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Numerical Analysis of an Artificial Compression Method for Magnetohydrodynamic Flows at Low Magnetic Reynolds Numbers

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Abstract Magnetohydrodynamics (MHD) is the study of the interaction of electrically conducting fluids in the presence of magnetic fields. MHD applications require substantially more efficient numerical methods than currently exist. In this paper, we construct two decoupled methods based on the artificial compression method (uncoupling the pressure and velocity) and partitioned method (uncoupling the velocity and electric potential) for magnetohydrodynamics flows at low magnetic Reynolds numbers. The methods we study allow us at each time step to solve linear problems, uncoupled by physical processes, per time step, which can greatly improve the computational efficiency. This paper gives the stability and error analysis, presents a brief analysis of the non-physical acoustic waves generated, and provides computational tests to support the theory.

Keywords Artificial compression method · Partitioned method · Finite element method · Magnetohydrodynamics

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1 Introduction

We consider the time-dependent magnetohydrodynamic (MHD) flows at low magnetic Reynolds numbers (denoted by R_m). The low-R_m MHD model (typical for terrestrial applications, see [12,20,23]) is given by: find fluid velocity $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, pressure $p : \Omega \times [0, T] \to \mathbb{R}$ and electric potential $\phi : \Omega \times [0, T] \to \mathbb{R}$ satisfying

$$\begin{aligned} &\frac{1}{N}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1}{M^2}\Delta\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{B}\times\nabla\phi + (\mathbf{u}\times\mathbf{B})\times\mathbf{B},\\ &\nabla\cdot\mathbf{u} = 0,\\ &\Delta\phi = \nabla\cdot(\mathbf{u}\times\mathbf{B}). \end{aligned} \tag{1.1}$$

For (1.1) impose homogeneous Dirichlet boundary conditions and the initial condition

$$\begin{aligned} \mathbf{u} &= 0 \qquad \text{on } \partial\Omega\times[0,T],\\ \phi &= 0 \qquad \text{on } \partial\Omega\times[0,T],\\ \mathbf{u}(x,0) &= \mathbf{u}_0(x) \quad \forall\, x\in\Omega. \end{aligned} \tag{1.2}$$

Here, body force $\mathbf{f}$, external magnetic field $\mathbf{B}$, and final time $T > 0$ are known. The domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is a convex polygon or polyhedra. N is interaction parameter and M is Hartmann number, $\frac{N}{M^2} = \frac{1}{Re}$, where Re is the Reynolds number. Further, $\mathbf{u}_0(x) \in H_0^1(\Omega)^d$ and $\nabla\cdot\mathbf{u}_0 = 0$.

In this report, we give an analysis of a classical artificial compression scheme from [28] adapted from the Navier–Stokes equations (NSE) to the model (1.1). Combined with two partitioned methods from [23], we gave two fully-decoupled methods. The schemes (*Algorithms* 1 and 2) are based on (i) replacing $\nabla\cdot\mathbf{u}$ by $\varepsilon p_t + \nabla\cdot\mathbf{u} = 0$, (ii) time discretization by the implicit methods (Backward-Euler and BDF2) and (iii) treating the magnetic field terms explicitly to further uncouple the system into components. Theorem 4.3 below shows that for smooth solutions the error of *Algorithm* 1 is $\mathcal{O}(\Delta t + \varepsilon)$. Numerical tests in Sect. 6 also confirm that the error of *Algorithms* 1 and 2 are $\mathcal{O}(\Delta t + \varepsilon)$ and $\mathcal{O}(\Delta t^2 + \varepsilon)$, respectively.

1.1 Previous Work

MHD describes the behavior of the electrically conducting fluids in the presence of an external magnetic field. The study of MHD, initiated by Alfvén [1], has been widely developed in many fields of science including astrophysics, geophysics, engineering, and metallurgy. Applications include the studying of sunspots and solar flares, pumping and stirring of liquid metals, liquid metals cooling of nuclear reactors, forecasting of climate change, controlled thermonuclear fusion and sea water propulsion, see [2–6]. Most terrestrial applications, such as liquid metals, involve small magnetic Reynolds numbers, $R_m \ll 1$. In these cases, the magnetic field influences the conducting fluid via the Lorentz force, but the conducting fluid does not significantly perturb the magnetic field. Thus the magnetic field induced by the electrically conducting fluid motion is small and can be negligible compared with the imposed magnetic field. Neglecting the induced magnetic field, the general MHD flows can be simplified to the low-R_m MHD model considered herein.

In the recent years, there are many works on the MHD equations. For instance, Refs. [13,15–19] studied some effective iterative methods in finite element approximation for the steady MHD equations. For the time-dependent MHD equations, He [33] discussed an Euler semi-implicit scheme for the three-dimensional MHD equations. The decoupled fully discrete finite element schemes for the unsteady MHD equations were analyzed in [31,32,39]. Zhang et al. [14] analyzed a partitioned scheme based on Gauge-Uzawa finite element method for the 2D time-dependent MHD equations. The mathematical structure of the steady low-R_m MHD model was established by Peterson [12]. Numerical analysis of the evolutionary problem (1.1) was performed by Yuksel and Isik [25] (coupled implicit method), Yuksel and Ingram [20] (coupled Crank–Nicolson method), and Rong et al. [35] (coupled spectral deferred correction method). Partitioned methods uncoupling the fluid velocity from the electric potential were analyzed in [23,24,36]. The *Algorithms* 1 and 2 herein continue this development uncoupling electric potential, pressure and individual components of the velocity.

1.2 The Slightly Compressible Model

There are two forms of coupling in the above Eq. (1.1). One is the coupling between the fluid velocity $\mathbf{u}$ and the electric potential ϕ. The other is that the fluid velocity $\mathbf{u}$ and the pressure p are coupled by the incompressibility restriction $\nabla \cdot \mathbf{u} = 0$. Both couplings increase memory requirements, make the equations more difficult to solve numerically, and reduce computational efficiency. In the existing papers on numerical analysis of the MHD flows at low magnetic Reynolds numbers, most methods considered to solve the problem are monolithic methods in which the coupled problem is solved iteratively at each time step. Therefore, we study uncoupling methods for the time-dependent MHD flows at low magnetic Reynolds numbers.

As to the the coupling between $\mathbf{u}$ and p via $\nabla \cdot \mathbf{u} = 0$, the general idea to deal with this coupling is to relax the incompressibility constraint. There have been some such methods: the artificial compression method, penalty method, projection method, and pressure stabilization method (see [9,10,13,17,28–30,34,37,38]). The artificial compression method (ACM), which was introduced by Chorin [9] and Temam [40], breaks the incompressibility restriction by adding a slightly compressible term εp_t ($\varepsilon > 0$ small) in $\nabla \cdot \mathbf{u} = 0$. This allows the pressure to be advanced in time explicitly. Using ACM, the slightly compressible model of (1.1) and (1.2) is given as follows.

$$\begin{aligned}
&\frac{1}{N}\left(\mathbf{u}_t^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \frac{1}{2}\left(\nabla \cdot \mathbf{u}^\varepsilon\right)\mathbf{u}^\varepsilon\right) - \frac{1}{M^2}\Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} + \mathbf{B} \times \nabla \phi^\varepsilon + (\mathbf{u}^\varepsilon \times \mathbf{B}) \times \mathbf{B},\\
&\varepsilon p_t^\varepsilon + \nabla \cdot \mathbf{u}^\varepsilon = 0,\\
&\Delta \phi^\varepsilon = \nabla \cdot (\mathbf{u}^\varepsilon \times \mathbf{B}),
\end{aligned} \tag{1.3}$$

with the conditions

$$\begin{aligned}
&\mathbf{u}^\varepsilon = 0 \quad \text{on } \partial\Omega \times [0, T],\\
&\phi^\varepsilon = 0 \quad \text{on } \partial\Omega \times [0, T],\\
&\mathbf{u}^\varepsilon(0) = \mathbf{u}_0, \quad p^\varepsilon(0) = p_0,
\end{aligned} \tag{1.4}$$

where typically $\varepsilon = \mathcal{O}(\Delta t)$ or $\mathcal{O}(\Delta t^2)$. The function $p_0 \in L^2(\Omega)$ is arbitrarily chosen but independent of ε. The term $\frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon$ preserves skew-symmetry of the trilinear form. Since $\frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{u} = 0$ for the true solution $\mathbf{u}$ of (1.1) and (1.2) and $\varepsilon p_t = \mathcal{O}(\varepsilon)$, the consistency error of model (1.3) and (1.4) is clearly $\mathcal{O}(\varepsilon)$.

1.3 The Artificial Compression Schemes

Implicit-explicit (IMEX) methods have been widely used in to reduce the cost per time step in solving coupled systems of partial differential equations, see [8,11,31–33,39]. In

Algorithms 1 and 2 below, the coupling between the velocity $\mathbf{u}$ and electric potential ϕ are treated explicitly to uncouple the systems. The fluid velocity $\mathbf{u}$ and pressure p are further uncoupled using artificial compression method. The nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u}$ is treated linearly implicitly to reduce complexity. Let $B(\mathbf{u},\mathbf{v}) := \mathbf{u}\cdot\nabla\mathbf{v}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}$. Based on the IMEX partitioned schemes in [23], the following first order (Backward-Euler) and second order (BDF2) artificial compression schemes are introduced.

Algorithm 1 Given $\mathbf{u}^n$, p^n, ϕ^n, find $\mathbf{u}^{n+1}$, p^{n+1}, ϕ^{n+1} satisfying

$$\begin{aligned}
&\frac{1}{N}\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}+B\left(\mathbf{u}^n,\mathbf{u}^{n+1}\right)\right)-\frac{1}{M^2}\Delta\mathbf{u}^{n+1}+\nabla p^{n+1}\\
&\quad=\mathbf{f}^{n+1}+\mathbf{B}\times\nabla\phi^n+(\mathbf{u}^{n+1}\times\mathbf{B})\times\mathbf{B},\\
&\quad\varepsilon\frac{p^{n+1}-p^n}{\Delta t}+\nabla\cdot\mathbf{u}^{n+1}=0,\\
&\quad\Delta\phi^{n+1}=\nabla\cdot(\mathbf{u}^n\times\mathbf{B}).
\end{aligned}\tag{1.5}$$

Algorithm 2 Given $\mathbf{u}^{n-1}$, $\mathbf{u}^n$, p^n, ϕ^{n-1}, ϕ^n, find $\mathbf{u}^{n+1}$, p^{n+1}, ϕ^{n+1} satisfying

$$\begin{aligned}
&\frac{1}{N}\left(\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t}+B\left(2\mathbf{u}^n-\mathbf{u}^{n-1},\mathbf{u}^{n+1}\right)\right)-\frac{1}{M^2}\Delta\mathbf{u}^{n+1}+\nabla p^{n+1}\\
&=\mathbf{f}^{n+1}+\mathbf{B}\times\nabla\left(2\phi^n-\phi^{n-1}\right)+(\mathbf{u}^{n+1}\times\mathbf{B})\times\mathbf{B},\\
&\varepsilon\frac{p^{n+1}-p^n}{\Delta t}+\nabla\cdot\mathbf{u}^{n+1}=0,\\
&\Delta\phi^{n+1}=\nabla\cdot(\mathbf{u}^{n+1}\times\mathbf{B}).
\end{aligned}\tag{1.6}$$

In both algorithms, the term ∇p^{n+1} can be eliminated by using $p^{n+1}=p^n-\frac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^{n+1}$. Thus, the calculation for *Algorithm* 1 (and similarly for *Algorithm* 2) proceeds as follows. Given $\mathbf{u}^n$, p^n, ϕ^n, solve for $\mathbf{u}^{n+1}$:

$$\begin{aligned}
&\frac{1}{N}\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}+B\left(\mathbf{u}^n,\mathbf{u}^{n+1}\right)\right)-\frac{1}{M^2}\Delta\mathbf{u}^{n+1}-\frac{\Delta t}{\varepsilon}\nabla\nabla\cdot\mathbf{u}^{n+1}-\left(\mathbf{u}^{n+1}\times\mathbf{B}\right)\times\mathbf{B}\\
&=\mathbf{f}^{n+1}+\mathbf{B}\times\nabla\phi^n-\nabla p^n.
\end{aligned}\tag{1.7}$$

Perform an algebraic update of p^{n+1}:

$$p^{n+1}=p^n-\frac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^{n+1}.\tag{1.8}$$

Solve for ϕ^{n+1}:

$$\Delta\phi^{n+1}=\nabla\cdot(\mathbf{u}^n\times\mathbf{B}).\tag{1.9}$$

In *Algorithm* 1, the explicit treatment of the coupling terms $\mathbf{B}\times\nabla\phi$ and $\nabla\cdot(\mathbf{u}\times\mathbf{B})$ preserves unconditional stability, Sect. 4. In *Algorithm* 2, the coupling term $\nabla\cdot(\mathbf{u}\times\mathbf{B})$ is treated implicitly preserving stability (conditionally stable, Sect. 4) and higher accuracy. The derivation of a fully uncoupled, unconditionally, long time stable, second order method for (1.1) is an open problem. Since decoupling is through time discretization and can be applied to various space discretizations, we focus on analyzing the time discretization scheme for the slightly compressible model. For the numerical tests in Sect. 6, we use a standard finite element method for space discretizations.

