Coprime Factors Model Reduction of Linear Systems Interconnected over Arbitrary Graphs with Communication Latency

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***Abstract*— This paper is on the coprime factors model reduction of spatially distributed systems consisting of heterogeneous, discrete-time, linear time-varying subsystems interconnected over arbitrary directed graphs. The communication between the subsystems is subjected to a one-step time-delay. We review the balanced truncation scheme for this class of systems, and then generalize the coprime factors model reduction method.**

I. Introduction

In this paper, we treat heterogenous, discrete-time, linear time-varying (LTV) subsystems interconnected over arbitrary directed graphs. We assume that the data transfer between the subsystems is subjected to a one-step time-delay. We use the model formulation of [1], which is based on [2] and [3].

Model reduction is desirable for spatially distributed systems. In fact, the dimension of the overall system scales with the size of the subsystems, the number of subsystems, the size of spatial states associated with the interconnections, and the complexity of the interconnection structure.

In [1], the balanced truncation method [4]–[16] is extended to the class of distributed systems over arbitrary directed graphs with communication latency. The method allows the preservation of the interconnection structure [15], [16], and further allows its simplification [16]. In particular, the order of the spatial states can be reduced. The method even allows the removal of a whole interconnection if deemed insignificant. Furthermore, the method guarantees the stability of the reduced order system and an upper bound on the ℓ_2-induced norm of the error system resulting from the reduction process, which generalizes the "twice the sum of the distinct truncated singular values" bound. However, the method is based on the existence of block-diagonal solutions, called generalized gramians, to the generalized Lyapunov inequalities. It is shown in [17] that the existence of such gramians is a sufficient, yet non-necessary, condition for stability of distributed systems. In [1], stable systems with structured gramians are referred to as strongly stable, and the balanced truncation scheme proposed therein only applies to strongly stable systems. See [18] for a group of systems that are guaranteed to have block-diagonal gramians.

In this work, we generalize the coprime factors reduction (CFR) method to the class of distributed systems. CFR provides a partial remedy to the conservatism of balanced truncation by being applicable to strongly stabilizable and detectable systems. However, the resulting error bound no longer captures the mismatch between the original and reduced order models. In the journal version of this work, we show how to interpret the error bound in terms of robust stability of the closed-loop system. The CFR method was introduced for linear time-invariant (LTI) systems in [19]. In [20], [21], the method was extended to systems described in a linear fractional transformation framework, e.g., uncertain systems and stationary linear parameter varying (LPV) systems. The works of [12] and [22] have since generalized the method to nonstationary LPV and Markovian jump linear systems, respectively. Additionally, [23] applies the CFR method for structure-preserving model reduction of continuous-time LTI systems.

This paper is organized as follows. In Section II, we gather the relevant notations. We present, in Section III, the model formulation of [1]. Balanced truncation is briefly summarized in Section IV. Section V is devoted for the coprime factors reduction method. The paper concludes with Section VI.

II. Preliminaries

We denote by $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{R}$ the sets of nonnegative integers, positive integers, and real numbers, respectively. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$ and that of $n \times n$ symmetric matrices by $\mathbb{S}^n$. $(v_i)_{i\in S}$ and $(M_i)_{i\in S}$ are, respectively, the vector and matrix sequences associated with S, an ordered subset of $\mathbb{N}$. The elements of sequences $(v_i)_{i\in S}$ and $(M_i)_{i\in S}$ follow the same ordering as the elements of the index set S. The elements of $(v_i)_{i\in S}$ can be vertically concatenated to form the augmented vector $\text{vec}(v_i)_{i\in S}$, whereas, the elements of $(M_i)_{i\in S}$ can be block-diagonally augmented to form $\text{diag}(M_i)_{i\in S}$. As an example, let $S = \{1, 2, 4\}$. Then, $(v_i)_{i\in S} = (v_1, v_2, v_4)$, $(M_i)_{i\in S} = (M_1, M_2, M_4)$, $\text{vec}(v_i)_{i\in S} = [v_1^T, v_2^T, v_4^T]^T$, and $\text{diag}(M_i)_{i\in S} = \text{diag}(M_1, M_2, M_4)$.

$\mathcal{G}(V, E)$ denotes a directed graph with set of vertices V and set of directed edges E. We restrict our discussion to directed graphs with a finite number N of vertices. For simplicity, we take the vertex set as $V = \{1, \dots, N\}$. We use the ordered pair (i, j) to represent an element of E directed from vertex $i \in V$ to vertex $j \in V$. With each $k \in V$, we associate the sets $E_{\text{in}}^{(k)} := \{i \in V \,|\, (i, k) \in E\}$ and $E_{\text{out}}^{(k)} := \{j \in V \,|\, (k, j) \in E\}$. The elements of these sets are sorted in an ascending order. We define the indegree, $m(k)$, and outdegree, $p(k)$, of vertex $k \in V$ as the number of elements in $E_{\text{in}}^{(k)}$ and $E_{\text{out}}^{(k)}$, respectively. For example, consider Fig. 1, which shows a directed graph with 5 vertices and 12 directed edges. $V = \{1, 2, 3, 4, 5\}$ and

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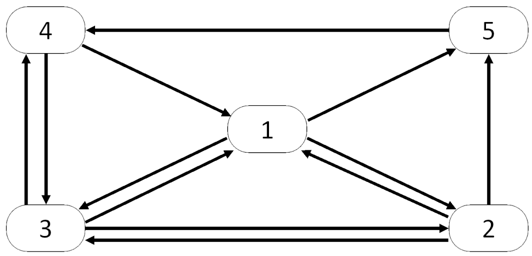

Fig. 1. Example of a Directed Graph.

