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Integrable Modules Over Affine Lie Superalgebras $\mathfrak{sl}(1|n)^{(1)}$

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Abstract: We describe the category of integrable $\mathfrak{sl}(1|n)^{(1)}$-modules with the positive level and show that the irreducible modules provide the full set of irreducible representations for the corresponding simple vertex algebra.

1. Introduction

Let $\mathfrak{g}$ be the Kac–Moody superalgebra $\mathfrak{sl}(1|n)^{(1)}$, $n \geq 2$. Recall that $\mathfrak{g}_{\overline{0}} = \mathfrak{gl}_n^{(1)}$. We call a $\mathfrak{g}$-module *integrable* if it is integrable over the affine Lie algebra $\mathfrak{sl}_n^{(1)}$, locally finite over the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_n^{(1)}$ and with finite-dimensional generalized $\mathfrak{h}$-weight spaces.

We normalize the invariant bilinear form on $\mathfrak{g}$ in the usual way ($(\alpha, \alpha) = 2$ for the non-isotropic roots α). We say that a module N has level k if the canonical central element K in $\mathfrak{sl}(1|n)^{(1)}$ acts on N by kId. Let $\mathcal{F}_k$ be the category of the finitely generated integrable $\mathfrak{g}$-modules of level k. This category is empty for $k \notin \mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{>0}$, the irreducible objects in $\mathcal{F}_k$ are highest weight modules, which were classified in [15] (see [7], Theorem C). In this paper we study the category $\mathcal{F}_k$ for $k \in \mathbb{Z}_{>0}$: we describe the blocks (see Corollary 3.4.1 and Theorem 3.5.4) in terms of quivers with relations and show that Duflo–Serganova functor provides an invariant for the atypical blocks (see Corollary 4.5) and this invariant separates the blocks.

In Sect. 5 we study modules over the simple affine vertex superalgebra $V_k(\mathfrak{g})$. Recall that the modules over affine vertex algebra $V^k(\mathfrak{g})$ are the restricted $\mathfrak{g}$-modules of level k.

Let $\mathfrak{u}$ be an affine Lie algebra, $V^k(\mathfrak{u})$ be the universal affine vertex algebra associated with $\mathfrak{u}$ at level k; let $V_k(\mathfrak{u})$ be the unique simple quotient of $V^k(\mathfrak{u})$. Let k be such that $V_k(\mathfrak{u})$ is integrable (as a $\mathfrak{u}$-module); for an appropriate normalization of the bilinear

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form, this means that k is a non-negative integer. In this case, the $V_k(\mathfrak{u})$-modules are the restricted integrable $[\mathfrak{u}, \mathfrak{u}]$-modules of level k. For $k \geq 0$ the irreducible restricted integrable $[\mathfrak{u}, \mathfrak{u}]$-modules of level k are highest weight modules. In particular, the vertex algebra $V_k(\mathfrak{u})$ is rational and regular:

(a) there are finitely many (up to isomorphism) irreducible $V_k(\mathfrak{u})$-modules;
(b) any representation is completely reducible.

For bounded $V_k(\mathfrak{u})$-modules these results were proven in [6]. In [4] it is shown that any $V_k(\mathfrak{u})$-module is a direct sum of bounded modules, which implies the general result.

Let $\mathfrak{g}$ be an (untwisted) affine Lie superalgebra and $\mathfrak{g}_{\bar{1}} \neq 0$. Let $\mathfrak{g}^\#$ be the "largest affine subalgebra" of $\mathfrak{g}_{\bar{0}}$ (see Sect. 5.2). Let k be such that $V_k(\mathfrak{g})$ is integrable as a $\mathfrak{g}^\#$-module (for an appropriate normalization of the bilinear form, this means that k is a non-negative integer). In Theorem 5.3.1 we prove that the $V_k(\mathfrak{g})$-modules are the restricted $[\mathfrak{g}, \mathfrak{g}]$-modules of level k which are $\mathfrak{g}^\#$-integrable. In addition, we show that the irreducible bounded $V_k(\mathfrak{g})$-modules are highest weight modules if and only if the Dynkin diagram of $\mathfrak{g}_{\bar{0}}$ is connected ($\mathfrak{g}$ is $\mathfrak{sl}(1|n)^{(1)}$ or $\mathfrak{osp}(n|m)^{(1)}$ for $n = 1, 2$), see Sect. 5.6.

For $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}$ one has $\mathfrak{g}^\# = \mathfrak{sl}_n^{(1)}$. For a positive integer k the category of finitely generated $V_k(\mathfrak{g})$-modules with finite-dimensional generalized weight spaces is the full subcategory of $\mathcal{F}_k$ with the objects annihilated by the Casimir operator.

In Appendix we recall the definition of DS-functor and prove several properties used in Sect. 4.

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2. Preliminaries

Let $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}$. Recall that by definition an integrable $\mathfrak{g}$-module is integrable over the affine Lie subalgebra $\mathfrak{sl}_n^{(1)} \subset \mathfrak{g}_{\bar{0}}$ and locally finite over the Cartan subalgebra $\mathfrak{h}$. Recall also that $\mathfrak{h} \cap [\mathfrak{sl}_n^{(1)}, \mathfrak{sl}_n^{(1)}]$ acts diagonally on any integrable $\mathfrak{sl}_n^{(1)}$-module.

Note that $\mathcal{F}_k$ is the full subcategory in the category $\mathcal{O}$. In particular, it is equipped with a covariant duality functor $\mathcal{D}$ inherited from the contragredient duality in category $\mathcal{O}$. For any simple object L we have $\mathcal{D}(L) \simeq L$. In particular, $\operatorname{Ext}^1(L, L') = \operatorname{Ext}^1(L', L)$ for any two simple objects L and L'.

2.1. Roots and sets of simple roots. View $\mathfrak{g}$ as the affinization of $\dot{\mathfrak{g}} = \mathfrak{sl}(1|n)$. Choose a basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\dot{\mathfrak{h}}^*$ such that the invariant form is given by

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} -1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}.$$

Even roots of $\dot{\mathfrak{g}}$ are of form

$$\{\varepsilon_i - \varepsilon_j, \,|, i \neq j, \; i, j = 1, \ldots, n\},$$

and all odd roots are isotropic

$$\{\pm\varepsilon_i \,|, i = 1, \ldots, n\}.$$

By δ we denote the smallest positive imaginary root of $\mathfrak{g}$. Then all real roots of $\mathfrak{g}$ are of the form $\alpha + j\delta$, where $j \in \mathbb{Z}$ and α is a root of $\dot{\mathfrak{g}}$.

For a set of simple roots Σ we consider the standard partial order on $\mathfrak{h}^*$ given by $\lambda \geq_\Sigma \mu$ if and only if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Sigma$. We denote by ρ the Weyl vector of Σ (i.e., $\rho \in \mathfrak{h}^*$ such that $2(\rho, \alpha) = (\alpha, \alpha)$ for $\alpha \in \Sigma$).

We fix a triangular decomposition of $\mathfrak{g}_{\bar{0}}$ and consider only triangular decompositions of $\mathfrak{g}$ which are compatible with it (i.e., $\Delta^+_{\bar{0}}$ is fixed). We denote such sets of simple roots by Σ, Σ', etc. In fact, the category $\mathcal{O}$ depends only on a triangular decomposition of the even part.

2.1.1. We fix a set of simple roots Π_0 of $\Delta^+_{\bar{0}}$ and let $\mathcal{B}$ denote the set of all sets of simple roots Σ such that $\Pi_0 \subset \mathbb{Z}_{\geq 0}\Sigma$. To be precise let

$$\Pi_0 = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \varepsilon_1 + \delta\}.$$

For example,

$$\Sigma = \{-\varepsilon_1 + \delta, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\} \in \mathcal{B}. \tag{1}$$

For any odd root β there exists a unique $\alpha \in \Pi_0$ such that $(\alpha, \beta) = -1$ and a unique $\alpha' \in \Pi_0$ such that $(\beta, \alpha') = 1$; the set

$$\Sigma = \{\beta, \alpha' - \beta\} \cup (\Pi_0 \backslash \{\alpha'\})$$

is a unique set of simple roots containing β.

Note that all $\Sigma \in \mathcal{B}$ have the same Dynkin diagram. Every Σ contains exactly two odd roots β_1 and β_2, $(\beta_1, \beta_2) = 1$ and all roots of Π_0 are even roots of Σ and $\beta_1 + \beta_2$. The Dynkin diagram is a cycle with $n + 1$ nodes: there are two nodes which correspond to the odd isotropic roots and these nodes are adjacent. The minimal imaginary positive root δ is the sum of all simple roots.

2.1.2. Odd reflections. Recall that for an odd root β belonging to a set of simple roots Σ, the odd reflection r_β gives another set of simple roots $r_\beta\Sigma$ which contains $-\beta$, the roots $\alpha \in \Sigma \backslash \{\beta\}$, which are orthogonal to β, and the roots $\alpha + \beta$ for $\alpha \in \Sigma$ which are not orthogonal to β. One has

$$\Delta^+(r_\beta\Sigma) = (\Delta^+(\Sigma)\backslash\{\beta\}) \cup \{-\beta\}. \tag{2}$$

Any two sets of simple roots in $\mathcal{B}$ are connected by a chain of odd reflections. We call a chain "proper" if it does not have loops (i.e. subsequences of the form $r_\beta r_{-\beta}$). Two sets of simple roots are connected by a unique "proper" chain of odd reflections. We call Σ and Σ' *adjacent* if they are obtained from each other by one odd reflection. The adjacency graph with vertices in $\mathcal{B}$ is an infinite string (every vertex has two adjacent vertices). For any two $\Sigma, \Sigma' \in \mathcal{B}$ there is a unique proper chain of odd reflections connecting them. We denote by $d(\Sigma, \Sigma')$ the number of odd reflections in this chain.

Let Σ be a set of simple roots. One readily sees that the chain $r_{\beta_s} r_{\beta_{s-1}} \ldots r_{\beta_1}\Sigma$ is proper if and only if $\beta_1, \ldots, \beta_s \in \Delta^+(\Sigma)$. Let $\beta \notin \Sigma$ be an odd root and Σ' be the set of simple roots containing β (by above, Σ' is unique). If $\beta \in \Delta^+(\Sigma)$, then the proper chain which connects Σ and Σ' does not contain the reflections $r_{\pm\beta}$; if $\beta \in -\Delta^+(\Sigma)$, then the proper chain is of the form $\Sigma' = r_{\beta_s} r_{\beta_{s-1}} \ldots r_{\beta_1}\Sigma$, where $\beta_s = \beta$.

