Distributed Control of Systems with Uncertain Initial Conditions

Devaprakash Muniraj and Mazen Farhood

***Abstract*— This paper focuses on the control of distributed systems with uncertain initial conditions, where the constituent subsystems are interconnected over directed graphs and represented by discrete-time, linear time-varying models. Specifically, we consider distributed systems where the individual subsystems are eventually time-periodic, by which we mean that the state-space matrices of the subsystems are aperiodic for an initial amount of time and then become time-periodic afterwards. The information transfer between the subsystems is subjected to a delay of one sampling period. Independent norm constraints are placed on the disturbance input and the uncertain initial state. We present convex synthesis conditions for control design in this setting, employing a square ℓ_2 induced norm as the performance measure. The synthesis conditions become finite-dimensional when the underlying graph has a finite set of vertices. An illustrative example on formation control of fixed-wing unmanned aircraft systems (UAS) is also provided.**

I. INTRODUCTION

In this paper, we deal with distributed systems where the individual subsystems are modeled as discrete-time, linear time-varying systems. Each subsystem has sensing and actuation capabilities, and the subsystems are interconnected over arbitrary directed graphs. The states of the individual subsystems are referred to as *temporal states* and the interconnections between the subsystems are modeled as states, which we call the *spatial states*. The focus of this paper is on the control of distributed systems with uncertain initial conditions, meaning that the temporal and spatial states are allowed to have non-zero values at time $t = 0$.

Specifically, the distributed systems of interest belong to the class of *eventually time-periodic* (ETP) distributed systems, where the state-space matrices of the constituent subsystems are aperiodic for an initial amount of time and then become time-periodic afterwards. Data transfer between the subsystems is subjected to a delay of one sampling period. The uncertain initial state is constrained to lie in a norm ball of some radius and the exogenous disturbance inputs are required to satisfy an independent norm constraint. The control design problem involves developing a distributed controller with the same interconnection structure as the nominal plant, which renders the closed-loop system stable and guarantees some performance criterion provided in terms of a square ℓ_2 induced norm performance measure [1], [2].

The primary motivation for this work is controlling the transient response of a multi-mission network of agents. It is conceivable that a multi-agent system would be required to switch missions, which may be triggered by environmental factors, direct commands, or situational awareness. During the switching process, the multi-agent system experiences a transient behavior which could adversely affect the mission performance. At the time of switching, the soon-to-be-active distributed controller has to deal with a bounded, uncertain initial (error) state of the network, which lies in some ellipsoid and has to do with the previous mission operation. The proposed approach incorporates this uncertainty in the initial state into the control design process to design a distributed controller, which would successfully recover the system from this initial state and ultimately force the network to exhibit the new desired behavior. This work has application in consensus problems under switching topologies [3], [4], and formation control involving concatenated trajectories, which was studied in [5] for the case of a single UAS.

The approach employs the operator theoretic machinery for distributed systems developed in [6] and builds upon the work in [7] on control of linear time-varying systems with uncertain initial conditions. The outline of the paper is as follows. We gather relevant notations in Section II and formulate the uncertain initial condition problem for distributed systems in Section III. Section IV presents synthesis results for control design in this setting. We conclude in Section V with an illustrative example on formation control of a fixed-wing UAS network.

II. PRELIMINARIES

We denote the set of integers by $\mathbb{Z}$ and the set of nonnegative integers by $\mathbb{N}_0$. If M_i is a sequence of matrices, then $\text{diag}(M_i)$ denotes their block-diagonal augmentation. We use I_n to denote an $n \times n$ identity matrix. Given Hilbert spaces H, F and W, $H \oplus W$ refers to the Hilbert space direct sum of H and W. The space of linear bounded operators mapping H to F is denoted by $\mathcal{L}(H, F)$; when H and F are equal, we shorten this to $\mathcal{L}(H)$. Given X in $\mathcal{L}(H, F)$, the H to F induced norm of X is denoted by $\|X\|_{H \to F}$. We use $X \prec 0$ to mean that a self-adjoint operator $X \in \mathcal{L}(H)$ is negative definite, that is, for all nonzero $h \in H$, there exists a positive scalar α such that $\langle h, Xh \rangle < -\alpha \|h\|^2$.

Given an integer sequence $n(t, k)$ mapping $\mathbb{Z} \times V$ to $\mathbb{N}_0$, we define $\ell(\{\mathbb{R}^{n(t,k)}\})$ to be the vector space of mappings $w\colon (t, k) \in \mathbb{Z} \times V \mapsto w(t, k) \in \mathbb{R}^{n(t,k)}$. We denote by $\ell_2(\{\mathbb{R}^{n(t,k)}\})$ the subspace of $\ell(\{\mathbb{R}^{n(t,k)}\})$ which is a Hilbert space under the inner product $\langle w, v \rangle := \sum_{(t,k)} w(t, k)^T v(t, k) < \infty$. When the dimensions are clear from context, we abbreviate the notations to simply ℓ and ℓ_2.

