Balanced Truncation of Spatially Distributed Nonstationary LPV Systems

Dany Abou Jaoude and Mazen Farhood

***Abstract*— This work is on the balanced truncation of distributed nonstationary linear parameter-varying (NSLPV) systems. These systems are composed of heterogeneous, discrete-time, NSLPV subsystems, formulated in a linear fractional transformation (LFT) framework, interconnected over arbitrary directed graphs, and subjected to a communication latency of one time-step. The proposed method is applied for the model reduction of strongly stable systems, i.e., stable systems with structured generalized gramians. The method guarantees the strong stability of the reduced order system, provides an upper bound on the error induced by the reduction process, and allows for the preservation and simplification of the interconnection and uncertainty structures.**

I. Introduction

This work is on the balanced truncation of heterogeneous, discrete-time, nonstationary linear parameter-varying (NSLPV) subsystems formulated in a linear fractional transformation (LFT) framework. The subsystems are interconnected over arbitrary directed graphs and are subjected to a communication latency one time-step. NSLPV subsystems [1], [2] are linear parameter-varying (LPV) subsystems where the state-space matrix-valued functions have an explicit dependence on a priori known time-varying terms in addition to their dependence on the scheduling parameters. This class of models includes the classes of linear time-invariant (LTI), linear time-varying (LTV), and standard/stationary LPV models. The need for distributed NSLPV systems arises when the subsystems require NSLPV models, or when some subsystems require LTV models while other subsystems require LPV models.

Model reduction is desirable for interconnected systems since the corresponding models grow in size with the state dimension of the subsystems as well as the complexity of the interconnection structure. Balanced truncation (BT) is of particular interest because it guarantees the stability of the resulting reduced order system as well as an upper bound on the norm of the error system, see [3], [4], [5], [6]. When applying model reduction schemes to interconnected systems, it is usually desirable to preserve the interconnection structure. Various works, such as [7], [8], [9], [10], [11], have appeared that treat the problem of structure-preserving model reduction, focusing primarily on BT and coprime factors reduction (CFR). These works are based on finding block-diagonal solutions to linear matrix inequalities (LMIs) [12]. Most closely related to the current work are the works of [9], [10], [11], where the interconnections between the subsystems are modeled as spatial states, in addition to the states of the subsystems, which are called the temporal states. In particular, [9] applies BT to homogeneous subsystems interconnected over a grid and described by either an LTI model or a stationary LPV model. The temporal states of all the subsystems, the forward spatial states, and the backward spatial states are then reduced in a uniform way, respectively. [10], [11] treat heterogeneous, LTV subsystems interconnected over arbitrary directed graphs and subjected to a communication latency. The methods therein preserve the interconnection structure and further allow for its simplification: the temporal and spatial states are truncated individually; and when deemed negligible, whole interconnections are removed from the interconnection structure. The BT method of [10] applies to stable systems which possess structured solutions, called generalized gramians, to the generalized Lyapunov inequalities. Such systems are called strongly stable. In [11], CFR extends the range of applicability of BT to stabilizable and detectable systems which can be represented using strongly stable coprime factorizations.

The current work is a sequel to [10], with the subsystems having NSLPV models formulated in an LFT framework instead of LTV models. In addition to the interconnection structure, the systems considered here have an uncertainty structure due to the LFT formulation. Thus, the proposed BT method needs to preserve/simplify both the interconnection structure and the uncertainty structure.

The paper is organized as follows. In Section II, we define the notation to be used throughout the paper. In Section III, we summarize the framework used to describe distributed NSLPV systems. In Section IV, we present the BT method. We conclude the paper with Section V.

II. Notations

We denote by $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{R}$ the sets of nonnegative integers, integers, and real numbers, respectively. We denote an $i \times i$ identity matrix by I_i. $\text{diag}(M_i)$ denotes the block-diagonal augmentation of the elements of the sequence of operators M_i.

A directed graph is denoted by $\mathcal{G}(V, E)$, where the countable set V is the set of vertices, and E is the set of directed edges. An element of E directed from $i \in V$ to $j \in V$ is denoted by (i, j). We assume that the graph under consideration is d-regular, i.e., for each vertex in V, the numbers of incoming and outgoing edges are equal to the positive integer d. Note that any arbitrary directed graph can be turned into a regular one via the addition of the necessary virtual edges and/or nodes. The graph-regularity assumption is needed for the development of the operator

The authors are with the Department of Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, VA 24061, USA. Email: {danyabj, farhood}@vt.edu. This material is based upon work supported by the National Science Foundation under Grant Number CMMI-1333785.

theoretic framework of Section III because it allows for the definition of d permutations, $\rho_1, \ldots, \rho_d$, of the set of vertices according to the edges. These permutations are chosen such that if $(i, j) \in E$, then one $e \in \{1, \ldots, d\}$ satisfies $\rho_e(i) = j$ and $\rho_e^{-1}(j) = i$. See [13] for more details.

Let J_1 and J_2 be vector spaces, and H and F be Hilbert spaces. We denote the vector space direct sum of J_1 and J_2 by $J_1 \oplus J_2$. . The inner product and norm associated with H are denoted by $\langle \cdot, \cdot \rangle_H$ and $\|\cdot\|_H$, respectively. The subscript is dropped when H is clear from context. We denote the space of bounded linear operators mapping H to F by $\mathcal{L}(H, F)$, and the space of bounded linear causal operators mapping H to F by $\mathcal{L}_c(H, F)$. These notations simplify to $\mathcal{L}(H)$ and $\mathcal{L}_c(H)$ when $H = F$. Let $X \in \mathcal{L}(H, F)$. $\|X\|$ denotes the H to F induced norm of X, and X^* denotes the adjoint of X. We say that a self-adjoint operator $X \in \mathcal{L}(H)$ is negative definite ($X \prec 0$) if, for all nonzero $x \in H$, there exists $\alpha > 0$ such that $\langle x, Xx \rangle < -\alpha \|x\|^2$.