The rest of this paper is organized as follows. Section 2 introduces some notations and preliminaries. In Sect. 3, we give *á* priori estimates for the slightly compressible model and show its convergence. We analyze the stability and convergence in Sect. 4. Artificial compression methods are fast, efficient and provide effective velocity approximations. However, the pressure can suffer from spurious acoustic oscillations. Section 5 gives a preliminary analysis of the non-physical acoustic waves generated. In Sect. 6, numerical examples are given to test the convergence rates of *Algorithms* 1 and 2, and the non-physical acoustic waves. Section 7 presents the conclusion and open questions.

2 Notations and Preliminaries

In this paper, we partition the time interval $[0, T]$ into m elements (t^n, t^{n+1}) for $n = 0, 1, \cdots, m-1$ where $t^n := n\Delta t$. The time step $\Delta t := \frac{T}{m}$. We denote the $L^2(\Omega)$ inner products by $(\cdot, \cdot)$ and its corresponding norms by $\|\cdot\|$. Let $\|\cdot\|_{L^p}$ denote the $L^p(\Omega)$ norms. We use the standard notations $H^k(\Omega)$ and $H_0^k(\Omega)$ to denote the usual Sobolev spaces over Ω, see [7]. Denote $\|\cdot\|_k$ as the norms in $H^k(\Omega)$. The spaces $H^{-k}(\Omega)$ denote the dual spaces of $H_0^k(\Omega)$. C is a positive constant which differs in different places but independent of mesh size and time step. In addition, we define

$$\begin{aligned} X &:= H_0^1(\Omega)^d = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = 0\}, \\ Q &:= L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}, \\ S &:= H_0^1(\Omega) = \{\psi \in H^1(\Omega) : \psi|_{\partial\Omega} = 0\}, \end{aligned}$$

as the velocity, pressure, and electric potentials spaces, respectively. The divergence free space V is given by

$$V := \{\mathbf{v} \in X : (\nabla \cdot \mathbf{v}, q) = 0 \quad \forall q \in Q\}.$$

We define the trilinear form by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := (B(\mathbf{u}, \mathbf{v}), \mathbf{w}).$$

Integrating by parts, we have

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \tfrac{1}{2}\left(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}\right) - \tfrac{1}{2}\left(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}\right), \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in X, \\ b\left(\mathbf{u}, \mathbf{v}, \mathbf{v}\right) = 0, \quad \forall \mathbf{u}, \mathbf{v} \in X. \end{aligned}$$

The following inequalities will be used frequently in our later analysis.

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &\le C\|\nabla \mathbf{u}\|\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\| \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in X, \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &\le C\|\mathbf{u}\|\|\mathbf{v}\|_2\|\nabla \mathbf{w}\| \quad \forall \mathbf{u}, \mathbf{w} \in X, \mathbf{v} \in X \cap H^2(\Omega). \end{aligned}$$

Furthermore, we introduce the following function spaces and their norms. For $1 \le p < \infty$, $1 \le s \le \infty$,

$$\begin{aligned} L^p(0, T; L^s(\Omega)) &:= \left\{ \mathbf{v}(\mathbf{x}, t) : \left(\int_0^T \|\mathbf{v}(\cdot, t)\|_{L^s}^p dt \right)^{\frac{1}{p}} < \infty \right\}, \\ L^p(0, T; H^k(\Omega)) &:= \left\{ \mathbf{v}(\mathbf{x}, t) : \left(\int_0^T \|\mathbf{v}(\cdot, t)\|_k^p dt \right)^{\frac{1}{p}} < \infty \right\}, \\ L^\infty(0, T; L^s(\Omega)) &:= \left\{ \mathbf{v}(\mathbf{x}, t) : EssSup_{[0,T]}\|\mathbf{v}(\cdot, t)\|_{L^s} < \infty \right\}, \end{aligned}$$

$$L^\infty(0,T;H^k(\Omega)) := \left\{\mathbf{v}(\mathbf{x},t) : EssSup_{[0,T]}\|\mathbf{v}(\cdot,t)\|_k < \infty\right\},$$
$$\|\mathbf{v}\|_{p,k} := \left(\int_0^T \|\mathbf{v}(\cdot,t)\|_k^p dt\right)^{\frac{1}{p}} \qquad for\ \mathbf{v}\in L^p(0,T;H^k(\Omega)).$$

3 Analysis of the Slightly Compressible Model

In this section, we analyze the slightly compressible model (1.3) and (1.4). Firstly, we give *á* prior estimates for its solution $(\mathbf{u}^\varepsilon, p^\varepsilon, \phi^\varepsilon)$. Then, we show that $(\mathbf{u}^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ is an approximation to the true solution of the simplified MHD Eqs. (1.1) and (1.2) when ε goes to 0.

The electric current density $\mathrm{J} := \sigma(-\nabla\phi+\mathbf{u}\times\mathbf{B})$ is an important electromagnetic quantity in MHD flows, see [21,22]. Here the electrical conductivity σ is a constant. For convenient analysis, we define $\mathbf{j} := -\nabla\phi + \mathbf{u}\times\mathbf{B}$ and $\mathbf{j}^\varepsilon := -\nabla\phi^\varepsilon + \mathbf{u}^\varepsilon \times \mathbf{B}$.

Theorem 3.1 *Let* $(\mathbf{u}^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ *be the solution of model* (1.3) *and* (1.4)*, then, with all bounds uniform in* ε*, we have*

$$\begin{aligned}
&\mathbf{u}^\varepsilon \in L^\infty(0,T;L^2(\Omega))\bigcap L^2(0,T;H_0^1(\Omega))\bigcap L^2(0,T;L^6(\Omega)),\\
&\sqrt{\varepsilon}p^\varepsilon \in L^\infty(0,T;L^2(\Omega)),\ \ \phi^\varepsilon \in L^2(0,T;H_0^1(\Omega)),\\
&\mathbf{j}^\varepsilon \in L^2(0,T;L^2(\Omega)),\\
&\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon \text{ and } (\nabla\cdot\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon \in L^2(0,T;L^1(\Omega))\bigcap L^1(0,T;L^{\frac{3}{2}}(\Omega)).
\end{aligned} \tag{3.1}$$

Proof Taking the inner product of the three equations in (1.3) with $\mathbf{u}^\varepsilon$, p^ε, and ϕ^ε, respectively, then summing up the three new equations, we obtain

$$\begin{aligned}
&\frac{1}{2N}\frac{d}{dt}\|\mathbf{u}^\varepsilon\|^2 + \frac{1}{M^2}\|\nabla\mathbf{u}^\varepsilon\|^2 + \|-\nabla\phi^\varepsilon + \mathbf{u}^\varepsilon\times\mathbf{B}\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|p^\varepsilon\|^2\\
&= (\mathbf{f},\mathbf{u}^\varepsilon) \le \frac{1}{2M^2}\|\nabla\mathbf{u}^\varepsilon\|^2 + \frac{M^2}{2}\|\mathbf{f}\|_{-1}^2.
\end{aligned} \tag{3.2}$$

Thus, we have

$$\frac{1}{N}\frac{d}{dt}\|\mathbf{u}^\varepsilon\|^2 + \frac{1}{M^2}\|\nabla\mathbf{u}^\varepsilon\|^2 + \|-\nabla\phi^\varepsilon + \mathbf{u}^\varepsilon\times\mathbf{B}\|^2 + \varepsilon\frac{d}{dt}\|p^\varepsilon\|^2 \le M^2\|\mathbf{f}\|_{-1}^2. \tag{3.3}$$

Integration of (3.3) from 0 to t shows that

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{u}^\varepsilon(t)\|^2 + \frac{1}{M^2}\int_0^t \|\nabla\mathbf{u}^\varepsilon(s)\|^2 ds + \int_0^t \|-\nabla\phi^\varepsilon(s) + \mathbf{u}^\varepsilon(s)\times\mathbf{B}\|^2 ds + \varepsilon\|p^\varepsilon(t)\|^2\\
&\le M^2\|\mathbf{f}\|_{L^2(0,T;H^{-1})}^2 + \frac{1}{N}\|\mathbf{u}^\varepsilon(0)\|^2 + \varepsilon\|p^\varepsilon(0)\|^2, \quad 0 < t \le T.
\end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned}
&\sup_{t\in[0,T]}\frac{1}{N}\|\mathbf{u}^\varepsilon(t)\|^2 + \varepsilon\|p^\varepsilon(t)\|^2 \le c_1,\\
&c_1 = M^2\|\mathbf{f}\|_{L^2(0,T;H^{-1})}^2 + \frac{1}{N}\|\mathbf{u}_0\|^2 + \|p_0\|^2.
\end{aligned} \tag{3.5}$$

Here, since we are interested in small values of ε, we assume $\varepsilon \le 1$.

We also have

$$\int_0^T \|\nabla \mathbf{u}^\varepsilon(s)\|^2 ds \le M^2 c_1, \qquad \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds \le c_1. \tag{3.6}$$

Since

$$\nabla \phi^\varepsilon(s) = -\mathbf{j}^\varepsilon(s) + \mathbf{u}^\varepsilon(s) \times \mathbf{B}, \tag{3.7}$$

using the inequality $\|\mathbf{v_1} \times \mathbf{v_2}\| \le 2\|\mathbf{v_1}\|_{L^\infty}\|\mathbf{v_2}\|$ and the Poincaré inequality, we can get

$$\begin{aligned}\int_0^T \|\nabla \phi^\varepsilon(s)\|^2 ds &\le \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds + \int_0^T \|\mathbf{u}^\varepsilon(s) \times \mathbf{B}\|^2 ds\\ &\le \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds + 4\|\mathbf{B}\|^2_{L^\infty}\int_0^T \|\mathbf{u}^\varepsilon(s)\|^2 ds\\ &\le \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds + C\|\mathbf{B}\|^2_{L^\infty}\int_0^T \|\nabla\mathbf{u}^\varepsilon(s)\|^2 ds\\ &\le (1 + CM^2\|\mathbf{B}\|^2_{L^\infty})c_1.\end{aligned} \tag{3.8}$$

The remaining conditions follow from Hölder's inequality and the Sobolev embedding theorem. □

Next, we derive an error estimate for the sightly compressible model. Denote the error $\mathbf{e_u} = \mathbf{u} - \mathbf{u}^\varepsilon$, $e_\phi = \phi - \phi^\varepsilon$, $e_p = p - p^\varepsilon$, and $\mathbf{e_j} = \mathbf{j} - \mathbf{j}^\varepsilon$. Thus, we have $\mathbf{e_j} = -\nabla \mathbf{e}_\phi + \mathbf{e_u} \times \mathbf{B}$. The following theorem shows that $|\mathbf{u} - \mathbf{u}^\varepsilon|$ tends to zero in $L^\infty(0, T; L^2(\Omega))$ as $\varepsilon \to 0$. The order of convergence is at least $\mathcal{O}(\sqrt{\varepsilon})$.

