ψ^* act isometrically on $\pi(L^2(\mathfrak{D}))$. There is some freedom here; one can left-multiply Δ by any operator-valued function ϕ such that $\phi(z)$ acts isometrically on $\text{Ran } \Delta(z)$. However, picking just Δ is the canonical choice for the Sz.-Nagy–Foiaş transcription, and we will follow it.

The incoming and outgoing spaces are given by

$$\mathcal{G}_* := \begin{pmatrix} H_-^2(\mathfrak{D}_*) \\ 0 \end{pmatrix}, \quad \mathcal{G} := \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2(\mathfrak{D}).$$

The model space is given by

$$\mathcal{K}_\theta := \begin{pmatrix} L^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \ominus (\mathcal{G}_* \oplus \mathcal{G}) = \begin{pmatrix} H^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2(\mathfrak{D}). \quad (3-16)$$

Remark. While the orthogonal projection from

$$\begin{pmatrix} L^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} L^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \ominus \mathcal{G}_*$$

is rather simple, the one from

$$\begin{pmatrix} L^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} L^2(\mathfrak{D}_*) \\ \text{clos } \Delta L^2(\mathfrak{D}) \end{pmatrix} \ominus \mathcal{G}$$

involves the range of a Toeplitz operator.

3C. De Branges–Rovnyak transcription. This transcription looks very complicated, but its advantage is that both coordinates are analytic functions. To describe this transcription, we use the auxiliary weight $W = W(z)$ as in the Pavlov transcription; see (3-10). The model space is the subspace of $L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W^{[-1]})$, where for a self-adjoint operator A the symbol $A^{[-1]}$ denotes its Moore–Penrose (pseudo)inverse, i.e., $A^{[-1]} = 0$ on $\text{Ker } A$ and $A^{[-1]}$ is the left inverse of A on $(\text{Ker } A)^\perp$.

The operator $\Psi^* : \mathcal{H} \rightarrow L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W^{[-1]})$ is defined by

$$\Psi^*(\pi f) = W \begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} \theta f \\ f \end{pmatrix}, \quad f \in L^2(\mathfrak{D}),$$

$$\Psi^*(\pi_* f) = W \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ \theta^* f \end{pmatrix}, \quad f \in L^2(\mathfrak{D}_*).$$

The incoming and outgoing spaces are

$$\mathcal{G}_* := \begin{pmatrix} I \\ \theta^* \end{pmatrix} H^2(\mathfrak{D}_*), \quad \mathcal{G} := \begin{pmatrix} \theta \\ I \end{pmatrix} H^2(\mathfrak{D}),$$

and the model space is defined as

$$\mathcal{K}_\theta := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in H^2(\mathfrak{D}_*), g \in H^2_-(\mathfrak{D}), g - \theta^* f \in \Delta L^2(\mathfrak{D}) \right\};$$

see [Nikolski and Vasyunin 1998, Section 3.7] for the details (there is a typo in that paper; in the definition of $\mathcal{K}_\theta$ on p. 251 it should be $f \in H^2(E_*)$, $g \in H^2(E)$).

3D. Parametrizing operators for the model, agreeing with coordinate operators. The parametrizing operators that agree with the coordinate operators V and V_* are described in the following lemma, which holds for any transcription of the model.

Let T be a c.n.u. contraction, and let $V : \mathfrak{D}_T \rightarrow \mathfrak{D}$ and $V_* : \mathfrak{D}_{T^*} \rightarrow \mathfrak{D}_*$ be coordinate operators for the defect spaces of T . Let $\theta = \theta_T = \theta_{T,V,V_*} \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$ be the characteristic function of T , defined by (3-7), and let $\mathcal{M}_\theta$ be the corresponding model operator (in any transcription).

Recall that Ψ is a unitary operator intertwining the minimal unitary dilation $\mathcal{U}$ of T and the multiplication operator M_z in the corresponding function space; see (3-8). The operator Ψ determines the transcription of the model, so for any particular transcription it is known.

Define

$$\tilde{e} := \Psi^* e, \quad \tilde{e}_* := \Psi^* e_*, \quad (3-17)$$

where the embeddings e and e_* are defined by (3-4), (3-5).

Lemma 3.2. *Under the above assumptions the parametrizing operators $C_* : \mathfrak{D}_* \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ and $C : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$ given by*

$$C_* e_* = (D_{\mathcal{M}_\theta^*} |_{\mathfrak{D}_{\mathcal{M}_\theta^*}})^{-1} P_{\mathcal{K}_\theta} M_z \tilde{e}_*(e_*), \quad e_* \in \mathfrak{D}_*, \quad (3-18)$$

$$C e = (D_{\mathcal{M}_\theta} |_{\mathfrak{D}_{\mathcal{M}_\theta}})^{-1} P_{\mathcal{K}_\theta} M_{\bar{z}} \tilde{e}(e), \quad e \in \mathfrak{D}, \quad (3-19)$$

agree with the coordinate operators V and V_* .

Remark. It follows from (3-20) below that $P_{\mathcal{K}_\theta} M_z \tilde{e}_*(e_*) \in \text{Ran } D_{\mathcal{M}_\theta^*}$ as well as $P_{\mathcal{K}_\theta} M_{\bar{z}} \tilde{e}(e) \in \text{Ran } D_{\mathcal{M}_\theta}$, so everything in (3-18), (3-19) is well defined.

Proof of Lemma 3.2. Right- and left-multiplying (3-6) by Ψ and Ψ^* respectively, we get

$$\Psi^* \mathcal{U} \Psi = \begin{pmatrix} \tilde{\mathcal{E}}_*^* & 0 & 0 \\ D_{\mathcal{M}_\theta^*} C_* \tilde{e}_*^* & \mathcal{M}_\theta & 0 \\ -\tilde{e} C^* \mathcal{M}_\theta^* C_* \tilde{e}_*^* & \tilde{e} C^* D_{\mathcal{M}_\theta} & \tilde{\mathcal{E}} \end{pmatrix}, \quad (3-20)$$

where $\tilde{\mathcal{E}} = \Psi^* \mathcal{E} \Psi$, $\tilde{\mathcal{E}}_* = \Psi \mathcal{E}_* \Psi$, $C^* = V \Psi$, $C_*^* = V_* \Psi$, $\tilde{e} = \Psi^* e$, $\tilde{e}_* = \Psi^* e_*$.

The operators $\tilde{e}$ and $\tilde{e}_*$ are the canonical embeddings of $\mathfrak{D}$ and $\mathfrak{D}_*$ into $\mathcal{G}$ and $\mathcal{G}_*$ that agree with the canonical embeddings e and e_* . The operators C and C_* are the parametrizing operators for the defect

spaces of the model operator $\mathcal{M}_\theta$ that agree with the coordinate operators V and V_* for the defect spaces of the operator T .

In any particular transcription of the model, the operator $\Psi^* \mathcal{U} \Psi$ is known (it is just the multiplication by z in an appropriate function space), so we get from the decomposition (3-20)

$$D_{\mathcal{M}_\theta^*} \mathbf{C}_* \tilde{\mathbf{e}}_*^* = P_{\mathcal{K}_\theta} M_z|_{\mathcal{G}_*}, \quad D_{\mathcal{M}_\theta} \mathbf{C} \tilde{\mathbf{e}}^* = P_{\mathcal{K}_\theta} M_{\bar{z}}|_{\mathcal{G}_*}.$$

Right- and left-multiplying the first identity by $\mathbf{e}_*$ and $(D_{\mathcal{M}_\theta^*}|_{\mathfrak{D}_{\mathcal{M}_\theta^*}})^{-1}$ respectively, we get (3-18). Similarly, to get (3-19) we just right- and left-multiply the second identity by $\mathbf{e}$ and $(D_{\mathcal{M}_\theta}|_{\mathfrak{D}_{\mathcal{M}_\theta}})^{-1}$. $\square$

Applying the above Lemma 3.2 to a particular transcription of the model, we can get more concrete formulas for $\mathbf{C}$, $\mathbf{C}_*$ just in terms of the characteristic function θ . For example, the following lemma gives formulas for $\mathbf{C}$ and $\mathbf{C}_*$ in the Sz.-Nagy–Foiaş transcription.

Lemma 3.3. *Let T be a c.n.u. contraction, and let $\mathcal{M}_\theta$ be its model in the Sz.-Nagy–Foiaş transcription, with the characteristic function $\theta = \theta_{T,V,V_*}$, $\theta \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$.*

Then the maps $\mathbf{C}_ : \mathfrak{D}_* \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ and $\mathbf{C} : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$ given by*

$$\mathbf{C}_* \mathbf{e}_* = \begin{pmatrix} \mathbf{I} - \theta(z)\theta^*(0) \\ -\Delta(z)\theta^*(0) \end{pmatrix} (\mathbf{I} - \theta(0)\theta^*(0))^{-1/2} \mathbf{e}_*, \quad \mathbf{e}_* \in \mathfrak{D}_*, \quad (3-21)$$

$$\mathbf{C} \mathbf{e} = \begin{pmatrix} z^{-1}(\theta(z) - \theta(0)) \\ z^{-1}\Delta(z) \end{pmatrix} (\mathbf{I} - \theta^*(0)\theta(0))^{-1/2} \mathbf{e}, \quad \mathbf{e} \in \mathfrak{D}, \quad (3-22)$$

agree with the coordinate operators V and V_ .*

Proof. To prove (3-21) we will use (3-18). It follows from (3-12) that

$$\tilde{\mathbf{e}}_*(\mathbf{e}_*) = z^{-1} \begin{pmatrix} \mathbf{e}_* \\ 0 \end{pmatrix},$$

so by (3-18)

$$\mathbf{C}_* \mathbf{e}_* = (\mathbf{I} - \mathcal{M}_\theta \mathcal{M}_\theta^*)|_{\mathfrak{D}_{\mathcal{M}_\theta^*}}^{-1/2} P_{\mathcal{K}_\theta} \begin{pmatrix} \mathbf{e}_* \\ 0 \end{pmatrix}, \quad \mathbf{e}_* \in \mathfrak{D}_*. \quad (3-23)$$

It is not hard to show that

$$P_{\mathcal{K}_\theta} \begin{pmatrix} \mathbf{e}_* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \theta\theta(0)^* \\ -\Delta\theta(0)^* \end{pmatrix} \mathbf{e}_*. \quad (3-24)$$

One also can compute

$$(\mathbf{I} - \mathcal{M}_\theta \mathcal{M}_\theta^*) \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \theta\theta(0)^* \\ -\Delta\theta(0)^* \end{pmatrix} f(0), \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{K}_\theta. \quad (3-25)$$

Combining the above identities we get

$$(\mathbf{I} - \mathcal{M}_\theta \mathcal{M}_\theta^*) P_{\mathcal{K}_\theta} \begin{pmatrix} \mathbf{e}_* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \theta\theta(0)^* \\ -\Delta\theta(0)^* \end{pmatrix} (\mathbf{e}_* - \theta(0)\theta^*(0)\mathbf{e}_*). \quad (3-26)$$

As we discussed above just after (3-19), $P_{\mathcal{K}_\theta} \begin{pmatrix} \mathbf{e}_* \\ 0 \end{pmatrix} \in \text{Ran } D_{\mathcal{M}_\theta^*}$, so in (3-26) we can replace $(\mathbf{I} - \mathcal{M}_\theta \mathcal{M}_\theta^*)$ by its restriction onto $\mathfrak{D}_{\mathcal{M}_\theta^*}$.

Applying $(I - \mathcal{M}_\theta \mathcal{M}_\theta^*)|_{\mathfrak{D}_{\mathcal{M}_\theta^*}}$ to (3-26) (with $(I - \mathcal{M}_\theta \mathcal{M}_\theta^*)$ replaced by its restriction onto $\mathfrak{D}_{\mathcal{M}_\theta}$) and using (3-25) we get

$$((I - \mathcal{M}_\theta \mathcal{M}_\theta^*)|_{\mathfrak{D}_{\mathcal{M}_\theta^*}})^2 P_{\mathcal{K}_\theta} \begin{pmatrix} e_* \\ 0 \end{pmatrix} = \begin{pmatrix} I - \theta \theta(0)^* \\ -\Delta \theta(0)^* \end{pmatrix} (I_{\mathfrak{D}_*} - \theta(0) \theta^*(0))^2 e_*.$$

Applying $(I - \mathcal{M}_\theta \mathcal{M}_\theta^*)|_{\mathfrak{D}_{\mathcal{M}_\theta^*}}$ to the above identity, and using again (3-25), we get by induction that

$$\varphi((I - \mathcal{M}_\theta \mathcal{M}_\theta^*)|_{\mathfrak{D}_{\mathcal{M}_\theta^*}}) P_{\mathcal{K}_\theta} \begin{pmatrix} e_* \\ 0 \end{pmatrix} = \begin{pmatrix} I - \theta \theta(0)^* \\ -\Delta \theta(0)^* \end{pmatrix} \varphi(I_{\mathfrak{D}_*} - \theta(0) \theta^*(0)) e_* \quad (3-27)$$

for any monomial φ , $\varphi(x) = x^n$, $n \geq 0$ (the case $n = 0$ is just the identity (3-24)).

Linearity implies that (3-27) holds for any polynomial φ . Using standard approximation reasoning we get that φ in (3-27) can be any measurable function. In particular, we can take $\varphi(x) = x^{-1/2}$, which together with (3-23) gives us (3-21).

To prove (3-22) we proceed similarly. Equation (3-13) implies

$$\tilde{e}(e) = \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e,$$

so by (3-19)

$$C e = ((I - \mathcal{M}_\theta^* \mathcal{M}_\theta)|_{\mathfrak{D}_{\mathcal{M}_\theta}})^{-1/2} P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e, \quad e \in \mathfrak{D}. \quad (3-28)$$

One can see that

$$P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e = M_{\bar{z}} \begin{pmatrix} \theta - \theta(0) \\ \Delta \end{pmatrix} e,$$

so

$$\mathcal{M}_\theta P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e = P_{\mathcal{K}_\theta} \begin{pmatrix} \theta - \theta(0) \\ \Delta \end{pmatrix} e = -P_{\mathcal{K}_\theta} \begin{pmatrix} \theta(0) \\ 0 \end{pmatrix} e.$$

Combining this with (3-24), we get

$$\mathcal{M}_\theta P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e = \begin{pmatrix} \theta \theta(0)^* - I \\ \Delta \theta(0)^* \end{pmatrix} \theta(0) e.$$

Using the fact that

$$\mathcal{M}_\theta^* \begin{pmatrix} f \\ g \end{pmatrix} = M_{\bar{z}} \begin{pmatrix} f - f(0) \\ g \end{pmatrix},$$

we arrive at

$$\mathcal{M}_\theta^* \mathcal{M}_\theta P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e = M_{\bar{z}} \begin{pmatrix} \theta - \theta(0) \\ \Delta \end{pmatrix} \theta(0)^* \theta(0) e,$$

so

$$(I - \mathcal{M}_\theta^* \mathcal{M}_\theta) P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e = M_{\bar{z}} \begin{pmatrix} \theta - \theta(0) \\ \Delta \end{pmatrix} (I - \theta(0)^* \theta(0)) e.$$

Using the same reasoning as in the above proof of (3-21) we get

$$\varphi((I - \mathcal{M}_\theta^* \mathcal{M}_\theta)|_{\mathfrak{D}_{\mathcal{M}_\theta}}) P_{\mathcal{K}_\theta} M_{\bar{z}} \begin{pmatrix} \theta \\ \Delta \end{pmatrix} e = M_{\bar{z}} \begin{pmatrix} \theta - \theta(0) \\ \Delta \end{pmatrix} \varphi(I - \theta(0)^* \theta(0)) e, \quad (3-29)$$

first with φ being a polynomial, and then any measurable function.