Consider the integer sequence $n : (t, k) \in \mathbb{Z} \times V \rightarrow n(t, k) \in \mathbb{N}_0$. We define $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ as the vector space of mappings $w : (t, k) \in \mathbb{Z} \times V \rightarrow w(t, k) \in \mathbb{R}^{n(t,k)}$. The Hilbert subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ that consists of mappings w having a finite norm $\|w\| := \sqrt{\sum_{(t,k)} w(t, k)^* w(t, k)}$ is denoted by $\ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right)$. We also define $\ell_{2e}\left(\{\mathbb{R}^{n(t,k)}\}\right)$ as the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ consisting of mappings w that satisfy the inequality $\sum_k w(t, k)^* w(t, k) < \infty$, for each $t \in \mathbb{Z}$. We will frequently use the abbreviated symbols ℓ, ℓ_2, and ℓ_{2e} when the dimensions are clear from context.

We now summarize the needed operator machinery from [13]. We say that Q is a graph-diagonal operator on ℓ_2 if it satisfies $(Qv)(t, k) = Q(t, k)v(t, k)$, for all $(t, k) \in \mathbb{Z} \times V$. An operator $W = [W_{ij}]$ is said to be partitioned graph-diagonal if each block W_{ij} is a graph-diagonal operator. The mapping $[\![W]\!](t, k) = [W_{ij}(t, k)]$ is a homomorphism from the space of partitioned graph-diagonal operators to the space of graph-diagonal operators. This mapping is isometric and preserves products, addition, and ordering. We also introduce the unitary temporal-shift operator $S_0 : \ell_2 \rightarrow \ell_2$ and the unitary spatial-shift operators $S_i : \ell_2 \rightarrow \ell_2$, for $i = 1, \ldots, d$, as follows:

$$
\begin{aligned}
(S_0 v)(t, k) &= v(t - 1, k), \quad (S_0^* v)(t, k) = v(t + 1, k), \\
(S_i v)(t, k) &= v(t, \rho_i^{-1}(k)), \quad (S_i^* v)(t, k) = v(t, \rho_i(k)).
\end{aligned}
$$

The definitions of graph-diagonal operators and of shift operators naturally extend to ℓ and ℓ_{2e}.

Consider a graph-diagonal operator X, where $X(t, k)$ is a diagonal matrix, for all $(t, k) \in \mathbb{Z} \times V$. We denote by $\phi(X)$ the sum of distinct entries of X, i.e., $\phi(X)$ is the sum of the distinct diagonal entries in $\text{diag}(X(t, k))_{(t,k) \in \mathbb{Z} \times V}$. For example, assume that $X(t, k) = 0$, for all $(t, k) \in \mathbb{Z} \times V$, except for some (t_0, k_0), (t_0, k_1), (t_1, k_2), where $X(t_0, k_0) = \text{diag}(w_1, w_1, w_2, w_2)$, $X(t_0, k_1) = \text{diag}(w_1, w_3, w_3, w_4)$, $X(t_1, k_2) = \text{diag}(w_3, w_3, w_4)$. Then, $\phi(X) = w_1 + w_2 + w_3 + w_4$. Now, let W be a partitioned graph-diagonal operator with a block-diagonal structure, i.e., $W = \text{diag}(W_i)$, where W_i are graph-diagonal operators. Assume that, for all $(t, k) \in \mathbb{Z} \times V$, $[\![W]\!](t, k)$ is a diagonal matrix, i.e., $W_i(t, k)$ are diagonal matrices for all i. We define $\Phi(W)$ as the sum of distinct entries of W, i.e., $\Phi(W) = \phi([\![W]\!])$.

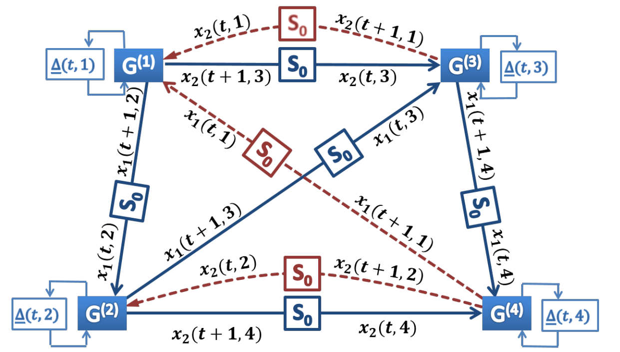

Fig. 1. Distributed system consisting of NSLPV subsystems formulated in an LFT framework.

III. Operator-Based Description

The discussion and results in this section are a summary of [14]. The interconnection structure of a distributed system is given by a directed graph, $\mathcal{G}(V, E)$, where each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$, and the interconnections between the subsystems are described by the directed edges. Each subsystem has a discrete-time NSLPV model, formulated in an LFT framework. We denote the standard states of subsystem $G^{(k)}$ by $x_T(t, k)$, where t is the discrete time-step, and refer to them as the temporal states. We denote the possibly time-varying dimension of $x_T(t, k)$ by $n_T(t, k)$. We denote the states which are due to the LFT formulation by $\beta(t, k)$ and $\alpha(t, k)$, and hereafter refer to them as the parameter states. The interconnections between the subsystems are also modeled as states denoted by $x_i(t, k)$, and of corresponding dimensions $n_i^S(t, k)$. We refer to these states as the spatial states. Namely, the spatial state $x_i(t, k)$ is associated with the interconnection $(\rho_i^{-1}(k), k)$, i.e., the edge incoming to vertex k along permutation ρ_i. Similarly, the spatial state $x_i(t, \rho_i(k))$ denotes the state associated with the interconnection $(k, \rho_i(k))$, i.e., the edge outgoing from vertex k along permutation ρ_i. Note that the dimensions of the spatial states associated with the virtual interconnections are zeros. Due to the communication latency between the subsystems, the data sent by a subsystem at the current time-step reaches the target subsystem at the next time-step. We denote the control inputs and the output measurements of subsystem $G^{(k)}$ by $u(t, k)$ and $y(t, k)$, and denote their corresponding dimensions by $n_u(t, k)$ and $n_y(t, k)$. Then, for all $(t, k) \in \mathbb{Z} \times V$, the state-space equations are