We denote a directed graph with set of vertices V and set of directed edges E by $\mathcal{G}(V, E)$. For a vertex $k \in V$, the vertex degree, denoted by $v(k)$, is given by $v(k)$=max$\{m(k), p(k)\}$, where $m(k)$ and $p(k)$ denote the

The authors are with the Kevin T. Crofton Department of Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, VA 24061, USA. Email: (devapm@vt.edu, farhood@vt.edu)

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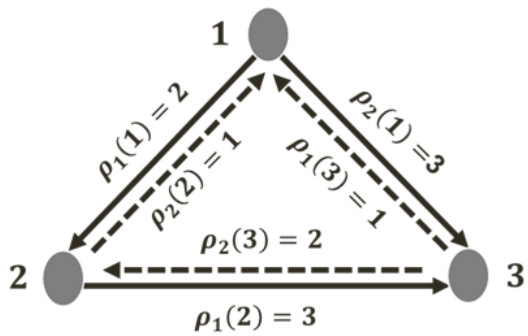

Fig. 1: A 2-regular directed graph with dashed arrows representing the virtual edges

indegree and outdegree of k, respectively. The maximum over the vertex degrees of the graph is denoted by $s(\mathcal{G})$. We call a directed graph d-regular if $m(k){=}p(k){=}d$ for each $k \in V$. A directed graph $\mathcal{G}(V, E)$ can always be transformed into an $s(\mathcal{G})$-regular directed graph by adding, if necessary, virtual edges and/or virtual nodes. We then define $s(\mathcal{G})$, or simply s, permutations $\rho_1, \rho_2, \ldots, \rho_s$ of the set of vertices V. The permutations satisfy the property that if $(i, j) \in E$, then there exists r such that $\rho_r(i) = j$ and $\rho_r^{-1}(j) = i$. For example, the directed graph shown in Fig. 1 is converted to a 2-regular directed graph by adding virtual edges which are represented in the figure by dashed arrows. Also, the two permutations ρ_1 and ρ_2 are defined as follows: $\rho_1(1) = 2$, $\rho_1(2) = 3$, $\rho_1(3) = 1$, $\rho_2(1) = 3$, $\rho_2(3) = 2$ and $\rho_2(2) = 1$. In this work, without any loss of generality, we assume that the underlying graph structure is d-regular. Any virtual edges, for instance, added to ensure graph regularity will correspond to spatial states with zero dimensions, and the developed results will still apply. The advantage of working with a d-regular directed graph is that it enables us to express the state-space equations of the distributed system in a compact operator form as will be seen in Section III.

Given integer sequences $r(t, k)$ and $m(t, k)$ mapping $\mathbb{Z}{\times}V$ to $\mathbb{N}_0$, an operator $X{\in}\mathcal{L}(\ell_2(\{\mathbb{R}^{r(t,k)}\}), \ell_2(\{\mathbb{R}^{m(t,k)}\}))$ is called a graph-diagonal operator if there exists a uniformly bounded sequence of matrices $X(t, k){\in}\mathbb{R}^{m(t,k)\times r(t,k)}$ such that, for all $(t, k){\in}\mathbb{Z}{\times}V$ and $w{\in}\ell_2(\{\mathbb{R}^{r(t,k)}\})$, the equality $(Xw)(t, k){=}X(t, k)w(t, k)$ holds. An operator $W{=}[W_{ij}]$ is said to be a partitioned graph-diagonal operator if each constituent block W_{ij} is graph-diagonal. We define the graph-diagonal realization of the partitioned graph-diagonal operator W as $([\![W]\!]x)(t, k){=}\,[W_{ij}(t, k)]x(t, k)$. $[\![\cdot]\!]$ is a homomorphism from the space of partitioned graph-diagonal operators to that of graph-diagonal operators, which is isometric and preserves products, addition, and ordering.

We define the following unitary shift operators mapping ℓ to ℓ, $(S_0w)(t, k){=}w(t{-}1, k)$ and $(S_jw)(t, k){=}w(t, \rho_j^{-1}(k))$, for $j = 1, \ldots, s$. If Q is a graph-diagonal operator, then $S_i^*QS_i$ is also graph-diagonal, namely,

$(S_0^*QS_0)(t, k){=}Q(t{+}1, k)$ and $(S_j^*QS_j)(t, k){=}Q(t, \rho_j(k))$,

for $j = 1, \ldots, s$. The composite shift operator S is defined as

$$S := \mathrm{diag}(S_0, S_0S_1, \ldots, S_0S_s).$$

III. PROBLEM FORMULATION

A. Operator-theoretic framework

We consider a discrete-time distributed system G, whose interconnection structure is described by a directed graph with a finite set of vertices. The vertices of the graph correspond to the individual subsystems, $G^{(k)}$, and the directed edges describe the interconnections between the subsystems. We assume a delay of one sampling period in the information transfer between the subsystems, meaning that the information sent by a subsystem at the current time step reaches the target subsystem at the next time step. The state space equations of the subsystem $G^{(k)}$ can be written as

$$\begin{bmatrix} x_0(t+1,k) \\ x_1(t+1,\rho_1(k)) \\ \vdots \\ x_s(t+1,\rho_s(k)) \end{bmatrix} = \bar{A}(t,k) \begin{bmatrix} x_0(t,k) \\ x_1(t,k) \\ \vdots \\ x_s(t,k) \end{bmatrix} + \bar{B}(t,k) \begin{bmatrix} w(t,k) \\ u(t,k) \end{bmatrix},$$

$$\begin{bmatrix} z(t,k) \\ y(t,k) \end{bmatrix} = \begin{bmatrix} \bar{C}_1(t,k) \\ \bar{C}_2(t,k) \end{bmatrix} \begin{bmatrix} x_0(t,k) \\ x_1(t,k) \\ \vdots \\ x_s(t,k) \end{bmatrix} + \bar{D}(t,k) \begin{bmatrix} w(t,k) \\ u(t,k) \end{bmatrix}, \quad (1)$$

where $\bar{B}(t, k) = \begin{bmatrix} \bar{B}_1(t, k) & \bar{B}_2(t, k) \end{bmatrix}$ and

$$\bar{D}(t,k) = \begin{bmatrix} D_{11}(t,k) & D_{12}(t,k) \\ D_{21}(t,k) & D_{22}(t,k) \end{bmatrix}, \text{ for all } t \in \mathbb{Z} \text{ and } k \in V.$$