Using (3-29) with $\varphi(x) = x^{-1/2}$ and taking (3-28) into account, we get (3-22). $\square$

3E. An auxiliary lemma. We already used, and we will also need later, the following simple lemma.

Lemma 3.4. *Let $\mathcal{M} = \mathcal{M}_\theta$ be model operator on a model space $\mathcal{K}_\theta \subset L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$, and let $C : \mathfrak{D}_* \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$, $C_* : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ be bounded operators.*

If C and C_ are the operator-valued functions defined by*

$$\begin{aligned} C(z)e &= Ce(z), & z \in \mathbb{T}, e \in \mathfrak{D}, \\ C_*(z)e_* &= C_*e_*(z), & z \in \mathbb{T}, e_* \in \mathfrak{D}_*, \end{aligned}$$

then the functions $W^{1/2}C$ and $W^{1/2}C_$ are bounded,*

$$\|W^{1/2}C\|_{L^\infty} = \|C\|, \quad \|W^{1/2}C_*\|_{L^\infty} = \|C_*\|.$$

Proof. It is well known and is not hard to show that if T is a contraction and $\mathcal{U}$ is its unitary dilation, then the subspaces $\mathcal{U}^n \mathfrak{D}_T$, $n \in \mathbb{Z}$ (where recall $\mathfrak{D}_T$ is the defect space of T) are mutually orthogonal, and similarly for subspaces $\mathcal{U}^n \mathfrak{D}_{T^*}$, $n \in \mathbb{Z}$.

Therefore, the subspaces $z^n \mathfrak{D}_{\mathcal{M}}$, $n \in \mathbb{Z}$, are mutually orthogonal in $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$, and the same holds for the subspaces $z^n \mathfrak{D}_{\mathcal{M}^*}$, $n \in \mathbb{Z}$.

The subspaces $z^n \mathfrak{D} \subset L^2(\mathbb{T}; \mathfrak{D})$ are mutually orthogonal, and since

$$C(z) \sum_{n \in \mathbb{Z}} z^n \hat{f}(n) = \sum_{n \in \mathbb{Z}} z^n C f_n, \quad \hat{f}(n) \in \mathfrak{D},$$

we conclude that the operator $f \mapsto Cf$ is a bounded operator acting from $L^2(\mathfrak{D})$ to $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$, and its norm is exactly $\|C\|$.

But that means the multiplication operator $f \mapsto W^{1/2}f$ between the nonweighted spaces $L^2(\mathfrak{D})$ and $L^2(\mathfrak{D}_* \oplus \mathfrak{D})$ is bounded with the same norm, which immediately implies $\|W^{1/2}C\|_{L^\infty} = \|C\|$.

The proof for C_* follows similarly. □

4. Characteristic function

We now derive formulas for the (matrix-valued) characteristic function θ_Γ ; see [Theorem 4.2](#) below.

4A. An inverse of a perturbation. We begin with an auxiliary result.

Lemma 4.1. *Let D be an operator in an auxiliary Hilbert space $\mathfrak{R}$ and let $B, C : \mathfrak{R} \rightarrow \mathcal{H}$. Then $I_{\mathcal{H}} - CDB^*$ is invertible if and only if $I_{\mathfrak{R}} - DB^*C$ is invertible, and if and only if $I_{\mathfrak{R}} - B^*CD$ is invertible.*

Moreover, in this case

$$\begin{aligned} (I_{\mathcal{H}} - CDB^*)^{-1} &= I_{\mathcal{H}} + C(I_{\mathfrak{R}} - DB^*C)^{-1}DB^* \\ &= I_{\mathcal{H}} + CD(I_{\mathfrak{R}} - B^*CD)^{-1}B^*. \end{aligned} \tag{4-1}$$

We will apply this lemma for $D : \mathbb{C}^d \rightarrow \mathbb{C}^d$, so in this case the inversion of $I_{\mathcal{H}} - CDB$ is reduced to inverting a $d \times d$ matrix.

This lemma can be obtained from the Woodbury inversion formula [1950], although formally in [Woodbury 1950] only the matrix case was treated.

Proof of Lemma 4.1. First let us note that it is sufficient to prove the lemma with $D = I_{\mathfrak{H}}$, because D can be incorporated either into C or into B^* .

One could guess the formula by writing the power series expansion of $I_{\mathcal{H}} - CDB^*$, and we can get the result for the case when the series converges. This method can be made rigorous for finite-rank perturbations by considering the family $(I_{\mathcal{H}} - \lambda CDB^*)^{-1}$, $\lambda \in \mathbb{C}$, and using analytic continuation.

However, the simplest way to prove the formula is just by performing multiplication,

$$\begin{aligned} (I_{\mathcal{H}} - CB^*)(I_{\mathcal{H}} + C(I_{\mathfrak{H}} - B^*C)^{-1}B^*) &= I_{\mathcal{H}} - CB^* + C(I_{\mathfrak{H}} - B^*C)^{-1}B^* - CB^*C(I_{\mathfrak{H}} - B^*C)^{-1}B^* \\ &= I_{\mathcal{H}} + C(-I_{\mathfrak{H}}(I_{\mathfrak{H}} - B^*C) + I_{\mathfrak{H}} - B^*C)(I_{\mathfrak{H}} - B^*C)^{-1}B^* \\ &= I_{\mathcal{H}}. \end{aligned}$$

Thus, when $I_{\mathfrak{H}} - B^*C$ is invertible, the operator $I_{\mathcal{H}} + C(I_{\mathfrak{H}} - B^*C)^{-1}B^*$ is the right inverse of $I_{\mathcal{H}} - CB^*$. To prove that it is also a left inverse we even do not need to perform the multiplication: we can just take the adjoint of the above identity and then interchange B and C .

So, the invertibility of $I_{\mathfrak{H}} - B^*C$ implies the invertibility of $I_{\mathcal{H}} - CB^*$ and the formula for the inverse. To prove the “if and only if” statement we just need to change the roles of $\mathcal{H}$ and $\mathfrak{H}$ and express, using the just proved formula, the inverse of $I_{\mathfrak{H}} - B^*C$ in terms of $(I_{\mathcal{H}} - CB^*)^{-1}$. $\square$

4B. Computation of the characteristic function. We turn to computing the characteristic function of $T = U + B(\Gamma - I_{\mathbb{C}^d})B^*U$, $\|\Gamma\| < 1$, where U is the multiplication operator M_{ξ} in $L^2(\mu; E)$.

We will use formula (3-7) with $V = B^*U$, $V_* = B^*$, $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$.

Let us first calculate for $|z| < 1$,

$$\begin{aligned} (I_{\mathcal{H}} - zT^*)^{-1} &= [(I_{\mathcal{H}} - zU^*)(I_{\mathcal{H}} - z(I_{\mathcal{H}} - zU^*)^{-1}U^*B(\Gamma^* - I_{\mathbb{C}^d})B^*))^{-1}]^{-1} \\ &= [I_{\mathcal{H}} - z(I_{\mathcal{H}} - zU^*)^{-1}U^*B(\Gamma^* - I_{\mathbb{C}^d})B^*]^{-1}(I_{\mathcal{H}} - zU^*)^{-1} \\ &=: X(z)(I_{\mathcal{H}} - zU^*)^{-1}. \end{aligned}$$

To compute the inverse $X(z)$ we use Lemma 4.1 with $z(I_{\mathcal{H}} - zU^*)^{-1}U^*B$ instead of C , $\Gamma^* - I_{\mathbb{C}^d}$ instead of D and B instead of B . Together with the first identity in (4-1) we get

$$X(z) = I_{\mathcal{H}} + z(I_{\mathcal{H}} - zU^*)^{-1}U^*B(I_{\mathbb{C}^d} - zDB^*(I_{\mathcal{H}} - zU^*)^{-1}U^*B)^{-1}DB^*, \quad (4-2)$$

where $D = \Gamma^* - I_{\mathbb{C}^d}$.

Now, let us express $zB^*(I_{\mathcal{H}} - zU^*)^{-1}U^*B$ as a Cauchy integral of some matrix-valued measure. Recall that U is a multiplication by the independent variable ξ in $\mathcal{H} \subset L^2(\mu; E)$ and that $b_1, b_2, \dots, b_d \in \mathcal{H}$ denote the “columns” of B (i.e., $b_k = Be_k$, where $e_1, e_2, \dots, e_d$ is the standard basis in $\mathbb{C}^d$), and $B(\xi) = (b_1(\xi), b_2(\xi), \dots, b_d(\xi))$ is the matrix with columns $b_k(\xi)$. Then

$$b_j^*(I_{\mathbb{C}^d} - zU^*)^{-1}U^*b_k = \int_{\mathbb{T}} \frac{\bar{\xi}}{1 - z\bar{\xi}} b_j(\xi)^* b_k(\xi) d\mu(\xi),$$

so

$$z\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B} = \int_{\mathbb{T}} \frac{z\bar{\xi}}{1 - z\xi} M(\xi) d\mu(\xi) =: \mathcal{C}_1[M\mu](z) =: F_1(z), \quad (4-3)$$

where M is the matrix-valued function $M(\xi) = B(\xi)^*B(\xi)$, or equivalently $M_{j,k}(\xi) = b_j(\xi)^*b_k(\xi)$, $1 \leq j, k \leq d$.

Using (4-3) and denoting $D := \Gamma^* - \mathbf{I}_{\mathbb{C}^d}$ we get from the above calculations that

$$(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1} = (\mathbf{I}_{\mathcal{H}} - zU^*)^{-1} + z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}(\mathbf{I}_{\mathbb{C}^d} - DF_1(z))^{-1}D\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}.$$

Applying formula (3-7), with $V = \mathbf{B}^*U$, $V_* = \mathbf{B}_*$, $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$, we see that the characteristic function is an analytic function $\theta = \theta_T$ whose values are bounded linear operators acting on $\mathfrak{D}$, defined by the formula

$$\theta_T(z) = \mathbf{B}^*(-T + zD_{T^*}(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1}D_T)U^*\mathbf{B}|_{\mathfrak{D}}, \quad z \in \mathbb{D}. \quad (4-4)$$

We can see from (1-3) that the defect operators D_T and D_{T^*} are given by

$$D_T = U^*\mathbf{B}D_{\Gamma}\mathbf{B}^*U, \quad D_{T^*} = \mathbf{B}D_{\Gamma^*}\mathbf{B}^*.$$

We can also see from (1-3) that the term $-T$ in (4-4) contributes $-\Gamma$ to the matrix θ_T . The rest can be obtained from the above representation formula for $(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1}$. Thus, recalling the definition (4-3) of $\mathcal{C}_1M\mu$ we get, defining $F_1(z) := (\mathcal{C}_1M\mu)(z)$, that

$$\begin{aligned} \theta_T(z) &= -\Gamma + D_{\Gamma^*}[F_1(z) + F_1(z)(\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z))^{-1}(\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)]D_{\Gamma} \\ &= -\Gamma + D_{\Gamma^*}F_1(z)(\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z))^{-1}D_{\Gamma}. \end{aligned}$$

In the above computation to compute $X(z)$ we can use the second formula in (4-1). We get instead of (4-2) an alternative representation

$$X(z) = \mathbf{I}_{\mathcal{H}} + z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}D(\mathbf{I}_{\mathfrak{D}} - z\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}D)^{-1}\mathbf{B}^*.$$

Repeating the same computations as above we get another formula for θ_T ,

$$\theta_T(z) = -\Gamma + D_{\Gamma^*}(\mathbf{I}_{\mathfrak{D}} - F_1(z)(\Gamma^* - \mathbf{I}_{\mathfrak{D}}))^{-1}F_1(z)D_{\Gamma}.$$

To summarize we have proved two representations of the characteristic operator-valued function.

Theorem 4.2. *Let $T = T_{\Gamma}$ be the operator given in (1-3), with Γ being a strict contraction. Then the characteristic function $\theta_T = \theta_{T_{\Gamma}} \in H^{\infty}(\mathfrak{D} \rightarrow \mathfrak{D}_*)$, with coordinate operators $V = \mathbf{B}^*U$, $V_* = \mathbf{B}^*$ (and with $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$), is given by*

$$\begin{aligned} \theta_{T_{\Gamma}}(z) &= -\Gamma + D_{\Gamma^*}F_1(z)(\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z))^{-1}D_{\Gamma} \\ &= -\Gamma + D_{\Gamma^*}(\mathbf{I}_{\mathfrak{D}} - F_1(z)(\Gamma^* - \mathbf{I}_{\mathfrak{D}}))^{-1}F_1(z)D_{\Gamma}, \end{aligned}$$

where $F_1(z)$ is the matrix-valued function given by (4-3).

In these formulas, the inverse is taken of a $d \times d$ matrix-valued function, which is much simpler than computing the inverse in (4-4).

4C. Characteristic function and the Cauchy integrals of matrix-valued measures. For a (possibly complex-valued) measure τ on $\mathbb{T}$ and $z \notin \mathbb{T}$, define the Cauchy-type transforms $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$\mathcal{C}\tau(z) := \int_{\mathbb{T}} \frac{d\tau(\xi)}{1 - \bar{\xi}z}, \quad \mathcal{C}_1\tau(z) := \int_{\mathbb{T}} \frac{\bar{\xi}z d\tau(\xi)}{1 - \bar{\xi}z}, \quad \mathcal{C}_2\tau(z) := \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z} d\tau(\xi).$$

Performing the Cauchy transforms componentwise we can define them for matrix-valued measures as well.

Thus F_1 from the above [Theorem 4.2](#) is given by $F_1 = \mathcal{C}_1[M\mu]$, where $M(\xi) = B^*(\xi)B(\xi)$. We would like to give the representation of θ_{T_Γ} in terms of function $F_2 := \mathcal{C}_2[M\mu]$.

Slightly abusing notation we will write θ_Γ instead of θ_{T_Γ} .

Corollary 4.3. *For $\theta_0 := \theta_{T_0}$ we have*

$$\theta_0(z) = F_1(z)(I + F_1(z))^{-1} = (I + F_1(z))^{-1}F_1(z) \quad (4-5)$$

$$= (F_2(z) - I)(F_2(z) + I)^{-1} = (F_2(z) + I)^{-1}(F_2(z) - I). \quad (4-6)$$

Proof. The identity (4-5) is a direct application of [Theorem 4.2](#). The identity (4-6) follows immediately from the trivial relation

$$F_2(z) = \int_{\mathbb{T}} M d\mu + 2F_1(z) = I_{\mathfrak{D}} + 2F_1(z);$$

the equality $\int_{\mathbb{T}} M d\mu = I_{\mathfrak{D}} = I_{\mathbb{C}^d}$ is just a restatement of the fact that the functions $b_1, b_2, \dots, b_d$ form an orthonormal basis in $\mathcal{H}$. $\square$

5. Relations between characteristic functions θ_Γ

5A. Characteristic functions and linear fractional transformations. When $d = 1$, it is known that the characteristic functions are related by a linear fractional transformation

$$\theta_\gamma(z) = \frac{\theta_0(z) - \gamma}{1 - \bar{\gamma}\theta_0(z)};$$

see [\[Liaw and Treil 2016, equation \(2.9\)\]](#).

It turns out that a similar formula holds for finite-rank perturbations.