$$
\left[\begin{array}{c}
x_T(t+1, k) \\
x_1(t+1, \rho_1(k)) \\
\vdots \\
x_d(t+1, \rho_d(k)) \\
\alpha(t, k) \\
\hline
y(t, k)
\end{array}\right]
=
\left[\begin{array}{c|c}
\bar{A}(t, k) & \bar{B}(t, k) \\
\hline
\bar{C}(t, k) & \bar{D}(t, k)
\end{array}\right]
\left[\begin{array}{c}
x_T(t, k) \\
x_1(t, k) \\
\vdots \\
x_d(t, k) \\
\beta(t, k) \\
\hline
u(t, k)
\end{array}\right],
$$

$$\beta(t,k) = \text{diag}\Big(\delta_1(t,k) I_{n_1^P(t,k)}, ..., \delta_r(t,k) I_{n_r^P(t,k)}\Big)\alpha(t,k)$$
$$= \underline{\Delta}(t,k)\ \alpha(t,k). \quad (1)$$

The parameter states evolve according to the feedback channel $\beta(t,k) = \underline{\Delta}(t,k)\alpha(t,k)$, where, for $j = 1,\dots,r$, the parameter values $\delta_j(t,k)$ are scalars which are not known a priori, but are assumed to be measurable at each time-step. We partition vectors $\beta(t,k)$ and $\alpha(t,k)$ into r vector-valued channels conformably with the blocks of $\underline{\Delta}(t,k)$, e.g., $\alpha(t,k) = [\alpha_1^*(t,k), \alpha_2^*(t,k), \dots, \alpha_r^*(t,k)]^*$. The partitions $\alpha_j(t,k)$ and $\beta_j(t,k)$ share the dimension $n_j^P(t,k)$. The dependence of the subsystems on the parameters is local, i.e., different subsystems may depend on different parameters, and even if two subsystems are affected by the same parameters, the evolution of these parameters is assumed to be independent in each subsystem. Let r_k be the number of parameters affecting subsystem $G^{(k)}$. Then, $r = \max_{k\in V} r_k$. If $G^{(k_0)}$ is affected by a number of parameters $r_{k_0} < r$, then we equate the corresponding $\delta_j(t,k_0)$ and $n_j^P(t,k_0)$ to 0, for all time-steps $t \in \mathbb{Z}$ and indices $j = r_{k_0}+1,\dots,r$.

Figure 1 shows a distributed system having NSLPV subsystems formulated in an LFT framework. The dashed red arrows correspond to the virtual interconnections added to render the graph 2-regular. The permutations ρ_1 and ρ_2 are defined as follows: $\rho_1(1) = 2$, $\rho_1(2) = 3$, $\rho_1(3) = 4$, $\rho_1(4) = 1$, $\rho_2(1) = 3$, $\rho_2(3) = 1$, $\rho_2(2) = 4$, and $\rho_2(4) = 2$. The spatial states associated with each interconnection are also specified in the figure. Operator S_0 marks the delay on the information transfer between the subsystems.

The state-space matrices, i.e., $\bar{A}(t,k)$, $\bar{B}(t,k)$, $\bar{C}(t,k)$, $\bar{D}(t,k)$, are known a priori and are assumed to be uniformly bounded. They can be conveniently partitioned as follows:

$$\bar{A}(t,k) = \begin{bmatrix} A_{TT}(t,k) & A^{TS}(t,k) & A^{TP}(t,k) \\ A^{ST}(t,k) & A^{SS}(t,k) & A^{SP}(t,k) \\ A^{PT}(t,k) & A^{PS}(t,k) & A^{PP}(t,k) \end{bmatrix},$$
$$\bar{B}(t,k) = \begin{bmatrix} B_T(t,k)^* & B^S(t,k)^* & B^P(t,k)^* \end{bmatrix}^*,$$
$$\bar{C}(t,k) = \begin{bmatrix} C_T(t,k) & C^S(t,k) & C^P(t,k) \end{bmatrix}.$$

With this partitioning, one can see that the equations in (1) look similar to the state-space equations for LPV interconnected subsystems. The difference is that in (1), the state-space matrices are allowed to be dependent on t as opposed to being constants as in the standard LPV case.

The blocks of the state-space matrices can be further partitioned conformably with the permutations, $\rho_1,\dots,\rho_d$, and the blocks of the feedback channel $\underline{\Delta}(t,k)$, e.g.,

$$A^{TS}(t,k) = \begin{bmatrix} A_1^{TS}(t,k) & \cdots & A_d^{TS}(t,k) \end{bmatrix},$$
$$A^{TP}(t,k) = \begin{bmatrix} A_1^{TP}(t,k) & \cdots & A_r^{TP}(t,k) \end{bmatrix},$$
$$A^{SS}(t,k) = \begin{bmatrix} A_{ie}^{SS}(t,k) \end{bmatrix}_{i=1,\dots,d;e=1,\dots,d},$$
$$A^{PP}(t,k) = \begin{bmatrix} A_{jf}^{PP}(t,k) \end{bmatrix}_{j=1,\dots,r;f=1,\dots,r}.$$

These partitions, e.g., $A_{TT}(t,k)$, $A_1^{TS}(t,k)$, define graph-diagonal operators, e.g., A_{TT}, A_1^{TS}, which in turn, when augmented in the obvious way, form partitioned graph-diagonal operators A, B, and C that satisfy $[\![A]\!](t,k) = \bar{A}(t,k)$, $[\![B]\!](t,k) = \bar{B}(t,k)$, and $[\![C]\!](t,k) = \bar{C}(t,k)$. The matrices $\bar{D}(t,k)$ define the graph-diagonal operator D such that $\bar{D}(t,k) = D(t,k) = [\![D]\!](t,k)$.