2.2. Simple modules. For any $\Sigma \in \mathcal{B}$ we denote by $L_\Sigma(\lambda)$ the irreducible module of highest weight λ with respect to the Borel subalgebra corresponding to Σ. Given an irreducible highest weight module L and $\Sigma \in \mathcal{B}$ we set $\rho wt_\Sigma L := \lambda$ if $L = L_\Sigma(\lambda - \rho_\Sigma)$ (where ρ_Σ is the Weyl vector for Σ). For an odd root $\alpha \in \Sigma$ one has

$$\rho wt_{r_\alpha \Sigma} L = \begin{cases} \rho wt_\Sigma L & \text{if} \quad (\lambda, \alpha) \neq 0, \\ \rho wt_\Sigma L + \alpha & \text{if} \quad (\lambda, \alpha) = 0. \end{cases} \tag{3}$$

From (3), it follows that $L_\Sigma(\lambda)$ is integrable if and only if $(\lambda, \alpha) \in \mathbb{Z}_{\geq 0}$ for every even $\alpha \in \Sigma$, and for two odd roots $\beta_1, \beta_2 \in \Sigma$ one has either $(\lambda, \beta_1 + \beta_2) \in \mathbb{Z}_{>0}$ or $(\lambda, \beta_1) = (\lambda, \beta_2) = 0$. Since δ is the sum of simple roots, a highest weight module $L_\Sigma(\lambda)$ has level $k = (\lambda, \sum_{\alpha \in \Sigma} \alpha)$. In particular, if $L_\Sigma(\lambda)$ is not one-dimensional, its level is a positive integer.

2.2.1. Example. Let Σ be as in (1) and let $b_i = (\lambda + \rho_\Sigma, \varepsilon_i)$ for $i = 1, \dots, n$. Then $L_\Sigma(\lambda)$ is integrable if and only if

(1) $b_1 - b_2, \dots, b_{n-1} - b_n \in \mathbb{Z}_{>0}$;
(2) $b_n - b_1 + n + k - 1 \in \mathbb{Z}_{>0}$ or $b_n = 0, b_1 = n + k - 1$.

2.3. Lemma. *If $L_\Sigma(\lambda)$ is integrable, then there exists at most one positive real even root α such that $(\lambda + \rho_\Sigma, \alpha) = 0$. Moreover, in this case $\alpha \in \Pi_0$ and α is a sum of the two odd roots $\beta_1, \beta_2 \in \Sigma$ and $(\lambda + \rho, \beta_1) = (\lambda + \rho, \beta_2) = 0$.*

Proof. If β_1, β_2 denote two odd roots of Σ, then integrability condition implies that $(\lambda + \rho_\Sigma, \alpha)$ is a positive integer for all even $\alpha \in \Sigma$ and $(\lambda + \rho, \beta_1 + \beta_2)$ is a non-negative integer. Since every positive even real root is a non-negative linear combination of even $\alpha \in \Sigma$ and $\beta_1 + \beta_2$, the statement follows. □

2.3.1. We fix a set of simple roots $\Sigma = \{\alpha_i\}_{i=0}^n$, where α_1, α_2 are odd. Note that $(\alpha_1, \alpha_2) = 1$. We set

$$\begin{aligned} P^+(\Sigma) := \{\lambda \in \mathfrak{h}^* |\ & (\lambda, \gamma) \in \mathbb{Z}_{\geq 0} \text{ for all } \gamma \in \Pi_0 \text{ and} \\ & (\lambda, \alpha_1 + \alpha_2) = 0 \implies (\lambda, \alpha_1) = (\lambda, \alpha_2) = 0\}. \end{aligned}$$

By above, the irreducible objects of $\mathcal{F}_k$ are the highest weight modules $L_\Sigma(\lambda)$, where $\lambda \in P^+(\Sigma)$; in other words, setting $a_i := (\rho wt_\Sigma L, \alpha_i)$, we have

(i) $a_i \in \mathbb{Z}_{>0}$ for $i = 0$ or $i = 3, \dots, n$;
(ii) $a_1 + a_2 \in \mathbb{Z}_{>0}$ or $a_1 = a_2 = 0$;
(iii) $a_0 + a_1 + \dots + a_n = k + n - 1$.

Notice that the numbers $\{a_i\}_{i=0}^n$ determine L as a $[\mathfrak{g}, \mathfrak{g}]$-module; for the $\mathfrak{g}$-modules $L(\lambda)$, $L(\lambda + s\delta)$ the numbers $\{a_i\}_{i=0}^n$ are the same, however the Casimir element acts on $L(\lambda)$ and on $L(\lambda + s\delta)$ by different scalars.

2.3.2. **Lemma.** *Let $L_\Sigma(L)$ be an integrable highest weight module and (λ, α) is real for each $\alpha \in \Sigma$. Then there exists a set of simple roots Σ' such that $(\lambda + \rho_\Sigma, \alpha) \geq 0$ for every $\alpha \in \Sigma'$.*

Proof. Recall that $(\lambda+\rho_\Sigma, \delta) = k+n-1$. Note that $(\Sigma\setminus\{\alpha_1, \alpha_2\}) \cup \{\alpha_1 + s\delta, \alpha_2 - s\delta\}$ is a set of simple roots for any $s \in \mathbb{Z}$. Therefore, without loss of generality we may assume that

$$0 \leq (\lambda + \rho_\Sigma, \alpha_1) < k + n - 1. \tag{4}$$

If $0 \leq (\lambda+\rho_\Sigma, \alpha_2)$, we take $\Sigma' = \Sigma$. Assume that $(\lambda+\rho_\Sigma, \alpha_2) < 0$. For $r = 2, \dots, n+1$ set $\beta_r := \sum_{i=2}^{r} \alpha_i$ (where $\alpha_{n+1} := \alpha_0$). Then $\delta = \beta_{n+1} + \alpha_1$, so (4) gives

$$(\lambda + \rho_\Sigma, \beta_2) < 0, \quad (\lambda + \rho_\Sigma, \beta_{n+1}) > 0.$$

Let s be maximal such that $(\lambda + \rho_\Sigma, \beta_s) < 0$. For $\Sigma' := r_{\beta_s} \dots r_{\beta_2} \Sigma$ the isotropic roots are $-\beta_s$ and β_{s+1}. Since $(\lambda + \rho_\Sigma, -\beta_s), (\lambda + \rho_\Sigma, \beta_{s+1}) \geq 0$, Σ' is as required. □

2.3.3. Definitions. From now on L stands for an *irreducible integrable highest weight module on non-zero level*.

Recall that L is called *typical* if $(\rho wt_\Sigma L, \alpha) \neq 0$ for any (isotropic) odd root α and *atypical* otherwise. From (3), it follows that this notion does not depend on the choice of Σ and, moreover, $\rho wt_\Sigma L$ does not depend on Σ for typical L.

We say that L is Σ-*tame* if $(\rho wt_\Sigma L, \beta) = 0$ for some odd $\beta \in \Sigma$. Any atypical L (for $\mathfrak{sl}(1, n)^{(1)}$) is tame with respect to some Σ.

Let β be an odd root. We call an odd reflection r_β *L-typical* if for Σ containing β one has $(\rho wt_\Sigma L, \beta) \neq 0$ (by 2.1.1, Σ is unique for given β). Note that if Σ and Σ' are connected by a chain of odd L-typical reflections, then $\rho wt_\Sigma(L) = \rho wt_{\Sigma'}(L)$.

We say that $\lambda \in \mathfrak{h}^*$ is *integral* if (λ, α) is integral for each $\alpha \in \Pi_0$. We say that $\lambda \in \mathfrak{h}^*$ is *regular* if $(\lambda, \alpha) \neq 0$ for any even real root and that λ is *singular* otherwise. Note that if λ integral, then there exists a unique $\bar{\lambda} \in W\lambda$ such that $(\bar{\lambda}, \alpha) \in \mathbb{N}$ for all $\alpha \in \Pi_0$. Obviously $\bar{\lambda}$ is regular if and only if λ is regular.

We say that L is Σ-*regular* if $\rho wt_\Sigma L$ is regular and that L is *regular* if it is Σ-regular for each Σ. We say that L is Σ-*singular* if it is not Σ-regular and that L is *singular* if it is not regular. By 2.3.1, L is Σ-singular if and only if $(\rho wt L, \alpha) = 0$ for both odd roots $\alpha \in \Sigma$ (in particular, in this case L is Σ-tame).

2.3.4. **Lemma.** *Let L be a simple integrable module and $HW(L)$ denote the set of all $\rho wt_\Sigma L$ for all $\Sigma \in \mathcal{B}$.*

(i) *If L is typical then $|HW(L)| = 1$;*
(ii) *If L is regular atypical then $|HW(L)| = 2$;*
(iii) *Let L be singular atypical and $\Sigma = \{\alpha_0, \dots, \alpha_n\}$ with odd α_0, α_1 be such that $\rho wt_\Sigma L$ is singular. Let m, l be such that*

$$(\rho wt_\Sigma L, \alpha_m) \geq 2 \text{ and } (\rho wt_\Sigma L, \alpha_i) = 1, \text{ for each } i \text{ such that } m < i \leq n,$$
$$(\rho wt_\Sigma L, \alpha_l) \geq 2 \text{ and } (\rho wt_\Sigma L, \alpha_i) = 1 \text{ for each } i \text{ such that } 2 \leq i < l.$$

Then

$$\begin{aligned} HW(L) = \{&\rho wt_\Sigma L, \rho wt_\Sigma L + \alpha_1, \rho wt_\Sigma L + 2\alpha_1 + \alpha_2, \dots, \\ &\rho wt_\Sigma L + (l-1)\alpha_1 + (l-2)\alpha_2 + \dots + \alpha_{l-1}\} \cup \{\rho wt_\Sigma L + \alpha_0, \dots, \\ &\rho wt_\Sigma L + (n-m+1)\alpha_0 + (n-m)\alpha_n + \dots + \alpha_{m+1}\} \end{aligned}$$

or equivalently

$$HW(L) = \{\overline{\rho wt_\Sigma L + s\alpha_0} \,|\, s = 1-l, \ldots, n-m+1\}.$$

Remark. The existence of l, m follows from the assumption that $k \neq 0$; one has $2 \leq l \leq m \leq n$.

Proof. The first assertion follows from (3) as any odd reflection is L typical. Now assume that L is atypical and regular. Then there exists exactly one positive odd root such that $(\rho wt_\Sigma L, \alpha) = 0$. Let $\Sigma' \in \mathcal{B}$ be such that $\alpha \in \Sigma'$. Let $\Sigma'' = r_\alpha \Sigma'$. Since both $\rho wt_{\Sigma'} L$ and $\rho wt_{\Sigma''} L$ are regular, all other odd reflections are L-typical. Hence $HW(L) = \{\rho wt_{\Sigma'} L, \rho wt_{\Sigma''} L\}$.

