CHARACTERIZING THE CONTINUOUS DEGREES

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Abstract. The continuous degrees measure the computability-theoretic content of elements of computable metric spaces. They properly extend the Turing degrees and naturally embed into the enumeration degrees. Although nontotal (i.e., non-Turing) continuous degrees exist, they are all very close to total: joining a continuous degree with a total degree that is not below it always results in a total degree. We call this property *almost totality*.

We prove that the almost total degrees coincide with the continuous degrees. Since the total degrees are definable in the partial order of enumeration degrees [1], we see that the continuous degrees are also definable. Applying earlier work on the continuous degrees [10], this shows that the relation "PA above" on the total degrees is definable in the enumeration degrees.

In order to prove that every almost total degree is continuous, we pass through another characterization of the continuous degrees that slightly simplifies one of Kihara and Pauly [7]. We prove that the enumeration degree of A is continuous if and only if A is *codable*, meaning that A is enumeration above the complement of an infinite tree, every path of which enumerates A.

1. Introduction

The continuous degrees were introduced by Miller [10] as a natural measure of the computability-theoretic content of elements of computable metric spaces. A *computable metric space* $\mathcal{M}$ is a metric space with a distinguished countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega} \subseteq \mathcal{M}$ on which the metric is computable, meaning that there is an effective way to approximate the distance between any two elements in the sequence with arbitrary precision. Common separable metric spaces can usually be given a computable structure. As a simple example, the real line $\mathbb{R}$ with the usual metric and $Q^{\mathbb{R}} = \mathbb{Q}$ is a computable metric space. For $\mathcal{C}[0,1]$, the space of continuous functions on the unit interval under the uniform metric, we can take $Q^{\mathcal{C}[0,1]}$ to consist of all rational polynomials on the unit interval.

If $\mathcal{M}$ is a computable metric space, a *name* for a point $x \in \mathcal{M}$ is a function that takes as input a rational number ε and outputs the index of a member of the sequence $Q^{\mathcal{M}}$ that is within distance ε of x. Note that names are discrete objects; they fall within the scope of classical computability theory. By naming points in computable metric spaces, we can compare the computability-theoretic content of points from different spaces: a point x in one computable metric space has at least as much computability-theoretic content as a point y in some (possibly different)

Date: January 17, 2019.

2010 *Mathematics Subject Classification.* Primary 03D30; Secondary 03D78, 54E35.

The first author was partially supported by NSF grant DMS1600228. The third author was partially supported by grant #358043 from the Simons Foundation. The fourth author was partially supported by National Science Fund of Bulgaria grant #01/18 from 23.07.2017 and by National Science Foundation grant DMS1762648.

computable metric space if every name for x computes a name for y. Miller [10] observed that every point in a computable metric space is equivalent to a point in $\mathcal{C}[0,1]$. For this reason, he called the induced degree structure the *continuous degrees*. We write $\mathcal{D}_r$ for the structure of the continuous degrees.

The Turing degrees embed into the continuous degrees; they are the continuous degrees of points in $\mathbb{R}$, or more naturally, Cantor space with an appropriate metric and computable structure. On the other hand, the continuous degrees embed into the *enumeration degrees*. An enumeration reduction determines the positive information about one set from the positive information about another. It is closely connected to the idea of computing with partial oracles. Equivalent forms of this reducibility have been introduced several times over the years: Kleene [8] and Myhill [11] in the partial oracles version, Friedberg and Rogers [3] and Selman [13] in the version we discuss here. For sets of natural numbers A and B, we say that A is enumeration reducible to B ($A \leqslant_e B$) if every function that enumerates A (uniformly) computes an enumeration of B. The enumeration degrees $\mathcal{D}_e$ arise from enumeration reducibility in the standard way. Thus, we have three structures: $\mathcal{D}_T \hookrightarrow \mathcal{D}_r \hookrightarrow \mathcal{D}_e$. The copy of $\mathcal{D}_T$ in the enumeration degrees is the set of *total* enumeration degrees and the copy of $\mathcal{D}_r$ in the enumeration degrees is the set of *continuous* enumeration degrees. Miller [10] proved that each of the embeddings is proper: there are nontotal continuous degrees and there are noncontinuous enumeration degrees.

The continuous degrees are, however, in some sense very close to total enumeration degrees.

Definition 1.1. We say that an enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \not\leqslant \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

In words, an enumeration degree is almost total if adding any new total information takes it to a total degree. This is true of total degrees because the join of any two total degrees is total; it is much harder to see that there is a nontotal enumeration degree with this property. In 2014, Cai, Lempp, Miller, and Soskova (unpublished) observed that continuous enumeration degrees are almost total (see Section 3 for a proof). Together with the fact that there are nontotal continuous degrees, which is itself nontrivial [10], this proves that there are nontotal almost total enumeration degrees. Surprisingly, we show that this is the only possible source of examples: an enumeration degree is almost total if and only if it is continuous.

Remark 1.2. It is worth saying a few words about the known proofs that nontotal continuous degrees exist. Miller's [10] proof relies on a generalization of Brouwer's fixed point theorem for multivalued functions on $[0,1]^\omega$, the Hilbert cube. Day and Miller [2] noted that Levin's [9] *neutral measures* (informally, measures relative to which every sequence is Martin-Löf random) must have nontotal continuous degree. Levin constructed a neutral measure using Sperner's lemma, which is a combinatorial version of Brouwer's fixed point theorem (see Gács [4] for the construction). Finally, Kihara and Pauly [7] and, independently, Mathieu Hoyrup (unpublished) realized that the existence of nontotal continuous degrees follows from the fact that the Hilbert cube $[0,1]^\omega$ is strongly infinite dimensional, hence not a countable union of zero dimensional subspaces. It is unlikely to be a coincidence that every known proof that nontotal continuous degrees exist—hence, that nontotal almost

total degrees exist—has a nontrivial topological component. However, we do not know how to formalize this observation into a conjecture.

In order to prove that every almost total degree is continuous, we establish a couple of other characterizations of the continuous degrees. In Section 4, we introduce *codable sets*. A set A is *codable* if it is enumeration above the complement of an infinite tree, every path of which computes an enumeration of A. Kihara and Pauly [7] introduced a very similar, but slightly more complicated notion and showed that it is equivalent to having continuous degree. We prove that codability is equivalent to its uniform version, and that if A has almost total enumeration degree, then it is uniformly codable. In Section 5, we define *holistic sets*, which are subsets of $\omega^{<\omega}$ with special combinatorial properties. We prove that the enumeration degrees of holistic sets and the enumeration degrees of uniformly codable sets coincide. In Section 6, we introduce a topology on the set of holistic sets, giving rise to the *holistic space*, a computable second countable Hausdorff space. Schröder [12] proved an effective version of Urysohn's metrization theorem.[1] We show that the holistic space is *computably regular*, which allows us to apply Schröder's theorem and conclude that the holistic space is a computable metric space. This gives us the final link in our chain of characterizations: we prove that holistic sets have continuous enumeration degree.

In Section 3, we also consider a uniform version of almost totality and give a short proof that uniformly almost total enumeration degrees are continuous. It should be noted that this uniform version could be avoided in the proof that almost total degrees are continuous, as could the nonuniform version of codability. We include both for completeness. Summarizing all of our characterizations:

Theorem 1.3. *Let* $\mathbf{a}$ *be an enumeration degree. The following are equivalent:*

(1) $\mathbf{a}$ *is (uniformly) almost total,*
(2) *The sets in* $\mathbf{a}$ *are (uniformly) codable,*
(3) $\mathbf{a}$ *contains a holistic set,*
(4) $\mathbf{a}$ *is continuous.*

The equivalence of the first and last statements has an important consequence for the structure of the enumeration degrees. In recent years, there has been a sequence of advances in understanding which relations are first order definable in $\mathcal{D}_e$ (as a partial order). There is a natural jump operator in the enumeration degrees that agrees with the Turing jump operator under the embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. Kalimullin [6] proved that the enumeration jump operator is first order definable in the enumeration degrees by a very simple structural property. Building on this work, Cai, Ganchev, Lempp, Miller, and Soskova [1] proved that the total enumeration degrees are first order definable. This, combined with the characterization of the continuous enumeration degrees as almost total, gives us a new first order definable class of enumeration degrees.

