Distributed Control of LPV Systems over Arbitrary Graphs: A Parameter-Dependent Lyapunov Approach

Mazen Farhood

Abstract— **The paper focuses on spatially distributed control systems where the controller sensing and actuation topology is inherited from that of the plant. Specifically, the paper considers distributed systems composed of discrete-time linear parameter-varying subsystems interconnected over general graph structures. These distributed systems are subject to a communication latency of one sampling period, where the information sent by a subsystem at the current time step is received by the target subsystem at the next time step. The paper provides analysis and synthesis conditions for control design in this setting, employing a parameter-dependent Lyapunov approach with the ℓ_2-induced norm as the performance measure. The paper also gives a fast and easy-to-implement algorithm for constructing the controller in real-time.**

I. Introduction

The paper deals with the control of distributed systems which are composed of interacting subsystems interconnected over general graph structures. Each subsystem has sensing and actuation capabilities, and the controller topology is the same as that of the plant. Specifically, the distributed systems of interest consist of discrete-time linear parameter-varying (LPV) subsystems interconnected over arbitrary directed graphs. The transfer of data among the subsystems is subject to a communication latency, namely the data sent by a subsystem at the current time step arrives to the target subsystem at the next time step.

The synthesis objective is to find a feedback distributed controller that has the same topological structure as the plant such that the closed-loop system is asymptotically stable and the ℓ_2-induced norm of the closed-loop input-output mapping is less than some ℓ_2-gain performance level γ for all permissible parameter trajectories. The approach proposed to solve this problem involves the use of a parameter-dependent Lyapunov function. Consequently, the analysis and synthesis results are given in terms of parameterized linear matrix inequalities (PLMIs), and hence, relaxation techniques, such as the sum of squares (SOS) decomposition method [1] and the multiconvexity technique [2], among others [3]–[5], are needed to render the analysis and synthesis convex problems computationally tractable. The parameter-dependent Lyapunov approach also leads to synthesis solutions that are dependent on the parameters, which are not known a priori. Thus, the controller realization has to be constructed online at each discrete instant, and a fast algorithm based on the results of [6], [7] is provided for this purpose.

M. Farhood is with the Department of Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, VA 24061, USA. Email: farhood@vt.edu

This material is based upon work supported by the National Science Foundation under Grant Number CMMI-1333785.

One of the applications of this work that is of particular interest pertains to the cooperative control of multi-agent systems with intermittent communications, where parameters are used to quantify the viability and importance of connections by scheduling the penalty weights on the output errors and the scalings on the measurement noise. For this problem, the use of a parameter-independent Lyapunov function is unfavorable because then all scheduled controllers will be inclined for worst-case-scenario behavior. This behavior could correspond to a sparse interconnection structure, which would ultimately dictate the performance of the controller for all the permissible interconnection structures. Thus, a parameter-dependent Lyapunov function should be used in order to optimize the control strategy according to the present interconnection structure.

The research on distributed control can be categorized according to the temporal and spatial structures of the considered distributed systems. As expected, the simpler the structures are, the more likely these structures can be exploited to reduce the computational complexity of the control design problem. Specifically, the subsystems can be homogeneous (all described by the same model) [8]–[10] or heterogeneous (different models) [11]–[15]. The interconnection topology can be of varying complexity, for instance, highly structured [9], [12] or arbitrary graph [11], [13], [15]. This work considers heterogeneous subsystems and extends the results of [16] on linear time-varying (LTV) subsystems to the LPV setting. Similar work on distributed LPV systems include [17]–[19]. To the best of the author's knowledge, however, this paper is the only work in the literature that uses parameter-dependent Lyapunov functions to solve discrete-time distributed LPV control problems.

The outline of the paper is as follows. Section II introduces the notation and provides a precise formulation of the distributed control problem in question, along with prior control results on distributed LTV systems that are directly relevant to this work. Section III gives analysis and synthesis results for distributed control of LPV systems, and Section IV presents an algorithm for online controller construction. A brief summary in Section V concludes the paper.

II. Preliminaries

A. Notation

We denote the set of non-negative integers by $\mathbb{N}_0$, the set of real $n \times m$ matrices by $\mathbb{R}^{n\times m}$, and that of symmetric $n\times n$ matrices by $\mathbb{S}^n$. Given a sequence of column vectors $(v_i)_{i\in S}$, where the index set S is a subset of the set of integers, the notation $\mathrm{vec}(v_i)_{i\in S}$ denotes a vector composed by stacking

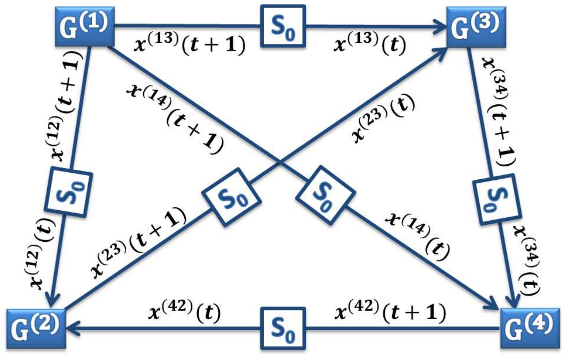

Fig. 1. Distributed system with delays

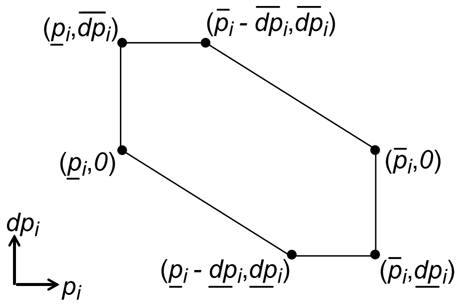

Fig. 2. Parameter space in (p_i, dp_i)-plane; the superscript (k) is dropped for simplicity