Finally, let assume that L is atypical and singular. Then by Lemma 2.3 there exists Σ_0 such that $(\rho wt_\Sigma L, \alpha_1) = (\rho wt_\Sigma L, \alpha_2) = 0$ for both simple odd roots $\alpha_1, \alpha_2 \in \Sigma_0$. Moreover, m and l exist as follows from integrability condition. If we set

$$\beta_1 = \alpha_0 + \alpha_n, \ldots, \beta_{n-m} = \alpha_0 + \alpha_n + \cdots + \alpha_{n-m+1},$$

then the reflection r_{β_i} are all L-atypical and we obtain that $HW(L)$ contains $\rho wt_{r_{\alpha_0}\Sigma} L$ and $\rho wt_{r_{\beta_i} \ldots r_{\beta_1} r_{\alpha_0}\Sigma} L$ for all $i = 1, \ldots, n-m$. Similarly, if we set

$$\gamma_1 = \alpha_1 + \alpha_2, \ldots, \gamma_{l-2} = \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1},$$

then $HW(L)$ contains $\rho wt_{r_{\alpha_1}\Sigma} L$ and $\rho wt_{r_{\gamma_i} \ldots r_{\gamma_1} r_{\alpha_1}\Sigma} L$ for all $i = 1, \ldots, l-2$. All other odd reflections are L typical and do not add new weights to $HW(L)$.

The last formula follows form the identity

$$\begin{aligned} r_{\alpha_j} \ldots r_{\alpha_2} r_{\alpha_1+\alpha_0}(\rho wt_\Sigma L - j\alpha_0) &= r_{\alpha_j} \ldots r_{\alpha_2}(\rho wt_\Sigma L + j\alpha_1) \\ &= \rho wt_\Sigma L + j\alpha_1 + (j-1)\alpha_2 + \cdots + \alpha_j. \end{aligned}$$

□

2.3.5. One readily sees that $\rho wt_\Sigma L + j\alpha_0$ is not regular for $1 - l < j < n - m + 1$.

Corollary. *If L is atypical, then $HW(L)$ contains exactly two regular weights.*

2.4. Character formulae. If $L(\lambda)$ is typical, then ch $L(\lambda)$ is given by the Kac–Weyl character formula; if $L(\lambda)$ is atypical and Σ-tame, ch $L(\lambda)$ is given by Kac–Wakimoto formula, see [15,17].

2.5. Adjacency relation on atypical simple modules. Note that if L is atypical, then all weights of L are integral. We say that L' is adjacent to L if there exist $\Sigma \in \mathcal{B}$ and an odd $\alpha \in \Sigma$ such that $(\rho wt_\Sigma L, \alpha) = 0$ and $\rho wt_\Sigma L' = \rho wt_\Sigma L - \alpha$. Note that if L' is adjacent to L, then L is adjacent to L', as for $\Sigma' = r_\alpha \Sigma$ we have $\rho wt_{\Sigma'} L = \rho wt_{\Sigma'} L' - (-\alpha)$ and $-\alpha \in \Sigma'$. Therefore the adjacency relation defines the adjacency graph Γ with vertices enumerated by isomorphism classes of atypical integrable simple $\mathfrak{g}$-modules of a fixed level k and a fixed eigenvalue of the Casimir operator (i.e., the value $(\rho wt_\Sigma L, \rho wt_\Sigma L)$ is fixed (this value does not depend on Σ)).

We denote this graph by Γ. It is important to characterize the connected components of Γ.

2.5.1. **Lemma.** *Let $\Sigma \in \mathcal{B}$ and α be an odd root of Σ such that $(\rho wt_{\Sigma} L, \alpha) = 0$. Then $\rho wt_{\Sigma} L - \alpha$ is integrable if and only if $\rho wt_{\Sigma} L$ is regular.*

Proof. If $\beta \in \Sigma$ is an even root, then $(-\alpha, \beta) \geq 0$ and hence $(\lambda - \alpha, \beta) \in \mathbb{Z}_{>0}$. If $\beta \in \Sigma$ is the second odd root, then $(\alpha + \beta, \alpha) = 1$ and therefore $(\lambda - \alpha, \beta + \alpha) \in \mathbb{Z}_{\geq 0}$ if and only if $(\lambda, \beta) \geq 1$. Hence λ is regular. □

2.5.2. **Corollary.** *An atypical simple integrable module L has exactly two adjacent L' and L''. To construct them recall that by Lemma 2.3.4 there exist exactly two Σ_1 and Σ_2 in $\mathcal{B}$ such that L is Σ_i-tame and $\rho wt_{\Sigma_i} L$ is regular. Then $\rho wt_{\Sigma_1} L' = \rho wt_{\Sigma_1} - \alpha_1$ and $\rho wt_{\Sigma_2} L'' = \rho wt_{\Sigma_2} - \alpha_2$ where α_i is the unique odd root in Σ_i such that $(\rho wt_{\Sigma_i} L, \alpha_i) = 0$.*

2.5.3. *Remark.* Let us fix $\Sigma \in \mathcal{B}$. It follows from Corollary 2.5.2 that

$$\rho wt_{\Sigma} L' <_{\Sigma} \rho wt_{\Sigma} L <_{\Sigma} \rho wt_{\Sigma} L''$$

if $\rho wt_{\Sigma} L$ is regular. If $\rho wt_{\Sigma} L$ is singular, we have

$$\rho wt_{\Sigma} L <_{\Sigma} \rho wt_{\Sigma} L' \text{ and } \rho wt_{\Sigma} L <_{\Sigma} \rho wt_{\Sigma} L''.$$

2.5.4. **Theorem.** *Fix $\Sigma \in \mathcal{B}$. Then every connected component Γ' of Γ contains exactly one L_0 such that $\lambda := \rho wt_{\Sigma} L_0$ is singular. Let β be any of two odd roots of Σ and*

$$S := \{s \in \mathbb{Z} \mid \lambda + s\beta \text{ is regular}\}.$$

Then $L' \in \Gamma'$ if and only if $L_0 \simeq L'$ or $\rho wt_{\Sigma} L' = \overline{\lambda + s\beta}$ for some $s \in S$. Enumerate elements of $S \cup \{0\}$ in increasing order assuming $s_0 = 0$ and set $L_i := L_{\Sigma}(\overline{\lambda + s_i \beta} - \rho_{\Sigma})$. Then every L_i is adjacent to L_{i-1} and L_{i+1}.

Proof. Uniqueness of L_0 follows from Remark 2.5.3 and Corollary 2.5.2. Let us prove the existence. Start with some L such that $\rho wt_{\Sigma} L$ is regular. There exists Σ' such that L is Σ'-tame. Let us pick up L with minimal $d(\Sigma, \Sigma')$. We claim that for such L, $\Sigma = \Sigma'$. Indeed, assume $\Sigma \neq \Sigma'$. By Lemma 2.3.4, $\mu = \rho wt_{\Sigma} L = \rho wt_{\Sigma'} L$ is regular. There is a unique odd $\alpha \in \Sigma'$ for which $(\mu, \alpha) = 0$. Consider the smallest $p > 0$ such that $\mu - p\alpha$ is not dominant. Then $\mu - (p-1)\alpha$ is singular. Let $L' = L_{\Sigma'}(\mu - (p-1)\alpha - \rho_{\Sigma'})$. If α' is the second odd root of Σ', then $d(\Sigma, r_{\alpha'}\Sigma) = d(\Sigma, \Sigma') - 1$ and this contradicts minimality of $d(\Sigma, \Sigma')$. To finish the proof of existence of L_0 take odd $\beta \in \Sigma$ such that $(\mu, \beta) = 0$ and consider the smallest $q \geq 0$ such that $\mu - q\beta$ is singular. Then L_0 is the simple module with $\rho wt_{\Sigma} L_0 = \mu - q\beta$.

The last assertion of the theorem follows fom the description of $HW(L)$ given in Lemma 2.3.4. □

For a fixed level k and a fixed eigenvalue of the Casimir element, a singular integrable weight λ is determined by non-negative integers $(\lambda, \alpha_2), \ldots, (\lambda, \alpha_n)$ such that $\sum_{i=2}^{n} (\lambda, \alpha_i) = k$.

2.5.5. **Corollary.** *For each level k and each eigenvalue of the Casimir element, the graph Γ has finitely many connected components. They are enumerated by singular integrable weights of level k.*

3. The Category of Integrable $sl(1|n)^{(1)}$-Modules at Non-zero Level

In this section we will describe $\mathcal{F}_k$ for $k > 0$.

3.1. Maximal integrable quotient of a Verma module. Let $\Sigma \in \mathcal{B}$. We denote by $M_\Sigma(\lambda)$ the Verma module with highest weight λ for the Borel subalgebra corresponding to Σ. The Verma module $M_\Sigma(\lambda)$ has a unique simple quotient $L_\Sigma(\lambda)$. If $L_\Sigma(\lambda)$ is integrable, then we denote by $V_\Sigma(\lambda)$ the maximal integrable quotient of $M_\Sigma(\lambda)$. Clearly we have a surjection $V_\Sigma(\lambda) \to L_\Sigma(\lambda)$.

In [17] the following lemma is proved.

3.1.1. **Lemma.** *Let $L = L_\Sigma(\lambda - \rho_\Sigma)$ be an integrable module.*

(i) *If L is typical, then $V_\Sigma(\lambda - \rho_\Sigma) = L$.*
(ii) *If L is atypical and λ is singular, then $V_\Sigma(\lambda - \rho_\Sigma) = L$.*
(iii) *If λ is regular, then the character of $V_\Sigma(\lambda - \rho_\Sigma)$ is given by typical formula*

$$\operatorname{ch} V_\Sigma(\lambda - \rho_\Sigma) = \sum_{w \in W} \operatorname{sgn}(w) \operatorname{ch} M_\Sigma(w(\lambda) - \rho_\Sigma).$$

Moreover if $(\lambda, \alpha) = 0$ for some odd $\alpha \in \Sigma$, then $V_\Sigma(\lambda - \rho_\Sigma)$ has length two and can be described by the following exact sequence

$$0 \to L_\Sigma(\lambda - \alpha - \rho_\Sigma) \to V_\Sigma(\lambda - \rho_\Sigma) \to L_\Sigma(\lambda - \rho_\Sigma) \to 0.$$

(iv) *For any Σ and Σ' in $\mathcal{B}$, such that $\lambda = \rho wt_\Sigma L = \rho wt_{\Sigma'} L$, we have $V_\Sigma(\lambda - \rho_\Sigma) = V_{\Sigma'}(\lambda - \rho_{\Sigma'})$.*

3.1.2. **Lemma.** *Let L and L' be two non-isomorphic simple integrable modules. Then* $\operatorname{Ext}^1(L, L') \neq 0$ *if and only of L and L' are two adjacent atypical modules. In this case* $\operatorname{Ext}^1(L, L') = \mathbb{C}$.