Theorem 3.2 *Assume that the true solution* $\mathbf{u} \in L^2(0, T; H^2(\Omega))$ *and* $p_t \in L^2(0, T; L^2(\Omega))$, *then we have the following estimate*

$$\begin{aligned}&\|\mathbf{e_u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{e_u}\|_{L^2(0,T;H_0^1(\Omega))} + \|\mathbf{e}_\phi\|_{L^2(0,T;H_0^1(\Omega))} + \|\sqrt{\varepsilon} e_p\|_{L^\infty(0,T;L^2(\Omega))}\\ &\quad \le C\sqrt{\varepsilon}.\end{aligned} \tag{3.9}$$

Proof Subtracting (1.3) from (1.1), we obtain

$$\begin{aligned}&\frac{1}{N}\frac{\partial}{\partial t}\mathbf{e_u} + \frac{1}{N}B(\mathbf{e_u}, \mathbf{u}) + \frac{1}{N}B\left(\mathbf{u}^\varepsilon, \mathbf{e_u}\right) - \frac{1}{M^2}\Delta \mathbf{e_u} + \nabla e_p = \mathbf{B} \times \nabla e_\phi + (\mathbf{e_u} \times \mathbf{B}) \times \mathbf{B},\\ &\varepsilon\frac{\partial}{\partial t}e_p + \nabla\cdot\mathbf{e_u} = \varepsilon p_t,\\ &\Delta e_\phi = \nabla\cdot(\mathbf{e_u} \times \mathbf{B}).\end{aligned} \tag{3.10}$$

Taking the inner product of the three equations in (3.10) with $\mathbf{e_u}$, e_p, and e_ϕ, respectively, then summing up the three new equations, we obtain

$$\begin{aligned}&\frac{1}{2N}\frac{d}{dt}\|\mathbf{e_u}\|^2 + \frac{1}{N}b(\mathbf{e_u}, \mathbf{u}, \mathbf{e_u}) + \frac{1}{M^2}\|\nabla\mathbf{e_u}\|^2 + \|-\nabla e_\phi + \mathbf{e_u} \times \mathbf{B}\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|e_p\|^2\\ &\quad = (\varepsilon p_t, e_p).\end{aligned} \tag{3.11}$$

Thus,

$$\begin{aligned}
&\frac{1}{2N}\frac{d}{dt}\|\mathbf{e_u}\|^2+\frac{1}{M^2}\|\nabla\mathbf{e_u}\|^2+\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2+\frac{\varepsilon}{2}\frac{d}{dt}\|e_p\|^2\\
&=(\varepsilon p_t,e_p)-\frac{1}{N}b(\mathbf{e_u},\mathbf{u},\mathbf{e_u})\\
&\le\varepsilon\|p_t\|\|e_p\|+\frac{C}{N}\|\mathbf{e_u}\|\|\mathbf{u}\|_2\|\nabla\mathbf{e_u}\|\\
&\le\frac{\varepsilon}{2}\|p_t\|^2+\frac{\varepsilon}{2}\|e_p\|^2+\frac{1}{2M^2}\|\nabla\mathbf{e_u}\|^2+\frac{CM^2}{2N^2}\|\mathbf{e_u}\|^2\|\mathbf{u}\|_2^2.
\end{aligned}\tag{3.12}$$

We have

$$\begin{aligned}
&\frac{1}{N}\frac{d}{dt}\|\mathbf{e_u}\|^2+\frac{1}{M^2}\|\nabla\mathbf{e_u}\|^2+\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2+\varepsilon\frac{d}{dt}\|e_p\|^2\\
&\le C\|\mathbf{u}\|_2^2\|\mathbf{e_u}\|^2+\varepsilon\|e_p\|^2+\varepsilon\|p_t\|^2.
\end{aligned}\tag{3.13}$$

Integrate (3.13) from 0 to t to obtain

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{e_u}(t)\|^2+\varepsilon\|e_p(t)\|^2+\frac{1}{M^2}\int_0^t\|\nabla\mathbf{e_u}\|^2ds+\int_0^t\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2ds\\
&\le\frac{1}{N}\|\mathbf{e_u}(0)\|^2+\varepsilon\|e_p(0)\|^2+C\int_0^t\|\mathbf{u}\|_2^2\|\mathbf{e_u}\|^2ds+\int_0^t\varepsilon\|e_p\|^2ds+\int_0^t\varepsilon\|p_t\|^2ds.
\end{aligned}\tag{3.14}$$

Using the Gronwall lemma, we get

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{e_u}\|^2+\varepsilon\|e_p\|^2+\frac{1}{M^2}\int_0^t\|\nabla\mathbf{e_u}\|^2ds+\int_0^t\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2ds\\
&\le C(\frac{1}{N}\|\mathbf{e_u}(0)\|^2+\varepsilon\|e_p(0)\|^2+\int_0^t\varepsilon\|p_t\|^2ds)\\
&\le C\varepsilon(\|e_p(0)\|^2+\int_0^t\|p_t\|^2ds),\ 0<t\le T.
\end{aligned}\tag{3.15}$$

Thus, we have, as $\varepsilon\to 0$,

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\mathbf{e_u}\|\le C\sqrt{\varepsilon}\to 0,\quad\int_0^T\|\nabla\mathbf{e_u}\|^2ds\le C\varepsilon\to 0,\\
&\int_0^T\|\mathbf{e_j}\|^2ds\le C\varepsilon\to 0,\quad\sup_{t\in[0,T]}\|\sqrt{\varepsilon}e_p\|\le C\sqrt{\varepsilon}\to 0.
\end{aligned}\tag{3.16}$$

Since

$$\nabla\mathbf{e}_\phi=-\mathbf{e_j}+\mathbf{e_u}\times\mathbf{B},\tag{3.17}$$

we can also deduce that

$$\int_0^T\|\nabla\mathbf{e}_\phi\|^2ds\le C\varepsilon\to 0,\quad as\ \varepsilon\to 0,\tag{3.18}$$

which completes the proof. □

4 Stability and Error Analysis

In this section, we analyze stability of *Algorithms* 1 and 2, then give an *á* priori error estimate for *Algorithm* 1. Theorem 4.1 below shows that *Algorithm* 1 is unconditionally stable.

Theorem 4.1 *For $(\mathbf{u}^n, p^n, \phi^n)$ satisfying Algorithm 1, we have the following unconditional stability.*

$$\frac{1}{N}\|\mathbf{u}^m\|^2+\frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2+\varepsilon\|p^m\|^2+\varepsilon\sum_{n=0}^{m-1}\|p^{n+1}-p^n\|^2+\frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2$$
$$+\Delta t\|\mathbf{u}^m\times\mathbf{B}\|^2+\Delta t\|\nabla\phi^m\|^2+\Delta t\sum_{n=0}^{m-1}\left(\|-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B}\|^2\right) \quad (4.1)$$
$$\leq M^2\Delta t\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2+\frac{1}{N}\|\mathbf{u}^0\|^2+\varepsilon\|p^0\|^2+\Delta t\|\mathbf{u}^0\times\mathbf{B}\|^2+\Delta t\|\nabla\phi^0\|^2.$$

Proof Taking the inner product of the three equations in (1.5) with $\mathbf{u}^{n+1}$, p^{n+1}, and ϕ^{n+1}, respectively, and multiplying it by $2\Delta t$, we can obtain

$$\begin{aligned}&\frac{1}{N}\left(\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2\right)+\frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2\\&\quad-2\Delta t\left(p^{n+1},\nabla\cdot\mathbf{u}^{n+1}\right)+2\Delta t(-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B},\mathbf{u}^{n+1}\times\mathbf{B})\\&\quad=2\Delta t\left(\mathbf{f}^{n+1},\mathbf{u}^{n+1}\right),\\&\quad\varepsilon\left(\|p^{n+1}\|^2-\|p^n\|^2+\|p^{n+1}-p^n\|^2\right)+2\Delta t\left(p^{n+1},\nabla\cdot\mathbf{u}^{n+1}\right)=0,\\&\quad 2\Delta t\left(-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B},-\nabla\phi^{n+1}\right)=0.\end{aligned} \quad (4.2)$$

Then summing up the three equations in (4.2), we have

$$\begin{aligned}&\frac{1}{N}\left(\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2\right)+\frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2\\&\quad+2\Delta t\left(-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B},\mathbf{u}^{n+1}\times\mathbf{B}\right)+2\Delta t\left(-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B},-\nabla\phi^{n+1}\right)\\&\quad+\varepsilon\left(\|p^{n+1}\|^2-\|p^n\|^2+\|p^{n+1}-p^n\|^2\right)=2\Delta t\left(\mathbf{f}^{n+1},\mathbf{u}^{n+1}\right).\end{aligned} \quad (4.3)$$

By using the following identity

$$2(a+b,b)+2(c+d,c)=c^2-a^2+b^2-d^2+(a+b)^2+(c+d)^2, \quad (4.4)$$

we get

$$\begin{aligned}&2\Delta t\left(-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B},\mathbf{u}^{n+1}\times\mathbf{B}\right)+2\Delta t\left(-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B},-\nabla\phi^{n+1}\right)\\&\quad=\Delta t\|\mathbf{u}^{n+1}\times\mathbf{B}\|^2-\Delta t\|\mathbf{u}^n\times\mathbf{B}\|^2+\Delta t\|\nabla\phi^{n+1}\|^2-\Delta t\|\nabla\phi^n\|^2\\&\quad+\Delta t\left(\|-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B}\|^2\right).\end{aligned} \quad (4.5)$$

Thus, (4.3) can be rewritten as

$$\begin{aligned}
&\frac{1}{N}\left(\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2\right)+\varepsilon\left(\|p^{n+1}\|^2-\|p^n\|^2+\|p^{n+1}-p^n\|^2\right)\\
&+\frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2+\Delta t\|\mathbf{u}^{n+1}\times\mathbf{B}\|^2-\Delta t\|\mathbf{u}^n\times\mathbf{B}\|^2+\Delta t\|\nabla\phi^{n+1}\|^2-\Delta t\|\nabla\phi^n\|^2\\
&+\Delta t\left(\|-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B}\|^2\right)\\
&=2\Delta t\left(\mathbf{f}^{n+1},\mathbf{u}^{n+1}\right)\le M^2\Delta t\|\mathbf{f}^{n+1}\|_{-1}^2+\frac{\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2.
\end{aligned}\tag{4.6}$$

Finally, summing (4.6) from $n=0$ to $n=m-1$ completes the proof. □

The following theorem shows that $Algorithm$ 2 is stable with a condition, relating the time step Δt with the problem data. Recall $\mathbf{j}^n := -\nabla\phi^n+\mathbf{u}^n\times\mathbf{B}$.