The state vector in (1) is denoted by $x(t, k)$, which is partitioned into $(s + 1)$ separate vector-valued channels, namely, $x(t, k) = (x_0(t, k), \ldots, x_s(t, k))$. $x_0(t, k)$ denotes the state vector of the subsystem $G^{(k)}$ and is called the temporal state. The vector $x_i(t, k)$, which we call the i^{th} spatial state, is associated with the interconnection $(\rho_i^{-1}(k), k)$ at time t. The spatial states representing the virtual interconnections, which are added to make the graph structure regular, have zero dimensions. For instance, the spatial states $x_1(t, 1)$, $x_2(t, 1)$, and $x_2(t, 2)$ of the distributed system with an underlying graph structure as in Fig. 1 have zero dimensions. The vectors $w(t, k)$, $u(t, k)$, $z(t, k)$, and $y(t, k)$ denote the exogenous disturbance, control input, performance output and the measurements, respectively. The vectors $x_i(t, k)$, $w(t, k)$, $u(t, k)$, $z(t, k)$, and $y(t, k)$ are real and have time-varying dimensions which we denote by $n_i(t, k)$, $n_w(t, k)$, $n_u(t, k)$, $n_z(t, k)$, and $n_y(t, k)$, respectively. We define $n(t, k){=}\sum_{i=0}^{s} n_i(t, k)$. It is noted that the system matrices are also partitioned according to the partitions in $x(t, k)$. In the sequel, we assume $D_{22}(t, k){=}0$ for all $(t, k){\in}\mathbb{Z}{\times}V$. The matrix sequences $A_{ij}(t, k)$, $B_{i1}(t, k)$, $B_{i2}(t, k)$, $C_{1j}(t, k)$, $C_{2j}(t, k)$, $D_{11}(t, k)$, $D_{12}(t, k)$, and $D_{21}(t, k)$, for $i, j = 0, \ldots, s$, define the graph-diagonal operators A_{ij}, B_{i1}, B_{i2}, C_{1j}, C_{2j}, D_{11}, D_{12}, and D_{21}, respectively. These graph-diagonal operators are then used to form partitioned graph-diagonal operators, A, B_i, and C_i, for $i = 1, 2$, given by

$$A = \begin{bmatrix} A_{00} & \cdots & A_{0s} \\ \vdots & \ddots & \vdots \\ A_{s0} & \cdots & A_{ss} \end{bmatrix}, \; B_i = \begin{bmatrix} B_{0i} \\ \vdots \\ B_{si} \end{bmatrix} \text{ and } C_i = \begin{bmatrix} C_{i0}^* \\ \vdots \\ C_{is}^* \end{bmatrix}^*.$$

Then, the system equations (1) of G can be equivalently expressed in the following compact operator form:

$$\begin{bmatrix} x \\ z \\ y \end{bmatrix} = \begin{bmatrix} SA & SB_1 & SB_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \qquad (2)$$

where S is the composite shift operator defined earlier. The matrix sequences used in (1) satisfy $[\![A]\!](t,k)=\bar{A}(t,k)$, $[\![B_1]\!](t,k)=\bar{B}_1(t,k)$, $[\![C_1]\!](t,k)=\bar{C}_1(t,k)$, and so on.

We develop our results for a special class of distributed systems, namely, the class of eventually time-periodic (ETP) distributed systems, where the subsystems are assumed to be eventually periodic. Given integers $h_k \geq 0$ and $q_k \geq 1$, we say the subsystem $G^{(k)}$ is (h_k, q_k)-eventually periodic if its state-space matrices are aperiodic over an initial finite horizon of length h_k and then become periodic afterwards with periodicity q_k. For example, the matrix sequence $A_{ij}(t,k)$ satisfies

$$A_{ij}(t + h_k + mq_k, k) = A_{ij}(t + h_k, k), \text{ for } t, m \in \mathbb{N}_0.$$

If the subsystems $G^{(k)}$ are (h_k, q_k)-eventually periodic, then the distributed system G is (h, q)-ETP with $h = \max_k h_k$ and q being the least common multiple of the integers q_k.

This paper concerns the control synthesis problem for system G, where $x(0,k) \neq 0$ for some $k \in V$, $x(0,k)$ being the state vector of the subsystem $G^{(k)}$ at time $t = 0$. This uncertain initial state is modeled as an exogenous disturbance acting on the system at time $t = -1$. The matrix sequences defining the system operators of G and the exogenous disturbances for $t < 0$ are given as follows:

$$\begin{aligned} &\bar{A}(t,k) = 0,\ \bar{B}_2(t,k) = 0,\ \bar{D}_{12}(t,k) = 0,\ \bar{D}_{i1}(t,k) = 0, \\ &\bar{C}_i(t,k) = 0, \text{ for } t < 0,\ k \in V \text{ and } i = 1, 2, \\ &\bar{B}_1(t,k) = \begin{cases} 0 & \text{for } t < -1,\ k \in V, \\ I_{n(0,k)} & \text{for } t = -1,\ k \in V, \end{cases} \qquad (3) \\ &w(t,k) = \begin{cases} 0 & \text{for } t < -1,\ k \in V, \\ x(0,k) & \text{for } t = -1,\ k \in V. \end{cases} \end{aligned}$$

The preceding formulation allows the state vector at $t=0$ to have non-zero values for some $k \in V$. We note that the distributed system G with the uncertain initial state, $x(0,k)$, is well-posed since $\bar{A}(t,k)=0$ for all $t < 0$ and $k \in V$. For a detailed discussion on well-posedness and stability, see [6].

