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Relative entropy in CFT

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ABSTRACT

Inspired by Edward Witten's questions, we compute the mutual information associated with free fermions, and we deduce many results about entropies for chiral CFT's which are embedded into free fermions, and their extensions. Such relative entropies in CFT are here computed explicitly for the first time in a mathematical rigorous way, and Our results agree with previous computations by physicists based on heuristic arguments; in addition we uncover a surprising connection with the theory of subfactors, in particular by showing that a certain duality, which is argued to be true on physical grounds, is in fact violated if the global dimension of the conformal net is greater than 1.

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1. Introduction

In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; see the references in [12] for a partial list of references. However, some very basic mathematical questions remain open. For example, most of the entropies computed in the physics literature are infinite, so the singularity structures, and sometimes the cut off independent quantities, are of most interest. Often, the mutual information is argued to be finite based on heuristic physical arguments, and one can derive the singularities of the entropies from the mutual information by taking singular limits. But it is not even clear that such mutual information, which is well defined as a special case of Araki's relative entropy, is indeed finite.

In this paper we begin to address some of these fundamental mathematical questions motivated by the physicists' work on entropy. For related works, see [12], [19] and [25]. Unlike the main focus in [12], the mutual information considered in our paper can be computed explicitly in many cases and satisfies many conditions, but not all, proposed by physicists such as those in [8]. Our project is strongly motivated by Edward Witten's questions, in particular his question to make physicists' entropy computations rigorous. In this paper we focus on the Chiral CFT in two dimensions, where the results we have obtained are most explicit and have interesting connections to subfactor theory, even though some of our results, such as Theorem 4.4, do not depend on conformal symmetries and apply to more general QFT. The main results are:

1) Theorem 3.18: Exact computation of the mutual information (through the relative entropy as defined by Araki for general states on von Neumann algebras) for free fermions. Note that this was not even known to be finite, for example the main quantity defined in [12] is smaller and does not seem to verify the conditions in the physical literature. Our proof uses Lieb's convexity and the theory of singular integrals; to the best of our knowledge, by Theorem 3.18, Theorem 4.2 and examples in Section 4.4, this is the first time that such relative entropies are computed in a mathematical rigorous way. The results verify earlier computations by physicists based on heuristic arguments, such as P. Calabrese and J. Cardy in [5] and H. Casini and M. Huerta in [9].

In particular, for the free chiral net $\mathcal{A}_r$ associated with r fermions, and two intervals $A = (a_1, b_1)$, $B = (a_2, b_2)$ of the real line, where $b_1 < a_2$, the mutual information associated with A, B is

$$F(A, B) = -\frac{r}{6} \ln \eta \ ,$$

where $\eta = \frac{(b_1-a_2)(b_2-a_1)}{(b_1-a_1)(b_2-a_2)}$ is the cross ratio of A, B, $0 < \eta < 1$.

2) It follows from 1) and the monotonicity of the relative entropy that any chiral CFT in two dimensions that embeds into free fermions, and their finite index extensions, verify most of the conditions (not all, see Section 4.2.1) discussed for example in [8], see Theorem 4.1. This includes a large family of chiral CFTs. Much more can be obtained if

the embedding has finite index as in Theorem 4.2. In this case, we also verify a proposal (cf. (1) of Theorem 4.4) in [8] about an entropy formula related to a derivation of the c theorem. Theorem 4.2 also connects relative entropy and index of subfactors in an interesting and unexpected way. There is one bit of surprise: it is usually postulated that the mutual information of a pure state such as vacuum state for complementary regions should be the same. But in the Chiral case this is not true, and the violation is measured by global dimension of the chiral CFT as will be seen in Section 4.2.1.

The physical meaning of the last part of (2) is not clear to us. The violation, which is in some sense proportional to the logarithm of global index, also turns out to be what is called topological entanglement entropy (cf. Remark 4.3). In [13] the authors discuss chiral theories where entanglement entropy cannot be defined with the expected properties due to anomalies. The relation to our work is not clear. On the other hand, when considering full CFT, one does have global dimension equal to 1, and it remains an interesting question to investigate entropies in the full CFT framework.

The rest of this paper is structured as follows. After a preliminary section on von Neumann entropy, Araki's relative entropy, graded nets and subnets, we consider the computation of mutual information in §3. In §4, we derive many of the properties of the mutual information in the vacuum state for all Chiral CFT which are embedded into free fermions, and their extensions, from the results of §3. In the last section we supply with two families of chiral CFT where our main results apply.

2. Preliminaries

2.1. Entropy and relative entropy

von Neumann entropy is the quantity associated with a density matrix ρ on a Hilbert space $\mathcal{H}$ by

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \ .$$

Von Neumann entropy can be viewed as a measure of the lack of information about a system to which one has ascribed the state. This interpretation is in accord for instance with the facts that $S(\rho) \geq 0$ and that a pure state $\rho = |\Psi\rangle\langle\Psi|$ has vanishing von Neumann entropy.

A related notion is that of the relative entropy. It is defined for two density matrices ρ, ρ' by

$$S(\rho, \rho') = \mathrm{Tr}(\rho \log \rho - \rho \log \rho') \ . \tag{1}$$

Like $S(\rho)$, $S(\rho, \rho')$ is non-negative, and can be infinite.

A generalization of the relative entropy in the context of von Neumann algebras of arbitrary type was found by Araki [2] and is formulated using modular theory. Given two faithful, normal states ω, ω' on a von Neumann algebra $\mathcal{A}$ in standard form, we choose the vector representatives in the natural cone $\mathcal{P}^\sharp$, called $|\Omega\rangle, |\Omega'\rangle$. The anti-linear operator $S_{\omega,\omega'} a|\Omega'\rangle = a^*|\Omega\rangle$, $a \in \mathcal{A}$, is closable and one considers again the polar decomposition of its closure $\bar{S}_{\omega,\omega'} = J\Delta^{1/2}_{\omega,\omega'}$. Here J is the modular conjugation of $\mathcal{A}$ associated with $\mathcal{P}^\sharp$ and $\Delta_{\omega,\omega'} = S^*_{\omega,\omega'}\bar{S}_{\omega,\omega'}$ is the relative modular operator w.r.t. $|\Omega\rangle, |\Omega'\rangle$. Of course, if $\omega = \omega'$ then $\Delta_\omega = \Delta_{\omega,\omega'}$ is the usual modular operator.

A related object is the Connes cocycle (Radon–Nikodym derivative) defined as $[D\omega : D\omega']_t = \Delta^{it}_{\omega,\psi}\Delta^{it}_{\psi,\omega'} \in \mathcal{A}$, where ψ is an arbitrary auxiliary faithful normal state on $\mathcal{A}'$.

Definition 2.1. The relative entropy w.r.t. ω and ω' is defined by

$$S(\omega, \omega') = \langle\Omega|\log\Delta_{\omega,\omega'}\,\Omega\rangle = \lim_{t\to 0}\frac{\omega([D\omega : D\omega']_t - 1)}{it}\ , \tag{2}$$

S is extended to positive linear functionals that are not necessarily normalized by the formula $S(\lambda\omega, \lambda'\omega') = \lambda S(\omega, \omega') + \lambda\log(\lambda/\lambda')$, where $\lambda, \lambda' > 0$ and ω, ω' are normalized. If ω' is not normal, then one sets $S(\omega, \omega') = \infty$.

For a type I algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, states ω, ω' correspond to density matrices ρ, ρ'. The square root of the relative modular operator $\Delta^{1/2}_{\omega,\omega'}$ corresponds to $\rho^{1/2} \otimes \rho'^{-1/2}$ in the standard representation of $\mathcal{A}$ on $\mathcal{H} \otimes \bar{\mathcal{H}}$; namely $\mathcal{H} \otimes \bar{\mathcal{H}}$ is identified with the Hilbert–Schmidt operators $HS(\mathcal{H})$ with the left/right multiplication of $\mathcal{A}/\mathcal{A}'$. In this representation, ω corresponds to the vector state $|\Omega\rangle = \rho^{1/2} \in \mathcal{H} \otimes \bar{\mathcal{H}}$, and the abstract definition of the relative entropy in (2) becomes

$$\langle\Omega|\log\Delta_{\omega,\omega'}\,\Omega\rangle = \mathrm{Tr}_{\mathcal{H}}\rho^{\frac{1}{2}}\left(\log\rho\otimes 1 - 1\otimes\log\rho'\right)\rho^{\frac{1}{2}} = \mathrm{Tr}_{\mathcal{H}}(\rho\log\rho - \rho\log\rho')\ . \tag{3}$$

As another example, let us consider a bi-partite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and observable algebra $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$. A normal state ω_{AB} on $\mathcal{A}$ corresponds to a density matrix ρ_{AB}. One calls $\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho_{AB}$ the "reduced density matrix", which defines a state ω_A on $\mathcal{B}(\mathcal{H}_A)$ (and similarly for system B). The mutual information is given in our example system by

$$S(\rho_{AB}, \rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})\ . \tag{4}$$

For tri-partite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and observable algebra $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) \otimes \mathcal{B}(\mathcal{H}_C)$, we have the following strong subadditivity (cf. [17]):

$$S(\rho_{AB}) + S(\rho_{AC}) - S(\rho_A) - S(\rho_{ABC}) \geq 0\ . \tag{5}$$

A list of properties of relative entropies that will be used later can be found in [24] (cf. Th. 5.3, Th. 5.15 and Cor. 5.12 [24]):

Theorem 2.2. *(1) Let M be a von Neumann algebra and M_1 a von Neumann subalgebra of M. Assume that there exists a faithful normal conditional expectation E of M onto M_1. If ψ and ω are states of M_1 and M, respectively, then $S(\omega, \psi \cdot E) = S(\omega \restriction M_1, \psi) + S(\omega, \omega \cdot E)$;*

(2) Let be M_i an increasing net of von Neumann subalgebras of M with the property $(\bigcup_i M_i)'' = M$. Then $S(\omega_1 \restriction M_i, \omega_2 \restriction M_i)$ converges to $S(\omega_1, \omega_2)$ where ω_1, ω_2 are two normal states on M;

(3) Let ω and ω_1 be two normal states on a von Neumann algebra M. If $\omega_1 \geq \mu\omega$, then $S(\omega, \omega_1) \leq \ln \mu^{-1}$;

(4) Let ω and ϕ be two normal states on a von Neumann algebra M, and denote by ω_1 and ϕ_1 the restrictions of ω and ϕ to a von Neumann subalgebra $M_1 \subset M$ respectively. Then $S(\omega_1, \phi_1) \leq S(\omega, \phi)$.

For type III factors, the von Neumann entropy is always infinite, but we shall see that in many cases mutual information is finite. By taking singular limits, we can also explore the singularities of von Neumann entropy from mutual information (cf. 4.2 for an example) which is important from physicists' point of view. The formal properties of von Neumann entropies are useful in proving properties of mutual information, see the proof of Th. 4.1.

2.2. Graded nets and subnets

This section is contained in [7]. We refer to [7] for more details and proofs.

We shall denote by $\mathsf{M\ddot{o}b}$ the Möbius group, which is isomorphic to $SL(2, \mathbb{R})/\mathbb{Z}_2$ and acts naturally and faithfully on the circle S^1.

By an interval of S^1 we mean, as usual, a non-empty, non-dense, open, connected subset of S^1 and we denote by $\mathcal{I}$ the set of all intervals. If $I \in \mathcal{I}$, then also $I' \in \mathcal{I}$ where I' is the interior of the complement of I. Intervals are disjoint if their closure are disjoint. We will denote by $\mathcal{PI}$ the set which consists of disjoint union of intervals.

A *net $\mathcal{A}$ of von Neumann algebras on S^1* is a map

$$I \in \mathcal{I} \mapsto \mathcal{A}(I)$$

from the set of intervals to the set of von Neumann algebras on a (fixed) Hilbert space $\mathcal{H}$ which verifies the *isotony property*:

$$I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$$

where $I_1, I_2 \in \mathcal{I}$.