$(1,2)$, $(1,3)$, and $(1,5)$ are some elements in E. Moreover, $E_{\text{in}}^{(1)} = \{2,3,4\}$, $m(1) = 3$, $E_{\text{out}}^{(1)} = \{2,3,5\}$, $p(1) = 3$, etc.

Given an integer sequence $n : (t,k) \in \mathbb{N}_0 \times V \to n^{(k)}(t) \in \mathbb{N}_0$, we define $\ell(\{\mathbb{R}^{n^{(k)}(t)}\})$, or shortly ℓ, as the vector space of mappings $w : (t,k) \in \mathbb{N}_0 \times V \to w^{(k)}(t) \in \mathbb{R}^{n^{(k)}(t)}$. The Hilbert space ℓ_2 is the subspace of ℓ consisting of mappings w that have a finite ℓ_2-norm defined as $\|w\|_2^2 := \sum_{(t,k)} w^{(k)}(t)^T w^{(k)}(t)$. The dimensions associated with ℓ and ℓ_2 should be clear from context.

III. Distributed System Model

In this section, we present the state-space equations for heterogeneous, discrete-time, LTV subsystems interconnected over arbitrary directed graphs and subjected to a communication latency, as well as the relevant analysis results, as were given in [1] and the references therein.

The interconnection structure of a distributed system is described using a directed graph, where each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$ and the interconnections are described by the directed edges in E. The temporal states of subsystem $G^{(k)}$ are denoted by $x^{(k)}(t)$. We model the interconnections between the subsystems as spatial states, and denote the states associated with edge (i,j) by $x^{(ij)}(t)$. Vectors $x_{\text{in}}^{(k)}(t) = \text{vec}(x^{(ik)}(t))_{i \in E_{\text{in}}^{(k)}}$ and $x_{\text{out}}^{(k)}(t) = \text{vec}(x^{(kj)}(t))_{j \in E_{\text{out}}^{(k)}}$ represent the total information received and sent by subsystem $G^{(k)}$, respectively. Each subsystem $G^{(k)}$ has its own inputs $u^{(k)}(t)$ and outputs $y^{(k)}(t)$. For all $(t,k) \in \mathbb{N}_0 \times V$, the state-space equations are given by

$$\begin{bmatrix} x^{(k)}(t+1) \\ x_{\text{out}}^{(k)}(t+1) \end{bmatrix} = \bar{A}^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\text{in}}^{(k)}(t) \end{bmatrix} + \bar{B}^{(k)}(t)\, u^{(k)}(t),$$
$$y^{(k)}(t) = \bar{C}^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\text{in}}^{(k)}(t) \end{bmatrix} + \bar{D}^{(k)}(t)\, u^{(k)}(t), \quad (1)$$

with zero initial conditions, i.e., $x^{(k)}(0) = 0$ and $x_{\text{in}}^{(k)}(0) = 0$. The sequences of the state-space matrices are assumed to be uniformly bounded. $n^{(k)}(t), n^{(ik)}(t), n^{(kj)}(t), n_u^{(k)}(t)$, and $n_y^{(k)}(t)$ denote the dimensions of $x^{(k)}(t)$, $x^{(ik)}(t)$, $x^{(kj)}(t)$, $u^{(k)}(t)$, and $y^{(k)}(t)$, respectively, for $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$. The state vectors $[x^{(k)}(t+1)^T, x_{\text{out}}^{(k)}(t+1)^T]^T$ and $[x^{(k)}(t)^T, x_{\text{in}}^{(k)}(t)^T]^T$ are partitioned into $p(k)+1$ and $m(k)+1$ vector-valued channels, conformably with which we partition the state-space matrices. For example, the state-space matrices of subsystem $G^{(1)}$ in Fig. 1 are partitioned as follows:

$$\bar{A}^{(1)}(t) = \begin{bmatrix} A_{00}^{(1)}(t) & A_{02}^{(1)}(t) & A_{03}^{(1)}(t) & A_{04}^{(1)}(t) \\ A_{20}^{(1)}(t) & A_{22}^{(1)}(t) & A_{23}^{(1)}(t) & A_{24}^{(1)}(t) \\ A_{30}^{(1)}(t) & A_{32}^{(1)}(t) & A_{33}^{(1)}(t) & A_{34}^{(1)}(t) \\ A_{50}^{(1)}(t) & A_{52}^{(1)}(t) & A_{53}^{(1)}(t) & A_{54}^{(1)}(t) \end{bmatrix},$$
$$\bar{B}^{(1)}(t) = \begin{bmatrix} B_0^{(1)}(t) \\ B_2^{(1)}(t) \\ B_3^{(1)}(t) \\ B_5^{(1)}(t) \end{bmatrix}, \quad \bar{C}^{(1)}(t) = \begin{bmatrix} C_0^{(1)}(t)^T \\ C_2^{(1)}(t)^T \\ C_3^{(1)}(t)^T \\ C_4^{(1)}(t)^T \end{bmatrix}^T.$$

Definition 1: The distributed system (1) is well-posed if, given inputs in ℓ, the state-space equations admit unique solutions in ℓ and define a linear causal mapping on ℓ. Moreover, the system is stable if it is well-posed and if, given inputs in ℓ_2, the state-space equations admit unique solutions in ℓ_2 and define a linear causal mapping on ℓ_2.