B. Isomorphic distributed system with a zero initial state

The synthesis results in [6], which form the basis for this work, are developed for distributed linear time-varying systems with zero initial conditions, i.e. $x(0,k)=0$ for all $k \in V$. Therefore, it is helpful to construct from G an $(h+1, q)$-ETP distributed system $\tilde{G}$ having a zero initial state. We denote the state space operators of $\tilde{G}$ by $\tilde{A}$, $\tilde{B}_i$, $\tilde{C}_i$, $\tilde{D}_{i1}$, and $\tilde{D}_{12}$ for $i=1,2$. The state-space equations of $\tilde{G}$ are written in operator form as

$$\begin{bmatrix} \tilde{x} \\ \tilde{z} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} S\tilde{A} & S\tilde{B}_1 & S\tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{w} \\ \tilde{u} \end{bmatrix}. \qquad (4)$$

The matrix sequences that define the state-space operators and the inputs of $\tilde{G}$ are related to (3) as follows:

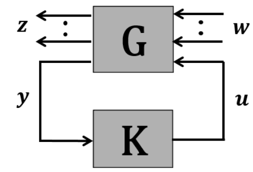

Fig. 2: Closed-loop system with partitions in the exogenous input and performance output channels

$$\begin{aligned} &[\![\tilde{A}]\!](t,k) = \bar{A}(t-1,k),\ [\![\tilde{B}_i]\!](t,k) = \bar{B}_i(t-1,k), \\ &[\![\tilde{C}_i]\!](t,k)=\bar{C}_i(t-1,k),\ [\![\tilde{D}_{12}]\!](t,k)=D_{12}(t-1,k), \qquad (5) \\ &[\![\tilde{D}_{i1}]\!](t,k) = D_{i1}(t-1,k),\ \tilde{w}(t,k) = w(t-1,k), \text{ and} \\ &\tilde{u}(t,k) = u(t-1,k), \text{ for } t \in \mathbb{Z},\ k \in V, \text{ and } i = 1, 2. \end{aligned}$$

The above transformation ensures that the system $\tilde{G}$ has a zero initial state, namely, $\tilde{x}(0,k) = 0$.

This paper extends the work of [7], which was developed for eventually periodic systems with uncertain initial conditions, to the class of ETP distributed systems with an uncertain initial state. The uncertain initial state $x(0,k)$ for $k \in V$ consists of the temporal and spatial states of all the subsystems at time $t = 0$. Therefore, in our controller synthesis, we have different possibilities of placing the norm constraints on the initial temporal and spatial states of the subsystems. The performance of the resulting controller, thereby, depends on the choice of combining the initial temporal and spatial states of the subsystems. In this paper, the norm constraints are placed on the combined uncertain spatial and temporal initial states of all the subsystems. Other possibilities include placing independent norm constraints on the uncertain spatial and temporal initial states of the distributed system, or considering the uncertain initial states of each of the subsystems separately and placing independent norm constraints on each of them.

IV. MAIN RESULTS

We first gather some definitions and results based on [2]. Consider the distributed system in Fig. 2, where the exogenous input signal w and the performance output z are partitioned into N_w and N_z channels, respectively.

Definition 1: Let F and T be partitioned graph-diagonal operators. Then, for a given γ, F and T are said to be γ-admissible scales if they are of the form

$$\begin{aligned} [\![F]\!](t,k) &= \text{diag}(f_1 I, \ldots, f_{N_w} I), \\ [\![T]\!](t,k) &= \text{diag}(t_1 I, \ldots, t_{N_z} I), \end{aligned}$$

where $f_i > 0$, $t_j > 0$, and $\sum_{i=1}^{N_w} f_i + \sum_{j=1}^{N_z} t_j < 2\gamma$.

This definition is an extension of its LTV counterpart in [2] to the class of distributed systems. For each partition in the input channel, w, we define an operator P_i, which projects onto the i^{th} vector-valued channel of w. Likewise, we define the operator Q_j, which projects onto the j^{th} vector-valued channel of z. Let the closed-loop system of Fig. 2 be denoted by M, where M is an operator mapping $\oplus_{i=1}^{N_w} \ell_2$ to $\oplus_{j=1}^{N_z} \ell_2$.

Definition 2: For the system depicted in Fig. 2, the square ℓ_2 induced norm of M is defined as

$$\|M\|_{sq} = \sup_{\|P_i w\| \leq 1} \sum_{j=1}^{N_z} \|Q_j M w\|.$$

Lemma 1 ([2]): For a given γ and M, $\|M\|_{sq} < \gamma$ if there exists γ-admissible scales F and T such that

$$\|T^{-\frac{1}{2}} M F^{-\frac{1}{2}}\|_{\ell_2 \to \ell_2} < 1.$$

Lemma 2 ([2]): Suppose that M satisfies $\|M\|_{sq} < \gamma$. If (a) M is time-periodic, or (b) the product $N_w.N_z \leq 2$, then there exists γ-admissible scales F and T such that $\|T^{-\frac{1}{2}} M F^{-\frac{1}{2}}\|_{\ell_2 \to \ell_2} < 1$.

We now state our synthesis objective.