Proof. Consider an extension given by a non-split exact sequence

$$0 \to L' \to M \to L \to 0.$$

Choose some $\Sigma \in \mathcal{B}$ and let $\lambda = \rho wt_\Sigma L, \mu = \rho wt_\Sigma L'$. If λ and μ are incomparable with respect to $\leq_\Sigma$, then the above sequence splits since a vector of weight $\lambda - \rho_\Sigma$ generates a submodule isomorphic to L in M. Note that duality implies $\operatorname{Ext}^1(L, L') = \operatorname{Ext}^1(L', L)$. Therefore without loss of generality we may assume that $\mu <_\Sigma \lambda$. But then M is a quotient of $V_\Sigma(\lambda - \rho_\Sigma)$. By Lemma 3.1.1 we know that the length of $V_\Sigma(\lambda - \rho_\Sigma)$ is at most 2. Hence, $M \simeq V_\Sigma(\lambda - \rho_\Sigma)$, L is atypical and λ is regular. Then there exists $\Sigma' \in \mathcal{B}$ such that L is Σ'-tame and $\rho wt_\Sigma L = \rho wt_{\Sigma'} L$. By Lemma 3.1.1 (iv) we obtain $\rho wt_{\Sigma'} L' = \rho wt_{\Sigma'} L - \alpha$ for odd $\alpha \in \Sigma'$ such that $(\lambda, \alpha) = 0$. Hence, by definition L and L' are adjacent. That proves the statement. $\square$

3.2. Self-extensions of simple modules.

3.2.1. Recall that $\mathfrak{g}$ is the affinization of $\dot{\mathfrak{g}} = \mathfrak{sl}(1|n)$. Fix $\Sigma \in \mathcal{B}$ and $\alpha \in \Sigma$. Note that $\Sigma \setminus \{\alpha\}$ is the set of simple roots of some subalgebra isomorphic to $\dot{\mathfrak{g}} \simeq \mathfrak{sl}(1|n)$. Let $h \in \mathfrak{h}^*$ be such that

$$\beta(h) = \begin{cases} 0 \text{ if } \beta \in \Sigma, \beta \neq \alpha \\ 1 \text{ if } \beta = \alpha. \end{cases}$$

Let N be such that h acts locally finitely and the eigenvalues are bounded: there exists a "maximal" eigenvalue, i.e. an eigenvalue a such that $a + j$ is not an eigenvalue for any positive integer j. In this case we denote by N^{top} the generalized h-eigenspace with the maximal eigenvalue.

Observe that if $L = L_\Sigma(\lambda)$ is simple, then L^{top} is a simple $\mathfrak{sl}(1|n)$-module with highest weight $\lambda|_{\dot{\mathfrak{h}}}$ where $\dot{\mathfrak{h}}$ is the Cartan subalgebra of $\dot{\mathfrak{g}}$. If L is integrable, then L^{top} is finite-dimensional.

The centre of $\mathfrak{sl}(1|n)^{(1)}_{\bar{0}}$ is two-dimensional: it is spanned by K and z, where z is a central element in $\mathfrak{sl}(1|n)_{\bar{0}} = \mathfrak{gl}_n$.

3.2.2. **Lemma.** *Let L be a simple module and*

$$0 \to L \to M \to L \to 0$$

be a non-split exact sequence, then the sequence

$$0 \to L^{top} \to M^{top} \to L^{top} \to 0$$

also does not split.

Proof. If $M^{top} \simeq L^{top} \oplus L^{top}$, the two copies of M^{top} generate two proper distinct submodules of M. Since M has length 2, it is a direct sum of two simple modules. □

3.3. Recall now the following result from [8].

Lemma. *If N is a simple $\dot{\mathfrak{g}}$-module, then* $\mathrm{Ext}^1(N, N) = 0$ *if N is atypical and* $\mathrm{Ext}^1(N, N) = \mathbb{C}$ *if N is typical.*

3.3.1. **Corollary.** *For a simple atypical $\mathfrak{g}$-module L,* $\mathrm{Ext}^1(L, L) = 0$.

Proof. It is easy to find Σ and $\alpha \in \Sigma$ such that L^{top} is atypical. Then the statement follows from Lemmas 3.2.2 and 3.3. □

3.3.2. In [17] the following statements are proved (Lemma 4.13).

Lemma. *Let L be a simple module and $\Sigma \in \mathcal{B}$ be such that $\rho wt_\Sigma L$ is regular. Let ω be a weight such that $(\omega, \alpha) = 1$ for some odd $\alpha \in \Sigma$ and $(\omega, \beta) = -1$ for another odd $\beta \in \Sigma$ and $(\omega, \gamma) = 0$ for all even $\gamma \in \Sigma$. Then $V_\Sigma(\lambda - \rho_\Sigma + t\omega)$ is a flat deformation of $V_\Sigma(\lambda - \rho_\Sigma)$.*

3.3.3. **Corollary.** *Under assumptions of Lemma 3.3.2 the module $V_\Sigma(\lambda-\rho_\Sigma+t\omega)/(t^p)$ is an indecomposable module which has a filtration with associated quotients isomorphic to $V_\Sigma(\lambda - \rho_\Sigma)$.*

3.4. Typical blocks in $\mathcal{F}_k$. Let $\dot{L}$ be a typical finite-dimensional $\mathfrak{sl}(1|n)$-module of highest weight $\dot{\lambda}$ and let $\mathcal{F}(\dot{L})$ be the block containing $\dot{L}$ in the category of finitely generated $\mathfrak{sl}(1|n)$-modules. It is easy to deduce from [8] that the functor $N \mapsto N_\lambda$ (here N_λ is the subspace with generalized weight λ) provides an equivalence between $\mathcal{F}(\dot{L})$ and the category of finite-dimensional $\mathbb{C}[z]$-modules with nilpotent action of $z - \lambda(z)$.

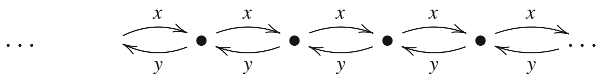
3.4.1. Retain notation of Sect. 3.2.1.

Corollary. *For any typical simple module L in $\mathcal{F}_k$ there exists a block $\mathcal{F}_k(L)$ of $\mathcal{F}_k$ which has one up to isomorphism simple module L. The functor $N \mapsto N^{top}$ provides an equivalence between $\mathcal{F}_k(L)$ and the typical block of the category of finitely generated $\mathfrak{sl}(1|n)$-modules. The functor $N \to N_\lambda$ provides an equivalence between $\mathcal{F}_k(L)$ and the category of finite-dimensional $\mathbb{C}[z]$-modules with nilpotent action of $z - \lambda(z)$.*

3.5. Atypical blocks of $\mathcal{F}_k$.

3.5.1. The following theorem is a direct consequence of Lemma 3.1.2, Corollary 3.3.1, and Theorem 2.5.4.

Theorem. *The Ext quiver of an atypical block in $\mathcal{F}_k$ coincides with a connected component of the graph Γ and is of the form*



3.5.2. **Lemma.** *There is no indecomposable module M in $\mathcal{F}_k$ such that $M/radM = L_1$, $radM/rad^2M = L_2$, $rad^2M = L_3$ for pairwise non-isomorphic simple modules L_1, L_2, L_3.*

Proof. Assume that such module exists. Then the vertices corresponding to $\{L_1, L_2, L_3\}$ generate a connected subgraph of Γ. It follows from Theorem 2.5.4 and Lemma 2.3.4 that there exist Σ and $\alpha \in \Sigma$ such that L_i^{top} has the same h-eigenvalue for $i = 1, 2, 3$ (see Sect. 3.2.1 for the notation h). It was shown in [8] that there is no similar indecomposable $\dot{\mathfrak{g}}$-module M^{top} with $M^{top}/radM^{top} = L_1^{top}$, $radM^{top}/rad^2M^{top} = L_2$, $rad^2M^{top} = L_3^{top}$. The statement follows. □

3.5.3. Let $\mathcal{F}_k^1$ be the full subcategory of $\mathcal{F}_k$ consisting of the modules with diagonal action of $\mathfrak{h}$.

Theorem. *The typical blocks in $\mathcal{F}_k^1$ are completely reducible with a unique irreducible module. Any atypical block in $\mathcal{F}_k^1$ is equivalent to the category of finite-dimensional representations of the quiver of Theorem 3.5.1 with relations $xy + yx = 0$ and $x^2 = y^2 = 0$.*

Proof. The statement about a typical block is a consequence of Corollary 3.4.1.

Now we prove the statement for an atypical block. We use the same argument as in the proof of Lemma 3.5.2. This lemma implies the relation $x^2 = y^2 = 0$. Consider a module M with a simple cosocle L_i. Then $radM/rad^2M = L_{i-1}^{\oplus a} \oplus L_{i+1}^{\oplus b}$ with $a, b \in \{0, 1\}$. Then the next layer of the radical filtration rad^2M/rad^3M is isomorphic to $L_i^{\oplus c}$ for $c \in \{0, 1, 2\}$. By induction we obtain that all even layers of the radical filtration are direct sums of several copies of L_i and odd layers are direct sums of several copies of L_{i-1} and L_{i+1}.

Consider the full subcategory $\mathcal{C}$ of $\mathcal{F}_k^1$ which contains only modules with semisimple subquotients isomorphic to L_i, L_{i-1} or L_{i+1} and let $\mathcal{C}'$ be the full subcategory of $\dot{\mathfrak{g}}$-modules which contains only modules with semisimple subquotients isomorphic to

L_i^{top}, L_{i-1}^{top} or L_{i+1}^{top}. We claim that the functor $?^{top}$ defines an equivalence between $\mathcal{C}$ and $\mathcal{C}'$. Indeed, $?^{top}$ is exact and provides a bijection on isomorphism classes of simple modules. To construct the left adjoint functor Φ consider the parabolic subalgebra $\mathfrak{p} := \mathfrak{b} + \dot{\mathfrak{g}}$ and set $\Phi(?)$ to be the maximal quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} ?$ which lie in $\mathcal{C}$. We leave it to the reader to check that Φ is also exact. Now theorem follows from the analogous results in [8] for $\dot{\mathfrak{g}}$. □

3.5.4. **Theorem.** *Any atypical block in $\mathcal{F}_k$ is equivalent to the category of finite-dimensional representations of the quiver of Theorem 3.5.1 with relations $x^2 = y^2 = 0$ and nilpotent action of $xy + yx$.*

Proof. The relations $x^2 = y^2 = 0$ follow again from Lemma 3.5.2.

Let $\mathcal{F}_k^l$ denote the full subcategory of $\mathcal{F}_k$ whose objects has a filtration of length $\leq l$ with adjoint quotients from $\mathcal{F}_k^1$. Then $\mathcal{F}_k = \lim\limits_{\to} \mathcal{F}_k^l$. Thus, the statement follows from Theorem 3.5.3. □

4. Invariants of Simple Objects in the Same Block

Let $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}$ with $n > 2$. The reader can find the definition and properties of the functor DS_x in Sect. 6. Take a non-zero $x \in \mathfrak{g}_\beta$, where β is an odd isotropic root; then $[x, x] = 0$.