We define graph-diagonal operators Δ_j, for $j = 1,\dots,r$, such that $\Delta_j(t,k) = \delta_j(t,k) I_{n_j^P(t,k)}$. These operators are augmented to form $\Delta_P = \text{diag}(\Delta_1,\dots,\Delta_r)$. Note that $[\![\Delta_P]\!](t,k) = \underline{\Delta}(t,k)$. Then, we define the composite-shift operator $S = \text{diag}\left(S_0, S_0S_1, \dots, S_0S_d, I^{(n_1^P,\dots,n_r^P)}\right)$ and the operator $\Delta = \text{diag}\left(I^{n_T}, I^{(n_1^S,\dots,n_d^S)}, \Delta_P\right)$, where the graph-diagonal operator I^q satisfies $[\![I^q]\!](t,k) = I_{q(t,k)}$, and $I^{(q_1,\dots,q_m)} = \text{diag}(I^{q_1},\dots,I^{q_m})$. We assume that Δ is in $\boldsymbol{\Delta} = \{\Delta : \|\Delta\| \le 1\}$. Then, we rewrite (1) as

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \Delta S A \begin{bmatrix} x \\ \beta \end{bmatrix} + \Delta S B\, u, \quad y = C\begin{bmatrix} x \\ \beta \end{bmatrix} + D\, u, \quad (2)$$

where $x = \left[x_T^*, x_1^*, \dots, x_d^*\right]^*$. For a fixed $\Delta \in \boldsymbol{\Delta}$, and assuming that the relevant inverse exists, the input-output map of the system can be written as

$$G_\delta = \Delta \star G = \Delta \star \left[\begin{array}{c|c} SA & SB \\ \hline C & D \end{array}\right] = C(I - \Delta SA)^{-1}\Delta SB + D.$$

The distributed NSLPV system $\mathcal{G}_\delta$ is then defined as $\mathcal{G}_\delta = \boldsymbol{\Delta} \star G = \{\Delta \star G : \Delta \in \boldsymbol{\Delta}\}$.

System $\mathcal{G}_\delta$ is said to be well-posed if G_δ defines a linear causal mapping on ℓ_{2e}, for all $\Delta \in \boldsymbol{\Delta}$. From [14], we know that the well-posedness of $\mathcal{G}_\delta$ is ensured if $\bar{A}(t,k) = 0$, for $t < 0$, and if $I - \Delta_P \tilde{A}^{PP}$ has a causal inverse on $\oplus_{j=1}^r \ell_2\left(\mathbb{R}^{n_j^P(t,k)}\right)$, for all $\Delta \in \boldsymbol{\Delta}$, where $\tilde{A}^{PP} = \left[A_{jf}^{PP}\right]$. We assume hereafter that all the state-space matrices are zeros for $t < 0$.

Definition 1: System $\mathcal{G}_\delta$ is said to be stable if $G_\delta \in \mathcal{L}_c\left(\ell_2\left(\mathbb{R}^{n_u(t,k)}\right), \ell_2\left(\mathbb{R}^{n_y(t,k)}\right)\right)$, for all $\Delta \in \boldsymbol{\Delta}$.

Before we give a sufficient condition for stability, we need to define the following sets:

$$\mathcal{T} = \big\{X : X = \text{diag}\left(X_T, X_1^S, ..., X_d^S, X_1^P, ..., X_r^P\right), X^{-1} \in \mathcal{L}(\ell_2(\mathbb{R}^{n_T(t,k)}) \oplus (\oplus_{i=1}^d \ell_2(\mathbb{R}^{n_i^S(t,k)})) \oplus (\oplus_{j=1}^r \ell_2(\mathbb{R}^{n_j^P(t,k)})))\big\}$$
$$\mathcal{X} = \{X : X = X^* \in \mathcal{T},\ X \succ 0\},$$

where X_T, X_i^S, X_j^P, for $i = 1,\dots,d$ and $j = 1,\dots,r$, are bounded graph-diagonal operators. Note that $\mathcal{T}$ and $\mathcal{X}$ are commutants of $\boldsymbol{\Delta}$. The symbols $\boldsymbol{\Delta}$ and $\mathcal{X}$ are used irrespectively of their associated dimensions.

Lemma 1: System $\mathcal{G}_\delta$ is stable if there exists $X \in \mathcal{X}$, or equivalently $X \succ 0$ in the commutant of $\boldsymbol{\Delta}$, such that

$$A^* S^* X S A - X \prec 0. \quad (3)$$

Assume $X \in \mathcal{X}$. Due to the block-diagonal structure of X, S^*XS is also a partitioned graph-diagonal operator with a block-diagonal structure. The blocks of S^*XS satisfy $(S_0^* X_T S_0)(t,k) = X_T(t+1,k)$, $(S_i^* S_0^* X_i^S S_0 S_i)(t,k) = X_i^S(t+1,\rho_i(k))$, and $(X_j^P)(t,k) = X_j^P(t,k)$. Note that the sequences of LMIs equivalent to (3) are trivial for $t < 0$ since the state-space matrices are assumed to be zeros for negative time-steps. In the sequel, we write $t \in \mathbb{Z}$ to allow for the operator theoretic framework, but we are only interested in $t \in \mathbb{N}_0$. Since (3) is only a sufficient condition for stability, we refer to systems that satisfy it as strongly stable. In other words, a strongly stable system $\mathcal{G}_\delta$ is a stable system which has a solution $X \in \mathcal{X}$ to (3). As will become apparent shortly, BT is only applicable to strongly stable systems, and so, suffers from the conservatism due to the imposed structure on X. It is, however, this structure that allows for the preservation and even the simplification of the interconnection and the uncertainty structures. The next result will be of use in the proof of Theorem 1.