Definition 3: A controller K, with zero initial state and having the same interconnection structure as the nominal plant G, is a γ-admissible synthesis for G if the closed-loop system M is stable and satisfies the performance inequality

$$\|M\|_{sq} = \sup_{\alpha \leq 1, \|w\| \leq 1} \|Mw\| < \gamma,$$

where $\alpha = (\sum_{k \in V} x^T(0,k) x(0,k))^{\frac{1}{2}}$.

Specifically, our goal is to develop a distributed controller K composed of (N,q)-ETP subcontrollers, where $N \geq h$, and having the same interconnection structure as the nominal plant G. As reasoned earlier, it is convenient to work with the $(h+1,q)$-ETP distributed system $\tilde{G}$ which has a zero initial state. We now rephrase the synthesis objective for the distributed system, $\tilde{G}$. Namely, a feedback controller $\tilde{K}$ is a γ-admissible synthesis for $\tilde{G}$ if the closed-loop system, denoted by $\tilde{M}$, is stable and further satisfies the performance criterion $\|\tilde{M}\|_{sq} < \gamma$, where $\|\tilde{M}\|_{sq} = \sup_{\alpha \leq 1, \|w\| \leq 1} \|\tilde{M}\tilde{w}\|$.

The following theorem connects the square ℓ_2 induced norm measure to the standard ℓ_2 induced norm measure.

Theorem 1: The closed-loop performance inequality $\|\tilde{M}\|_{sq} < \gamma$ holds, for some γ, if and only if there exist positive scalars t_1, f_1, f_2, and associated graph-diagonal operators $\tilde{T}$ and $\tilde{F}$, defined by the matrix sequences

$$\tilde{T}(t,k) = t_1 I, \quad \text{for } t \in \mathbb{Z}, \ k \in V,$$

$$\tilde{F}(t,k) = \begin{cases} f_1 I & \text{for } t = 0, \ k \in V, \\ f_2 I & \text{for } t \neq 0, \ k \in V, \end{cases}$$

such that $t_1 + f_1 + f_2 < 2\gamma$ and $\|\tilde{T}^{-\frac{1}{2}} \tilde{M} \tilde{F}^{-\frac{1}{2}}\|_{\ell_2 \to \ell_2} < 1$.

Proof: The proof makes use of Lemmas 1 and 2. To have independent norm constraints on the uncertain initial state and the disturbance input, we equivalently reformulate the distributed system $\tilde{G}$ into a distributed system with two exogenous input channels (N_w=2) and one performance output channel (N_z=1). The exogenous input channels consist of the uncertain initial state channel and the disturbance input channel. The input channel corresponding to the uncertain initial state is relevant only at time t=0, where it has a value $x(0,k)$; at all other times, this channel has zero dimensions. On the other hand, the disturbance input channel has zero dimensions at t=0, and at all other times, it has a value of $w(t,k)$. Whenever a particular channel is irrelevant, that is, has zero dimensions, the matrix blocks corresponding to that input channel will also have conformable zero dimensions. Then, the proof of the "if" direction follows immediately from Lemma 1. Since N_w=2 and N_z=1, Lemma 2 can be invoked to prove the "only if" direction. ■

Before we state the main result, let us define some important sets. Let $\mathcal{P}$ denote the set of partitioned graph-diagonal operators. Given a partitioned graph-diagonal operator X with the following structure X=diag$(X_0, \ldots, X_s)$, where X_i is graph-diagonal, then $S^* X S$ is in $\mathcal{P}$. We now define the subset $\mathcal{X}$ of $\mathcal{P}$ as

$$\mathcal{X} = \{X \in \mathcal{P} : X = \text{diag}(X_0, \ldots, X_s) = X^*, \\ X^{-1} \in \mathcal{L}(\oplus_{j=0}^{s} \ell_2), \text{ and } X_j \succ 0 \text{ for } j = 0, \ldots, s\}.$$

Theorem 2: Suppose G is an (h,q)-ETP distributed system with an uncertain initial state. Then, there exists an (N,q)-ETP γ-admissible synthesis K, for some γ, if there exist positive scalars t_1, f_1, f_2, p, r, and positive definite matrices $X_i(t,k)$ and $Y_i(t,k)$ satisfying the LMIs

$$t_1 + f_1 + f_2 < 2\gamma, \quad [\![X]\!](0,k) \prec f_1 I, \tag{6}$$

$$\left[\!\!\left[N_X^* \left\{ L^* \begin{bmatrix} S^* X S & 0 \\ 0 & R \end{bmatrix} L - \begin{bmatrix} X & 0 \\ 0 & F \end{bmatrix} \right\} N_X \right]\!\!\right](t,k) \prec 0, \tag{7}$$

$$\left[\!\!\left[N_Y^* \left\{ L \begin{bmatrix} Y & 0 \\ 0 & P \end{bmatrix} L^* - \begin{bmatrix} S^* Y S & 0 \\ 0 & T \end{bmatrix} \right\} N_Y \right]\!\!\right](t,k) \prec 0, \tag{8}$$

$$\left[\!\!\left[\begin{matrix} X_i & I \\ I & Y_i \end{matrix} \right]\!\!\right](t,k) \succeq 0, \ \begin{bmatrix} p & 1 \\ 1 & f_2 \end{bmatrix} \succeq 0, \ \begin{bmatrix} r & 1 \\ 1 & t_1 \end{bmatrix} \succeq 0, \tag{9}$$

for $t = 0, 1, \ldots, N+q-1$, $k \in V$, and $i = 0, \ldots, s$, where

$$\text{Im}\, N_Y = \ker \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}, \quad \text{Im}\, N_X = \ker \begin{bmatrix} C_2 & D_{21} \end{bmatrix},$$

and $L = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$. The matrix sequences defining the graph-diagonal operators T, F, P, and R are given by $T(t,k)$=$t_1 I$, $F(t,k)$=$f_2 I$, $P(t,k)$=pI, $R(t,k)$=rI, and $X, Y \in \mathcal{X}$ are (N,q)-ETP, namely, $X_i(N+q,k)$=$X_i(N,k)$ and $Y_i(N+q,k)$=$Y_i(N,k)$.

