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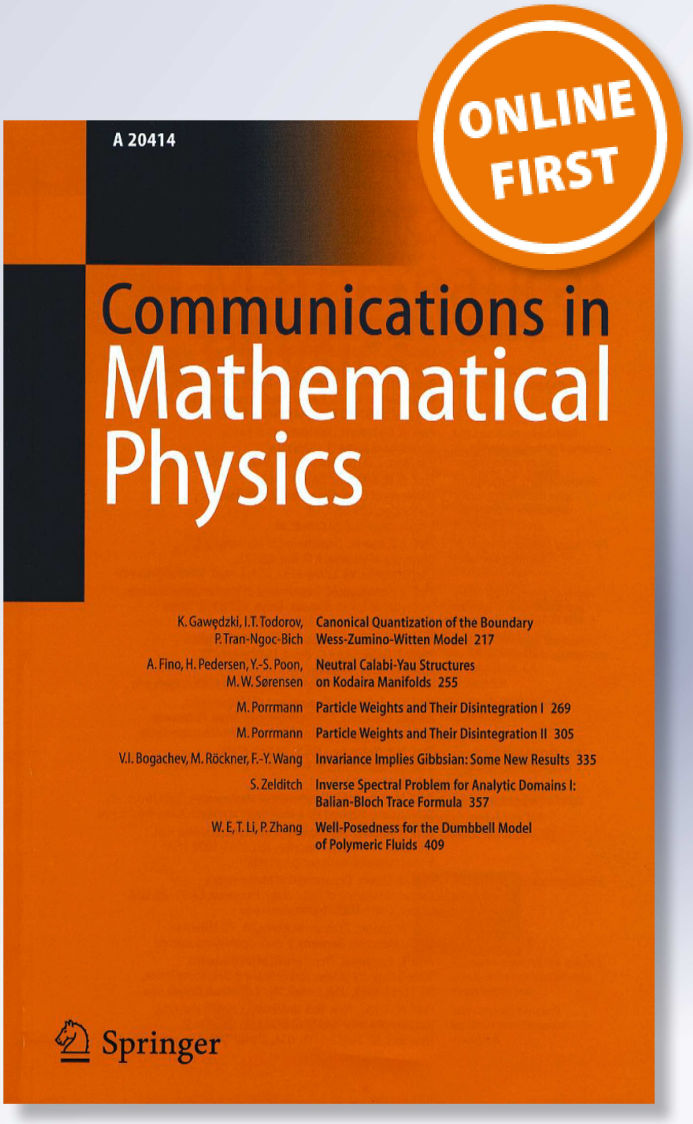

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Some Results On Relative Entropy in Quantum Field Theory

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Abstract: We prove that the mutual information for vacuum state as defined by Araki is finite for general Dirac Quantum Fields in Minkowski spacetime of any dimension. In the case of two dimensional chiral conformal field theory (CFT) we use our previous results for the free fermions to show that for a large class of chiral CFT the mutual information is finite. We also provide a direct relation between relative entropy and the index of a representation of conformal net.

1. Introduction

In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; see the references in [17] for a partial list of references. However, some very basic mathematical questions remain open. Often, the mutual information is argued to be finite based on heuristic physical arguments, and one can derive the singularities of the entropies from the mutual information by taking singular limits. But it is not even clear that such mutual information, which is well defined as a special case of Araki's relative entropy, is indeed finite. All the heuristic computations such those done in [10] and [9] take this for granted and these papers contain a number of amazing results about the nature of such mutual information. It is clear that there should be a rich mathematical theory behind these physical considerations. See [16,18–21,24,29] for a partial list of recent mathematical work.

In [21] we showed that mutual information for massless free fermions is finite, and in [21] we calculate its value for all cases. In fact this is the only example where all mutual information is known (see [4] for recent computations in the case of massless bosons with two intervals). The method in §3 of [21] uses the explicit resolvent formula for free fermions which unfortunately is not known in other cases such as free massive

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fermions. One of the goals on this paper is to improve on some of the estimates in §3 of [21] so that we can obtain finiteness of mutual information for all free fermion theories (cf. Corollary 3.7) without having the explicit resolvent formula available. The main results which lead to Corollary 3.7 are Theorem 3.4 and Proposition 3.6.

We then consider more such finiteness results for chiral CFT in two dimensions by embedding into free fermions and using monotonicity of relative entropy. First we show that every lattice contains a finite index sublattice which can be embedded into free fermions in Corollary 4.3. As a consequence we show that mutual information is finite for all conformal net coming from lattices in Corollary 4.4. These immediately show that all conformal nets which can be embedded as a subnet of conformal nets associated with a lattice, and with a simply connected Lie group G at level k or so called Wess-Zumino-Witten models, their cosets, orbifolds, simple current extensions and combinations of such constructions, the mutual information is always finite. Our last result Theorem 4.5 gives a direct relation between relative entropy and the index of a representation of conformal net, in a similar spirit to a result in §4 of [21].

The rest of this paper is structured as follows. After a preliminary section on von Neumann entropy, Araki's relative entropy, we consider the mutual information in an algebraic quantum field theory with split property, and use free fermion theory as an example. Then we consider a general problem motivated by computations of mutual information in Sect. 3, where we prove a few keys results such as Thereom 3.4, Proposition 3.6. The finiteness of mutual information in free fermion QFT is obtained as a consequence in Corollary 3.7. In Sect. 4 we first show that a conformal net $\mathcal{A}_L$ associated with a lattice has a finite index subnet which can be embedded into free fermions. It follows by monotonicity of relative entropy that mutual information for $\mathcal{A}_L$ is finite. From this we derive the finiteness of mutual information for a large class of chiral CFTs. In the last subsection we prove Theorem 4.5.