Lemma 2: System $\mathcal{G}_\delta$ is strongly stable and satisfies $\|G_\delta\| < \gamma$, for all $\Delta \in \mathbf{\Delta}$, if there exists $X \in \mathcal{X}$, or equivalently, $X \succ 0$ in the commutant of $\mathbf{\Delta}$, such that

$$\begin{bmatrix} -\begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} & \begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \\ \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} & -\begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} \prec 0. \tag{4}$$

IV. Balanced Truncation

Definition 2: A realization of system $\mathcal{G}_\delta$, which we denote by $(A, B, C, D, \mathbf{\Delta})$, is said to be balanced if there exists an operator $\Sigma = X = Y \in \mathcal{X}$ that satisfies

$$AXA^* - S^*XS + BB^* \prec 0, \tag{5}$$

$$A^*S^*YSA - Y + C^*C \prec 0, \tag{6}$$

where $[\![\Sigma]\!](t,k)$ is a diagonal matrix, for all $(t,k) \in \mathbb{Z} \times V$. Σ is called the balanced generalized gramian.

(5) and (6) are called the generalized Lyapunov inequalities. These inequalities can be solved separately for the generalized controllability and observability gramians, $X \in \mathcal{X}$ and $Y \in \mathcal{X}$, respectively. Note that $[\![X]\!](t,k)$ and $[\![Y]\!](t,k)$ need not be diagonal matrices.

Lemma 3: A strongly stable system $\mathcal{G}_\delta$ admits a balanced realization.

Proof: For a strongly stable system, there exists a solution $P \in \mathcal{X}$ to (3). By scalability and homogeneity of (3), there exist solutions X and $Y \in \mathcal{X}$ to (5) and (6), respectively. These generalized gramians can be used to construct a balanced realization for system $\mathcal{G}_\delta$ as follows. First, we perform the Cholesky factorizations $X = R^*R$ and $Y = H^*H$. Then, we perform the singular value decomposition $HR^* = U\Sigma V^*$, where U and V are in $\mathcal{T}$, and Σ is the balanced generalized gramian. Then, we define the balancing transformation $T = \Sigma^{-1/2}U^*H \in \mathcal{T}$ and its inverse $T^{-1} = R^*V\Sigma^{-1/2} \in \mathcal{T}$. Thus, the balanced generalized gramian can be expressed as $\Sigma = TXT^* = (T^*)^{-1}YT^{-1}$. Moreover, $((S^*TS)AT^{-1}, (S^*TS)B, CT^{-1}, D, \mathbf{\Delta})$ can be easily verified to be a balanced realization of $\mathcal{G}_\delta$. Note that because of the structure imposed on $\mathcal{T}$ and $\mathcal{X}$, the previous computations are performed block-wise, e.g.,

$$X_T = (R_T)^* R_T,\ Y_T = (H_T)^* H_T,\ H_T(R_T)^* = U_T \Sigma_T (V_T)^*.$$

■

The obtained balanced generalized gramian and balanced realization of system $\mathcal{G}_\delta$ depend on the gramians X and Y used in the balancing procedure. To obtain useful results for BT, we usually seek gramians with minimum traces.

Now, consider a strongly stable system $\mathcal{G}_\delta$ with a balanced realization $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian $\Sigma = \text{diag}\left(\Sigma_T, \Sigma_1^S, ..., \Sigma_d^S, \Sigma_1^P, ..., \Sigma_r^P\right)$. We want to apply BT to this system to reduce its order. To do so, we look at the entries of Σ. Based on the relative order of these entries, the upper bound on the ℓ_2-induced norm of the system given in Lemma 2, and the upper bound on the error induced by BT given in Theorem 2, we decide which entries are negligible, and thus, decide which state variables can be truncated without heavily altering the behavior of the system. Since Σ has a block-diagonal structure, the reduction process is applied individually to each of its blocks, i.e., Σ_T, Σ_i^S, Σ_j^P, and their corresponding temporal, spatial, and parameter state variables.

Let us focus on Σ_T. For each $(t,k) \in \mathbb{Z} \times V$, $\Sigma_T(t,k)$ is an $n_T(t,k) \times n_T(t,k)$ positive definite, diagonal matrix. We assume without loss of generality that the entries of $\Sigma_T(t,k)$ are ordered in a decreasing fashion with the largest entry as the $(1,1)$-term. Suppose we want to reduce the dimensions of the temporal states from $n_T(t,k)$ to $m_T(t,k)$, where $0 \leq m_T(t,k) \leq n_T(t,k)$. We start by partitioning $\Sigma_T(t,k)$ into two blocks $\Gamma_T(t,k)$ and $\Omega_T(t,k)$, with dimensions $m_T(t,k)$ and $n_T(t,k) - m_T(t,k)$, respectively, such that $\Sigma_T(t,k) = \text{diag}\left(\Gamma_T(t,k), \Omega_T(t,k)\right)$. Note that if either $m_T(t_0,k_0)$ or $n_T(t_0,k_0) - m_T(t_0,k_0)$ is zero for some (t_0,k_0), then, correspondingly, either $\Gamma_T(t_0,k_0)$ or $\Omega_T(t_0,k_0)$ is a matrix of zero dimensions, i.e., nonexistent. $\Gamma_T(t,k)$ and $\Omega_T(t,k)$ define graph-diagonal operators Γ_T and Ω_T, respectively. We repeat the partitioning procedure for Σ_i^S and Σ_j^P, for all $i = 1, \ldots, d$ and $j = 1, \ldots, r$, and augment the resulting graph-diagonal operators as in

$$\Gamma = \text{diag}\left(\Gamma_T, \Gamma_1^S, \ldots, \Gamma_d^S, \Gamma_1^P, \ldots, \Gamma_r^P\right) \in \mathcal{X},$$
$$\Omega = \text{diag}\left(\Omega_T, \Omega_1^S, \ldots, \Omega_d^S, \Omega_1^P, \ldots, \Omega_r^P\right) \in \mathcal{X}.$$