Proof: Consider the $(h+1,q)$-ETP distributed system $\tilde{G}$ defined in (4). The existence of a γ-admissible $(N+1,q)$-ETP controller $\tilde{K}$ for $\tilde{G}$ is equivalent to the existence of a γ-admissible (N,q)-ETP controller K for the (h,q)-ETP distributed system G defined in (3). By Theorem 1, a γ-admissible $(N+1,q)$-ETP synthesis $\tilde{K}$ for $\tilde{G}$ renders the closed-loop system $\tilde{M}$ stable and achieves the performance inequality $\|\tilde{T}^{-\frac{1}{2}} \tilde{M} \tilde{F}^{-\frac{1}{2}}\|_{\ell_2 \to \ell_2} < 1$ for some graph-diagonal operators $\tilde{T}$ and $\tilde{F}$, as defined in Theorem 1, with positive scalars t_1, f_1, and f_2 satisfying $t_1 + f_1 + f_2 < 2\gamma$.
Now, combining the above two statements we can say that a γ-admissible synthesis K for the distributed system G exists if and only if a 1-admissible synthesis $\tilde{K}$ exists for the scaled $(h+1,q)$-ETP system $\tilde{G}^s$ which has the following representation:

$$\tilde{G}^s = \begin{bmatrix} \tilde{A}^s & \tilde{B}_1^s & \tilde{B}_2^s \\ \tilde{C}_1^s & \tilde{D}_{11}^s & \tilde{D}_{12}^s \\ \tilde{C}_2^s & \tilde{D}_{21}^s & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 \tilde{F}^{-\frac{1}{2}} & \tilde{B}_2 \\ \tilde{T}^{-\frac{1}{2}} \tilde{C}_1 & \tilde{T}^{-\frac{1}{2}} \tilde{D}_{11} \tilde{F}^{-\frac{1}{2}} & \tilde{T}^{-\frac{1}{2}} \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} \tilde{F}^{-\frac{1}{2}} & 0 \end{bmatrix},$$

where the operators $\tilde{A}$, $\tilde{B}_i$, $\tilde{C}_i$, $\tilde{D}_{1i}$, and $\tilde{D}_{21}$, for $i = 1, 2$, are defined in (5). Since the distributed system $\tilde{G}^s$ has a zero initial state, by combining Theorem 24 and Corollary 22 in [6], we can say that a distributed controller $\tilde{K}$ composed of $(N+1,q)$-ETP subcontrollers and having the same structure as $\tilde{G}^s$ exists if, for all t=$0, 1, \ldots, N+q$,

$k \in V$, and $i=0,1,\ldots,s$, there exist positive definite matrices $X_i^s(t,k)$ and $Y_i^s(t,k)$, with $X_i^s(N+q+1,k)=X_i^s(N+1,k)$ and $Y_i^s(N+q+1,k)=Y_i^s(N+1,k)$, satisfying the LMIs

$$\left[\!\!\left[\tilde{N}_X^{s^*}\left\{\tilde{L}^{s^*}\begin{bmatrix}S^*X^sS & 0\\ 0 & I\end{bmatrix}\tilde{L}^s - \begin{bmatrix}X^s & 0\\ 0 & I\end{bmatrix}\right\}\tilde{N}_X^s\right]\!\!\right](t,k) \prec 0, \quad (10)$$

$$\left[\!\!\left[\tilde{N}_Y^{s^*}\left\{\tilde{L}^{s}\begin{bmatrix}Y^s & 0\\ 0 & I\end{bmatrix}\tilde{L}^{s^*} - \begin{bmatrix}S^*Y^sS & 0\\ 0 & I\end{bmatrix}\right\}\tilde{N}_Y^s\right]\!\!\right](t,k) \prec 0, \quad (11)$$

$$\left[\!\!\left[\begin{matrix}X_i^s & I\\ I & Y_i^s\end{matrix}\right]\!\!\right](t,k) \succeq 0, \quad (12)$$

where $\tilde{L}^s = \begin{bmatrix}\tilde{A}^s & \tilde{B}_1^s\\ \tilde{C}_1^s & \tilde{D}_{11}^s\end{bmatrix}$, $\operatorname{Im}\tilde{N}_Y^s = \ker\begin{bmatrix}\tilde{B}_2^{s^*} & \tilde{D}_{12}^{s^*}\end{bmatrix}$, and $\operatorname{Im}\tilde{N}_X^s = \ker\begin{bmatrix}\tilde{C}_2^s & \tilde{D}_{21}^s\end{bmatrix}$. From inequality (11), it is not difficult to obtain

$$\left[\!\!\left[\left(\begin{bmatrix}I & 0\\ 0 & \tilde{T}^{-\frac{1}{2}}\end{bmatrix}\tilde{N}_Y^s\right)^*\left\{\begin{bmatrix}\tilde{A} & \tilde{B}_1\\ \tilde{C}_1 & \tilde{D}_{11}\end{bmatrix}\begin{bmatrix}Y^s & 0\\ 0 & \tilde{F}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{A} & \tilde{B}_1\\ \tilde{C}_1 & \tilde{D}_{11}\end{bmatrix}^* - \begin{bmatrix}S^*Y^sS & 0\\ 0 & \tilde{T}\end{bmatrix}\right\}\left(\begin{bmatrix}I & 0\\ 0 & \tilde{T}^{-\frac{1}{2}}\end{bmatrix}\tilde{N}_Y^s\right)\right]\!\!\right](t,k) \prec 0. \quad (13)$$

