Balanced Truncation of Linear Systems Interconnected over Arbitrary Graphs with Communication Latency

Dany Abou Jaoude and Mazen Farhood

Abstract— **This paper deals with the model reduction of spatially distributed systems. In particular, we treat discrete-time linear time-varying subsystems interconnected over arbitrary graph structures and subjected to communication latency, a one step time delay on the information exchange between the source and the target subsystems. We propose a generalization of the balanced truncation scheme, apply it to the model reduction of distributed systems, and derive a guaranteed upper bound on the resultant error. This bound reduces to a finite sum in the case of periodic and time-invariant subsystems.**

I. INTRODUCTION

The study of interconnected systems is of interest because a collective and synchronized behavior of multiple agents is observed in natural systems as well as engineering systems [1]. Additionally, many applications of distributed controllers have emerged in recent years, such as cooperative control of multi-vehicle systems, multi-point surveillance, and mobile sensor networks [1]. Moreover, distributed controller approaches are appealing as they boast many advantages over centralized ones: simplicity, flexibility, scalability, and robustness to name a few [2].

In this paper, we treat heterogenous discrete-time linear time-varying (LTV) subsystems interconnected over arbitrary directed graphs, and assume that the data transfer between subsystems is subjected to a uniform one step time delay. We formulate our model similarly to the work of [3], [4]. In [3], an operator theoretic framework is introduced to allow for a succinct representation of distributed systems and a facilitated development of the results. Such a framework is possible because any arbitrary directed graph can be transformed into a regular one by adding, if necessary, nonexistent nodes and edges. A directed graph is said to be regular when the indegree of a vertex is equal to its outdegree and to the indegrees of all other vertices. In this work, however, we deal with the actual interconnection structure of the system, as in [4], and hence, shun the need for the graph regularity assumption.

After formulating the model, we propose a balanced truncation scheme to reduce its order. Model reduction is desirable in the case of interconnected systems because the order of the global system scales with the number of subsystems and the complexity of the interconnection structure, and thus, can be very large. The use of balanced realizations for model order reduction was first proposed in [5]. Then, bounds on the error induced by balanced truncation for linear time-invariant (LTI) systems were computed in [6]–[9]. A survey of the literature reveals that multiple works have since extended the balanced truncation scheme and its resulting error bound to more general classes of systems [10]–[17].

As with control design, one can treat interconnected systems as a single global system when applying balanced truncation, however, such an approach does not preserve the structure of the network. The work in [18] proposes a coprime factors approach for model reduction that preserves the partitioning of the states. In [19], model reduction is discussed for interconnected systems where the input to each subsystem is a combination of the outputs of the other subsystems and an externally applied input, and the output of the global system is a weighted average of the outputs of all subsystems. The proposed algorithm therein unifies and extends the frequency-weighted and closed-loop balanced truncations, previously applied to the model reduction of interconnected systems. However, it fails to guarantee the stability of the reduced order model and a bound on the error induced by the reduction process. The work of [20] uses a linear fractional transformation framework, and suggests a model reduction method for continuous-time LTI interconnected systems based on the block-diagonal solutions of the Lyapunov inequalities. This method addresses the issues of its antecedent; however, it suffers from some conservatism because of the structure imposed on the solutions to the Lyapunov inequalities. In our work, we generalize the method of [20] to the class of discrete-time LTV subsystems interconnected over arbitrary graphs.

The paper is organized as follows. In Section II, we define the notation to be used throughout the paper. We then summarize, in Section III, the formulation of the model and the results on well-posedness, stability, and performance analysis. In Section IV, we present the balanced truncation method and its resultant error upper bound. We devote a subsection to the special case of time-periodic systems. The paper concludes with Section V.

II. NOTATION

Symbols $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of nonnegative integers, integers, and real numbers, respectively. We refer to the set of $n \times m$ real matrices by $\mathbb{R}^{n\times m}$ and that of $n \times n$ symmetric matrices by $\mathbb{S}^n$. Given an ordered subset S of the set of positive integers, the notations $(v_i)_{i\in S}$ and $(M_i)_{i\in S}$ denote its associated vector and matrix sequences. The ordering of the elements in these sequences corresponds to the ordering of the elements in S. $\text{vec}(v_i)_{i\in S}$ denotes the vertical concatenation of the

The authors are with the Department of Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, VA 24061, USA. Email: {danyabj, farhood}@vt.edu. This material is based upon work supported by the National Science Foundation under Grant Number CMMI-1333785.

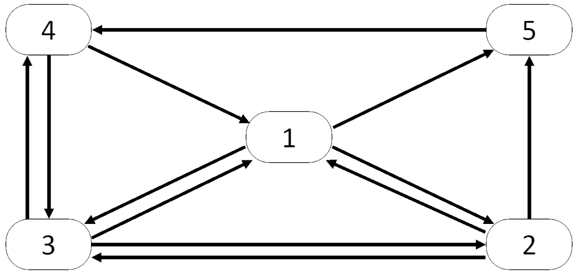

Fig. 1. Example of a Directed Graph

elements of $(v_i)_{i\in S}$, whereas, $\mathrm{diag}(M_i)_{i\in S}$ designates the block-diagonal augmentation of the elements of $(M_i)_{i\in S}$. For example, let $S = \{1, 2, 4\}$, then $(v_i)_{i\in S} = (v_1, v_2, v_4)$, $(M_i)_{i\in S} = (M_1, M_2, M_4)$, $\mathrm{vec}(v_i)_{i\in S} = [v_1^T, v_2^T, v_4^T]^T$, and $\mathrm{diag}(M_i)_{i\in S} = \mathrm{diag}(M_1, M_2, M_4)$.