2. Preliminaries

2.1. Entropy and relative entropy. von Neumann entropy is the quantity associated with a density matrix ρ on a Hilbert space $\mathcal{H}$ by

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho).$$

von Neumann entropy can be viewed as a measure of the lack of information about a system to which one has ascribed the state. This interpretation is in accord for instance with the facts that $S(\rho) \geq 0$ and that a pure state $\rho = |\Psi\rangle\langle\Psi|$ has vanishing von Neumann entropy.

A related notion is that of the relative entropy. It is defined for two density matrices ρ, ρ' by

$$S(\rho, \rho') = \mathrm{Tr}(\rho \log \rho - \rho \log \rho'). \tag{1}$$

Like $S(\rho)$, $S(\rho, \rho')$ is non-negative, and can be infinite.

A generalization of the relative entropy in the context of von Neumann algebras of arbitrary type was found by Araki [1,2] and is formulated using modular theory. Given two faithful, normal states ω, ω' on a von Neumann algebra $\mathcal{A}$ in standard form, we choose the vector representatives in the natural cone $\mathcal{P}^\sharp$, called $|\Omega\rangle, |\Omega'\rangle$. The anti-linear operator $S_{\omega,\omega'} a|\Omega'\rangle = a^*|\Omega\rangle$, $a \in \mathcal{A}$, is closable and one considers again the polar decomposition of its closure $\bar{S}_{\omega,\omega'} = J\Delta^{1/2}_{\omega,\omega'}$. Here J is the modular conjugation

of $\mathcal{A}$ associated with $\mathcal{P}^\sharp$ and $\Delta_{\omega,\omega'} = S^*_{\omega,\omega'}\bar{S}_{\omega,\omega'}$ is the relative modular operator w.r.t. $|\Omega\rangle, |\Omega'\rangle$. Of course, if $\omega = \omega'$ then $\Delta_\omega = \Delta_{\omega,\omega'}$ is the usual modular operator.

A related object is the Connes cocycle (Radon–Nikodym derivative) defined as $[D\omega : D\omega']_t = \Delta^{it}_{\omega,\psi}\Delta^{it}_{\psi,\omega'} \in \mathcal{A}$, where ψ is an arbitrary auxiliary faithful normal state on $\mathcal{A}'$.

Definition 2.1. The relative entropy w.r.t. ω and ω' is defined by

$$S(\omega, \omega') = \langle\Omega| \log \Delta_{\omega,\omega'}\, \Omega\rangle = \lim_{t\to 0} \frac{\omega([D\omega : D\omega']_t - 1)}{it}, \tag{2}$$

S is extended to positive linear functionals that are not necessarily normalized by the formula $S(\lambda\omega, \lambda'\omega') = \lambda S(\omega, \omega') + \lambda \log(\lambda/\lambda')$, where $\lambda, \lambda' > 0$ and ω, ω' are normalized. If ω' is not normal, then one sets $S(\omega, \omega') = \infty$.

For a type I algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, states ω, ω' correspond to density matrices ρ, ρ'. The square root of the relative modular operator $\Delta^{1/2}_{\omega,\omega'}$ corresponds to $\rho^{1/2} \otimes \rho'^{-1/2}$ in the standard representation of $\mathcal{A}$ on $\mathcal{H} \otimes \bar{\mathcal{H}}$; namely $\mathcal{H} \otimes \bar{\mathcal{H}}$ is identified with the Hiilbert-Schmidt operators $HS(\mathcal{H})$ with the left/right multiplication of $\mathcal{A}/\mathcal{A}'$. In this representation, ω corresponds to the vector state $|\Omega\rangle = \rho^{1/2} \in \mathcal{H} \otimes \bar{\mathcal{H}}$, and the abstract definition of the relative entropy in (2) becomes

$$\begin{aligned}\langle\Omega| \log \Delta_{\omega,\omega'}\, \Omega\rangle &= \mathrm{Tr}_{\mathcal{H}} \rho^{\frac{1}{2}} \left(\log \rho \otimes 1 - 1 \otimes \log \rho'\right) \rho^{\frac{1}{2}} \\ &= \mathrm{Tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \rho'). \end{aligned} \tag{3}$$

As another example, let us consider a bi-partite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and observable algebra $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$. A normal state ω_{AB} on $\mathcal{A}$ corresponds to a density matrix ρ_{AB}. One calls $\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho_{AB}$ the "reduced density matrix", which defines a state ω_A on $\mathcal{B}(\mathcal{H}_A)$ (and similarly for system B). The mutual information is given in our example system by

$$S(\rho_{AB}, \rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{4}$$

A list of properties of relative entropies that will be used later can be found in [23] (cf. Th. 5.3, Th. 5.15 and Cor. 5.12 [23]):

Theorem 2.2. (1) *Let M be a von Neumann algebra and M_1 a von Neumann subalgebra of M. Assume that there exists a faithful normal conditional expectation E of M onto M_1. If ψ and ω are states of M_1 and M, respectively, then $S(\omega, \psi \cdot E) = S(\omega \restriction M_1, \psi) + S(\omega, \omega \cdot E)$;*

(2) *Let be M_i an increasing net of von Neumann subalgebras of M with the property $(\bigcup_i M_i)'' = M$. Then $S(\omega_1 \restriction M_i, \omega_2 \restriction M_i)$ converges to $S(\omega_1, \omega_2)$ where ω_1, ω_2 are two normal states on M;*

(3) *Let ω and ω_1 be two normal states on a von Neumann algebra M. If $\omega_1 \geq \mu\omega$, then $S(\omega, \omega_1) \leq \ln \mu^{-1}$;*

(4) *Let ω and ϕ be two normal states on a von Neumann algebra M, and denote by ω_1 and ϕ_1 the restrictions of ω and ϕ to a von Neumann subalgebra $M_1 \subset M$ respectively. Then $S(\omega_1, \phi_1) \leq S(\omega, \phi)$.*

For type III factors, the von Neumann entropy is always infinite, but we shall see that in many cases mutual information is finite.