A *Möbius covariant net* $\mathcal{A}$ of von Neumann algebras on S^1 is a net of von Neumann algebras on S^1 such that the following properties 1–4 hold:

1. MÖBIUS COVARIANCE: *There is a strongly continuous unitary representation* U *of* Möb *on* $\mathcal{H}$ *such that*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \ , \qquad g \in \text{Möb}, \ I \in \mathcal{I} \ .$$

2. POSITIVITY OF THE ENERGY: *The generator of the rotation one-parameter subgroup* $\theta \mapsto U(\mathrm{rot}(\theta))$ *(conformal Hamiltonian) is positive*, namely U is a positive energy representation.
3. EXISTENCE AND UNIQUENESS OF THE VACUUM *There exists a unit* U*-invariant vector* Ω *(vacuum vector), unique up to a phase, and* Ω *is cyclic for the von Neumann algebra* $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$.

A $\mathbb{Z}_2$-grading on $\mathcal{A}$ is an involutive automorphism $\mathbf{g} = \mathrm{Ad}\Gamma$ of $\mathcal{A}$, such that $\Gamma^2 = 1$, $\Gamma\Omega = \Omega, \Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$ for all I.

Given the grading $\mathbf{g}$, an element x of $\mathcal{A}$ such that $\mathbf{g}(x) = \pm x$ is called homogeneous, indeed a Bose or Fermi element according to the $\pm$ alternative, or simply even or odd elements. We shall say that the degree ∂x of the homogeneous element x is 0 in the Bose case and 1 in the Fermi case.

A *Möbius covariant graded net* $\mathcal{A}$ on S^1 is a $\mathbb{Z}_2$-graded Möbius covariant net satisfying graded locality, namely a Möbius covariant net of von Neumann algebras on S^1 such that the following holds:

4. GRADED LOCALITY: *There exists a grading automorphism* $\mathbf{g}$ *of* $\mathcal{A}$ *such that, if* I_1 *and* I_2 *are disjoint intervals,*

$$[x, y] = 0, \quad x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2) \ .$$

Here $[x, y]$ is the graded commutator with respect to the grading automorphism $\mathbf{g}$ defined as follows: if x, y are homogeneous then

$$[x, y] \equiv xy - (-1)^{\partial x \cdot \partial y} yx$$

and, for the general elements x, y, it is extended by linearity. When the grading is trivial, i.e., when $\Gamma = 1$, we shall refer to $\mathcal{A}$ as a *local net*.

Note the *Bose subnet* $\mathcal{A}_b$, namely the $\mathbf{g}$-fixed point subnet $\mathcal{A}^{\mathbf{g}}$ of degree zero elements, is local.

Moreover, setting

$$Z \equiv \frac{1 - i\Gamma}{1 - i} \ ,$$

we have that the unitary Z fixes Ω and

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*$$

(twisted locality w.r.t. Z).

Theorem 2.3. *Let $\mathcal{A}$ be a Möbius covariant Fermi net on S^1. Then Ω is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.*

If $I \in \mathcal{I}$, we shall denote by Λ_I the one parameter subgroup of Möb of "dilation associated with I".

Theorem 2.4. *Let $I \in \mathcal{I}$ and Δ_I, J_I be the modular operator and the modular conjugation of $(\mathcal{A}(I), \Omega)$. Then we have:*
(i):

$$\Delta_I^{it} = U(\Lambda_I(-2\pi t)),\ t \in \mathbb{R}, \tag{6}$$

(ii): U extends to an (anti-)unitary representation of Möb $\ltimes \mathbb{Z}_2$ *determined by*

$$U(r_I) = ZJ_I,\ I \in \mathcal{I},$$

acting covariantly on $\mathcal{A}$, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(\dot{g}I) \quad g \in \text{Möb} \ltimes \mathbb{Z}_2\ I \in \mathcal{I}\ .$$

Here $r_I : S^1 \to S^1$ is the reflection mapping I onto I'.

Corollary 2.5. (Additivity) *Let I and I_i be intervals with $I \subset \cup_i I_i$. Then $\mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$.*

Theorem 2.6. *For every $I \in \mathcal{I}$, we have:*

$$\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*\ .$$

In the following corollary, the grading and the graded commutator is considered on $B(\mathcal{H})$ w.r.t. AdΓ.

Corollary 2.7. $\mathcal{A}(I') = \{x \in B(\mathcal{H}) :\ [x, y] = 0\ \forall y \in \mathcal{A}(I)\}$.

Let now G be a simply connected compact Lie group. By Th. 3.2 of [11], the vacuum positive energy representation of the loop group LG (cf. [26]) at level k gives rise to an irreducible local net denoted by $\mathcal{A}_{G_k}$. By Th. 3.3 of [11], every irreducible positive energy representation of the loop group LG at level k gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$. When no confusion arises we will write $\mathcal{A}_{G_k}$ simply as G_k as in the last section 4.4.

Next we recall some definitions from [14]. Recall that $\mathcal{I}$ denotes the set of intervals of S^1. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of I in S^1. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be a graded Möbius covariant net. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := \mathcal{A}(I_1) \vee \mathcal{A}(I_2), \quad \hat{\mathcal{A}}(E) := (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))'.$$

Note that $\mathcal{A}(E) \subset Z\hat{\mathcal{A}}(E)Z^{-1}$, and its index will be denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$ or global index of $\mathcal{A}$. This generalizes the usual μ-index of $\mathcal{A}$ when $\mathcal{A}$ is local.

Let $\mathcal{A}$ be a graded Möbius net. By a *Möbius subnet* (cf. [18]) we shall mean a map

$$I \in \mathcal{I} \to \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$, which is isotonic

$$\mathcal{B}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and Möbius covariant with respect to the representation U, namely

$$U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$$

for all $g \in \mathsf{Möb}$ and $I \in \mathcal{I}$, and we also require that $\mathrm{Ad}\Gamma$ preserves $\mathcal{B}$ as a set. Note that by Lemma 13 of [18] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{A}(I) \to \mathcal{B}(I)$ such that E_I preserves the vector state given by the vacuum of $\mathcal{A}$. Let P be the projection onto the closed subspace spanned by $\mathcal{B}(I)\Omega$.

Definition 2.8. Let $\mathcal{A}$ be a graded Möbius covariant net and $\mathcal{B} \subset \mathcal{A}$ a subnet. We say $\mathcal{B} \subset \mathcal{A}$ is of finite index if $\mathcal{B}(I) \subset \mathcal{A}(I)$is of finite index for some (and hence all) interval I. The index will be denoted by $[\mathcal{A} : \mathcal{B}]$.

Assume that $\mathcal{B} \subset \mathcal{A}$ has finite index and $[\mathcal{A} : \mathcal{B}] = \lambda^{-1}$. Let I_1 and I_2 be two intervals obtained from an interval I by removing an interior point, and let $J_1 \subset I_2$. By [20] there are isometries $w_1 \in \mathcal{A}(I_1), v_1 \in \mathcal{B}(I_1)$ such that $a = \lambda^{-1}E_I(aw_1^*)w_1, \forall a \in \mathcal{A}(I)$. Let $e_1 = w_1w_1^*$. Then

$$Pe_1P = \lambda P, \quad e_1v_1v_1^*e_1 = \lambda e_1, \quad \lambda^{-1}v_1^*e_1v_1 = 1 \ .$$

Similarly we have $w_2 \in \mathcal{A}(J_1), v_2 \in \mathcal{B}(J_1)$ and $e_2 = w_2w_2^*$ which verify same relations as above. $e_1 \in \mathcal{A}(I_1), e_2 \in \mathcal{A}(J_1)$ are known as Jones projections for $\mathcal{B}(I_1) \subset \mathcal{A}(I_1)$ and

$\mathcal{B}(J_1) \subset \mathcal{A}(J_1)$ respectively. They are related by an inner automorphism of $\mathcal{B}(I)$, which is the following Lemma:

Lemma 2.9. *Let $u = \lambda^{-1}E_I(w_1 w_2^*) \in \mathcal{B}(I)$. Then u is unitary and we have $e_1 = ue_2u^*$.*

Proof. First we have $w_1 = uw_2$ and so $e_1 = ue_2u^*$. Now compute

$$uu^* = \lambda^{-2}E_I(w_1w_2^*)E_I(w_2w_1^*) = \lambda^{-2}E_I(w_1w_2^*E_I(w_2w_1^*)) = \lambda^{-1}E_I(w_1w_1^*) = 1 \ ,$$

where in the third equality we have used that $w_2^*E_I(w_2w_1^*) = \lambda w_1^*$ and in the last equality that $E_I(e_1) = \lambda$. □

The following is proved in exactly the same way as in [14]:

Lemma 2.10. *If $\mathcal{B} \subset \mathcal{A}$ is a Möbius subnet such that $\mu_{\mathcal{A}}$ is finite and $[\mathcal{A} : \mathcal{B}] < \infty$. Then $\mu_{\mathcal{B}} = \mu_{\mathcal{A}}[\mathcal{A} : \mathcal{B}]^2$.*

3. Mutual information in the case of free fermions

3.1. Basic representation of LU_r and free fermion net

Let H denote the Hilbert space $L^2(S^1; \mathbb{C}^r)$ of square-summable $\mathbb{C}^r$-valued functions on the circle. The group LU_r of smooth maps $S^1 \to U_r$, with U_r the unitary group on $\mathbb{C}^r$, acts on H multiplication operators.

Let us decompose $H = H_+ \oplus H_-$, where

$$H_+ = \{\text{functions whose negative Fourier coefficients vanish}\} .$$

We denote by p the Hardy projection from H onto H_+.

Denote by $U_{\text{res}}(H)$ the group consisting of unitary operator A on H such that the commutator $[p, A]$ is a Hilbert–Schmidt operator. Denote by $\text{Diff}^+(S^1)$ the group of orientation preserving diffeomorphism of the circle. It follows from Proposition 6.3.1 and Proposition 6.8.2 in [26] that LU_r and $\text{Diff}^+(S^1)$ are subgroups of $U_{\text{res}}(H)$. The basic representation of LU_r is the representation on Fermionic Fock space $F_p = \Lambda(pH) \otimes \Lambda((1-p)H)^*$ as defined in §10.6 of [26]. For more details, see [26] or [30]. Such a representation gives rise to a graded net as follows. Denote by $\mathcal{A}_r(I)$ the von Neumann algebra generated by $c(\xi)'s$, with $\xi \in L^2(I, \mathbb{C}^r)$. Here $c(\xi) = a(\xi) + a(\xi)^*$ and $a(\xi)$ is the creation operator defined as in Chapter 1 of [30]. Let $Z : F_p \to F_p$ be the Klein transformation given by multiplication by 1 on even forms and by i on odd forms. It follows from §15 of chapter 2 of [30] that $\mathcal{A}_r$ is a graded Möbius covariant net, and $\mathcal{A}_r$ will be called the *net of r free fermions*. It follows from Prop. 1.3.2 of [29] that $\mathcal{A}_r$ is strongly additive and §15 of chapter 2 of [30] that $\mu_{\mathcal{A}_r} = 1$.

Fix $I_i \in \mathcal{PI}, i = 1, 2$, and I_1, I_2 disjoint, that is $\bar{I_1} \cap \bar{I_2} = \emptyset$, and $I = I_1 \cup I_2$.

For bounded operators $A, B : F_p \to F_p$, we define $A^+ = \Gamma A \Gamma,\ A^- = A - A^+$, where Γ is an operator on F_p given by multiplication by 1 on even forms and -1 on odd forms. An operator A is called even (resp. odd) if $A = A^+$ (resp. $A = A^-$).

We define a graded tensor product $\otimes_2$ by the following formula:

$$A \otimes_2 B = A \otimes B^+ + A\Gamma \otimes B^- \ ,$$

where $A \otimes_2 B$ is considered as an operator on Hilbert space tensor product $F_p \otimes F_p$.