We refer to a directed graph with set of vertices V and set of directed edges E by $\mathcal{G}(V, E)$. We only consider directed graphs with a finite number of vertices, say, N. We choose the corresponding vertex set as $V = \{1, \ldots, N\}$, for simplicity. An element of E directed from vertex $i \in V$ to vertex $j \in V$ is represented by the ordered pair (i, j). With each k in V, we associate the sets $E_{\mathrm{in}}^{(k)} := \{i \in V \,|\, (i, k) \in E\}$ and $E_{\mathrm{out}}^{(k)} := \{j \in V \,|\, (k, j) \in E\}$. For convenience, we sort the elements of these sets in ascending order. We also define $m(k)$ and $p(k)$ as the indegree and outdegree of a vertex $k \in V$. As an illustration, consider Figure 1, which shows a directed graph with 5 vertices and 12 directed edges. Clearly, $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 2), (3, 4), (4, 1), (4, 3), (5, 4)\}$. Furthermore, $E_{\mathrm{in}}^{(1)} = \{2, 3, 4\}$, $m(1) = 3$, $E_{\mathrm{out}}^{(1)} = \{2, 3, 5\}$, $p(1) = 3$, $E_{\mathrm{in}}^{(2)} = \{1, 3\}$, $m(2) = 2$, $E_{\mathrm{out}}^{(2)} = \{1, 3, 5\}$, $p(2) = 3$, etc.

Given a symmetric matrix X, $X \prec 0$ means it is negative definite. Given an integer sequence $n : (t, k) \in \mathbb{N}_0 \times V \to n(t, k) \in \mathbb{N}_0$, we define $\ell(\{\mathbb{R}^{n(t,k)}\})$ as the vector space of mappings $w : (t, k) \in \mathbb{N}_0 \times V \to w(t, k) \in \mathbb{R}^{n(t,k)}$. Note that since the number of vertices is finite, then $\sum_{k\in V} w(t, k)^T w(t, k) < \infty$ for each $t \in \mathbb{N}_0$. The Hilbert space $\ell_2(\{\mathbb{R}^{n(t,k)}\})$ is the subspace of $\ell(\{\mathbb{R}^{n(t,k)}\})$ consisting of mappings w that have a finite ℓ_2-norm $\|w\|_2^2 := \sum_{(t,k)} w(t, k)^T w(t, k)$. We will adopt the abbreviated notations ℓ and ℓ_2 when the dimensions are clear from context.

III. DISTRIBUTED SYSTEM MODEL

A. State-Space Representation

In this section, we develop the state-space equations for discrete-time LTV subsystems interconnected over arbitrary directed graphs with a communication latency, namely, the data sent by a subsystem at the current time step reaches the target subsystem at the next time step. The notation we use is reminiscent of [4], and hence, the state-space equations of the distributed system are given in terms of the original graph setting, unlike in [3], where the graph needs to be transformed into a regular one through the addition, if necessary, of virtual interconnections and/or nodes. Yet, our framework is equivalent to that of [3] because the virtual interconnections correspond to signals of zero dimensions, and thus, their associated blocks in the state-space matrices are nonexistent. Therefore, we can adopt the analysis results on well-posedness, stability, and performance from [3] without difficulty.

The interconnection structure of a distributed system can be represented using a directed graph, where each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$ and the interconnections between the subsystems correspond to the directed edges. We denote the states of subsystem $G^{(k)}$ by $x^{(k)}(t)$. These states are referred to as the temporal states. We also define $x^{(ij)}(t)$ as the information sent from subsystem $G^{(i)}$ at time t to subsystem $G^{(j)}$. These states are called spatial states. With each $G^{(k)}$, we associate the following vectors:

$$x_{\mathrm{in}}^{(k)}(t) = \mathrm{vec}(x^{(ik)}(t))_{i\in E_{\mathrm{in}}^{(k)}},$$
$$x_{\mathrm{out}}^{(k)}(t) = \mathrm{vec}(x^{(kj)}(t))_{j\in E_{\mathrm{out}}^{(k)}}.$$

Subsystem $G^{(k)}$ has its own inputs $u^{(k)}(t)$ and outputs $y^{(k)}(t)$. Then, the state-space equations are, for all $(t, k) \in \mathbb{N}_0 \times V$:

$$\begin{bmatrix} x^{(k)}(t+1) \\ x_{\mathrm{out}}^{(k)}(t+1) \end{bmatrix} = \bar{A}^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\mathrm{in}}^{(k)}(t) \end{bmatrix} + \bar{B}^{(k)}(t)\, u^{(k)}(t),$$
$$y^{(k)}(t) = \bar{C}^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\mathrm{in}}^{(k)}(t) \end{bmatrix} + \bar{D}^{(k)}(t)\, u^{(k)}(t), \quad (1)$$
$$\text{with} \quad \begin{bmatrix} x^{(k)}(0) \\ x_{\mathrm{in}}^{(k)}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix-valued sequences $\bar{A}^{(k)}(t)$, $\bar{B}^{(k)}(t)$, $\bar{C}^{(k)}(t)$, and $\bar{D}^{(k)}(t)$ are known a priori and are assumed to be uniformly bounded. Signals $x^{(k)}(t)$, $x^{(ik)}(t)$, $x^{(kj)}(t)$, $u^{(k)}(t)$, and $y^{(k)}(t)$ are real with possibly time-varying dimensions, denoted by $n^{(k)}(t), n^{(ik)}(t), n^{(kj)}(t), n_u^{(k)}(t)$, and $n_y^{(k)}(t)$, respectively, for $(t, k) \in \mathbb{N}_0 \times V$, $i \in E_{\mathrm{in}}^{(k)}$, and $j \in E_{\mathrm{out}}^{(k)}$. Vectors $[x^{(k)}(t+1)^T, x_{\mathrm{out}}^{(k)}(t+1)^T]^T$ and $[x^{(k)}(t)^T, x_{\mathrm{in}}^{(k)}(t)^T]^T$ are naturally partitioned into $p(k)+1$ and $m(k)+1$ vector-valued channels, respectively. We partition the state-space matrices conformably with the partitioning of these two vectors. To illustrate this process, consider a distributed system represented by the directed graph in Figure 1. The state-space matrices of subsystem $G^{(1)}$ are partitioned as follows:

$$\bar{A}^{(1)}(t) = \begin{bmatrix} A_{00}^{(1)}(t) & A_{02}^{(1)}(t) & A_{03}^{(1)}(t) & A_{04}^{(1)}(t) \\ A_{20}^{(1)}(t) & A_{22}^{(1)}(t) & A_{23}^{(1)}(t) & A_{24}^{(1)}(t) \\ A_{30}^{(1)}(t) & A_{32}^{(1)}(t) & A_{33}^{(1)}(t) & A_{34}^{(1)}(t) \\ A_{50}^{(1)}(t) & A_{52}^{(1)}(t) & A_{53}^{(1)}(t) & A_{54}^{(1)}(t) \end{bmatrix},$$
$$\bar{B}^{(1)}(t) = \begin{bmatrix} B_0^{(1)}(t)^T \; B_2^{(1)}(t)^T \; B_3^{(1)}(t)^T \; B_5^{(1)}(t)^T \end{bmatrix}^T,$$
$$\bar{C}^{(1)}(t) = \begin{bmatrix} C_0^{(1)}(t) & C_2^{(1)}(t) & C_3^{(1)}(t) & C_4^{(1)}(t) \end{bmatrix}.$$

B. Analysis Results

We now give some analysis results from [3], reformulated for the original graph setting.