In this section we will show that DS_x is an invariant for atypical blocks, i.e. for irreducible modules $L, L' \in \mathcal{F}_k$ one has

(i) $DS_x(L) = 0$ if and only if L is typical;
(ii) if L is atypical, then $DS_x(L) \cong DS_x(L')$ if and only if L and L' lie in the same block.

For $n = 2$ $\mathfrak{g}_x$ is a commutative two-dimensional Lie algebra (spanned by d and K) and (i), (ii) also hold. Note that in this case the graph Γ is connected.

4.1. Fix a set of simple roots Σ; let $\alpha_1, \alpha_2 \in \Sigma$ be odd roots. Since for any odd root β the orbit $W\beta$ contains either α_2 or $-\alpha_2$, in the light of Proposition 6.4 we may assume that $x \in \mathfrak{g}_{\alpha_2}$ or $x \in \mathfrak{g}_{-\alpha_2}$. Then $\mathfrak{g}_x \cong \mathfrak{sl}_{n-1}^{(1)}$ with the set of simple roots

$$\Sigma_x := \{\alpha_0, \alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \dots, \alpha_n\}.$$

4.2. **Lemma.** *Let N be an integrable quotient of $M_\Sigma(\lambda)$ and let L' be a simple subquotient of $DS_x(N)$. Then there exists a weight μ of N such that the restriction of μ to $\mathfrak{h}_x := \mathfrak{g}_x \cap \mathfrak{h}$ is the highest weight of L' and $(\mu, \alpha_2) = 0$, $(\mu + \rho_\Sigma, \mu + \rho_\Sigma) = (\lambda + \rho_\Sigma, \lambda + \rho_\Sigma)$.*

Proof. This lemma is a consequence of Proposition 6.3. Indeed, let ν be the highest weight of L'. Then ν is the restriction of some weight μ to $\mathfrak{h}_x$. The condition $L_\mu \cap Ker x \neq L_\mu \cap Im x$ implies $(\mu, \alpha_2) = 0$. It is easy to see that the restriction of ρ_Σ to $\mathfrak{h}_x$ equal to ρ_{Σ_x}. If we denote by $(-, -)_{\mathfrak{g}_x}$ the invariant scalar product on $\mathfrak{h}_x^*$, then $(\nu, \nu)_{\mathfrak{g}_x} = (\mu, \mu)$ as $\mathfrak{h}_x^* = \alpha_2^\perp / \mathbb{C}\alpha_2$. Thus, by Proposition 6.3 we obtain

$$(\lambda + \rho_\Sigma, \lambda + \rho_\Sigma) = (\nu + \rho_{\Sigma_x}, \nu + \rho_{\Sigma_x}) = (\mu + \rho_\Sigma, \mu + \rho_\Sigma).$$

□

4.3. **Proposition.** *Let L be an irreducible typical integrable highest weight module. Then $DS_x(L) = 0$ for any non-zero $x \in \mathfrak{g}_\beta$, where β is an odd isotropic root.*

Proof. Set $\lambda := \rho wt_\Sigma L$; since L is typical, λ does not depend on Σ. First, nore that if λ is non-integral, then $(\mu, \beta) \notin \mathbb{Z}$ for each weght μ of L and each odd root β. Hence, $DS_x(L) = 0$ by Lemma 4.2. Thus, we may assume that λ is integral. By Lemma 2.3.2, we can (and will) assume that $(\lambda, \alpha) > 0$ for each $\alpha \in \Sigma$.

Let $\Sigma = \{\alpha_i\}_{i=0}^n$ and α_1, α_2 are odd. By Proposition 6.4 it sufficies to show that $DS_x(L) = 0$ for $x \in \mathfrak{g}_{\pm\alpha_2}$. Assume that $DS_x(L) \neq 0$. Then by Lemma 4.2 there exists a weight μ in L such that $(\mu, \alpha_2) = 0$ and $(\mu + \rho_\Sigma, \mu + \rho_\Sigma) = (\lambda, \lambda)$, or equivalently $(\lambda - \mu - \rho_\Sigma, \lambda + \mu + \rho_\Sigma) = 0$. On the other hand, since $\lambda - \rho_\Sigma$ is the highest weight of L we have

$$\lambda - \mu - \rho_\Sigma = \sum_{i=0}^n k_i \alpha_i$$

for some non-negative integers $k_0, \ldots, k_n$. Set $a_i := (\mu + \rho_\Sigma, \alpha_i)$. Combining Lemma 4.2 and Proposition 6.4, we conclude that $\mu|_{\mathfrak{h}_x}$ is an integrable weight, that is

$$a_2 = 0, \quad a_1 + a_3 \geq 0, \quad a_i > 0 \text{ for } \quad i \neq 1, 2, 3. \tag{5}$$

Set $\lambda' := \mu + \rho_\Sigma - a_1\alpha_2, \quad \nu := \lambda - \lambda'$. One has

$$(\lambda', \alpha_1) = (\lambda', \alpha_2) = 0, \quad (\lambda', \alpha_i) \geq 0 \text{ for } \quad i = 0, \ldots, n.$$

By above, $(\mu + \rho_\Sigma, \mu + \rho_\Sigma) = (\lambda, \lambda)$ and $(\mu, \alpha_2) = 0$, so $(\mu + \rho_\Sigma, \mu + \rho_\Sigma) = (\lambda', \lambda')$. Thus, $(\lambda, \lambda) = (\lambda', \lambda')$ or, equivalently, $(\nu, \lambda + \lambda') = 0$. Since $a_1 = (\lambda, \alpha_1) + k_0 - k_2$, one has $k_2 + a_1 \geq 0$ and therefore $\nu \in \mathbb{Z}_{\geq 0}\Sigma$.

Since $(\lambda, \alpha_i) > 0$ and $(\lambda', \alpha_i) \geq 0$ for each $i = 0, \ldots, n$ we obtain $\lambda = \lambda'$. However, $(\lambda', \alpha_2) = 0$, a contradiction. □

Recall that, by Lemma 3.1.1, a Verma module $M(\lambda)$ has at most two integrable quotients: $L(\lambda)$ and $V(\lambda)$ such that $V(\lambda)/L(\lambda - \beta) = L(\lambda)$.

4.4. **Proposition.** *Let N be an integrable quotient of an atypical Verma module $M(\lambda)$.*

(i) *$DS_x(N) \cong L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})^{\oplus s}$, where $s = 1$ if $N = L(\lambda)$ and $s = 0$ or $s = 2$ otherwise.*
(ii) *Let $(\lambda, \beta) = 0$ for an isotropic simple root β. Then*

$$DS_x(N) \cong L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})^{\oplus s} \text{ where } \begin{cases} s = 1 & \text{if } N = L(\lambda), \\ s = 0 & \text{if } x \in \mathfrak{g}_{-\beta}, \ N \neq L(\lambda), \\ s = 2 & \text{if } x \in \mathfrak{g}_\beta, \ N \neq L(\lambda). \end{cases}$$

Proof. By 3.1, $M(\lambda) = M_{\Sigma'}(\lambda')$, where $(\lambda', \alpha) = 0$ for some isotropic $\alpha \in \Sigma'$. Thus for (i) we can assume that $(\lambda, \beta) = 0$ for an isotropic simple root β. By above, we have $DS_x(N) = DS_y(N)$, where y in $\mathfrak{g}_\beta$ or in $\mathfrak{g}_{-\beta}$. Therefore (i) is reduced to (ii). Let us prove (ii).

By Proposition 6.4 $DS_x(N)$ is $\mathfrak{g}_x$-integrable (where $\mathfrak{g}_x = \mathfrak{sl}_n^{(1)}$), so completely reducible. Let L' be a simple submodule of $DS_x(N)$. By Lemma 4.2 there exists a weight

μ in N such that $\mu|_{\mathfrak{h}_x}$ is the highest weight of L' and $(\mu,\beta)=0$, $(\mu+\rho_\Sigma,\mu+\rho_\Sigma)=(\lambda+\rho_\Sigma,\lambda+\rho_\Sigma)$. Set $\nu:=\lambda-\mu$. Then

$$(\nu,\beta)=0,\quad (\lambda+\rho_\Sigma,\nu)+(\lambda+\rho_\Sigma-\nu,\nu)=0$$

and $\nu\in\mathbb{Z}_{\geq 0}\Sigma$ that is $\nu\in\mathbb{Z}_{\geq 0}\Sigma_x+\mathbb{Z}\beta$.

Since N is integrable and $(\lambda,\beta)=0$, we get $(\lambda,\alpha)\geq 0$ for each $\alpha\in\Sigma$. Thus $(\lambda+\rho_\Sigma,\nu)\geq 0$ and so $(\lambda+\rho_\Sigma-\nu,\nu)\leq 0$.

Since L' is $\mathfrak{g}_x$-integrable and $\nu\in\mathbb{Z}_{\geq 0}\Sigma_x+\mathbb{Z}\beta$, one has $(\lambda+\rho-\nu,\nu)\geq 0$ and the equality holds if and only if $\nu\in\mathbb{Z}\beta$. Therefore, $\nu\in\mathbb{Z}\beta$. Since $\lambda-\nu$ is a weight of N, one has $\nu\in\{0,\beta\}$. Hence,

$$DS_x(N)=L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})^{\oplus s},\quad \text{where } s:=\dim DS_x(N_\lambda\oplus N_{\lambda-\beta}).$$

Note that $N':=N_\lambda\oplus N_{\lambda-\beta}$ is a module over a copy of $\mathfrak{sl}(1|1)$ generated by $\mathfrak{g}_{\pm\beta}$ (one has $x\in\mathfrak{sl}(1|1)$). If $N=L(\lambda)$, then N' is a trivial $\mathfrak{sl}(1|1)$-module; and if $N/L(\lambda-\beta)=L(\lambda)$, then N' is a Verma $\mathfrak{sl}(1|1)$-module of highest weight zero. The assertion follows. □

4.5. **Corollary.** *Let $L\in\mathcal{F}_k$ be an irreducible module. Then $DS_x(L)=0$ if and only if L is typical. For atypical L, $DS_x(L)$ is integrable $\mathfrak{sl}_{n-1}^{(1)}$-module and $DS_x(L)\cong DS_x(L')$ if and only L and L' lie in the same block.*

Proof. Retain notation of Theorem 2.5.4. If L_j, L_{j+1} are simple objects in an atypical block $\mathcal{B}$ and $j\geq 0$ (resp. $j<-1$), then there exists a Verma module $M(\lambda)$ such that its maximal integrable quotient $V(\lambda)$ such that $V(\lambda)/L_j\cong L_{j+1}$ (resp., $V(\lambda)/L_{j+1}\cong L_j$). From Proposition 4.4, we get $DS_x(L_j)\cong DS_x(L_{j+1})$, so $DS_x(L)$ is a non-zero invariant of an atypical block.