Theorem 1.4. *The property "*$\mathbf{a}$ *is a continuous enumeration degree" is first order definable in* $\mathcal{D}_e$*.*

Miller [10] studied the structural relationship between total and nontotal continuous degrees. He proved that nontotal continuous degrees can be used to characterize the relation "PA above" between (pre-images of) total enumeration degrees.

[1] See [5] for an outline of the proof of Schröder's theorem.

For Turing degrees $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{y}$ is *PA above* $\mathbf{x}$ ($\mathbf{x} \ll \mathbf{y}$) if every infinite tree computable in $\mathbf{x}$ has a path computable in $\mathbf{y}$. We transfer the relation "PA above" to total enumeration degrees in a natural way: the enumeration degree $\mathbf{b}$ is *PA above* the enumeration degree $\mathbf{a}$ ($\mathbf{a} \ll \mathbf{b}$) if $\mathbf{a}$ and $\mathbf{b}$ are the images of Turing degrees $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \ll \mathbf{y}$. Miller showed that $\mathbf{a} \ll \mathbf{b}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are total and there is a nontotal continuous enumeration degree $\mathbf{c}$, such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$. Thus, we have another definable relation in the enumeration degrees.

Theorem 1.5. *The relation "$\mathbf{a} \ll \mathbf{b}$" is first order definable in $\mathcal{D}_e$.*

This nicely complements a result by Cai et. al [1] that shows that if we transfer the relation "c.e. in" from pairs of Turing degrees to pairs of total enumeration degrees, then once again we obtain a first order definable relation. Definable relations are natural obstacles to nontrivial automorphisms. The question about the existence of a nontrivial automorphism in the Turing degrees has remained impenetrable, in part due to the lack of many examples of definable relations. Neither "c.e. in", nor "PA above" are known to be first order definable relations in the Turing degrees. The total enumeration degrees are a definable automorphism base for the enumeration degrees. This means that a nontrivial automorphism of $\mathcal{D}_e$ would induce a nontrivial automorphism of $\mathcal{D}_T$. We now know that this automorphism must preserve the jump, the relation "c.e. in", and the relation "PA above". Could such an automorphism be nontrivial?

2. Preliminaries

Enumeration degrees. Enumeration reducibility captures the notion of relative enumerability between sets of natural numbers. We present it in the form first given by Friedberg and Rogers [3]. Let $\{D_v\}_{v\in\omega}$ be a computable listing of all finite sets.

Definition 2.1. A set $A \subseteq \omega$ is *enumeration reducible* to a set $B \subseteq \omega$ ($A \leqslant_e B$) if there is a c.e. set W such that

$$A = \{x \colon (\exists v)\, \langle x, v\rangle \in W \text{ and } D_v \subseteq B\}.$$

A c.e. set W can, in this sense, be treated as an operator mapping sets of natural numbers to sets of natural numbers. We write $A = W(B)$ and call W an *enumeration operator*. The notation $A = W^B$ is reserved to mean that A is c.e. in B via the *c.e. operator* W. Enumeration reducibility and the relation "c.e. in" are closely connected. In particular, the relation "c.e. in" can be expressed using enumeration reducibility.

Proposition 2.2. *Let A and B be sets of natural numbers.*

1. *A is c.e. in B if and only if $A \leqslant_e B \oplus \overline{B}$.*
2. *$A \leqslant_T B$ if and only if $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.*

In both cases, there is a uniform way to compute an index for the operator witnessing one relation from an index witnessing the other.

On the other hand, Selman [13] proved that enumeration reducibility can be captured in terms of the relation "c.e. in".

Theorem 2.3 (Selman 1971)**.** *If A and B be sets of natural numbers, then $A \leqslant_e B$ if and only if $(\forall X)[B$ is c.e. in $X \Rightarrow A$ is c.e. in $X]$.*

We associate a degree structure to enumeration reducibility in the standard way: A is *enumeration equivalent* to B ($A \equiv_e B$) if $A \leqslant_e B$ and $B \leqslant_e A$. The *enumeration degree* of A is the equivalence class of A under enumeration equivalence; we write it as $d_e(A)$. The enumeration degrees inherit an order from the reduction: we let $d_e(A) \leqslant d_e(B)$ if and only if $A \leqslant_e B$. The disjoint union of two sets $A \oplus B$ gives rise to a least upper bound operation $d_e(A) \vee d_e(B) = d_e(A \oplus B)$. The resulting upper semilattice is the structure of the enumeration degrees $\mathcal{D}_e$.

Part (2) of Proposition 2.2 implies that the map taking $d_T(A)$ to $d_e(A \oplus \overline{A})$, for all $A \subseteq \omega$, is an embedding of the Turing degrees into the enumeration degrees. Its range is an important class of degrees.

Definition 2.4. A set $A \subseteq \omega$ is *total* if $\overline{A} \leqslant_e A$. An enumeration degree is *total* if it contains a total set.

Examples of total sets include graphs of total functions, sets of the form $A \oplus \overline{A}$, and for every $X \in 2^\omega$ the set $(X)_\leq = \{\sigma \in 2^{<\omega} \colon \sigma \preceq X\}$. It is easy to see that an enumeration degree is total if and only if it contains a set of the form $A \oplus \overline{A}$, so the total degrees are an embedded copy of the Turing degrees.

Selman's theorem tells us that we can view the enumeration degree of a set A as the collection of Turing degrees that can enumerate A. It turns out that an enumeration degree is total if and only if this set of degrees has a least element. If A is total, then every Turing degree that enumerates A also enumerates $\overline{A}$, hence computes A. So $d_T(A)$ is the least Turing degree that enumerates A. For the other direction, if there is a least Turing degree $\mathbf{x}$ that enumerates A, then fix $X \in \mathbf{x}$. We know that A is c.e. in X, and so $A \leqslant_e X \oplus \overline{X}$. On the other hand, every Turing degree that enumerates A computes X, and so enumerates $X \oplus \overline{X}$. By Selman's theorem, $X \oplus \overline{X} \leqslant_e A$, hence A is of total degree.

Continuous degrees. The continuous degrees were introduced by Miller [10]. As we noted in the introduction, a *computable metric space*[2] is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n \in \omega} \subseteq \mathcal{M}$ on which the metric is computable, meaning that there is a computable function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $n, m, \varepsilon \in \mathbb{N} \times \mathbb{N} \times \mathbb{Q}^+$, $|d_{\mathcal{M}}(q_n^{\mathcal{M}}, q_m^{\mathcal{M}}) - f(n, m, \varepsilon)| < \varepsilon$.

Definition 2.5. A *name* for a point x in a computable metric space $\mathcal{M}$ is a function $\lambda : \mathbb{Q}^+ \to \mathbb{N}$, such that $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$ for every $\varepsilon \in \mathbb{Q}^+$.

If x is a point in a computable metric space $\mathcal{M}$ and y is a point in a computable metric space $\mathcal{N}$, then x is *representation reducible to* y ($x \leqslant_r y$) if every name for y computes a name for x.

Representation reducibility induces a degree structure in the same way that enumeration reducibility does. We call this structure the *continuous degrees* $\mathcal{D}_r$. The continuous degree of a point x in a computable metric space can, in this case as well, be view as a set of Turing degrees: the Turing degrees of names for x.