Definition 1: A distributed system is well-posed if, given inputs in ℓ, the state equations admit unique solutions in ℓ and further define a linear causal mapping on ℓ. Moreover, the system is stable if it is well-posed and if, given inputs in ℓ_2, the state equations admit unique solutions in ℓ_2 and further define a linear causal mapping on ℓ_2.

The equations, as given in (1), are defined for $t \in \mathbb{N}_0$ with zero initial conditions. The definition can be alternatively extended to $t \in \mathbb{Z}$ with the state-space matrices set equal to zero for negative times. Using the result [21, Lemma 6] and a similar argument as in the proof of [21, Lemma 8], we can show that a distributed system with $\bar{A}^{(k)}(t) = 0$ for $t < 0$ is well-posed. Thus, the distributed system described by (1) is well-posed. Next, we give a Lyapunov-based test for stability. This result constitutes the basis for the balanced truncation method. However, it is only sufficient in nature, and hence, introduces conservatism into the approach. We refer to systems that satisfy this test as strongly stable ones.

Lemma 1: A distributed system is strongly stable if there exist uniformly bounded, positive definite, matrix-valued functions $X^{(k)}(t) \in \mathbb{S}^{n^{(k)}(t)}$, $X^{(ik)}(t) \in \mathbb{S}^{n^{(ik)}(t)}$, and $X^{(kj)}(t) \in \mathbb{S}^{n^{(kj)}(t)}$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\left(\bar{A}^{(k)}(t)\right)^T \begin{bmatrix} X^{(k)}(t+1) & 0 \\ 0 & X_{\text{out}}^{(k)}(t+1) \end{bmatrix} \bar{A}^{(k)}(t) - \begin{bmatrix} X^{(k)}(t) & 0 \\ 0 & X_{\text{in}}^{(k)}(t) \end{bmatrix} \prec -\beta I, \quad (2)$$

for some positive scalar β, where

$$X_{\text{in}}^{(k)}(t) = \text{diag}\left(X^{(ik)}(t)\right)_{i \in E_{\text{in}}^{(k)}},$$
$$X_{\text{out}}^{(k)}(t) = \text{diag}\left(X^{(kj)}(t)\right)_{j \in E_{\text{out}}^{(k)}}.$$

Hereafter, we no longer specify the dimensions of $X^{(k)}(t)$, $X^{(ik)}(t)$, $X^{(kj)}(t)$, and any similar matrix-valued function. Also, the definitions of $X_{\text{in}}^{(k)}(t)$ and $X_{\text{out}}^{(k)}(t)$ are extended to similar notations, e.g., $Y_{\text{in}}^{(k)}(t)$ and $Y_{\text{out}}^{(k)}(t)$. The next result gives an upper bound $\gamma > 0$ on the ℓ_2-induced norm of a strongly stable distributed system.

Lemma 2: A distributed system is strongly stable and satisfies $\|u \to y\| < \gamma$, for some $\gamma > 0$, if there exist uniformly bounded, positive definite, matrix-valued functions $X^{(k)}(t)$, $X^{(ik)}(t)$, and $X^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\left(F^{(k)}(t)\right)^T \begin{bmatrix} X^{(k)}(t+1) & 0 & 0 \\ 0 & X_{\text{out}}^{(k)}(t+1) & 0 \\ 0 & 0 & I \end{bmatrix} F^{(k)}(t) - \begin{bmatrix} X^{(k)}(t) & 0 & 0 \\ 0 & X_{\text{in}}^{(k)}(t) & 0 \\ 0 & 0 & \gamma^2 I \end{bmatrix} \prec -\beta I, \quad (3)$$

for some positive scalar β, where

$$F^{(k)}(t) = \begin{bmatrix} \bar{A}^{(k)}(t) & \bar{B}^{(k)}(t) \\ \bar{C}^{(k)}(t) & \bar{D}^{(k)}(t) \end{bmatrix}.$$

IV. BALANCED TRUNCATION MODEL REDUCTION

A. Balanced Realization

Definition 2: A distributed system is said to be balanced if there exist uniformly bounded, diagonal, positive definite, matrix-valued functions $\Sigma^{(k)}(t)$, $\Sigma^{(ik)}(t)$, and $\Sigma^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\bar{A}^{(k)}(t) \begin{bmatrix} \Sigma^{(k)}(t) & 0 \\ 0 & \Sigma_{\text{in}}^{(k)}(t) \end{bmatrix} \left(\bar{A}^{(k)}(t)\right)^T - \begin{bmatrix} \Sigma^{(k)}(t+1) & 0 \\ 0 & \Sigma_{\text{out}}^{(k)}(t+1) \end{bmatrix} + \bar{B}^{(k)}(t)\left(\bar{B}^{(k)}(t)\right)^T \prec -\beta I, \quad (4)$$

$$\left(\bar{A}^{(k)}(t)\right)^T \begin{bmatrix} \Sigma^{(k)}(t+1) & 0 \\ 0 & \Sigma_{\text{out}}^{(k)}(t+1) \end{bmatrix} \bar{A}^{(k)}(t) - \begin{bmatrix} \Sigma^{(k)}(t) & 0 \\ 0 & \Sigma_{\text{in}}^{(k)}(t) \end{bmatrix} + \left(\bar{C}^{(k)}(t)\right)^T \bar{C}^{(k)}(t) \prec -\beta I, \quad (5)$$

for some positive scalar β.