Let us show that this invariant separates blocks. Fix a set of simple roots Σ and take $x\in\mathfrak{g}_{-\alpha_2}$. Let $\lambda^\#\in\mathfrak{h}_x$ be the highest weight of $DS_x(L)$, $DS_x(L')$. Let us show that L, L' are in the same block. Indeed, each block contains a unique Σ-singular irreducible module. Thus we can (and will) assume that L, L' are Σ-singular. Let $L=L(\lambda)$, $L'=L(\lambda')$. One has $\lambda^\#=\lambda|_{\mathfrak{h}_x}=\lambda'|_{\mathfrak{h}_x}$. Since λ,λ' are Σ-singular, $\lambda=\lambda'$, that is $L\cong L'$ as required. □

5. Modules Over Simple Affine Vertex Superalgebras

In this section

$$\mathfrak{g}=\mathbb{C}d\oplus\mathbb{C}K\oplus\bigoplus_{n\in\mathbb{Z}}\dot{\mathfrak{g}}\otimes t^n$$

is an untwisted affine Lie superalgebra, i.e. the affinization of a finite-dimensional Kac–Moody Lie superalgebra $\dot{\mathfrak{g}}$ and $k\neq -h^\vee$. Here $d\in\mathfrak{h}$ is the standard element ($[d,xt^s]=sxt^s$ for $x\in\dot{\mathfrak{g}}$) and $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]\oplus\mathbb{C}d$.

5.1. Definitions. Recall that a $[\mathfrak{g},\mathfrak{g}]$-module (resp., $\mathfrak{g}$-module) N is called *restricted* if for every $a \in \dot{\mathfrak{g}}$, $v \in N$ there exists n such that $(at^m)v = 0$ for each $m > n$. A particular case of the restricted $\mathfrak{g}$-modules are the *bounded modules*, i.e. the modules where d acts diagonally with integral eigenvalues bounded from above; as before, denote by N^{top} the eigenspace with the maximal eigenvalue. A bounded module N is called *almost irreducible* if any nontrivial submodule of N has a non-zero intersection with N^{top}.

Let N be a restricted $[\mathfrak{g},\mathfrak{g}]$-module of level k with $k \neq -h^\vee$. The Sugawara construction equips N with an action of the Virasoro algebra $\{L_n\}_{n\in\mathbb{Z}}$, see [13], 12.8 for details. Moreover, the $[\mathfrak{g},\mathfrak{g}]$-module structure on N can be extended to a $\mathfrak{g}$-module structure by setting $d|_N := -L_0|_N$.

For a restricted $\mathfrak{g}$-module the action of L_0 and the Casimir element Ω are related by the formula $\Omega = 2(K + h^\vee)(d + L_0)$. Therefore, the above procedure assigns to a restricted $[\mathfrak{g},\mathfrak{g}]$-module of level $k \neq -h^\vee$ a restricted $\mathfrak{g}$-module with the zero action of the Casimir operator.

5.2. The subalgebra $\mathfrak{g}^\#$. Recall that (for affine $\mathfrak{g}$) the $\mathfrak{g}_{\bar{0}}$-integrable modules exist only at level zero or in the case when the Dynkin diagram of $\mathfrak{g}_{\bar{0}}$ is connected, see [15]. We consider the integrability with respect to the "largest affine subalgebra" of $\mathfrak{g}_{\bar{0}}$, see below.

Recall that $\dot{\mathfrak{g}}_{\bar{0}}$ is a reductive Lie algebra and it can be decomposed as $\dot{\mathfrak{g}}_{\bar{0}} = \dot{\mathfrak{g}}^\# \times \dot{\mathfrak{t}}$, where $\dot{\mathfrak{g}}^\#$ is a simple Lie algebra (the "largest part" of $\dot{\mathfrak{g}}_{\bar{0}}$) and $\mathfrak{t}$ is a reductive Lie algebra:

for $\dot{\mathfrak{g}} = \mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|n)$ with $n \geq m$ one has $\dot{\mathfrak{g}}^\# = \mathfrak{sl}_n, \mathfrak{sp}_n$ respectively;

for $\dot{\mathfrak{g}} = \mathfrak{osp}(m|n)$ with $m > n$ one has $\dot{\mathfrak{g}}^\# = \mathfrak{so}_m$;

for the exceptional Lie superalgebras $F(4)$, $G(3)$ one has $\dot{\mathfrak{g}}^\# = B_3, G_2$ respectively; for $D(2,1,a)$ we have $\dot{\mathfrak{g}}_{\bar{0}} = A_1 \times A_1 \times A_1$ with the corresponding roots $\alpha_1, \alpha_2, \alpha_3$ subject to the relation $||\alpha_1||^2 : ||\alpha_2||^2 : ||\alpha_3||^2 = 1 : a : (-a-1)$; we take $\dot{\mathfrak{g}}^\# = A_1$, which corresponds any copy of A_1 if $a \notin \mathbb{Q}$ and the copy with the root α_i ($i \in \{1,2,3\}$) such that $|||\alpha_i||^2|$ is maximal (see [10], 6.1).

We have a natural embedding of the affine algebra $\mathfrak{g}^\#$ (which is the affinization of $\dot{\mathfrak{g}}^\#$) to $\mathfrak{g}_{\bar{0}}$.

5.3. Modules over affine vertex superalgebras. Let $V^k(\mathfrak{g})$ be the affine vertex algebra and $V_k(\mathfrak{g})$ be its simple quotient.

There is a natural equivalence between the categories of $V^k(\mathfrak{g})$-modules and the restricted $[\mathfrak{g},\mathfrak{g}]$-modules of level k if $k \neq -h^\vee$, see [6], Thm. 2.4.3.

If $\mathfrak{g}$ is a Lie algebra and $k \neq 0$ is such that $V_k(\mathfrak{g})$ is $[\mathfrak{g},\mathfrak{g}]$-integrable, then the $V_k(\mathfrak{g})$-modules correspond to the integrable $[\mathfrak{g},\mathfrak{g}]$-modules, see [6], Thm. 3.1.3 and [4], Thm. 3.7.

5.3.1. **Theorem.** *If $V_k(\mathfrak{g})$ is integrable as a $\mathfrak{g}^\#$-module and $k \neq 0$, then the $V_k(\mathfrak{g})$-modules are the restricted $[\mathfrak{g},\mathfrak{g}]$-modules of level k which are integrable over $\mathfrak{g}^\#$. As $\mathfrak{g}^\#$-modules these modules are direct sums of irreducible integrable highest weight modules.*

The $V_0(\mathfrak{g})$-modules are trivial.

5.3.2. Remark. Normalize the non-degenerate bilinear form by the condition $(\alpha,\alpha) = 2$, where α is the longest root in $\dot{\mathfrak{g}}^\#$. Then $V_k(\mathfrak{g})$ is integrable over $\mathfrak{g}^\#$ if and only if k is a non-negative integer.

5.3.3. Using Sect. 5.1 we obtain the following corollary.

Corollary. *For $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}$ and a positive integer k, the category of finitely generated $V_k(\mathfrak{g})$-modules with finite-dimensional weight spaces is the full subcategory of $\mathcal{F}_k$ whose objects are annihilated by the Casimir operator.*

5.4. Proof of Theorem 5.3.1. Introduce the vacuum $\mathfrak{g}$-module of level k:

$$V^k := \mathrm{Ind}^{\mathfrak{g}}_{\dot{\mathfrak{g}}+\mathfrak{n}+\mathfrak{h}} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the trivial $\dot{\mathfrak{g}} + \mathfrak{n}$-module with K acting by kId and d acting by zero. As a $[\mathfrak{g}, \mathfrak{g}]$-module $V^k(\mathfrak{g})$ is isomorphic to V^k.

Let $\Lambda_0 \in \mathfrak{h}^*$ be such that $\Lambda_0(K) = 1$, $\Lambda_0(\dot{\mathfrak{h}}) = \Lambda_0(d) = 0$. Note that V^k is a $\dot{\mathfrak{g}}_{\bar{0}}$-integrable quotient of the Verma module $M(k\Lambda_0)$ and that $L(k\Lambda_0)$ is a unique simple quotient of V^k. As a $[\mathfrak{g}, \mathfrak{g}]$-module $V_k(\mathfrak{g})$ is isomorphic to $L(k\Lambda_0)$.

Recall that for a given non-degenerate bilinear form $h^\vee = (\rho, \delta)$, where ρ is the Weyl vector and δ is the minimal imaginary root.

***5.4.1.* Theorem.** *Let $k \neq -h^\vee$ be such that $L(k\Lambda_0)$ is $\mathfrak{g}^\#$-integrable. Then $L(k\Lambda_0)$ is a unique $\mathfrak{g}^\#$-integrable quotient of V^k.*

Proof. Let N be some non-zero integrable quotient of V^k. From [10] it follows that the character of N is given by the KW-character formula (see Sect. 4 for the cases $h^\vee \neq 0$ and for $A(n, n)^{(1)}$, and Sect. 6 for the remaining cases). Hence N is irreducible. □

5.4.2. We denote by $|0\rangle$ the highest weight vector of V^k (and its image in $L(k\Lambda_0)$).

We normalize the bilinear form as in Remark 5.3.2 and fix a triangular decomposition in $\dot{\mathfrak{g}}$ in such a way that the maximal root θ lies in the root system of $\dot{\mathfrak{g}}^\#$. Then $\alpha_0 = \delta - \theta$ is a simple root and $(\alpha_0, \alpha_0) = 2$. Let f_0 be a non-zero element in $\mathfrak{g}_{-\alpha_0}$; note that $f_0 \in \mathfrak{g}^\#$.

***5.4.3.* Corollary.** *Let $k \neq -h^\vee$ be such that $L(k\Lambda_0)$ is $\mathfrak{g}^\#$-integrable. Then $L(k\Lambda_0) = V^k/I$, where the submodule I is generated by $f_0^{k+1}|0\rangle$.*

Proof. Since $L(k\Lambda_0)$ is $\mathfrak{g}^\#$-integrable, $f_0^{k+1}|0\rangle$ is a singular vector in V^k. Let I be the submodule of V^k generated by this vector. By Theorem 5.4.1, it is enough to show that V^k/I is $\mathfrak{g}^\#$-integrable. From [13], Lemmas 3.4, 3.5, it suffices to check that for each α in the set of simple roots of $\mathfrak{g}^\#$ the root spaces $\mathfrak{g}_{\pm\alpha}$ act nilpotently on v, where v is the image of $|0\rangle$ in V^k/I. Clearly, $\mathfrak{g}_{\pm\alpha}|0\rangle = 0$ for $\alpha \neq \alpha_0$ and $\mathfrak{g}_{\alpha_0} v = \mathfrak{g}^{k+1}_{-\alpha_0} v = 0$. The assertion follows. □

5.4.4. Remark. Theorem 5.4.1 and Corollary 5.4.3 hold also in the case when $\mathfrak{g}$ is a twisted affinization ($\mathfrak{g}$ is any symmetrizable affine Lie superalgebra). In Corollary 5.4.3 the following change should be done if $\frac{\alpha_0}{2} \in \Delta$: f_0 should be chosen in $\mathfrak{g}_{-\alpha_0/2}$ and I is generated by $f_0^{2k+1}|0\rangle$. The proofs are the same.