The equations in (1) are defined for $t \in \mathbb{N}_0$ with zero initial conditions. The definition can be extended to $t \in \mathbb{Z}$ with zero state-space matrices for $t < 0$, and the resultant system is well-posed [17]. Next, we give the stability condition upon which balanced truncation is based. Distributed systems that satisfy this condition are called strongly stable.

Lemma 1: A distributed system is strongly stable if there exist a positive scalar β and uniformly bounded, positive definite matrix-valued functions $X^{(k)}(t) \in \mathbb{S}^{n^{(k)}(t)}$, $X^{(ik)}(t) \in \mathbb{S}^{n^{(ik)}(t)}$, and $X^{(kj)}(t) \in \mathbb{S}^{n^{(kj)}(t)}$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\bar{A}^{(k)}(t)^T \begin{bmatrix} X^{(k)}(t+1) & 0 \\ 0 & X_{\text{out}}^{(k)}(t+1) \end{bmatrix} \bar{A}^{(k)}(t) - \begin{bmatrix} X^{(k)}(t) & 0 \\ 0 & X_{\text{in}}^{(k)}(t) \end{bmatrix} \prec -\beta I. \quad (2)$$

The notations $X_{\text{in}}^{(k)}(t)$ and $X_{\text{out}}^{(k)}(t)$ are defined as $\text{diag}(X^{(ik)}(t))_{i \in E_{\text{in}}^{(k)}}$ and $\text{diag}(X^{(kj)}(t))_{j \in E_{\text{out}}^{(k)}}$, respectively. These definitions are extended to similar notations, e.g., $Y_{\text{in}}^{(k)}(t)$ and $Y_{\text{out}}^{(k)}(t)$. Hereafter, we no longer specify the dimensions of $X^{(k)}(t)$, $X^{(ik)}(t)$, $X^{(kj)}(t)$ and similar matrix-valued functions as they can be determined from context.

IV. Balanced Truncation Model Reduction

In this section, we summarize the balanced truncation method for distributed systems as presented in [1].

Definition 2: A realization of a distributed system is said to be balanced if there exist a positive scalar β and uniformly bounded, diagonal, and positive definite matrix-valued functions $\Sigma^{(k)}(t)$, $\Sigma^{(ik)}(t)$, and $\Sigma^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$,

$i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\bar{A}^{(k)}(t)\text{diag}(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t))\bar{A}^{(k)}(t)^T + \bar{B}^{(k)}(t)\bar{B}^{(k)}(t)^T - \text{diag}(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1)) \prec -\beta I, \quad (3)$$

$$\bar{A}^{(k)}(t)^T\text{diag}(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1))\bar{A}^{(k)}(t) - \text{diag}(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t)) + \bar{C}^{(k)}(t)^T\bar{C}^{(k)}(t) \prec -\beta I. \quad (4)$$

Inequalities (3) and (4) are the generalized Lyapunov inequalities [8]. For a strongly stable system, they can be solved separately, and the resultant solutions, which are not necessarily diagonal, are the generalized gramians [9]. These gramians are used to construct a balanced realization as outlined next. Assume that $X^{(k)}(t)$, $X^{(ik)}(t)$, $X^{(kj)}(t)$ and $Y^{(k)}(t)$, $Y^{(ik)}(t)$, $Y^{(kj)}(t)$ satisfy (3) and (4), respectively. We focus on the functions associated with the temporal states and repeat a similar procedure for the functions associated with the spatial states. First, we compute the Cholesky factorizations $X^{(k)}(t) = R^{(k)}(t)^T R^{(k)}(t)$ and $Y^{(k)}(t) = H^{(k)}(t)^T H^{(k)}(t)$. Then, performing the singular value decomposition

$$H^{(k)}(t)R^{(k)}(t)^T = U^{(k)}(t)\Sigma^{(k)}(t)V^{(k)}(t)^T,$$

we define the balancing transformation and its inverse as

$$T^{(k)}(t) = \Sigma^{(k)}(t)^{-\frac{1}{2}}U^{(k)}(t)^T H^{(k)}(t),$$
$$T^{(k)}(t)^{-1} = R^{(k)}(t)^T V^{(k)}(t)\Sigma^{(k)}(t)^{-\frac{1}{2}},$$

respectively. Finally, the balanced realization is defined as

$$\bar{A}_{\text{bal}}^{(k)}(t) = T_{\text{pre}}^{(k)}(t+1)\bar{A}^{(k)}(t)T_{\text{post}}^{(k)}(t),$$
$$\bar{B}_{\text{bal}}^{(k)}(t) = T_{\text{pre}}^{(k)}(t+1)\bar{B}^{(k)}(t),$$
$$\bar{C}_{\text{bal}}^{(k)}(t) = \bar{C}^{(k)}(t)T_{\text{post}}^{(k)}(t),$$

where $T_{\text{pre}}^{(k)}(t) = \text{diag}(T^{(k)}(t), T_{\text{out}}^{(k)}(t))$ and $T_{\text{post}}^{(k)}(t) = \text{diag}(T^{(k)}(t)^{-1}, T_{\text{in}}^{(k)}(t)^{-1})$.