These LMIs are the generalized Lyapunov inequalities [9] for distributed systems. They can be solved separately, and the resultant solutions are called the generalized gramians [10]. If existent, the generalized gramians can be used to construct a balanced realization for the distributed system. A strongly stable system has generalized gramians and, hence, a balanced realization, as shown in the next lemma.

Lemma 3: For a strongly stable system, there exist uniformly bounded, diagonal, positive definite, matrix-valued functions $\Sigma^{(k)}(t)$, $\Sigma^{(ik)}(t)$, and $\Sigma^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, that satisfy (4) and (5), and hence, the system has a balanced realization.

Proof: A strongly stable system satisfies condition (2). Then, by scaling and homogeneity, there exist uniformly bounded, positive definite functions $X^{(k)}(t)$, $X^{(ik)}(t)$, $X^{(kj)}(t)$ and $Y^{(k)}(t)$, $Y^{(ik)}(t)$, $Y^{(kj)}(t)$ that satisfy (4) and (5), respectively. We focus on the functions associated with the temporal states. First, we perform a singular value decomposition, $(X^{(k)}(t))^{1/2} Y^{(k)}(t) (X^{(k)}(t))^{1/2} = U^{(k)}(t)(\Sigma^{(k)}(t))^2 (U^{(k)}(t))^T$. We define the balancing transformations as

$$T^{(k)}(t) = (\Sigma^{(k)}(t))^{1/2} (U^{(k)}(t))^T (X^{(k)}(t))^{-1/2}.$$

We can thus express $\Sigma^{(k)}(t)$ as $T^{(k)}(t) X^{(k)}(t) (T^{(k)}(t))^T$ or as $(T^{(k)}(t)^{-1})^T Y^{(k)}(t) (T^{(k)}(t))^{-1}$. We repeat the same procedure for $X^{(ik)}(t)$ $\left(X^{(kj)}(t)\right)$ and $Y^{(ik)}(t)$ $\left(Y^{(kj)}(t)\right)$ to obtain the balancing transformations $T^{(ik)}(t)$ $(T^{(kj)}(t))$. Then, the system realization with the following state-space matrices is balanced: $\bar{A}_{\text{bal}}^{(k)}(t) = T_{\text{pre}} \bar{A}^{(k)}(t) T_{\text{post}}$, $\bar{B}_{\text{bal}}^{(k)}(t) = T_{\text{pre}} \bar{B}^{(k)}(t)$, and $\bar{C}_{\text{bal}}^{(k)}(t) = \bar{C}^{(k)}(t) T_{\text{post}}$, where

$$T_{\text{pre}} = \text{diag}\left(T^{(k)}(t+1), T_{\text{out}}^{(k)}(t+1)\right),$$
$$T_{\text{post}} = \text{diag}\left(T^{(k)}(t)^{-1}, T_{\text{in}}^{(k)}(t)^{-1}\right).$$

■

The previous proof includes a procedure to obtain balanced gramians and balancing transformations from the solutions of (4) and (5). An alternative procedure is outlined here. We focus on the functions associated with the temporal states as a similar procedure can repeated for the others. Let $X^{(k)}(t) = (R^{(k)}(t))^T R^{(k)}(t)$ and $Y^{(k)}(t) = (H^{(k)}(t))^T H^{(k)}(t)$ be the Cholesky factorizations of $X^{(k)}(t)$ and $Y^{(k)}(t)$, respectively. Performing a singular value decomposition on $H^{(k)}(t)(R^{(k)}(t))^T$, we obtain $U^{(k)}(t)\Sigma^{(k)}(t)(V^{(k)}(t))^T$. The balancing transformation and its inverse are then defined as $T^{(k)}(t) = (\Sigma^{(k)}(t))^{-\frac{1}{2}}(U^{(k)}(t))^T H^{(k)}(t)$ and $(T^{(k)}(t))^{-1} = (R^{(k)}(t))^T V^{(k)}(t)(\Sigma^{(k)}(t))^{-\frac{1}{2}}$. Having found all the balancing transformations, we define the balanced realization as before.

B. Balanced Truncation

Having proved the existence of a balanced realization for a strongly stable distributed system, we now proceed onto the formulation of the balanced truncation problem. Let G be a balanced distributed system. By Definition 2, there exist diagonal generalized gramians that simultaneously satisfy (4) and (5). We can assume, without loss of generality, that the diagonal entries of the generalized gramians are ordered in a decreasing fashion. We partition the diagonal gramians into two blocks: one associated with the truncated states and the other with the nontruncated ones. We illustrate the process for the gramians associated with the temporal states, i.e., $\Sigma^{(k)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$. Given integers $r^{(k)}(t)$ such that $0 \leq r^{(k)}(t) \leq n^{(k)}(t)$, we partition $\Sigma^{(k)}(t)$ as follows:

$$\Sigma^{(k)}(t) = \begin{bmatrix} \Gamma^{(k)}(t) & 0 \\ 0 & \Omega^{(k)}(t) \end{bmatrix},$$

where $\Gamma^{(k)}(t) \in \mathbb{S}^{r^{(k)}(t)}$ and $\Omega^{(k)}(t) \in \mathbb{S}^{n^{(k)}(t)-r^{(k)}(t)}$. $\Omega^{(k)}(t)$ corresponds to the truncated states. Then, we partition the blocks of the state-space matrices in accordance with the partitioning of $\text{diag}(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1))$ and $\text{diag}(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t))$. We revisit subsystem $G^{(1)}$ in our example. $A_{00}^{(1)}(t)$ is partitioned conformably with $\Sigma^{(1)}(t+1)$ and $\Sigma^{(1)}(t)$, i.e.,

$$A_{00}^{(1)}(t) = \begin{bmatrix} \hat{A}_{00}^{(1)}(t) & A_{00_{12}}^{(1)}(t) \\ A_{00_{21}}^{(1)}(t) & A_{00_{22}}^{(1)}(t) \end{bmatrix},$$