5.4.5. For each $a \in V^k(\mathfrak{g})$ let $Y(a, z)$ be the corresponding vertex operator. The following lemma is standard (see, for example, [1], Prop. 3.4).

Lemma. *Let $I \subset V^k(\mathfrak{g})$ be a cyclic submodule generated by a vector $a \in V^k(\mathfrak{g})$. A $V^k(\mathfrak{g})$-module N is a $V^k(\mathfrak{g})/I$-module if and only if $Y(a, z)N = 0$.*

5.4.6. By [9], Thm. 3.2.1 any restricted integrable $[\mathfrak{g}^\#, \mathfrak{g}^\#]$-module is completely reducible. Let us show that $V_k(\mathfrak{g})$-modules are restricted $[\mathfrak{g}, \mathfrak{g}]$-modules of level k which are integrable over $[\mathfrak{g}^\#, \mathfrak{g}^\#]$.

Take $k = 0$. Then $V_k(\mathfrak{g})$ is one-dimensional. Hence, $V_k(\mathfrak{g})$-modules are restricted $[\mathfrak{g}, \mathfrak{g}]$-modules of zero level which are annihilated by $[\mathfrak{g}, \mathfrak{g}]$.

Take $k \neq 0$. From Lemma 5.4.5 and Corollary 5.4.3, we conclude that $V_k(\mathfrak{g})$-modules are restricted $[\mathfrak{g}, \mathfrak{g}]$-modules which are annihilated by $Y(f_0^{k+1}|0\rangle, z)$. Note that $Y(f_0^{k+1}|0\rangle, z) \in V^k(\mathfrak{g}^\#)$ and $V_k(\mathfrak{g}^\#) := V^k(\mathfrak{g}^\#)/I'$, where I' is the $\mathfrak{g}^\#$-submodule of $V^k(\mathfrak{g}^\#)$ which is generated by $f_0^{k+1}|0\rangle$. In particular, $V_k(\mathfrak{g}^\#)$ is a subalgebra of $V_k(\mathfrak{g})$. By [4], Thm. 3.7, the $V_k(\mathfrak{g}^\#)$-modules are direct sums of irreducible integrable highest weight $[\mathfrak{g}^\#, \mathfrak{g}^\#]$-modules of level k. We conclude that the $V_k(\mathfrak{g})$-modules are the restricted integrable $[\mathfrak{g}^\#, \mathfrak{g}^\#]$-modules of level k as required. This completes the proof of Theorem 5.3.1. □

5.5. Integrable bounded $\mathfrak{g}$-modules. If $\mathfrak{g}$ is an affine Lie algebra, then, by [9], Thm. 3.2.1, the restricted integrable $[\mathfrak{g}, \mathfrak{g}]$-modules are completely reducible and the irreducible ones are highest weight modules. The situation is similar for $\mathfrak{g} = \mathfrak{osp}(1|2n)^{(1)}$, but is different for other affine Lie superalgebras, see Sect. 5.6.4 below.

5.5.1. **Proposition.** *If N is a bounded $\mathfrak{g}$-module which is $[\dot{\mathfrak{g}}_{\bar{0}}, \dot{\mathfrak{g}}_{\bar{0}}]$-integrable, then $N^{\mathfrak{n}} \neq 0$.*

Proof. Set $E := N^{top}$. Since $E^{\dot{\mathfrak{n}}} \subset N^{\mathfrak{n}}$, it is enough to show that $E^{\dot{\mathfrak{n}}} \neq 0$.

Note that $\mathfrak{s} := [\dot{\mathfrak{g}}_{\bar{0}}, \dot{\mathfrak{g}}_{\bar{0}}]$ is a semimple Lie algebra. Note that E is a $\dot{\mathfrak{g}}$-module which is $\mathfrak{s}$-integrable. Therefore E is a direct sum of finite-dimensional $\mathfrak{s}$-modules. In particular, $\dot{\mathfrak{n}}_{\bar{0}}$ acts locally nilpotently on E. Therefore, $\dot{\mathfrak{n}}$ acts locally nilpotently on E. Let $0 = \dot{\mathfrak{n}}^0 \subset \dot{\mathfrak{n}}^1 \subset \ldots \subset \dot{\mathfrak{n}}^s = \dot{\mathfrak{n}}$ be the derived series of $\dot{\mathfrak{n}}$ ($\dot{\mathfrak{n}}^i = [\dot{\mathfrak{n}}^{i+1}, \dot{\mathfrak{n}}^{i+1}]$). Set $E(0) := E$ and $E(i) := E(i-1)^{\dot{\mathfrak{n}}^i}$ for $i = 1, \ldots, s$. By induction $E(i) \neq 0$, since $\mathfrak{n}^i/\mathfrak{n}^{i-1}$ is a finite-dimensional abelian Lie superalgebra which acts locally nilpotently on $E(i-1)$. Hence, $E^{\dot{\mathfrak{n}}} = E(s) \neq 0$ as required. □

5.5.2. Remark. From Proposition 5.5.1 a bounded irreducible $\mathfrak{g}$-module which is integrable over $[\dot{\mathfrak{g}}_{\bar{0}}, \dot{\mathfrak{g}}_{\bar{0}}]$ is a highest weight module. In particular, a bounded irreducible $\mathfrak{sl}(1|n)^{(1)}$-module which is $\mathfrak{sl}_n^{(1)}$-integrable is an irreducible highest weight module.

5.6. Bounded $V_k(\mathfrak{g})$-modules. A $V^k(\mathfrak{g})$-module is called *positive energy* (see [3]) if it is $\mathbb{Z}$-graded $[\mathfrak{g}, \mathfrak{g}]$-module of level k: $M = \oplus_{m\in\mathbb{Z}} M_m$ with $(at^n)M_m \subset M_{m-n}$ with the grading bounded from below. For such a module we extend the $[\mathfrak{g}, \mathfrak{g}]$-action to the $\mathfrak{g}$-action by $dv := -mv$ for $v \in M_m$. Thus the positive energy $V^k(\mathfrak{g})$-modules correspond to the bounded $\mathfrak{g}$-modules of level k. (In [4] a similar object is called an admissible module; in [6] all modules are assumed to be of this form.)

A positive energy $V^k(\mathfrak{g})$-module is *ordinary* (see [4]) if the grading is given by the action of L_0 and the homogeneous components are finite-dimensional. Thus the ordinary modules are the bounded $\mathfrak{g}$-modules of level k with the zero action of the Casimir operator.

5.6.1. Let V be a vertex operator algebra and $A(V)$ be its Zhu algebra.

Thm. 2.30 in [3] (see also Thm. 2.2.1 in [18]) for the trivial twisting gives

Proposition. *The restriction functor $N \mapsto N^{top}$ is a functor from the category of positive energy V-modules up to a shift of grading to the category of $A(V)$-modules, which is inverse to the induction functor $E \mapsto V(E)$ from the $A(V)$-modules to the full subcategory of almost irreducible V-modules (up to a shift of grading). In particular, these functors establish a bijective correspondence between the irreducible positive energy V-modules and the irreducible $A(V)$-modules.*

For $V = V^k(\mathfrak{g})$ the positive energy V-modules correspond to the bounded $\mathfrak{g}$-modules of level k; the ordinary (see [4]) modules correspond to the bounded modules of level k with the zero action of the Casimir element.

5.6.2. As in [6] Thm. 3.1.1, 3.1.2, the Zhu algebra of $V^k(\mathfrak{g})$ is $\mathcal{U}(\dot{\mathfrak{g}})$ and the Zhu algebra of $V_k(\mathfrak{g})$ is $\mathcal{U}(\mathfrak{g})/(e_\theta^{k+1})$, where $f_0 = e_\theta t^{-1}$ ($e_\theta \in \dot{\mathfrak{g}}_\theta$). This implies the following corollary.

Corollary. *Let k be a non-negative integer and let E be a $\dot{\mathfrak{g}}$-module satisfying $e_\theta^{k+1}E = 0$. There exists a unique almost irreducible $\mathfrak{g}^{\#}$-integrable $\mathfrak{g}$-module $N = \oplus_{i=0}^{\infty} N^i$ of level k such that N^i is the ith eigenspace of $-d$ and $N^0 = E$. This module has a natural structure of V_k-module. Moreover, N is irreducible if and only if E is irreducible.*

5.6.3. If $\mathfrak{g}$ is such that the Dynkin diagram of $\mathfrak{g}_0$ is connected, then $[\dot{\mathfrak{g}}_{\bar{0}}, \dot{\mathfrak{g}}_{\bar{0}}] = \dot{\mathfrak{g}}^{\#}$. Combining Lemma 5.5.1 and Theorem 5.3.1 we obtain the following corollary.

Corollary. *Let $\mathfrak{g}$ be such that the Dynkin diagram of $\mathfrak{g}_0$ is connected and k be a non-negative integer. Then a bounded $V_k(\mathfrak{g})$-module contains a singular vector (v such that $\mathfrak{n}v = 0$). In particular, the irreducible bounded $V_k(\mathfrak{g})$-modules are the $\mathfrak{g}^{\#}$-integrable highest weight $\mathfrak{g}$-modules of level k.*

5.6.4. Below we give an example of a cyclic bounded $\mathfrak{sl}(1|2)^{(1)}$-module which is $\mathfrak{sl}_2^{(1)}$-integrable, but is not $\mathfrak{sl}(1|2)^{(1)}$-integrable (the action of $\mathfrak{h}$ is not locally finite).

Consider the usual $\mathbb{Z}$-grading on $\mathfrak{sl}(1|2)$: $\dot{\mathfrak{g}} = \dot{\mathfrak{g}}_{-1} \oplus \dot{\mathfrak{g}}_0 \oplus \dot{\mathfrak{g}}_1$, where $\dot{\mathfrak{g}}_0 = \dot{\mathfrak{g}}_{\bar{0}} = \mathfrak{sl}_2 \times \mathbb{C}z$ and $\dot{\mathfrak{g}}_{\pm 1}$ are irreducible $\mathfrak{sl}_2$-modules. Let f, h, e be the standard generators of $\mathfrak{sl}_2$. Consider the triangular decomposition of $\mathfrak{g}$ with $\mathfrak{n} = \mathbb{C}e + \dot{\mathfrak{g}}_1 + \sum_{s=1}^{\infty} \dot{\mathfrak{g}}t^s$.