Theorem 4.2 *For* $(\mathbf{u}^n, p^n, \phi^n)$ *satisfying* $Algorithm$ 2*, if time step* Δt *satisfies*

$$\Delta t<\frac{1}{2N\left(1+C_p^2M^2\|\mathbf{B}\|_{L^\infty}^2\right)\|\mathbf{B}\|_{L^\infty}^2},\tag{4.7}$$

we have the following stability.

$$\begin{aligned}
&\frac{1}{2N}\|\mathbf{u}^m\|^2+\frac{1}{2N}\|2\mathbf{u}^m-\mathbf{u}^{m-1}\|^2+\frac{\Delta t}{2M^2}\sum_{n=1}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2\\
&+\Delta t\sum_{n=1}^{m-1}\|\mathbf{j}^{n+1}\|^2+\Delta t\sum_{n=1}^{m-1}\|2\mathbf{j}^n-\mathbf{j}^{n-1}\|^2\\
&\le\frac{1}{2N}\|\mathbf{u}^1\|^2+\frac{1}{2N}\|2\mathbf{u}^1-\mathbf{u}^0\|^2+2M^2\Delta t\sum_{n=1}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2.
\end{aligned}\tag{4.8}$$

Proof Taking the inner product of the three equations in (1.6) with $\mathbf{u}^{n+1}$, p^{n+1}, and ϕ^{n+1}, respectively, and multiplying it by $2\Delta t$, we can obtain

$$\begin{aligned}
&\frac{1}{2N}\left(3\|\mathbf{u}^{n+1}\|^2-4\|\mathbf{u}^n\|^2+\|\mathbf{u}^{n-1}\|^2\right)+\frac{1}{N}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2-\frac{1}{N}\|\mathbf{u}^n-\mathbf{u}^{n-1}\|^2\\
&+\frac{1}{2N}\|\mathbf{u}^{n+1}-2\mathbf{u}^n+\mathbf{u}^{n-1}\|^2+\frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2-2\Delta t\left(p^{n+1},\nabla\cdot\mathbf{u}^{n+1}\right)\\
&+2\Delta t\left(-\nabla\left(2\phi^n-\phi^{n-1}\right)+\mathbf{u}^{n+1}\times\mathbf{B},\mathbf{u}^{n+1}\times\mathbf{B}\right)\\
&=2\Delta t\left(\mathbf{f}^{n+1},\mathbf{u}^{n+1}\right),\\
&\varepsilon\left(\|p^{n+1}\|^2-\|p^n\|^2+\|p^{n+1}-p^n\|^2\right)+2\Delta t\left(p^{n+1},\nabla\cdot\mathbf{u}^{n+1}\right)=0,\\
&2\Delta t\left(-\nabla\phi^{n+1}+\mathbf{u}^{n+1}\times\mathbf{B},-\nabla\phi^{n+1}\right)=0,
\end{aligned}\tag{4.9}$$

where we use the identity

$$\frac{1}{2}(3a-4b+c)a=\frac{1}{4}\left(3a^2-4b^2+c^2\right)+\frac{1}{2}(a-b)^2-\frac{1}{2}(b-c)^2+\frac{1}{4}(a-2b+c)^2.$$

The key issue is to deal with the term $2\Delta t(-\nabla(2\phi^n - \phi^{n-1}) + \mathbf{u}^{n+1} \times \mathbf{B}, \mathbf{u}^{n+1} \times \mathbf{B})$. We have

$$
\begin{aligned}
&2\Delta t \left(-\nabla(2\phi^n - \phi^{n-1}) + \mathbf{u}^{n+1} \times \mathbf{B}, \mathbf{u}^{n+1} \times \mathbf{B}\right) \\
&\quad = 2\Delta t \left(2\mathbf{j}^n - \mathbf{j}^{n-1} + \left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}, \mathbf{u}^{n+1} \times \mathbf{B}\right) \\
&\quad = 2\Delta t \left(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1} \times \mathbf{B}\right) + 2\Delta t \left(\left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}, \mathbf{u}^{n+1} \times \mathbf{B}\right) \\
&\quad = 2\Delta t \left(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1} \times \mathbf{B}\right) \\
&\qquad + \Delta t \left(\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 - \|\left(2\mathbf{u}^n - \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2 + \|\left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2\right).
\end{aligned} \tag{4.10}
$$

From the third equation in (1.6), we have

$$
\left(2\mathbf{j}^n - \mathbf{j}^{n-1}, -\nabla\psi\right) = 0, \quad \forall \psi \in S. \tag{4.11}
$$

Taking $\psi = \phi^{n+1}$ and adding it to the term $2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1} \times \mathbf{B})$ gives

$$
\begin{aligned}
&2\Delta t \left(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1} \times \mathbf{B}\right) = 2\Delta t \left(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{j}^{n+1}\right) \\
&\quad = \Delta t \left(\|\mathbf{j}^{n+1}\|^2 + \|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 - \|\mathbf{j}^{n+1} - 2\mathbf{j}^n + \mathbf{j}^{n-1}\|^2\right) \\
&\quad = \Delta t \left(\|\mathbf{j}^{n+1}\|^2 + \|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 - \|(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) \times \mathbf{B}\|^2\right. \\
&\qquad \left. + \|\nabla\phi^{n+1} - 2\nabla\phi^n + \nabla\phi^{n-1}\|^2\right),
\end{aligned} \tag{4.12}
$$

where we use $\|\mathbf{j}^i\|^2 = \|\mathbf{u}^i \times \mathbf{B}\|^2 - \|\nabla\phi^i\|^2, \ \forall i \leq n+1$. Combining (4.9), (4.10) and (4.12) yields

$$
\begin{aligned}
&\frac{1}{2N}\left(3\|\mathbf{u}^{n+1}\|^2 - 4\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n-1}\|^2\right) + \frac{1}{N}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 - \frac{1}{N}\|\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 \\
&\quad + \frac{1}{2N}\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 + \frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2 \\
&\quad + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 + \Delta t\|\mathbf{j}^{n+1}\|^2 + \Delta t\|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 \\
&\quad = 2\Delta t \left(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}\right) + \Delta t\|\left(2\mathbf{u}^n - \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2.
\end{aligned} \tag{4.13}
$$

For an arbitrary $\delta > 0$, the term $\Delta t\|(2\mathbf{u}^n - \mathbf{u}^{n-1}) \times \mathbf{B}\|^2$ can be bounded by

$$
\begin{aligned}
\Delta t\|\left(2\mathbf{u}^n - \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2 &= \Delta t\|\left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2 + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 \\
&\quad -2\Delta t \left(\left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}, \mathbf{u}^{n+1} \times \mathbf{B}\right) \\
&\leq \Delta t\|\left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2 + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 \\
&\quad + \frac{\Delta t}{\delta^2}\|\left(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right) \times \mathbf{B}\|^2 + \Delta t\delta^2\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 \\
&\leq \Delta t\left(1 + \frac{1}{\delta^2}\right)\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 \\
&\quad + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 + \Delta t C_p^2\delta^2\|\mathbf{B}\|_{L^\infty}^2\|\nabla\mathbf{u}^{n+1}\|^2,
\end{aligned} \tag{4.14}
$$

where we use the Poincaré inequality $\|\mathbf{u}\| \leq C_p\|\nabla\mathbf{u}\|,\ \forall\mathbf{u}\in X$. By taking $\delta = \frac{1}{C_p M\|\mathbf{B}\|_{L^\infty}}$, we can obtain

$$\begin{aligned}
&\frac{1}{2N}\left(3\|\mathbf{u}^{n+1}\|^2 - 4\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n-1}\|^2\right) + \frac{1}{N}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2 - \frac{1}{N}\|\mathbf{u}^n - \mathbf{u}^{n-1}\|^2\\
&\quad + \left(\frac{1}{2N} - \Delta t\left(1+\frac{1}{\varepsilon^2}\right)\|\mathbf{B}\|^2_{L^\infty}\right)\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2\\
&\quad + \frac{\Delta t}{2M^2}\|\nabla\mathbf{u}^{n+1}\|^2 + \Delta t\|\mathbf{j}^{n+1}\|^2 + \Delta t\|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2\\
&\leq 2M^2\Delta t\|\mathbf{f}^{n+1}\|^2_{-1}.
\end{aligned} \tag{4.15}$$

Finally, under the condition (4.7), summing (4.15) from $n = 1$ to $n = m-1$ completes the proof. □

Next, we analyze the convergency of $Algorithm$ 1 and give an $\acute{a}$ priori error estimate for $Algorithm$ 1. Since the error analysis of $Algorithm$ 2 is similar to that of $Algorithm$ 1 but considerably longer, we omit it. Denote $\mathbf{e}^n_{\mathbf{u}} = \mathbf{u}(t^n) - \mathbf{u}^n$, $e^n_p = p(t^n) - p^n$, and $e^n_\phi = \phi(t^n) - \phi^n$. The following theorem not only shows the convergence order of $Algorithm$ 1 but can also indicates a way to choose the value of ε.

Theorem 4.3 *Assume that the true solution* $(\mathbf{u}, p, \phi)$ *satisfies the following regularity*

$$\begin{aligned}
&\mathbf{u}\in L^\infty(0,T;H^2(\Omega)),\quad \mathbf{u}_t\in L^2(0,T;H^1(\Omega)),\quad \mathbf{u}_{tt}\in L^2(0,T;H^{-1}(\Omega)),\\
&p_t\in L^\infty(0,T;L^2(\Omega)),\quad p_{tt}\in L^2(0,T;L^2(\Omega)),\quad \phi_t\in L^2(0,T;H^1(\Omega)).
\end{aligned} \tag{4.16}$$

For $(\mathbf{u}^n, p^n, \phi^n)$ *satisfying* $Algorithm$ 1*, we have the following estimate*

$$\begin{aligned}
&\frac{1}{2N}\|\mathbf{e}^m_{\mathbf{u}}\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{e}^{n+1}_{\mathbf{u}}\|^2 + \Delta t\sum_{n=0}^{m-1}\|\nabla e^{n+1}_\phi\|^2\\
&+ \Delta t\sum_{n=0}^{m-1}\left(\|-\nabla e^n_\phi + \mathbf{e}^{n+1}_{\mathbf{u}}\times\mathbf{B}\|^2 + \|-\nabla e^{n+1}_\phi + \mathbf{e}^n_{\mathbf{u}}\times\mathbf{B}\|^2\right) \leq C\left(\Delta t^2 + \varepsilon^2\right),
\end{aligned} \tag{4.17}$$

for sufficiently small Δt.