Let A_1, A_2, B_1, B_2 be even or odd operators, i.e. $\Gamma A_i \Gamma = A_i$ or $-A_i,\ \Gamma B_i \Gamma = B_i$ or $-B_i,\ i = 1, 2$. Define the degree $d(A) = 0$ or 1 if A is even or odd.

It follows from the definition of $\otimes_2$ that:

$$(A_1 \otimes_2 B_1)^* = (-1)^{d(A_1)d(B_1)} A_1^* \otimes_2 B_1^* \ ,$$
$$(A_1 \otimes_2 B_1) \cdot (A_2 \otimes_2 B_2) = (-1)^{d(B_1)d(A_2)} A_1 A_2 \otimes_2 B_1 B_2 \ .$$

For $A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2)$, we define

$$\omega(A \otimes_2 B) = \langle \Omega, AB\,\Omega \rangle$$

where Ω is the vacuum vector in F_p.

Lemma 3.1. (1) *ω extends to a normal faithful state on the von Neumann algebra $\{A \otimes_2 B,\ A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2)\}''$ (denoted by $\mathcal{A}_r(I_1)\hat{\otimes}_2\mathcal{A}_r(I_2)$) on $F_p \otimes F_p$. There exists a unitary operator $U_1 : F_p \to F_p \otimes F_p$ such that:*

$$U_1 AB U_1^* = A \otimes_2 B \qquad \text{for every} \qquad A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2) \ .$$

(2) *The unitary operator U_1 in (1) can be chosen such that $U_1^*(\Gamma \otimes \Gamma)U_1 = \Gamma$, hence $U_1^*(\mathcal{B}(F_p) \otimes 1)U_1$ commutes with $Z\mathcal{A}_r(I_2)Z^{-1}$ and therefore is* AdΓ *invariant as a set and lies in $\mathcal{A}_r(I_2')$ when I_2 is an interval.*

Proof. (1) is proved in Prop. 2.3.1 of [31]. We note that by (2) the state $\omega_1 \otimes_2 \omega_2$ defined in Definition 3.3 is a normal state on type III factor $\mathcal{A}_r(I_1) \vee \mathcal{A}_r(I_2)$, and hence can be represented by a unique vector ψ in the positive cone associated with vector state ω on F_p. Since both $\omega_1 \otimes_2 \omega_2$ and ω are AdΓ invariant, it follows that AdΓ preserves the positive cone, and $\omega_1 \otimes_2 \omega_2$ is also represented by $\Gamma\psi$. By uniqueness we must have $\Gamma\psi = \psi$. Now U_1 in (2) is uniquely fixed by the condition $U_1\psi = \Omega \otimes \Omega$, and it follows that $U_1^*\Gamma \otimes \Gamma U_1 = \Gamma$, hence $U_1^*\mathcal{B}(F_p) \otimes 1 U_1$ is AdΓ invariant as a set, graded commuting with $\mathcal{A}_r(I_2)$ and therefore lies in $\mathcal{A}_r(I_2')$ when I_2 is an interval by Corollary 2.7. □

Remark 3.2. If $\mathcal{B} \subset \mathcal{A}$ is a graded subnet, the proof of (3) then applies to $\mathcal{B}$, and for any interval I, by choosing $I_{1n} \subset I_{2n}^c \subset I$ with $\cup_{n=1}^{\infty} I_{1n} = I$, we can get an increasing sequence of AdΓ invariant (as a set) finite dimensional type I factors B_n such that $\cup_n B_n$ is strongly dense in $\mathcal{B}(I)$.

3.2. Mutual information for free fermions

Let $I_1, I_2 \in \mathcal{PI}$ and $I = I_1 \cap I_2$ as above.

Definition 3.3. We set

$$\omega_1 \otimes_2 \omega_2(AB) = \langle \Omega \otimes \Omega, A \otimes_2 B\, \Omega \otimes \Omega \rangle, \quad \forall A \in \mathcal{A}_r(I_1),\ B \in \mathcal{A}_r(I_2)\ .$$

By (1) Lemma 3.1 $\omega_1 \otimes_2 \omega_2$ defines a normal state on $\mathcal{A}_r(I)$. We note that the restriction of $\omega_1 \otimes_2 \omega_2$ to $\mathcal{A}_r(I_1)$ and $\mathcal{A}_r(I_2)$ is the same as ω.

The mutual information we will compute is $S(\omega, \omega_1 \otimes_2 \omega_2)$. When we wish to emphasize the underlying net, we will also write the mutual information as $S_{\mathcal{A}_r}(\omega, \omega_1 \otimes_2 \omega_2)$. When $\mathcal{B} \subset \mathcal{A}_r$ is a subnet, we write $S_{\mathcal{B}}(\omega, \omega_1 \otimes_2 \omega_2)$ the mutual information for the net $\mathcal{B}$ obtained by restricting $\omega, \omega_1 \otimes_2 \omega_2$ from $\mathcal{A}_r$ to $\mathcal{B}$. Note that by (4) of Th. 2.2 $S_{\mathcal{B}}(\omega, \omega_1 \otimes_2 \omega_2) \leq S_{\mathcal{A}_r}(\omega, \omega_1 \otimes_2 \omega_2)$.

ω on $\mathcal{A}_r(I)$ is quasi-free state as studied by Araki in [1]. To describe this state, it is convenient to use Cayley transform $V(x) = (x-i)/(x+i)$, which carries the (one point compactification of the) real line onto the circle and the upper half plane onto the unit disk. It induces a unitary map

$$Uf(x) = \pi^{-\frac{1}{2}}(x+i)^{-1} f(V(x))$$

of $L^2(S^1, \mathbb{C}^r)$ onto $L^2(\mathbb{R}, \mathbb{C}^r)$. The operator U carries the Hardy space on the circle onto the Hardy space on the real line (cf. [28]). We will use the Cayley transform to identify intervals on the circle with one point removed to intervals on the real line. Under the unitary transformation above, the Hardy projection on $L^2(S^1, \mathbb{C}^r)$ is transformed to the Hardy projection on $L^2(\mathbb{R}, \mathbb{C}^r)$ given by:

$$Pf(x) = \frac{1}{2} f(x) + \int \frac{i}{2\pi} \frac{1}{(x-y)} f(y)dy\ ,$$

where the singular integral is (proportional to) the Hilbert transform.

We write the kernel of the above integral transformation as C:

$$C(x,y) = \frac{1}{2}\delta(x-y) - \frac{i}{2\pi} \frac{1}{(x-y)}\ . \tag{7}$$

The quasi free state ω is determined by

$$\omega\big(a(f)^* a(g)\big) = \langle g, Pf \rangle.$$

Slightly abusing our notations, we will identify P with its kernel C and simply write

$$\omega\big(a(f)^* a(g)\big) = \langle g, Cf\rangle.$$

C will be called *covariance operator*.

Recall $I_i \in \mathcal{PI}, i = 1, 2$, and I_1, I_2 are disjoint, that is $\bar{I_1} \cap \bar{I_2} = \emptyset$, and $I = I_1 \cup I_2$. We assume that $I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n)$ in increasing order.

3.3. Computation of mutual information in finite dimensional case

Choose finite dimensional subspaces H_i of $L^2(I_i, \mathbb{C}_r), i = 1, 2$, and denote by $\mathrm{CAR}(H_i) \subset \mathcal{A}(I_i)$ the corresponding finite dimensional factors of dimensions $2^{2\dim H_i}$ generated by $a(f), f \in H_i$. Let ρ_{12}, ρ_1, ρ_2 be the density matrices of the restriction of ω to $\mathrm{CAR}(H_1) \otimes_2 \mathrm{CAR}(H_2)$, $\mathrm{CAR}(H_1)$, $\mathrm{CAR}(H_2)$ respectively, and $\rho_1 \otimes_2 \rho_2$ of the restriction of $\omega_1 \otimes_2 \omega_2$ to $\mathrm{CAR}(H_1) \otimes_2 \mathrm{CAR}(H_2)$. Our goal in this section is to compute the relative entropy $S(\rho_{12}, \rho_1 \otimes_2 \rho_2)$.

Note that since $\mathrm{CAR}(H_1)$ is type I factor, $\mathrm{Ad}\Gamma$ acts on $\mathrm{CAR}(H_1)$ by an inner automorphism $\mathrm{Ad}u, u \in \mathrm{CAR}(H_1)$. Since $\mathrm{Ad}u$ has order two, by suitably choosing phase factor we can assume that $u^2 = 1$. Note that $\Gamma u \Gamma = u^3 = u$, so u is even, and Γu commutes with $\mathrm{CAR}(H_1)$. So $\Gamma u \otimes B^-, 1 \otimes B^+$ generates a type I factor $\widetilde{\mathrm{CAR}}(H_2)$ isomorphic to $\mathrm{CAR}(H_2)$, and commuting with $\mathrm{CAR}(H_1) \otimes 1$. It follows that $\mathrm{CAR}(H_1) \otimes_2 \mathrm{CAR}(H_2) = \mathrm{CAR}(H_1) \otimes \widetilde{\mathrm{CAR}}(H_2)$. Let us show that $\omega_1 \otimes_2 \omega_2$, when restricting to $\mathrm{CAR}(H_1) \otimes \widetilde{\mathrm{CAR}}(H_2)$, is the tensor product state $\rho_1 \otimes \rho_2'$, where ρ_1, ρ_2' denote the restriction of ω to $\mathrm{CAR}(H_1)$, $\widetilde{\mathrm{CAR}}(H_2)$ respectively. Since $\omega_1 \otimes_2 \omega_2$ clearly agrees with $\rho_1 \otimes \rho_2'$ on $A \otimes B^+$, it is sufficient to check that

$$\omega_1 \otimes_2 \omega_2(a\tilde{b}) = \omega(a)\omega(\tilde{b}), \quad \forall a \in \mathrm{CAR}(H_1) \otimes 1,\ \tilde{b} = \Gamma u \otimes b^-,\ b^- \in \mathrm{CAR}(H_1)^- \ .$$

The left-hand side of the above is

$$\langle \Omega, au\Omega\rangle\langle \Omega, b^-\Omega\rangle = 0$$

and the right-hand side is

$$\langle \Omega, a\Omega\rangle\langle \Omega, ub^-\Omega\rangle = 0$$

since u is even. We also note that ω restricted to $\widetilde{\mathrm{CAR}}(H_2)$ is the same as ω restricted to $\mathrm{CAR}(H_2)$ under the natural isomorphism of $\widetilde{\mathrm{CAR}}(H_2)$ with $\mathrm{CAR}(H_2)$.

So we have shown the analog of (4) in this graded local context:

Proposition 3.4.

$$S(\rho_{12}, \rho_1 \otimes_2 \rho_2) = S(\rho_1) + S(\rho_2) - S(\rho_{12}) \ .$$

Now we turn to the computation of von Neumann entropy $S(\rho_1)$. Let p_1 be the projection onto the finite dimensional subspace H_1 of $L^2(I_1, \mathbb{C}_r)$. ρ_1 on $\mathrm{CAR}(H_1)$ is quasi free state given by covariance operator $C_{p_1} = p_1 C p_1$. Let K be the operator such that

$$(1 + \exp(-K)) = C_{p_1} \ .$$

Since K is self adjoint, we can choose an orthonormal basis ψ_i, $1 \leq i \leq \dim H_1$ of H_1 such that $K\psi_i = \lambda_i \psi$, where λ_i are real eigenvalues of K.

$\mathrm{CAR}(H_1)$ acts on the Fermionic Fock space $F(H_1)$. Let

$$K_1 := \sum_i \lambda_i a(\psi_i)^* a(\psi_i) \ .$$

According to [1] and [10], the density matrix of ρ_1 (still denoted by ρ_1) as an operator on $F(H_1)$ is given by the following

$$\rho_1 = c \exp(-K_1) \ ,$$

where $c^{-1} = \mathrm{Tr}\big(\exp(-K_1)\big)$.