View $\mathbb{C}[z]$ as a module over $\mathfrak{p} := \mathfrak{h} + \dot{\mathfrak{g}}_0 + \mathfrak{n}$ by the trivial action of $\mathbb{C}d + \dot{\mathfrak{g}}_0 + \mathfrak{n}$ and K acting by Id. Consider the induced module $M := Ind_{\mathfrak{p}}^{\mathfrak{g}} \mathbb{C}[z]$. Then M has level 1 and M^{top} is a free $\mathbb{C}[z]$-module. As an $\mathfrak{sl}_2$-module M^{top} is a direct sum of countably many copies of $\Lambda\mathfrak{g}_{-1}$, so $e^2 M^{top} = 0$.

From Corollary 5.6.2 it follows that M has an almost irreducible quotient N which is $\mathfrak{sl}_2^{(1)}$-integrable and $N^{top} = M^{top}$. Since z acts freely on M^{top}, N is not $\mathfrak{sl}(1|n)^{(1)}$-integrable. Note that M is bounded and cyclic (generated by the image of $1 \in \mathbb{C}[z]$), so N is also bounded and cyclic. It is not hard to see that the Casimir acts freely on N.

5.6.5. Remark. The example in Sect. 5.6.4 gives a cyclic $\mathfrak{g}$-bounded $\mathfrak{g}^{\#}$-integrable module of level 1 with a free action of the Casimir operator. In the light of Theorem 5.3.1, this module is a cyclic almost irreducible positive energy $V_1(\mathfrak{g})$-module with a free action of L_0 (in particular, this module is not ordinary, see Sect. 5.6 for definition).

5.6.6. Let us show that if the Dynkin diagram of $\mathfrak{g}_0$ is not connected, then for sufficiently large integral k there exists an irreducible bounded $V_k(\mathfrak{g})$-module which is not a highest weight module.

Let $\mathfrak{g}$ be such that the Dynkin diagram of $\mathfrak{g}_{\bar{0}}$ is not connected. In this case $\dot{\mathfrak{g}}_{\bar{0}} = \dot{\mathfrak{g}}^{\#} \times \mathfrak{t}$, where $\mathfrak{t}$ is semisimple. Take any irreducible $\mathfrak{t}$-module E and view it as $\dot{\mathfrak{g}}_{\bar{0}}$-module via the trivial action of $\dot{\mathfrak{g}}^{\#}$. Set $E' := \mathrm{Ind}_{\dot{\mathfrak{g}}_{\bar{0}}}^{\dot{\mathfrak{g}}} E$. As $\dot{\mathfrak{g}}^{\#}$-module E' is a direct sum of copies of $\Lambda \dot{\mathfrak{g}}_{\bar{1}}$, so there exists m such that $e_{\theta}^{m} E' = 0$. Let L be an irreducible quotient of E'. By Corollary 5.6.2, for each integral $k \geq m - 1$ there exists an irreducible bounded $\mathfrak{g}^{\#}$-integrable $\mathfrak{g}$-module N of level k such that $N^{top} = L$. Note that E is an $\mathfrak{t}$-quotient of L. In particular, $\mathfrak{h}$ acts locally finitely on N if and only if the Cartan algebra of $\mathfrak{t}$ acts locally finitely on E.

6. Appendix: The Functor DS_x

In this section we assume that $\mathfrak{g}$ is a Kac–Moody Lie superalgebra.

Take $x \in \mathfrak{g}_{\bar{1}}$ satisfying $[x, x] = 0$. The following construction is due to Duflo and Serganova, see [5]. For a $\mathfrak{g}$-module N introduce

$$DS_x(N) := Ker_N x / Im_N x.$$

Let $\mathfrak{g}^x$ be the centralizer of x in $\mathfrak{g}$. We view $DS_x(N)$ as a module over $\mathfrak{g}^x$. Note that $[x, \mathfrak{g}] \subset \mathfrak{g}^x$ acts trivially on $DS_x(N)$ and that $\mathfrak{g}_x := DS_x(\mathfrak{g}) = \mathfrak{g}^x/[x, \mathfrak{g}]$ is a Lie superalgebra. Thus, $DS_x(N)$ is a $\mathfrak{g}_x$-module and DS_x is a functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.

In [5,16] the functor DS_x was studied for finite-dimensional $\mathfrak{g}$. However, certain properties can be easily generalized to the affine case. In particular, DS_x is a tensor functor, i.e. there is a canonical isomorphism $DS_x(N_1 \otimes N_2) \simeq DS_x(N_1) \otimes DS_x(N_2)$.

***6.1.* Proposition.** *Let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ be the affinization of a Lie superalgebra $\dot{\mathfrak{g}}$ and assume that $x \in \dot{\mathfrak{g}}$. If $\dot{\mathfrak{g}}_x \neq 0$, then $\mathfrak{g}_x$ is the affinization of $\dot{\mathfrak{g}}_x$, If $\dot{\mathfrak{g}}_x = 0$ then $\mathfrak{g}_x$ is the abelian two-dimensional Lie algebra generated by K and d.*

Proof. Since

$$\mathfrak{g} = \mathbb{C}d \oplus \mathbb{C}K \oplus \bigoplus_{n \in \mathbb{Z}} \dot{\mathfrak{g}} \otimes t^n$$

and $\dot{\mathfrak{g}} \otimes t^n$ is isomorphic to the adjoint representation of $\dot{\mathfrak{g}}$ for every n, the statement follows. □

6.2. Let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ be the affinization of a Lie superalgebra $\dot{\mathfrak{g}}$ and assume that $x \in \dot{\mathfrak{g}}$. Let $\dot{\Sigma}$ (resp., Σ) be the set of simple roots of $\dot{\mathfrak{g}}$ (resp., $\mathfrak{g}$).

Let $\beta_1, \dots \beta_r \in \dot{\Sigma}$ be a set of mutually orthogonal isotopic simple roots, fix non-zero root vectors $x_i \in \mathfrak{g}_{\beta_i}$ for all $i = 1, \dots, r$. Let $x := x_1 + \dots + x_r$. It is shown in [5] that $\dot{\mathfrak{g}}_x$ is a finite-dimensional Kac–Moody superalgebra with roots

$$\dot{\Delta}^{\perp} := \{\alpha \in \dot{\Delta} | \ (\alpha, \beta_i) = 0, \alpha \neq \pm\beta_i \ i = 1, \dots, r\}$$

and the Cartan subalgebra

$$\mathfrak{h}_x := (\beta_1^{\perp} \cap \dots \cap \beta_r^{\perp})/(\mathbb{C}h_{\beta_1} \oplus \dots \oplus \mathbb{C}h_{\beta_r}).$$

Assume that $\dot{\Delta}^{\perp}$ is not empty, then $\dot{\Delta}^{\perp}$ is the root system of the Lie superalgebra $\dot{\mathfrak{g}}_x$. One can choose a set of simple roots $\dot{\Sigma}_x$ such that $\Delta^+(\dot{\Sigma}_x) = \Delta^+ \cap \dot{\Delta}^{\perp}$. Let $\mathfrak{g}_x \subset \mathfrak{g}$ be the affinization of $\dot{\mathfrak{g}}_x$: the affine Lie superalgebra with a set of simple roots Σ_x containing $\dot{\Sigma}_x$ such that $\Delta^+(\Sigma_x) \subset \Delta^+$.

For example, if $\dot{\mathfrak{g}} = A(m|n)$, $B(m|n)$ or $D(m|n)$, then $\dot{\mathfrak{g}}_x = A(m-r|n-r)$, $B(m-r|n-r)$ or $D(m-r|n-r)$. If $\dot{\mathfrak{g}} = C(n)$, G_3 or F_4, then $r = 1$ and $\dot{\mathfrak{g}}_x$ is the Lie algebra of type C_{n-1}, A_1 and A_2 respectively. If $\dot{\mathfrak{g}} = D(2, 1; \alpha)$, then $r = 1$ and $\mathfrak{g}_x = \mathbb{C}$.

***6.3.* Proposition.** *Let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ be the affinization of a Lie superalgebra $\dot{\mathfrak{g}}$ and assume that $x \in \dot{\mathfrak{g}}$. Let $x \in \dot{\mathfrak{g}}$ and N be a restricted $\mathfrak{g}$-module. If the Casimir element $\Omega_{\mathfrak{g}}$ acts on a N by a scalar C, then the Casimir element $\Omega_{\mathfrak{g}_x}$ acts on the $\mathfrak{g}_x$-module $DS_x(N)$ by the same scalar C.*

Proof. Let us write the Casimir element $\Omega_{\mathfrak{g}}$ in the following form (see [13], (12.8.3))

$$\Omega_{\mathfrak{g}} = 2(h^{\vee} + K)d + \Omega_0 + 2\sum_{i=1}^{\infty} \Omega(i),$$

where $\Omega(i) = \sum v_j v^j$ for some basis $\{v_j\}$ in $\dot{\mathfrak{g}} \otimes t^{-i}$ and the dual basis $\{v^j\}$ in $\dot{\mathfrak{g}} \otimes t^i$. Similarly we have

$$\Omega_{\mathfrak{g}_x} = 2(h^{\vee} + K)d + \Omega_0 + 2\sum_{i=1}^{\infty} \Omega_x(i).$$

We claim that $\Omega_x(i) \equiv \Omega(i)(\mathrm{mod}[x, U(\mathfrak{g})])$. Indeed, we use the decomposition $\dot{\mathfrak{g}} = \dot{\mathfrak{g}}_x \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a free $\mathbb{C}[x]$-module. Using a suitable choice of bases we can write

$$\Omega(i) = \Omega_x(i) + \sum u_s u^s$$

for the pair of dual bases $\{u_s\}$ in $\mathfrak{m} \otimes t^{-i}$ and $\{u^s\}$ in $\mathfrak{m} \otimes t^i$. If $i > 0$, then $\sum u_s u^s$ is x-invariant element via the embedding $\mathfrak{m} \otimes \mathfrak{m} \hookrightarrow U(\mathfrak{g})$. If $i = 0$, then $\sum u_s u^s$ is x-invariant element via the embedding $S^2(\mathfrak{m}) \hookrightarrow U(\mathfrak{g})$. Since $\mathfrak{m} \otimes \mathfrak{m}$ and $S^2(\mathfrak{m})$ are free $\mathbb{C}[x]$-modules, we obtain in both cases that $\sum u_s u^s$ lies in the image of $\operatorname{ad} x$.

Now the statement follows from the fact that $[x, U(\mathfrak{g})]$ annihilates $DS_x(N)$. □

***6.4.* Proposition.** *If N is an integrable $\mathfrak{g}$-module, then $DS_x(N)$ is an integrable $\mathfrak{g}_x$-module. Moreover, if $x = w(y)$ for some element y of the Weyl group of $\mathfrak{g}_{\bar{0}}$, then $\mathfrak{g}_x \cong \mathfrak{g}_y$ and $DS_x(N) \cong DS_y(N)$.*

Proof. The first statement is obvious and the second is an immediate consequence of the identities $\mathfrak{g}_x = w(\mathfrak{g}_y)$, $DS_x(N) = w(DS_y(N))$. □

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