Theorem 5.1. *Let T be the operator given in (1-3), with Γ being a strict contraction. Then the characteristic functions $\theta_\Gamma := \theta_{T_\Gamma}$ and $\theta_0 = \theta_{T_0}$ are related via linear fractional transformation,*

$$\theta_\Gamma = D_{\Gamma^*}^{-1}(\theta_0 - \Gamma)(I_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}D_\Gamma = D_{\Gamma^*}(I_{\mathfrak{D}} - \theta_0\Gamma^*)^{-1}(\theta_0 - \Gamma)D_\Gamma^{-1}.$$

Remark. At first sight, this formula looks like a formula in [\[Nikolski and Vasyunin 1998, p. 234\]](#). However, their result expresses the characteristic function in terms of a linear fractional transformation in T , whereas, here we have a linear fractional transformation in Γ .

Theorem 5.2. *Under assumptions of the above [Theorem 5.1](#)*

$$\theta_0 = D_{\Gamma^*}(\mathbf{I} + \theta_{\Gamma}\Gamma^*)^{-1}(\theta_{\Gamma} + \Gamma)D_{\Gamma}^{-1} = D_{\Gamma^*}^{-1}(\theta_{\Gamma} + \Gamma)(\mathbf{I} + \Gamma^*\theta_{\Gamma})^{-1}D_{\Gamma}.$$

To prove [Theorem 5.1](#) we start with the following simpler statement.

Proposition 5.3. *The matrix-valued characteristic functions θ_{Γ} and θ_0 are related via*

$$\theta_{\Gamma} = -\Gamma + D_{\Gamma^*}\theta_0(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}D_{\Gamma} = -\Gamma + D_{\Gamma^*}(\mathbf{I}_{\mathfrak{D}} - \theta_0\Gamma^*)^{-1}\theta_0D_{\Gamma}.$$

Proof. Solving (4-5) for F_1 we get

$$F_1(z) = \theta_0(z)[\mathbf{I} - \theta_0(z)]^{-1}.$$

Substituting this expression into the formula for the characteristic function from [Theorem 4.2](#), we see that

$$\theta_{\Gamma} = -\Gamma + D_{\Gamma^*}\theta_0[\mathbf{I}_{\mathfrak{D}} - \theta_0]^{-1}\{\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})\theta_0[\mathbf{I}_{\mathfrak{D}} - \theta_0]^{-1}\}^{-1}D_{\Gamma}. \quad (5-1)$$

We manipulate the term inside the curly brackets

$$\begin{aligned} \mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})\theta_0[\mathbf{I}_{\mathfrak{D}} - \theta_0]^{-1} &= (\mathbf{I}_{\mathfrak{D}} - \theta_0 - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})\theta_0)[\mathbf{I}_{\mathfrak{D}} - \theta_0]^{-1} \\ &= (\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)[\mathbf{I}_{\mathfrak{D}} - \theta_0]^{-1}, \end{aligned}$$

so that

$$\{\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})\theta_0[\mathbf{I}_{\mathfrak{D}} - \theta_0]^{-1}\}^{-1} = [\mathbf{I}_{\mathfrak{D}} - \theta_0](\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}.$$

Substituting this back into (5-1), we get the first equation in the proposition.

The second equation is obtained similarly. □

Lemma 5.4. *For $\|\Gamma\| < 1$ we have for all $\alpha \in \mathbb{R}$*

$$D_{\Gamma^*}^{\alpha}\Gamma = \Gamma D_{\Gamma}^{\alpha}, \quad (5-2)$$

$$D_{\Gamma}^{\alpha}\Gamma^* = \Gamma^* D_{\Gamma^*}^{\alpha}, \quad (5-3)$$

where, recall $D_{\Gamma} := (\mathbf{I} - \Gamma^*\Gamma)^{1/2}$, $D_{\Gamma^*} := (\mathbf{I} - \Gamma\Gamma^*)^{1/2}$ are the defect operators.

Proof. Let us prove (5-2). It is trivially true for $\alpha = 2$, and by induction we get that it is true for $\alpha = 2n$, $n \in \mathbb{N}$. Since $\|\Gamma\| < 1$, the spectrum of D_{Γ} lies in the interval $[a, 1]$, $a = (1 - \|\Gamma\|^2)^{1/2} > 0$.

Approximating $\varphi(x) = x^{\alpha}$ uniformly on $[a, 1]$ by polynomials of x^2 we get (5-2).

Applying (5-2) to Γ^* we get (5-3). □

Proof of Theorem 5.1. From (5-2) we get $D_{\Gamma^*}^{-1}\Gamma D_{\Gamma}^{-1} = D_{\Gamma^*}^{-2}\Gamma$, so

$$\begin{aligned} \theta_{\Gamma} &= -\Gamma + D_{\Gamma^*}\theta_0(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}D_{\Gamma} \\ &= D_{\Gamma^*}[-D_{\Gamma^*}^{-2}\Gamma + \theta_0(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}]D_{\Gamma} \\ &= D_{\Gamma^*}^{-1}[-\Gamma + D_{\Gamma^*}^2\theta_0(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}]D_{\Gamma} \\ &= D_{\Gamma^*}^{-1}[-\Gamma(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0) + (\mathbf{I} - \Gamma\Gamma^*)\theta_0](\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}D_{\Gamma} \\ &= D_{\Gamma^*}^{-1}[-\Gamma + \theta_0](\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}D_{\Gamma}, \end{aligned}$$

which is exactly the first identity.

The second identity is obtained similarly, using the formula $D_{\Gamma^*}^{-1}\Gamma D_{\Gamma}^{-1} = \Gamma D_{\Gamma}^{-2}$ and taking the factor $(I_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}$ out of brackets on the left. $\square$

Proof of Theorem 5.2. Right-multiplying the first identity in Theorem 5.1 by $D_{\Gamma}^{-1}(I - \Gamma^*\theta_0)$ we get

$$\theta_{\Gamma} D_{\Gamma}^{-1} - \theta_{\Gamma} D_{\Gamma}^{-1} \Gamma^* \theta_0 = D_{\Gamma^*}^{-1} \theta_0 - D_{\Gamma^*}^{-1} \Gamma.$$

Using identities $D_{\Gamma^*}^{-1} \Gamma = \Gamma D_{\Gamma}^{-1}$ and $D_{\Gamma}^{-1} \Gamma^* = \Gamma^* D_{\Gamma^*}^{-1}$, see Lemma 5.4, we rewrite the above equality as

$$\theta_{\Gamma} D_{\Gamma}^{-1} + \Gamma D_{\Gamma}^{-1} = \theta_{\Gamma} \Gamma^* D_{\Gamma^*}^{-1} \theta_0 + D_{\Gamma^*}^{-1} \theta_0.$$

Right-multiplying both sides by $D_{\Gamma^*}(\theta_{\Gamma} \Gamma^* + I)^{-1}$ we get the first equality in the theorem.

The second one is proved similarly. $\square$

5B. The defect functions Δ_{Γ} and relations between them. Recall that every strict contraction Γ yields a characteristic matrix-valued function θ_{Γ} through association with the c.n.u. contraction U_{Γ} . The definition of the Sz.-Nagy–Foiş model space, see, e.g., formula (3-16), reveals immediately that the defect functions $\Delta_{\Gamma} = (I - \theta_{\Gamma}^* \theta_{\Gamma})^{1/2}$ are central objects in model theory. We express the defect function Δ_{Γ} in terms of Δ_0 (and Γ and θ_0).

Theorem 5.5. *The defect functions of θ_{Γ} and θ_0 are related by*

$$\Delta_{\Gamma}^2 = D_{\Gamma}(I - \theta_0^* \Gamma)^{-1} \Delta_0^2 (I - \Gamma^* \theta_0)^{-1} D_{\Gamma}.$$

Proof. By Theorem 5.1

$$\theta_{\Gamma} = D_{\Gamma^*}^{-1}(\theta_0 - \Gamma)(I_{\mathfrak{D}} - \Gamma^* \theta_0)^{-1} D_{\Gamma},$$

so $\theta_{\Gamma}^* \theta_{\Gamma} = A^* B A$, where

$$A = (I - \Gamma^* \theta_0) D_{\Gamma}, \quad B = (\theta_0^* - \Gamma^*) D_{\Gamma^*}^{-2} (\theta_0 - \Gamma).$$

Then $\Delta_{\Gamma} = I - \theta_{\Gamma}^* \theta_{\Gamma} = A^* X A$, where

$$\begin{aligned} X &= (A^*)^{-1} A^{-1} - B = (I - \theta_0^* \Gamma) D_{\Gamma}^{-2} (I - \Gamma^* \theta_0) - (\theta_0^* - \Gamma^*) D_{\Gamma^*}^{-2} (\theta_0 - \Gamma) \\ &= D_{\Gamma}^{-2} - \theta_0^* \Gamma D_{\Gamma}^{-2} - D_{\Gamma}^{-2} \Gamma^* \theta_0 + \theta_0^* \Gamma D_{\Gamma}^{-2} \Gamma^* \theta_0 - \theta_0^* D_{\Gamma^*}^{-2} \theta_0 + \Gamma^* D_{\Gamma^*}^{-2} \theta_0 + \theta_0^* D_{\Gamma^*}^{-2} \Gamma - \Gamma^* D_{\Gamma^*}^{-2} \Gamma. \end{aligned}$$

It follows from Lemma 5.4 that $D_{\Gamma}^{-2} \Gamma^* = \Gamma^* D_{\Gamma^*}^{-2}$ and that $\Gamma^* D_{\Gamma}^{-2} = D_{\Gamma^*}^{-2} \Gamma$, so in the above identity we have cancellation of nonsymmetric terms,

$$-\theta_0^* \Gamma D_{\Gamma}^{-2} - D_{\Gamma}^{-2} \Gamma^* \theta_0 + \Gamma^* D_{\Gamma^*}^{-2} \theta_0 + \theta_0^* D_{\Gamma^*}^{-2} \Gamma = 0.$$

Therefore

$$\begin{aligned} X &= D_{\Gamma}^{-2} + \theta_0^* \Gamma D_{\Gamma}^{-2} \Gamma^* \theta_0 - \theta_0^* D_{\Gamma^*}^{-2} \theta_0 - \Gamma^* D_{\Gamma^*}^{-2} \Gamma \\ &= D_{\Gamma}^{-2} + \theta_0^* D_{\Gamma^*}^{-2} \Gamma \Gamma^* \theta_0 - \theta_0^* D_{\Gamma^*}^{-2} \theta_0 - D_{\Gamma}^{-2} \Gamma^* \Gamma \\ &= D_{\Gamma}^{-2} (I - \Gamma^* \Gamma) + \theta_0^* D_{\Gamma^*}^{-2} (\Gamma^* \Gamma - I) \theta_0 = I - \theta_0^* \theta_0 = \Delta_0. \end{aligned}$$

Thus we get $\Delta_{\Gamma} = A^* \Delta_0 A$, which is exactly the conclusion of the theorem. $\square$

5C. Multiplicity of the absolutely continuous spectrum. It is well known that the Sz.-Nagy–Foiaş model space reduces to the familiar one-story setting with $\mathcal{K}_\theta = H^2(\mathfrak{D}_*) \ominus \theta H^2(\mathfrak{D})$ when θ is inner. Indeed, for inner θ the nontangential boundary values of the defect $\Delta(\xi) = (\mathbf{I} - \theta^*(\xi)\theta(\xi))^{1/2} = 0$ for Lebesgue a.e. $\xi \in \mathbb{T}$. So, the second component of the Sz.-Nagy–Foiaş model space collapses completely.

Here we provide a finer result that reveals the matrix-valued weight function and the multiplicity of U 's absolutely continuous part.

Before we formulate the statement, we recall some terminology. First, we Lebesgue decompose the (scalar) measure $d\mu = d\mu_{ac} + d\mu_{sing}$. The absolutely continuous part of U is unitarily equivalent to the multiplication by the independent variable ξ on the von Neumann direct integral $\mathcal{H}_{ac} = \int_{\mathbb{T}}^{\oplus} E(\xi) d\mu_{ac}(\xi)$. Note that the dimension of $E(\xi)$ is the multiplicity function of the spectrum.

Let w denote the density of the absolutely continuous part of μ , i.e., $d\mu_{ac}(\xi) = w(\xi) dm(\xi)$. Then the matrix-valued function $\xi \mapsto B^*(\xi)B(\xi)w(\xi)$ is the absolutely continuous part of the matrix-valued measure $B^*B\mu$.

Theorem 5.6. *The defect function Δ_0 of θ_0 and the absolutely continuous part B^*Bw of the matrix-valued measure $B^*B\mu$ are related by*

$$(\mathbf{I} - \theta_0^*(\xi))B^*(\xi)B(\xi)w(\xi)(\mathbf{I} - \theta_0(\xi)) = (\Delta_0(\xi))^2 \quad (5-4)$$

for Lebesgue a.e. $\xi \in \mathbb{T}$.

The function $\mathbf{I} - \theta_0$ is invertible a.e. on $\mathbb{T}$, so the multiplicity of the absolutely continuous part of μ is given by

$$\dim E(\xi) = \text{rank}(\mathbf{I} - \theta_0^*(\xi)\theta_0(\xi)) = \text{rank } \Delta_0(\xi), \quad (5-5)$$

of course, with respect to Lebesgue a.e. $\xi \in \mathbb{T}$.

Combining (5-5) with Theorem 5.5 we obtain:

Corollary 5.7. *For Lebesgue a.e. $\xi \in \mathbb{T}$ we have $\dim E(\xi) = \text{rank } \Delta_\Gamma(\xi)$ for all strict contractions Γ .*

Another immediate consequence is the following:

Corollary 5.8. *The operator U has no absolutely continuous part on a Borel set $B \subset \mathbb{T}$ if and only if $\theta_0(\xi)$ (or, equivalently, $\theta_\Gamma(\xi)$ for all strict contractions Γ) is unitary for Lebesgue a.e. $\xi \in B$.*

This corollary is closely related to the main result of [Douglas and Liaw 2013, Theorem 3.1]. Interestingly, it appears that their proof of that result cannot be refined to yield our current result (Theorem 5.6).

Corollary 5.9. *In particular, we confirm that the following are equivalent:*

- (i) U is purely singular.
- (ii) $\theta_\Gamma(\xi)$ is inner for one (equivalently any) strict contraction Γ .
- (iii) $\Delta_\Gamma \equiv \mathbf{0}$ for one (equivalently any) strict contraction Γ .
- (iv) The second story of the Sz.-Nagy–Foiaş model space collapses (and we are dealing with the model space $\mathcal{K}_{\theta_\Gamma} = H^2(\mathbb{C}^d) \ominus \theta_\Gamma H^2(\mathbb{C}^d)$ for one (equivalently any) strict contraction Γ).

Proof of Theorem 5.6. Take $\Gamma \equiv \mathbf{0}$. Solving (4-6) for F_2 we see

$$F_2(z) = [I + \theta_0(z)][I - \theta_0(z)]^{-1}.$$

Let $\mathcal{P}(B^*B\mu)$ denote the Poisson extension of the matrix-valued measure $B^*B\mu$ to the unit disc $\mathbb{D}$. Since $F_2 = \mathcal{C}_2 B^*B\mu$, we can see that $\mathcal{P}(B^*B\mu) = \operatorname{Re} F_2$ on $\mathbb{D}$, so

$$\mathcal{P}(B^*B\mu) = \operatorname{Re} F_2 = \operatorname{Re}[(I + \theta_0)(I - \theta_0)^{-1}].$$

Standard computations yield

$$\begin{aligned} \mathcal{P}(B^*B\mu) &= \operatorname{Re}[(I + \theta_0)(I - \theta_0)^{-1}] = \frac{1}{2}[(I + \theta_0)(I - \theta_0)^{-1} + (I - \theta_0^*)^{-1}(I + \theta_0^*)] \\ &= \frac{1}{2}(I - \theta_0^*)^{-1}[(I - \theta_0^*)(I + \theta_0) + (I + \theta_0^*)(I - \theta_0)](I - \theta_0)^{-1} \\ &= \frac{1}{2}(I - \theta_0^*)^{-1}[I - \theta_0^*\theta_0](I - \theta_0)^{-1} = (I - \theta_0^*)^{-1} \operatorname{Re}[I - \theta_0^*\theta_0](I - \theta_0)^{-1} \\ &= (I - \theta_0^*)^{-1}[I - \theta_0^*\theta_0](I - \theta_0)^{-1} \end{aligned}$$

on $\mathbb{D}$. Note that for any characteristic function θ and $z \in \mathbb{D}$ the matrix $\theta(z)$ is a strict contraction, so in our case $I - \theta_0$ is invertible on $\mathbb{D}$, and all computations are justified.