By a simple computation we find that $\mathrm{Tr}\big(\exp(-K_1)\big) = \det(1 + e^{-K})$ and

$$S(\rho_1) = \mathrm{Tr}(\rho_1 \ln \rho_1) = \mathrm{Tr}\big((1 - C_{p_1}) \log(1 - C_{p_1}) + C_{p_1} \log C_{p_1}\big) \ . \tag{8}$$

Definition 3.5. Let $\mathbf{P}_i$ be projections from $L^2(I, \mathbb{C}^r)$ onto $L^2(I_i, \mathbb{C}^r)$, and $C_i = \mathbf{P}_i C \mathbf{P}_i, i = 1, 2$.

Let

$$\sigma_C = \mathbf{P_1}\big(C \ln C + (1 - C)\ln(1 - C)\big)\mathbf{P_1} - \big(C_1 \ln C_1 + (\mathbf{P_1} - C_1)\ln(\mathbf{P_1} - C_1)\big) + \mathbf{P_2}\big(C \ln C + (1 - C)\ln(1 - C)\big)\mathbf{P_2} - \big(C_2 \ln C_2 + (\mathbf{P_2} - C_2)\ln(\mathbf{P_2} - C_2)\big)$$

and σ_{C_p} be the same as in the definition of σ_C with C replaced by $C_p = pCp$, if p is a projection commuting with $\mathbf{P}_1$.

Denote by p the projection from $L^2(I, \mathbb{C}^r)$ onto $H_1 \oplus H_2$. By Proposition 3.4 and equation (8) we have proved the following

Proposition 3.6.

$$S(\rho_{12}, \rho_1 \otimes_2 \rho_2) = \mathrm{Tr}(\sigma_{C_p}) \ .$$

3.4. Inequality from operator convexity

The proof of the following result can be found in [6] (See Th. 2.6 and Th. 4.19 of [6]):

Theorem 3.7. *(1) For all operator convex functions f on $\mathbb{R}$, and all orthogonal projections p, we have $pf(pAp)p \leq pf(A)p$ for every selfadjoint operator A; (2) $f(t) = t\ln(t)$ is operator convex.*

(1) of the above Theorem is known as Sherman–Davis Inequality. It in instructive to review the idea of the proof of (1) which is also used in the proof of Th. 3.12: Consider the selfadjoint unitary operator $U^p = 2p - I$; by operator convexity we have

$$f\big(\frac{1}{2}A + \frac{1}{2}U^p A U^p\big) \leq \frac{1}{2}f(A) + \frac{1}{2}f(U^p A U^p)\ .$$

Now notice that

$$\frac{1}{2}A + \frac{1}{2}U^p A U^p = A_p + A_{1-p}, \quad f(U^p A U^p) = U^p f(A) U^p\ ,$$

where $A_p = pAp$, and the inequality follows.

For (2), see e.g. [6].

Lemma 3.8. *(1)*

$$S(\omega, \omega_1 \otimes_2 \omega_2) = \lim_{p\to 1} \mathrm{Tr}(\sigma_{C_p}) \geq \mathrm{Tr}(\sigma_C)$$

where $p \to 1$ strongly.

(2) The mutual information for r free fermion net is r times the mutual information for 1 free fermion net.

Proof. (1): The first follows from Proposition 3.6 and (2) of Th. 2.2. To prove the inequality, we use the fact that $x\ln x$ is operator convex, and so $\mathbf{P_1} C\ln C\mathbf{P_1} \geq C_1 \ln C_1$, and similarly with C replaced by $1 - C$ by Th. 3.7. It follows that $\sigma \geq 0, \sigma_p \geq 0$. Since σ_p goes to σ strongly as $p \to 1$ strongly, the inequality follows.

(2): For the case of r free fermions, the trace in (1) is over $L^2(\mathbb{R}, \mathbb{C}^r)$ which is r direct sum of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$, and (2) follows. □

We shall prove later that the inequality in the above Lemma is actually an equality. It would follow if one can show that σ_{C_p} goes to σ_C in tracial norm. This is not so easy, and we note that $\mathbf{P_1}\big(C\ln C + (1 - C)\ln(1 - C)\big)\mathbf{P_1}$ is not trace class. To overcome this difficulty and to compute the mutual information we prove the reverse inequality by applying Lieb's joint convexity and regularized kernel as in the next two sections.

3.5. Reversed inequality from Lieb's joint convexity

We begin with the following Lieb's Concavity Theorem:

Theorem 3.9. *(1) For all $m \times n$ matrices K, and all $0 \leq t \leq 1$, the real valued map given by $(A, B) \to \mathrm{Tr}(K^* A^{1-t} K B)$ is concave where A, B are non-negative $m \times m$ and $n \times n$ matrices respectively;*

(2) If $A \geq 0, B \geq 0$ and K is trace class, then

$$(A, B) \to \mathrm{Tr}(K^* A^{1-t} K B), \quad 0 \leq t \leq 1,$$

is jointly concave;

(3) If $A \geq \epsilon I, B \geq \epsilon I, \epsilon > 0$ and K is trace class, then

$$(A, B) \to \mathrm{Tr}(K^* A \ln A K - K^* A K \ln B)$$

is jointly convex.

Proof. (1) is proved in Th. 6.1 of [6]. (2) follows from (2) by functional calculus. To prove (3), we note that

$$\mathrm{Tr}(K^* A \ln A K - K^* A K \ln B) = \lim_{t \to 0} \frac{\mathrm{Tr}(K^* A^{1-t} K B) - \mathrm{Tr}(K^* A K)}{t - 1}$$

and (3) follows from (2). □

Lemma 3.10. *Assume that S is trace class, then $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$ where T is any bounded operator, and if the sequence of bounded operators $T_n \to T$ strongly, then $\mathrm{Tr}(ST_n) \to \mathrm{Tr}(ST)$.*

Proof. The equality is proved in [27]. Let e_i be an orthonormal basis, and $S = U|S|$ be the polar decomposition of S. Then

$$\mathrm{Tr}(T_n S) = \sum_i \langle e_i, T_n U |S|^{1/2} |S|^{1/2} e_i \rangle \ .$$

Note that

$$|\langle e_i, T_n U |S|^{1/2} |S|^{1/2} e_i \rangle| \leq ||T_n U |S|^{1/2} e_i|| \, |||S|^{1/2} e_i\| \leq c \langle e_i, |S| e_i \rangle, \ \forall i \ ,$$

where c is a constant, so the last part of the Lemma follows by Lebesgue dominated convergence theorem. □

Lemma 3.11. *Suppose that $K \geq \epsilon I, L \geq \epsilon I, \epsilon > 0$ and $K - L$ is trace class. Then $\ln K - \ln L$ is trace class.*

Proof. Note that

$$\ln K = -\int_0^\infty \left(\frac{1}{K+t} - \frac{1}{1+t}\right) dt, \quad \ln L = -\int_0^\infty \left(\frac{1}{L+t} - \frac{1}{1+t}\right) dt \ .$$

Hence

$$\ln K - \ln L = -\int_0^\infty \left(\frac{1}{K+t} - \frac{1}{L+t}\right) dt = \int_0^\infty \left(\frac{1}{L+t}(K-L)\frac{1}{K+t}\right) dt \ .$$

We have

$$||\ln K - \ln L||_1 \leq \int_0^\infty \left\|\frac{1}{L+t}(K-L)\frac{1}{K+t}\right\|_1 dt$$

$$\leq \int_0^\infty \left\|\frac{1}{L+t}\right\| \|K-L\|_1 \left\|\frac{1}{K+t}\right\| dt \leq ||K-L||_1\, \epsilon^{-1} \ ,$$

where $||\cdot||_1$ denotes tracial norm. □

Theorem 3.12. *Let $A \geq \epsilon, \epsilon > 0, B := \mathbf{P_1} A \mathbf{P_1} + \mathbf{P_2} A \mathbf{P_2}$, where $\mathbf{P_1}$ is a projection, $\mathbf{P_1} + \mathbf{P_2} = 1$, and p is a finite rank projection commuting with $\mathbf{P_1}$. Assume that $A - B$ is trace class. Then*

$$\mathrm{Tr}\big(A(\ln A - \ln B)\big) \geq \mathrm{Tr}\big(A_p(\ln A_p - \ln B_p)\big) \ .$$

Proof. Apply Th. 3.9 to A, B and unitary $U^p = 2P - I$, with $f(A, B, K) = \mathrm{Tr}(K^* A \ln AK - K^* AK \ln B)$, K is a finite rank projection, we have

$$f\left(\frac{1}{2}(A + U^p A U^p), \frac{1}{2}(B + U^p B U^p), K\right) \leq \frac{1}{2} f(A, B, K) + \frac{1}{2} f(U^p A U^p, U^p B U^p, K) \ .$$

Note that

$$f\left(\frac{1}{2}(A + U^p A U^p), \frac{1}{2}(B + U^p B U^p), K\right) = f(A_p + A_{1-p}, B_p + B_{1-p}, K)$$

and

$$f(A_p + A_{1-p}, B_p + B_{1-p}, K)$$
$$= \mathrm{Tr}\big(K(A_p \ln A_p + A_{1-p} \ln A_{1-p})K - K(A_p + A_{1-p})K \ln(B_p + B_{1-p})\big)$$

and

$$\frac{1}{2}f(A,B,K)+\frac{1}{2}f(U^pAU^p,U^pBU^p,K)=$$
$$\frac{1}{2}\mathrm{Tr}(KA\ln A\,K-KAK\ln B)+\frac{1}{2}\mathrm{Tr}(KU^pA\ln A\,U^pK-KU^pAU^pKU^p\ln B\,U^p)\ .$$

Observe that $KU^p=[K,2p]+U^pK$ and $K\ln B=[K,\ln B]+\ln B\,K$, $K\ln(B_p+B_{1-p})=[K,\ln(B_p+B_{1-p})]+\ln(B_p+B_{1-p})K$. We will let $K\to 1$ strongly eventually. Up to terms that go to 0 as $K\to 1$ strongly, we can freely permute K and U^p. By permuting K with $\ln(B_p+B_{1-p})$ and $\ln B$ on the left hand side and righthand side of the above inequality respectively, we get terms on the left hand side of the above inequality

$$-\mathrm{Tr}\big(K(A_p+A_{1-p})[K,\ln(B_p+B_{1-p})]\big)$$

and on the right hand side of the above inequality

$$-\frac{1}{2}\mathrm{Tr}\big(KA[K,\ln B]+KU^pA[K,\ln B]U^p\big)$$
$$=-\mathrm{Tr}\big(K(pA[K,\ln B]p+(1-p)A[K,\ln B](1-p)\big).$$

Up to terms that go to 0 as $K\to 1$ strongly, we have that

$$-\mathrm{Tr}\Big(K\big(pA[K,\ln B]p+(1-p)A[K,\ln B](1-p)\big)\Big)$$

is equal to

$$-\mathrm{Tr}\Big(K\big(pAp[K,\ln B]+(1-p)A(1-p)[K,\ln B](1-p)\big)\Big)\ .$$

This is the same as

$$-\mathrm{Tr}\big(K(A_p+A_{1-p})[K,\ln(B_p+B_{1-p})]\big)$$

up to terms that go to 0 as $K\to 1$ strongly since $\ln B-\ln(B_p+B_{1-p})$ is trace class by Lemma 3.11, and p is of finite rank.