Let us describe the setting where the relative entropy we are interested in computing. We consider the formulation of algebraic quantum field theory on a $D = d + 1$ dimensional Minkowski spacetime (cf. [14]). Let DO be an open subset of space time such that the closure of DO is compact. Let $\mathcal{A}(DO)$ be the algebra of observable associated with DO, and ω the vector state given by the vacuum vector. For simplicity we will assume that DO is the double cone generated by an open set O on the time zero slice $\mathbb{R}^d$. We shall assume that O has smooth boundary and the closure of O in $\mathbb{R}^d$ is compact. Slightly abusing our notation we denote $\mathcal{A}(DO)$ simply by $\mathcal{A}(O)$. O_1 and O_2 are disjoint if $\bar{O}_1 \cap \bar{O}_2 = \emptyset$. Denote by $\mathcal{P}O$ the set which consists of finite union of disjoint Os. Let $O_1, O_2 \subset \mathcal{P}O$, with $\bar{O}_1 \cap \bar{O}_2 = \emptyset$. Let ω_1, ω_2 be the restriction of ω to $\mathcal{A}(O_1), \mathcal{A}(O_2)$ respectively.

We shall assume that our theory is split (cf. [6] for bosonic case, [28] and [12] for fermionic case), which means that $\omega_1 \otimes \omega_2$, which is defined on elements of the form $xy, x \in \mathcal{A}(O_1), y \in \mathcal{A}(O_2)$ by $\omega_1 \otimes \omega_2(xy) = \omega_1(x)\omega_2(y)$, extends to a normal faithful state of the von Neumann algebra generated by $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$. The basic quantity we are interested in is the relative entropy (also called mutual information) $S(\omega, \omega_1 \otimes \omega_2)$.

As an example let us consider chiral free fermion CFT as discussed in details in §3 of [21]. We will describe the formula for mutual information which is proved in Th. 3.18 of [21], and refer reader to §3 of [21] for more details.

Let H denote the Hilbert space $L^2(S^1; \mathbb{C}^r)$ of square-summable $\mathbb{C}^r$-valued functions on the circle.

Fix $I_i \in \mathcal{P}I$, $i = 1, 2$, and I_1, I_2 disjoint, that is $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, and $I = I_1 \cup I_2$. Denote by $\mathcal{A}_r$ the graded conformal net associated with r chiral free fermions. We will write the normal faithful state $\omega_1 \otimes_2 \omega_2$ with graded tensor product in §3 of [21] simply as $\omega_1 \otimes \omega_2$, and the mutual information we are interested is now $S_{\mathcal{A}_r}(\omega, \omega_1 \otimes \omega_2)$.

The vacuum state ω on $\mathcal{A}_r(I)$ is a quasi-free state as studied by Araki in [3]. To describe this state, it is convenient to use Cayley transform $V(x) = (x - i)/(x + i)$, which carries the (one point compactification of the) real line onto the circle and the upper half plane onto the unit disk. It induces a unitary map

$$Uf(x) = \pi^{-\frac{1}{2}}(x + i)^{-1} f(V(x))$$

of $L^2(S^1, \mathbb{C}^r)$ onto $L^2(\mathbb{R}, \mathbb{C}^r)$. The operator U carries the Hardy space on the circle onto the Hardy space on the real line . We will use the Cayley transform to identify intervals on the circle with one point removed to intervals on the real line. Under the unitary transformation above, the Hardy projection on $L^2(S^1, \mathbb{C}^r)$ is transformed to the Hardy projection on $L^2(\mathbb{R}, \mathbb{C}^r)$ given by

$$Pf(x) = \frac{1}{2} f(x) + \int \frac{i}{2\pi} \frac{1}{(x - y)} f(y) dy,$$

where the singular integral is (proportional to) the Hilbert transform.

We write the kernel of the above integral transformation as C:

$$C(x, y) = \frac{1}{2}\delta(x - y) - \frac{i}{2\pi} \frac{1}{(x - y)}. \tag{5}$$

The quasi free state ω is determined by

$$\omega\big(a(f)^* a(g)\big) = \langle g, Pf \rangle.$$

Slightly abusing our notations, we will identify P with its kernel C and simply write

$$\omega\big(a(f)^* a(g)\big) = \langle g, Cf\rangle.$$

C will be called *covariance operator*.

Definition 2.3. Let $\mathbf{P}_i$ be projections from $L^2(I, \mathbb{C}^r)$ onto $L^2(I_i, \mathbb{C}^r)$, and $C_i = \mathbf{P}_i C \mathbf{P}_i$, $i = 1, 2$.

Let

$$\begin{aligned}\sigma_C = {} & \mathbf{P_1}\big(C \ln C + (1-C)\ln(1-C)\big)\mathbf{P_1} - \big(C_1 \ln C_1 + (\mathbf{P_1} - C_1)\ln(\mathbf{P_1} - C_1)\big) \\ & + \mathbf{P_2}\big(C \ln C + (1-C)\ln(1-C)\big)\mathbf{P_2} - \big(C_2 \ln C_2 + (\mathbf{P_2} - C_2)\ln(\mathbf{P_2} - C_2)\big)\end{aligned}$$

and σ_{C_p} be the same as in the definition of σ_C with C replaced by $C_p = pCp$, if p is a projection commuting with $\mathbf{P}_1$.

As a consequence of Theorem 3.18 of [21] we have

Proposition 2.4.

$$S(\omega, \omega_1 \otimes \omega_2) = \lim_{p \to 1} \mathrm{Tr}(\sigma_{C_p}) = \mathrm{Tr}(\sigma_C)$$

where $p \to 1$ strongly.