We can rewrite the above identity as

$$(I - \theta_0)^*\mathcal{P}(B^*B\mu)(I - \theta_0) = I - \theta_0^*\theta_0,$$

and taking the nontangential boundary values we get (5-4). Here we used the Fatou lemma, see, e.g., [Nikolski 2002a, Theorem 3.11.7], which says that for a complex measure τ the nontangential boundary values of its Poisson extension $\mathcal{P}\tau$ coincide a.e. with the density of the absolutely continuous part of τ ; applying this lemma entrywise we get what we need in the left-hand side.

To see that the boundary values of $I - \theta_0$ are invertible a.e. on $\mathbb{T}$ we notice that $z \mapsto \det(I - \theta_0(z))$ is a bounded analytic function on $\mathbb{D}$, so its boundary values are nonzero a.e. on $\mathbb{T}$. $\square$

6. What is wrong with the universal representation formula and what to do about it?

There are several things that are not completely satisfactory with the universal representation formula given by Theorem 2.4.

First of all, it is defined only on functions of form $h b$, where $h \in C^1$ is a scalar function and $b \in \operatorname{Ran} B$. Of course, one can then define it on a dense set, for example on the dense set of linear combinations $f = \sum_k h_k b_k$, where b_k are columns of the matrix B , $b_k = B e_k$, and $h_k \in C^1(\mathbb{T})$. But the use of functions b (or b_k) in the representation is a bit bothersome, especially taking into account that the representation $f = \sum_k h_k b_k$ is not always unique. So, it would be a good idea to get rid of the function b .

The second thing is that while the representation formula looks like a singular integral operator (Cauchy transform), it is not represented as a classical singular integral operator, so it is not especially clear if the (well-developed) theory of such operators applies in our case. So, we would like to represent the operator in more classical way.

Defining $C_1(z) := C_*(z) - zC(z)$ and using the formal Cauchy-type expression

$$(T^{B^* \mu} f)(z) = \int_{\mathbb{T}} \frac{1}{1 - z\bar{\xi}} B^*(\xi) f(\xi) d\mu(\xi),$$

we can, performing formal algebraic manipulations, rewrite (2-10) as

$$(\Phi^* h b)(z) = C_1(z)(T^{B^* \mu} h b)(z) + h(z)[C_*(z)B^* b - C_1(z)(T^{B^* \mu} b)(z)], \quad z \in \mathbb{T}. \quad (6-1)$$

So, is it possible to turn these formal manipulations into meaningful mathematics? And the answer is “yes”: the formula (6-1) gives the representation of Φ^* if one interprets $T^{B^* \mu} f$ as the boundary values of the Cauchy transform $\mathcal{C}[B^* f \mu](z)$, $z \notin \mathbb{T}$; see the definition in the next section.

In the next section (Section 7) we present necessary facts about the (vector-valued) Cauchy transform and its regularization that will allow us to interpret and justify the formal expression (6-1). We will complete this justification in Section 8; see (8-12). This representation is a universal one, meaning that it works in any transcription of the model, but still involves the function $b \in \text{Ran } B$.

The function b is kind of eliminated in Proposition 8.4 below, and as it is usually happens in the theory of singular integral operators, the operator Φ^* splits into the singular integral part (weighted boundary values of the Cauchy transform) and the multiplication part. The function b becomes hidden in the multiplication part, and at first glance it is not clear why this part is well-defined.

Thus the representation given by Proposition 8.4 is still not completely satisfactory (the price one pays for the universality), but it is a step towards obtaining a nice representation for a fixed transcription of a model. Thus we were able to obtain a precise and unambiguous representation of Φ^* in the Sz.-Nagy–Foiaş transcription; see Theorem 8.1 which is the main result of Section 8.

7. Singular integral operators

7A. Cauchy-type integrals. For a finite (signed or even complex-valued) measure ν on $\mathbb{T}$, its Cauchy transform $\mathcal{C}\nu$ is defined as

$$\mathcal{C}\nu(z) = \mathcal{C}[\nu](z) = \int_{\mathbb{T}} \frac{d\nu(\xi)}{1 - \bar{\xi}z}, \quad z \in \mathbb{C} \setminus \mathbb{T}.$$

It is a classical fact that $\mathcal{C}\nu(z)$ has nontangential boundary values a.e. on $\mathbb{T}$ as $z \rightarrow z_0 \in \mathbb{T}$ from the inside and from the outside of the disc $\mathbb{D}$. So, given a finite positive Borel measure μ one can define operators $T_{\pm}^{\mu}$ from $L^1(\mu; E)$ to the space of measurable functions on $\mathbb{T}$ as the nontangential boundary values from inside and outside of the unit disc $\mathbb{D}$,

$$(T_{+}^{\mu} f)(z_0) = \text{n.t.-} \lim_{\substack{z \rightarrow z_0 \\ z \in \mathbb{D}}} \mathcal{C}[f\mu](z), \quad (T_{-}^{\mu} f)(z_0) = \text{n.t.-} \lim_{\substack{z \rightarrow z_0 \\ z \notin \mathbb{D}}} \mathcal{C}[f\mu](z).$$

One can also define the regularized operators T_r^{μ} , $r \in (0, \infty) \setminus \{1\}$, and the restriction of $\mathcal{C}[f\mu]$ to the circle of radius r ,

$$T_r^{\mu} f(z) = \mathcal{C}[f\mu](rz).$$

Everything can be extended to the case of vector- and matrix-valued measures; there are some technical details that should be taken care of in the infinite-dimensional case, but in our case everything is finite-dimensional ($\dim E \leq d < \infty$), so the generalization is pretty straightforward.

So, given a (finite, positive) scalar measure μ and a matrix-valued function B^* (with entries in $L^2(\mu)$) and vector-valued function $f \in L^2(\mu; E)$ we can define $T_{\pm}^{B^*\mu} f$ and $T_r^{B^*\mu} f$ as the nontangential boundary values and the restriction to the circle of radius r respectively of the Cauchy transform $\mathcal{C}[B^* f \mu](z)$. Modulo slight abuse of notation this notation agrees with the accepted notation for the scalar case.

In what follows the function B^* will be the function B^* from [Theorem 2.4](#).

7B. Uniform boundedness of the boundary Cauchy operator and its regularization. For a finite Borel measure ν on $\mathbb{T}$ and $n \in \mathbb{Z}$ define

$$P_n \nu(z) = \begin{cases} \sum_{k=0}^n \hat{\nu}(k) z^k, & n \geq 0, \\ \sum_{k=n}^{-1} \hat{\nu}(k) z^k, & n < 0; \end{cases}$$

here $\hat{\nu}(k)$ is the Fourier coefficient of ν , $\hat{\nu}(k) = \int_{\mathbb{T}} \xi^{-k} d\nu(\xi)$.

Recall that $C_1(z) := C_*(z) - zC(z)$, where C_* and C are from [Theorem 2.4](#).

Recall that if W is a matrix-valued weight (i.e., a function whose values $W(\xi)$ are positive semidefinite operators on a finite-dimensional space H), then the norm in the weighted space $L^2(W; H)$ is defined as

$$\|f\|_{L^2(W; H)}^2 = \int_{\mathbb{T}} (W(\xi) f(\xi), f(\xi))_H d\mu(\xi).$$

We are working with the model space $\mathcal{K}_{\theta}$ which is a subspace of a weighted space $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$ (the weight could be trivial, $W \equiv I$, as in the case of Sz.-Nagy–Foiş model).

Define $\tilde{C}_1 := W^{1/2} C_1$. The function $\tilde{C}_1^* \tilde{C}_1$ is a matrix-valued weight, whose values are operators on $\mathfrak{D}_* \oplus \mathfrak{D}$, so we can define the weighted space $L^2(\tilde{C}_1^* \tilde{C}_1) = L^2(\tilde{C}_1^* \tilde{C}_1; \mathfrak{D}_* \oplus \mathfrak{D})$. Note that

$$\|f\|_{L^2(\tilde{C}_1^* \tilde{C}_1)} = \|\tilde{C}_1 f\|_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})} = \|C_1 f\|_{L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})}.$$

Lemma 7.1. *The operators $P_n^{B^*\mu} : \mathcal{H} \subset L^2(\mu; E) \rightarrow L^2(\tilde{C}_1^* \tilde{C}_1; \mathfrak{D}_* \oplus \mathfrak{D})$ defined by*

$$P_n^{B^*\mu} f := P_n(B^* f \mu), \quad n \in \mathbb{Z},$$

are bounded uniformly in n with norm at most 2; i.e.,

$$\|\tilde{C}_1 P_n(B^* f \mu)\|_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})} \leq 2\|f\|_{L^2(\mu; E)}.$$

Proof. The columns b_k of B are in $\mathcal{H} \subset L^2(\mu; E)$, so $B^* f \mu \in L^1(\mu; \mathfrak{D})$, and therefore the operators $P_n^{B^*\mu}$ are bounded operators $\mathcal{H} \rightarrow L^2(\mathfrak{D})$. It follows from [Lemma 3.4](#) that $\|\tilde{C}_1\|_{\infty} \leq 2$, so the operators $f \mapsto \tilde{C}_1 P_n^{B^*\mu} f$ are bounded operators $\mathcal{H} \rightarrow L^2(\mathfrak{D}_* \oplus \mathfrak{D})$ (notice that we do not claim the uniform-in- n bound here). Therefore, it is sufficient to check the uniform boundedness on a dense set.

Take $f = hb$, where $b \in \text{Ran } B$ and $h \in C^1(\mathbb{T})$ is scalar-valued. Then for $n \in \mathbb{Z}$ we have by [Theorem 2.4](#)

$$\begin{aligned} \Phi^* f - z^n \Phi^*(\bar{\xi}^n f) &= C_1(z) \int_{\mathbb{T}} \frac{h(\xi) - h(z)}{1 - \bar{\xi}z} B^* b \, d\mu(\xi) - z^n C_1(z) \int_{\mathbb{T}} \frac{\bar{\xi}^n h(\xi) - \bar{z}^n h(z)}{1 - \bar{\xi}z} B^* b \, d\mu(\xi) \\ &= C_1(z) \int_{\mathbb{T}} \frac{1 - (\bar{\xi}z)^n}{1 - \bar{\xi}z} B^* h b \, d\mu(\xi). \end{aligned}$$

Expressing $(1 - (\bar{\xi}z)^n)/(1 - \bar{\xi}z)$ as a sum of geometric series we get that for $f = hb$, $h \in C^1(\mathbb{T})$,

$$\Phi^* f - z^n \Phi^*(\bar{\xi}^n f) = \begin{cases} C_1 P_{n-1}(B^* f \mu), & n \geq 1, \\ -C_1 P_n(B^* f \mu), & n < 0. \end{cases}$$

By linearity the above identity holds for a dense set of linear combinations $f = \sum_k h_k b_k$, $h_k \in C^1(\mathbb{T})$. The operators $\Phi^* : \mathcal{H} \rightarrow \mathcal{K}_\theta \subset L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$ are bounded (unitary) operators, so the desired estimate holds on the above dense set. $\square$

For a measure ν on $\mathbb{T}$ let $T_r \nu$ be the restriction of the Cauchy transform of ν to the circle of radius $r \neq 1$,

$$T_r \nu(z) = \int_{\mathbb{T}} \frac{d\nu(\xi)}{1 - r\bar{\xi}z}, \quad z \in \mathbb{T}.$$

Define operators $T_r^{B^* \mu}$ on $L^2(\mu; E)$ as

$$T_r^{B^* \mu} f = T_r(B^* f \mu).$$

The lemma below is an immediate corollary of the above [Lemma 7.1](#).

Lemma 7.2. *The operators $T_r^{B^* \mu} : \mathcal{H} \subset L^2(\mu; E) \rightarrow L^2(\tilde{C}_1^* \tilde{C}_1; \mathfrak{D}_* \oplus \mathfrak{D})$ are bounded uniformly in r with norm at most 2; i.e.,*

$$\|\tilde{C}_1 T_r^{B^* \mu} f\|_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})} \leq 2 \|f\|_{L^2(\mu; E)}.$$

Proof. The result follows immediately from [Lemma 7.1](#), since the operators $T_r^{B^* \mu}$ can be represented as averages of operators $P_n^{B^* \mu}$,

$$T_r^{B^* \mu} = \begin{cases} \sum_{n=0}^{\infty} (r^n - r^{n+1}) P_n^{B^* \mu}, & 0 < r < 1, \\ \sum_{n=1}^{\infty} (r^{-n+1} - r^{-n}) P_{-n}^{B^* \mu}, & r > 1. \end{cases} \quad \square$$

Using uniform boundedness of the operators $\tilde{C}_1 T_r^{B^* \mu}$ ([Lemma 7.2](#)) and existence of nontangential boundary values $T_{\pm}^{B^* \mu} f$ we can get the convergence of operators $\tilde{C}_1 T_r^{B^* \mu}$ in the weak operator topology.

Proposition 7.3. *The operators $\tilde{C}_1 T_{\pm}^{B^* \mu} : \mathcal{H} \subset L^2(\mu; E) \rightarrow L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$ are bounded and*

$$C_1 T_{\pm}^{B^* \mu} = \text{w.o.t.-} \lim_{r \rightarrow 1^{\mp}} C_1 T_r^{B^* \mu}.$$

Proof. We want to show that for any $f \in \mathcal{H} \subset L^2(\mu; E)$

$$C_1 T_{\pm}^{B^* \mu} f = \text{w-} \lim_{r \rightarrow 1^{\mp}} C_1 T_r^{B^* \mu} f,$$

where the limit is in the weak topology of $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$. This is equivalent to

$$\tilde{C}_1 T_{\pm}^{B^* \mu} f = \text{w-} \lim_{r \rightarrow 1^{\mp}} \tilde{C}_1 T_r^{B^* \mu} f,$$

with the limit being in the weak topology of $L^2(\mathfrak{D}_* \oplus \mathfrak{D})$.

Let us prove this identity for $\tilde{C}_1 T_+^{B^* \mu} f$. Assume that for some $f \in L^2(\mu; E)$

$$\tilde{C}_1 T_+^{B^* \mu} f \neq \text{w-} \lim_{r \rightarrow 1^-} \tilde{C}_1 T_r^{B^* \mu} f.$$

Then for some $h \in L^2(\mathfrak{D}_* \oplus \mathfrak{D})$

$$(\tilde{C}_1 T_r^{B^* \mu} f, h)_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})} \not\rightarrow (\tilde{C}_1 T_+^{B^* \mu} f, h)_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})} \quad \text{as } r \rightarrow 1^-, \quad (7-1)$$

so there exists a sequence $r_k \nearrow 1$ such that

$$\lim_{k \rightarrow \infty} (\tilde{C}_1 T_{r_k}^{B^* \mu} f, h)_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})} \neq (\tilde{C}_1 T_+^{B^* \mu} f, h)_{L^2(\mathfrak{D}_* \oplus \mathfrak{D})};$$

note that taking a subsequence we can assume without loss of generality that the limit in the left-hand side exists.