Now, suppose that we are given a distributed system with a balanced realization and associated diagonal gramians. We assume that the diagonal entries of the gramians are ordered in a decreasing fashion. Balanced truncation is based on removing the states that correspond to the negligible entries. Given integers $r^{(k)}(t)$, such that $0 \le r^{(k)}(t) \le n^{(k)}(t)$, we partition $\Sigma^{(k)}(t)$ as $\Sigma^{(k)}(t) = \text{diag}(\Gamma^{(k)}(t), \Omega^{(k)}(t))$, where $\Gamma^{(k)}(t) \in \mathbb{S}^{r^{(k)}(t)}$ corresponds to the non-truncated states and $\Omega^{(k)}(t)$ corresponds to the truncated states. A similar procedure is repeated for the gramians associated with the spatial states. Then, we partition the state-space matrices conformably with the partitioning of $\text{diag}(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1))$ and $\text{diag}(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t))$. We revisit subsystem $G^{(1)}$ in our example. $A_{00}^{(1)}(t)$ is partitioned according to the partitioning of $\Sigma^{(1)}(t+1)$ and $\Sigma^{(1)}(t)$ as

$$A_{00}^{(1)}(t) = \begin{bmatrix} \hat{A}_{00}^{(1)}(t) & A_{00_{12}}^{(1)}(t) \\ A_{00_{21}}^{(1)}(t) & A_{00_{22}}^{(1)}(t) \end{bmatrix},$$

where $\hat{A}_{00}^{(1)}(t)$ is an $r^{(1)}(t+1) \times r^{(1)}(t)$ matrix. Similarly, $B_0^{(1)}(t)$ is partitioned as $B_0^{(1)}(t) = \begin{bmatrix} \hat{B}_0^{(1)}(t)^T & B_{0_2}^{(1)}(t)^T \end{bmatrix}^T$, where $\hat{B}_0^{(1)}(t)$ is an $r^{(1)}(t+1) \times n_u^{(1)}(t)$ matrix, and $C_0^{(1)}(t)$ is partitioned as $C_0^{(1)}(t) = \begin{bmatrix} \hat{C}_0^{(1)}(t) & C_{0_2}^{(1)}(t) \end{bmatrix}$, where $\hat{C}_0^{(1)}(t)$ is an $n_y^{(1)}(t) \times r^{(1)}(t)$ matrix.

After completing the partitioning process, we form the reduced order model G_r by defining its state-space matrices $A_r^{(k)}(t)$, $B_r^{(k)}(t)$, and $C_r^{(k)}(t)$. We only keep the blocks that correspond to the non-truncated states, i.e., the partitions marked with a hat, and set $D_r^{(k)}(t) = D^{(k)}(t)$. The obtained realization for the reduced order system is balanced. The upper bound on the ℓ_2-induced norm of the error system resulting from the reduction process is given by

$$\|G - G_r\| < 2\sum_{(t,k)} \left(\sum_{j_1} w_{j_1}^{(k)}(t) + \sum_{i \in E_{\text{in}}^{(k)}} \sum_{j_2} w_{j_2}^{(ik)}(t) \right), \quad (5)$$

where $w_{j_1}^{(k)}(t)$ and $w_{j_2}^{(ik)}(t)$ denote the distinct diagonal entries of $\Omega^{(k)}(t)$ and $\Omega^{(ik)}(t)$, respectively. We do not consider the entries of $\Omega^{(kj)}(t)$ in order to avoid double counting, as the input to a subsystem is an output of another.

V. Coprime Factors Model Reduction

We now extend the CFR method to the class of distributed systems treated in this paper. We start by precisely defining the notions of strong stabilizability and strong detectability. Then, we give a Lyapunov-like test to check whether a system is strongly stabilizable, and further propose a feedback control law that renders the closed-loop system strongly stable. The counterpart of these results pertaining to the class of uncertain linear systems can be found in [24].