where $\hat{A}_{00}^{(1)}(t)$ is an $r^{(1)}(t+1) \times r^{(1)}(t)$ matrix. Similarly, $A_{02}^{(1)}(t)$ is partitioned according to $\Sigma^{(1)}(t+1)$ and $\Sigma^{(21)}(t)$, $A_{20}^{(1)}(t)$ according to $\Sigma^{(12)}(t+1)$ and $\Sigma^{(1)}(t)$, etc. The blocks of $\bar{B}^{(1)}(t)$ and $\bar{C}^{(1)}(t)$ are partitioned likewise. For instance, $B_2^{(1)}(t)$ and $C_2^{(1)}(t)$ are partitioned conformably with $\Sigma^{(12)}(t+1)$ and $\Sigma^{(21)}(t)$, respectively. Namely, $B_2^{(1)}(t) = \begin{bmatrix} \hat{B}_2^{(1)}(t)^T & B_{2_2}^{(1)}(t)^T \end{bmatrix}^T$ where $\hat{B}_2^{(1)}(t)$ is an $r^{(12)}(t+1) \times n_u^{(1)}(t)$ matrix, whereas, $C_2^{(1)}(t) = \begin{bmatrix} \hat{C}_2^{(1)}(t) & C_{2_2}^{(1)}(t) \end{bmatrix}$ where $\hat{C}_2^{(1)}(t)$ is an $n_y^{(1)}(t) \times r^{(21)}(t)$ matrix.

After completing the partitioning process, we form the reduced order model G_r by defining its state-space matrices $A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t)$, and $D_r^{(k)}(t)$. To do this, we keep the blocks that correspond to the nontruncated states, i.e., the partitions marked by a hat. For example, the reduced order state-space matrices of subsystem $G^{(1)}$ become:

$$A_r^{(1)}(t) = \begin{bmatrix} \hat{A}_{00}^{(1)}(t) & \hat{A}_{02}^{(1)}(t) & \hat{A}_{03}^{(1)}(t) & \hat{A}_{04}^{(1)}(t) \\ \hat{A}_{20}^{(1)}(t) & \hat{A}_{22}^{(1)}(t) & \hat{A}_{23}^{(1)}(t) & \hat{A}_{24}^{(1)}(t) \\ \hat{A}_{30}^{(1)}(t) & \hat{A}_{32}^{(1)}(t) & \hat{A}_{33}^{(1)}(t) & \hat{A}_{34}^{(1)}(t) \\ \hat{A}_{50}^{(1)}(t) & \hat{A}_{52}^{(1)}(t) & \hat{A}_{53}^{(1)}(t) & \hat{A}_{54}^{(1)}(t) \end{bmatrix},$$

$$B_r^{(1)}(t) = \begin{bmatrix} \hat{B}_0^{(1)}(t)^T \ \hat{B}_2^{(1)}(t)^T \ \hat{B}_3^{(1)}(t)^T \ \hat{B}_5^{(1)}(t)^T \end{bmatrix}^T,$$

$$C_r^{(1)}(t) = \begin{bmatrix} \hat{C}_0^{(1)}(t) & \hat{C}_2^{(1)}(t) & \hat{C}_3^{(1)}(t) & \hat{C}_4^{(1)}(t) \end{bmatrix},$$

$$D_r^{(1)}(t) = \bar{D}^{(1)}(t).$$

It is useful at this point to permute the original state-space matrices and the balanced gramians to group the reduced states together as follows:

$$A_b^{(k)}(t) = \begin{bmatrix} A_r^{(k)}(t) & \bar{A}_{12}^{(k)}(t) \\ \bar{A}_{21}^{(k)}(t) & \bar{A}_{22}^{(k)}(t) \end{bmatrix}, \quad B_b^{(k)}(t) = \begin{bmatrix} B_r^{(k)}(t) \\ \bar{B}_2^{(k)}(t) \end{bmatrix},$$

$$\Gamma_k^{\text{in}}(t) = \begin{bmatrix} \Gamma^{(k)}(t) & 0 \\ 0 & \Gamma_{\text{in}}^{(k)}(t) \end{bmatrix}, \quad C_b^{(k)}(t) = \begin{bmatrix} C_r^{(k)}(t)^T \\ \bar{C}_2^{(k)}(t)^T \end{bmatrix}^T.$$

$\Omega_k^{\text{in}}(t)$, $\Gamma_k^{\text{out}}(t)$, $\Omega_k^{\text{out}}(t)$ are defined similarly to $\Gamma_k^{\text{in}}(t)$.

Lemma 4: The reduced order system G_r, as defined above, is strongly stable and balanced.

Proof: Since system G is balanced, there exist diagonal gramians $\Sigma^{(k)}(t)$, $\Sigma^{(ik)}(t)$, and $\Sigma^{(kj)}(t)$ that satisfy (4) and (5). By applying appropriate permutations, we obtain:

$$A_b^{(k)}(t) \begin{bmatrix} \Gamma_k^{\text{in}}(t) & 0 \\ 0 & \Omega_k^{\text{in}}(t) \end{bmatrix} \left(A_b^{(k)}(t)\right)^T - \begin{bmatrix} \Gamma_k^{\text{out}}(t+1) & 0 \\ 0 & \Omega_k^{\text{out}}(t+1) \end{bmatrix} + B_b^{(k)}(t)\left(B_b^{(k)}(t)\right)^T \prec -\beta I, \tag{6}$$

$$\left(A_b^{(k)}(t)\right)^T \begin{bmatrix} \Gamma_k^{\text{out}}(t+1) & 0 \\ 0 & \Omega_k^{\text{out}}(t+1) \end{bmatrix} A_b^{(k)}(t) - \begin{bmatrix} \Gamma_k^{\text{in}}(t) & \\ & \Omega_k^{\text{in}}(t) \end{bmatrix} + \left(C_b^{(k)}(t)\right)^T C_b^{(k)}(t) \prec -\beta I. \tag{7}$$

From (6), we infer that

$$A_r^{(k)}(t) \begin{bmatrix} \Gamma^{(k)}(t) & 0 \\ 0 & \Gamma_{\text{in}}^{(k)}(t) \end{bmatrix} \left(A_r^{(k)}(t)\right)^T - \begin{bmatrix} \Gamma^{(k)}(t+1) & 0 \\ 0 & \Gamma_{\text{out}}^{(k)}(t+1) \end{bmatrix} + B_r^{(k)}(t)\left(B_r^{(k)}(t)\right)^T \prec -\beta I.$$

By a similar inference from (7), we conclude that G_r is balanced and, hence, strongly stable. ■

C. Error Bound

In this part, we seek to find an upper bound on the ℓ_2-induced norm of the error system $G - G_r$.