So by permuting K with U^p and $\ln B$, $\ln(B_p+B_{1-p})$ in the inequalities above, by Lemma 3.10, use $\ln B-\ln(B_p+B_{1-p})$ is trace class by Lemma 3.11, and p is finite rank, we have that up to terms which go to zero as $K\to 1$ strongly,

$$\mathrm{Tr}\Big(K\big(A_p(\ln A_p-\ln B_p)+A_{1-p}(\ln A_{1-p}-\ln B_{1-p})\big)\Big)$$
$$\le\mathrm{Tr}\Big(K\big(pA(\ln A-\ln B)p+(1-p)A(\ln A-\ln B)(1-p)\big)\Big)\ .$$

By Lemma 3.11, $\ln A_p-\ln B_p,\ln A_{1-p}-\ln B_{1-p},\ln A-\ln B$ are trace class operators, and by Lemma 3.10 let $K\to 1$ strongly, we get

$$\mathrm{Tr}\big(A_p(\ln A_p-\ln B_p)\big)+\mathrm{Tr}\big(A_{1-p}(\ln A_{1-p}-\ln B_{1-p})\big)\le\mathrm{Tr}A(\ln A-\ln B)\ .$$

As in the proof of Lemma 3.8, by operator convexity of $xlnx$ we have

$$(1-p)A_{1-p}\ln A_{1-p}(1-p) \geq B_{1-p}\ln B_{1-p}$$

and so

$$\mathrm{Tr}\big(A_{1-p}(\ln A_{1-p} - \ln B_{1-p})\big) = \mathrm{Tr}\big(A_{1-p}\ln A_{1-p} - B_{1-p}\ln B_{1-p}\big) \geq 0$$

and the theorem is proved. □

3.6. Regularized Kernel for one free fermion case

Note that, by Lemma 3.8, the mutual information for r free fermion net is r times the mutual information for 1 free fermion net. In this section we will determine the mutual information for 1 free fermion net.

This section is inspired by formal computations in [10]. The regularization is also motivated by Th. 3.12 which applies to strictly positive operators.

Recall that the Hardy projection on $L^2(\mathbb{R},\mathbb{C})$ is given by:

$$Pf(x) = \frac{1}{2}f(x) + \int \frac{i}{2\pi}\frac{1}{(x-y)}f(y)dy\ ,$$

where the integral is the singular integral or Hilbert transform.

We write the kernel of the above integral transformation as C.

$$C(x,y) = \frac{1}{2}\delta(x-y) - \frac{i}{2\pi}\frac{1}{(x-y)}\quad . \tag{9}$$

Recall $I_i \in \mathcal{PI}, i = 1,2$, and I_1, I_2 are disjoint, that is $\bar{I_1} \cap \bar{I_2} = \emptyset$, and $I = I_1 \cup I_2$. We assume that $I = (a_1,b_1) \cup (a_1,b_1) \cup ... \cup (a_n,b_n)$ in increasing order.

Then resolvent of C as restriction of an operator on $L^2(I,\mathbb{C})$

$$R^0(\beta) = (C - 1/2 + \beta)^{-1} \equiv \left(-\frac{i}{2\pi}\frac{1}{x-y} + \beta\,\delta(x-y)\right)^{-1} \tag{10}$$

has the following expression ([22] or Page 133 of [21]):

$$R^0(\beta) = \left(\beta^2 - 1/4\right)^{-1}\left(\beta\,\delta(x-y)\ + \frac{i}{2\pi}\frac{e^{-\frac{i}{2\pi}\log\left(\frac{\beta-1/2}{\beta+1/2}\right)(Z(x)-Z(y))}}{x-y}\right)\ , \tag{11}$$

where

$$Z(x) = \log\left(-\frac{\prod_{i=1}^n(x-a_i)}{\prod_{i=1}^n(x-b_i)}\right)\ . \tag{12}$$

It is useful to consider the following regularized operator: Let $\epsilon_0 > 0$, and $E := \frac{C+\epsilon_0}{1+2\epsilon_0}$. Note that

$$\frac{1+\epsilon_0}{1+2\epsilon_0} \geq E \geq \frac{\epsilon_0}{1+2\epsilon_0} \ .$$

Then we have

$$E \ln E + (1-E)\ln(1-E) = \int_{\frac{1}{2}}^{\infty} \left[\big(\beta - \frac{1}{2}\big)\big(R_E(\beta) - R_E(-\beta)\big) - \frac{2\beta}{\beta + \frac{1}{2}} \right] d\beta \ ,$$

where $R_E(\beta) = \frac{1}{E - \frac{1}{2} + \beta}$.

We note that the integral above is absolutely convergent in norm. This can be seen as follows: the integrand is

$$\big(\beta - \frac{1}{2}\big)\big(R_E(\beta) - R_E(-\beta)\big) - \frac{2\beta}{\beta + \frac{1}{2}} = \frac{\beta/2 - 2\beta(E - \frac{1}{2})^2}{[(E-\frac{1}{2})^2 - \beta^2](\beta + \frac{1}{2})} \ .$$

For $1/2 \leq \beta \leq 1$, since

$$\frac{1+\epsilon_0}{1+2\epsilon_0} \geq E \geq \frac{\epsilon_0}{1+2\epsilon_0} \ ,$$

we have

$$\left\| \frac{2\beta}{\beta + \frac{1}{2}} = \frac{\beta/2 - 2\beta(E - \frac{1}{2})^2}{[(E-\frac{1}{2})^2 - \beta^2](\beta + \frac{1}{2})} \right\| \leq \frac{3\beta}{\beta + 1/2} \left(\frac{\epsilon_0}{1+2\epsilon_0} \right)^{-2} \ .$$

On the other hand

$$\left\| \frac{2\beta}{\beta + \frac{1}{2}} = \frac{\beta/2 - 2\beta(E - \frac{1}{2})^2}{[(E-\frac{1}{2})^2 - \beta^2](\beta + \frac{1}{2})} \right\|$$

is bounded by $\frac{1}{\beta^2}$ when β is large.

To evaluate the above integral using resolvent, let $t = \beta(1+2\epsilon_0)$ we get

$$\begin{aligned} &E \ln E + (1-E)\ln(1-E) \\ &\quad = \int_{\frac{1}{2}(1+2\epsilon_0)}^{\infty} \left[\left(\frac{t}{1+2\epsilon_0} - \frac{1}{2} \right) \big(R(t) - R(-t)\big) - \frac{2t}{t + \frac{1}{2}(1+2\epsilon_0)} \frac{1}{1+2\epsilon_0} \right] d\beta. \end{aligned}$$

Now we determine the kernel $K_1^{\epsilon_0}(x,y), x, y \in I_1$ of

$$\mathbf{P}_1 E \ln E + (1-E)\ln(1-E)\mathbf{P}_1 - E_1 \ln E_1 - (\mathbf{P}_1 - E_1)\ln(\mathbf{P}_1 - E_1) \ .$$

Lemma 3.13. *Suppose $f \in C^1(I_1 \times I_1)$ and $f(x,y) = -f(y,x)$. Let $g(x,y) = \frac{f(x,y)}{x-y}$ if $x \neq y$ and $g(x,x) = \frac{\partial f}{\partial x}$. Then $g(x,y)$ is continuous on $I_1 \times I_1$.*

Proof. It is enough to check continuity at $(y,y), y \in I_1$. Since $f \in C^1(I_1 \times I_1)$, we can write $f(x',y') = \frac{\partial f}{\partial x}(y,y)(x'-x)+\frac{\partial f}{\partial y}(y,y)(y'-y)+o(x'-y') = \frac{\partial f}{\partial x}(y,y)(x'-y')+o(x'-y')$ where in the second $=$ we have used $f(x,y) = -f(y,x)$ and hence $\frac{\partial f}{\partial x}(y,y) = -\frac{\partial f}{\partial y}(y,y)$. It follows that $\lim_{(x',y')\to(y,y)} g(x',y') = g(y,y)$. □

We shall denote by $Z_{I,I_1}(x) = Z_I(x) - Z_{I_1}(x)$. Even though both $Z_I(x)$ and $Z_{I_1}(x)$ are singular when x is close to the boundary of its domain, it is crucial that $Z_{I,I_1}(x)$ is a smooth function on the closure of $\bar{I}_1$.

Lemma 3.14. *Let*

$$G(t,x,y) = \frac{\sin\left(\frac{1}{2\pi}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)\left(Z_I(x) - Z_I(y)\right)\right) - \sin\left(\frac{1}{2\pi}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)\left(Z_{I_1}(x) - Z_{I_1}(y)\right)\right)}{x-y}$$

if $x \neq y$ and $G(t,x,x) = \frac{1}{2\pi}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)\left(Z_I'(x) - Z_{I_1}'(x)\right)$, $t > \frac{1}{2}$.

Then $G(t,x,y)$ is continuous on $(\frac{1}{2},\infty) \times I_1 \times I_1$ and

$$|G(t,x,y)| \leq \left|\frac{1}{2\pi}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)\right| M, \quad (t,x,y) \in (\frac{1}{2},\infty) \times I_1 \times I_1 \ ,$$

where M is a constant.

Proof. The continuity of G follows from Lemma 3.13. To prove the inequality, we note that

$$|G(t,x,y)| \leq \left|\frac{1}{2\pi}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)\right| \left|\frac{Z_{I,I_1}(x) - Z_{I,I_1}(y)}{x-y}\right| .$$

We note that $Z_{I,I_1}(x) - Z_{I,I_1}(y)$ is smooth on $\bar{I}_1 \times \bar{I}_1$, and apply Lemma 3.13 we have proved the inequality. □

By Lemma 3.14, we have that the kernel before Lemma 3.13 is given by

$$K_1^{\epsilon_0}(x,y) = \frac{-1}{\pi}\int_{\frac{1}{2}(1+2\epsilon_0)}^{\infty} \frac{\left(\frac{t}{1+2\epsilon_0} - \frac{1}{2}\right)}{t^2 - 1/4} G(t,x,y)dt \ .$$

Lemma 3.15. *(1) $K_1^{\epsilon_0}(x,y)$ is continuous, uniformly bounded and converges uniformly on $I_1 \times I_1$ to $K_1^0(x,y)$ as ϵ_0 goes to 0;*

(2) The kernel of

$$\mathbf{P}_1 C \ln C + (1-C)\ln(1-C)\mathbf{P}_1 - C_1 \ln C_1 - (\mathbf{P_1} - C_1)\ln(\mathbf{P}_1 - C_1)$$

is given by the bounded continuous function $K_1^0(x,y)$, and moreover its trace is given by

$$\int_{I_1} K_1^0(x,x)dx = \lim_{\epsilon_0 \to 0} \int_{I_1} K_1^{\epsilon_0}(x,x)dx;$$

(3)

$$\int_{I_1} K_1^0(x,x)dx = \frac{1}{12} \sum_{(a_i,b_i)\in I_2,(a_j,b_j)\in I_1} \ln\left(\frac{(a_j-a_i)(b_j-b_i)}{(b_j-a_i)(a_j-b_i)}\right) .$$

Proof. (1): It is clear $K_1^{\epsilon_0}(x,y)$ is continuous and uniformly bounded by Lemma 3.14. By Lemma 3.14 again

$$|K_1^{\epsilon_0}(x,y) - K_1^0(x,y)| \le \frac{1}{2\pi^2} M \int_{1/2}^{\infty} \left| \frac{\left(\frac{t}{1+2\epsilon_0} - \frac{1}{2}\right)}{t^2 - 1/4} \chi_{(\frac{1}{2}(1+2\epsilon_0),\infty)} - \frac{1}{t+1/2} \right| \left| \ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right) \right| dt$$

where $\chi_{(\frac{1}{2}(1+2\epsilon_0),\infty)}$ denotes the characteristic function. We note that the integrand above is bounded and when t is large decays like a constant multiply by $\frac{1}{t^2}$.

The uniform convergence now follows by Lebesgue's dominated convergence theorem.

(2): Note that as ϵ_0 goes to 0, $\mathbf{P}_1 E \ln E + (1-E)\ln(1-E)\mathbf{P}_1 - E_1 \ln E_1 - (\mathbf{P}_1 - E_1)\ln(\mathbf{P}_1 - E_1)$ converges to

$$\mathbf{P}_1 C \ln C + (1-C)\ln(1-C)\mathbf{P}_1 - C_1 \ln C_1 - (\mathbf{P}_1 - C_1)\ln(\mathbf{P}_1 - C_1)$$

strongly. (2) now follows from (1) and [3] which contains more general results on the trace of operators with integrable kernels.