A. Strong Stabilizability and Strong Detectability

Definition 3: A well-posed distributed system is strongly stabilizable if there exist uniformly bounded, matrix-valued functions $F^{(k)}(t)$ such that the resulting closed-loop system is strongly stable, i.e., there exist a positive scalar β and uniformly bounded, positive definite matrix-valued functions $P^{(k)}(t)$, $P^{(ik)}(t)$, and $P^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\left(\bar{A}^{(k)}(t) + \bar{B}^{(k)}(t)F^{(k)}(t)\right)\begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t) \end{bmatrix}\left(\bar{A}^{(k)}(t) + \bar{B}^{(k)}(t)F^{(k)}(t)\right)^T - \begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1) \end{bmatrix} \prec -\beta I. \quad (6)$$

Similarly, we say that a well-posed distributed system is strongly detectable if there exist uniformly bounded, matrix-valued functions $K^{(k)}(t)$ such that the resulting closed-loop system is strongly stable, i.e., there exist a positive scalar β and uniformly bounded, positive definite matrix-valued functions $Q^{(k)}(t)$, $Q^{(ik)}(t)$, and $Q^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\left(\bar{A}^{(k)}(t) + K^{(k)}(t)\bar{C}^{(k)}(t)\right)^T\begin{bmatrix} Q^{(k)}(t+1) & 0 \\ 0 & Q_{\text{out}}^{(k)}(t+1) \end{bmatrix} \times\left(\bar{A}^{(k)}(t) + K^{(k)}(t)\bar{C}^{(k)}(t)\right) - \begin{bmatrix} Q^{(k)}(t) & 0 \\ 0 & Q_{\text{in}}^{(k)}(t) \end{bmatrix} \prec -\beta I.$$

Theorem 1: A distributed system is strongly stabilizable by uniformly bounded, matrix-valued functions $F^{(k)}(t)$ if and only if there exist a positive scalar β and uniformly bounded, positive definite matrix-valued functions $P^{(k)}(t)$, $P^{(ik)}(t)$, and $P^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, such that

$$\bar{A}^{(k)}(t)\begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t)\end{bmatrix}\bar{A}^{(k)}(t)^T - \begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix} - \bar{B}^{(k)}(t)\bar{B}^{(k)}(t)^T \prec -\beta I. \quad (7)$$

Furthermore, $F^{(k)}(t)$ can be chosen as the following quantity, if well-defined:

$$-\left(\bar{B}^{(k)}(t)^T\begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix}^{-1}\bar{B}^{(k)}(t)\right)^{-1} \times \bar{B}^{(k)}(t)^T\begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix}^{-1}\bar{A}^{(k)}(t).$$

Proof: We assume that $\operatorname{rank}\bar{B}^{(k)}(t) = n_u^{(k)}(t) < \left(n^{(k)}(t) + \sum_{j\in E_{\text{out}}^{(k)}} n^{(kj)}(t)\right)$ without loss of generality. This is because if rank of $\bar{B}^{(k)}(t)$ is strictly less than $n_u^{(k)}(t)$, then there exist redudant controls which we can easily remove. As for the case where $\bar{B}^{(k)}(t)$ is square and nonsingular, the proof follows immediately. Then, we can always find uniformly bounded, matrix-valued functions $\bar{B}_\perp^{(k)}(t)$ such that $\bar{B}_\perp^{(k)}(t)^T\bar{B}_\perp^{(k)}(t) = I$, $\bar{B}^{(k)}(t)^T\bar{B}_\perp^{(k)}(t) = 0$, and the inverses of the matrix-valued functions $\begin{bmatrix}\bar{B}^{(k)}(t) & \bar{B}_\perp^{(k)}(t)\end{bmatrix}$ exist and are uniformly bounded. Applying the Schur complement formula to (6), we obtain

$$\overbrace{\begin{bmatrix} -\begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t)\end{bmatrix}^{-1} & \bar{A}^{(k)}(t)^T \\ \bar{A}^{(k)}(t) & -\begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix}+\beta I\end{bmatrix}}^{\Psi} + \begin{bmatrix} 0 \\ \bar{B}^{(k)}(t)\end{bmatrix}\underbrace{F^{(k)}(t)}_{\Theta}\underbrace{\begin{bmatrix} I & 0\end{bmatrix}}_{M} + \begin{bmatrix} I \\ 0\end{bmatrix}F^{(k)}(t)^T\underbrace{\begin{bmatrix} 0 & \bar{B}^{(k)}(t)^T\end{bmatrix}}_{N} \prec 0.$$

As shown in [25], the previous inequality has a solution Θ if and only if $W_M^T\Psi W_M \prec 0$ and $W_N^T\Psi W_N \prec 0$ for any matrices W_M and W_N whose columns form bases for the null spaces of M and N, respectively. We choose W_M and W_N, respectively, as $\begin{bmatrix} 0 & I\end{bmatrix}^T$ and $\operatorname{diag}(I, \bar{B}_\perp^{(k)}(t))$. The condition $W_M^T\Psi W_M \prec 0$ is trivially satisfied. So, we focus on $W_N^T\Psi W_N \prec 0$. By applying the Schur complement formula, we get the inequality

$$\bar{B}_\perp^{(k)}(t)^T\left(-\begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix}+\beta I + \bar{A}^{(k)}(t)\begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t)\end{bmatrix}\bar{A}^{(k)}(t)^T\right)\bar{B}_\perp^{(k)}(t) \prec 0,$$

which is equivalent by Finsler's Lemma to (7). We still need to show that the given choice of $F^{(k)}(t)$ renders the closed-loop system strongly stable. First, note that, for each $k \in V$, $F^{(k)}(t)$ is well-defined if $\bar{B}^{(k)}(t)^T\bar{B}^{(k)}(t)$ is a uniformly invertible function. This is ensured by removing all redundant controls and properly perturbing the sequence $\bar{B}^{(k)}(t)$, if necessary, to ensure that the product has a uniform full rank. We apply the Schur complement formula twice to (7). For simplicity, we take $\beta = 0$, but stress that the ensuing sequence of negative definite matrices is uniformly bounded. We get the inequality

$$\bar{A}^{(k)}(t)^T\left(\begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix}+\bar{B}^{(k)}(t)\bar{B}^{(k)}(t)^T\right)^{-1} \times \bar{A}^{(k)}(t) - \begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t)\end{bmatrix}^{-1} \prec 0.$$