Γ is associated with the non-truncated blocks of Σ, and Ω is associated with the truncated blocks of Σ. We now partition the blocks of the states-space matrices into non-truncated and truncated portions conformably with the partitioning of the blocks of Σ. The non-truncated portions are marked with a hat. For instance,

$$A^{TS}(t,k) = \left[\begin{bmatrix} \hat{A}_1^{TS}(t,k) & A_{1,12}^{TS}(t,k) \\ A_{1,21}^{TS}(t,k) & A_{1,22}^{TS}(t,k) \end{bmatrix} \cdots \begin{bmatrix} \hat{A}_d^{TS}(t,k) & A_{d,12}^{TS}(t,k) \\ A_{d,21}^{TS}(t,k) & A_{d,22}^{TS}(t,k) \end{bmatrix}\right]$$

$$A^{TP}(t,k) = \left[\begin{bmatrix} \hat{A}_1^{TP}(t,k) & A_{1,12}^{TP}(t,k) \\ A_{1,21}^{TP}(t,k) & A_{1,22}^{TP}(t,k) \end{bmatrix} \cdots \begin{bmatrix} \hat{A}_r^{TP}(t,k) & A_{r,12}^{TP}(t,k) \\ A_{r,21}^{TP}(t,k) & A_{r,22}^{TP}(t,k) \end{bmatrix}\right]$$

$$A^{SS}(t,k) = \left[\begin{bmatrix} \hat{A}_{ie}^{SS}(t,k) & A_{ie,12}^{SS}(t,k) \\ A_{ie,21}^{SS}(t,k) & A_{ie,22}^{SS}(t,k) \end{bmatrix}\right]_{i=1,\ldots,d;e=1,\ldots,d}$$

$$A^{PP}(t,k) = \left[\begin{bmatrix} \hat{A}_{jf}^{PP}(t,k) & A_{jf,12}^{PP}(t,k) \\ A_{jf,21}^{PP}(t,k) & A_{jf,22}^{PP}(t,k) \end{bmatrix}\right]_{j=1,\ldots,r;f=1,\ldots,r}$$

where $\hat{A}_i^{TS}(t,k)$ is an $m_T(t+1,k)\times m_i^S(t,k)$ matrix, $\hat{A}_j^{TP}(t,k)$ is an $m_T(t+1,k)\times m_j^P(t,k)$ matrix, $\hat{A}_{ie}^{SS}(t,k)$ is an $m_i^S(t+1,\rho_i(k))\times m_e^S(t,k)$ matrix, and $\hat{A}_{jf}^{PP}(t,k)$ is an $m_j^P(t,k)\times m_f^P(t,k)$ matrix.

Notice that the partitioning of the state-space matrices into non-truncated and truncated portions is performed at the level of the most elemental blocks. For instance, we do not partition $\bar{A}(t,k)$ nor $A^{TS}(t,k)$, but rather, we partition $A_i^{TS}(t,k)$ into a non-truncated block $\hat{A}_i^{TS}(t,k)$ and three truncated blocks $A_{i,12}^{TS}(t,k)$, $A_{i,21}^{TS}(t,k)$, and $A_{i,22}^{TS}(t,k)$. Thus, the proposed procedure allows for the preservation of the interconnection and uncertainty structures of the system. The non-truncated blocks, e.g., $\hat{A}_{TT}(t,k)$, $\hat{A}_i^{TS}(t,k)$, define graph-diagonal operators, e.g., $\hat{A}_{TT}$, $\hat{A}_i^{TS}$, which when augmented in the obvious way, define the operators $A_{\rm red}$, $B_{\rm red}$, and $C_{\rm red}$. $D_{\rm red}$ is set equal to D. We also define

$$\Delta_{\rm red}=\mathrm{diag}\big(I^{m_T},I^{(m_1^S,\ldots,m_d^S)},\hat{\Delta}_P\big)\in\mathbf{\Delta},$$

where $\hat{\Delta}_P=\mathrm{diag}\big(\hat{\Delta}_1,\ldots,\hat{\Delta}_r\big)$, and $\hat{\Delta}_j$ are graph-diagonal operators that satisfy $\hat{\Delta}_j(t,k)=\delta_j(t,k)I_{m_j^P(t,k)}$, for $j=1,\ldots,r$. Notice that the same parameter values $\delta_j(t,k)$ affect both $\Delta_j(t,k)$ and $\hat{\Delta}_j(t,k)$. Thus, the realization of the reduced order system $\mathcal{G}_{{\rm red},\delta}$ is given by $(A_{\rm red},B_{\rm red},C_{\rm red},D_{\rm red},\mathbf{\Delta})$. For a fixed $\Delta_{\rm red}\in\mathbf{\Delta}$, the input-output map of $\mathcal{G}_{{\rm red},\delta}$ is given by

$$G_{{\rm red},\delta}=C_{\rm red}(I-\Delta_{\rm red}SA_{\rm red})^{-1}\Delta_{\rm red}SB_{\rm red}+D_{\rm red}.$$

We do not distinguish between the shift operators for ℓ_2 (resp. ℓ, ℓ_{2e}) with various associated dimensions. The interconnection structure of $\mathcal{G}_{{\rm red},\delta}$ is the same as the interconnection structure of $\mathcal{G}_\delta$, with the spatial states having smaller or equal dimensions. Moreover, the subsystems in $\mathcal{G}_{{\rm red},\delta}$ have NSLPV models formulated in an LFT framework, where the feedback operator $\Delta_{\rm red}$ has the same structure as Δ with the feedback channels having smaller or equal dimensions.

The method permits the removal of a whole interconnection if all its associated entries in Σ are deemed negligible. For example, if $m_{i_0}^S(t,k_0)=0$, for all $t\in\mathbb{Z}$, then the interconnection $(\rho_{i_0}^{-1}(k_0),k_0)$ is altogether removed from the graph of the reduced order system. Similarly, if $m_{j_0}^P(t,k_0)=0$, for all $t\in\mathbb{Z}$, then $\hat{\Delta}_{j_0}(t,k_0)$ is removed from the $\Delta_{\rm red}$ operator.

Lemma 4: The reduced order system $\mathcal{G}_{{\rm red},\delta}$ is strongly stable, and its given realization is balanced.