Once again, the continuous degrees that correspond to Turing cones form an isomorphic copy of $\mathcal{D}_T$ in $\mathcal{D}_r$. Recall our first example of a computable metric

[2]In [10], Miller includes the nonstandard requirement that $\mathcal{M}$ is a complete metric space. Since every metric space has a unique completion, this does not limit the collection of continuous degrees. On the other hand, by requiring completeness, the computable structure determines the underlying metric space, so there are only countably many computable metric spaces. Under the standard definition, which we use in this paper, there are $2^{2^{\aleph_0}}$ computable metric spaces.

space: $\mathbb{R}$ with $Q^{\mathbb{R}} = \mathbb{Q}$. We map the Turing degree of a set A to the continuous degree of real $r_A \in [0,1]$ whose binary expansion is given by A. A real number r can be identified with its *left Dedekind cut* $L_r = \{q \in \mathbb{Q} \colon q < r\}$. Any name for r computes L_r and the set L_r computes a name for r, so the continuous degree of r corresponds to Turing cone above $d_T(L_r)$. Of course, for every set A we have $L_{r_A} \equiv_T A$, so the mapping we defined is, in fact, an embedding. We could have used the *right Dedekind cut* $R_r = \{q \in \mathbb{Q} \colon q > r\}$ instead of the left. The sets L_r and R_r are Turing equivalent. In fact, if r is not a rational then $L_r = \overline{R}_r$ and if r is rational then $L_r = \overline{R}_r \setminus \{r\}$.

Another example of a computable metric space is the Hilbert cube $[0,1]^\omega$ under the metric given by $d(\alpha,\beta) = \sum_{n\in\omega} |\alpha(n)-\beta(n)|/2^n$. In this case, we can take $Q^{[0,1]^\omega}$ to be a listing of the set of finitely nonzero sequences of rationals in the interval $[0,1]$. Miller [10] proved that every continuous degree contains an element of Hilbert cube. This allowed him to embed the continuous degrees into the enumeration degrees. Let $\alpha \in [0,1]^\omega$ and let $C_\alpha = \bigoplus_{n\in\omega} \left[L_{\alpha(n)} \oplus R_{\alpha(n)}\right]$. Every name for α can enumerate the set C_α, and conversely, any set that can enumerate C_α can compute a name for α. Therefore, by Selman's theorem (Theorem 2.3), the mapping that sends the continuous degree of α to the enumeration degree of C_α is an embedding of $\mathcal{D}_r$ into $\mathcal{D}_e$.

Definition 2.6. An enumeration degree $\mathbf{a}$ is *continuous* if it contains the set C_α for some sequence $\alpha \in [0,1]^\omega$.

We now have two ways to embed the Turing degrees into the enumeration degrees: directly, mapping $d_T(A)$ to $d_e(A \oplus \overline{A})$, or via the continuous degrees. It is straightforward to see that, for every set A, the continuous degree of r_A is mapped to the enumeration degree of $L_{r_A} \oplus R_{r_A} \equiv_e A \oplus \overline{A}$. Thus the two methods of embedding the Turing degrees produce the same result.

3. Almost totality

As we have defined above, an enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \not\leqslant \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total. In this section, we show that continuous degrees are almost total. In fact, we will see that the almost totality of continuous degrees is witnessed uniformly. It turns out to be easy to show that this uniform version of almost totality is equivalent to being continuous. We provide this proof, even though we are primarily interested in the nonuniform version.

Definition 3.1. A set $A \subseteq \omega$ is *uniformly almost total* if there is an enumeration operator Γ and a c.e. operator W such that

$$(\forall X \subseteq \omega)\left[X \oplus \overline{X} \not\leqslant_e A \Rightarrow (\exists Y \subseteq \omega)\left[\Gamma(A \oplus X \oplus \overline{X}) = Y \oplus \overline{Y} \text{ and } W^Y = A\right]\right].$$

Note first that uniform almost totality is an enumeration degree notion. It is also not hard to see that it implies almost totality: we could require, without strengthening the property, that Γ preserve $X \oplus \overline{X}$ as part of its output, so that $A \oplus X \oplus \overline{X} \equiv_e Y \oplus \overline{Y}$.

Lemma 3.2. *Continuous enumeration degrees are uniformly almost total.*

Proof. Fix $\mathbf{a}$ and let α be a sequence of real numbers such that $C_\alpha \in \mathbf{a}$. Let X be the binary expansion of a real number $r_X \in [0,1]$. Consider the sequence β, defined by $\beta(n) = (\alpha(n) + r_X)/2$. Addition on real numbers is a computable operation, as

is division by 2, so from any name for α and any name for r_X, we can compute a name for β. To be explicit: if $q_1 < \alpha(n)$ and $q_2 < r_X$, then $(q_1 + q_2)/2 < \beta(n)$, and if $q_1 > \alpha(n)$ and $q_2 > r_X$ then $(q_1 + q_2)/2 > \beta(n)$. So from an enumeration of C_α and $L_{r_X} \oplus R_{r_X}$, we can uniformly compute an enumeration of C_β. It follows that $C_\beta \leqslant_e C_\alpha \oplus X \oplus \overline{X}$. The choice of X did not matter to our enumeration procedure, so there is an enumeration operator Γ that works for all X.

If $\beta(n)$ is a rational number q_X, then $r_X \leqslant_r \alpha(n)$. To see this, note that if $d(\alpha, \{q_n\}_{n\in\omega}) < \varepsilon/2^n$, then $d(\alpha(n), q_n) = |\alpha(n) - q_n| < \varepsilon$, and so $|r_X - (2q_X - q_n)| = |(2q_X - \alpha(n)) - (2q_X - q_n)| < \varepsilon$. Thus we can compute a name for r_X from a name for α. So if $X \oplus \overline{X} \not\leqslant_e C_\alpha$, then C_β is a total set: for every n, the real $\beta(n)$ is not rational and hence $R_{\beta(n)} = \overline{L_{\beta(n)}}$.

To complete the proof, we modify the operator Γ, as described above, so that $\Gamma(C_\alpha \oplus X \oplus \overline{X}) = Y \oplus \overline{Y}$, where $Y \oplus \overline{Y}$ is obtained by rearranging the set $C_\beta \oplus (X \oplus \overline{X})$. This rearrangement is, of course, computable and uniform. Finally, we use the fact $\alpha = 2\beta - r_X$ to obtain a c.e. operator W such that $C_\alpha = W^Y$. This shows that C_α, and hence $\mathbf{a}$, is uniformly almost total. □

Proposition 3.3. *A set $A \subseteq \omega$ is uniformly almost total if and only if it has continuous degree.*

Proof. One direction was proved above. So now assume that $A \subseteq \omega$ is uniformly almost total as witnessed by Γ and W. Furthermore, assume without loss of generality that A is nonempty. Note that $\mu\{X \in 2^\omega : X \oplus \overline{X} \leqslant_e A\} = 0$, where μ is Lebesgue measure. Thus, for almost all X, we have $W^Y = A$, where $\Gamma(A \oplus X \oplus \overline{X}) = Y \oplus \overline{Y}$.

The power of uniform almost totality is that, by combining Γ and W, we can take a total set $X \oplus \overline{X} \not\leqslant_e A$ and any enumeration of A to a fixed enumeration of A that only depends on X. To make this more explicit, fix a Turing functional Φ such that $\Phi^Y : \omega \to \omega$ has range W^Y and is total as long as $W^Y \neq \varnothing$. We define a sequence of reals $\alpha \in [0, 1]^\omega$ as follows. Let

$$\alpha(\langle k, n\rangle) = \mu\{X \in 2^\omega : \Phi^Y(k) = n, \text{ where } \Gamma(A \oplus X \oplus \overline{X}) = Y \oplus \overline{Y}\}.$$

Note that we can uniformly compute α from any enumeration of A: to approximate $\alpha(\langle k, n\rangle)$ to within ε, it is enough to wait for a stage when we see

$$\mu\{X \in 2^\omega : \Phi^Y(k) \downarrow, \text{ where } \Gamma(A \oplus X \oplus \overline{X}) = Y \oplus \overline{Y}\} > 1 - \varepsilon.$$

Finally, we claim that it is easy to enumerate A from (a name for) α. Note that if $n \in A$, then $\sum_{k\in\omega} \alpha(\langle k, n\rangle) \geqslant 1$. On the other hand, if $n \notin A$, then $\sum_{k\in\omega} \alpha(\langle k, n\rangle) = 0$. So $n \in A$ if and only if there is a $k \in \omega$ such that $\alpha(\langle k, n\rangle) > 0$, which proves the claim. Therefore, A has continuous degree; it has the same degree as α. □

In what follows, we will work considerably harder to show that every (not necessarily uniformly) almost total degree is continuous. We do not have a direct proof that uniform almost totality and almost totality are equivalent. One reason for our specific interest in almost totality, as opposed to its uniform version, was mentioned in the introduction: it provides a definition of the continuous degrees in the partial order of enumeration degrees. This is immediate from the fact that the total degrees are definable [1].