3. Estimation of Relative Entropy

Proposition 2.4 suggests that it is useful to study the following general problem. Let $\mathcal{H}$ be a Hilbert space of countable dimension, and P be a projection on $\mathcal{H}$. Let A be a positive bounded operator on $\mathcal{H}$ and $B := PAP + (1-P)A(1-P)$. It is useful to note that if $U := 2P - 1$, then $U^2 = 1$ and $B = \frac{1}{2}(A + UAU)$. In particular $B \geq \frac{1}{2}A$. Let $\tau_A := PA \ln AP + (1-P)A \ln A(1-P) - B \ln B$. Then the problem is to compute/estimate $\mathrm{Tr}(\tau_A)$.

Proposition 3.1. *(1)*

$$\tau_A = \int_0^\infty t\left(P\frac{1}{t+A}P + (1-P)\frac{1}{t+A}(1-P) - \frac{1}{t+B}\right)dt;$$

(2)

$$\tau_{A+\epsilon} \leq \tau_A, \forall \epsilon > 0.$$

Proof. Ad (1): We note that $||t(P\frac{1}{t+A}P + (1-P)\frac{1}{t+A}(1-P) - \frac{1}{t+B})|| \leq 3$, and when t is large

$$\frac{t}{t+A} - \left(1 - \frac{A}{t}\right) = \frac{A^2/t^2}{1 + A/t}.$$

Note that

$$P\left(1 - \frac{A}{t}\right)P + (1-P)\left(1 - \frac{A}{t}\right)(1-P) = \left(1 - \frac{B}{t}\right).$$

Hence

$$t\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)$$
$$=P\frac{A^2/t^2}{1+A/t}P+(1-P)\frac{A^2/t^2}{1+A/t}(1-P)-\frac{B^2/t^2}{1+B/t}$$

and its norm is bounded by constant multiplied by $1/t^2$ when t goes to infinity, hence the improper integral is absolutely convergent in operator norm, and (1) follows by functional calculus. Ad (2): By (1) we have

$$\tau_{A+\epsilon}=\int_0^\infty t\left(P\frac{1}{t+A+\epsilon}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B+\epsilon}\right)dt$$
$$=\int_\epsilon^\infty (t-\epsilon)\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)dt.$$

So

$$\tau_A-\tau_{A+\epsilon}=\int_0^\epsilon t\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)dt$$
$$+\int_\epsilon^\infty \epsilon\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)dt.$$

Since $\frac{1}{x}$ is operator convex (cf. [8]), $P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\geq 0$ and (2) is proved. □

As a consequence of (2) of the above Proposition, we have the following improvement of Theorem 3.12 of [21]:

Proposition 3.2. *Let p be a finite rank projection commuting with P, and A, B as above. Assume that $A-B$ is trace class. Then*

$$\mathrm{Tr}(\tau_A)\geq \mathrm{Tr}\big(\tau_{A_p}\big).$$

Proof. When $A\geq\epsilon>0$ the proposition is Theorem 3.12 of [21]. Now replace A by $A+\epsilon$ and use (2) of Proposition 3.1, we have $\mathrm{Tr}(\sigma_{A+\epsilon})\leq \mathrm{Tr}(\sigma_A)$. On the other hand since $\sigma_{A+\epsilon}$ converges to σ_A strongly, it follows that

$$\lim_{\epsilon\to 0}\mathrm{Tr}(\sigma_{A+\epsilon})\geq \mathrm{Tr}(\sigma_A)$$

and so we have

$$\lim_{\epsilon\to 0}\mathrm{Tr}(\sigma_{A+\epsilon})=\mathrm{Tr}(\sigma_A)$$

and the proposition follows by Theorem 3.12 of [21]. □

As an application of Proposition 3.2, we specialize A and $\mathcal{H}$ as follows. We take $\mathcal{H}=L^2(O,\mathbb{C})$, $O=O_1\cup O_2\in\mathcal{P}O$. P is the projection onto $L^2(O_1,\mathbb{C})$, and A is given by a kernel $K(x,y)=K(x-y)$ which is singular when $x=y$ but smooth when $x\neq y$.

It is instructive to review how $S(\omega,\omega_1\otimes\omega_2)=\mathrm{Tr}(\sigma_C)$ is proved. Choose a sequence of finite rank projections p_n which converges strongly to 1 and commute with P. Then

by the property of relative entropy $S(\omega, \omega_1 \otimes \omega_2) = \lim_n \mathrm{Tr}(\sigma_{C_{p_n}})$. Since $C_{p_n} \geq 0$ converges to C strongly, we have $S(\omega, \omega_1 \otimes \omega_2) \geq \mathrm{Tr}(\sigma_C)$. The reversed inequality would follow from Theorem 3.12 of [21] if we can drop the assumption that A is strictly positive. In [21] we use regularized kernel and explicit form of resolvent in the chiral free fermion case (cf. (1) of Lemma 3.15 in [21]) to prove the reversed inequality. Now with Proposition 3.2 we will always have $S(\omega, \omega_1 \otimes \omega_2) = \mathrm{Tr}(\sigma_C)$ even without knowing the explicit form of the resolvent of C. In particular this identity is also true for free massive fermions, where the corresponding covariance operator C is given by formula 187 in [10].

To motivate the next result, note that our goal is to estimate $A \ln A - B \ln B$ when $A - B$ is trace class. The derivative of $x \ln x$ is singular at $x = 0$, this explains that when A, B has 0 in their spectrum one needs additional conditions. Note that the derivative of $x^2 \ln x$ is bounded when x is close to zero, and when A, B are positive we can consider $A \ln \sqrt{A} - B \ln \sqrt{B}$ with condition that $\sqrt{A} - \sqrt{B}$ being trace class. It is more convenient in applications to replace last condition by $|A - B|^{1/2}$ being trace class, and that is the condition we impose in Theorem 3.4.