Taking a subsequence again, we can assume without loss of generality that $\tilde{C}_1 T_{r_k}^{B^* \mu} f \rightarrow g$ in the weak topology, and (7-1) implies $g \neq \tilde{C}_1 T_+^{B^* \mu} f$.

The existence of nontangential boundary values and the definition of $T_+^{B^* \mu}$ implies $\tilde{C}_1 T_{r_k}^{B^* \mu} f \rightarrow \tilde{C}_1 T_+^{B^* \mu} f$ a.e. on $\mathbb{T}$. But as [Liaw and Treil 2009, Lemma 3.3] asserts, if $f_n \rightarrow f$ a.e. and $f_n \rightarrow g$ in the weak topology of L^2 , then $f = g$, so we arrive at a contradiction.

Note, that in [Liaw and Treil 2009, Lemma 3.3] everything was stated for scalar functions, but applying this scalar lemma componentwise we immediately get the same result for $L^2(\mu; E)$ with values in a separable Hilbert space. $\square$

8. Adjoint Clark operator in Sz.-Nagy–Foiaş transcription

The main result of this section is Theorem 8.1 below, giving a formula for the adjoint Clark operator Φ^* .

Denote by F the Cauchy transform of the matrix-valued measure $B^* B \mu$,

$$F(z) = \mathcal{C}[B^* B \mu](z) = \int_{\mathbb{T}} \frac{1}{1 - z \bar{\xi}} B^*(\xi) B(\xi) d\mu(\xi), \quad z \in \mathbb{D}, \quad (8-1)$$

and let us use the same symbol for its nontangential boundary values, which exist a.e. on $\mathbb{T}$. Using the operator $T_+^{B^* \mu}$ introduced in the previous section, we give the following formula for Φ^* .

Theorem 8.1. *The adjoint Clark operator in Sz.-Nagy–Foiaş transcription reduces to*

$$\Phi^* f = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} f + \begin{pmatrix} (I + \theta_{\Gamma} \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta_{\Gamma} D_{\Gamma}^{-1} (\Gamma^* - I) \end{pmatrix} T_+^{B^* \mu} f, \quad f \in \mathcal{H}, \quad (8-2)$$

with $\Psi_2(z) = \tilde{\Psi}_2(z)R(z)$, where

$$\begin{aligned}\tilde{\Psi}_2(z) &= \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* + (\mathbf{I} - \Gamma^*)F(z)) \\ &= \Delta_\Gamma D_\Gamma^{-1}(\mathbf{I} - \Gamma^*\theta_0(z))F(z) \quad \text{a.e. on } \mathbb{T},\end{aligned}\tag{8-3}$$

and R is a measurable right inverse for the matrix-valued function B .

Remark. When $d = 1$, this result reduces to [Liaw and Treil 2016, equation (4.5)].

Remark 8.2. As one should expect, the matrix-valued function Ψ_2 does not depend on the choice of the right inverse R . To prove this, it is sufficient to show that $\ker B(z) \subset \ker \tilde{\Psi}_2(z)$ a.e., which follows from the proposition below.

Proposition 8.3. For $\tilde{\Psi}_2$ defined above in (8-16) and w being the density of μ_{ac} we have

$$\tilde{\Psi}_2(\xi)^* \tilde{\Psi}_2(\xi) = F(\xi)^* \Delta_0(\xi)^2 F(\xi) = B(\xi)^* B(\xi) w(\xi) \quad \mu_{ac}\text{-a.e.},\tag{8-4}$$

and so

$$\Psi_2(\xi)^* \Psi_2(\xi) = w(\xi) \mathbf{I}_{E(\xi)} \quad \mu_{ac}\text{-a.e.}\tag{8-5}$$

Proof. Since $\Psi_2 = \tilde{\Psi}_2 R$, (8-5) follows immediately from (8-4).

To prove (8-4), consider first the case, $\Gamma = \mathbf{0}$. In this case $\tilde{\Psi} = \Delta_0 F$, so

$$\begin{aligned}\tilde{\Psi}_2^* \tilde{\Psi}_2 &= F^* \Delta_0^2 F = (\mathbf{I} - \theta_0^*)^{-1} \Delta_0^2 (\mathbf{I} - \theta_0)^{-1} \\ &= B^* B w \quad \text{by (5-4)}.\end{aligned}\tag{8-6}$$

Consider now the case of general Γ . We get

$$\begin{aligned}\tilde{\Psi}_2^* \tilde{\Psi}_2 &= F^* (\mathbf{I} - \theta_0^* \Gamma) D_\Gamma^{-1} \Delta_\Gamma^2 D_\Gamma^{-1} (\mathbf{I} - \Gamma^* \theta_0) F \\ &= F^* \Delta_0^2 F \quad \text{by Theorem 5.5} \\ &= B^* B w \quad \text{by (8-6)}.\end{aligned}\tag{8-7}$$

□

8A. A preliminary formula. We start proving Theorem 8.1 by first proving this preliminary result, which holds for any transcription of the model. Below, the matrix-valued functions C_* and C are from Theorem 2.4, and $C_1(z) := C_*(z) - zC(z)$.

Proposition 8.4. The adjoint Clark operator is represented for $f \in \mathcal{H} \subset L^2(\mu; E)$ by

$$(\Phi^* f)(z) = C_1(z)(T_\pm^{B^* \mu} f)(z) + \Psi_\pm(z)f(z), \quad z \in \mathbb{T},\tag{8-7}$$

where the matrix-functions $\Psi_\pm, \Psi_\pm(z) : E(z) \rightarrow \mathbb{C}^{2d} = \mathfrak{D}_* \oplus \mathfrak{D}$ are defined via the identities

$$\Psi_\pm(z)b(z) := C_*(z)B^*b - C_1(z)(T_\pm^{B^* \mu} b)(z), \quad b \in \text{Ran } B;\tag{8-8}$$

here two choices of sign (the same sign for all terms) give two different representation formulas.

Remark. When $d = 1$ and $b \equiv 1$ this alternative representation formula reduces to a formula that occurs in the proof of [Liaw and Treil 2016, Theorem 4.7].

Remark. It is clear that relations (8-8) with $b = b_k$, $k = 1, 2, \dots, d$, completely define the matrix-valued function Ψ . However, it is not immediately clear that such a function Ψ exists; the existence of Ψ will be shown in the proof.

Recalling the definition (8-1) of the function F , we can see that $\Psi(z)b_k(z)$ can be given as the (nontangential) boundary values of the vector-valued function

$$C_*(z)e_k - C_1(z)F(z)e_k, \quad z \in \mathbb{D}, \quad (8-9)$$

where $e_1, e_2, \dots, e_d$ is the standard orthonormal basis in $\mathbb{C}^d$.

Proof of Proposition 8.4. Let us first show the result for functions of the form $f = hb \in L^2(\mu; E)$, where $b \in \text{Ran } \mathbf{B}$ and h is a scalar function. We want to show that

$$(\Phi^*hb)(z) = C_1(z)(T_{\pm}^{B^*\mu}hb)(z) + h(z)\psi_b^{\pm}(z), \quad z \in \mathbb{T}, \quad (8-10)$$

where

$$\psi_b^{\pm}(z) := C_*(z)\mathbf{B}^*b - C_1(z)(T_{\pm}^{B^*\mu}b)(z).$$

First note that (2-10) implies that for $b \in \text{Ran } \mathbf{B}$

$$\Phi^*b(z) = C_*(z)\mathbf{B}^*b.$$

Observe that for (scalar) $h \in C^1$ we have uniform-on- z convergence, $z \in \mathbb{T}$, as $r \rightarrow 1^{\mp}$:

$$\int_{\mathbb{T}} \frac{h(\xi) - h(z)}{1 - rz\bar{\xi}} B^*(\xi)b(\xi) d\mu(\xi) \Rightarrow \int_{\mathbb{T}} \frac{h(\xi) - h(z)}{1 - z\bar{\xi}} B^*(\xi)b(\xi) d\mu(\xi). \quad (8-11)$$

Multiplying both sides by $C_1(z)$ we get in the left-hand side $C_1(z)(T_r^{B^*\mu}hb)(z) - h(z)C_1(z)(T_r^{B^*\mu}b)(z)$, and in the right-hand side the part with the integral in the representation (2-10).

Recall that the model space $\mathcal{K}_{\theta_r}$ is a subspace of a weighted space $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$. Uniform convergence in (8-11) implies the convergence in $L^2(\mathfrak{D}_* \oplus \mathfrak{D})$, and by Lemma 3.4 the multiplications by C_* and C_1 are bounded operators from $L^2(\mathfrak{D})$ to $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$. Thus (because h is bounded)

$$hC_*\mathbf{B}^*b + C_1T_r^{B^*\mu}hb - hC_1T_r^{B^*\mu}b \rightarrow \Phi^*hb$$

as $r \rightarrow 1^{\mp}$ in the norm of $L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$. By Proposition 7.3, $C_1T_r^{B^*\mu} \rightarrow C_1T_{\pm}^{B^*\mu}$ in weak operator topology as $r \rightarrow 1^{\mp}$, so

$$\Phi^*hb = C_1T_{\pm}^{B^*\mu}hb + hC_*\mathbf{B}^*b - hC_1T_{\pm}^{B^*\mu}b, \quad (8-12)$$

which immediately implies (8-10). Thus, (8-10) is proved for $h \in C^1(\mathbb{T})$.

To get (8-12), and so (8-10) for general h such that $hb \in L^2(\mu; E)$ (recall that $b \in \text{Ran } \mathbf{B}$), we use the standard approximation argument: the operators $\Phi^*, C_1T_{\pm}^{B^*\mu} : \mathcal{H} \rightarrow L^2(W; \mathfrak{D}_* \oplus \mathfrak{D})$ are bounded, and therefore for a fixed $b \in \text{Ran } \mathbf{B}$ the operators $hb \mapsto h\psi_b^{\pm}$ (which are defined initially on a submanifold of $\mathcal{H}$ consisting of functions of the form hb , $h \in C^1(\mathbb{T})$) are bounded (as a difference of two bounded operators). Approximating in $L^2(\mu; E)$ the function hb by functions h_nb , $h_n \in C^1(\mathbb{T})$ we get (8-12) and (8-10) for general h .

Let us now prove the existence of Ψ . Consider the (bounded) linear operator $\Phi^* - C_1 T^{B^* \mu}$. We know that for $f = hb \in L^2(\mu; E)$ with $b \in \text{Ran } B$ and scalar h

$$(\Phi^* - C_1 T_{\pm}^{B^* \mu})hb = h\psi_b^{\pm},$$

so on functions $f = hb$ the operators $\Phi^* - C_1 T_{\pm}^{B^* \mu}$ intertwine the multiplication operators M_{ξ} and M_z . Since linear combinations of functions $h_k b_k$ are dense in $\mathcal{H}$, we conclude that the operators $\Phi^* - C_1 T_{\pm}^{B^* \mu}$ intertwine M_{ξ} and M_z on all $\mathcal{H}$, and so these operators are the multiplications by some matrix functions $\Psi_{\pm}$.

Using (8-12) with $h = 1$ we can see that

$$\Psi_{\pm} b = \Phi^* b - C_1 T_{\pm}^{B^* \mu} b = C_* B^* b - C_1 T_{\pm}^{B^* \mu} b,$$

so $\Psi_{\pm}$ are defined exactly as stated in the proposition. $\square$

8B. Some calculations. Let us start with writing more-detailed formulas for the matrix functions C_* and C_1 from Proposition 8.4.

Lemma 8.5. *We have*

$$C_*(z) = \begin{pmatrix} I + \theta_{\Gamma}(z)\Gamma^* \\ \Delta_{\Gamma}(z)\Gamma^* \end{pmatrix} D_{\Gamma^*}^{-1}, \quad C_1(z) = \begin{pmatrix} I \\ 0 \end{pmatrix} D_{\Gamma^*}^{-1}(I - \Gamma) + \begin{pmatrix} \theta_{\Gamma}(z) \\ \Delta_{\Gamma}(z) \end{pmatrix} D_{\Gamma}^{-1}(\Gamma^* - I).$$

Proof. The formula for $C_*(z)$ is just (3-21) and the identity $\theta_{\Gamma}(0) = -\Gamma$. Similarly, equation (3-22) gives us

$$C(z) = \begin{pmatrix} z^{-1}(\theta_{\Gamma}(z) + \Gamma) \\ z^{-1}\Delta_{\Gamma}(z) \end{pmatrix} D_{\Gamma}^{-1}.$$

Substituting these expressions into $C_1(z) = C_*(z) - zC(z)$ and applying the commutation relations from Lemma 5.4 we see

$$\begin{aligned} C_1(z) &= \begin{pmatrix} D_{\Gamma^*}^{-1} + \theta_{\Gamma}\Gamma^*D_{\Gamma^*}^{-1} - \theta_{\Gamma}D_{\Gamma}^{-1} - \Gamma D_{\Gamma}^{-1} \\ \Delta_{\Gamma}\Gamma^*D_{\Gamma^*}^{-1} - \Delta_{\Gamma}D_{\Gamma}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} D_{\Gamma^*}^{-1} + \theta_{\Gamma}D_{\Gamma}^{-1}\Gamma^* - \theta_{\Gamma}D_{\Gamma}^{-1} - D_{\Gamma^*}^{-1}\Gamma \\ \Delta_{\Gamma}D_{\Gamma}^{-1}\Gamma^* - \Delta_{\Gamma}D_{\Gamma}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} D_{\Gamma^*}^{-1}(I - \Gamma) + \theta_{\Gamma}D_{\Gamma}^{-1}(\Gamma^* - I) \\ \Delta_{\Gamma}D_{\Gamma}^{-1}(\Gamma^* - I) \end{pmatrix} \\ &= \begin{pmatrix} I \\ 0 \end{pmatrix} D_{\Gamma^*}^{-1}(I - \Gamma) + \begin{pmatrix} \theta_{\Gamma} \\ \Delta_{\Gamma} \end{pmatrix} D_{\Gamma}^{-1}(\Gamma^* - I), \end{aligned}$$

and the second statement in the lemma is verified. $\square$

Recall that $F(z)$, $z \in \mathbb{D}$, is the matrix-valued Cauchy transform of the measure $B^*B\mu$, see (8-1), and that for $z \in \mathbb{T}$ the symbol $F(z)$ denotes the nontangential boundary values of F . We need the following simple relations between F and θ_0 .

Lemma 8.6. *For all $z \in \mathbb{D}$ and a.e. on $\mathbb{T}$*

$$F(z) = (\mathbf{I} - \theta_0(z))^{-1};$$

note that for all $z \in \mathbb{D}$ the matrix $\theta_0(z)$ is a strict contraction, so $\mathbf{I} - \theta_0(z)$ is invertible.

Proof. Recall that the function F_1 was defined by $F_1(z) = C_1[B^*B\mu](z)$. Since $F(z) = \mathbf{I} + F_1(z)$, we get from (4-5) that

$$\theta_0(z) = F_1(z)(\mathbf{I} + F_1(z))^{-1} = (F(z) - \mathbf{I})F(z)^{-1}.$$

Solving for F we get the conclusion of the lemma. $\square$

8C. Proof of Theorem 8.1. Let us first prove the second identity in (8-3). Using the identity $F = (\mathbf{I} - \theta_0)^{-1}$ we compute

$$\Gamma^* + (\mathbf{I} - \Gamma^*)F = (\Gamma^*(\mathbf{I} - \theta_0) + \mathbf{I} - \Gamma^*)F = (\mathbf{I} - \Gamma^*\theta_0)F,$$

which is exactly what we need.