Using the matrix inversion lemma, we can verify that

$$-\begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t)\end{bmatrix}^{-1} + \left(\bar{A}^{(k)}(t) + \bar{B}^{(k)}(t)F^{(k)}(t)\right)^T \times \begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix}^{-1}\left(\bar{A}^{(k)}(t)+\bar{B}^{(k)}(t)F^{(k)}(t)\right) \prec 0.$$

By applying the Schur complement formula twice to the previous inequality, we retrieve (6), which concludes the proof of the theorem. ∎

A similar result exists for strong detectability. Next, we extend the notion of a right coprime factorization (RCF) to the class of distributed systems considered in this paper. We then show that a strongly stabilizable and strongly detectable distributed system is guaranteed to have an RCF. We omit similar definitions and results for the left coprime factoriozation (LCF) of the system due to space restrictions.

B. Coprime Factorizations

Definition 4: Two linear causal mappings on ℓ_2, M and N, are right coprime if there exist two linear causal mappings on ℓ_2, X and Y, such that $YM + XN = I$, where I is the identity map on ℓ_2.

Definition 5: We say that a strongly stable pair (N, M) is an RCF for a distributed system G if M has a causal inverse on ℓ, M and N are right coprime, and $G = NM^{-1}$.

Lemma 2: Every strongly stabilizable and strongly detectable distributed system G has an RCF.

Proof: Consider the distributed system G, defined in (1), with its state-space realization denoted for simplicity by the quadruple $(\bar{A}^{(k)}(t), \bar{B}^{(k)}(t), \bar{C}^{(k)}(t), \bar{D}^{(k)}(t))$. Suppose that the matrix-valued functions $F^{(k)}(t)$ are strongly stabilizing. Define distributed systems M and N with state-space realizations $(\bar{A}^{(k)}(t)+\bar{B}^{(k)}(t)F^{(k)}(t), \bar{B}^{(k)}(t), F^{(k)}(t), I)$ and $(\bar{A}^{(k)}(t)+\bar{B}^{(k)}(t)F^{(k)}(t), \bar{B}^{(k)}(t), \bar{C}^{(k)}(t)+\bar{D}^{(k)}(t)F^{(k)}(t), \bar{D}^{(k)}(t))$, respectively. Since the feedback gains are strongly stabilizing, then M and N are linear causal mappings on ℓ_2. To show that M has a causal inverse on ℓ, consider the distributed system R with state-space realization

$(\bar{A}^{(k)}(t), \bar{B}^{(k)}(t), -F^{(k)}(t), I)$. Clearly, R defines a linear causal mapping on ℓ because we assume zero state-space matrices for negative times. So, we just need to prove that $MR = RM = I$. We focus on $MR = I$; the proof of $RM = I$ follows similarly. Noting that $u_M^{(k)}(t) = y_R^{(k)}(t)$, we write the state-space equations for systems M and R, for all $(t,k) \in \mathbb{N}_0 \times V$:

$$y_M^{(k)}(t) = F^{(k)}(t)\left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{\text{in}_M}^{(k)}(t)\end{bmatrix} - \begin{bmatrix} x_R^{(k)}(t) \\ x_{\text{in}_R}^{(k)}(t)\end{bmatrix}\right) + u_R^{(k)}(t),$$

$$\begin{bmatrix} x_M^{(k)}(t+1) \\ x_{\text{out}_M}^{(k)}(t+1)\end{bmatrix} - \begin{bmatrix} x_R^{(k)}(t+1) \\ x_{\text{out}_R}^{(k)}(t+1)\end{bmatrix} =$$
$$\left(\bar{A}^{(k)}(t) + \bar{B}^{(k)}(t)F^{(k)}(t)\right)\left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{\text{in}_M}^{(k)}(t)\end{bmatrix} - \begin{bmatrix} x_R^{(k)}(t) \\ x_{\text{in}_R}^{(k)}(t)\end{bmatrix}\right).$$