the elements of the sequence below each other. For instance, if $S = \{1, 3, 5\}$, the sequence would be (v_1, v_3, v_5), that is, the order of the elements in the sequence is conformable with the order of the elements in the index set; also, in this case, $\mathrm{vec}(v_1, v_3, v_5) = [v_1^T,\, v_3^T,\, v_5^T]^T$, where v_i^T stands for the transpose of v_i. Similarly, if $(M_i)_{i \in S}$ is a sequence of matrices, then $\mathrm{diag}(M_i)_{i \in S}$ denotes their block-diagonal augmentation. We use $0_{i \times j}$ to denote an $i \times j$ zero matrix and $I_{i \times j}$ to denote an $i \times j$ matrix with 1's on the diagonal and zeros elsewhere. Given a symmetric matrix X, i.e., $X = X^T$, we use $X \prec 0$ to mean it is negative definite. The linear space of elements $x = (x(0), x(1), x(2), \ldots)$, with each $x(t) \in \mathbb{R}^n$, is denoted by $\ell(\mathbb{R}^n)$. The Hilbert space $\ell_2(\mathbb{R}^n)$ is a subspace of $\ell(\mathbb{R}^n)$ consisting of elements $x \in \ell(\mathbb{R}^n)$ that have a finite ℓ_2-norm $\|x\|_{\ell_2}$ defined by $\|x\|_{\ell_2}^2 = \sum_{t=0}^{\infty} x(t)^T x(t) < \infty$. The Hilbert space direct sum $\ell_2(\mathbb{R}^{n_1}) \oplus \ell_2(\mathbb{R}^{n_2}) \oplus \cdots \oplus \ell_2(\mathbb{R}^{n_p})$ consists of elements $(x_1, x_2, \ldots, x_p)$, with each $x_i \in \ell_2(\mathbb{R}^{n_i})$. In the sequel, we will use the denotations ℓ and ℓ_2 irrespective of the spatial structure and dimensions; for instance, we will simply abbreviate $\oplus_{i=1}^{p} \ell_2(R^{n_i})$ by ℓ_2 for any integer $p \geq 1$ and dimensions $n_1, \ldots, n_p$.

B. Distributed LPV Plant Model and Controller

The distributed systems of interest consist of interacting subsystems interconnected over arbitrary graph structures. Discrete-time LPV models will be used to describe the dynamics of the generally nonlinear subsystems over some operating envelopes, and a directed graph will be used to define the interconnection structure of the distributed system. The vertices of this graph correspond to the subsystems, and the directed edges describe the interconnections between these subsystems. We assume the number of subsystems is finite, say equal to N, and the interconnection structure is subject to a communication latency of one sampling period, that is, the information sent by a subsystem at the current time step is received by the target subsystem at the next time step. An example of such a distributed system is shown in Fig. 1, where S_0 denotes the delay operator.

Before giving the state-space equations of the distributed systems of interest, we make the following definitions. We denote a digraph with set of vertices V and set of directed edges E by $\mathcal{G}(V, E)$. For simplicity, for a graph with N vertices we define V as $V = \{1, 2, \ldots, N\}$. We use the ordered-pair (i, j) to denote the element of E corresponding to an edge directed from vertex $i \in V$ to vertex $j \in V$. For every $k \in V$, we define the index sets $E_{\mathrm{in}}^{(k)} := \{i \in V \,|\, (i, k) \in E\}$ and $E_{\mathrm{out}}^{(k)} := \{j \in V \,|\, (k, j) \in E\}$, where the elements in these sets are sorted in ascending order. For instance, for the digraph defining the interconnection structure of the distributed system $G = \{G^{(1)}, \ldots, G^{(4)}\}$ in Fig. 1, $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4), (4, 2)\}$, $E_{\mathrm{in}}^{(1)} = \{\}$, $E_{\mathrm{out}}^{(1)} = \{2, 3, 4\}$, $E_{\mathrm{in}}^{(4)} = \{1, 3\}$, etc.

As mentioned before, we assume that each of the subsystems can be modeled as a discrete-time LPV system; specifically, the state-space model of $G^{(k)}$ is given by the system equations in (1). The variable $t \in \mathbb{N}_0$ is discrete time, and $\delta^{(k)}(t) = (\delta_1^{(k)}(t), \ldots, \delta_{r_k}^{(k)}(t))$ is a vector of real scalar parameters, where r_k is the number of scheduling parameters in the LPV model of $G^{(k)}$ (clearly, different subsystem models may have different numbers of scheduling parameters). The signals $w^{(k)}(t)$ and $z^{(k)}(t)$ denote the exogenous disturbances and errors, respectively, whereas $u^{(k)}(t)$ denotes the applied control and $y^{(k)}(t)$ the measurements. The matrix-valued functions $A^{(k)}(\cdot)$, $B_1^{(k)}(\cdot)$, $B_2^{(k)}(\cdot)$, etc., are known a priori. We assume that these matrix-valued functions have continuous dependence on the parameters $\delta_i^{(k)}(t)$ and are uniformly bounded for all admissible values of the parameters. The signal $x^{(k)}(t)$ designates the state vector of the subsystem $G^{(k)}$ in the standard state-space representation. The vector-valued functions

$$x_{\mathrm{in}}^{(k)}(t) = \mathrm{vec}\left(x^{(ik)}(t)\right)_{i \in E_{\mathrm{in}}^{(k)}}$$

$$x_{\mathrm{out}}^{(k)}(t) = \mathrm{vec}\left(x^{(kj)}(t)\right)_{j \in E_{\mathrm{out}}^{(k)}}$$

denote the data transferred to and from, respectively, the subsystem $G^{(k)}$ at time t, where $x^{(ij)}(t)$ designates the information sent from subsystem $G^{(i)}$ to subsystem $G^{(j)}$ at time t ($G^{(j)}$ will receive this info at time $t+1$). Notice that we regard the interconnections between the subsystems as states when formulating the system equations. For example, in Fig. 1, $x_{\mathrm{out}}^{(1)}(t) = \mathrm{vec}(x^{(12)}(t), x^{(13)}(t), x^{(14)}(t))$, $x_{\mathrm{in}}^{(1)}(t)$ is an empty vector, $x_{\mathrm{in}}^{(4)}(t) = \mathrm{vec}(x^{(14)}(t), x^{(34)}(t))$, etc. The vectors $x^{(k)}(t)$, $x^{(ij)}(t)$, $w^{(k)}(t)$, $z^{(k)}(t)$, $u^{(k)}(t)$, and $y^{(k)}(t)$ are real and have dimensions denoted by $n^{(k)}$, $n^{(ij)}$, $n_w^{(k)}$, $n_z^{(k)}$, $n_u^{(k)}$, and $n_y^{(k)}$, respectively.