We now prove that Ψ from Proposition 8.4 is given by $\Psi = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix}$, with Ψ_2 defined above in Theorem 8.1. Since $R(z)b_k(z) = e_k$, it is sufficient to show that $\Psi = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix}$ and that

$$\Psi_2(z)b_k(z) = \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* + (\mathbf{I} - \Gamma^*)F(z))e_k, \quad k = 1, 2, \dots, d. \quad (8-13)$$

Using the formulas for C_* and C_1 provided in Lemma 8.5 we get from (8-9)

$$\Psi(z)b_k(z) = C_*(z)e_k - C_1(z)F(z)e_k = \begin{pmatrix} (\mathbf{I} + \theta_\Gamma \Gamma^*)D_{\Gamma^*}^{-1} - [D_{\Gamma^*}^{-1}(\mathbf{I} - \Gamma) + \theta_\Gamma D_\Gamma^{-1}(\Gamma^* - \mathbf{I})]F \\ \Delta_\Gamma \Gamma^* D_{\Gamma^*}^{-1} - \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* - \mathbf{I})F \end{pmatrix} e_k.$$

Note that it is clear from the representation (8-7) that the top entry of Ψ should disappear, i.e., that

$$(\mathbf{I} + \theta_\Gamma \Gamma^*)D_{\Gamma^*}^{-1} = [D_{\Gamma^*}^{-1}(\mathbf{I} - \Gamma) + \theta_\Gamma D_\Gamma^{-1}(\Gamma^* - \mathbf{I})]F. \quad (8-14)$$

Indeed, by the definition of $\mathcal{K}_\theta$ in the Sz.-Nagy–Foiş transcription the top entry of Φ^*f belongs to $H^2(\mathcal{D}_*)$. One can see from Lemma 8.5, for example, that the top entry of C_1 belongs to the matrix-valued H^∞ , so the top entry of $C_1 T_+^{B^*\mu} f$ is also in $H^2(\mathcal{D}_*)$. Therefore the top entry of Ψf must be in $H^2(\mathcal{D}_*)$ for all f . But that is impossible, because f can be any function in $L^2(\mu; E)$.

For a reader that is not comfortable with such “soft” reasoning, we present a “hard” computational proof of (8-14). This computation also helps to assure the reader that the previous computations were correct.

To do the computation, consider the term in the square brackets in the right-hand side of (8-14). Using the commutation relations from Lemma 5.4 in the second equality, we get

$$\begin{aligned} D_{\Gamma^*}^{-1}(\mathbf{I} - \Gamma) + \theta_\Gamma D_\Gamma^{-1}(\Gamma^* - \mathbf{I}) &= D_{\Gamma^*}^{-1} + \theta D_\Gamma^{-1} \Gamma^* - \theta D_\Gamma^{-1} - D_{\Gamma^*}^{-1} \Gamma \\ &= D_{\Gamma^*}^{-1} + \theta \Gamma^* D_{\Gamma^*}^{-1} - \theta D_\Gamma^{-1} - \Gamma D_\Gamma^{-1} \\ &= (\mathbf{I} + \theta_\Gamma \Gamma^*)D_{\Gamma^*}^{-1} \{ \mathbf{I} - D_{\Gamma^*}(\mathbf{I} + \theta_\Gamma \Gamma^*)^{-1}(\theta_\Gamma + \Gamma)D_\Gamma^{-1} \} \\ &= (\mathbf{I} + \theta_\Gamma \Gamma^*)D_{\Gamma^*}^{-1} \{ \mathbf{I} - \theta_0 \}; \end{aligned}$$

the last equality holds by Theorem 5.2.

By Lemma 8.6 we have $I - \theta_0 = F^{-1}$, so we have for the term in the square brackets

$$[D_{\Gamma^*}^{-1}(I - \Gamma) + \theta_\Gamma D_\Gamma^{-1}(\Gamma^* - I)] = (I + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1},$$

which proves (8-14).

To deal with the bottom entry of Ψ we use the commutation relations from Lemma 5.4,

$$\begin{aligned} \Delta_\Gamma \Gamma^* D_{\Gamma^*}^{-1} - \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* - I)F &= \Delta_\Gamma D_\Gamma^{-1} \Gamma^* - \Delta_\Gamma D_\Gamma^{-1} \Gamma^* F + \Delta_\Gamma D_\Gamma^{-1} F \\ &= \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* + (I - \Gamma^*)F), \end{aligned}$$

which gives the desired formula (8-13) for Ψ_2 .

Finally, let us deal with the second term in the right-hand side of (8-2). We know from Proposition 8.4 that the term in front of $T_+^{B^* \mu} f$ is given by C_1 . From Lemma 8.5 we get

$$C_1 = \begin{pmatrix} D_{\Gamma^*}^{-1}(I - \Gamma) + \theta_\Gamma D_\Gamma^{-1}(\Gamma^* - I) \\ \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* - I) \end{pmatrix}.$$

But the top entry of C_1 here is the expression in brackets in the right-hand side of (8-14), so it is equal to $(I + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1}$. Therefore

$$C_1 = \begin{pmatrix} (I + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* - I) \end{pmatrix},$$

which is exactly what we have in (8-2). □

8D. Representation of Φ^* using matrix-valued measures. The above Theorem 8.1 is more transparent if we represent the direct integral $\mathcal{H}$ as the weighted L^2 space with a matrix-valued measure.

Namely, consider the weighted space $L^2(B^* B \mu)$,

$$\|f\|_{L^2(B^* B \mu)}^2 = \int_{\mathbb{T}} (B(\xi)^* B(\xi) f(\xi), f(\xi))_{\mathbb{C}^d} d\mu(\xi) = \int_{\mathbb{T}} \|B(\xi) f(\xi)\|_{\mathbb{C}^d}^2 d\mu(\xi)$$

(of course one needs to take the quotient space over the set of functions with norm 0).

Then for all scalar functions φ_k we have

$$\left\| \sum_{k=1}^d \varphi_k e_k \right\|_{L^2(B^* B \mu)} = \left\| \sum_{k=1}^d \varphi_k b_k \right\|_{L^2};$$

recall that $e_1, e_2, \dots, e_d$ is the standard basis in $\mathbb{C}^d$ and $b_k(\xi) = B(\xi) e_k$. Then the map $\mathcal{U}$,

$$\mathcal{U} \left(\sum_{k=1}^d \varphi_k e_k \right) = \sum_{k=1}^d \varphi_k b_k, \quad \text{or, equivalently, } \mathcal{U} f = B f,$$

defines a unitary operator from $L^2(B^* B \mu)$ to $\mathcal{H}$.

The inverse operator $\mathcal{U}^*$ is given by $\mathcal{U}^* f(\xi) = R(\xi) f(\xi)$, where, recall, R is a measurable pointwise right inverse of B , $B(\xi) R(\xi) = I_{E(\xi)}$ μ -a.e.

We define $\tilde{\Phi} := \mathcal{U}^* \Phi$, so $\tilde{\Phi}^* = \Phi^* \mathcal{U}$, and denote by $T_+^{B^* B \mu} f$ the nontangential boundary values of the Cauchy integral $\mathcal{C}[B^* B f \mu](z)$, $z \in \mathbb{D}$. Substituting $f = Bg$ into (8-2) we can restate Theorem 8.1 as follows.

Theorem 8.7. *The adjoint Clark operator $\tilde{\Phi}^* : L^2(B^* B \mu) \rightarrow \mathcal{K}_\theta$ in the Sz.-Nagy–Foiaş transcription is given by*

$$\tilde{\Phi}^* g = \begin{pmatrix} 0 \\ \tilde{\Psi}_2 \end{pmatrix} g + \begin{pmatrix} (I + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta_\Gamma D_\Gamma^{-1} (\Gamma^* - I) \end{pmatrix} T_+^{B^* B \mu} g, \quad g \in L^2(B^* B \mu), \quad (8-15)$$

where the matrix-valued function $\tilde{\Psi}_2(z)$ is defined as

$$\tilde{\Psi}_2(z) = \Delta_\Gamma D_\Gamma^{-1} (\Gamma^* + (I - \Gamma^*) F(z)). \quad (8-16)$$

8E. A generalization of the normalized Cauchy transform. Consider the case when the unitary operator U has purely singular spectrum. By virtue of Corollary 5.9, the second component of the Sz.-Nagy–Foiaş model space collapses, i.e., $\mathcal{K}_{\theta_\Gamma} = H^2(\mathbb{C}^d) \ominus \theta_\Gamma H^2(\mathbb{C}^d)$ for all strict contractions Γ .

The representation formula (8-2) then reduces to a generalization of the well-studied normalized Cauchy transform.

Corollary 8.8. *If $\theta = \theta_0$ is inner, then*

$$(\Phi^* f)(z) = (I - \theta(z))(T_+^{B^* \mu} f)(z) = (F(z))^{-1} (T_+^{B^* \mu} f)(z)$$

for $z \in \mathbb{D}$, $f \in L^2(\mu; E)$.

The first equation was also obtained in [Kapustin and Poltoratski 2006, Theorem 1].

Here we used $\Gamma = \mathbf{0}$ only for simplicity. With the linear fractional relation in Theorem 5.2, it is not hard to write the result in terms of θ_Γ for any strict contraction Γ .

Proof. Theorem 8.1 for inner θ and $\Gamma = \mathbf{0}$ immediately reduces to the first statement.

The equality of the second expression follows immediately from Lemma 8.6. $\square$

9. The Clark operator

Let $f \in \mathcal{H} \subset L^2(\mu; E)$ and let

$$\Phi^* f = h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{K}_\theta. \quad (9-1)$$

From the representation (8-15) we get, subtracting from the second component the first component multiplied by an appropriate matrix-valued function, that

$$\Psi_2 f = h_2 - \Delta_\Gamma D_\Gamma^{-1} (\Gamma^* - I) F D_{\Gamma^*} (I + \theta_\Gamma \Gamma^*)^{-1} h_1.$$

Right-multiplying this identity by Ψ_2^* , and using Proposition 8.3 and formulas for $\Psi_2, \tilde{\Psi}_2$ from Theorem 8.1, we get an expression for the density of the absolutely continuous part of μ_{ac} . Namely, we find that

a.e. (with respect to Lebesgue measure on $\mathbb{T}$)

$$\begin{aligned} wf &= R^* F^* (I - \theta_0^* \Gamma) D_{\Gamma}^{-1} \Delta_{\Gamma} h_2 R^* F^* (I - \theta_0^* \Gamma) D_{\Gamma}^{-1} \Delta_{\Gamma}^2 D_{\Gamma}^{-1} (\Gamma^* - I) F D_{\Gamma^*} (I + \theta_{\Gamma} \Gamma^*)^{-1} h_1 \\ &= R^* F^* (I - \theta_0^* \Gamma) D_{\Gamma}^{-1} \Delta_{\Gamma} h_2 - R^* F^* \Delta_0^2 (I - \Gamma^* \theta_0)^{-1} (\Gamma^* - I) F D_{\Gamma^*} (I + \theta_{\Gamma} \Gamma^*)^{-1} h_1. \end{aligned} \quad (9-2)$$

In the case $\Gamma = \mathbf{0}$ the above equation simplifies:

$$\begin{aligned} wf &= R^* F^* \Delta_0 h_2 + R^* F^* \Delta_0^2 F h_1 \\ &= R^* F \Delta_0 h_2 + w B h_1; \end{aligned} \quad (9-3)$$

in the second equality we used (8-4).

The above formulas (9-2), (9-3) determine the absolutely continuous part of f .

The singular part of f was in essence computed in [Kapustin and Poltoratski 2006]. Formally it was computed there only for inner functions θ , but using the ideas and results from that paper it is easy to get the general case from our Theorem 8.1.

For the convenience of the reader, we give a self-contained presentation.

Lemma 9.1. *Let $f \in L^2(\mathbb{T}, \mu; \mathbb{C}^d)$. Then μ_s -a.e. the nontangential boundary values of $\mathcal{C}[f\mu](z)/\mathcal{C}[\mu](z)$, $z \in \mathbb{D}$, exist and equal $f(\xi)$, $\xi \in \mathbb{T}$.*

This lemma was proved in [Kapustin and Poltoratski 2006] even for the more general case of $f \in L^2(\mu; E)$, where E is a separable Hilbert space. Note that our case $E = \mathbb{C}^d$ follows trivially by applying the corresponding scalar result ($E = \mathbb{C}$) proved in [Poltoratskiĭ 1993] to entries of the vector f .

Applying the above lemma to the representation giving by the first coordinate of (8-2) from Theorem 8.1, we get that for f and h related by (9-1) we have

$$B^* f = \frac{1}{\mathcal{C}[\mu]} F D_{\Gamma^*} (I + \theta_{\Gamma} \Gamma^*)^{-1} h_1 \quad \mu_s\text{-a.e.}$$

Left-multiplying this identity by R^* we get

$$\Phi h = f = \frac{1}{\mathcal{C}[\mu]} R^* F D_{\Gamma^*} (I + \theta_{\Gamma} \Gamma^*)^{-1} h_1 \quad \mu_s\text{-a.e.} \quad (9-4)$$

Summarizing, we get the following theorem, describing the direct Clark operator Φ .

Theorem 9.2. *If $\Phi^* f = h$ as in (9-1), so $f = \Phi h$, then the absolutely continuous part of f is given by (9-2) and the singular part of f is given by (9-4).*

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ON THE MAXIMAL RANK PROBLEM FOR THE COMPLEX HOMOGENEOUS MONGE–AMPÈRE EQUATION

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We give examples of regular boundary data for the Dirichlet problem for the complex homogeneous Monge–Ampère equation over the unit disc, whose solution is completely degenerate on a nonempty open set and thus fails to have maximal rank.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and B be a Riemann surface with boundary ∂B . Suppose $(\phi_\tau)_{\tau \in \partial B}$ is a smooth family of Kähler potentials on X ; so each ϕ_τ is a smooth function on X , varying smoothly in τ , that satisfies

$$\omega + dd^c \phi_\tau > 0.$$

Then let Φ be the solution to the Dirichlet problem for the complex homogeneous Monge–Ampère equation (HMAE) with this boundary data, so Φ is a function on $X \times B$ that satisfies

$$\begin{aligned} \Phi(\cdot, \tau) &= \phi_\tau(\cdot) \quad \text{for } \tau \in \partial B, \\ \pi_X^* \omega + dd^c \Phi &\geq 0, \\ (\pi_X^* \omega + dd^c \Phi)^{n+1} &= 0, \end{aligned} \tag{1}$$

where $\pi_X : X \times B \rightarrow X$ is the projection. From standard pluripotential theory we know there exists a unique weak solution Φ to this equation. The *maximal rank problem* in this setting asks whether the current

$$\pi_X^* \omega + dd^c \Phi$$

has maximal rank in the fibre directions, that is, whether the current $\omega + dd^c \Phi(\cdot, \tau)$ on X is strictly positive for each $\tau \in B$. Said another way, this asks if the rank of $\pi_X^* \omega + dd^c \Phi$ is precisely n at every point in $X \times B$, which is the maximum possible since $(\pi_X^* \omega + dd^c \Phi)^{n+1} = 0$. Similarly one has the *constant-rank problem* in which one asks if the rank of $\pi_X^* \omega + dd^c \Phi$ is the same at every point. The purpose of this note is to answer this question negatively, giving an explicit example in which the rank fails to be maximal.