(3): By Lemma 3.14 and (2) we have

$$\int_{I_1} K_1^0(x,x)dx =$$
$$\frac{-1}{2\pi^2} \int_{\frac{1}{2}}^{1} \frac{1}{t+1/2} \ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right) \left(\sum_{(a_i,b_i)\in I_2,(a_j,b_j)\in I_1} \ln\left(\frac{(a_j-a_i)(b_j-b_i)}{(b_j-a_i)(a_j-b_i)}\right)\right) dt .$$

To finish the proof we just need to show $\frac{-1}{2\pi^2}\int_{\frac{1}{2}}^{1} \frac{1}{t+1/2} \ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right) = 1/12$. By change of integration variable to $u = \ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)$ it is sufficient to check that

$$\int_{-\infty}^{0} \frac{ue^u}{1-e^u} du = \frac{-1}{6\pi^2} \ .$$

Since the anti-derivative of $\frac{ue^u}{1-e^u}$ is $-\mathbf{Li}_2(e^u) - u\ln(1-e^u)$ where $\mathbf{Li}_2(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is the dilogarithm, the desired equality follows from

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \ . \quad \square$$

Remark 3.16. We note that the previous Lemma works in exactly the same way when we replace I_1 by I_2, and $\mathbf{P}_1$ by $\mathbf{P}_2$.

3.7. The proof of Theorem 3.18

Definition 3.17. If $I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n)$ in increasing order, define

$$G(I) := \frac{1}{6}\left(\sum_{i,j} \log|b_i - a_j| - \sum_{i<j} \log|a_i - a_j| - \sum_{i<j} \log|b_i - b_j|\right) \ .$$

Theorem 3.18. *Let $I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n) \in \mathcal{PI}$ and $I_1 \cup I_2 = I, \bar{I}_1 \cap \bar{I}_2 = \emptyset$. Then*

$$S_{\mathcal{A}_r}(\omega, \omega_1 \otimes_2 \omega_2) = r\big(G(I_1) + G(I_2) - G(I_1 \cup I_2)\big) \ .$$

Proof. By Lemma 3.8 it is sufficient to prove $r = 1$ case.

Recall that $E := \frac{C+\epsilon_0}{1+2\epsilon_0}$. Apply Theorem 3.12 to $A = E$ and $A = (1 - E)$ respectively, we have

$$\mathrm{Tr}\sigma_E \geq \mathrm{Tr}\sigma_{E_p} \ .$$

Now let ϵ_0 go to 0 and by (2), (3) of Lemma 3.15, Lemma 3.8 and Remark 3.16, Theorem 3.18 is proved. $\square$

4. Subnets of free fermion nets and their finite index extensions

4.1. Formal properties of entropy for free fermion nets and their subnets

In the previous section we use Cayley transformation to identify punctured circle with real line as a tool to compute relative entropy. Now we return to general discussion about formal properties of entropy, and it is now convenient to be back to intervals on the circle. Let $I \in \mathcal{PI}$ be disjoint union of intervals on the circle. Explicitly we write

$I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n)$ in anti-clockwise order on the unit circle. We note that relative entropies as computed in Th. 3.18 is invariant under Möb transformations on the circle. The results of this section are inspired by [8].

By Theorem 3.18, we have $F_{\mathcal{A}_r}(A, B) := S(\omega, \omega_A \otimes_2 \omega_B) < \infty$ where A, B are union of disjoint intervals. When no confusion arises, we will simply write $F_{\mathcal{A}_r}(A, B)$ as $F(A, B)$.

We can extend the definition mutual information to more general union of disjoint intervals by the following

$$F(A \cup B, A \cup C) = F(A, B \cup C) + F(B, C) - F(A, C) - F(A, B) \ .$$

Theorem 4.1. *(1)*

$$F(A \cup B, A \cup C) \geq 0;$$

$F(A \cup B, A \cup C)$ is continuous from inside in the following sense: if $A_n \subset A, B_n \subset B, C_n \subset C$ is an increasing sequence of intervals such that $\cup_n A_n = A, \cup_n B_n = B, \cup_n C_n = C$, then $\lim_n F(A_n \cup B_n, A_n \cup C_n) = F(A \cup B, A \cup C)$;

(2)

$$F(A, B) + F(A, C) + F(A \cup B, A \cup C) + F(A \cap C, A \cap B) \\ = F(B, C) + F(A, B \cup C) + F(A, B \cap C).$$

(3) There exists function $G : \mathcal{PI} \to \mathbb{R}$ such that

$$F(A, B) = G(A) + G(B) - G(A \cup B) - G(A \cap B) \ .$$

Such G is uniquely determined by its value on connected open intervals;

(4) One can choose $G(a, b) = \frac{r}{6} \ln |b - a|$ in (3) for the r free fermion net $\mathcal{A}_r$, and such a choice determines

$$G(I) = \frac{r}{6} \left(\sum_{i,j} \ln |b_i - a_j| - \sum_{i<j} \ln |a_i - a_j| - \sum_{i<j} \ln |b_i - b_j| \right)$$

for $I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n)$ on unit circle with anti-clockwise order;

(5) $F(A \cup B, A \cup C) = F(A \cup B, C) - F(A, C) = F(B, A \cup C) - F(B, A)$; In particular $F(A \cup B, A \cup C)$ increases with B, C;

(6) If $\mathcal{B} \subset \mathcal{A}$ is a graded subnet, then (1), (2), (3) is also true for the system of mutual information associated with $\mathcal{B}$.

Proof. (1) and (5) for free fermions can be checked by using explicit formulas in Th. 3.18, but here we present general arguments which will also works for other cases such as subnets of free fermions.

Choose increasing sequence of finite dimensional factors I_{A_n}, I_{B_n}, invariant under the conjugate action of Γ such that $(\bigcup_n I_{A_n})'' = \mathcal{A}_r(A)$, $(\bigcup_n I_{B_n})'' = \mathcal{A}_r(B)$, and denote by $\rho_{A_nB_n}, \rho_{A_n} \otimes_2 \rho_{B_n}$ the restrictions of ω and $\omega_1 \otimes_2 \omega_2$ to $I_{A_n} \vee I_{B_n}$ respectively. Let ρ_{A_n} and ρ_{B_n} be the restrictions of ω to I_{A_n} and I_{B_n} respectively.

By Proposition 3.6

$$S(\rho_{A_nB_n}, \rho_{A_n} \otimes_2 \rho_{B_n}) = S(\rho_{A_n}) + S(\rho_{B_n}) - S(\rho_{A_nB_n}) \ .$$

To simplify notations, let us write $S(A_n) := S(\rho_{A_n}), S(A_n \cup B_n) := S(\rho_{A_nB_n})$. Then we have

$$F(A,B) = \lim_{n\to\infty} S(A_n) + S(B_n) - S(A_n \cup B_n) \ .$$

It follows that

$$F(A \cup B, A \cup C) = \lim_{n\to\infty} \big(S(A_n \cup B_n) + S(A_n \cup C_n) - S(A_n) - S(A_n \cup B_n \cup C_n)\big).$$

Note that

$$S(A_n \cup B_n) + S(A_n \cup C_n) - S(A_n) - S(A_n \cup B_n \cup C_n) \geq 0$$

by strong subadditivity of von Neumann entropy, (1) follows and (2) also follows from the limit formula and the fact that $F(A,B)$ is finite by Theorem 3.18.

(3): Starting with arbitrary real valued function G defined on open connected intervals of S^1, we can define $G(A)$ for any $A \in \mathcal{PI}$ as follows: define $G(A \cup B) = G(A) + G(B) - F(A,B)$ when A and B are disjoint. It is easy to see that such $G(A \cup B)$ is well defined and only depends on $A \cup B$ thanks to (2).

(4): This follows from Theorem 3.18, (1) and direct computations.

(5): The identities follow from (3).

(6): We note that by Theorem 3.18 and monotonicity of relative entropy in (4) of Th. 2.2 that for $\mathcal{B}$, $F_{\mathcal{B}}(A,B) \leq F_{\mathcal{A}_r}(A,B) < \infty$. For (1) and (2) we can use Remark 3.2 and proceed in exactly the same way as in free fermion net case. (3) and (5) are proved in the same way as in free fermion net case. □

4.2. Structure of singularities in the finite index case

G from (3) in Th. 4.1 can be thought as "regularized" version of von Neumann entropy which is always infinite in our case (cf. [23]). From (3) of the above Theorem we see that if we only allow G to be defined on $\mathcal{PI}$ then G is highly non unique. Due to the continuity properties of $F(A,B)$, we require that $G(A)$ depends continuously only on the length r_A of interval A. In addition we require that $G(A) = G(A^c)$ for a connected interval, and we set $G(\emptyset) = 0$. Still such G is highly non unique. However, we shall impose further conditions coming from studying the singularities of relative entropy when we allow intervals to approach each other. Let $B_\epsilon = (a_1, a_{2\epsilon}) \cup C = (a_2, b_2) \in \mathcal{PI}$, with $|a_{2\epsilon} - a_2| = \epsilon > 0$. We shall consider the singular limit when ϵ goes to zero while fixing a_1 and C. Let $B_0 = (a_1, a_2)$. We will denote by $B_0 \bar{\cup} C = (a_1, b_2)$, i.e., $B_0 \bar{\cup} C$ is obtained from $B_0 \cup C$ by adding the point a_2: notice in the process the number of components decrease by 1.

To probe the singularity structure of von Neumann entropy, we can consider $F(B_\epsilon, C)$ which goes to ∞ as $\epsilon \to 0$ while fixing a_1 and C. As an example, by Th. 3.18

$$F_{\mathcal{A}_r}(B_\epsilon, C) = \frac{r}{6}\big(\ln|a_2 - a_1| + \ln|b_2 - a_2| - \ln|b_2 - a_1| - \ln(\epsilon)\big) + o(\epsilon) \ .$$

Since $G(B_\epsilon \cup C) = G(B_\epsilon) + G(C) - F(B_\epsilon, C)$, the singularity structure of $G(B_\epsilon \cup C)$ is the same as that of $-F(B_\epsilon, C)$ as $\epsilon \to 0$. In fact this is also true for general case: consider

$$G(A \cup B_\epsilon \cup C) = G(A) + G(B_\epsilon \cup C) - F(A, B_\epsilon \cup C) \ .$$

One can see that the singularity structure of $G(A \cup B_\epsilon \cup C)$ is the same as that of $G(B_\epsilon \cup C)$ as $\epsilon \to 0$, since the rest of terms are bounded. So we can not expect $G(B_\epsilon \cup C)$ to be close to

$$G(B_0 \bar{\cup} C)$$

when $\epsilon \to 0$, but we may demand that

$$\lim_{\epsilon \to 0} G(B_\epsilon \cup C) - P(\epsilon) = G(B_0 \bar{\cup} C) \tag{13}$$

for some function $P(\epsilon)$ which is independent of B, C. The equation is a condition that connects the value of G for different components and as we shall see is a very useful condition. Equation (13) is of course equivalent to

$$G(B_0 \bar{\cup} C) = G(B_0) + G(C) - \lim_{\epsilon \to 0}\big(P(\epsilon) + F(B_\epsilon, C)\big) \ . \tag{14}$$

In general we may take multiple singular limits. Equation (13) allows us to evaluate such limits. Let us consider such an example in details. Let $A = (a_2, b_2)$, $B_{\epsilon_1} = (a_1, a_{2\epsilon_1})$, $C_{\epsilon_2} = (b_{2\epsilon_2}, b_3)$, $|a_{2\epsilon_1} - a_2| = \epsilon_1 > 0$, $|b_{2\epsilon_2} - b_2| = \epsilon_2 > 0$. Let ϵ_1 goes to 0 first, we find

$$F(A \bar{\cup} B_0, A \cup C_{\epsilon_2}) = G(A \bar{\cup} B_0) + G(A \cup C_{\epsilon_2}) - G(A \bar{\cup} B_0 \cup C_{\epsilon_2}) - G(A)$$

since the same function $P(\epsilon_1)$ appears in both $G(A \cup B_{\epsilon_1})$ and $G(A \cup B_{\epsilon_1} \cup C_{\epsilon_2})$ with opposite signs. Then let ϵ_1 goes to 0 we get by the same argument

$$F(A \bar{\cup} B_0, A \bar{\cup} C_0) = G(A \bar{\cup} B_0) + G(A \bar{\cup} C_0) - G(A \bar{\cup} B_0 \bar{\cup} C_0) - G(A) \ .$$

It is easy to see that the result is independent of the order of taking limits, and this way we can extend the definition of $F(A,B)$ to any $F(A,B)$ with $A \in \mathcal{PI}, B \in \mathcal{PI}$. Such $F(A,B)$ is used in [8]. In the case of free fermions, by Th. 3.18 we have that $P(\epsilon) = r/6 \ln \epsilon + o(\epsilon)$, and we have

$$F(A\bar{\cup}B_0, A\bar{\cup}C_0) = -\frac{r}{6} \ln \left| \frac{(b_2 - a_2)(b_3 - a_1)}{(b_3 - a_2)(b_2 - a_1)} \right| .$$

Now we will show that equation (13) is also true for a large class of examples. We assume that $\mathcal{B} \subset \mathcal{A}_r$ has finite index.