$$\begin{bmatrix} \begin{bmatrix} x^{(k)}(t+1) \\ x^{(k)}_{\text{out}}(t+1) \end{bmatrix} \\ z^{(k)}(t) \\ y^{(k)}(t) \end{bmatrix} = \begin{bmatrix} A^{(k)}(\delta^{(k)}(t)) & B^{(k)}_1(\delta^{(k)}(t)) & B^{(k)}_2(\delta^{(k)}(t)) \\ C^{(k)}_1(\delta^{(k)}(t)) & D^{(k)}_{11}(\delta^{(k)}(t)) & D^{(k)}_{12}(\delta^{(k)}(t)) \\ C^{(k)}_2(\delta^{(k)}(t)) & D^{(k)}_{21}(\delta^{(k)}(t)) & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x^{(k)}(t) \\ x^{(k)}_{\text{in}}(t) \end{bmatrix} \\ w^{(k)}(t) \\ u^{(k)}(t) \end{bmatrix}, \quad x^{(k)}(0) = 0 \tag{1}$$

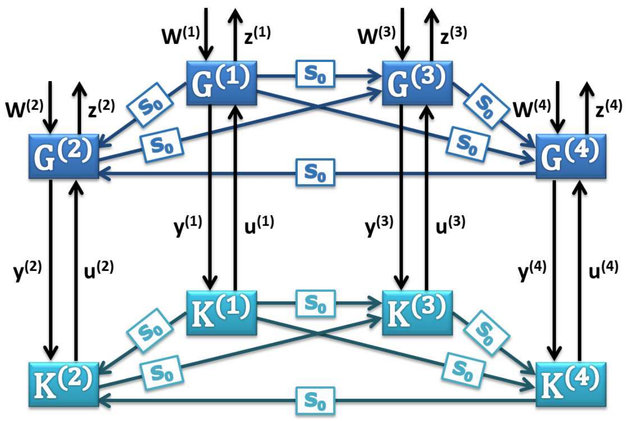

Fig. 3. Distributed closed-loop system with delays

As in [7], [20], [21], we assume the parameters $\delta^{(k)}(t) = (\delta^{(k)}_1(t), \dots, \delta^{(k)}_{r_k}(t))$ and parameter increments $d\delta^{(k)}(t) = \delta^{(k)}(t+1) - \delta^{(k)}(t)$ such that $(\delta^{(k)}(t), d\delta^{(k)}(t)) \in \Gamma^{(k)}$ for all $t \in \mathbb{N}_0$ and $k \in V$, where $\Gamma^{(k)}$ is a polytope defined as

$$\Gamma^{(k)} = \{(p, dp) \in \mathbb{R}^{r_k} \times \mathbb{R}^{r_k} \mid f^{(k)}_{i,j}(p_i, dp_i) \geq 0 \text{ for all } i = 1, \dots, r_k \text{ and } j = 1, 2, 3\}, \tag{2}$$

with
$$\begin{aligned} f^{(k)}_{i,1} &= (p_i - \underline{p}^{(k)}_i)(\overline{p}^{(k)}_i - p_i), \\ f^{(k)}_{i,2} &= (dp_i - \underline{dp}^{(k)}_i)(\overline{dp}^{(k)}_i - dp_i), \\ f^{(k)}_{i,3} &= (p_i + dp_i - \underline{p}^{(k)}_i)(\overline{p}^{(k)}_i - p_i - dp_i), \\ &\underline{p}^{(k)}_i, \overline{p}^{(k)}_i, \underline{dp}^{(k)}_i, \overline{dp}^{(k)}_i \in \mathbb{R},\ \underline{dp}^{(k)}_i \leq 0,\ \overline{dp}^{(k)}_i \geq 0. \end{aligned}$$

Notice that, for each $i = 1, \dots, r_k$, the set of points satisfying $f^{(k)}_{i,j}(p_i, dp_i) \geq 0$ for $j = 1, 2, 3$ defines a polygon which constitutes the projection of polytope $\Gamma^{(k)}$ on the (p_i, dp_i)-plane, as shown in Fig. 2. Thus, the allowable parameter trajectories $\delta^{(k)}$ reside in the set $\mathbf{\Delta}^{(k)} =$

$\{\delta^{(k)} : \mathbb{N}_0 \to \mathbb{R}^{r_k} \mid (\delta^{(k)}(t), d\delta^{(k)}(t)) \in \Gamma^{(k)} \text{ for all } t \in \mathbb{N}_0\}$,

and so trajectories $\delta = (\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(N)})$ belong to the set

$$\mathbf{\Delta} = \{\delta = (\delta^{(1)}, \dots, \delta^{(N)}) \mid \delta^{(k)} \in \mathbf{\Delta}^{(k)}\}. \tag{3}$$

The parameters $\delta^{(k)}(t)$ and parameter increments $d\delta^{(k)}(t)$ are assumed to be available for measurement at each instant t. If the parameter trajectories are continuously differentiable and the sampling time T is sufficiently small, the parameter increments may be approximated as $d\delta^{(k)}(t) \approx T\dot{\delta}^{(k)}(t)$.

The distributed system, as formulated in the preceding, is well-posed, that is, for each $\delta \in \mathbf{\Delta}$, given inputs $w^{(k)}$ and $u^{(k)}$ in ℓ, the state-space equations (1) admit unique solutions of the state variables for all $k \in V$, and further define a causal linear mapping on ℓ. The distributed system is said to be stable if it is well-posed and, for each $\delta \in \mathbf{\Delta}$, the state-space equations (1) admit unique solutions in ℓ_2 given inputs in ℓ_2 for all $k \in V$, that is, the system equations define a linear causal mapping on ℓ_2. In this paper, we consider a controller synthesis problem, where the controller has the same structure as the plant, as illustrated in Fig. 3. Each subcontroller $K^{(k)}$ is defined by the state-space equations in (4), where $x^{(k)}_K(0) = 0$, $\delta^{(k)}_{\text{in}}(t) = (\delta^{(i)}(t))_{i \in E^{(k)}_{\text{in}}}$, and the dimensions of the vectors $x^{(k)}_K(t)$ and $x^{(ij)}_K(t)$ are $m^{(k)}$ and $m^{(ij)}$, respectively. The state-space matrix-valued functions of the subcontrollers will be constructed from the solutions of the synthesis conditions, as discussed in Section IV, and will be uniformly bounded with continuous dependence on the parameters and parameter increments by construction. It is not surprising that the parameters $\delta^{(k)}(t)$ (and their increments $d\delta^{(k)}(t)$) that affect the subsystem $G^{(k)}$ also affect the associated subcontroller $K^{(k)}$. What is distinct to this controller formulation, though, is that the scheduling parameters of the subsystems that send data to $G^{(k)}$ also appear in the state-space equations of the subcontroller $K^{(k)}$. With this said, this configuration is still easily implementable. For instance, focusing on some $i \in E^{(k)}_{\text{in}}$, the subsystem $G^{(i)}$ has access to the values of the parameters $\delta^{(i)}$ and their increments $d\delta^{(i)}$ at time t and, hence, the value of $\delta^{(i)}(t+1) = \delta^{(i)}(t) + d\delta^{(i)}(t)$, which can be sent at time t, along with the data $x^{(ik)}(t+1)$, to the subsystem $G^{(k)}$. Factoring in the delay operation, the subsystem $G^{(k)}$ and, ultimately, its subcontroller will then have access to the value of $\delta^{(i)}$ at each time t.