Proof At time t^{n+1}, the true solution $(\mathbf{u}, p, \phi)$ satisfies

$$\begin{aligned}
&\frac{1}{N}\left(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} + B\left(\mathbf{u}\left(t^{n+1}\right), \mathbf{u}(t^{n+1})\right)\right) - \frac{1}{M^2}\Delta\mathbf{u}\left(t^{n+1}\right) + \nabla p\left(t^{n+1}\right)\\
&\quad - \mathbf{B}\times\nabla\phi\left(t^{n+1}\right) - \left(\mathbf{u}\left(t^{n+1}\right)\times\mathbf{B}\right)\times\mathbf{B} = \mathbf{f}^{n+1} + \frac{1}{N}\left(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t\left(t^{n+1}\right)\right),\\
&\varepsilon\frac{p\left(t^{n+1}\right) - p\left(t^n\right)}{\Delta t} + \nabla\cdot\mathbf{u}\left(t^{n+1}\right) = \frac{\varepsilon}{\Delta t}\int_{t^n}^{t^{n+1}} p_t dt,\ \Delta\phi\left(t^{n+1}\right) = \nabla\cdot\left(\mathbf{u}(t^{n+1}\right)\times\mathbf{B}).
\end{aligned} \tag{4.18}$$

Subtract (1.5) from (4.18) to obtain

$$
\begin{aligned}
&\frac{1}{N}\left(\frac{\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^{n}}{\Delta t}+B\left(\mathbf{e}_{\mathbf{u}}^{n},\mathbf{u}(t^{n+1})\right)+B\left(\mathbf{u}^{n},\mathbf{e}_{\mathbf{u}}^{n+1}\right)+B\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}),\mathbf{u}(t^{n+1})\right)\right)\\
&\quad-\frac{1}{M^2}\Delta\mathbf{e}_{\mathbf{u}}^{n+1}+\nabla e_p^{n+1}-\mathbf{B}\times\nabla e_\phi^{n}-\mathbf{B}\times\left(\nabla\phi(t^{n+1})-\nabla\phi(t^{n})\right)-\left(\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\right)\times\mathbf{B}\\
&=\frac{1}{N}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n})}{\Delta t}-\mathbf{u}_t(t^{n+1})\right),\\
&\varepsilon\frac{e_p^{n+1}-e_p^{n}}{\Delta t}+\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1}=\frac{\varepsilon}{\Delta t}\int_{t^n}^{t^{n+1}}p_t\,dt,\\
&\Delta e_\phi^{n+1}=\nabla\cdot\left(\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\right)+\nabla\cdot\left(\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n})\right)\times\mathbf{B}\right).
\end{aligned}
\tag{4.19}
$$

Taking the inner product of the three equations in (4.19) with $\mathbf{e}_{\mathbf{u}}^{n+1}$, e_p^{n+1}, and e_ϕ^{n+1}, respectively, then summing up the three new equations and multiplying it by $2\Delta t$, we obtain

$$
\begin{aligned}
&\frac{1}{N}\left(\|\mathbf{e}_{\mathbf{u}}^{n+1}\|^2-\|\mathbf{e}_{\mathbf{u}}^{n}\|^2+\|\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^{n}\|^2\right)+\varepsilon\left(\|e_p^{n+1}\|^2-\|e_p^{n}\|^2+\|e_p^{n+1}-e_p^{n}\|^2\right)\\
&\quad+\frac{2\Delta t}{M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2+\Delta t\|\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2-\Delta t\|\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\|^2+\Delta t\|\nabla e_\phi^{n+1}\|^2-\Delta t\|\nabla e_\phi^{n}\|^2\\
&\quad+\Delta t(\|-\nabla e_\phi^{n}+\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2+\|-\nabla e_\phi^{n+1}+\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\|^2)\\
&=\frac{2\Delta t}{N}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n})}{\Delta t}-\mathbf{u}_t(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)-\frac{2\Delta t}{N}b\left(\mathbf{e}_{\mathbf{u}}^{n},\mathbf{u}(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad-\frac{2\Delta t}{N}b\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}),\mathbf{u}(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)+2\Delta t\left(\nabla\phi(t^{n+1})-\nabla\phi(t^{n}),\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\right)\\
&\quad+2\Delta t\left((\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}))\times\mathbf{B},\nabla e_\phi^{n+1}\right)+2\varepsilon\left(\int_{t^n}^{t^{n+1}}p_t\,dt,e_p^{n+1}\right),
\end{aligned}
\tag{4.20}
$$

where we use the identity (4.4) again. We have that

$$
2\Delta t\|\nabla e_\phi^{n+1}\|^2=2\Delta t\left(\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n})\right)\times\mathbf{B},\nabla e_\phi^{n+1}\right)+2\Delta t\left(\nabla e_\phi^{n+1},\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\right)
\tag{4.21}
$$

Adding (4.20) and (4.21), we obtain

$$
\begin{aligned}
&\frac{1}{N}\left(\|\mathbf{e}_{\mathbf{u}}^{n+1}\|^2-\|\mathbf{e}_{\mathbf{u}}^{n}\|^2+\|\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^{n}\|^2\right)+\varepsilon\left(\|e_p^{n+1}\|^2-\|e_p^{n}\|^2+\|e_p^{n+1}-e_p^{n}\|^2\right)\\
&\quad+\frac{2\Delta t}{M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2+2\Delta t\|\nabla e_\phi^{n+1}\|^2+\Delta t\|\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2-\Delta t\|\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\|^2\\
&\quad+\Delta t\|\nabla e_\phi^{n+1}\|^2-\Delta t\|\nabla e_\phi^{n}\|^2+\Delta t\left(\|-\nabla e_\phi^{n}+\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2+\|-\nabla e_\phi^{n+1}+\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\|^2\right)\\
&=\frac{2\Delta t}{N}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n})}{\Delta t}-\mathbf{u}_t(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)-\frac{2\Delta t}{N}b(\mathbf{e}_{\mathbf{u}}^{n},\mathbf{u}(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1})\\
&\quad-\frac{2\Delta t}{N}b\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}),\mathbf{u}(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)+2\Delta t\left(\nabla\phi(t^{n+1})-\nabla\phi(t^{n}),\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\right)\\
&\quad+4\Delta t\left((\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}))\times\mathbf{B},\nabla e_\phi^{n+1}\right)+2\Delta t\left(\nabla e_\phi^{n+1},\mathbf{e}_{\mathbf{u}}^{n}\times\mathbf{B}\right)\\
&\quad+2\varepsilon\left(\int_{t^n}^{t^{n+1}}p_t\,dt,e_p^{n+1}\right).
\end{aligned}
\tag{4.22}
$$

Next, we need to bound all the terms on the RHS of (4.22). For arbitrary $\sigma > 0, \sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$, we have the following estimates. The first term can be bounded as

$$\frac{2\Delta t}{N}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t}-\mathbf{u}_t(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)\leq C\Delta t\|\frac{\mathbf{u}\left(t^{n+1}\right)-\mathbf{u}\left(t^n\right)}{\Delta t}-\mathbf{u}_t\left(t^{n+1}\right)\|_{-1}^2 + \sigma\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2. \tag{4.23}$$

The nonlinear terms can be bounded as

$$\begin{aligned}
&-\frac{2\Delta t}{N}b\left(\mathbf{e}_{\mathbf{u}}^n,\mathbf{u}(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)\leq C\Delta t\|\mathbf{e}_{\mathbf{u}}^n\|\|\mathbf{u}(t^{n+1})\|_2\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|\\
&\leq C\Delta t\|\mathbf{u}(t^{n+1})\|_2^2\|\mathbf{e}_{\mathbf{u}}^n\|^2+\sigma\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2, &(4.24)\\
&-\frac{2\Delta t}{N}b\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n),\mathbf{u}(t^{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)\leq C\Delta t\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|\|\mathbf{u}(t^{n+1})\|_2\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|\\
&\leq C\Delta t\|\mathbf{u}(t^{n+1})\|_2^2\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2+\sigma\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2. &(4.25)
\end{aligned}$$

We bound the term $2\Delta t(\nabla\phi(t^{n+1})-\nabla\phi(t^n),\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B})$ as follows.

$$\begin{aligned}
&2\Delta t\left(\nabla\phi(t^{n+1})-\nabla\phi(t^n),\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\right)\leq 4\Delta t\|\mathbf{e}_{\mathbf{u}}^{n+1}\|\|\mathbf{B}\|_{L^\infty}\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|\\
&\leq C\Delta t\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|^2+\sigma\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2.
\end{aligned}\tag{4.26}$$

The term $4\Delta t((\mathbf{u}(t^{n+1})-\mathbf{u}(t^n))\times\mathbf{B},\nabla e_\phi^{n+1})$ is bounded as

$$\begin{aligned}
&4\Delta t\left(\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\right)\times\mathbf{B},\nabla e_\phi^{n+1}\right)\leq 8\Delta t\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|\|\mathbf{B}\|_{L^\infty}\|\nabla e_\phi^{n+1}\|\\
&\leq C\Delta t\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2+\sigma_1\Delta t\|\nabla e_\phi^{n+1}\|^2.
\end{aligned}\tag{4.27}$$

We also bound the term $2\Delta t(\nabla e_\phi^{n+1},\mathbf{e}_{\mathbf{u}}^n\times\mathbf{B})$ by

$$2\Delta t\left(\nabla e_\phi^{n+1},\mathbf{e}_{\mathbf{u}}^n\times\mathbf{B}\right)\leq 4\Delta t\|\mathbf{e}_{\mathbf{u}}^n\|\|\mathbf{B}\|_{L^\infty}\|\nabla e_\phi^{n+1}\|\leq C\Delta t\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{e}_{\mathbf{u}}^n\|^2+\sigma_1\Delta t\|\nabla e_\phi^{n+1}\|^2. \tag{4.28}$$

Lastly, we need to bound the remaining term $2\varepsilon(\int_{t^n}^{t^{n+1}}p_t dt, e_p^{n+1})$. It is known (see [28]) that if $p_t(t), p_{tt}(t)\in L^2(\Omega)/\mathbb{R}$, there exists an unique $\varphi(t)\in H_0^1(\Omega)$, such that

$$\nabla\cdot\varphi(t)=p_t(t),\quad \nabla\cdot\varphi_t(t)=p_{tt}(t) \tag{4.29}$$

and

$$\|\varphi(t)\|_1\leq C\|p_t(t)\|,\quad \|\varphi_t(t)\|_1\leq C\|p_{tt}(t)\|\qquad for\ all\ t\in[0,T]. \tag{4.30}$$

From (4.19), we have, at time t^{n+1},

$$\begin{aligned}
\nabla e_p^{n+1}=\;&-\frac{1}{N}\frac{\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^n}{\Delta t}+\frac{1}{M^2}\Delta\mathbf{e}_{\mathbf{u}}^{n+1}-\frac{1}{N}B\left(\mathbf{e}_{\mathbf{u}}^n,\mathbf{u}(t^{n+1})\right)-\frac{1}{N}B\left(\mathbf{u}^n,\mathbf{e}_{\mathbf{u}}^{n+1}\right)\\
&-\frac{1}{N}B\left(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n),\mathbf{u}(t^{n+1})\right)+\mathbf{B}\times\nabla e_\phi^n+\mathbf{B}\times\left(\nabla\phi(t^{n+1})-\nabla\phi(t^n)\right)\\
&+(\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B})\times\mathbf{B}+\frac{1}{N}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t}-\mathbf{u}_t(t^{n+1})\right).
\end{aligned}\tag{4.31}$$

Thus, we obtain

$$
\begin{aligned}
&2\varepsilon\left(\int_{t^n}^{t^{n+1}} p_t dt, e_p^{n+1}\right) = 2\varepsilon\left(\int_{t^n}^{t^{n+1}} \nabla\cdot\varphi(t)dt, e_p^{n+1}\right) = -2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \nabla e_p^{n+1}\right)\\
&= -2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}))\right)\\
&\quad -2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{M^2}\Delta \mathbf{e}_{\mathbf{u}}^{n+1}\right) - 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B}\times\nabla e_\phi^n)\\
&\quad -2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B}\times(\nabla\phi(t^{n+1})-\nabla\phi(t^n))\right)\\
&\quad -2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, (\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B})\times\mathbf{B}\right) + 2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\frac{\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^n}{\Delta t}\right)\\
&\quad +2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{u}^n, \mathbf{e}_{\mathbf{u}}^{n+1})\right) + 2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1}))\right)\\
&\quad +2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n), \mathbf{u}(t^{n+1}))\right).
\end{aligned} \tag{4.32}
$$

Next, we need to bound all the terms on the RHS of (4.32). First, we have

$$
\begin{aligned}
&-2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\right)\right)\\
&\le \frac{2\varepsilon}{N}\|\int_{t^n}^{t^{n+1}} \varphi dt\|_1 \|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}\\
&\le C\varepsilon\sqrt{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|\varphi\|_1^2 dt\right)^{\frac{1}{2}} \|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}\\
&\le C\varepsilon^2\int_{t^n}^{t^{n+1}} \|\varphi\|_1^2 dt + \Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2\\
&\le C\varepsilon^2\int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2.
\end{aligned} \tag{4.33}
$$