4. Codability

In order to prove that almost total degrees are continuous, we pass through another property that turns out to be a characterization of the continuous degrees. We start by relativizing the notion of "Π^0_1 class" to an enumeration oracle. We use $\langle A\rangle$ to signify that we are treating A as an enumeration oracle, rather than a Turing oracle.

Definition 4.1. Let $A \subseteq \omega$. Call $U \subseteq 2^\omega$ a $\Sigma^0_1\langle A\rangle$ *class* if there is a set $W \leqslant_e A$, such that $U = [W]^\prec = \{X \in 2^\omega \colon (\exists \sigma \in W)\ X \succeq \sigma\}$. A $\Pi^0_1\langle A\rangle$ *class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Note that a $\Pi^0_1\big\langle A \oplus \overline{A}\big\rangle$ class is just a $\Pi^0_1[A]$ class in the usual sense.

Definition 4.2. A set $A \subseteq \omega$ is *codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class $P \subseteq 2^\omega$ such that for every $X \in P$, A is c.e. relative to X. If there is a c.e. operator W such that $A = W^X$ for every $X \in P$, then A is *uniformly codable*.

First note that codable and uniformly codable are enumeration degree properties. It should also be clear that every total degree is uniformly codable; indeed, $\{A \oplus \overline{A}\}$ is a $\Pi^0_1\big\langle A \oplus \overline{A}\big\rangle$ class.

Remark 4.3. Uniform codability first arose as a potentially interesting property in a 2014 attempt by Mingzhong Cai, Steffen Lempp, Miller, and Soskova to understand almost totality. They proved that if A has almost total degree *and* there is a nonempty $\Pi^0_1\langle A\rangle$ class Q such that no path in Q is below the enumeration degree of A, then A is uniformly codable. The proof was never published.

Uniform codability can also be found in a characterization of the continuous degrees that was recently given by Kihara and Pauly [7, Section 7.1]. In fact, their result motivated us to revisit almost totality. Translating from their notation, they prove that A has continuous degree if and only if A is uniformly codable via the $\Pi^0_1\langle A\rangle$ class P *and* there is a uniform way to compute a path in P from an enumeration of A. It should be noted that it is not clear that they use the extra hypothesis in an essential way, and in light of our results, we expect that their proof can be modified to do without it.

We will prove in Lemma 4.5 that if A has almost total enumeration degree, then it is uniformly codable. So for the other results in this paper, we could avoid the nonuniform version of codability altogether. However, it is simple enough to prove that codability implies uniform codability.

Proposition 4.4. *If $A \subseteq \omega$ is codable, then it is uniformly codable.*

Proof. We prove the contrapositive. Assume that A is not uniformly codable and let $P \subseteq 2^\omega$ be a nonempty $\Pi^0_1\langle A\rangle$ class. We will construct an $X \in P$ that does not enumerate A. Since P is arbitrary, this proves that A is not codable. We construct X by "forcing with $\Pi^0_1\langle A\rangle$ classes". In other words, we let $P_0 = P$ and construct a sequence $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ of nonempty $\Pi^0_1\langle A\rangle$ classes such that any $X \in \bigcap_{e\in\omega} P_e$ is sufficient.

Say that we have constructed P_e. Let W_e be the eth c.e. operator; we want to ensure that $A \neq W_e^X$. Since A is not uniformly codable, there is a $Z \in P_e$ such that $A \neq W_e^Z$. There are two possibilities.

Case 1. If there is an $n \in W_e^Z \smallsetminus A$, then let $\sigma \prec Z$ be long enough that $n \in W_e^\sigma$. Set $P_{e+1} = P_e \cap [\sigma]^\prec$. So if $X \in P_{e+1}$, we have $n \in W_e^X \smallsetminus A$, hence $A \neq W_e^X$.

Case 2. If there is an $n \in A \smallsetminus W_e^Z$, then let $P_{e+1} = \{Z \in P_e \colon n \notin W_e^Z\}$. By assumption, P_{e+1} is nonempty. Also, if $X \in P_{e+1}$, we have $A \neq W_e^X$. □

As promised, almost total degrees consist of uniformly codable sets.

Lemma 4.5. *If $A \subseteq \omega$ has almost total enumeration degree, then it is uniformly codable.*

Proof. Assume that A has almost total enumeration degree. The proof of this lemma consists of two parts: we first use a failed forcing argument to construct an enumeration operator Γ with specific properties. We then use this operator to define a $\Pi^0_1\langle A\rangle$ class, witnessing that A is uniformly codable.

Recall that $(X)_\preceq = \{\sigma \in 2^{<\omega} \colon \sigma \preceq X\}$ was one of our examples of a total set. In this proof, it is convenient to use $(X)_\preceq$ instead of the enumeration equivalent set $X \oplus \overline{X}$. We will also use $(\tau)_\preceq$, for $\tau \in 2^{<\omega}$, to denote the set $\{\sigma \in 2^{<\omega} \colon \sigma \preceq \tau\}$. We want to build an enumeration operator Γ such that if $X \in 2^\omega$ is sufficiently generic, then $\Gamma(A \oplus (X)_\preceq)$ is the graph of an enumeration of A. In particular, the operator Γ will have the following properties:

(1) If $\sigma \in 2^{<\omega}$, then $\Gamma(A \oplus (\sigma)_\preceq)$ is the graph of a partial function with range contained in A.
(2) For every $n \in \mathbb{N}$ and every $\sigma \in 2^{<\omega}$, there is an extension $\tau \succeq \sigma$ such that the domain of $\Gamma(A \oplus (\tau)_\preceq)$ contains n.
(3) For every $a \in A$ and every $\sigma \in 2^{<\omega}$, there is an extension $\tau \succeq \sigma$ such that the range of $\Gamma(A \oplus (\tau)_\preceq)$ contains a.

To find such an enumeration operator, we consider the following attempt to construct an element $X \in 2^\omega$ that witnesses that A is not almost total.

Construction. We build X by initial segments as $\bigcup_{s\in\omega} \sigma_s$. Let $\sigma_0 = \varnothing$. We use even stages to ensure that $(X)_\preceq \not\leqslant_e A$. At stage $s = 2e$, we diagonalize against $\Gamma_e(A)$. If $\sigma_s\frown 0 \in \Gamma_e(A)$, then let $\sigma_{s+1} = \sigma_s\frown 1$. Otherwise, $\sigma_{s+1} = \sigma_s\frown 0$. Since only one of $\sigma_s\frown 0$ or $\sigma_s\frown 1$ is in $(X)_\preceq$, this ensures that $\Gamma_e(A) \neq (X)_\preceq$.