The feedback interconnection of $G = \{G^{(1)}, \dots, G^{(N)}\}$ and $K = \{K^{(1)}, \dots, K^{(N)}\}$ results in the closed-loop system $L = \{L^{(1)}, \dots, L^{(N)}\}$, where the realization for each $L^{(k)}$ can be written as in (5). The vectors $x^{(k)}_L(t) = \text{vec}(x^{(k)}(t), x^{(k)}_K(t))$, $x^{(k)}_{L,\text{str}}(t) = \text{vec}(x^{(k)}_{\text{str}}(t), x^{(k)}_{K,\text{str}}(t))$, for $\text{str} = \text{in}, \text{out}$, and the closed-loop state-space matrices are defined in the obvious way. The closed-loop system can be viewed as a map from the exogenous disturbances $w = (w^{(1)}, \dots, w^{(N)})$ to the exogenous errors to be controlled $z = (z^{(1)}, \dots, z^{(N)})$. Clearly, as z represents the errors caused by the disturbances, we would like to design a stabilizing controller that would minimize the effect of the disturbances w on z. In other words, we would like a controller that would make the map $w \mapsto z$ "small" according to some measure for all $\delta \in \mathbf{\Delta}$. We use herein the popular ℓ_2 induced norm performance measure, specifically:

$$\sup_{\delta \in \mathbf{\Delta}} \|G\|_{\ell_2 \to \ell_2} = \sup_{\delta \in \mathbf{\Delta}} \|w \mapsto z\|_{\ell_2 \to \ell_2} = \sup_{\delta \in \mathbf{\Delta}} \sup_{w \neq 0} \frac{\|z\|_{\ell_2}}{\|w\|_{\ell_2}}.$$

We say a controller K, as described before, is a γ-*admissible* synthesis for plant G if it leads to a stable closed-loop system L that satisfies the inequality $\sup_{\delta \in \mathbf{\Delta}} \|w \mapsto z\|_{\ell_2 \to \ell_2} < \gamma$.

$$\begin{bmatrix}\begin{bmatrix}x_K^{(k)}(t+1)\\ x_{K,\text{out}}^{(k)}(t+1)\end{bmatrix}\\ u^{(k)}(t)\end{bmatrix}=\begin{bmatrix}A_K^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t)) & B_K^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t))\\ C_K^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t)) & D_K^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t))\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x_K^{(k)}(t)\\ x_{K,\text{in}}^{(k)}(t)\end{bmatrix}\\ y^{(k)}(t)\end{bmatrix} \tag{4}$$

$$\begin{bmatrix}\begin{bmatrix}x_L^{(k)}(t+1)\\ x_{L,\text{out}}^{(k)}(t+1)\end{bmatrix}\\ z^{(k)}(t)\end{bmatrix}=\begin{bmatrix}A_L^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t)) & B_L^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t))\\ C_L^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t)) & D_L^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t))\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x_L^{(k)}(t)\\ x_{L,\text{in}}^{(k)}(t)\end{bmatrix}\\ w^{(k)}(t)\end{bmatrix} \tag{5}$$

$$\left(\bar{F}_L^{(k)}(t)\right)^T\begin{bmatrix}X^{(k)}(t+1) & 0 & 0\\ 0 & X_{\text{out}}^{(k)}(t+1) & 0\\ 0 & 0 & \frac{1}{\gamma^2}I\end{bmatrix}\bar{F}_L^{(k)}(t)-\begin{bmatrix}X^{(k)}(t) & 0 & 0\\ 0 & X_{\text{in}}^{(k)}(t) & 0\\ 0 & 0 & I\end{bmatrix}\prec-\beta I \tag{6}$$

$$\begin{gathered}\mathcal{F}_1(R,R^+)=\begin{bmatrix}N_R & 0\\ 0 & I\end{bmatrix}^T\begin{bmatrix}ARA^T-R^+ & ARC_1^T & B_1\\ C_1RA^T & -\gamma I+C_1RC_1^T & D_{11}\\ B_1^T & D_{11}^T & -\gamma I\end{bmatrix}\begin{bmatrix}N_R & 0\\ 0 & I\end{bmatrix}\\ \mathcal{F}_2(S,S^+)=\begin{bmatrix}N_S & 0\\ 0 & I\end{bmatrix}^T\begin{bmatrix}A^TS^+A-S & A^TS^+B_1 & C_1^T\\ B_1^TS^+A & -\gamma I+B_1^TS^+B_1 & D_{11}^T\\ C_1 & D_{11} & -\gamma I\end{bmatrix}\begin{bmatrix}N_S & 0\\ 0 & I\end{bmatrix}\qquad \mathcal{F}_3(R,S)=\begin{bmatrix}R & I\\ I & S\end{bmatrix}\end{gathered} \tag{7}$$

Remark 1: The state-space matrix-valued functions of subsystem $G^{(k)}$ may have explicit dependence on $\delta_{\text{in}}^{(k)}(t)$, in addition to $\delta^{(k)}(t)$, if required. The results of this paper will still apply in this case as long as the same controller formulation as the one given in (4) is used.