For the term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{M^2}\Delta \mathbf{e}_{\mathbf{u}}^{n+1})$, we have

$$
\begin{aligned}
-2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{M^2}\Delta \mathbf{e}_{\mathbf{u}}^{n+1}\right) &= 2\varepsilon\left(\int_{t^n}^{t^{n+1}} \nabla\varphi dt, \frac{1}{M^2}\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right)\\
&\le \frac{2\varepsilon}{M^2}\|\int_{t^n}^{t^{n+1}} \nabla\varphi dt\|\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2\\
&\le C\varepsilon\sqrt{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|\nabla\varphi\|^2 dt\right)^{\frac{1}{2}} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2\\
&\le C\varepsilon^2\int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma\Delta t\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2.
\end{aligned} \tag{4.34}
$$

For the term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times \nabla e_\phi^n)$, we have

$$
\begin{aligned}
&-2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times \nabla e_\phi^n\right) \le 4\varepsilon\|\mathbf{B}\|_{L^\infty}\|\int_{t^n}^{t^{n+1}} \varphi dt\|\|\nabla e_\phi^n\| \\
&\le C\varepsilon\sqrt{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|\varphi\|^2 dt\right)^{\frac{1}{2}} \|\nabla e_\phi^n\| \le C\varepsilon^2\int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma_2\Delta t\|\nabla e_\phi^n\|^2.
\end{aligned} \tag{4.35}
$$

The term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times (\nabla\phi(t^{n+1}) - \nabla\phi(t^n)))$ is bounded as

$$
\begin{aligned}
&-2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times (\nabla\phi(t^{n+1})\right. \\
&\left.-\nabla\phi(t^n))\right) \le C\varepsilon\sqrt{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|\varphi\|^2 dt\right)^{\frac{1}{2}} \|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\| \\
&\le C\varepsilon^2\int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \Delta t\|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\|^2.
\end{aligned} \tag{4.36}
$$

The term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, (\mathbf{e}_\mathbf{u}^{n+1} \times \mathbf{B}) \times \mathbf{B})$ is by

$$
\begin{aligned}
&-2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, (\mathbf{e}_\mathbf{u}^{n+1} \times \mathbf{B}) \times \mathbf{B}\right) \le C\varepsilon\sqrt{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|\varphi\|^2 dt\right)^{\frac{1}{2}} \|\mathbf{e}_\mathbf{u}^{n+1}\| \\
&\le C\varepsilon^2\int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma\Delta t\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|^2.
\end{aligned} \tag{4.37}
$$

We bound the term $2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t})$ by

$$
\begin{aligned}
&2\varepsilon\left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t}\right) \\
&= \frac{2\varepsilon}{N\Delta t}\left[\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}\right) - \left(\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n\right)\right] - \frac{2\varepsilon}{N\Delta t}\left(\int_{t^n}^{t^{n+1}} \varphi dt - \int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n\right) \\
&\le \frac{2\varepsilon}{N\Delta t}\left[\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}\right) - \left(\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n\right)\right] + \frac{2\varepsilon\Delta t}{N}|(\varphi_t(\xi_n), \mathbf{e}_\mathbf{u}^n)| \\
&\le \frac{2\varepsilon}{N\Delta t}\left[\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}\right) - \left(\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n\right)\right] + \frac{2\varepsilon\Delta t}{N}\|\varphi_t(\xi_n)\|\|\mathbf{e}_\mathbf{u}^n\| \\
&\le \frac{2\varepsilon}{N\Delta t}\left[\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}\right) - \left(\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n\right)\right] + C\varepsilon^2\Delta t\|p_{tt}(\xi_n)\|^2 + \sigma_3\Delta t\|\nabla\mathbf{e}_\mathbf{u}^n\|^2,
\end{aligned} \tag{4.38}
$$

where $\xi_n \in (t^{n-1}, t^{n+1})$. We bound the term $2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N} B(\mathbf{u}^n, \mathbf{e}_{\mathbf{u}}^{n+1}))$ as

$$\begin{aligned} 2\varepsilon \left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N} B(\mathbf{u}^n, \mathbf{e}_{\mathbf{u}}^{n+1}) \right) &\leq C\varepsilon \| \int_{t^n}^{t^{n+1}} \nabla \varphi dt \| \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\| \|\nabla \mathbf{u}^n\| \\ &\leq C\varepsilon \sqrt{\Delta t} \left(\int_{t^n}^{t^{n+1}} \|\nabla \varphi\|^2 dt \right)^{\frac{1}{2}} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\| \|\nabla \mathbf{u}^n\| \\ &\leq C\varepsilon^2 \|\nabla \mathbf{u}^n\|^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma \Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2. \end{aligned} \tag{4.39}$$

The term $2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N} B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1})))$ is bounded as

$$\begin{aligned} 2\varepsilon \left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N} B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1})) \right) &\leq C\varepsilon \| \int_{t^n}^{t^{n+1}} \nabla \varphi dt \| \|\nabla \mathbf{e}_{\mathbf{u}}^n\| \|\nabla \mathbf{u}(t^{n+1})\| \\ &\leq C\varepsilon^2 \|\nabla \mathbf{u}(t^{n+1})\|^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma_3 \Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^n\|^2. \end{aligned} \tag{4.40}$$

For the last term, we have

$$\begin{aligned} &2\varepsilon \left(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N} B(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1})) \right) \\ &\leq C\varepsilon \sqrt{\Delta t} \left(\int_{t^n}^{t^{n+1}} \|\nabla \varphi\|^2 dt \right)^{\frac{1}{2}} \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\| \|\mathbf{u}(t^{n+1})\|_2 \\ &\leq C\varepsilon^2 \|\mathbf{u}(t^{n+1})\|_2^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \Delta t \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|^2. \end{aligned} \tag{4.41}$$

Combining (4.33) and (4.41), we have

$$\begin{aligned} 2\varepsilon \left(\int_{t^n}^{t^{n+1}} p_t dt, e_p^{n+1} \right) &\leq 3\sigma \Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \sigma_2 \Delta t \|\nabla e_\phi^n\|^2 + 2\sigma_3 \Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^n\|^2 \\ &+ \Delta t \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|^2 + \Delta t \left\| \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}) \right\|_{-1}^2 \\ &+ \Delta t \|\nabla \phi(t^{n+1}) - \nabla \phi(t^n)\|^2 + \frac{2\varepsilon}{N\Delta t} \left[\left(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_{\mathbf{u}}^{n+1} \right) - \left(\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_{\mathbf{u}}^n \right) \right] \\ &+ C\varepsilon^2 \Delta t \|p_{tt}(\xi_n)\|^2 + C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + C\varepsilon^2 \|\nabla \mathbf{u}^n\|^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt. \end{aligned} \tag{4.42}$$

Set $\sigma = \frac{1}{14M^2}, \sigma_1 = \frac{1}{4}, \sigma_2 = \frac{1}{2}, \sigma_3 = \frac{1}{4M^2}$. Combining (4.22)–(4.28) and (4.42), we have

$$\begin{aligned} &\frac{1}{N} \left(\|\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{e}_{\mathbf{u}}^n\|^2 + \|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n\|^2 \right) + \varepsilon \left(\|e_p^{n+1}\|^2 - \|e_p^n\|^2 + \|e_p^{n+1} - e_p^n\|^2 \right) \\ &+ \frac{\Delta t}{M^2} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t \|\nabla e_\phi^{n+1}\|^2 + \Delta t \|\mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 - \Delta t \|\mathbf{e}_{\mathbf{u}}^n \times \mathbf{B}\|^2 \\ &+ \frac{\Delta t}{2M^2} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 - \frac{\Delta t}{2M^2} \|\nabla \mathbf{e}_{\mathbf{u}}^n\|^2 + \frac{3}{2} \Delta t \|\nabla e_\phi^{n+1}\|^2 - \frac{3}{2} \Delta t \|\nabla e_\phi^n\|^2 \end{aligned}$$

$$+\Delta t\left(\|-\nabla e_\phi^n+\mathbf{e}_\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla e_\phi^{n+1}+\mathbf{e}_\mathbf{u}^n\times\mathbf{B}\|^2\right)$$
$$\leq C\Delta t\left(\|\mathbf{u}(t^{n+1})\|_2^2+\|\mathbf{B}\|_{L^\infty}^2\right)\|\mathbf{e}_\mathbf{u}^n\|^2+C\Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t}-\mathbf{u}_t(t^{n+1})\|_{-1}^2$$
$$+C\Delta t\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|^2+C\Delta t(1+\|\mathbf{u}(t^{n+1})\|_2^2+\|\mathbf{B}\|_{L^\infty}^2)\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2$$
$$+\frac{2\varepsilon}{N\Delta t}\left[\left(\int_{t^n}^{t^{n+1}}\varphi dt,\mathbf{e}_\mathbf{u}^{n+1}\right)-\left(\int_{t^{n-1}}^{t^n}\varphi dt,\mathbf{e}_\mathbf{u}^n\right)\right]$$
$$+C\varepsilon^2\Delta t\|p_{tt}(\xi_n)\|^2+C\varepsilon^2\int_{t^n}^{t^{n+1}}\|p_t\|^2dt+C\varepsilon^2\|\nabla\mathbf{u}^n\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2dt. \tag{4.43}$$

Summing (4.43) from $n=0$ to $n=m-1$, we obtain

$$\frac{1}{N}\|\mathbf{e}_\mathbf{u}^m\|^2+\frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{e}_\mathbf{u}^{n+1}-\mathbf{e}_\mathbf{u}^n\|^2+\varepsilon\|e_p^m\|^2+\sum_{n=0}^{m-1}\|e_p^{n+1}-e_p^n\|^2+\frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|^2$$
$$+\Delta t\sum_{n=0}^{m-1}\|\nabla e_\phi^{n+1}\|^2+\Delta t\|\mathbf{e}_\mathbf{u}^m\times\mathbf{B}\|^2+\frac{3}{2}\Delta t\|\nabla e_\phi^m\|^2+\frac{\Delta t}{2M^2}\|\nabla\mathbf{e}_\mathbf{u}^m\|^2$$
$$+\Delta t\sum_{n=0}^{m-1}\left(\|-\nabla e_\phi^n+\mathbf{e}_\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla e_\phi^{n+1}+\mathbf{e}_\mathbf{u}^n\times\mathbf{B}\|^2\right)$$
$$\leq\frac{1}{N}\|\mathbf{e}_\mathbf{u}^0\|^2+\varepsilon\|e_p^0\|^2+\Delta t\|\mathbf{e}_\mathbf{u}^0\times\mathbf{B}\|^2+\frac{\Delta t}{2M^2}\|\nabla\mathbf{e}_\mathbf{u}^0\|^2+\frac{3}{2}\Delta t\|\nabla e_\phi^0\|^2+C\Delta t\sum_{n=0}^{m-1}\|\mathbf{e}_\mathbf{u}^n\|^2$$
$$+C\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t}-\mathbf{u}_t(t^{n+1})\|_{-1}^2+C\Delta t\sum_{n=0}^{m-1}\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|^2$$
$$+C\Delta t\sum_{n=0}^{m-1}\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2+\frac{2\varepsilon}{N\Delta t}\left(\int_{t^{m-1}}^{t^m}\varphi dt,\mathbf{e}_\mathbf{u}^m\right)+C\varepsilon^2\Delta t\sum_{n=0}^{m-1}\|p_{tt}(\xi_n)\|^2$$
$$+C\varepsilon^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|p_t\|^2dt+C\varepsilon^2\sum_{n=0}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2dt. \tag{4.44}$$

Next, we bound the terms in the right hand side of (4.44). First, we have

$$\begin{aligned}C\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t}-\mathbf{u}_t(t^{n+1})\|_{-1}^2&\leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt\\&\leq C\Delta t^2\|\mathbf{u}_{tt}\|_{2,-1}^2.\end{aligned} \tag{4.45}$$

We also have

$$C\Delta t\sum_{n=0}^{m-1}\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|^2\leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\nabla\phi_t\|^2dt\leq C\Delta t^2\|\phi_t\|_{2,1}^2, \tag{4.46}$$

and

$$C\Delta t\sum_{n=0}^{m-1}\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2\leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2dt\leq C\Delta t^2\|\mathbf{u}_t\|_{2,0}^2. \tag{4.47}$$

Moreover,

$$\begin{aligned}\frac{2\varepsilon}{N\Delta t}\left(\int_{t^{m-1}}^{t^m}\varphi dt,\mathbf{e}_{\mathbf{u}}^m\right)&=\frac{2\varepsilon}{N}(\varphi(\eta^m),\mathbf{e}_{\mathbf{u}}^m)\le\frac{2\varepsilon^2}{N}\|\varphi(\eta^m)\|^2+\frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2\\&\le C\varepsilon^2\|p_t(\eta^m)\|^2+\frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2\le C\varepsilon^2\|p_t\|_{\infty,0}^2+\frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2,\end{aligned}\tag{4.48}$$

where $\eta^m\in(t^{m-1},t^m)$. Lastly, it gives that

$$C\varepsilon^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|p_t\|^2dt\le C\varepsilon^2\|p_t\|_{2,0}^2,\tag{4.49}$$

and

$$C\varepsilon^2\sum_{n=0}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2dt\le C\varepsilon^2\|p_t\|_{\infty,0}^2\sum_{n=0}^{m-1}\Delta t\|\nabla\mathbf{u}^{n+1}\|^2\le C\varepsilon^2\|p_t\|_{\infty,0}^2,\tag{4.50}$$

where we use the result in Theorem 4.1.