The second equation tells us that the difference between state vectors of systems R and M evolves independently of the applied input, and since it is zero at time $t = 0$, then it remains zero at all time steps. Then, $y_M^{(k)}(t) = u_R^{(k)}(t)$, i.e., $MR = I$. We now show that $G = NM^{-1} = NR$. Specifically, we write the equations for N and R, with $u_N = y_R$ and $u_R = u_G$. Following a similar argument as before, we show that $y_G = y_N$. The final step is to prove that N and M are right coprime. To this end, we define Y as a distributed system with state-space realization $(\bar{A}^{(k)}(t) + K^{(k)}(t)\bar{C}^{(k)}(t), \bar{B}^{(k)}(t) + K^{(k)}(t)\bar{D}^{(k)}(t), -F^{(k)}(t), I)$, and X as a distributed system with state-space realization $(\bar{A}^{(k)}(t) + K^{(k)}(t)\bar{C}^{(k)}(t), K^{(k)}(t), F^{(k)}(t), 0)$. Since G is strongly detectable, then $K^{(k)}(t)$ can be chosen so that X and Y define bounded linear causal mappings. To prove that $YM + XN = I$, we write the state-space equations of $YM : u \to y_1$ and $XN : u \to y_2$, then we define $y = y_1 + y_2$ and show that $y = u$. The equations are

$$y_1^{(k)}(t) = -F^{(k)}(t)\begin{bmatrix} x_Y^{(k)}(t) \\ x_{\text{in}_Y}^{(k)}(t)\end{bmatrix} + F^{(k)}(t)\begin{bmatrix} x_M^{(k)}(t) \\ \\ x_{\text{in}_M}^{(k)}(t)\end{bmatrix} + u^{(k)}(t),$$
$$y_2^{(k)}(t) = F^{(k)}(t)\begin{bmatrix} x_X^{(k)}(t) \\ x_{\text{in}_X}^{(k)}(t)\end{bmatrix},$$

$$y^{(k)}(t) = F^{(k)}(t)\left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{\text{in}_M}^{(k)}(t)\end{bmatrix} - \begin{bmatrix} x_Y^{(k)}(t) \\ x_{\text{in}_Y}^{(k)}(t)\end{bmatrix} + \begin{bmatrix} x_X^{(k)}(t) \\ x_{\text{in}_X}^{(k)}(t)\end{bmatrix}\right) + u^{(k)}(t).$$

By an argument similar to the one above, we can see that $x_M^{(k)}(t) = x_N^{(k)}(t)$ and $x_{\text{in}_M}^{(k)}(t) = x_{\text{in}_N}^{(k)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$. Thus,

$$\begin{bmatrix} x_M^{(k)}(t+1) \\ x_{\text{out}_M}^{(k)}(t+1)\end{bmatrix} - \begin{bmatrix} x_Y^{(k)}(t+1) \\ x_{\text{out}_Y}^{(k)}(t+1)\end{bmatrix} + \begin{bmatrix} x_X^{(k)}(t+1) \\ x_{\text{out}_X}^{(k)}(t+1)\end{bmatrix} =$$

$$\left(\bar{A}^{(k)}(t) + K^{(k)}(t)\bar{C}^{(k)}(t)\right)\left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{\text{in}_M}^{(k)}(t)\end{bmatrix} - \begin{bmatrix} x_Y^{(k)}(t) \\ x_{\text{in}_Y}^{(k)}(t)\end{bmatrix} + \begin{bmatrix} x_X^{(k)}(t) \\ x_{\text{in}_X}^{(k)}(t)\end{bmatrix}\right).$$

Then, the quantity in parentheses remains equal to zero for all times, which implies that $y = u$ and concludes the proof. ■

C. Coprime Factors Model Reduction Algorithm

Given a strongly stabilizable and strongly detectable distributed system G with RCF (N, M), we define the strongly stable system $H = \begin{bmatrix} N \\ M \end{bmatrix}$. $H_r = \begin{bmatrix} N_r \\ M_r \end{bmatrix}$ is the reduced order model of H obtained from balanced truncation. We use the factorization (N_r, M_r) to define the reduced order model $G_r = N_r M_r^{-1}$ which approximates G. G_r is always well-posed since we are assuming zero state-space matrices for negative times and a communication latency between the subsystems. Also, N_r and M_r are bounded linear causal mappings because H_r is strongly stable. We use the upper bound on $\|H - H_r\|$ as a guideline for the reduction process. However, unlike balanced truncation, this measure is no longer related to $\|G - G_r\|$.

We now detail the steps of the previous outline. First, we solve for uniformly bounded, positive definite matrix-valued functions $P^{(k)}(t)$, $P^{(ik)}(t)$, and $P^{(kj)}(t)$, for all $(t,k) \in \mathbb{N}_0 \times V$, $i \in E_{\text{in}}^{(k)}$, and $j \in E_{\text{out}}^{(k)}$, that satisfy

$$\bar{A}^{(k)}(t)\begin{bmatrix} P^{(k)}(t) & 0 \\ 0 & P_{\text{in}}^{(k)}(t)\end{bmatrix}\bar{A}^{(k)}(t)^T - \begin{bmatrix} P^{(k)}(t+1) & 0 \\ 0 & P_{\text{out}}^{(k)}(t+1)\end{bmatrix} - \bar{B}^{(k)}(t)\bar{B}^{(k)}(t)^T \prec 0,$$

while ensuring the uniform boundedness of the left-hand side. Then, we define the strongly stabilizing feedback gains $F^{(k)}(t)$ according to Theorem 1. Removing all control redundancies, and slightly perturbing the product $\bar{B}^{(k)}(t)^T\bar{B}^{(k)}(t)$ when needed to ensure uniform invertibility, guarantee that $F^{(k)}(t)$ are well-defined functions. We then construct system H such that

$$\bar{A}_H^{(k)}(t) = \bar{A}^{(k)}(t) + \bar{B}^{(k)}(t)F^{(k)}(t), \quad \bar{B}_H^{(k)}(t) = \bar{B}^{(k)}(t),$$
$$\bar{C}_H^{(k)}(t) = \begin{bmatrix} \bar{C}^{(k)}(t) + \bar{D}^{(k)}(t)F^{(k)}(t) \\ F^{(k)}(t)\end{bmatrix}, \quad \bar{D}_H^{(k)}(t) = \begin{bmatrix} \bar{D}^{(k)}(t) \\ I \end{bmatrix}.$$