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Theorem 1.1. *Let $B = \bar{\mathbb{D}} \subset \mathbb{C}$ be the closed unit disc and $(X, \omega) = (\mathbb{P}^1, \omega_{\text{FS}})$, where ω_{FS} denotes the Fubini–Study form. Then there exists a smooth family of Kähler potentials $(\phi_\tau)_{\tau \in \partial \mathbb{D}}$ on $\mathbb{P}^1$ such that the solution Φ to the HMAE (1) is completely degenerate on some nonempty open subset $S \subset \mathbb{P}^1 \times \mathbb{D}$, i.e.,*

$$\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \Phi|_S = 0.$$

A more precise version of this statement is provided in [Theorem 2.1](#). The motivation and ideas build on previous work of the authors [[Ross and Witt Nyström 2015a; 2015b; 2015c](#)] in which we understand the solution to the HMAE of a certain kind through a free boundary problem in the plane called the Hele–Shaw flow. But rather than expecting the reader to be an expert in this topic we have chosen to give a direct proof, which can be found in [Section 2](#), that is both self-contained and rather simple. Then in [Section 3](#) we explain the motivation behind our construction, as well as give a second (but essentially equivalent) proof that relies on more machinery. We then end with some questions and possible extensions.

Of course in the above theorem, $\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \Phi$ is not identically zero, and so does not have constant rank. In fact we can say more, and it is possible to arrange so that there is a nonempty open set in $\mathbb{P}^1 \times \mathbb{D}$ on which $\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \Phi$ is regular (i.e., smooth and of maximal rank). It is worth commenting from the outset that we do not expect the solution we have here to be everywhere smooth, but it should be possible to describe precisely where it is regular and where it is degenerate. All of this will be discussed in more detail in [Section 3](#).

1A. Comparison with other work. It is known that convex solutions to elliptic partial differential equations have a constant-rank property. Early works of this include [[Caffarelli and Friedman 1985; Singer, Wong, Yau, and Yau 1985](#)]. These have since been built upon by many others, and it is now known that the constant-rank property holds for a wide class of elliptic equations; see, for instance, [[Korevaar and Lewis 1987; Bian and Guan 2009; 2010; Caffarelli, Guan and Ma 2007; Székelyhidi and Weinkove 2016](#)]. In this paper we are interested in the complex degenerate situation, about which much less has been written. The most famous result along these lines, and in the positive direction, is that of Lempert [[1981](#)] who proved that on a convex domain in $\mathbb{C}^n$ the solution to the complex HMAE with prescribed singularity at an interior point (the pluricomplex Green function) is smooth and of maximal rank. The maximal rank problem for other partial differential equations in the complex case has also been taken up by Guan, Li and Zhang [[2009](#)] and by Li [[2009](#)].

The closest previous work to that of this paper is probably that of Guan and Phong [[2012](#)], who studied the problem of finding uniform lower bounds for the eigenvalues of the solution to the (nondegenerate) Monge–Ampère equation in the limit as the equation becomes degenerate. Moreover, they asked whether solutions to the complex HMAE have maximal rank [[Guan and Phong 2012](#), discussion after Theorem 4]. The idea of maximal rank for the complex HMAE also appears in the ideas of Chen and Tian [[2008](#)] through the concept of an almost-regular solution to the HMAE, which fails to have maximal rank only on a set which is small in a precise sense. The kinds of envelopes that we use in our proof also can be defined more generally, and even in higher dimensions, which is the topic of previous work of the authors [[Ross and Witt Nyström 2017b](#)], in which we prove a constant-rank theorem, Theorem 6.2 of that paper, that we call “optimal regularity”.

Questions concerning the regularity of the solution to the Dirichlet problem for the kind of complex HMAE we consider here go back at least as far as [Semmes 1992; Donaldson 2002], and this HMAE has been the focus of much interest due to it being the geodesic equation in the space of Kähler metrics. By [Chen 2000] with complements by Błocki [2012] we know such a solution always has bounded Laplacian (so in particular is $C^{1,\alpha}$ for any $\alpha < 1$). In fact in our case, since we are working on $\mathbb{P}^1$, the results of [Błocki 2012] imply that Φ is $C^{1,1}$. (We observe that we do not actually need to know this regularity for the direct proof of our main theorem). Donaldson [2002] gives examples of boundary data for which the solution is not regular, but the nature of the irregularity there is left unknown (for instance Donaldson’s example may have maximal rank but fail to be everywhere smooth).

2. Main theorem

2A. Notation. We let $\mathbb{D}_r$ be the open disc of radius r in the complex plane about the origin, $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$. Throughout we consider the standard cover of $\mathbb{P}^1$ by two charts equal to the complex plane with coordinates z and $w = 1/z$. We shall denote these two charts by $\mathbb{C}_z$ and $\mathbb{C}_w$ respectively. We use the convention $d^c = \frac{1}{2\pi}(\bar{\partial} - \partial)$ so $dd^c \log |z|^2 = \delta_0$, and normalise the Fubini–Study form ω_{FS} so $\int_{\mathbb{P}^1} \omega_{\text{FS}} = 1$. Thus $\omega_{\text{FS}} = dd^c \log(1 + |z|^2)$ locally on $\mathbb{C}_z$.

2B. Statement of the main theorem. The following is a precise version of our main theorem. By an *arc* in $\mathbb{C}$ we mean the image γ of a smooth map $[0, 1] \rightarrow \mathbb{C}$ that does not intersect itself. From now on $B = \overline{\mathbb{D}}$ is the closed unit disc and $(X, \omega) = (\mathbb{P}^1, \omega_{\text{FS}})$.

Theorem 2.1. *Suppose that $\phi \in C^\infty(\mathbb{P}^1)$ satisfies:*

- (1) $\omega_{\text{FS}} + dd^c \phi > 0$.
- (2) *On $\mathbb{C}_w \subset \mathbb{P}^1$ it holds that*

$$\phi(w) \geq -\ln(1 + |w|^2)$$

with equality precisely on an arc in $\mathbb{C}_w$.

Then setting

$$\phi_\tau(z) := \phi(\tau z) \quad \text{for } \tau \in \partial\mathbb{D},$$

the solution Φ to the HMAE (1) does not have maximal rank. In fact there is a nonempty open subset $S \subset \mathbb{P}^1 \times \mathbb{D}$ such that

$$\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \Phi|_S = 0.$$

2C. Envelopes. For the proof we need some background concerning envelopes of subharmonic functions. Fix a potential $\phi \in C^\infty(\mathbb{P}^1)$ so $\omega_{\text{FS}} + dd^c \phi > 0$. For a topological space X let

$$\text{USC}(X) = \{\psi : X \rightarrow \mathbb{R} \cup \{-\infty\} \text{ such that } \psi \text{ is upper semicontinuous}\}.$$

Definition 2.2. For $t \in (0, 1]$ set

$$\psi_t := \sup\{\psi \in \text{USC}(\mathbb{P}^1) : \psi \leq \phi \text{ and } \omega_{\text{FS}} + dd^c \psi \geq 0 \text{ and } \nu_{z=0}(\psi) \geq t\}.$$

Here $v_{z=0}$ denotes the Lelong number at the point $z = 0$, so $v_{z=0}(\psi) \geq t$ means $\psi(z) \leq t \ln |z|^2 + O(1)$ near $z = 0$. As the upper-semicontinuous regularisation of ψ_t is itself a candidate for the envelope defining ψ_t , we see that ψ_t is itself upper-semicontinuous.

Definition 2.3. For $t \in (0, 1]$ set

$$\Omega_t := \Omega_t(\phi) := \{z \in \mathbb{P}^1 : \psi_t(z) < \phi(z)\}. \quad (2)$$

Clearly if $t \leq t'$ then $\psi_{t'} \leq \psi_t$ and so $\Omega_t \subset \Omega_{t'}$. Now, unless one assumes some additional symmetry of ϕ , it is generally quite hard to describe the sets Ω_t . However, as the next lemma shows, it is possible, under a suitable hypothesis, to describe the largest one Ω_1 by looking at the level set on which ϕ takes its minimum value.

Lemma 2.4. *Let $\phi \in C^\infty(\mathbb{P}^1)$ be such that $\omega_{\text{FS}} + dd^c \phi > 0$ and $\phi(w) \geq -\ln(1 + |w|^2)$ on $\mathbb{C}_w$ with equality precisely on some nonempty subset $\gamma \subset \mathbb{C}_w$ containing $w = 0$. Then*

$$\psi_1(z) = \ln \left(\frac{|z|^2}{1 + |z|^2} \right)$$

and

$$\Omega_1(\phi) = \mathbb{P}^1 \setminus \gamma.$$

Proof. Observe first that the only upper-semicontinuous $\psi : \mathbb{P}^1 \rightarrow \mathbb{R} \cup \{-\infty\}$ with $\omega_{\text{FS}} + dd^c \psi \geq 0$ and $v_{z=0}(\psi) \geq 1$ is, up to an additive constant, equal to

$$\zeta(z) := \ln \left(\frac{|z|^2}{1 + |z|^2} \right) \quad \text{on } \mathbb{C}_z.$$

To see this observe first that we certainly cannot have $v_{z=0}(\psi) > 1$ since we have normalised so $\int_{\mathbb{P}^1} \omega_{\text{FS}} = 1$. Thus we may assume $v_{z=0} \psi = 1$. Then observe that ζ is ω_{FS} -harmonic on $\mathbb{C}_z \setminus \{0\}$, and that the difference $\psi - \zeta$ is bounded near 0. Thus $\psi - \zeta$ extends to a bounded subharmonic function on all of $\mathbb{C}_z$, and hence is constant by the Liouville property. Thus the envelope ψ_1 from [Definition 2.2](#) must be

$$\psi_1 = \zeta + C,$$

where C is the largest constant one can choose so that $\psi_1 \leq \phi$. Now on $\mathbb{C}_w$ we have

$$\psi_1(w) = -\ln(1 + |w|^2) + C$$

and so as γ is nontrivial our hypothesis forces $C = 0$. Thus

$$\Omega_1 = \{-\ln(1 + |w|^2) < \phi(w)\} = \mathbb{P}^1 \setminus \gamma. \quad \square$$

2D. Weak solutions to the HMAE. We now discuss the weak solution to two versions of the Dirichlet problem for the complex HMAE, first over the disc and second over the punctured disc; this follows the discussion in [\[Ross and Witt Nyström 2015b\]](#). Again we let $\phi \in C^\infty(\mathbb{P}^1)$ be such that $\omega_{\text{FS}} + dd^c \phi > 0$.

Definition 2.5. Let

$$\begin{aligned} \Phi &:= \sup\{\psi \in \text{USC}(\mathbb{P}^1 \times \bar{\mathbb{D}}) : \pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \psi \geq 0 \text{ and } \psi(z, \tau) \leq \phi(\tau z) \text{ for } (z, \tau) \in \mathbb{P}^1 \times \partial\mathbb{D}\}. \\ \text{and} \\ \tilde{\Phi} &:= \sup\{\psi \in \text{USC}(\mathbb{P}^1 \times \bar{\mathbb{D}}) : \pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \psi \geq 0 \\ &\quad \text{and } \psi(z, \tau) \leq \phi(z) \text{ for } (z, \tau) \in \mathbb{P}^1 \times \partial\mathbb{D} \text{ and } \nu_{(z=0, \tau=0)}(\psi) \geq 1\}. \end{aligned} \quad (3)$$

The function Φ is the weak solution to the complex HMAE with boundary data $\phi(\tau z)$ for $\tau \in \partial\mathbb{D}$, that is, the solution to (1). Similarly $\tilde{\Phi}$ is the weak solution to the Dirichlet problem with boundary data $\phi(z)$, but with the additional requirement of having a prescribed singularity at the point $p := (0, 0) \in \mathbb{C}_z \times \mathbb{D} \subset \mathbb{P}^1 \times \mathbb{D}$. That is, $\tilde{\Phi}$ is upper-semicontinuous, $\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \tilde{\Phi} \geq 0$ and $(\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \tilde{\Phi})^2 = 0$ away from p and $\tilde{\Phi}(z, \tau) = \phi(z)$ for $\tau \in \partial\mathbb{D}$. Moreover it is not hard to show that $\tilde{\Phi}$ is locally bounded away from p and $\nu_p \tilde{\Phi} = 1$. These two quantities carry the same information, as given by:

Proposition 2.6. *We have that*

$$\Phi(z, \tau) + \ln |\tau|^2 + \ln(1 + |z|^2) = \tilde{\Phi}(\tau z, \tau) + \ln(1 + |\tau z|^2) \quad \text{for } (z, \tau) \in \mathbb{P}^1 \times \bar{\mathbb{D}}^\times.$$

Proof. It is easily seen from the definition that $\Phi(z, \tau) + \ln |\tau|^2 + \ln(1 + |z|^2) - \ln(1 + |\tau z|^2)$ is a candidate for the envelope defining $\tilde{\Phi}(\tau z, \tau)$, giving one inequality and the other inequality is proved similarly. $\square$

2E. Proof of Theorem 2.1. Without loss of generality we assume the arc γ goes through the point $w = 0$. By Lemma 2.4

$$\psi_1(z) = \ln\left(\frac{|z|^2}{1 + |z|^2}\right)$$

and

$$\Omega_1 = \mathbb{P}^1 \setminus \gamma.$$

Looking at the other coordinate patch $\mathbb{C}_z$, we have that γ is a curve passing through infinity, and so $\mathbb{C}_z \setminus \gamma$ is an open, simply connected proper subset of $\mathbb{C}_z$. Hence by the Riemann mapping theorem there is a biholomorphism

$$f : \mathbb{D} \rightarrow \mathbb{C}_z \setminus \gamma \quad \text{with } f(0) = 0.$$

For $\tau \in \mathbb{D}^\times$ set

$$A_\tau := f(\mathbb{D}_{|\tau|}) \subset \mathbb{C}_z \subset \mathbb{P}^1.$$

Clearly each A_τ is a proper subset of $\mathbb{C}_z$ containing the origin, whose complement has nonempty interior.

Proposition 2.7. *We have*

$$\tilde{\Phi}(z, \tau) = \psi_1(z) \quad \text{for all } \tau \in \mathbb{D}^\times \text{ and } z \in \mathbb{P}^1 \setminus A_\tau.$$

Proof. By abuse of notation we write ψ_1 also for the pullback of ψ_1 to $\mathbb{P}^1 \times \bar{\mathbb{D}}$. Then

$$\tilde{\Phi}(z, \tau) \geq \psi_1(z) \quad \text{for } (z, \tau) \in \mathbb{P}^1 \times \bar{\mathbb{D}} \quad (4)$$

since ψ_1 is a candidate for the envelope (3) defining $\tilde{\Phi}$.

We next claim that

$$\tilde{\Phi}(f(\tau), \tau) = \psi_1(f(\tau)) \quad \text{for all } \tau \in \mathbb{D}. \quad (5)$$

To see this, observe that $\tau \mapsto \tilde{\Phi}(f(\tau), \tau)$ is $f^*\omega_{\text{FS}}$ -subharmonic and has Lelong number 1 at $\tau = 0$. On the other hand $\psi_1(f(\tau))$ is $f^*\omega_{\text{FS}}$ -harmonic except at $\tau = 0$ where it has Lelong number 1. But $\tilde{\Phi}(f(\tau), \tau)$ tends to $\psi_1(f(\tau))$ as $|\tau|$ tends to 1, and hence from the maximum principle along with (4), we get (5).