Lemma 3.3.

$$||\frac{1}{t+B}A|| \leq \frac{||A||^{1/2}}{t^{1/2}}, \quad \forall t > 0.$$

Proof. For any unit vector $\phi \in \mathcal{H}$ we have

$$||\frac{1}{t+B}A\phi||^2 = \left\langle A\frac{1}{(t+B)^2}A\phi, \phi \right\rangle.$$

Note that $(t+B)^2 = t^2 + 2tB + B^2 \geq t^2 + tA$, and so

$$A\frac{1}{(t+B)^2}A \leq A\frac{1}{t^2+tA}A$$

and

$$\left\langle A\frac{1}{(t+B)^2}A\phi, \phi \right\rangle \leq \frac{1}{t}\left\langle A\frac{1}{t+A}A\phi, \phi \right\rangle \leq ||A||\frac{1}{t}$$

and the Lemma follows. □

Theorem 3.4. *Suppose that $|A - B|^{\frac{1}{2}}$ is trace class, then τ_A is also trace class.*

Proof. By (1) of Proposition 3.1,

$$\tau_A = \int_0^1 t\left(P\frac{1}{t+A}P + (1-P)\frac{1}{t+A}(1-P) - \frac{1}{t+B}\right)dt$$
$$+ \int_1^\infty t\left(P\frac{1}{t+A}P + (1-P)\frac{1}{t+A}(1-P) - \frac{1}{t+B}\right)dt.$$

Note that

$$\int_1^\infty t(P\frac{1}{t+A}P + (1-P)\frac{1}{t+A}(1-P) - \frac{1}{t+B}dt$$
$$= -B\ln(B+1) + PA\ln(A+1)P + (1-P)A\ln(A+1)(1-P).$$

By Lemma 3.11 of [21] $\ln(A+1)-\ln(B+1)$ is trace class, it follows that

$$A\ln(A+1)-B\ln(B+1)=A(\ln(A+1)-\ln(B+1))+(A-B)\ln(B+1)$$

is trace class, and so is $-B\ln(B+1)+PA\ln(A+1)P+(1-P)A\ln(A+1)(1-P)$. Hence, it is sufficient to show that

$$\int_0^1 t\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)dt$$

is trace class. Let $0<\epsilon<1$ be a small number, and denote by

$$D_\epsilon:=\int_\epsilon^1 t\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)dt.$$

We note that $D_\epsilon\geq 0$ is an increasing sequence of positive trace class operators which converge in norm to

$$\int_0^1 t\left(P\frac{1}{t+A}P+(1-P)\frac{1}{t+A}(1-P)-\frac{1}{t+B}\right)dt,$$

it is sufficient to show that $\mathrm{Tr}(D_\epsilon)$ is bounded by a constant independent of ϵ.

By assumption $B-A$ is trace class, we can find an ONB of ψ_i of $\mathcal{H}$ which are the eigenvectors of $B-A$ with eigenvalues λ_i. We have

$$\begin{aligned}\mathrm{Tr}(D_\epsilon)=\int_\epsilon^1 \mathrm{Tr}\left(t\left(\frac{1}{t+A}-\frac{1}{t+B}\right)\right)dt&=\int_\epsilon^1 \mathrm{Tr}\left(t\left(\frac{1}{t+B}\frac{1}{t+A}(B-A)\right)\right)\\&=\sum_i\lambda_i\int_\epsilon^1\left\langle\frac{t}{t+A}\psi_i,\frac{1}{t+B}\psi_i\right\rangle\end{aligned}$$

where the interchange of sum and integral in the third equality follows since the integrand is a continuous function of t in tracial norm. First assume that $\lambda_i>0$. Then from $(t+B)\psi_i=A\psi+(t+\lambda_i)\psi_i$ we have

$$\frac{1}{t+B}\psi_i=\frac{1}{t+\lambda_i}\left(\psi_i-\frac{1}{t+B}A\psi_i\right).$$

Hence

$$\begin{aligned}\left\langle\frac{t}{t+A}\psi_i,\frac{1}{t+\lambda_i}\left(\psi_i-\frac{1}{t+B}A\psi_i\right)\right\rangle&\leq\frac{1}{t+\lambda_i}||\frac{t}{t+A}\psi_i||\left(||\frac{1}{t+B}A||+1\right)\\&\leq\frac{1}{t+\lambda_i}\left(||A||^{\frac{1}{2}}\frac{1}{t^{\frac{1}{2}}}+1\right)\end{aligned}$$

where in the last step we have used Lemma 3.3. So

$$\begin{aligned}\left|\int_\epsilon^1\left\langle\frac{t}{t+A}\psi_i,\frac{1}{t+B}\psi_i\right\rangle\right|dt&\leq\int_\epsilon^1\frac{1}{t+\lambda_i}\left(||A||^{\frac{1}{2}}\frac{1}{t^{\frac{1}{2}}}+1\right)dt\\&\leq\pi||A||^{\frac{1}{2}}\frac{1}{\lambda_i^{\frac{1}{2}}}+\ln(1+\lambda_i)-\ln\lambda_i.\end{aligned}$$

When $\lambda_i < 0$, exchanging the roles of A, B as above we have

$$\frac{1}{t+A}\psi_i = \frac{1}{t-\lambda_i}\left(\psi_i - \frac{1}{t+A}B\psi_i\right).$$