Theorem 1: If $\Omega^{(k)}(t), \Omega^{(ik)}(t)$, and $\Omega^{(kj)}(t)$ are equal to I, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, then

$$\|G - G_r\| < 2. \tag{8}$$

Proof: G and G_r are both strongly stable, and so is $\frac{1}{2}(G - G_r)$. Applying the Schur complement formula twice to (6) and invoking (7), we show that the following holds:

$$(K^{(k)}(t))^T (R_2^{(k)}(t+1))^{-1} K^{(k)}(t) - R_1^{(k)}(t) \prec -\beta I,$$

where

$$R_s^{(k)}(t) = \begin{bmatrix} (\Gamma_s^{(k)}(t))^{-1} & 0 & 0 & 0 & 0 \\ 0 & (\Omega_s^{(k)}(t))^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_s^{(k)}(t)} & 0 & 0 \\ 0 & 0 & 0 & \Gamma_s^{(k)}(t) & 0 \\ 0 & 0 & 0 & 0 & \Omega_s^{(k)}(t) \end{bmatrix}$$

with $I_{n_s^{(k)}(t)}$, $\Gamma_s^{(k)}(t)$, $\Omega_s^{(k)}(t)$ equal to $I_{n_u^{(k)}(t)}$, $\Gamma_{\text{in}}^{(k)}(t)$, $\Omega_{\text{in}}^{(k)}(t)$ for $s=1$, and $I_{n_y^{(k)}(t)}$, $\Gamma_{\text{out}}^{(k)}(t)$, $\Omega_{\text{out}}^{(k)}(t)$ for $s=2$, respectively, and

$$K^{(k)}(t) = \begin{bmatrix} 0 & 0 & 0 & A_r^{(k)}(t) & \bar{A}_{12}^{(k)}(t) \\ 0 & 0 & 0 & \bar{A}_{21}^{(k)}(t) & \bar{A}_{22}^{(k)}(t) \\ 0 & 0 & 0 & C_r^{(k)}(t) & \bar{C}_2^{(k)}(t) \\ A_r^{(k)}(t) & \bar{A}_{12}^{(k)}(t) & B_r^{(k)}(t) & 0 & 0 \\ \bar{A}_{21}^{(k)}(t) & \bar{A}_{22}^{(k)}(t) & \bar{B}_2^{(k)}(t) & 0 & 0 \end{bmatrix}.$$

We define permutation matrices P and L, respectively, as

$$\frac{1}{\sqrt{2}}\begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & \sqrt{2}I & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -I & 0 & 0 & I & 0 \\ I & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & \sqrt{2}I & 0 & 0 \\ 0 & -I & 0 & 0 & I \end{bmatrix}.$$

Pre- and post- multiplying the previous inequality by P^T and P, respectively, and inserting $L^T L = I$ as needed, we get

$$P^T (K^{(k)}(t))^T L^T L (R_2^{(k)}(t+1))^{-1} L^T L K^{(k)}(t) P - P^T R_1^{(k)}(t) P \prec -\beta I. \quad (9)$$

The matrices $P^T R_1^{(k)}(t) P$ and $L(R_2^{(k)}(t))^{-1}L^T$ have the same structure, namely $\frac{1}{2}\text{diag}(B1, B2)$, with blocks $B1$ and $B2$ given below:

$$B1 = \begin{bmatrix} (\Gamma_e^{(k)}(t))^{-1} + \Gamma_e^{(k)}(t) & (\Gamma_e^{(k)}(t))^{-1} - \Gamma_e^{(k)}(t) \\ (\Gamma_e^{(k)}(t))^{-1} - \Gamma_e^{(k)}(t) & (\Gamma_e^{(k)}(t))^{-1} + \Gamma_e^{(k)}(t) \end{bmatrix},$$

$$B2 = \begin{bmatrix} (\Omega_e^{(k)}(t))^{-1} + \Omega_e^{(k)}(t) & 0 & (\Omega_e^{(k)}(t))^{-1} - \Omega_e^{(k)}(t) \\ 0 & 2I & 0 \\ (\Omega_e^{(k)}(t))^{-1} - \Omega_e^{(k)}(t) & 0 & (\Omega_e^{(k)}(t))^{-1} + \Omega_e^{(k)}(t) \end{bmatrix},$$

where $\Gamma_e^{(k)}(t)$, $\Omega_e^{(k)}(t)$ refer to $\Gamma_{\text{in}}^{(k)}(t)$, $\Omega_{\text{in}}^{(k)}(t)$ in the blocks of $P^T R_1^{(k)}(t) P$, and $\Gamma_{\text{out}}^{(k)}(t)$, $\Omega_{\text{out}}^{(k)}(t)$ in the blocks of $L(R_2^{(k)}(t))^{-1}L^T$, respectively. Additionally,

$$LK^{(k)}(t)P = \begin{bmatrix} M^{(k)}(t) & N_{12}^{(k)}(t) \\ N_{21}^{(k)}(t) & \bar{A}_{22}^{(k)}(t) \end{bmatrix}, \quad \text{where}$$

$$M^{(k)}(t) = \begin{bmatrix} A_r^{(k)}(t) & 0 & 0 & \frac{1}{\sqrt{2}} B_r^{(k)}(t) \\ 0 & A_r^{(k)}(t) & \bar{A}_{12}^{(k)}(t) & \frac{1}{\sqrt{2}} B_r^{(k)}(t) \\ 0 & \bar{A}_{21}^{(k)}(t) & \bar{A}_{22}^{(k)}(t) & \frac{1}{\sqrt{2}} \bar{B}_2^{(k)}(t) \\ -\frac{1}{\sqrt{2}} C_r^{(k)}(t) & \frac{1}{\sqrt{2}} C_r^{(k)}(t) & \frac{1}{\sqrt{2}} \bar{C}_2^{(k)}(t) & 0 \end{bmatrix}.$$