Note that by Lemma 2.10 $\mu_{\mathcal{B}} = \mu_{\mathcal{A}_r}[\mathcal{A} : \mathcal{B}]^2 = [\mathcal{A} : \mathcal{B}]^2 = \lambda^{-2}$.

Let $F_1(A,B) := F_{\mathcal{A}_r}(A,B) - F_{\mathcal{B}}(A,B)$ and $G_1(A) = G_{\mathcal{A}_r}(A) - G_{\mathcal{B}}(A)$. Then $F_1(A,B)$ verifies (2) and (3) of Th. 4.1. Note that $F_1(A,B)$ is not non-negative in general, being the difference of two non-negative numbers, but is always bounded by finite index assumptions.

We examine possible solutions of equation (14) for G_1. Let B_ϵ, C be two connected intervals as in equation (14), and E the unique conditional expectation from $\mathcal{A}_r(B_\epsilon) \vee \mathcal{A}_r(C)$ to $\mathcal{B}(B_\epsilon) \vee \mathcal{B}(C)$ which preserves the state $\omega_1 \otimes_2 \omega_2$. Then $S_{\mathcal{A}}(\omega, \omega_1 \otimes_2 \omega_2) = S_{\mathcal{B}}(\omega, \omega_1 \otimes_2 \omega_2) + S(\omega, \omega \cdot E)$ by Th. 2.2. Note that by Pimsner–Popa inequality $E(x) \geq \lambda^{-2} x$ for positive x, and so $F_1(B_\epsilon, C) = S(\omega, \omega \cdot E) \leq \ln \lambda^{-2}$. By Th. 4.4 $\lim_{\epsilon \to 0} F_1(B_\epsilon, C) = \ln \lambda^{-1}$, and equation (14) is simply

$$G_1(B_0 \bar{\cup} C) = G_1(B_0) + G_1(C) - (P - \ln \lambda) ,$$

where P is a constant. Up to a constant in the definition of $G_1(A)$ we can set $P = \ln \lambda$, and it follows that $G_1(A)$ is a constant multiplied by the arc length of A. But since we also require $G_1(A) = G_1(A^c)$, $G_1(A) = 0$.

In this case we get $G_{\mathcal{B}}(A) = G_{\mathcal{A}}(A)$ for any connected interval A, and use $G_{\mathcal{B}} = G_{\mathcal{A}_r} - G_1$ the system of solutions of equation (14) for $\mathcal{B}$.

We have proved the following:

Theorem 4.2. *Assume that a subnet $\mathcal{B} \subset \mathcal{A}_r$ has finite index, then:*

(1): $G_{\mathcal{B}}((a,b)) = \frac{r}{6} \ln |b - a|$ and verifies equation (14) and (3) of Th. 4.1, and

$$F_{\mathcal{B}}(A,B) = -\frac{r}{6} |\ln \eta_{AB}| ,$$

where A, B are two overlapping intervals with cross ratio $0 < \eta_{AB} < 1$;

(2) Let $B = (a_1, a_{2\epsilon})$, $C = (a_2, b_2)$, $|a_{2\epsilon} - a_2| = \epsilon > 0$. Then:

$$F_{\mathcal{B}}(B,C) = \frac{r}{6}\big(\ln |a_2 - a_1| + \ln |b_2 - a_2| - \ln |b_2 - a_1| - \ln(\epsilon)\big) - \frac{1}{2} \ln \mu_{\mathcal{B}} + o(\epsilon)$$

as ϵ goes to 0.

In exactly the same way if $\mathcal{B} \subset \mathcal{C}$ is a subnet with finite index where $\mathcal{B}$ is as in the above theorem, then we also get a system of solutions of equation (14) for $\mathcal{C}$ as in the above theorem.

Remark 4.3. It is interesting to note that the constant term in (2) of Th. 4.2 seems to be related to the topological entropy discussed in [15] even with the right factor: in our case we have additional factor 1/2 since we are discussing chiral half of CFT.

We conjecture that the above theorem is true for any rational conformal net, where r is replaced by the central charge. More examples where Th. 4.2 applies are discussed in Section 4.4.

Notice also that the cross ratio enters in formulas concerning nuclearity (partition function) [4] and entanglement entropy [12], so we can infer relations about the mutual information and these quantities.

4.2.1. Failure of duality is related to global dimension

By Th. 3.18 for the free fermion net $\mathcal{A}_r$, and two intervals $A = (a_1, b_1)$, $B = (a_2, b_2)$, where $b_1 < a_2$, we have

$$F_{\mathcal{A}}(A, B) = \frac{-r}{6} \ln \eta \ ,$$

where $\eta = \frac{(b_1-a_2)(b_2-a_1)}{(b_1-a_1)(b_2-a_2)}$ is the cross ratio, $0 < \eta < 1$. For simplicity we denote by $F_{\mathcal{A}_r}(\eta) = F_{\mathcal{A}}(A, B)$.

One checks that $F_{\mathcal{A}_r}(A, B) = F_{\mathcal{A}_r}(A^c, B^c)$, which is in fact equivalent to

$$F_{\mathcal{A}_r}(\eta) - F_{\mathcal{A}_r}(1-\eta) = \frac{-r}{6} \ln\left(\frac{\eta}{1-\eta}\right) \ .$$

Similarly for $\mathcal{B} \subset \mathcal{A}_r$ with finite index, by Th. 4.2 $F_{\mathcal{B}}(A, B) = F_{\mathcal{B}}(A^c, B^c)$ is equivalent to

$$F_{\mathcal{B}}(\eta) - F_{\mathcal{B}}(1-\eta) = \frac{-r}{6} \ln\left(\frac{\eta}{1-\eta}\right) \ .$$

We note that $F_{\mathcal{A}_r}(A, B) = F_{\mathcal{A}_r}(A^c, B^c)$ for the free fermion net $\mathcal{A}_r$. However here we show that $F_{\mathcal{B}}(A, B) \neq F_{\mathcal{B}}(A^c, B^c)$ with $\mathcal{B} \subset \mathcal{A}_r$ has finite index $[\mathcal{A}_r : \mathcal{B}] = \lambda^{-1} > 1$. By Lemma 2.10 $\mu_{\mathcal{B}} = [\mathcal{A}_r : \mathcal{B}]^2$.

We note that, as before the proof of Th. 4.2, $S(\omega, \omega \cdot E) = F_1(\eta) = F_{\mathcal{A}}(\eta) - F_{\mathcal{B}}(\eta)$ is a decreasing function of η, and $0 \leq F_1(\eta) \leq F_{\mathcal{A}}(\eta)$. So we have

$$\lim_{\eta \to 1} F_1(\eta) = 0 \ .$$

On the other hand, by Th. 4.4

$$\lim_{\eta\to 0} F_1(\eta) = \ln[\mathcal{A}_r : \mathcal{B}] = \frac{1}{2}\ln \mu_{\mathcal{B}} \ .$$

It follows that $F_{\mathcal{B}}(A,B) \neq F_{\mathcal{B}}(A^c,B^c)$ due to the fact that $\mu_{\mathcal{B}} > 1$.

4.3. Computation of limit of relative entropy

In this section we determine the exact limit of relative entropies which are necessary for analyzing the singularity structures of entropies in Section 4.2. The goal is to prove the following:

Theorem 4.4. *Assume that subnet $\mathcal{B} \subset \mathcal{A}$ has finite index, $\mathcal{B}$ is strongly additive. Let I_1 and I_2 be two intervals obtained from an interval I by removing an interior point, and let $J_n \subset I_2, n \geq 1$ be an increasing sequence of intervals such that*

$$\bigcup_n J_n = I_2, \quad \bar{J}_n \cap \bar{I}_1 = \emptyset \ .$$

Let E_n be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$ to $\mathcal{A}(I_1) \vee \mathcal{B}(J_n)$ such that $E_n(xy) = xE_I(y), \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(J_n)$. Then

$$\lim_{n\to\infty} S(\omega, \omega \cdot E_n) = [\mathcal{A} : \mathcal{B}] \ .$$

4.3.1. Basic idea from Kosaki's formula

Denote by $\phi_n = \omega \cdot E_n$. By Kosaki's formula (cf. [16])

$$S(\omega, \omega \cdot E_n) = \sup_{m\in\mathbb{N}} \sup_{x_t+y_t=1} \left(\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big) dt \right) ,$$

where x_t is a step function which is equal to 0 when t is sufficiently large. To motivate the proof of Th. 4.4, it is instructive to see how we can get $S(\omega, \lambda\omega) = -\ln\lambda, 0 < \lambda < 1$ from Kosaki's formula. By tracing the proof in [16], one can see that the path which gives approximation to $-\ln\lambda$ is given by the following continuous path

$$x(t) = \frac{\lambda}{\lambda + t}, y(t) = \frac{t}{\lambda + t}, t \geq k^{-1}$$

and with such a choice we have

$$\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big) dt = -\ln(\lambda + 1/k)$$

which tends to $-\ln\lambda$ as k goes to ∞. This suggests that for the proof of Th. 4.4, we need to choose path x_t, y_t such that $\omega(x_t^* x_t)$ and $\phi_n(y_t y_t^*)$ are close to $\big(\frac{\lambda}{\lambda+t}\big)^2$ and $\lambda\big(\frac{t}{\lambda+t}\big)^2$ respectively, and this motivates our Proposition 4.5 and the proof of Th. 4.4.

4.3.2. A key step in the proof of Th. 4.4

Let $e_1 \in \mathcal{A}(I_1), e_2 \in \mathcal{A}(J_1)$ be Jones projections for $\mathcal{B}(I_1) \subset \mathcal{A}(I_1)$ and $\mathcal{B}(J_1) \subset \mathcal{A}(J_1)$ respectively as in Lemma 2.9. Let P be the projection from the vacuum representation of $\mathcal{A}$ onto the vacuum representation of $\mathcal{B}$. By Lemma 2.9, there is a unitary $u \in \mathcal{B}(I)$ such that $ue_1u^* = e_2$. Choose isometry $v_2 \in \mathcal{B}(J_1)$ such that $\lambda^{-1}v_2^*e_2v_2 = 1$. Note that $e_2v_2v_2^*e_2 = \lambda e_2$, and $Pe_2P = \lambda P$. It follows that $Pe_2^+P = \lambda P, Pe_2^-P = 0$ by our assumption that $[\Gamma, P] = 0$.

Since $\mathcal{B}$ is strongly additive, we can find a sequence of bounded operators $u_n \in \mathcal{B}(I_1) \vee \mathcal{B}(J_n), n \geq 2$ such that $u_n \to u, u_n^* \to u^*$ strongly. Let $e_{2n} := u_n e_1 u_n^*$. Then $e_{2n} \to e_2$ strongly.