We have

$$\begin{aligned}\left\langle \frac{1}{t+A}B\psi_i, \frac{t}{t+B}\psi_i\right\rangle &= \left\langle \frac{1}{t+A}(A+\lambda_i)\psi_i, \frac{t}{t+B}\psi_i\right\rangle \\ &= \left\langle \frac{1}{t+A}A\psi_i, \frac{t}{t+B}\psi_i\right\rangle + \left\langle \frac{t}{t+A}\psi_i, \frac{1}{t+B}(B-A)\psi_i\right\rangle\end{aligned}$$

and it follows that

$$\left|\left\langle \frac{1}{t+A}B\psi_i, \frac{t}{t+B}\psi_i\right\rangle\right| \le 2 + ||\frac{1}{t+B}A|| \le 2 + ||A||^{\frac{1}{2}}\frac{1}{t^{\frac{1}{2}}}.$$

It follows that

$$\begin{aligned}\left|\int_\epsilon^1 \left\langle \frac{t}{t+A}\psi_i, \frac{1}{t+B}\psi_i\right\rangle\right| dt &\le \int_\epsilon^1 \frac{1}{t-\lambda_i}\left(3 + ||A||^{\frac{1}{2}}\frac{1}{t^{\frac{1}{2}}}\right)dt \\ &\le \pi||A||^{\frac{1}{2}}\frac{1}{\lambda_i^{\frac{1}{2}}} + 3(\ln(1-\lambda_i) - \ln(-\lambda_i)).\end{aligned}$$

Putting these two cases together we have

$$\mathrm{Tr}(D_\epsilon) \le \sum_i |\lambda_i|(\pi||A||^{\frac{1}{2}}|\lambda_i|^{\frac{1}{2}} + 3(\ln(1+|\lambda_i|) - \ln(|\lambda_i|)).$$

By assumption $\sum_i |\lambda_i|^{\frac{1}{2}} < \infty$, it follows that $\mathrm{Tr}(D_\epsilon)$ is bounded by a number independent of ϵ and the proof is complete. □

To apply Theorem 3.4 to the computation of relative entropy in free QFT, we specialize A and $\mathcal{H}$ as follows. We take $\mathcal{H} = L^2(O, \mathbb{C})$, $O = O_1 \cup O_2 \in \mathcal{P}O$. P is the projection onto $L^2(O_1, \mathbb{C})$, and A is given by a kernel $K(x, y) = K(x-y)$ which is singular when $x = y$ but smooth when $x \neq y$.

Lemma 3.5. (1) *Suppose $F_1 = PF(1-P) + (1-P)FP$ where P is a projection, and $|F|^{\frac{1}{2}}$ is trace class, then $|F_1|^{\frac{1}{2}}$ is also trace class, and $|PFP|^{1/2}$ is also trace class.*

Proof. Let $U = 2p - 1$. Then $F_1 = \frac{1}{2}(F - UFU)$. For a compact operator T, we shall denote by $\mu_n(T)$ its n-th largest singular value among all nonzero eigenvalues. By Fan's Theorem (cf. §1 of [27]), we have

$$\mu_{n+m+1}(F - UFU) \le \mu_{n+1}(F) + \mu_{m+1}(UFU) = \mu_{n+1}(F) + \mu_{m+1}(F), \quad \forall n, m \ge 0.$$

Choose $n = m \ge 0$ we have $\mu_{2n+1}(F - UFU) \le 2\mu_{n+1}(F)$, and choose $n = m+1$ with $n \ge 1$ we have

$$\mu_{2n+1}(F - UFU) \le \mu_{n+1}(F) + \mu_n(F).$$

It follows that $(\mu_{2n+1}(F - UFU))^{1/2} \le 2^{1/2}\mu_{n+1}(F)^{1/2}$, $(\mu_{2n}(F - UFU))^{1/2} \le \mu_{n+1}(F)^{1/2} + \mu_n(F)^{1/2}$, and so $\mathrm{Tr}(|F_1|^{1/2}) = \sum_n \mu_n(F_1)^{1/2} \le (\sqrt{2}+1)\mathrm{Tr}(|F|^{1/2})$. The same argument with UFU replaced by $-UFU$ shows that $|F + UFU|^{1/2}$ is trace class. Note that $|PFP| \le \frac{1}{2}|F + UFU|$, the second statement in the Lemma also follows. □

Proposition 3.6. *Suppose $\mathcal{H} = L^2(O, \mathbb{C})$, $O = O_1 \cup O_2 \in \mathcal{P}O$, P is the projection onto $L^2(O_1, \mathbb{C})$, and A is given by a kernel $K(x, y) = K(x - y)$ which is singular when $x = y$ but smooth when $x \neq y$. Then $|A - B|^{1/2}$ is trace class.*

Proof. By assumption $A - B$ is given by a kernel $K(x, y) = K(x - y)$ which is a smooth function for $x \in O_1$, $y \in O_2$. Choose a large cube CU of length L centered at origin whose interior contains the closure of the union of O_1, O_2, $O_1 - O_2$, $O_2 - O_1$, and let $G(x_1, \ldots, x_d)$ be a smooth function on CU such that $G(x - y) = K(x - y)$ whenever $x \in O_1$, $y \in O_2$, $\tilde{G}(x) = G(-x)$ and G is periodic in each of its variables with period L. The operator T on $L^2(CU, \mathbb{C})$ given by kernel $G(x - y)$ can be diagonalized by Fourier transformation, and its eigenvalues as functions of $(n_1, \ldots, n_d)$ where n_i are integers go to zero faster than the inverse of any polynomial in $n_1, \ldots, n_d$. It follows that $|T|^{1/2}$ is trace class. $A - B$ is $PF(1 - P) + (1 - P)FP$, where F is the restriction of T to subspace $L^2(O_1 \cup O_2, \mathbb{C})$. By Lemma 3.5 the Proposition is proved. □

The split property for general Dirac Quantum Fields in curved space time is proved in [12]. Since the computation of relative entropy for general Dirac Quantum Fields in Minkowski spacetime reduces to finite linear combinations of traces of τ_A where A is as in Proposition 3.6, combined with Theorem 3.4 we have proved the following:

Corollary 3.7. *The mutual information for general Dirac Quantum Fields in Minkowski spacetime of any dimension is finite.*

Remark 3.8. For free boson case there is a formula (cf. equation (63) of [10]) for mutual information, but the corresponding operator C there is unbounded and does not seem to have a good kernel representations. In the case of chiral massless free bosons there has been recent computation of mutual information in the case of two intervals with diagolization of a non-symmetric operator (cf. §3 of [4]). We note that in the later case the mutual information is finite since it is embedded into free fermions. It is an interesting question to see if one can have a similar treatment of free bosons in general cases as in this section.