From (9), and using the fact that $\Omega^{(k)}(t)$, $\Omega^{(ik)}(t)$, and $\Omega^{(kj)}(t)$ are equal to I, we obtain

$$(M^{(k)}(t))^T \begin{bmatrix} V_2^{(k)}(t+1) & 0 \\ 0 & I \end{bmatrix} M^{(k)}(t) - \begin{bmatrix} V_1^{(k)}(t) & 0 \\ 0 & I \end{bmatrix} \prec -\beta I,$$

where, for the same references of $s \in \{1, 2\}$,

$$V_s^{(k)}(t) = \frac{1}{2}\begin{bmatrix} (\Gamma_s^{(k)}(t))^{-1} + \Gamma_s^{(k)}(t) & (\Gamma_s^{(k)}(t))^{-1} - \Gamma_s^{(k)}(t) & 0 \\ (\Gamma_s^{(k)}(t))^{-1} - \Gamma_s^{(k)}(t) & (\Gamma_s^{(k)}(t))^{-1} + \Gamma_s^{(k)}(t) & 0 \\ 0 & 0 & 2I \end{bmatrix}.$$

Note that $V_1^{(k)}(t)$ and $V_2^{(k)}(t)$ are positive definite. Then, performing some permutations to the previous inequality, and invoking Lemma 2 with $\gamma = 1$, we conclude that

$$\|(1/2)(G - G_r)\| < 1. \quad \blacksquare$$

Theorem 2: Given a balanced distributed system G and its balanced truncation G_r, then

$$\|G - G_r\| < 2 \sum_{(t,k)} \left(\sum_{j_1} w_{j_1}^{(k)}(t) + \sum_{i \in E_{\text{in}}^{(k)}} \sum_{j_2} w_{j_2}^{(ik)}(t) \right), \quad (10)$$

where $w_{j_1}^{(k)}(t)$ and $w_{j_2}^{(ik)}(t)$ are the distinct diagonal entries of $\Omega^{(k)}(t)$ and $\Omega^{(ik)}(t)$, respectively.

Proof: The proof follows from scaling and repeated applications of Theorem 1. Lemma 4 ensures that the intermediate realizations are strongly stable and balanced. We do not account for the distinct diagonal entries of $\Omega^{(kj)}(t)$ to avoid double counting, as the interconnection input to a subsystem is an output of another. ■

D. Periodic Subsystems

While (10) gives an upper bound on the error induced by balanced truncation, this bound might not always be convergent. The result is more useful when the subsystems are periodic because the existence of solutions to Lemma 1 (and hence the Lyapunov inequalities) is equivalent to the existence of periodic solutions. We will prove this equivalency for Lemma 2, which is a more general result than Lemma 1 and, hence, encompasses it. The existence of periodic solutions simplifies the error bound by restricting the sum to the first period of the system only.

Definition 3: A subsystem $G^{(k)}$ is said to be q time-periodic if $\bar{A}^{(k)}(t+q) = \bar{A}^{(k)}(t)$, $\bar{B}^{(k)}(t+q) = \bar{B}^{(k)}(t)$, $\bar{C}^{(k)}(t+q) = \bar{C}^{(k)}(t)$, and $\bar{D}^{(k)}(t+q) = \bar{D}^{(k)}(t)$, for all $t \in \mathbb{N}_0$. A distributed system G is said to be q time-periodic if all its subsystems are q time-periodic.

Theorem 3: Suppose G is a strongly stable, q time-periodic distributed system. Then, there exist q time-periodic, uniformly bounded, positive definite, matrix-valued functions $X_{\text{per}}^{(k)}(t)$, $X_{\text{per}}^{(ik)}(t)$, and $X_{\text{per}}^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, that satisfy

$$\left(F^{(k)}(t)\right)^T \begin{bmatrix} X_{\text{per}}^{(k)}(t+1) & 0 & 0 \\ 0 & X_{\text{out}_{\text{per}}}^{(k)}(t+1) & 0 \\ 0 & 0 & I \end{bmatrix} F^{(k)}(t) - \begin{bmatrix} X_{\text{per}}^{(k)}(t) & 0 & 0 \\ 0 & X_{\text{in}_{\text{per}}}^{(k)}(t) & 0 \\ 0 & 0 & \gamma^2 I \end{bmatrix} \prec -\beta I,$$

for some positive scalar β.

Proof: Since G is strongly stable, then there exist solutions to (3) for some $\gamma > 0$. From these, we construct the required q time-periodic solutions. We resort to averaging techniques similar to the ones used in [22] and [23]. Since the distributed system is q time-periodic, then $F^{(k)}(t+zq) = F^{(k)}(t)$ for any $z \in \mathbb{N}_0$. We fix t in $\mathbb{N}_0$, choose an integer $\lambda \geq 1$, and evaluate (3) at $(t+zq, k)$ for $z = 0, \ldots, \lambda - 1$. Averaging the resulting inequalities, we get

$$(F^{(k)}(t))^T \begin{bmatrix} Y_\lambda^{(k)}(t+1) & 0 & 0 \\ 0 & Y_{\text{out},\lambda}^{(k)}(t+1) & 0 \\ 0 & 0 & I \end{bmatrix} F^{(k)}(t) - \begin{bmatrix} Y_\lambda^{(k)}(t) & 0 & 0 \\ 0 & Y_{\text{in},\lambda}^{(k)}(t) & 0 \\ 0 & 0 & \gamma^2 I \end{bmatrix} \prec -\beta I,$$

where $Y_\lambda^{(k)}(t) = \frac{1}{\lambda}\sum_{z=0}^{\lambda-1} X^{(k)}(t+zq)$. Similar definitions apply to $Y_\lambda^{(ik)}(t)$ and $Y_\lambda^{(kj)}(t)$. Since the solutions to (3) are uniformly bounded, so are $Y_\lambda^{(k)}(t)$, $Y_\lambda^{(ik)}(t)$, and $Y_\lambda^{(kj)}(t)$. Then, there exist weakly convergent subsequences $Y_{\lambda_c}^{(k)}(t)$, $Y_{\lambda_c}^{(ik)}(t)$, and $Y_{\lambda_c}^{(kj)}(t)$ with limits $L^{(k)}(t)$, $L^{(ik)}(t)$, and $L^{(kj)}(t)$, respectively. We refer the reader to [24] for further details on convergence in weak topology. By construction, the limits are positive definite. We need to show that they are q time-periodic. We complete the proof for $L^{(k)}(t)$ (the others follow similarly): $L^{(k)}(t+q) - L^{(k)}(t) = \lim_{\lambda_c \to \infty} \frac{1}{\lambda_c}\sum_{z=0}^{\lambda_c-1}(X^{(k)}(t+(z+1)q) - X^{(k)}(t+zq)) = \lim_{\lambda_c \to \infty} \frac{1}{\lambda_c}(X^{(k)}(t+\lambda_c q) - X^{(k)}(t)) = 0$. We set $X_{\text{per}}^{(k)}(t) = L^{(k)}(t)$, $X_{\text{per}}^{(ik)}(t) = L^{(ik)}(t)$, and $X_{\text{per}}^{(kj)}(t) = L^{(kj)}(t)$. ■