At the odd stage $s = 2e + 1$, we want to ensure that $\Gamma_e(A \oplus (X)_\preceq)$ is not an enumeration of A. There are several ways in which this could be achieved. It might be possible to extend σ_s appropriately so that $\Gamma_e(A \oplus (\sigma_{s+1})_\preceq)$ is not the graph of a function: for some n we have two different elements $a \neq b$ such that $\langle n, a\rangle$ and $\langle n, b\rangle$ are both in $\Gamma_e(A \oplus (\sigma_{s+1})_\preceq)$. It might be that we can extend σ_s so that $\Gamma_e(A \oplus (\sigma_{s+1})_\preceq)$ does not have range contained in A: for some $b \notin A$ and some natural number n we have that $\langle n, b\rangle \in \Gamma_e(A \oplus (\sigma_{s+1})_\preceq)$. If these two attempts at achieving our goal fail, then there is still the possibility that we could find an extension σ_{s+1} of σ_s that forces $\Gamma_e(A \oplus (X)_\preceq)$ to not be a total function or to only enumerate a proper subset of A. In the first case, there would be an n for which there is no extension $\tau \succeq \sigma_{s+1}$ such that n is in the domain of $\Gamma_e(A \oplus (\tau)_\preceq)$. In the second, there would be an element $a \in A$ for which there is no extension $\tau \succeq \sigma_{s+1}$ such that a is in the range of $\Gamma_e(A \oplus (\tau)_\preceq)$. If none of these options are possible, we say that the construction fails at stage s. ⊣

The proposed construction must fail at some finite stage. Otherwise, we would build an $X \in 2^\omega$ such that $X \oplus \overline{X} \equiv_e (X)_\preceq \not\leqslant_e A$ and such that no enumeration of A is enumeration reducible to $A \oplus (X)_\preceq \equiv_e A \oplus X \oplus \overline{X}$. This contradicts the assumption that A has almost total degree. Even stages cannot cause any problems, so the failure must be at an odd stage, say $s = 2e + 1$, giving us an e-operator Γ_e

that works as requested for every $\tau \succeq \sigma_s$. To get Γ, we *hardcode* σ_s, i.e., Γ consists of the axioms $\langle n, D_A \oplus D_X \rangle$ in Γ_e such that D_X only contains strings that are comparable with σ_s.

Next, using the operator Γ we define a $\Pi^0_1\langle A\rangle$ class P such that every path in P uniformly enumerates A. If B is a superset of A and X is sufficiently generic, then $\Gamma(B \oplus (X)_{\leq}) \subseteq \Gamma(A \oplus (X)_{\leq})$, which is the graph of an enumeration of A. Of course, $\Gamma(B \oplus (X)_{\leq})$ may fail to be a function: there may be some n for which there are two numbers $a \neq b$ such that $\langle n, a\rangle$ and $\langle n, b\rangle$ are both in $\Gamma(B \oplus (X)_{\leq})$. We will let $P \subseteq 2^\omega$ be the set of all B such that $A \subseteq B$ and B is small enough so that there is no $X \in 2^\omega$ that causes $\Gamma(B \oplus (X)_{\leq})$ to be a proper multifunction. The set P is a $\Pi^0_1\langle A\rangle$ class because it is the complement of the $\Sigma^0_1\langle A\rangle$ class generated by all $\beta \in 2^{<\omega}$ such that

$$(\exists n)[\beta(n) = 0 \text{ and } n \in A], \text{ or}$$
$$(\exists \sigma \in 2^{<\omega})(\exists n)(\exists a)(\exists b)\left[a \neq b \text{ and } \{\langle n, a\rangle, \langle n, b\rangle\} \subseteq \Gamma(\{x\colon \beta(x) = 1\} \oplus (\sigma)_{\leq})\right].$$

Note that P is nonempty because it contains A.

Finally, to prove that A is uniformly codable, we must explain how to enumerate A from any $B \in P$. This is simple, because A is exactly the set of elements that appear in the range of $\Gamma(B \oplus (\sigma)_{\leq})$, as σ ranges over $2^{<\omega}$. □

5. Holistic sets

We have show that every almost total degree is uniformly codable. The next step in our proof that these degrees are continuous is to introduce a concrete combinatorial property that guarantees that a set is uniformly codable. Then we will prove that every uniformly codable degree contains such a set.

Definition 5.1. Say $S \subseteq \omega^{<\omega}$ is *holistic* if for every $\sigma \in \omega^{<\omega}$,

(1) $(\forall n)$ $\sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S,
(2) If $\sigma \notin S$, then $(\forall n)$ $\sigma^\frown(2n) \in S$,
(3) If $\sigma \in S$, then $(\exists n)$ $\sigma^\frown(2n+1) \in S$.

Proposition 5.2. *If $S \subseteq \omega^{<\omega}$ is holistic, then it is uniformly codable.*

Proof. We build a tree $T \subseteq 2^{<\omega}$. Every level of this tree corresponds to a specific pair $(\sigma^\frown(2n), \sigma^\frown(2n+1))$, where $\sigma \in \omega^{<\omega}$. If $\tau \in T$ and τ is of a level that corresponds to $(\sigma^\frown(2n), \sigma^\frown(2n+1))$ then $\tau^\frown 0 \in T$ if and only if $\sigma^\frown(2n) \notin S$ and $\tau^\frown 1 \in T$ if and only if $\sigma^\frown(2n+1) \notin S$. Clearly $\overline{T} \leqslant_e S$, and so T defines a $\Pi^0_1\langle S\rangle$ class P. As $\sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ cannot both be S by the first property of holistic sets, it follows that T has no dead ends and so $P \neq \varnothing$. Every $X \in P$ can enumerate S using the following procedure:

> *If $X(k) = 0$ and level k corresponds to $(\sigma^\frown(2n), \sigma^\frown(2n+1))$, then enumerate σ.*

We claim that this procedure works. If $X(k) = 0$, then by definition of T we have $\sigma^\frown(2n) \notin S$, and hence $\sigma \in S$ by the second property of holistic sets. On the other hand, if $\sigma \in S$, then by the third property of holistic sets there is an n such that $\sigma^\frown(2n+1) \in S$ (and hence $\sigma^\frown(2n) \notin S$). If level k of T corresponds to the pair $(\sigma^\frown(2n), \sigma^\frown(2n+1))$, then $X(k)$ must be 0, and hence X will enumerate σ. □

Holistic sets are not hard to construct. Consider the following easy examples of computable holistic sets, S_{out} and S_{in}. The first, S_{out}, does not contain the empty

string and is defined inductively as follows: for every $\sigma \in \omega^{<\omega}$ and every $n \in \omega$, if $\sigma \notin S_{out}$, then $\sigma^\frown(2n) \in S_{out}$ and $\sigma^\frown(2n+1) \notin S_{out}$; if $\sigma \in S_{out}$, then $\sigma^\frown(2n) \notin S_{out}$ and $\sigma^\frown(2n+1) \in S_{out}$. The set S_{in} contains the empty string, but otherwise follows the same inductive definition. With a little more work, we can define an infinite family of computable holistic sets, with all possible finite restrictions. This will be useful in Section 6.

Lemma 5.3. *The set of finite sets $D \subseteq \omega^{<\omega}$ such that D can be extended to a holistic set is computable. If $D \subseteq \omega^{<\omega}$ is such a finite set, then there is computable holistic set S_D such that $D \subseteq S_D$.*

Proof. Fix a finite set D and let k be an even number such that $D \subseteq k^k$. We search for a finite set F such that $D \subseteq F \subseteq k^k$ which satisfies the restrictions of a holistic set:

(1) $(\forall n)$ if $2n+1 < k$, then $\sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in F,
(2) If $\sigma \in k^{<k}$ and $\sigma \notin F$, then $(\forall n)$ if $2n < k$ then $\sigma^\frown(2n) \in F$.

If there is no such F, then D cannot be extended to a holistic set. If there is, then let F be the least one. We complete it to a holistic set by using essentially the same procedure as we used to define S_{in} and S_{out}. Start with $F \subseteq S_D$ and proceed by induction for every $\sigma \in \omega^{<\omega}$ and every $n \in \omega$:

(1) If $\sigma \notin S_D$, then $\sigma^\frown(2n) \in S_D$ and $\sigma^\frown(2n+1) \notin S_D$,
(2) If $\sigma \in S_D$ and if $\sigma^\frown(2n) \notin k^k$ (i.e., it is not determined by F), then $\sigma^\frown(2n) \notin S_D$ and $\sigma^\frown(2n+1) \in S_D$.