C. Prior Results

Our recent results on distributed control in [16] deal with LTV subsystems interconnected over arbitrary graph structures. These results are relevant to this paper since the LPV subsystems become simply LTV models when the parameter trajectories are known a priori and, hence, a distributed LPV system reduces to a distributed LTV one for each $\delta\in\mathbf{\Delta}$. In [16], it is argued that any digraph can be transformed into a regular digraph by adding, if necessary, non-existent, or "virtual," edges and nodes to the graph. Such a transformation is required strictly as a conceptual tool to ensure a succinct representation of the distributed system and ultimately facilitate the development of the theory. Thus, while the analysis and synthesis results in [16] are developed for regular digraphs, these results can be easily reformulated for the original graph, which we will do in this subsection.

Suppose that the parameter trajectory $\delta\in\mathbf{\Delta}$ is known a priori, and, hence, the state-space matrix-valued functions in the system equations of the plant G and controller K, for example, $A^{(k)}(\delta^{(k)}(t))$ and $A_K^{(k)}(\delta^{(k)}(t),d\delta^{(k)}(t),\delta_{\text{in}}^{(k)}(t))$, can be expressed as functions of time, namely, $\bar{A}^{(k)}(t)$ and $\bar{A}_K^{(k)}(t)$, respectively. Similar notations will be used for the rest of the state-space functions. We will also denote the resulting distributed LTV plant by $\bar{G}$, its distributed LTV controller by $\bar{K}$, and the closed-loop system by $\bar{L}$. The following theorems are reformulations of results from [16].

Theorem 1: Closed-loop distributed LTV system $\bar{L}$ is stable and $\|w\mapsto z\|_{\ell_2\to\ell_2}<\gamma$, for some scalar γ, if there exist uniformly bounded, positive definite matrix-valued functions $X^{(k)}(t)\in\mathbb{S}^{n^{(k)}+m^{(k)}}$, $X^{(ik)}(t)\in\mathbb{S}^{n^{(ik)}+m^{(ik)}}$, and $X^{(kj)}(t)\in\mathbb{S}^{n^{(kj)}+m^{(kj)}}$, for all $k\in V$, $i\in E_{\text{in}}^{(k)}$, $j\in E_{\text{out}}^{(k)}$, and $t\in\mathbb{N}_0$, such that inequality (6) holds for some positive scalar β, where $X_{\text{in}}^{(k)}(t)=\text{diag}(X^{(ik)}(t))_{i\in E_{\text{in}}^{(k)}}$, $X_{\text{out}}^{(k)}(t)=\text{diag}(X^{(kj)}(t))_{j\in E_{\text{out}}^{(k)}}$, and $\bar{F}_L^{(k)}(t)=\begin{bmatrix}\bar{A}_L^{(k)}(t) & \bar{B}_L^{(k)}(t)\\ \bar{C}_L^{(k)}(t) & \bar{D}_L^{(k)}(t)\end{bmatrix}$.

Before stating the next reformulated synthesis result from [16], we make the definitions in (7), where $\text{Im}\,N_R=\text{Ker}\,[B_2^T,\,D_{12}^T]$, $N_R^TN_R=I$, $\text{Im}\,N_S=\text{Ker}\,[C_2,\,D_{21}]$, and $N_S^TN_S=I$, with $\text{Im}\,P$ and $\text{Ker}\,P$ denoting the image and kernel of a linear map P, respectively. These functions will be used to present the synthesis conditions concisely. In the sequel, we write $\bar{\mathcal{F}}_1^{(k)}(R,R^+,t)$ to imply that the state-space matrices in the expression defining $\mathcal{F}_1$ are of the form $\bar{A}^{(k)}(t)$, $\bar{B}_1^{(k)}(t)$, $\bar{B}_2^{(k)}(t)$, etc., as opposed to A, B_1, B_2, etc.

Theorem 2: There exists a γ-admissible synthesis $\bar{K}$ to distributed LTV plant $\bar{G}$ for some scalar γ if there exist uniformly bounded matrix-valued functions $R^{(k)}(t),S^{(k)}(t)\in\mathbb{S}^{n^{(k)}}$, $R^{(ik)}(t),S^{(ik)}(t)\in\mathbb{S}^{n^{(ik)}}$, and $R^{(kj)}(t),S^{(kj)}(t)\in\mathbb{S}^{n^{(kj)}}$, for all $k\in V$, $i\in E_{\text{in}}^{(k)}$, $j\in E_{\text{out}}^{(k)}$, $t\in\mathbb{N}_0$, such that

$$\begin{aligned}&\bar{\mathcal{F}}_1^{(k)}(\text{diag}(R^{(k)}(t),R_{\text{in}}^{(k)}(t)),\ldots\\ &\qquad\text{diag}(R^{(k)}(t+1),R_{\text{out}}^{(k)}(t+1)),t)\prec-\beta I\\ &\bar{\mathcal{F}}_2^{(k)}(\text{diag}(S^{(k)}(t),S_{\text{in}}^{(k)}(t)),\ldots\\ &\qquad\text{diag}(S^{(k)}(t+1),S_{\text{out}}^{(k)}(t+1)),t)\prec-\beta I\\ &\mathcal{F}_3(R^{(k)}(t),S^{(k)}(t))\succeq0,\ \ \mathcal{F}_3(R_{\text{in}}^{(k)}(t),S_{\text{in}}^{(k)}(t))\succeq0,\end{aligned} \tag{8}$$

for some positive scalar β, where $\bar{\mathcal{F}}_1^{(k)}$, $\bar{\mathcal{F}}_2^{(k)}$, and $\mathcal{F}_3$ are as defined in (7), $Q_{\text{in}}^{(k)}(t)=\text{diag}(Q^{(ik)}(t))_{i\in E_{\text{in}}^{(k)}}$ and $Q_{\text{out}}^{(k)}(t)=\text{diag}(Q^{(kj)}(t))_{j\in E_{\text{out}}^{(k)}}$, for $Q=R,S$.

$$\left(F_L^{(k)}\right)^T \begin{bmatrix} X^{(k)}(p^{(k)}+dp^{(k)}) & 0 & 0 \\ 0 & X_{\text{out}}^{(k)}(p^{(k)}+dp^{(k)}) & 0 \\ 0 & 0 & \frac{1}{\gamma^2}I \end{bmatrix} F_L^{(k)}(p^{(k)}, dp^{(k)}, p_{\text{in}}^{(k)}) - \begin{bmatrix} X^{(k)}(p^{(k)}) & 0 & 0 \\ 0 & X_{\text{in}}^{(k)}(p_{\text{in}}^{(k)}) & 0 \\ 0 & 0 & I \end{bmatrix} \prec -\beta I \quad (9)$$

III. Analysis and Synthesis Results

We now state the following analysis and synthesis results.