4. Chiral CFT

We shall refer the reader to §2 of [21] for the definition of conformal net and its properties.

A positive lattice L of rank n is the $\mathbb{Z}$ span of a basis $\alpha_1, \ldots, \alpha_n$ in a vector space with a positive definite inner product $\langle , \rangle$ such that $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}, \forall 1 \leq i, j \leq n$. L is called even if $\langle \alpha_i, \alpha_i \rangle \in 2\mathbb{Z}, \forall 1 \leq i \leq n$. To each even positive lattice L one can associate a rational conformal net $\mathcal{A}_L$ (cf. [13]). The free fermion net $\mathcal{A}_r$ can be considered as conformal net associated with $\mathbb{Z}^r$ with its usual Euclidean inner product. $\mathcal{A}_r$ is not local, but graded local since $\mathbb{Z}^r$ is not even.

Lemma 4.1. *Let L be a positive lattice with a basis $\alpha_1, \ldots, \alpha_n$, and for k a positive integer, let kL be the $\mathbb{Z}$ span of a basis $k\alpha_1, \ldots, k\alpha_n$. Then $\mathcal{A}_{kL} \subset \mathcal{A}_L$ has index n^k.*

Proof. By [13] the vacuum representation of $\mathcal{A}_L$ decomposes into finitely many irreducible representations of $\mathcal{A}_{kL}$, which are in one to one correspondence with abelian group of L/kL, and all such representations have index 1. Note L/kL is isomorphic to direct product of n copies of $\mathbb{Z}/k\mathbb{Z}$, the Lemma follows. □

Proposition 4.2. *Let L positive lattice L of rank n with a basis $\alpha_1, \ldots, \alpha_n$. Then:*

(1) *There exists a $\mathbb{Z}$ linear injective map $\phi : L \to \mathbb{Q}^r$ for some positive integer r such that ϕ preserves inner product ;*

(2) *There exists a positive integer k such that the image of kL under ϕ lies in $\mathbb{Z}^r$.*

Proof. Ad (1) It is equivalent to show that for some positive integer r there exist vectors $A_i = (A_{1i}, \dots, A_{ri}) \in \mathbb{Q}^r$ such that

$$\sum_{1 \le k \le r} A_{ki} A_{kj} = \langle \alpha_i, \alpha_j \rangle, \forall 1 \le i, j \le n.$$

We prove this by induction on n. When $n = 1$, one can take $A_1 = (1, \dots, 1)$ with $r = \langle \alpha_1, \alpha_1 \rangle$. Assume that the Proposition is true for $n-1$, i.e., for some positive integer r there exist vectors $A_i = (A_{1i}, \dots, A_{ri}) \in \mathbb{Q}^r$ such that

$$\sum_{1 \le k \le r} A_{ki} A_{kj} = \langle \alpha_i, \alpha_j \rangle, \forall 1 \le i, j \le (n-1).$$

First, we choose a vector $\tilde{A_n}$ in the linear span of $A_1, \dots, A_{n-1}$ such that $\langle \tilde{A_n}, A_i \rangle = \langle \alpha_n, \alpha_i \rangle, \forall 1 \le i \le n-1$. Suppose that $\tilde{A_n} = \sum_{1 \le i \le n-1} x_i A_i$, then we have a system of linear equations

$$\sum_{1 \le j \le n-1} x_j \langle \alpha_j, \alpha_i \rangle = \langle \alpha_n, \alpha_i \rangle, \quad 1 \le i \le n-1.$$

Since $\langle \alpha_j, \alpha_i \rangle$ are integers, it follows that $x_i \in \mathbb{Q}$. Moreover, we note that $\tilde{\alpha_n} := \sum_{1 \le i \le n-1} x_i \alpha_i$ is exactly the projection of α_n onto the linear space spanned by α_i, $1 \le i \le n-1$, and it follows that

$$\langle \tilde{\alpha}_n, \tilde{\alpha}_n \rangle = \sum_{1 \le k \le r} \tilde{A}_{kn} \tilde{A}_{kn} \in \mathbb{Q}.$$

Since $\langle \tilde{\alpha}_n, \tilde{\alpha}_n \rangle < \langle \alpha_n, \alpha_n \rangle$, we have

$$\langle \alpha_n, \alpha_n \rangle - \sum_{1 \le k \le r} \tilde{A}_{kn} \tilde{A}_{kn} = \frac{p}{q}$$

with both p, q positive integers. Let $s = r + pq$ and A_n be a vector in $\mathbb{Q}^s$ whose first r entries are that of $\tilde{A_n}$, and the last pq entries are all $\frac{1}{q}$, and we embed A_i into $\mathbb{Q}^s$ by simply adding last pq zeros to the components of A_i, $1 \le i \le n-1$, and we have proved the Proposition for n. By induction the proof is complete.