Proof: There exists a unique operator Q such that $Q^*\Sigma Q=\mathrm{diag}(\Gamma,\Omega)$. It is not difficult to verify that Q also satisfies

$$Q^*SAQ{=}\begin{bmatrix}S&0\\0&S\end{bmatrix}\begin{bmatrix}A_{\rm red}&\bar{A}_{12}\\ \bar{A}_{21}&\bar{A}_{22}\end{bmatrix}{=}\tilde{S}\bar{\bar{A}},\;\; CQ{=}\begin{bmatrix}C_{\rm red}&\bar{C}_2\end{bmatrix}{=}\bar{\bar{C}},$$

$$Q^*SB{=}\begin{bmatrix}S&0\\0&S\end{bmatrix}\begin{bmatrix}B_{\rm red}\\ \bar{B}_2\end{bmatrix}{=}\tilde{S}\bar{\bar{B}},\;\; Q^*\Delta Q{=}\mathrm{diag}(\Delta_{\rm red},\bar{\Delta}_2){=}\bar{\bar{\Delta}},$$

where $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{C}_2$, $\bar{B}_2$, and $\bar{\Delta}_2$ are appropriately defined partitioned graph-diagonal operators. Namely, the operator Q is applied to the operators of $\mathcal{G}_\delta$ in order to group together the non-truncated blocks.

Since Σ satisfies (5) and (6), then using the operator Q it is not difficult to show that the following hold:

$$\bar{\bar{A}}\mathrm{diag}(\Gamma,\Omega)\bar{\bar{A}}^*-\tilde{S}^*\mathrm{diag}(\Gamma,\Omega)\tilde{S}+\bar{\bar{B}}\bar{\bar{B}}^*\prec 0, \tag{7}$$

$$\bar{\bar{A}}^*\tilde{S}^*\mathrm{diag}(\Gamma,\Omega)\tilde{S}\bar{\bar{A}}-\mathrm{diag}(\Gamma,\Omega)+\bar{\bar{C}}^*\bar{\bar{C}}\prec 0. \tag{8}$$

From the $(1,1)$-terms in (7) and (8), we get

$$A_{\rm red}\Gamma A_{\rm red}^*-S^*\Gamma S+B_{\rm red}B_{\rm red}^*\prec 0,$$
$$A_{\rm red}^*S^*\Gamma SA_{\rm red}-\Gamma+C_{\rm red}^*C_{\rm red}\prec 0.$$

That is, the reduced order system $\mathcal{G}_{{\rm red},\delta}$ is strongly stable, and the realization $(A_{\rm red},B_{\rm red},C_{\rm red},D_{\rm red},\mathbf{\Delta})$ is balanced with balanced generalized gramian Γ. ■

We now derive an upper bound on the ℓ_2-induced norm of the error system resulting from balanced truncation.

Theorem 1: If $\Omega=I$, i.e., for all $i=1,\ldots,d$, $j=1,\ldots,r$, and $(t,k)\in\mathbb{Z}\times V$, $\Omega_T(t,k)=I$, $\Omega_i^S(t,k)=I$, and $\Omega_j^P(t,k)=I$, then the reduced order system $\mathcal{G}_{{\rm red},\delta}$ satisfies $\|(G_\delta-G_{{\rm red},\delta})\|<2$, for all $\Delta\in\mathbf{\Delta}$.

Proof: Since $\mathcal{G}_\delta$ and $\mathcal{G}_{{\rm red},\delta}$ are strongly stable systems, then so is the error system $\mathcal{E}_\delta=\{\frac{1}{2}(G_\delta-G_{{\rm red},\delta}):\Delta\in\mathbf{\Delta}\}$. Notice that $\frac{1}{2}(G_\delta-G_{{\rm red},\delta})=$

$$\begin{bmatrix}\Delta_{\rm red}&0\\0&\bar{\bar{\Delta}}\end{bmatrix}\star\left[\begin{array}{cc|c}SA_{\rm red}&0&\frac{1}{\sqrt{2}}SB_{\rm red}\\0&\tilde{S}\bar{\bar{A}}&\frac{1}{\sqrt{2}}\tilde{S}\bar{\bar{B}}\\ \hline -\frac{1}{\sqrt{2}}C_{\rm red}&\frac{1}{\sqrt{2}}\bar{\bar{C}}&0\end{array}\right].$$

We want to construct an operator $\mathcal{V}\succ 0$ that commutes with every $\mathrm{diag}(\Delta_{\rm red},\bar{\bar{\Delta}})$ and satisfies (4) for the given realization of the error system $\mathcal{E}_\delta$.

By direct application of the Schur Complement Formula twice to (7) and (8), we can see that

$$\begin{bmatrix}-R_1&K^*\\K&-S_a^*R_2S_a\end{bmatrix}\prec 0,$$

where $S_a=\mathrm{diag}(\tilde{S},I,\tilde{S})$, $R_i=\mathrm{diag}(\Gamma^{-1},\Omega^{-1},I^{q_i},\Gamma,\Omega)$, $K=\begin{bmatrix}0&0&\bar{\bar{A}}\\0&0&\bar{\bar{C}}\\ \bar{\bar{A}}&\bar{\bar{B}}&0\end{bmatrix}$, with $q_1=n_u$ and $q_2=n_y$. We then define the operators L and P, respectively, as

$$L=\frac{1}{\sqrt{2}}\begin{bmatrix}-I&0&0&I&0\\I&0&0&I&0\\0&I&0&0&I\\0&0&\sqrt{2}I^{n_y}&0&0\\0&-I&0&0&I\end{bmatrix},$$

$$P=\frac{1}{\sqrt{2}}\begin{bmatrix}I&I&0&0&0\\0&0&I&0&I\\0&0&0&\sqrt{2}I^{n_u}&0\\-I&I&0&0&0\\0&0&I&0&-I\end{bmatrix}.$$