Theorem 3: Closed-loop system L, defined in (5), is stable and $\|w \mapsto z\|_{\ell_2 \to \ell_2} < \gamma$ for all $\delta \in \mathbf{\Delta}$, as defined in (3), if there exist uniformly bounded, positive definite matrix-valued functions $X^{(k)}(p^{(k)}) \in \mathbb{S}^{n^{(k)}+m^{(k)}}$, $X^{(ik)}(p^{(i)}) \in \mathbb{S}^{n^{(ik)}+m^{(ik)}}$, and $X^{(kj)}(p^{(k)}) \in \mathbb{S}^{n^{(kj)}+m^{(kj)}}$, continuous in $p^{(k)}$ and $p^{(i)}$, for all $k \in V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that inequality (9) holds for all $(p^{(k)}, dp^{(k)}) \in \Gamma^{(k)}$, as defined in (2), $p_v^{(i)} \in [\underline{p}_v^{(i)}, \overline{p}_v^{(i)}]$, $i \in E_{\text{in}}^{(k)}$, $v = 1, 2, \ldots, r_i$, and some positive scalar β, where $p_{\text{in}}^{(k)} = (p^{(i)})_{i \in E_{\text{in}}^{(k)}}$,

$$X_{\text{in}}^{(k)}(p_{\text{in}}^{(k)}) = \text{diag}(X^{(ik)}(p^{(i)}))_{i \in E_{\text{in}}^{(k)}},$$

$$X_{\text{out}}^{(k)}(p^{(k)}+dp^{(k)}) = \text{diag}(X^{(kj)}(p^{(k)}+dp^{(k)}))_{j \in E_{\text{out}}^{(k)}},$$

$$F_L^{(k)} = \begin{bmatrix} A_L^{(k)} & B_L^{(k)} \\ C_L^{(k)} & D_L^{(k)} \end{bmatrix}.$$

Proof: Given any trajectory $\delta \in \mathbf{\Delta}$, the distributed LPV system L reduces to the distributed LTV system $\bar{L}$, as discussed in Subsection II-C. Suppose inequality (9) holds for all $(p^{(k)}, dp^{(k)}) \in \Gamma^{(k)}$ and $p_v^{(i)} \in [\underline{p}_v^{(i)}, \overline{p}_v^{(i)}]$, where $i \in E_{\text{in}}^{(k)}$ and $v = 1, 2, \ldots, r_i$. Then, given $\delta \in \mathbf{\Delta}$, by replacing p with $\delta(t)$ in (9), the resulting inequality would still be valid for all $t \in \mathbb{N}_0$; this immediately follows from the definition of $\mathbf{\Delta}$, which ensures that $(\delta^{(k)}(t), d\delta^{(k)}(t)) \in \Gamma^{(k)}$ and $\delta_v^{(i)}(t) \in [\underline{p}_v^{(i)}, \overline{p}_v^{(i)}]$ for all $t \in \mathbb{N}_0$. Then, as $\delta^{(k)}(t+1) = \delta^{(k)}(t) + d\delta^{(k)}(t)$, we obtain that the matrix-valued functions $X^{(k)}(\delta^{(k)}(t)) \succ 0$, $X_{\text{in}}^{(k)}(\delta_{\text{in}}^{(k)}(t)) \succ 0$, and $X_{\text{out}}^{(k)}(\delta^{(k)}(t)) \succ 0$, bounded above and below, satisfy condition (6) for the distributed LTV system $\bar{L}$, which, by Theorem 1, implies that $\bar{L}$ is stable and $\|\bar{L}\|_{\ell_2 \to \ell_2} < \gamma$. ■

It will be convenient to define the following functions:

$$\mathcal{G}_1(R, R^+, \sigma) = \begin{bmatrix} ARA^T - R^+ & ARC_1^T & B_1 \\ C_1RA^T & -\gamma I + C_1RC_1^T & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} B_2^T & D_{12}^T & 0 \end{bmatrix}^T \begin{bmatrix} B_2^T & D_{12}^T & 0 \end{bmatrix}, \quad (10)$$

$$\mathcal{G}_2(S, S^+, \sigma) = \begin{bmatrix} A^TS^+A - S & A^TS^+B_1 & C_1^T \\ B_1^TS^+A & -\gamma I + B_1^TS^+B_1 & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix}^T \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix}. \quad (11)$$

These definitions are similar to those in (7), and are made to simplify the presentation of the synthesis conditions. In addition, as discussed before, we write $\mathcal{G}_1^{(k)}(R, R^+, \sigma, p^{(k)})$ to imply that the state-space matrices in the expression defining $\mathcal{G}_1$ are of the form $A^{(k)}(p^{(k)})$, $B_1^{(k)}(p^{(k)})$, $B_2^{(k)}(p^{(k)})$, etc., as opposed to simply A, B_1, B_2, etc.

Theorem 4: Given plant G defined in (1) with $\delta \in \mathbf{\Delta}$, suppose that the matrices $[B_2^{(k)}(\delta^{(k)}(t))^T, D_{12}^{(k)}(\delta^{(k)}(t))^T]$ and $[C_2^{(k)}(\delta^{(k)}(t)), D_{21}^{(k)}(\delta^{(k)}(t))]$ have full-row rank uniformly for all $k \in V$, $t \in \mathbb{N}_0$, and $\delta^{(k)} \in \mathbf{\Delta}^{(k)}$. Then there exists a γ-admissible distributed LPV synthesis K to G for some scalar γ if there exist uniformly bounded, positive definite matrix-valued functions $R^{(k)}(p^{(k)}), S^{(k)}(p^{(k)}) \in \mathbb{S}^{n^{(k)}}$, $R^{(ik)}(p^{(i)}), S^{(ik)}(p^{(i)}) \in \mathbb{S}^{n^{(ik)}}$, and $R^{(kj)}(p^{(k)}), S^{(kj)}(p^{(k)}) \in \mathbb{S}^{n^{(kj)}}$, continuous in $p^{(k)}$ and $p^{(i)}$, for all $k \in V$, $i \in E_{\text{in}}^{(k)}$, $j \in E_{\text{out}}^{(k)}$, such that