We note that

$$\operatorname{Im}\left(\begin{bmatrix}I & 0\\ 0 & \tilde{T}^{-\frac{1}{2}}\end{bmatrix}\tilde{N}_Y^s\right) = \operatorname{Im}\,\tilde{N}_Y = \ker\begin{bmatrix}\tilde{B}_2^* & \tilde{D}_{12}^*\end{bmatrix}.$$

Then, inequalities (13) and (10) reduce to

$$[\![\tilde{N}_Y^*]\!](t,k)\left\{[\![\tilde{L}]\!](t,k)\left[\!\!\left[\begin{matrix}Y^s & 0\\ 0 & \tilde{F}^{-1}\end{matrix}\right]\!\!\right](t,k)\ [\![\tilde{L}]\!]^*(t,k) - \left[\!\!\left[\begin{matrix}S^*Y^sS & 0\\ 0 & \tilde{T}\end{matrix}\right]\!\!\right](t,k)\right\}[\![\tilde{N}_Y]\!](t,k) \prec 0 \text{ and} \quad (14)$$

$$[\![\tilde{N}_X^*]\!](t,k)\left\{[\![\tilde{L}]\!]^*(t,k)\left[\!\!\left[\begin{matrix}S^*X^sS & 0\\ 0 & \tilde{T}^{-1}\end{matrix}\right]\!\!\right](t,k)\ [\![\tilde{L}]\!](t,k) - \left[\!\!\left[\begin{matrix}X^s & 0\\ 0 & \tilde{F}\end{matrix}\right]\!\!\right](t,k)\right\}[\![\tilde{N}_X]\!](t,k) \prec 0, \quad (15)$$

where $\tilde{L} = \begin{bmatrix}\tilde{A} & \tilde{B}_1\\ \tilde{C}_1 & \tilde{D}_{11}\end{bmatrix}$. Given that the synthesis conditions (6)-(9) hold, we need to show that the inequalities (12), (14), and (15) hold, where $X(t,k)=X^s(t+1,k)$ and $Y(t,k)=Y^s(t+1,k)$ for $t=0,\ldots,N+q-1$. Applying the Schur complement formula to the third condition in (9), we get $r \geq 1/t_1$. It is easy to see that the inequality (7), along with $r \geq 1/t_1$, results in

$$[\![N_X^*]\!](t,k)\left\{[\![L]\!]^*(t,k)\left[\!\!\left[\begin{matrix}S^*XS & 0\\ 0 & T^{-1}\end{matrix}\right]\!\!\right](t,k)\ [\![L]\!](t,k) - \left[\!\!\left[\begin{matrix}X & 0\\ 0 & F\end{matrix}\right]\!\!\right](t,k)\right\}[\![N_X]\!](t,k) \prec 0, \quad (16)$$

where $L = \begin{bmatrix}A & B_1\\ C_1 & D_{11}\end{bmatrix}$. Likewise, we note from (9) that $p \geq 1/f_2$ and rewrite inequality (8) as

$$[\![N_Y^*]\!](t,k)\left\{[\![L]\!](t,k)\left[\!\!\left[\begin{matrix}Y & 0\\ 0 & F^{-1}\end{matrix}\right]\!\!\right](t,k)\ [\![L]\!]^*(t,k) - \left[\!\!\left[\begin{matrix}S^*YS & 0\\ 0 & T\end{matrix}\right]\!\!\right](t,k)\right\}[\![N_Y]\!](t,k) \prec 0. \quad (17)$$

Using (5), we see that the inequalities (16), (17), and the first coupling condition in (9) evaluated at $t=0,1,\ldots,N+q-1$ are equivalent to the inequalities (15), (14), and (12) evaluated at $t=1,\ldots,N+q$, respectively. Evaluating the first coupling condition in (9) at $t=0$ and applying the Schur complement formula result in $X_i(0,k) \succeq Y_i^{-1}(0,k)$ for $i=0,\ldots,s$, which is equivalent to

$$X_i^s(1,k) \succeq Y_i^{s^{-1}}(1,k). \quad (18)$$

From the second inequality in (6), we can see that

$$X_i^s(1,k) \prec f_1 I. \quad (19)$$

Combining (18) and (19), we can write

$$X_i^s(1,k) \prec f_1 I \quad \text{and} \quad Y_i^s(1,k) \succ 1/f_1 I. \quad (20)$$

Using (5), it is not difficult to see that (18) and (20) imply (12), (14), and (15) at $t=0$. Notice that $X_i^s(0,k)$ and $Y_i^s(0,k)$ are inconsequential here, and are only required to satisfy the coupling condition (12), which is always possible. Thus, we have shown that the inequalities (12), (14), and (15) hold for $t=0,1,\ldots,N+q$, and thereby a γ-admissible synthesis for G exists if the inequalities (6)-(9) hold. ■

The solutions $X_i(t,k)$, $Y_i(t,k)$, t_1, and f_2 obtained from above can be used to construct an (N,q)-ETP controller with a zero initial condition. We first form an (h,q)-ETP distributed system from $\tilde{G}^s$ by discarding the inconsequential part at t=0. Then, using the solutions $X_i(t,k)$ and $Y_i(t,k)$, an (N,q)-ETP controller is constructed following the procedure outlined in [8], [6].