We pre- and post- multiply the previous inequality by $\mathrm{diag}(P^*,L)$ and $\mathrm{diag}(P,L^*)$, respectively, to obtain

$$\begin{bmatrix}-P^*R_1P&P^*K^*L^*\\LKP&-LS_a^*R_2S_aL^*\end{bmatrix}\prec 0, \tag{9}$$

where $LKP = \left[\begin{array}{c|c} M & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] =$

$$\left[\begin{array}{cccc|c} A_{\text{red}} & 0 & 0 & \frac{1}{\sqrt{2}}B_{\text{red}} & \bar{A}_{12} \\ 0 & A_{\text{red}} & \bar{A}_{12} & \frac{1}{\sqrt{2}}B_{\text{red}} & 0 \\ 0 & \bar{A}_{21} & \bar{A}_{22} & \frac{1}{\sqrt{2}}\bar{B}_2 & 0 \\ \frac{-1}{\sqrt{2}}C_{\text{red}} & \frac{1}{\sqrt{2}}C_{\text{red}} & \frac{1}{\sqrt{2}}\bar{C}_2 & 0 & \frac{-1}{\sqrt{2}}\bar{C}_2 \\ \hline \bar{A}_{21} & 0 & 0 & \frac{1}{\sqrt{2}}\bar{B}_2 & \bar{A}_{22} \end{array}\right].$$

Recalling that $\Omega = I$, one can see that

$$P^* R_1 P = \text{diag}\left(\frac{1}{2}\begin{bmatrix}(\Gamma^{-1}+\Gamma) & (\Gamma^{-1}-\Gamma) \\ (\Gamma^{-1}-\Gamma) & (\Gamma^{-1}+\Gamma)\end{bmatrix}, \text{diag}\left(I, I^{n_u}, I\right)\right).$$

Similarly, since $\Omega = I$, then $LS_a^* R_2 S_a L^*$ is equal to

$$\text{diag}\left(\frac{1}{2}\begin{bmatrix}S^*(\Gamma^{-1}+\Gamma)S & S^*(\Gamma-\Gamma^{-1})S \\ S^*(\Gamma-\Gamma^{-1})S & S^*(\Gamma^{-1}+\Gamma)S\end{bmatrix}, \text{diag}\left(I, I^{n_y}, I\right)\right).$$

If we define the desired operator $\mathcal{V}$ as

$$\mathcal{V} = \frac{1}{2}\begin{bmatrix}(\Gamma^{-1}+\Gamma) & (\Gamma^{-1}-\Gamma) & 0 \\ (\Gamma^{-1}-\Gamma) & (\Gamma^{-1}+\Gamma) & 0 \\ 0 & 0 & 2I\end{bmatrix} \succ 0,$$

then, from (9), we can see that

$$\begin{bmatrix} -\begin{bmatrix}\mathcal{V} & 0 \\ 0 & I^{n_u}\end{bmatrix} & M^* \\ M & -\begin{bmatrix}\begin{bmatrix}S & 0 \\ 0 & \tilde{S}\end{bmatrix}^* \mathcal{V}^{-1}\begin{bmatrix}S & 0 \\ 0 & \tilde{S}\end{bmatrix} & 0 \\ 0 & I^{n_y}\end{bmatrix}\end{bmatrix} \prec 0.$$

We can also easily verify that $\mathcal{V}$ commutes with every $\text{diag}(\Delta_{\text{red}}, \bar{\bar{\Delta}})$. Then, invoking Lemma 2 with $\gamma = 1$, we conclude that $\|\frac{1}{2}(G_\delta - G_{\text{red},\delta})\| < 1$, i.e., $\|(G_\delta - G_{\text{red},\delta})\| < 2$, for all $\Delta \in \boldsymbol{\Delta}$. ■

The error bound for the case of a general Ω is given next. The theorem follows by scaling and repeated application of Theorem 1. Lemma 4 ensures that we can apply BT to the resulting intermediate reduced order systems.

Theorem 2: The reduced order system $\mathcal{G}_{\text{red},\delta}$ satisfies $\|(G_\delta - G_{\text{red},\delta})\| < 2\Phi(\Omega)$, for all $\Delta \in \boldsymbol{\Delta}$.

The symbol $\Phi(\Omega)$ is defined at the end of Section II. The bound in Theorem 2 may become infinite when there are infinitely many distinct entries in Ω. However, if the graph is finite, i.e., V and E are finite sets, and if the subsystems are (h, q)-eventually time-periodic for some integers $h \geq 0$ and $q > 0$, i.e., the state-space matrices satisfy

$$[\![Z]\!](t + h + zq, k) = [\![Z]\!](t + h, k), \quad Z \in \{A, B, C, D\},$$

for all $t, z \in \mathbb{N}_0$ and $k \in V$, then the given bound is guaranteed to be finite. This is because, if the subsystems are (h, q)-eventually time-periodic, then, there exists a balanced generalized gramian Σ if and only if there exists an (h, q)-eventually time-periodic balanced generalized gramian Σ_{eper}; see the averaging techniques of [15], [16]. Thus, when evaluating $\Phi(\Omega_{\text{eper}})$, we restrict t to the finite time-horizon h and the first time-period truncation, i.e., $0 \leq t \leq h+q-1$. If in addition, the graph is finite, then $\Phi(\Omega_{\text{eper}})$ is guaranteed to be finite. Finally, we note that finite time-horizon subsystems, standard LPV subsystems, and time-periodic subsystems are special cases of eventually time-periodic subsystems.

V. Conclusion

BT is extended to the class of interconnected NSLPV subsystems modeled in an LFT framework. The proposed method exhibits the characteristics of BT for standard state-space systems, and allows for preserving and simplifying the interconnection structure and the uncertainty structure. However, the method suffers from the conservatism introduced by imposing a block-diagonal structure on the generalized gramians. Future research will focus on generalizing the CFR method to the class of systems treated here, as CFR presents a partial solution to the conservatism of BT. In addition, we plan to work on identifying classes of systems with guaranteed structured generalized gramians.

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