Combining (4.45)–(4.50) with (4.44), we can have

$$\begin{aligned}&\frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2+\frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2+\Delta t\sum_{n=0}^{m-1}\|\nabla e_\phi^{n+1}\|^2\\&\quad+\Delta t\sum_{n=0}^{m-1}\left(\|-\nabla e_\phi^n+\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2+\|-\nabla e_\phi^{n+1}+\mathbf{e}_{\mathbf{u}}^n\times\mathbf{B}\|^2\right)\\&\quad\le C\Delta t\sum_{n=0}^{m-1}\|\mathbf{e}_{\mathbf{u}}^n\|^2+C\left(\Delta t^2+\varepsilon^2\right).\end{aligned}\tag{4.51}$$

Finally applying the discrete Gronwall lemma completes the proof. □

5 Analysis of Non-physical Acoustic Waves

Artificial compression method can greatly speed up computations but can also introduce new physical flow behaviors associated with compressibility such as non-physical, fast, pressure oscillations (acoustic waves). These fast acoustic waves can yield restrictive time step conditions for explicit time discretization of the full system or pollute the pressure approximation. In this section, we analyze these non-physical acoustic waves through an acoustic waves equation for pressure. We rewrite the model (1.3) as follows.

$$\begin{aligned}&\frac{1}{N}\left(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u}\right)-\frac{1}{N\cdot Re}\Delta\mathbf{u}+\nabla p=\mathbf{f}+\mathbf{j}\times\mathbf{B},\\&\varepsilon p_t+\nabla\cdot\mathbf{u}=0,\\&\nabla\cdot\mathbf{j}=0.\end{aligned}\tag{5.1}$$

Taking the divergence of the first equation and $\partial/\partial t$ of the second equation in (5.1), we obtain

$$\begin{aligned}&\frac{1}{N}\nabla\cdot\mathbf{u}_t+\frac{1}{N}\nabla\cdot\left(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}\left(\nabla\cdot\mathbf{u}\right)\mathbf{u}-\frac{1}{\cdot Re}\Delta\mathbf{u}\right)+\Delta p=\nabla\cdot\mathbf{f}+\nabla\cdot\left(\mathbf{j}\times\mathbf{B}\right),\\&\varepsilon p_{tt}+\nabla\cdot\mathbf{u}_t=0.\end{aligned}\tag{5.2}$$

Assume $\nabla \cdot \mathbf{f} = 0$. Multiplying the first equation in (5.2) by N and eliminating $\nabla \cdot \mathbf{u}_t$ term, we can have

$$\varepsilon p_{tt} - N\Delta p = \nabla \cdot \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2} (\nabla \cdot \mathbf{u})\, \mathbf{u} - \frac{1}{\cdot Re} \Delta \mathbf{u} \right) - N\nabla \cdot (\mathbf{j} \times \mathbf{B}). \tag{5.3}$$

Since $\nabla \cdot (\mathbf{j} \times \mathbf{B}) = (\nabla \times \mathbf{j}) \cdot \mathbf{B}$ and $\nabla \times \mathbf{j} = \nabla \times (-\nabla\phi + \mathbf{u} \times \mathbf{B}) = \nabla \times (\mathbf{u} \times \mathbf{B})$, we have $\nabla \times \mathbf{j} = -(\nabla \cdot \mathbf{u})|\mathbf{B}|$ and $\nabla \cdot (\mathbf{j} \times \mathbf{B}) = -(\nabla \cdot \mathbf{u})|\mathbf{B}|^2$ in 2 dimension case. Thus Eq. (5.3) can be rewritten as

$$\varepsilon p_{tt} + \varepsilon N|\mathbf{B}|^2 p_t - N\Delta p = \nabla \cdot \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2} (\nabla \cdot \mathbf{u})\, \mathbf{u} - \frac{1}{\cdot Re} \Delta \mathbf{u} \right). \tag{5.4}$$

The pressure wave equation (5.4) shows that the non-physical oscillation in p will be damped by damping coefficient $N|\mathbf{B}|^2$. Thus, *the effect of magnetic field is to damp acoustic waves in slightly compressible flows*. The waves have energy input due to the right hand side terms. The speed of the non-physical acoustic wave is $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, which indicates that the wave speed goes up to ∞ as $\varepsilon \to 0$. We test the non-physical acoustic wave in Sect. 6.2. We compute and plot (Figs. 1, 2) the pressure at origin (0, 0) on a time interval after initial transients pass to test if the wave speed increases as the time step $\Delta t \downarrow 0$.

5.1 Nonlinear Acoustics

We consider the effect of the nonlinear term in the right hand side of (5.4) on acoustic waves. Let the usual Lighthill sound source (see, [26] and [27] for justification) be denoted by

$$Q(\mathbf{u}, \mathbf{u}) := \nabla \mathbf{u} : (\nabla \mathbf{u})^T = \sum_{i,j} \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i} \quad \text{for } d = 2 \text{ or } 3.$$

Since we have the identity that

$$\nabla \cdot \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2} (\nabla \cdot \mathbf{u})\, \mathbf{u} \right) = Q\,(\mathbf{u}, \mathbf{u}) - \frac{1}{2} \mathbf{u} \cdot \nabla\, (\nabla \cdot \mathbf{u}) + \frac{1}{2} |\nabla \cdot \mathbf{u}|^2,$$

if the effect of slight compressibility on acoustic waves is negligible, we can obtain

$$\nabla \cdot \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2} (\nabla \cdot \mathbf{u})\, \mathbf{u} - \frac{1}{\cdot Re} \Delta \mathbf{u} \right) \simeq Q\,(\mathbf{u}, \mathbf{u}) \quad \text{if } \nabla \cdot \mathbf{u} \simeq 0.$$

Recall (1.3) that $\nabla \cdot \mathbf{u} = -\varepsilon p_t$. Then we have

$$\nabla \cdot \left(\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2} (\nabla \cdot \mathbf{u})\, \mathbf{u} - \frac{1}{\cdot Re} \Delta \mathbf{u} \right) = Q(\mathbf{u}, \mathbf{u}) + \frac{\varepsilon}{2} \mathbf{u} \cdot \nabla p_t + \frac{\varepsilon^2}{2} |p_t|^2 + \frac{\varepsilon}{Re} \Delta p_t.$$

Thus (5.4) can be rewritten as

$$\varepsilon p_{tt} + \varepsilon N|\mathbf{B}|^2 p_t - N\Delta p = Q(\mathbf{u}, \mathbf{u}) + \frac{\varepsilon}{2} \mathbf{u} \cdot \nabla p_t + \frac{\varepsilon^2}{2} |p_t|^2 + \frac{\varepsilon}{Re} \Delta p_t. \tag{5.5}$$

$Q(\mathbf{u}, \mathbf{u})$ represents the physical sound source, i.e., the sound generated by the flow. The terms $\frac{\varepsilon}{2} \mathbf{u} \cdot \nabla p_t + \frac{\varepsilon^2}{2} |p_t|^2 + \frac{\varepsilon}{Re} \Delta p_t$ are the nonphysical sound sources and their values vanish as ε goes to 0. When $Re \gg 1$, the leading order term in nonphysical sound sources is $\frac{\varepsilon}{2} \mathbf{u} \cdot \nabla p_t$. In Sect. 6.2, we compute the relative size of the leading order term in non-physical sound sources to the Lighthill sound source

$$Ratio = \frac{\| \frac{\varepsilon}{2} \mathbf{u} \cdot \nabla p_t \|}{\| Q(\mathbf{u}, \mathbf{u}) \|}.$$

Table 1 Errors and convergence rates of $Algorithm$ 1

Δt	$\|\mathbf{e}_{\mathbf{u}}^m\|$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$	Rate
1/20	6.0467e−2		2.2961e−1		2.5699e−1	
1/30	4.3838e−2	0.79	1.4885e−1	1.07	1.7247e−1	0.98
1/40	3.3862e−2	0.90	1.0901e−1	1.08	1.2983e−1	0.99
1/50	2.7299e−2	0.97	8.6329e−2	1.05	1.0411e−1	0.99
1/60	2.2684e−2	1.02	7.2225e−2	0.98	8.6917e−2	0.99

The result is shown in Figs. 3 and 4. Looking at the vertical axis scales, we conclude that the nonphysical sound source is negligible compared to the physical one.

6 Numerical Test

In this section, we provide numerical experiments to test the convergence of $Algorithms$ 1 and 2, and the non-physical acoustic waves analyzed in Sect. 5. We utilize the P2-P1 Taylor-Hood mixed finite elements for fluid velocity and pressure and P2 finite element for electric potential. The software package *FreeFEM++*, see [41], are used for our simulation.

6.1 Testing Convergence

Let the domain $\Omega = [0, 1] \times [0, 1]$, $Re = 1$, $N = 1$, $M = 1$ and $\mathbf{B} = (0, 0, 1)$. Consider the true solution $(\mathbf{u}, p, \phi)$ given as follows.

$$\begin{aligned}
\mathbf{u}(x, y, t) &= (2\pi \cos(2\pi x)\sin(2\pi y), -2\pi \sin(2\pi x)\cos(2\pi y), 0)e^{-5t},\\
p(x, y, t) &= 0,\\
\phi(x, y, t) &= (\cos(2\pi x)\cos(2\pi y) + x^2 - y^2)e^{-5t}.
\end{aligned}$$

The body force $\mathbf{f}$, boundary condition and initial condition are determined by the true solution.

We firstly compute the rate of convergence to confirm the effectiveness of our theoretical analysis for $Algorithm$ 1. Set $\varepsilon = \Delta t$. Select $T = 1$, $h = \frac{1}{60}$ and then $\Delta t = \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{1}{50}, \frac{1}{60}$. We compute $\|\mathbf{e}_{\mathbf{u}}^m\|$, $(\sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$ and $(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$ to obtain the convergence rate.

Table 1 confirms that the rate of convergence is first order in accord with the theoretical result of Theorem 4.3.

We also compute the rate of convergence for $Algorithm$ 2. Since $Algorithm$ 2 is second order accurate, we set $\varepsilon = \Delta t^2$. Select $T = 1$, $h = \Delta t$ and then $\Delta t = \frac{1}{20}, \frac{1}{40}, \frac{1}{60}, \frac{1}{80}, \frac{1}{100}$. We also compute $\|\mathbf{e}_{\mathbf{u}}^m\|$, $(\sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$ and $(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$.