It is easy to see that the constructed set S_D is holistic. Fix σ and n. By our choice of k as even, either $2n+1 < k$ or $2n \geqslant k$. In the first case, the fact that $\sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S_D follows from our choice of F; in the second case, it follows from our inductive definition. If $\sigma \notin S_D$ and $2n < k$, then $\sigma^\frown(2n) \in S_D$ follows from $F \subseteq D$; if $2n \geqslant k$, then $\sigma^\frown(2n) \in S_D$ follows from our inductive definition. Finally, if $\sigma \in S_D$, then for all n such that $2n+1 > k$ we will have $\sigma^\frown(2n+1) \in S_D$. Therefore, all three properties of holistic sets are satisfied. □

The reason that the holistic sets are important for us is that they occupy *every* uniformly codable degree.

Lemma 5.4. *If $A \subseteq \omega$ is uniformly codable, then there is a holistic set $S \equiv_e A$.*

Proof. Fix a uniformly codable set A. Without loss of generality, we may assume that A is not c.e. We will build a holistic set S so that $A \equiv_e S$. For $\sigma \in \omega^{<\omega}$ and $Y \subseteq \omega^{<\omega}$, we use the notation $\sigma^\frown Y$ to denote the set of all strings in $\omega^{<\omega}$ obtained by concatenating σ with some member of Y. We start with $\varnothing \in S$. For every n, we use $(2n)^\frown S_{out}$ to define $S \cap (2n)^\frown(\omega^{<\omega})$. We put the string $(2n+1)$ into S if and only if $n \in A$, which ensures that $A \leqslant_e S$. Since A is not empty, as least one string of the form $(2n+1)$ will be in S, so the holistic set conditions are satisfied for $\varnothing$. The main difficulty is what we do with strings extending $(2n+1)$. In particular, we need to be careful when $n \notin A$ and hence $(2n+1) \notin S$. We need to find a way to transform this negative fact about A into a positive fact that will force us to enumerate into S every string $(2n+1)^\frown(2k)$, in order to make S a holistic set enumeration reducible to A. The following observation will facilitate this.

Let P and W be the $\Pi^0_1\langle A\rangle$ class and uniform c.e. procedure witnessing that A is uniformly codable. Let Γ be an enumeration operator witnessing that P is a $\Pi^0_1\langle A\rangle$

class, i.e., such that $\overline{P} = \{X \in 2^\omega \colon (\exists \sigma \in \Gamma(A))\ \sigma \preceq X\}$. Fix a finite set D (for instance, $D = \{2n+1\}$). If $D \subseteq A$, then by compactness there is some n such that for every $X \in 2^\omega$, either X extends a member of $\Gamma(A)$ of length less than n or D is enumerated by $W^{X\restriction n}$ in no more than n steps. Let $C = \{\tau \in 2^n \colon \tau \in \overline{\Gamma(A)}\}$. Then the finite set C generates a clopen set $[C]^\prec$ such that $D \subseteq W^\tau$ for every $\tau \in C$ (in $|\tau|$ many steps), and such that $P \subseteq [C]^\prec$. If, on the other hand, $D \nsubseteq A$ and C is a finite set of strings such that $D \subseteq W^\tau$ for every $\tau \in C$, then it must be that $[C]^\prec \cap P = \varnothing$. Otherwise, for some $X \in P$, we will have that $D \subseteq W^X$, contrary to our assumption that $A = W^X$. Now, if C is finite and $[C]^\prec \cap P = \varnothing$, then by compactness, there is a finite set $D_C \subseteq A$ such that $[C]^\prec \subseteq [\Gamma(D_C)]^\prec$, and this is seen in finite time. Thus, we have witnessed the negative fact $D \nsubseteq A$ by a positive fact $D_C \subseteq A$. We will use this idea to define an inductive procedure that decides, for every $\sigma \succeq (2n+1)$, whether or not $\sigma \in S$.

To every $\sigma \succeq (2n+1)$, for $n \in \omega$, we will assign a statement φ_σ so that $\sigma \in S$ if and only if φ_σ is true. Further, this statement will come with a uniformly c.e. sequence of finite sets $\{D^\sigma_i\}_{i\in\omega}$ such that φ_σ is true if and only if $(\exists i)\ D^\sigma_i \subseteq A$. Since A is not c.e., there is a $\hat{a} \notin A$ and a string $\hat{\tau}$ such that $\hat{a} \in W^{\hat{\tau}}$. We will always set $D^\sigma_0 = \{\hat{\tau}\}$. We need to handle three kinds of statements:

(1) The statement "$n \in A$" is assigned to the string $(2n+1)$. The sequence of finite sets $\{D^\sigma_i\}_{i\in\omega}$ is defined simply by setting $D^\sigma_i = \{n\}$ for every $i > 0$.
(2) Statements of the form "$[C]^\prec \cap P = \varnothing$", where C is a finite set of binary strings, are assigned to strings $\sigma \succ (2n+1)$ that end in an even number. As discussed above, this statement is true if and only if there is a finite set $D \subseteq A$ such that $[C]^\prec \subseteq [\Gamma(D)]^\prec$. In this case, we will let $\{D^\sigma_i\}_{i>0}$ be the sequence that lists, possibly with repetition, all finite sets D such that $[C]^\prec \subseteq [\Gamma(D)]^\prec$. It can happen that there are no such finite sets at all. To deal with this situation, we will pad by letting $D^\sigma_i = \{\hat{a}\}$.
(3) Statements of the form "$P \subseteq [C]^\prec$", where C is a finite set of binary strings, are assigned to strings $\sigma \succ (2n+1)$ that end in an odd number. Such a statement is true if and only if $(2^\omega \smallsetminus [C]^\prec) \cap P = \varnothing$. Let O be a finite set of strings such that $[O]^\prec = 2^\omega \smallsetminus [C]^\prec$; this can be found computably. Now, just like in the previous case, we let $\{D^\sigma_i\}_{i\in\omega}$ be a sequence of finite sets that would witness "$[O]^\prec \cap P = \varnothing$", were any of them subsets of A (along with the finite set $\{\hat{a}\}$).

The assignment is defined inductively: fix σ and suppose that we have assigned to it the statement φ_σ along with the sequence of finite sets $\{D^\sigma_i\}_{i\in\omega}$. Let $\{C_n\}_{n\in\omega}$ be a c.e. listing, allowing repetition, of all finite sets of strings such that for some i and every $\tau \in C_n$ we have $D^\sigma_i \subseteq W^\tau$. Note that this list is not empty because $\{\hat{\tau}\}$ will always appear. Next, for every n, we associate the statement "$[C_n]^\prec \cap P = \varnothing$" to $\sigma^\frown(2n) \in S$ and "$P \subseteq [C_n]^\prec$" to $\sigma^\frown(2n+1) \in S$.

Property (1) of holistic sets is clearly true at σ. Furthermore, by our earlier analysis we have:

(2) $\sigma \notin S \Rightarrow (\forall i)[D^\sigma_i \nsubseteq A] \Rightarrow (\forall n)[[C_n]^\prec \cap P = \varnothing] \Rightarrow (\forall n)[\sigma^\frown(2n) \in S]$,
(3) $\sigma \in S \Rightarrow (\exists i)[D^\sigma_i \subseteq A] \Rightarrow (\exists n)[P \subseteq [C_n]^\prec] \Rightarrow (\exists n)[\sigma^\frown(2n+1) \in S]$.

So S is a holistic set. Finally, we note that $S \leqslant_e A$ because $\sigma \in S$ if and only if $\sigma = \varnothing$, or $\sigma \in (2n)^\frown S_{out}$ for some n, or if σ is assigned the statement φ_σ with sequence $\{D^\sigma_i\}_{i\in\omega}$ and $(\exists i)\ D^\sigma_i \subseteq A$. □

6. The holistic space

In the previous section, we showed that every uniformly codable degree—hence every almost total enumeration degree—contains a holistic set. Our next step is to form a topological space from the holistic sets. It turns out to be a very well-behaved topological space: it is Hausdorff, second countable (by definition), and regular, so it satisfies the hypotheses of Urysohn's metrization theorem. In fact, it can be turned into a computable metric space, which is how we complete the chain of implications and prove that every almost total degree is continuous.