Proposition 4.5. *For any $\epsilon > 0$, one can find $n \geq 2$ and $e \in \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$ such that*

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon,\ |\omega(e^*e) - 1| < \epsilon,\ |\phi_n(ee^*) - \lambda| < \epsilon\ .$$

Proof. Let us first denote by $e = \lambda^{-1}v_2^*e_{2n}e_2v_2$. We will show that given $\epsilon > 0$, we can choose n sufficiently large such that e verifies the conditions in the Proposition. First we observe that since $e_{2n} \to e_2$ strongly, it follows that $e \to 1$ strongly, and hence by choosing n sufficiently large we can have

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, |\omega(e^*e) - 1| < \epsilon\ .$$

Now let us evaluate

$$\phi_n(ee^*) = \phi_n(\lambda^{-2}v_2^*e_{2n}e_2v_2v_2^*e_2e_{2n}v_2) = \lambda^{-1}\phi_n(v_2^*e_{2n}e_2e_{2n}v_2)\ .$$

Recall the definition of ϕ_n as a state on $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$: For any x, y with $x \in \mathcal{A}(I_1), y \in \mathcal{A}(J_n)$,

$$\phi_n(xy) = \langle \Omega, xPyP\Omega \rangle\ .$$

Recall that $e_2 = e_2^+ + e_2^-, Pe_2^+P = \lambda P, Pe_2^-P = 0$. To evaluate $\phi_n(v_2^*u_ne_1u_n^*e_2u_ne_1u_n^*v_2)$, we approximate u_n with finite linear combination of operator of the form $u_{1m}u_{2m}$ with $u_{1m} \in \mathcal{B}(I_1), u_{2m} \in \mathcal{B}(J_n)$, then we move those operators in $\mathcal{A}(I_1)$ to the left of those operators in $\mathcal{A}(J_n)$ using commuting or anti-commuting relations, and it is crucial to observe the operators that belong to $\mathcal{A}(J_n)$ has only one term e_2^+ or e_2^-, and the rest are in $\mathcal{B}(J_n)$. When compressed such term with P and acting on Ω, we see that e_2^+ is replaced with λ, and e_2^- is replaced with 0. We note that e_2^+ commuting with $\mathcal{A}(I_1)$. It follows that

$$\phi_n(ee^*) = \phi_n(\lambda^{-2}v_2^*e_{2n}e_2v_2v_2^*e_2e_{2n}v_2) = \lambda^{-1}\phi_n(v_2^*e_{2n}e_2e_{2n}v_2) = \langle \Omega, v_2^*(u_ne_1u_n^*)^2v_2\Omega \rangle\ .$$

Since $v_2^*(u_ne_1u_n^*)^2v_2$ goes to $v_2^*e_2v_2 = \lambda$ strongly, the Proposition is proved. □

4.3.3. The proof of Th. 4.4

Recall $\phi_n = \omega \cdot E_n$. By Pimsner–Popa inequality, $E_n(x) \geq \lambda x$ for any positive $x \in \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$, it follows that $\phi_n \geq \lambda\omega$, and hence by Th. 2.2

$$S(\omega, \omega \cdot E_n) \leq [\mathcal{A} : \mathcal{B}].$$

Note that by monotonicity of relative entropy $S(\omega, \omega \cdot E_n)$ increases with n, hence $\lim_{n\to\infty} S(\omega, \omega \cdot E_n)$ exists and is less or equal to $[\mathcal{A} : \mathcal{B}]$.

By Kosaki's formula

$$S(\omega, \omega \cdot E_n) = \sup_{m\in\mathbb{N}} \sup_{x_t+y_t=1} \left(\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big) dt \right) ,$$

where x_t is a step function which is equal to 0 when t is sufficiently large. Since we can approximate any continuous function with step functions in the strong topology and vice versa, we can assume that x_t is continuous and is equal to 0 when t is sufficiently large. Given $\epsilon > 0$, for fixed $k, m \in \mathbb{N}$ choose e as in Proposition 4.5 and

$$x_t = 1 - \frac{t}{\lambda + t} e, k^{-1} \leq t \leq m \ .$$

We have

$$\omega(x_t^* x_t) = 1 - \frac{t}{\lambda+t}\omega(e) - \frac{t}{\lambda+t}\omega(e^*) + \left(\frac{t}{\lambda+t}\right)^2 \omega(e^* e)$$

and

$$\phi_n(y_t y_t^*) = \left(\frac{t}{\lambda+t}\right)^2 \phi_n(e e^*) \ .$$

By Proposition 4.5 we can choose n large enough such that

$$\int_{k^{-1}}^{m} \Big|\omega(x_t x_t^*) - \Big(\frac{\lambda}{\lambda+t}\Big)^2\Big| \frac{dt}{t} \leq \epsilon \ ,$$

$$\int_{k^{-1}}^{m} \Big|\phi_n(y_t y_t^*) - \lambda\Big(\frac{t}{\lambda+t}\Big)^2\Big| \frac{dt}{t^2} \leq \epsilon \ ,$$

and with such a choice of n we have:

$$\ln k - \int_{k^{-1}}^{\infty} \Big(\omega(x_t^* x_t)\frac{1}{t} + \phi_n(y_t y_t^*)\frac{1}{t^2}\Big) dt \geq$$

$$\ln k - \int_{k^{-1}}^{m} \left(\left(\frac{\lambda}{\lambda + t} \right)^2 \frac{1}{t} + \left(\frac{t}{\lambda + t} \right)^2 \frac{\lambda}{t^2} \right) dt + 1/m - 2\epsilon$$

$$= \ln \left(\frac{k}{k\lambda + 1} \right) - \ln \left(\frac{m}{\lambda + m} \right) + 1/m - 2\epsilon \ .$$

It follows that

$$\lim_{n\to\infty} S(\omega, \omega \cdot E_n) \geq \ln \left(\frac{k}{k\lambda + 1} \right) - \ln \left(\frac{m}{\lambda + m} \right) + 1/m - 2\epsilon \ .$$

Let k, m go to ∞ and ϵ go to 0, we have proved theorem. □

4.4. More examples

4.4.1. Orbifold examples

Take $U(1)_{4k^2} \subset U(1)_1$. This is $\mathbb{Z}_{2k}$ orbifold of $U(1)_1$. So Th. 4.2 apply to the net $U(1)_{4k^2}$. Another special case is when $k = 1$, we can take a further $\mathbb{Z}_2$ orbifold of $U(1)_4$ which corresponds to complex conjugation on $U(1)$ to get a tensor product of two Ising model with central charge $\frac{1}{2}$. It follows that Ising model with central charge $\frac{1}{2}$ verifies Th. 4.2, and in particular violates duality discussed in Section 4.2.1.

More generally, we can take any finite subgroup of $U(n)$ which commutes with $\mathrm{Ad}\Gamma$ and obtain orbifold subnet of $U(n)_1$. This provides a large family of examples which verify Th. 4.2.

4.4.2. Conformal inclusions

By [32], we have the following inclusions with finite index:

$$SU(n)_m \times SU(m)_n \times U(1)_{mn(m+n)^2} \subset Spin(2mn)_1 \subset U(mn)_1 \ .$$

So Th. 4.2 apply to the net $SU(n)_m \times SU(m)_n \times U(1)_{mn(m+n)^2}$. If we take $m = n$, then since $U(1)_{(4n^4)}$ verifies Th. 4.2 by the example in previous section, it follows that the net associated with $SU(n)_n \times SU(n)_n$, and hence the net associated with $SU(n)_n$ also verifies Th. 4.2.

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References

[1] H. Araki, On quasifree states of the CAR and Bogoliubov automorphisms, Publ. Res. Inst. Math. Sci., Kyoto Univ. 6 (1970) 385–442.

[2] H. Araki, Relative entropy of states of von Neumann algebras, I, II, Publ. Res. Inst. Math. Sci., Kyoto Univ. 11 (1976) 809–833 13 (1977) 173–192.
[3] C. Brislawn, Kernels of trace class operators, Proc. Amer. Math. Soc. 104 (1988) 1181–1190.
[4] D. Buchholz, C. D'Antoni, R. Longo, Nuclearity and thermal states in conformal field theory, Comm. Math. Phys. 270 (2007) 267–293.
[5] P. Calabrese, J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A 42 (2009) 504005.
[6] E.A. Carlenl, Trace inequalities and entropy: an introductory course, http://www.ueltschi.org/AZschool/notes/EricCarlen.pdf.
[7] S. Carpi, Y. Kawahigashi, R. Longo, Structure and classification of superconformal nets, Ann. Henri Poincaré 9 (6) (2008) 1069–1121.
[8] H. Casini, M. Huerta, A finite entanglement entropy and the c-theorem, Phys. Lett. B 600 (2004) 142–150, arXiv:hep-th/0405111.
[9] H. Casini, M. Huerta, Entanglement entropy in free quantum field theory, J. Phys. A 42 (2009) 504007, arXiv:0905.2562.
[10] H. Casini, M. Huerta, Reduced density matrix and internal dynamics for multicomponent region, Classical Quantum Gravity 26 (2009) 185005, arXiv:0903.5284.
[11] J. Fröhlich, F. Gabbiani, Operator algebras and conformal field theory, Comm. Math. Phys. 155 (1993) 569–640.
[12] S. Hollands, K. Sanders, Entanglement measures and their properties in quantum field theory, arXiv:1702.04924.
[13] N. Iqbal, A.C. Wall, Anomalies of the entanglement entropy in chiral theories, J. High Energy Phys. 10 (2016) 111, arXiv:1509.04325.
[14] Y. Kawahigashi, R. Longo, M. Müger, Multi-interval subfactors and modularity of representations in conformal field theory, Comm. Math. Phys. 219 (2001) 631–669.
[15] A. Kitaev, J. Preskill, Topological entanglement entropy, Phys. Rev. Lett. 96 (2006) 110404.
[16] H. Kosaki, Relative entropy of states: a variational expression, J. Operator Theory 16 (2) (1986) 335–348.
[17] E.H. Lieb, M.B. Ruskai, Proof of the strong subadditivity of quantum-mechanical entropy, J. Math. Phys. 14 (1973) 1938.
[18] R. Longo, Conformal subnets and intermediate subfactors, Comm. Math. Phys. 237 (1–2) (2003) 7–30.
[19] R. Longo, On Landauer principle and bound for infinite systems, arXiv:1710.00910.
[20] R. Longo, K.-H. Rehren, Nets of subfactors, Rev. Math. Phys. 7 (1995) 567–597.
[21] S.G. Mikhlin, Integral Equations: And Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology, Elsevier, 2014.
[22] N.I. Muskhelishvili, Singular Integral Equations, 1953, Groningen–Holland, Chapter 14, 1997.
[23] H. Narnhofer, Entanglement, split, and nuclearity in quantum field theory, Rep. Math. Phys. 50 (2002) 307–347.
[24] M. Ohya, D. Petz, Quantum Entropy and Its Use, Theoretical and Mathematical Physics, Springer-Verlag, Berlin, Heidelberg, 1993.
[25] Y. Otani, Y. Tanimoto, Towards entanglement entropy with UV cutoff in conformal nets, arXiv:1701.01186.
[26] A. Pressley, G. Segal, Loop Groups, Oxford University Press, 1986.
[27] M. Reed, B. Simon, Methods of Modern Mathematical Physics II, Academic Press, 1975.
[28] M. Rosenblum, J. Rovnyak, Hardy Classes and Operator Theory, Dover, ISBN 0-486-69536-0, 1997.
[29] V. Toledano Laredo, Fusion of positive energy representations of $LSPin_{2n}$, arXiv:math/0409044 [math.OA].
[30] A. Wassermann, Operator algebras and conformal field theories III, Invent. Math. 133 (1998) 467–538.
[31] F. Xu, Jones-Wassermann subfactors for disconnected intervals, Commun. Contemp. Math. 02 (03) (2000) 307–347.
[32] F. Xu, On the equivalence of certain coset conformal field theories, Comm. Math. Phys. 228 (2002) 257–279.