V. ILLUSTRATIVE EXAMPLE

The control approach developed in the preceding is applied to the formation tracking problem of a network of three fixed-wing UAS. The interconnection structure of the network is the same as the one shown in Fig. 1. The UAS network is tasked to track a time-parameterized path consisting of a straight line segment followed by a circular trajectory. The information about the reference trajectory is known only to $G^{(1)}$, which acts as the leader and the other two aircraft in the network follow the leader while maintaining the formation. The dynamic model of each UAS is based on the commercially available 6-foot Telemaster radio-controlled aircraft. A nonlinear rigid-body model of the UAS is developed based on flight test data. The nonlinear model has twelve states, which include the three aircraft positions, the three linear velocities, the three Euler angles representing the orientation of the aircraft, and the three angular velocities. The UAS model has four inputs, namely, the elevator, aileron, rudder and throttle commands.

We adopt a decoupled control design approach and design two distributed controllers, one for the straight line trajectory and the other for the circular trajectory. The measurements consist of the airspeed, the GPS position, the orientation,

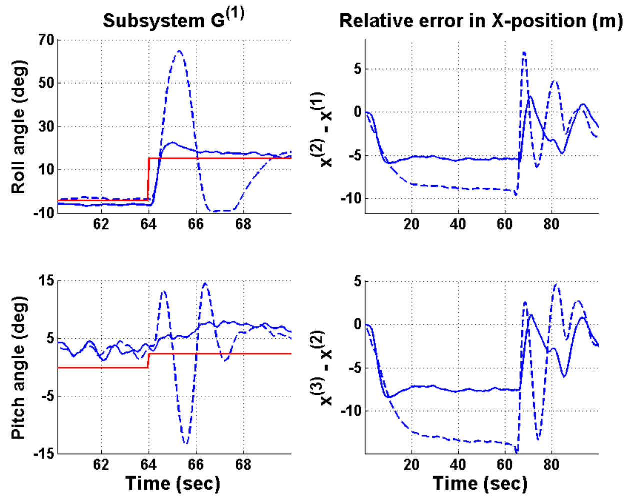

Fig. 3: The two plots on the left show the roll angle and the pitch angle variations for the leader during trajectory switching at $t = 64$ sec. The two plots on the right show the relative error in the X-position between the subsystems. (Solid blue line: UIC distributed controller; dotted blue line: LTI distributed controller; solid red line: reference trajectory)

and the angular velocities. We assume that each UAS is subjected to exogenous disturbances in the form of steady winds and atmospheric turbulence, which are modeled using the Dryden turbulence model [9], as well as sensor noise with covariance as described in [5]. The nonlinear system describing the UAS motion is linearized about the respective trim conditions, namely, straight and level flight for the straight line trajectory and steady right turn for the circular trajectory. The resulting linear time-invariant (LTI) system is discretized using zero-order hold with a sampling time of 40 ms. The subsystems share their position, airspeed, and the yaw angle over the communication network. Thus, each subsystem has twelve temporal states, five spatial states, four control inputs, ten measurements, and thirteen exogenous disturbance inputs. The performance output of the leader consists of the ten measurements and the four control inputs, whereas the performance channel of the followers consists of the relative errors in the position, airspeed, and the yaw angle. A distributed system comprised of the three LTI subsystems is formed for each of the two trim conditions.

We now design feedback controllers for the two distributed systems using the synthesis conditions provided in Theorem 2. For each of the three subsystems, uncertainty in all the twelve temporal and five spatial states is considered. Since an LTI distributed system is equivalent to a $(0, 1)$-ETP distributed system, we seek an $(N, 1)$-ETP distributed controller, where $N \geq 0$. The synthesis problem is solved using Yalmip/SDPT3 [10], [11] on a Dell desktop computer with a four core Intel Xeon, 3.07 GHz processor and 6 GB of RAM running Windows 7. For a finite horizon length of five (N=5), we obtain $\gamma_{\text{min}} \approx 60.8, 60.2$, for the straight line and circular controller synthesis problems, respectively. The wall clock times for solving the two optimization problems are 11.6 min and 18 min, respectively. The distributed controllers are then constructed using the solutions $X_i(t, k)$, $Y_i(t, k)$, t_1, and f_2 obtained from the synthesis optimization problem.

The performances of the resulting uncertain initial condition (UIC) distributed controllers are compared with distributed LTI controllers designed based on [6] in a realistic MATLAB based simulation environment. The simulation setup is characterized by a steady wind of 3 m/s magnitude, light turbulence, sensor noise, and second-order actuator dynamics. The UIC distributed controller exhibits better performance during the trajectory transition phase as seen from the variation of the roll and pitch angles of the leader in Fig. 3. We also observe that the relative position error between the subsystems is smaller with the UIC distributed controller. To summarize, the UIC distributed controller makes the transition to the new reference trajectory with less aggressive state variations and without experiencing significant degradation in the tracking performance.

VI. CONCLUSION

This paper solves the control synthesis problem for distributed systems with uncertain initial conditions, where the distributed controller inherits the interconnection topology of the plant. Uncertainty in both the temporal and spatial initial states of the system are considered and finite-dimensional convex synthesis conditions for the existence of a distributed controller are presented. The developed control approach provides a systematic method for improving the performance of a distributed system during switching between trajectories or network topologies. The effectiveness of the approach is demonstrated through an illustrative example.

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