Definition 6.1. Let

$$\mathcal{H} = \{S \subseteq \omega^{<\omega} \colon S \text{ is holistic}\}.$$

For each $\sigma \in \omega^{<\omega}$, let $O_\sigma = \{S \in \mathcal{H} \colon \sigma \in S\}$. These sets form a subbasis for the desired topology, i.e., their finite intersections form a basis. We call the resulting topological space the *holistic space*.

We extend the subbasis to an explicit basis as follows. By Lemma 5.3, there is a computable listing $\{D_n\}_{n\in\omega}$ of all finite subsets of $\omega^{<\omega}$ that extend to holistic sets. Define $\{B_n\}_{n\in\omega}$ by $B_n = \bigcap_{\sigma\in D_n} O_\sigma$. Note that for every n, the open set B_n is not empty; in particlar, it contains the computable holistic set extending D_n that was constructed in Lemma 5.3.

For computability on topological spaces, we essentially follow the definitions used by Grubba, Schröder, and Weihrauch [5].

Definition 6.2. A second countable T_0 space $\mathcal{X}$ is *computable* if it has a countable basis $\{B_n\}_{n\in\omega}$ of nonempty open sets on which intersection is a computable operation, meaning that there is a total computable function i such that $B_n \cap B_m = \bigcup_{k\in W_{i(n,m)}} B_k$.

Lemma 6.3. *The holistic space is a computable second countable Hausdorff space.*

Proof. The holistic space is clearly second countable. The basis defined above makes it computable: if $D_n \cup D_m$ extends to a holistic set, then $B_n \cap B_m = B_k$ for $D_k = D_n \cup D_m$, otherwise $B_n \cap B_m$ is empty. To see that $\mathcal{H}$ is Hausdorff, fix two different points $S_1 \neq S_2 \in \mathcal{H}$. Without loss of generality, fix $\sigma \in S_1 \smallsetminus S_2$. As $\sigma \in S_1$, by the third property of holistic sets there is a number n such that $\sigma^\frown(2n+1) \in S_1$. On the other hand, since $\sigma \notin S_2$, by the second property we have that $\sigma^\frown(2n) \in S_2$. So $S_1 \in O_{\sigma^\frown(2n+1)}$, $S_2 \in O_{\sigma^\frown(2n)}$, and by the first property of holistic sets, $O_{\sigma^\frown(2n+1)} \cap O_{\sigma^\frown(2n)} = \varnothing$. □

Definition 6.4. Let $\mathcal{X}$ be a computable topological space with basis $\{B_n\}_{n\in\omega}$. A *name* for a point $x \in \mathcal{X}$ is any enumeration of the set $\{n \colon x \in B_n\}$. A *name* for an open set $O \subseteq \mathcal{X}$ is any enumeration of the set $\{n \colon B_n \subseteq O\}$. A *name* for a closed set $F \subseteq \mathcal{X}$ is just a name for $\mathcal{X} \smallsetminus F$.

It is easy to see that from an enumeration of a holistic set S, we can compute a name for S as an element of $\mathcal{H}$, i.e., we can enumerate $\{n \colon S \in B_n\} = \{n \colon D_n \subseteq S\}$. Conversely, from a name for S we can enumerate S itself. This means that the degree of S as a point in $\mathcal{H}$, in the sense of Kihara and Pauly [7], is just the enumeration degree of S.

Recall that a topological space $\mathcal{X}$ is *regular* if whenever $F \subseteq \mathcal{X}$ is closed and $x \in \mathcal{X} \smallsetminus F$, there are disjoint open sets U, V such that $x \in U$ and $F \subseteq V$. If $\mathcal{X}$ has a

countable base $\{B_n\}_{n\in\omega}$, then one way to ensure that $\mathcal{X}$ is regular is to require that for every basic open set B_n and $x \in B_n$, there is a basic open set B_m and a closed set C such that $x \in B_m \subseteq C \subseteq B_n$. Indeed, if $\mathcal{X}$ has that property and $x \notin F$ are given, then we can find a basic open set B_n such that $x \in B_n \subseteq \mathcal{X} \smallsetminus F$, and then $x \in B_m$ and $F \subseteq \mathcal{X} \smallsetminus C$ witness that $\mathcal{X}$ is regular. If we have a computable space $\mathcal{X}$, then it is *computably regular* if this version of regularity is effective.

Definition 6.5. A computable topological space $\mathcal{X}$ with base $\{B_n\}_{n\in\omega}$ is *computably regular* if there is a c.e. set R and a computable function c such that:

(1) For all n, we have $B_n = \bigcup_{\langle n,m\rangle\in R} B_m$;
(2) If $\langle n, m\rangle \in R$, then $c(n, m)$ is the index of a c.e. set $C_{n,m}$ describing a closed set $F_{n,m} = \mathcal{X} \smallsetminus (\bigcup_{k\in C_{n,m}} B_k)$ such that $B_m \subseteq F_{n,m} \subseteq B_n$.

Lemma 6.6. *The holistic space is computably regular.*

Proof. If S is holistic and $\sigma^\frown(2k + 1) \in S$, then $\sigma \in S$. In terms of our topology, this means that $O_{\sigma^\frown(2k+1)} \subseteq O_\sigma$. If $\sigma^\frown(2k + 1) \in S$, then we also know that $\sigma^\frown(2k) \notin S$ and so $O_{\sigma^\frown(2k+1)} \cap O_{\sigma^\frown(2k)} = \varnothing$. Finally, $\sigma^\frown(2k) \notin S$ implies $\sigma \in S$, so

$$O_{\sigma^\frown(2k+1)} \subseteq \mathcal{H} \smallsetminus O_{\sigma^\frown(2k)} \subseteq O_\sigma.$$

Let R be the set of all pairs $\langle n, m\rangle$ such that if $D_n = \{\sigma_0, \sigma_1, \ldots, \sigma_i\}$, then $D_m = \{\sigma_0^\frown(2k_0 + 1), \sigma_1^\frown(2k_1 + 1), \ldots, \sigma_i^\frown(2k_i + 1)\}$ for some $k_0, k_1, \ldots, k_i \in \omega$. Clearly, $B_n = \bigcup_{\langle n,m\rangle\in R} B_m$: if $S \in B_n = \bigcap_{\sigma\in D_n} O_\sigma$ then $D_n \subseteq S$, and so by the third property of holistic sets, there are numbers $k_0, k_1, \ldots, k_i$ such that $D = \{\sigma_0^\frown(2k_0 + 1), \sigma_1^\frown(2k_1 + 1), \ldots, \sigma_i^\frown(2k_i + 1)\} \subseteq S$. It follows that D extends to a holistic set, hence $D = D_m$ for some m.

Fix $\langle n, m\rangle \in R$. Let C be the finite set of indices (in the listing $\{D_n\}_{n\in\omega}$) of sets of the form $\{\sigma^\frown(2k)\}$, where $\sigma^\frown(2k + 1) \in D_m$. Let $F_{n,m} = \mathcal{H} \smallsetminus (\bigcup_{k\in C} B_k)$. It is straightforward to check that $B_m \subseteq F_{n,m} \subseteq B_n$. Of course, the process of converting $\langle n, m\rangle$ to a c.e. index of the (finite) set C is computable, thus we have established that $\mathcal{H}$ is computably regular. □

The previous lemmas are important in light of Urysohn's metrization theorem [15], which states that every regular second countable Hausdorff space is metrizable.[3] Schröder [12] proved an effective version of Urysohn's theorem that holds in our case: there is a computable metric d on $\mathcal{H}$ that induces the given topology. This metric is actually computable in the sense we need, i.e., if $S, T \in \mathcal{H}$, then from enumerations of S and T we can compute $d(S, T)$. Moreover, we will produce a computable dense sequence of points in $\mathcal{H}$, making it a computable metric space. We will outline the steps in Schröder's proof following the exposition given in Grubba, Schröder, and Weihrauch [5].