$$\begin{aligned}
&\mathcal{G}_1^{(k)}(\text{diag}(R^{(k)}(p^{(k)}), R_{\text{in}}^{(k)}(p_{\text{in}}^{(k)})), \ldots \\
&\text{diag}(R^{(k)}(p^{(k)}+dp^{(k)}), R_{\text{out}}^{(k)}(p^{(k)}+dp^{(k)})), \sigma, p^{(k)}) \prec -\beta I \\
&\mathcal{G}_2^{(k)}(\text{diag}(S^{(k)}(p^{(k)}), S_{\text{in}}^{(k)}(p_{\text{in}}^{(k)})), \ldots \\
&\text{diag}(S^{(k)}(p^{(k)}+dp^{(k)}), S_{\text{out}}^{(k)}(p^{(k)}+dp^{(k)})), \sigma, p^{(k)}) \prec -\beta I \\
&\mathcal{F}_3(R^{(k)}(p^{(k)}), S^{(k)}(p^{(k)})) \succeq 0, \\
&\mathcal{F}_3(R_{\text{in}}^{(k)}(p_{\text{in}}^{(k)}), S_{\text{in}}^{(k)}(p_{\text{in}}^{(k)})) \succeq 0,
\end{aligned} \quad (12)$$

for all $(p^{(k)}, dp^{(k)}) \in \Gamma^{(k)}$, $p_v^{(i)} \in [\underline{p}_v^{(i)}, \overline{p}_v^{(i)}]$, $i \in E_{\text{in}}^{(k)}$, $v = 1, 2, \ldots, r_i$, and some positive scalar β, where $\mathcal{G}_1^{(k)}$, $\mathcal{G}_2^{(k)}$, and $\mathcal{F}_3$ are as defined in (10), (11), and (7), respectively, $p_{\text{in}}^{(k)} = (p^{(i)})_{i \in E_{\text{in}}^{(k)}}$, and, for $Q = R, S$,

$$Q_{\text{in}}^{(k)}(p_{\text{in}}^{(k)}) = \text{diag}(Q^{(ik)}(p^{(i)}))_{i \in E_{\text{in}}^{(k)}}, \quad (13)$$

$$Q_{\text{out}}^{(k)}(p^{(k)}+dp^{(k)}) = \text{diag}(Q^{(kj)}(p^{(k)}+dp^{(k)}))_{j \in E_{\text{out}}^{(k)}}.$$

Proof: As in the proof of Theorem 3, for each $\delta \in \mathbf{\Delta}$, distributed LPV system G reduces to a distributed LTV system. Then, invoking Theorem 2 along with applications of Finsler's lemma and a similar argument to that in the proof of [2, Theorem 5.2(iii)] complete the proof. ■

The synthesis conditions (12) are infinitely constrained and given in terms of PLMIs. We will assume that the state-space matrix-valued functions and synthesis solutions have polynomial dependence on the parameters. With that said, there are several relaxation techniques available in the literature that may render such infinite dimensional PLMI problems computationally tractable, such as the multiconvexity relaxation technique [2] and the SOS decomposition method [1]. The reader is referred to [3], [4] for some useful surveys and the papers [21], [22] for examples on how to apply the SOS and multiconvexity methods to solve stationary and nonstationary LPV problems. The latest features of YALMIP [5], [23] are also useful for solving PLMIs. The computational complexity will be dependent on the specific relaxation technique used and how conservative this relaxation is. From our experience, the multiconvexity technique seems to be quite effective when it comes to solving relatively large PLMI problems.

IV. Controller Construction

The controller is constructed online from the solutions, $R^{(k)}(p^{(k)})$, $R_{\text{in}}^{(k)}(p_{\text{in}}^{(k)})$, $R_{\text{out}}^{(k)}(p^{(k)})$, $S^{(k)}(p^{(k)})$, $S_{\text{in}}^{(k)}(p_{\text{in}}^{(k)})$, and $S_{\text{out}}^{(k)}(p^{(k)})$, of the synthesis conditions (12). In this section, we briefly present a fast algorithm for constructing the controller realization online at each discrete instant t. The algorithm is a generalized version of the ones given in [6], [7]. A detailed derivation of the algorithm will be provided in the journal version of this paper. For simplicity, in the following we will mostly suppress the dependence of the state-space matrix-valued functions of all systems on the parameters and their increments and further omit the superscript (k). We will also use the notations in (13) for any given matrix-valued function Q, and further introduce the following notations:

$$\tilde{Q}^{+} = \text{diag}(Q^{(k)}(p^{(k)} + dp^{(k)}), Q_{\text{out}}^{(k)}(p^{(k)} + dp^{(k)})),$$
$$\tilde{Q} = \text{diag}(Q^{(k)}(p^{(k)}), Q_{\text{in}}^{(k)}(p_{\text{in}}^{(k)})).$$

Using some relaxation technique, we solve for polynomial functions $D_K^{(k)}(p^{(k)}, dp^{(k)}, p_{\text{in}}^{(k)})$, $K_C^{(k)}(p^{(k)}, dp^{(k)}, p_{\text{in}}^{(k)})$, and $K_B^{(k)}(p^{(k)}, dp^{(k)}, p_{\text{in}}^{(k)})$ satisfying the PLMIs

$$\begin{bmatrix} -\tilde{R} & L_R^T \\ L_R & \Omega \end{bmatrix} \prec 0 \ \text{ and } \ \begin{bmatrix} -\tilde{S}^{+} & L_S^T \\ L_S & \Omega \end{bmatrix} \prec 0, \quad \text{where}$$

$$\Omega = \begin{bmatrix} -\tilde{R}^{+} & 0 & \underline{A} & \underline{B} \\ 0 & -\gamma I & \underline{C} & D_L \\ \underline{A}^T & \underline{C}^T & -\tilde{S} & 0 \\ \underline{B}^T & D_L^T & 0 & -\gamma I \end{bmatrix}, \quad \begin{aligned} \underline{A} &= A + B_2 D_K C_2, \\ \underline{B} &= B_1 + B_2 D_K D_{21}, \\ \underline{C} &= C_1 + D_{12} D_K C_2, \end{aligned}$$

$$L_R = \begin{bmatrix} A\tilde{R} + B_2 K_C \\ C_1 \tilde{R} + D_{12} K_C \\ -I \\ 0 \end{bmatrix}, \text{ and } L_S = \begin{bmatrix} -I \\ 0 \\ A^T \tilde{S}^{+} + C_2^T K_B \\ B_1^T \tilde{S}^{+} + D_{21}^T K_B \end{bmatrix}.$$