Table 2 shows that the rate of convergence of $Algorithm$ 2 is second order, as expected.

6.2 Testing Non-physical Acoustic Waves

We explore the non-physical acoustic waves by the following 2D test problem (Flow between offset circles).

Table 2 Errors and convergence rates of *Algorithm* 2

Δt	$\|\mathbf{e}_{\mathbf{u}}^m\|$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$	Rate
1/20	7.1314e−3		1.1478e−1		9.7290e−3	
1/40	1.7696e−3	2.01	3.6299e−2	1.66	2.9077e−3	1.72
1/60	7.6980e−4	2.05	1.7458e−2	1.81	1.3712e−3	1.85
1/80	4.2564e−4	2.06	1.0219e−2	1.86	7.9445e−4	1.90
1/100	2.6889e−4	2.06	6.6996e−3	1.89	5.1754e−4	1.92

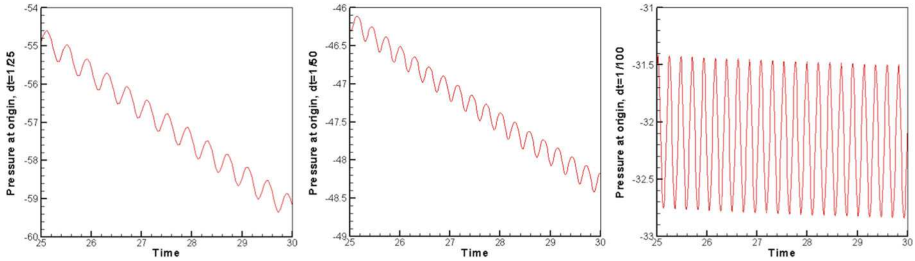

Fig. 1 Pressure at (0,0) versus time, *Algorithm* 1, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

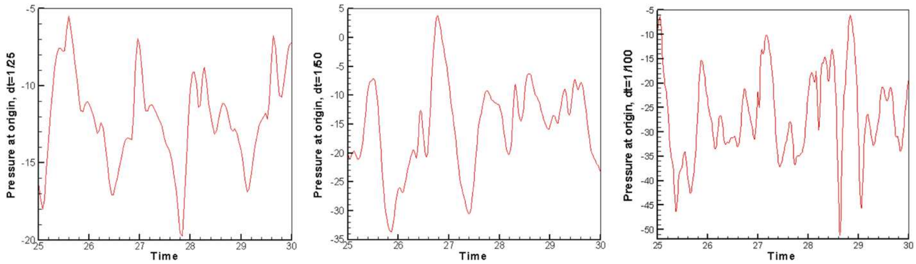

Fig. 2 Pressure at (0,0) versus time, *Algorithm* 2, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

Let the domain $\Omega = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } (x - 0.5)^2 + y^2 \geq 0.1^2\}$, $\mathbf{B} = 1 \cdot \mathbf{k}$, the final time $T = 30$, and the body force $\mathbf{f} = (-4y(1 - x^2 - y^2), 4x(1 - x^2 - y^2))^{\mathrm{T}}$. Set $Re = 1000$ and $N = 1$, thus $M = \sqrt{N \cdot Re} = \sqrt{1000}$. For velocity boundary conditions, let $\mathbf{u} = 0$ on both circles. Similarly set $\varepsilon = \Delta t$ for *Algorithm* 1 ($\varepsilon = \Delta t^2$ for *Algorithm* 2). Choose time step $\Delta t = \frac{1}{25}, \frac{1}{50}, \frac{1}{100}$.

Firstly, we plot the pressure versus t at $(0, 0)$. Figure 1 (Fig. 2) shows the results of *Algorithm* 1 (*Algorithm* 2) on a time interval [25, 30] after initial transients pass. We find that the time evolution of the pressure at one point in space varies greatly as Δt changes. However, the wave's frequency has a clear pattern consistent with (5.4): As Δt decreases, the wave's frequency increases.

Figures 3 (*Algorithm* 1) and 4 (*Algorithm* 2) present the relative size of the leading order term in non-physical sound sources to the Lighthill sound source: $\frac{\|\frac{\varepsilon}{2}\mathbf{u}\cdot\nabla p_t\|}{\|Q(\mathbf{u},\mathbf{u})\|}$. It shows that $Q(\mathbf{u}, \mathbf{u})$ is the dominant forcing for oscillations in p and $\nabla \cdot \mathbf{u}$.

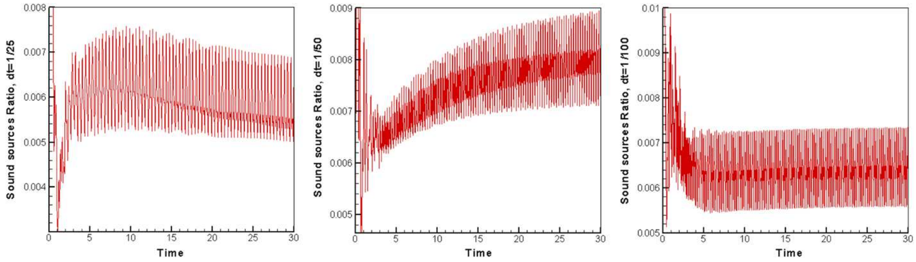

Fig. 3 Non-physical acoustic source / Lighthill source, *Algorithm* 1, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

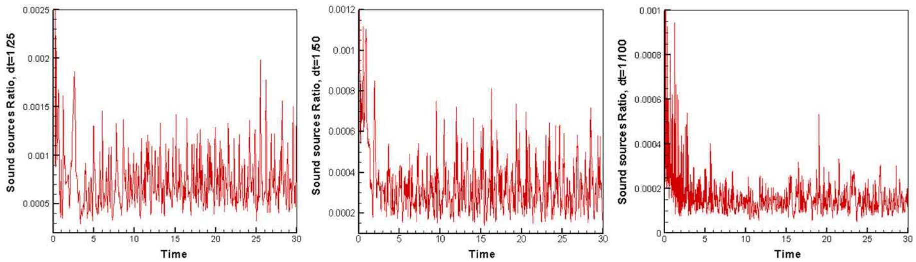

Fig. 4 Non-physical acoustic source / Lighthill source, *Algorithm* 2, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

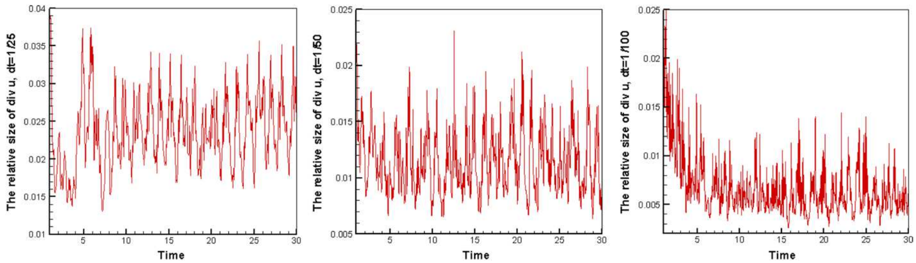

Fig. 5 $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$, *Algorithm* 2, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

Lastly, in order to study how close the computing solution is to incompressible, we compute $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$. As shown in Fig. 5 (*Algorithm* 2), the relative size of $\nabla \cdot \mathbf{u}$ decreases as Δt decreases. However, in Fig. 6 (*Algorithm* 1), $\nabla \cdot \mathbf{u}$ fails to decrease when Δt decreases, which implies that the selection of $\varepsilon = \Delta t$ or Δt^2 should influence the stability of the artificial compression method. For *Algorithm* 1, we could further stabilize this first order method. Thus, we consider to add a stabilization term $\gamma \nabla \nabla \cdot \mathbf{u}$ in *Algorithm* 1 ($\gamma = 10{,}000$) and then recompute the relative size of $\nabla \cdot \mathbf{u}$. Figure 7 presents the computing results and shows that the relative size of $\nabla \cdot \mathbf{u}$ decreases as Δt decreases when we select a large γ in the stabilization term $\gamma \nabla \nabla \cdot \mathbf{u}$. Compared with Fig. 6, when we apply the stabilization term, the oscillation of the relative size of $\nabla \cdot \mathbf{u}$ becomes weaker, which shows the stabilization term $\gamma \nabla \nabla \cdot \mathbf{u}$ might be an effective way to dampen non-physical acoustic waves. Meanwhile, we find that as $\Delta t \downarrow 0$, $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$ appears to be more oscillating, which is also consistent with our analysis of non-physical acoustic waves since $\nabla \cdot \mathbf{u} = -\varepsilon p_t$.

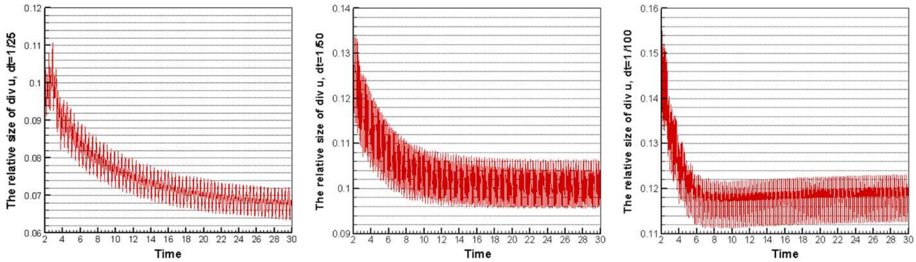

Fig. 6 $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$, *Algorithm* 1, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

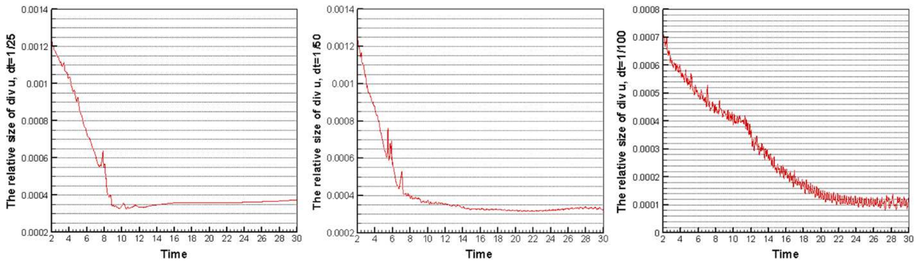

Fig. 7 $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$, with $\gamma \nabla\nabla \cdot \mathbf{u}$, *Algorithm* 1, dt = 1/25 (left), dt = 1/50 (middle) and dt = 1/100 (right)

7 Conclusion

In this paper, we construct two decoupled methods based on the artificial compression method and the partitioned method for the time-dependent magnetohydrodynamics flows at low magnetic Reynolds numbers. Theoretical analysis indicates that the error estimate of *Algorithm* 1 is $\|\mathbf{u}(t_n) - \mathbf{u}^n\| \leq C(\Delta t + \varepsilon)\ \forall n \leq T/\Delta t$. We also explore the non-physical acoustic waves that comes from the application of the artificial compression method and give a brief analysis for it. The numerical examples illustrate the correctness of our theoretical analysis.

An open question is that the error estimate given in Theorem 3.2 shows $\|\mathbf{u}(t) - \mathbf{u}^\varepsilon(t)\| \leq C\sqrt{\varepsilon}\ \forall t \in [0, T]$, which is not optimal in view of the error estimate in Theorem 4.3. The optimal error estimate for the slightly compressible model is necessary because it can indicate the relation between the coefficient ε and the time step Δt and thus suggest the optimal choice of ε according to different time-discretization schemes. The other open question is how to control the non-physical acoustic waves. Since the non-physical acoustic waves will increasingly influence the accuracy of computing solution as the time step goes to 0, it should be an interesting issue to study effective methods to solve this problem, such as adding stabilization terms (e.g., $\gamma \nabla\nabla \cdot \mathbf{u}$) or utilizing time filters.

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