The first step is to show that every computably regular space is computably normal. Recall that a space $\mathcal{X}$ is normal if every two disjoint closed subsets of $\mathcal{X}$ have disjoint open neighborhoods. There is a natural effective version of this definition:

Definition 6.7. A computable space $\mathcal{X}$ is *computably normal* if given names of two disjoint closed sets F_1 and F_2 there is a uniform way to compute names for disjoint open sets O_1 and O_2 so that $F_1 \subseteq O_1$ and $F_2 \subseteq O_2$.

[3]Actually, Urysohn assumed normality instead of regularity; following up on Urysohn's work, Tychonoff [14] showed that every regular second countable Hausdorff space is normal.

It is a classical theorem that every second countable regular space is normal. Grubba et al. [5] prove the effective version of this theorem: every computably regular space is computably normal. Therefore:

Corollary 6.8. *$\mathcal{H}$ is computably normal.*

An equivalent way to express that a space $\mathcal{X}$ is normal is to say that whenever $F_1 \subseteq O_1$ are a closed and an open set, we can find an open set O_2 and a closed set F_2 so that $F_1 \subseteq O_2 \subseteq F_2 \subseteq O_1$. Indeed, $\mathcal{X} \smallsetminus O_1$ is closed and disjoint from F_1, so there are disjoint open neighborhoods $\mathcal{X} \smallsetminus F_2$ and O_2 of $\mathcal{X} \smallsetminus O_1$ and F_1. Note, that we use the same set as a name for an open set and its complement, so in a computably normal space there is uniform way to obtain a description of the second pair of sets from the first. Grubba et al. [5] use this idea to prove the following: if A and B are disjoint closed sets, then given names for A and B there is a uniform way to compute a name for a continuous function $f_{AB} : \mathcal{X} \to [0, 1]$ so that $f_{AB}[A] = 0$ and $f_{AB}[B] = 1$. Using normality, they build a sequence of closed sets F_q and open sets $O_q \subseteq F_q$, where q ranges over all rational numbers in the unit interval. This sequence has the property that if $q < p$ then $F_q \subseteq O_p$, that $A \subseteq O_0$, and that $F_1 \subseteq \mathcal{X} \smallsetminus B$. Then if S is a holistic set,

$$f_{AB}(S) = \sup(\{q\colon S \notin F_q\} \cup \{0\}) = \inf(\{q\colon S \in O_q\} \cup \{1\}).$$

The distance function d on $\mathcal{H}$ can now be defined as follows: fix a listing $\{\langle n_i, m_i\rangle\}_{i\in\omega}$ of the c.e. set R witnessing that $\mathcal{H}$ is computably regular. For every i, let $f_i = f_{AB}$ for the sets $A = F_{n_i,m_i}$ (the closed set that sits between $B_{m_i} \subset B_{n_i}$) and $B = \mathcal{H} \smallsetminus B_{n_i}$. Finally if S and T are two holistic sets, then define

$$d(S, T) = \sum_{i\in\omega} 2^{-i}|f_i(S) - f_i(T)|.$$

Grubba et al. [5] prove that d is a metric on $\mathcal{H}$ that induces the original topology. They note two important properties of d (see [5, Lemma 4.7]):

(1) The metric is computable in the sense that from any name for S and any name for T, we can compute $d(S, T)$,
(2) For every pair $\langle n_i, m_i\rangle$, if $d(S, B_{m_i}) < 2^{-i}$, then $S \in B_{n_i}$.

We have that $\mathcal{H}$ is a metrizable space with a computable metric d. In order to show that it is a computable metric space, we need to define a dense set $Q^{\mathcal{H}}$ on which the metric remains computable. We use Lemma 5.3: let $S_k = S_{D_k}$ be the computable holistic set that contains the finite set D_k and let $Q^{\mathcal{H}} = \{S_k\}_{k\in\omega}$. The metric d is computable on $Q^{\mathcal{H}}$ because names for the points S_k are uniformly computable in k. We have shown:

Corollary 6.9. *$(\mathcal{H}, d)$ is a computable metric space.*

The final thing we need to check is that if $S \in \mathcal{H}$, then the continuous degree of S as a point in the computable metric space $(\mathcal{H}, d)$ is the same as its enumeration degree. Recall that a name for S from the point of view of a computable metric space is a function that takes as input a rational number ε and outputs the index of a member of $Q^{\mathcal{H}}$ that is within distance ε of S. Let us call such names for S *metric names*.

Lemma 6.10. *Let S be a point in $\mathcal{H}$.*

(1) *Every enumeration of S computes a metric name for S.*

(2) *Every metric name for S computes an enumeration of S.*

Proof. The first statement of the lemma follows easily from the fact that we have a computable metric. If we have an enumeration of S, then we can compute a name for S as a point in the computable topological space $\mathcal{H}$, so we can compute (a name for) the distance between S and any of the computable points S_k. This means that we can search through the list $Q^{\mathcal{H}}$ until we find an appropriate point that is at distance less than any fixed rational ε. Since the topology induced by the metric is the same as the original, we know that there are computable points arbitrarily close to S, i.e., in any open ball $B_\varepsilon = \{T\colon d(S,T) < \varepsilon\}$. This lets us compute a metric name for S.

For the second statement, we will use property (2) above to construct an enumeration of S given access to a metric name for S. We start enumerating elements S_k from the sequence $Q^{\mathcal{H}}$ that are closer and closer to S using the metric name for S. Recall, that $\{\langle n_i, m_i\rangle\}_{i\in\omega}$ is the listing of R that we used to define the metric d. If we enumerate a point S_k such that $d(S,S_k) < 2^{-i}$ and $S_k \in B_{m_i}$, then $d(S, B_{m_i}) < 2^{-i}$ and so S must be in B_{n_i}. This means that $D_{n_i} \subseteq S$, so we can safely enumerate the finite set D_{n_i}. We must show that this procedure will not miss any element of S. If $\sigma \in S$, then for some k we have that $\sigma^\frown(2k+1) \in S$. Let i be such that $B_{n_i} = \{\sigma\}$ and $B_{m_i} = \{\sigma^\frown(2k+1)\}$. Then $S \in B_{m_i}$. Now using the fact that the metric d induces the original topology on $\mathcal{H}$, for some rational ε we will have that the open ball $B_\varepsilon = \{T\colon d(S,T) < \varepsilon\} \subseteq B_{m_i}$. When we use the metric name for S to produce a point in $Q^{\mathcal{H}}$ at distance no more than $\min(2^{-i}, \varepsilon)$ it must give us a point in B_{m_i}, and hence our procedure will enumerate σ. □

By the lemma, the continuous degree of a holistic set S as a point in $(\mathcal{H}, d)$ coincides with its enumeration degree.

Corollary 6.11. *Holistic sets have continuous enumeration degree.*

This was the last step in the proof of our main result. We conclude with a summary of what we have shown.

Theorem 1.3. *Let* $\mathbf{a}$ *be an enumeration degree. The following are equivalent*

(1) $\mathbf{a}$ *is (uniformly) almost total,*
(2) *The sets in* $\mathbf{a}$ *are (uniformly) codable,*
(3) $\mathbf{a}$ *contains a holistic set,*
(4) $\mathbf{a}$ *is continuous.*

Proof. In Lemma 4.5, we proved that the nonuniform version of (1) implies the uniform version of (2). Proposition 4.4 established the equivalence of the two versions of (2). Lemma 5.4 showed that (2) implies (3), and we just finished proving that (3) implies (4) in Corollary 6.11. Finally, in Lemma 3.2 we proved that (4) implies the uniform version of (1). □

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