Corollary 1: Given a q time-periodic, balanced distributed system G and its balanced truncation G_r, then

$$\|G - G_r\| < 2\sum_k \sum_{t=0}^{q-1} \left(\sum_{j_1} w_{j_1}^{(k)}(t) + \sum_{i \in E_{\text{in}}^{(k)}} \sum_{j_2} w_{j_2}^{(ik)}(t) \right),$$

where $w_{j_1}^{(k)}(t)$ and $w_{j_2}^{(ik)}(t)$ are the distinct diagonal entries of $\Omega^{(k)}(t)$ and $\Omega^{(ik)}(t)$, respectively.

The previous result further simplifies when dealing with LTI subsystems ($q = 1$). In such cases, all the state-space functions become time-independent, e.g., $\bar{A}^{(k)}(t) \equiv \bar{A}_k$, for all $(t,k) \in \mathbb{N}_0 \times V$. Also, due to time-invariance, we drop the time parameter t from the gramians and only sum over the vertex indices $k \in V$ when computing the upper bound on the error.

V. Conclusion

Model reduction is desirable for distributed systems as a smaller scale model facilitates both system analysis and control synthesis problems. Balanced truncation is generalized and applied to the model reduction of linear systems interconnected over arbitrary graphs and subjected to a communication latency. The proposed scheme comes with an a priori known error bound, and preserves the interconnection structure between subsystems. However, it is not universally applicable because it is based on a sufficient, but not necessary, convex Lyapunov-type condition for stability.

References

[1] F. L. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, Eds., *Cooperative Control of Multi-Agent Systems Optimal and Adaptive Design Approaches*, ser. Communications and Control Engineering. London: Springer, 2014.

[2] W. Ren and Y. Cao, Eds., *Distributed Coordination of Multi-agents Networks Emergent Problems, Models, and Issues*, ser. Communications and Control Engineering. London: Springer, 2011.

[3] M. Farhood, Z. Di, and G. E. Dullerud, "Distributed control of linear time-varying systems interconnected over arbitrary graphs," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 2, pp. 179–206, 2015.

[4] M. Farhood, "Distributed control of LPV systems over arbitrary graphs: A parameter-dependent Lyapunov approach," in *Proceedings of the American Control Conference*, July 2015, pp. 1525–1530.

[5] B. C. Moore, "Principal component analysis in linear systems: controllability, observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 17–32, 1981.

[6] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds," *International Journal of Control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[7] D. F. Enns, "Model reduction with balanced realizations: an error bound and a frequency weighted generalization," in *The 23rd IEEE Conference on Decision and Control*, 1984, pp. 127–132.

[8] U. M. Al-Saggaf and G. F. Franklin, "An error bound for a discrete reduced order model of a linear multivariable system," *IEEE Transactions on Automatic Control*, vol. 32, no. 9, pp. 815–819, 1987.

[9] D. Hinrichsen and A. J. Pritchard, "An improved error estimate for reduced-order models of discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 3, pp. 317–320, 1990.

[10] C. L. Beck, J. Doyle, and K. Glover, "Model reduction of multidimensional and uncertain systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 10, pp. 1466–1477, 1996.

[11] S. Lall and C. Beck, "Error-bounds for balanced model-reduction of linear time-varying systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 946–956, 2003.

[12] H. Sandberg and A. Rantzer, "Balanced truncation of linear time-varying systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 217–229, 2004.

[13] M. Farhood and G. E. Dullerud, "On the balanced truncation of LTV systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 315–320, 2006.

[14] ——, "Model reduction of nonstationary LPV systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 181–196, 2007.

[15] T. Reis and E. Virnik, "Positivity preserving balanced truncation for descriptor systems," *SIAM Journal on Control and Optimization*, vol. 48, no. 4, pp. 2600–2619, 2009.

[16] G. Kotsalis, A. Megretski, and M. A. Dahleh, "Balanced truncation for a class of stochastic jump linear systems and model reduction for hidden markov models," *IEEE Transactions On Automatic Control*, vol. 53, no. 11, pp. 2543–2557, 2008.

[17] M. Farhood and C. Beck, "On the balanced truncation and coprime factors reduction of markovian jump linear systems," *Systems and Control Letters*, vol. 64, pp. 96–106, 2014.

[18] L. Li and F. Paganini, "Structured coprime factor model reduction based on LMIs," *Automatica*, vol. 41, no. 2, pp. 145–151, 2005.

[19] A. Vandendorpe and P. V. Dooren, *Model order reduction: theory, research aspects and applications*, ser. Mathematics in Industry Series. Berlin: Springer, 2008, no. 13, ch. Model reduction of interconnected systems, pp. 305–321.

[20] H. Sandberg and R. M. Murray, "Model reduction of interconnected linear systems," *Optimal Control Applications and Methods*, vol. 30, no. 3, pp. 225–245, 2009.

[21] G. E. Dullerud and R. DAndrea, "Distributed control of heterogeneous systems," *IEEE Transactions On Automatic Control*, vol. 49, no. 12, pp. 2113–2128, 2004.

[22] G. E. Dullerud and S. Lall, "A new approach for analysis and synthesis of time-varying systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 8, pp. 1486–1497, 1999.

[23] M. Farhood, "LPV control of nonstationary systems: a parameter-dependent Lyapunov approach," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 212–218, 2012.

[24] C. S. Kubrusly, Ed., *The elements of operator theory*, 2nd ed. Basel: Birkhauser, 2011.