Then, the online computation of the controller realization at time t can be summarized as follows:

1. Compute and evaluate the following factorizations at $(\delta^{(k)}(t), d\delta^{(k)}(t), \delta_{\text{in}}^{(k)}(t))$: $\tilde{M}\tilde{W}^T = I - \tilde{R}\tilde{S}$ and $\tilde{M}^{+}(\tilde{W}^{+})^T = I - \tilde{R}^{+}\tilde{S}^{+}$. For instance, choose $\tilde{W} = I$ and $\tilde{M} = I - \tilde{R}\tilde{S}$ (and similarly for $\tilde{W}^{+}$ and $\tilde{M}^{+}$); this choice of $\tilde{M}$ is invertible provided that the last two conditions in (12) hold with strict inequality.
2. Evaluate D_K, K_C, K_B at $(\delta^{(k)}(t), d\delta^{(k)}(t), \delta_{\text{in}}^{(k)}(t))$. Then, we have

$$C_K = \left(K_C - D_K C_2 \tilde{R}\right)\left(\tilde{M}^T\right)^{-1},$$
$$B_K = \left(\tilde{W}^{+}\right)^{-1}\left(K_B^T - \tilde{S}^{+} B_2 D_K\right),$$
$$A_K = \left(\tilde{W}^{+}\right)^{-1}\left(L_S^T \Omega^{-1} L_R - \tilde{S}^{+}\underline{A}\tilde{R} - \tilde{S}^{+} B_2 C_K \tilde{M}^T - \tilde{W}^{+} B_K C_2 \tilde{R}\right)\left(\tilde{M}^T\right)^{-1}.$$

V. Conclusions

The paper focuses on the control of distributed LPV systems. The distributed plant model in question consists of discrete-time LPV subsystems interconnected over a directed graph, where the interconnection structure is subject to a communication delay of one sampling period. The synthesis objective is to design a controller that has the same topological structure as the plant, which ensures closed-loop stability and a performance criterion given in terms of the ℓ_2-induced norm performance measure. The paper provides a solution to this control problem using a parameter-dependent Lyapunov approach, with the analysis and synthesis conditions given in terms of PLMIs. The paper also gives a fast and easy-to-implement algorithm for constructing the controller online.

References

[1] P. A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Math. Program. Ser. B*, vol. 96, no. 2, pp. 293–320, 2003.

[2] P. Apkarian and H. D. Tuan, "Parameterized LMIs in control theory," *SIAM J. Control Optim.*, vol. 38, no. 4, pp. 1241–1264, 2000.

[3] M. Laurent, "Sums of squares, moment matrices and optimization over polynomials," in *Emerging Applications of Algebraic Geometry*, vol. 149 of *IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds.)*, pp. 157–270, Springer, 2009.

[4] C. W. Scherer, "LMI relaxations in robust control," *European Journal of Control*, vol. 12, no. 1, 2006.

[5] J. Löfberg, "Pre- and post-processing sum-of-squares programs in practice," *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1007–1011, 2009.

[6] P. Gahinet, "Explicit controller formulas for LMI-based H_∞ synthesis," *Automatica*, vol. 32, pp. 1007–1014, July 1996.

[7] M. Farhood, "LPV control of nonstationary systems: A parameter-dependent Lyapunov approach," *IEEE Transactions on Automatic Control*, vol. 57, pp. 209–215, Jan. 2012.

[8] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *IEEE Transactions on Automatic Control*, vol. 47, pp. 1091–1107, July 2002.

[9] R. D'Andrea and G. E. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Transactions on Automatic Control*, vol. 48, pp. 1478–1495, Sept. 2003.

[10] B. Recht and R. D'Andrea, "Distributed control of systems over discrete groups," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1446–1452, Sept. 2004.

[11] C. Langbort, R. S. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1502–1519, 2004.

[12] G. E. Dullerud and R. D'Andrea, "Distributed control of heterogeneous systems," *IEEE Transactions on Automatic Control*, vol. 49, pp. 2113–2128, Dec. 2004.

[13] N. Motee and A. Jadbabaie, "Optimal control of spatially distributed systems," *IEEE Trans. Autom. Control*, vol. 53, pp. 1616–1629, 2008.

[14] H. Fang and P. J. Antsaklis, "Distributed control with integral quadratic constraints," in *Proceedings of the 17th IFAC World Congress*, 2008.

[15] S. Tonetti and R. M. Murray, "Stability and performance of non-homogeneous multi-agent systems on a graph," in *Proceedings of the 18th IFAC World Congress*, (Milano, Italy), 2011.

[16] M. Farhood, Z. Di, and G. E. Dullerud, "Distributed control of linear time-varying systems interconnected over arbitrary graphs," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 2, pp. 179–206, Jan. 2015.

[17] F. Wu, "Distributed control for interconnected linear parameter-dependent systems," *IEE Proceedings - Control Theory and Applications*, vol. 150, pp. 518–27, Sept. 2003.

[18] C. Hoffmann, A. Eichler, and H. Werner, "Distributed control of linear parameter-varying decomposable systems," in *American Control Conference (ACC)*, pp. 2380–2385, June 2013.

[19] C. Hoffmann, A. Eichler, and H. Werner, "Control of heterogeneous groups of LPV systems interconnected through directed and switching topologies," in *American Control Conference*, pp. 5156–5161, 2014.

[20] F. Wu and K. Dong, "Gain-scheduling control of LFT systems using parameter-dependent Lyapunov functions," *Automatica*, vol. 42, pp. 39–50, 2006.

[21] M. Farhood and E. Feron, "Obstacle-sensitive trajectory regulation via gain scheduling and semidefinite programming," *IEEE Transactions on Control Systems Technology*, vol. 20, pp. 1107–1115, July 2012.

[22] M. Farhood, "Nonstationary LPV control for trajectory tracking: A double pendulum example," *International Journal of Control*, vol. 85, no. 5, pp. 545–562, 2012.

[23] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, (Taipei, Taiwan), pp. 284–289, 2004.