Thirdly, we apply balanced truncation on H. We find uniformly bounded, positive definite $X^{(k)}(t)$, $X^{(ik)}(t)$, $X^{(kj)}(t)$ and $Y^{(k)}(t)$, $Y^{(ik)}(t)$, $Y^{(kj)}(t)$ that satisfy

$$\bar{A}_H^{(k)}(t)^T \text{diag}(Y^{(k)}(t+1), Y_{\text{out}}^{(k)}(t+1))\bar{A}_H^{(k)}(t) - \text{diag}(Y^{(k)}(t), Y_{\text{in}}^{(k)}(t)) + \bar{C}_H^{(k)}(t)^T\bar{C}_H^{(k)}(t) \prec 0,$$

$$\bar{A}_H^{(k)}(t)\text{diag}(X^{(k)}(t), X_{\text{in}}^{(k)}(t))\bar{A}_H^{(k)}(t)^T - \text{diag}(X^{(k)}(t+1), X_{\text{out}}^{(k)}(t+1)) + \bar{B}_H^{(k)}(t)\bar{B}_H^{(k)}(t)^T \prec 0.$$

We usually seek the solutions with minimum trace. Then, we obtain the balanced realization and balanced gramians as in Section IV. Based on the diagonal entries of the gramians, we choose which temporal and spatial states to reduce/truncate, and we compute the resultant error bound. We denote the state-space matrices of the resulting reduced order model

H_r by $(\bar{A}^{(k)}_{H_r}(t), \bar{B}^{(k)}_{H_r}(t), \bar{C}^{(k)}_{H_r}(t), \bar{D}^{(k)}_{H_r}(t))$. Then, the state-space matrices $(\bar{A}^{(k)}_r(t), \bar{B}^{(k)}_r(t), \bar{C}^{(k)}_r(t), \bar{D}^{(k)}_r(t))$ of G_r are defined according to

$$\bar{A}^{(k)}_{H_r}(t) = \bar{A}^{(k)}_r(t) + \bar{B}^{(k)}_r(t)F^{(k)}_r(t),\ \bar{B}^{(k)}_{H_r}(t) = \bar{B}^{(k)}_r(t),$$
$$\bar{C}^{(k)}_{H_r}(t) = \begin{bmatrix} \bar{C}^{(k)}_r(t) + \bar{D}^{(k)}_r(t)F^{(k)}_r(t) \\ F^{(k)}_r(t) \end{bmatrix}, \bar{D}^{(k)}_{H_r}(t) = \begin{bmatrix} \bar{D}^{(k)}_r(t) \\ I \end{bmatrix}.$$

Note that $F^{(k)}_r(t)$ strongly stabilizes G_r.

Remark 1: A distributed system G is said to be (h, q)-eventually time-periodic (ETP), for some $h \in \mathbb{N}_0$ and $q \in \mathbb{N}$, if the state-space matrices are aperiodic for an initial time h, and then become time-periodic with period q, i.e., for all $t, z \in \mathbb{N}_0$ and $k \in V$, $\bar{A}^{(k)}(t+h+zq) = \bar{A}^{(k)}(t+h)$, and so on. ETP systems include as special cases time-periodic, time-invariant, and finite horizon systems. In the case of an (h, q)-ETP system, there exist solutions to (2) (respectively, (3), (4), and (7)) if and only if there exist (h, q)-ETP solutions [1], [26]–[29]. Hence, $F^{(k)}(t)$ can be taken as (h, q)-ETP. As a result, the state-space matrices of H, H_r, and G_r become (h, q)-ETP, and the bound on $\|H - H_r\|$ obtained from (5) reduces to a finite sum. Namely,

$$2\sum_k \sum_{t=0}^{h+q-1} \Bigg(\sum_{j_1} w^{(k)}_{j_1}(t) + \sum_{i \in E^{(k)}_{\text{in}}} \sum_{j_2} w^{(ik)}_{j_2}(t) \Bigg).$$

VI. Conclusion

Model reduction is desirable for distributed systems. Balanced truncation is a systematic procedure that preserves the interconnection structure and even allows its simplification, i.e., the order reduction of spatial states and possibly the removal of whole interconnections. Balanced truncation comes with an upper bound on the norm of the error system. However, it only applies to strongly stable systems. CFR provides a partial solution to this limitation by applying to strongly stabilizable and strongly detectable systems. CFR retains some advantages of balanced truncation such as structure preservation and simplification. However, unlike balanced truncation which guarantees strong stability, CFR only guarantees the strong stabilizability and detectability of the reduced order system. Nevertheless, strongly stabilizing feedback gains are readily available from the method. The error bound associated with CFR has interpretations in terms of closed-loop robust stability.

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