Now fix some $\tau \in \mathbb{D}^\times$ and set

$$\phi_\tau(z) := \tilde{\Phi}(z, \tau).$$

Then the above says that $\phi_\tau = \psi_1$ on ∂A_τ . On the other hand by (4) we have $\phi_\tau \geq \psi_1$ everywhere. Moreover ϕ_τ is ω_{FS} -subharmonic on A_τ^c , whereas ψ_1 is bounded and ω_{FS} -harmonic on A_τ^c . Thus by the maximum principle we deduce $\phi_\tau = \psi_1$ on A_τ^c as required. $\square$

Proof of Theorem 2.1. Set

$$S := \{(z, \tau) \in \mathbb{P}^1 \times \mathbb{D}^\times : \tau z \in (A_\tau^c)^\circ\},$$

which is nonempty and open in $\mathbb{P}^1 \times \mathbb{D}^\times$. Then by Proposition 2.6 and then Proposition 2.7 if $(z, \tau) \in S$ we have

$$\Phi(z, \tau) = \tilde{\Phi}(\tau z, \tau) + \ln\left(\frac{1 + |\tau z|^2}{|\tau|^2(1 + |z|^2)}\right) = \psi_1(\tau z) + \ln\left(\frac{1 + |\tau z|^2}{|\tau|^2(1 + |z|^2)}\right).$$

Thus on S we have

$$\pi_{\mathbb{P}^1} \omega_{\text{FS}} + dd^c \Phi = \pi_{\mathbb{P}^1} \omega_{\text{FS}} + dd^c \psi_1(\tau z) = 0$$

as ψ_1 is ω_{FS} -harmonic away from $z = 0$. $\square$

2F. A specific example. We now construct a specific potential ϕ that satisfies the hypotheses of Theorem 2.1. Fix γ to be the interval $[-1, 1] \subset \mathbb{R} \subset \mathbb{C}_w$. Our goal is to find a $\phi \in C^\infty(\mathbb{P}^1)$ such that $\omega_{\text{FS}} + dd^c \phi > 0$ and $\phi \geq -\ln(1 + |w|^2)$ with equality precisely on γ .

To do so, let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative smooth nondecreasing convex function with $\alpha(t) = 0$ for $t \leq 1$ and $\alpha(t) > 0$ for $t > 1$. Let

$$u(w) := \alpha(|w|^2) + \text{Im}(w)^2.$$

Thus u is a smooth strictly subharmonic function on $\mathbb{C}_w$ that vanishes precisely on γ . Then $\epsilon u - \ln(1 + |w|^2)$ for some small constant $\epsilon > 0$ is essentially the function that we want; we simply need to adjust it to have the correct behaviour far away from γ .

To do so we shall use a regularised version of the maximum function, which can be explicitly given as follows: Let $|\cdot|_{\text{reg}}$ be a smooth convex function on $\mathbb{R}$ so that $|t|_{\text{reg}} = |t|$ for $|t| \geq 1$. Set $\max_{\text{reg}}(a, b) := \frac{1}{2}(|a - b|_{\text{reg}} + a + b)$ and for $\delta > 0$ put

$$\max_{\delta}(a, b) := \delta \max_{\text{reg}}(\delta^{-1}a, \delta^{-1}b). \quad (6)$$

Then $\max_\delta(\cdot, \cdot)$ is smooth, and satisfies

$$\max_\delta(a, b) = \begin{cases} a & \text{if } a > b + \delta, \\ b & \text{if } b > a + \delta. \end{cases}$$

Returning to the construction of ϕ , fix a sufficiently large constant C and a sufficiently small positive constant ϵ so that

$$\begin{aligned} \epsilon u &\geq \ln(1 + |w|^2) - C + 1 && \text{on } \mathbb{D}_2, \\ \epsilon u &\leq \ln(1 + |w|^2) - C - 1 && \text{on } \mathbb{D}_4 \setminus \mathbb{D}_3. \end{aligned}$$

Then for $0 < \delta \ll 1$ set

$$v := \max_\delta(\epsilon u, \ln(1 + |w|^2) - C).$$

So v is smooth, nonnegative, strictly subharmonic, equal to $\ln(1 + |w|^2) - C$ on $\mathbb{D}_4 \setminus \mathbb{D}_3$ and vanishes precisely on γ . We then put

$$\phi := v - \ln(1 + |w|^2)$$

and extend ϕ to take the constant value C in $\mathbb{C}_w \setminus \mathbb{D}_4$. So ϕ extends to a smooth function over $\mathbb{P}^1$ with the desired properties.

3. Discussion

3A. Context. Fix a $\phi \in C^\infty(\mathbb{P}^1)$ such that $\omega_{\text{FS}} + dd^c \phi > 0$. Then associated to ϕ we have two constructions:

- (1) The solution $\tilde{\Phi}$ to the complex HMAE on $\mathbb{P}^1 \times \bar{\mathbb{D}}$ with boundary data given by $\phi_\tau = \phi$ for all $\tau \in \partial\mathbb{D}$ and the requirement of having Lelong number 1 at the point $(z, \tau) = (0, 0) \in \mathbb{C}_z \times \mathbb{D} \subset \mathbb{P}^1 \times \bar{\mathbb{D}}$.
- (2) The envelopes ψ_t for $t \in (0, 1]$ and the associated sets $\Omega_t(\phi) = \{\psi_t < \phi\}$.

In previous work we showed that these sets of data are intimately connected. First, $\tilde{\Phi}$ and ψ_t are Legendre dual to each other [Ross and Witt Nyström 2015b, Theorem 2.7] in that

$$\psi_t(z) = \inf_{|\tau|>0} \{\tilde{\Phi}(z, \tau) - (1-t) \ln |\tau|^2\} \quad (7)$$

and

$$\tilde{\Phi}(z, \tau) = \sup_t \{\psi_t(z) + (1-t) \ln |\tau|^2\}. \quad (8)$$

Second, the collection of sets $\Omega_t(\phi)$ that are biholomorphic to a disc describes the harmonic discs of $\tilde{\Phi}$. That is, if t is such that $\Omega_t(\phi)$ is a proper simply connected subset of $\mathbb{C}_z$ and $f: \mathbb{D} \rightarrow \Omega_t$ is a Riemann map with $f(0) = 0$ then the restriction of $\tilde{\Phi}$ to the graph $\{(f(\tau), \tau) \in \mathbb{P}^1 \times \mathbb{D}\}$ is ω_{FS} -harmonic. Furthermore it is shown in [Ross and Witt Nyström 2015b, Theorem 3.1] these are (essentially) the only harmonic discs that occur.

We can say more. For $\tau \in \mathbb{D}^\times$ set

$$\phi_\tau(z) := \tilde{\Phi}(z, \tau).$$

If $\tilde{\Phi}$ is regular then each ϕ_τ will be a smooth Kähler potential, but in general this will not be the case. Nevertheless, by [Błocki 2012] we know ϕ_τ is $C^{1,1}$ and since $\pi_{\mathbb{P}^1}^* \omega_{\text{FS}} + dd^c \tilde{\Phi} \geq 0$, we know $\omega_{\text{FS}} + dd^c \phi_\tau$ is semipositive. We can then define the associated sets $\Omega_t(\phi_\tau)$ in exactly the same way as before.

Proposition 3.1. *Suppose t is such that $\Omega_t(\phi) \subset \mathbb{C}_z$ is proper and simply connected and let $f_t : \mathbb{D} \rightarrow \Omega_t(\phi)$ be a Riemann map with $f(0) = 0$. Then for each $\tau \in \mathbb{D}^\times$ we have*

$$f_t(\mathbb{D}_{|\tau|}) = \Omega_t(\phi_\tau).$$

We shall give a proof of this fact below, but assuming it for now we can give an alternative proof that, under the hypotheses of Theorem 2.1, for each $\tau \in \mathbb{D}^\times$ the current $\omega_{\text{FS}} + dd^c \tilde{\Phi}(\cdot, \tau)$ is degenerate on some nonempty open subset of $\mathbb{P}^1$. First Lemma 2.4 gives

$$\Omega_1(\phi) = \mathbb{P}^1 \setminus \gamma,$$

which is a simply connected proper subset of $\mathbb{C}_z$. We then take our Riemann map $f : \mathbb{D} \rightarrow \Omega_1(\phi)$ and consider the image

$$A_\tau := f(\mathbb{D}_{|\tau|}) = \Omega_1(\phi_\tau) \quad \text{for } \tau \in \mathbb{D}^\times.$$

As observed before, A_τ is a proper subset of $\mathbb{C}_z$ whose complement has nonempty interior.

On the other hand, it is a general fact that for each t the set $\Omega_t(\phi_\tau)$ has measure t with respect to the current $\omega_{\text{FS}} + dd^c \phi_\tau$. (If ϕ_τ is smooth and $\omega_{\text{FS}} + dd^c \phi_\tau$ is strictly positive, this is a standard piece of potential theory and is discussed in [Ross and Witt Nyström 2015b, Proposition 1.1]; when ϕ_τ is merely C^2 and $t < 1$ then this is proved in [Ross and Witt Nyström 2017a, Theorem 1.2] and the case $t = 1$ follows from this by continuity as $\Omega_1(\phi_\tau) = \bigcup_{t < 1} \Omega_t(\phi_\tau)$; finally when ϕ_τ is merely $C^{1,1}$ this is given in [Berman and Demailly 2012, Remark 1.19, Corollary 2.5].)

Therefore

$$\int_{A_\tau} (\omega_{\text{FS}} + dd^c \phi_\tau) = \int_{\Omega_1(\phi_\tau)} (\omega_{\text{FS}} + dd^c \phi_\tau) = 1.$$

But our normalisation is that $\int_{\mathbb{P}^1} (\omega_{\text{FS}} + dd^c \phi_\tau) = \int_{\mathbb{P}^1} \omega_{\text{FS}} = 1$ as well, and so the current $\omega_{\text{FS}} + dd^c \phi_\tau$ gives zero measure to the complement of A_τ , which is precisely what we were aiming to prove.

Proof of Proposition 3.1. Fix $\sigma \in \mathbb{D}^\times$ and set $r := |\sigma|$. Our aim is to show

$$f_t(\mathbb{D}_r) = \Omega_t(\phi_\sigma).$$

Consider the S^1 -action on $\mathbb{P}^1 \times \bar{\mathbb{D}}$ given by $e^{i\theta} \cdot (z, \tau) = (z, e^{i\theta} \tau)$, and observe that the boundary data used to define $\tilde{\Phi}$ from (3) is S^1 -invariant, which implies $\tilde{\Phi}$ is S^1 -invariant as well. Thus we may as well assume that σ is real, so $\phi_\sigma = \phi_r$.

For a function F on $\mathbb{P}^1 \times \bar{\mathbb{D}}$ and $D \subset \bar{\mathbb{D}}$ we write $F|_D$ for the restriction of F to $\mathbb{P}^1 \times D$. Then $\tilde{\Phi}|_{\bar{\mathbb{D}}_r}$ is the solution to the Dirichlet problem for the HMAE with boundary data $(\phi_\tau)_{\tau \in \partial \bar{\mathbb{D}}_r} = \phi_r$ and the requirement that $\tilde{\Phi}|_{\bar{\mathbb{D}}_r}$ has Lelong number 1 at the point $(0, 0) \in \mathbb{C}_z \times \bar{\mathbb{D}}_r \subset \mathbb{P}^1 \times \bar{\mathbb{D}}_r$.

Letting $s := -\ln |\tau|^2$, consider the function on $\mathbb{P}^1 \times \bar{\mathbb{D}}^\times$ given by

$$H(z, \tau) := \frac{\partial}{\partial s} \tilde{\Phi}(z, e^{-s/2})$$

(when $|\tau| = 1$ and thus $s = 0$, we take the right derivative). As $\tilde{\Phi}$ is $C^{1,1}$ on $\mathbb{P}^1 \times \bar{\mathbb{D}}^\times$, the function H is well-defined and Lipschitz. Clearly this is compatible with restriction; i.e.,

$$H|_{\bar{\mathbb{D}}^\times}(z, \tau) = \frac{\partial}{\partial s} \tilde{\Phi}|_{\bar{\mathbb{D}}_r}(z, e^{-s/2}).$$

Now, as discussed above, and proved in [Ross and Witt Nyström 2015b, Theorem 3.1], the graph $\{(f(\tau), \tau) : \tau \in \mathbb{D}\}$ of f is a harmonic disc for $\tilde{\Phi}$. What is also proved is that H takes the constant value $t - 1$ along this disc so

$$H(f(\tau), \tau) = t - 1 \quad \text{for } \tau \in \mathbb{D}^\times.$$

Now H is also S^1 -invariant and so this in particular implies

$$H(f(re^{i\theta}), r) = H(f(re^{i\theta}), re^{i\theta}) = t - 1 \quad \text{for all } \theta \in [0, 2\pi].$$

In other words, the function $H(\cdot, r)$ takes the value $t - 1$ on the boundary of $f(\mathbb{D}_r)$. On the other hand, we prove in [Ross and Witt Nyström 2015b, Proposition 2.9] that the function $H(\cdot, r)$ describes the set $\Omega_t(\phi_r)$, in that

$$H(z, r) + 1 = \sup\{s : z \notin \Omega_s(\phi_r)\}$$

(we remark that the proof of that proposition does not require any regularity or strict positivity assumptions on the potential ϕ_σ). Thus $\Omega_t(\phi_r)$ is the interior component of the curve $\theta \mapsto f(re^{i\theta})$ (that is, the component containing $z = 0$), which gives $\Omega_t(\phi_r) = f(\mathbb{D}_r)$ as desired. $\square$

3B. Extensions and questions. Under the hypotheses of Theorem 2.1 we have shown that the current $\omega_{\text{FS}} + dd^c \Phi(\cdot, \tau)$ fails to be strictly positive on any interior fibre (that is, for any τ with $0 < |\tau| < 1$). Furthermore we have no reason to expect our solution to be smooth everywhere. Thus the following two questions are natural:

Question 3.2. Does there exist a smooth family of potentials $(\phi_\tau)_{\tau \in \partial B}$ for which the solution to the complex HMAE (1) is everywhere smooth but not of maximal rank?

Question 3.3. Does there exist a smooth family of potentials $(\phi_\tau)_{\tau \in \partial B}$ for which the solution to the complex HMAE (1) such that $\omega + dd^c \Phi(\cdot, \tau)$ is a Kähler form for some τ with $0 < |\tau| < 1$ but not for others.

We are not currently able to answer these questions. However, we believe that the degenerate solutions we describe in this paper are actually regular in the interior of the complement of the degenerate set S (that is, they are smooth there and of maximal rank). In fact from our previous work in [Ross and Witt Nyström 2015b] we can understand the set on which our solution is regular in terms of the collection of sets $\Omega_t(\phi)$ that are simply connected. Now, our specific potential ϕ (Section 2F) was constructed to have curvature equal to ω_{FS} far away from the arc $\gamma = [-1, 1] \subset \mathbb{C}_w \subset \mathbb{P}^1$, from which one can see that $\Omega_t(\phi)$ is a disc for sufficiently small t . This gives an open set of $\mathbb{P}^1 \times \mathbb{D}$ for which the solution Φ is regular. Furthermore, by construction, $\Omega_1(\phi)$ is simply connected. We think it likely that $\Omega_t(\phi)$ is actually simply connected for all t , which would give rather precise information about the set on which our solution is regular, but it does not seem easy to prove that this is the case.

We furthermore believe that the fibrewise Laplacian of such a solution is uniformly bounded from below on the complement of S , and so has a discontinuity on the boundary ∂S where it jumps to zero. A somewhat bold conjecture would be that any solution to the HMAE is regular away from the set where it fails to have maximal rank, and is smooth away from the boundary of this set.

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