Ad (2): The image of each $\phi(\alpha_i)$ has components in $\mathbb{Q}$, we just have to choose an integer k such that k multiply each of these components are in $\mathbb{Z}$. □

By Proposition 4.2 we immediately have:

Corollary 4.3. *Let L be an even positive lattice and $\mathcal{A}_L$ the associated conformal net. Then there exist positive integers k, r such that $\mathcal{A}_{kL}$ is a subnet of free r fermion net $\mathcal{A}_r$.*

By Corollary 4.3 and Lemma 4.1 we have proved the following:

Corollary 4.4. *Let L be an even positive lattice and $\mathcal{A}_L$ the associated conformal net. The mutual information for $\mathcal{A}_L$ is finite.*

If G is a simply connected simply laced compact Lie group, it is known (cf. [26]) that $\mathcal{A}_{G_1}$ is conformal net associated with a lattice. When G is not simply laced, G

is of type B, C, F_4, G_2. Note that $SO(2n+1) \subset SO(2n+2)$, $G_2 \subset F_4 \subset E_6$, and $Sp(n) \subset SO(4n)$. So G_1 can always be embedded into H for some simply laced H.

Hence, the mutual information for $\mathcal{A}_{G_1}$ is finite by Corollary 4.4.

Since $\mathcal{A}_{G_k}$ is a subnet of k tensor product of $\mathcal{A}_{G_1}$, it follows that the mutual information for $\mathcal{A}_{G_k}$ is finite by Corollary 4.4. It is also clear that the same is true for all conformal nets that can be obtained from cosets, orbifolds, simple current extensions or combinations of these constructions starting with $\mathcal{A}_{G_k}$ or $\mathcal{A}_L$, and of course any subnets of such conformal nets.

4.1. A relation between relative entropy and index. Fix an interval $I_1 = (a, b)$ on a circle. Suppose ρ is an irreducible representation of a conformal net $\mathcal{A}$ localized on I_1 with finite index given by λ^{-1}. Let $J_n \subset I_1'$, $n \geq 1$ be an increasing sequence of intervals such that

$$\bigcup_n J_n = I_1', \quad \bar{J}_n \cap \bar{I}_1 = a.$$

Let E_n be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$ to $\rho(\mathcal{A}(I_1)) \vee \mathcal{A}(J_n)$. Clearly E_{n+1} is an extension of E_n.

We note that by Pimsner-Popa inequality we have $E_n(a) \geq \lambda a, \forall$ positive $a \in \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$. Denote by $\phi_n = \omega E_n$.

Theorem 4.5.

$$\lim_{n\to\infty} S(\omega, \phi_n) = \ln(\lambda^{-1})$$

where λ^{-1} is the index of representation ρ.

Proof. By Section 3.6.2 of [21] (We note that there is typo in the formula in Th. 3.24 of [21], there should be a natural log on the righthand side) it is sufficient to prove the following as in Proposition 3.25 of [21]: Given any $\epsilon > 0$, we need to find $e \in \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$, such that

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, \ |\omega(e^*e) - 1| < \epsilon, \ |\phi_n(ee^*) - \lambda| < \epsilon.$$

Let $e_1 \in \mathcal{A}(I_1)$ be the Jones projection for $\rho(\mathcal{A}(I_1)) \subset \mathcal{A}(I_1)$, and $v \in \rho(\mathcal{A}(I_1)$ be the isometry such that $\lambda v^* e_1 v = 1$. Since ρ is irreducible, $\rho(\mathcal{A}(I_1)) \vee \mathcal{A}(I_1') = \mathcal{B}(\mathcal{H})$. And since $\vee_n \mathcal{A}(J_n) = \mathcal{A}(I_1')$, we can find a sequence of elements $e_n \in \rho(\mathcal{A}(I_1)) \vee \mathcal{A}(J_n), n \geq 2$ which converges in strong star topology to e_1. Now choose $x_n = \lambda^{-1} v^* e_1 e_n v$. Then $x_n \to 1$ in strong star topology, and so $\omega(x_n)$, $\omega(x_n^* x_n)$ converges to 1. On the other hand by definition

$$E_n(x_n^* x_n) = v^* e_n^* e_n v$$

converges to $v^* e_1 v = \lambda^{-1}$ strongly. Hence given any $\epsilon > 0$, we can choose n sufficiently large such that if we set $e = x_n$, then $e \in \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$, and

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, \ |\omega(e^*e) - 1| < \epsilon, \ |\phi_n(ee^*) - \lambda| < \epsilon.$$

□

One way of interpreting Theorem 4.5 is the following: Let $I \subset I_1 \cup \bar{J}_n$ and denote by $\omega_I, \phi_{n,I}$ the restriction of ω, ϕ_n to $\mathcal{A}(I)$ respectively. When $I \subset I_1 \cup \bar{J}_n$ is disjoint from I_1, by definition $\omega = \phi_n$ and $S(\omega_I, \phi_{n,I}) = 0$. Then as I increases, $S(\omega_I, \phi_{n,I})$ will increase. When $I = I_1 \cup \bar{J}_n$ increases so that $\vee_n J_n = I_1'$, Theorem 4.5 states that the limiting value is $\ln(\lambda^{-1})$. This picture has some similarity (but not the same) to the result in [15].

5. Outlook

There are a number of questions which come naturally from our work. Does split property imply finiteness of mutual information? Or less strongly does nuclearity (cf. [5,22]) imply finiteness of mutual information? Having established finiteness of mutual information in Corollary 3.7 the next step will be to understand the singularity structure when the distance between regions goes to zero as in §4.2 of [21], and determine the regularized entropy. These may be related to the heuristic computations in §3 of [10], and may provide rigorous definition of C function (cf. [11]) starting with cut off independent relative entropy as in §4.2 of [21] for CFT. Finally, it will also be interesting to determine the asymptotics of the mutual information when the distance between regions goes to infinity and compare with the heuristic computations in [9]. We plan to address some of these questions in the future.

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