Mathematical Reasoning and Proving for Prospective Secondary Teachers

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The design-based research approach was used to develop and study a novel capstone course: Mathematical Reasoning and Proving for Secondary Teachers. The course aimed to enhance prospective secondary teachers' (PSTs) content and pedagogical knowledge by emphasizing reasoning and proving as an overarching approach for teaching mathematics at all levels. The course focused on four proof-themes: quantified statements, conditional statements, direct proof and indirect reasoning. The PSTs strengthened their own knowledge of these themes, and then developed and taught in local schools a lesson incorporating the proof-theme within an ongoing mathematical topic. Analysis of the first-year data shows enhancements of PSTs' content and pedagogical knowledge specific to proving.

Keywords: Reasoning and Proving, Preservice Secondary Teachers, Design-Based Research

Mathematics educators and education researchers agree on the importance of teaching mathematics in ways that emphasize sense making, reasoning and proving. In fact, teaching practices that emphasize mathematical reasoning have been shown to be more equitable, and associated with higher knowledge retention (Boaler & Staples, 2008; Harel, 2013). These practices encompass a wide range of processes such as exploring, conjecturing, generalizing, justifying and evaluating mathematical arguments (Hanna & deVillers, 2012; Ellis, Bieda & Knuth, 2012). Policy documents also encourage integrating reasoning and proving throughout grade levels and mathematical topics (CCSSI, 2010; NCTM, 2009). However, the reality of many mathematics classrooms has been far from the vision put forth by researchers and policy makers. Reasoning and proof is often confined to high school geometry, tied to a particular format and used to show already known results. Consequently, students and teachers alike often view proof as redundant, rather than a means of deepening mathematical understanding (Knuth, 2002). Although teachers often recognize the importance of reasoning and proof beyond high school geometry, they tend to choose skills-oriented activities over proof-oriented ones for their own classrooms (Kotelawala, 2016). One reason for this might be, as Bieda (2010) found, that even experienced teachers struggle to implement reasoning and proving in their classrooms.

These studies suggest that enacting teaching practices that emphasize reasoning and proof is a complex process that requires awareness and intentionality on behalf of the teacher. Preparing teachers who are capable of implementing such teaching practices and cultivating positive attitudes towards mathematical reasoning and proving is a critical objective of teacher preparation programs (AMTE, 2017). Yet, the theoretical and practical knowledge in this area of teacher preparation has been scarce (e.g., Ko, 2010; Stylianides & Stylianides, 2015).

To address this knowledge gap, we utilized a design-based-research approach (Edelson, 2000) to develop and study a novel capstone course *Mathematical Reasoning and Proving for Secondary Teachers*. The goals of the course were to improve the PSTs mathematical knowledge for teaching proving, including subject matter knowledge and pedagogical content knowledge. The goals of the research study were to (a) explore how PSTs' knowledge and dispositions towards the teaching and learning of reasoning and proof developed as a result of participating in the capstone course, and (b) to identify course design principles that afforded PSTs' learning. In this paper, we illustrate the overall structure of the course; we describe in greater detail one of the four modules, specifically, the module on Conditional Statements (CS); and we provide details on how PSTs interacted with the components of the CS module. We also show evidence of growth of PSTs' mathematical knowledge for teaching proof, following the completion of the course.

Theoretical Framework and the Course Design

Researchers have conjectured that engaging students in reasoning and proving, that is, exploring, generalizing, conjecturing and justifying, might require a special type of teacher knowledge: Mathematical Knowledge for Teaching of Proof (MKT-P) (e.g., Lesseig, 2016; Stylianides, 2011). Building on their work, we theorize that MKT-P consists of four interrelated types of knowledge, two specific to content, and two related to pedagogy (Fig. 1).

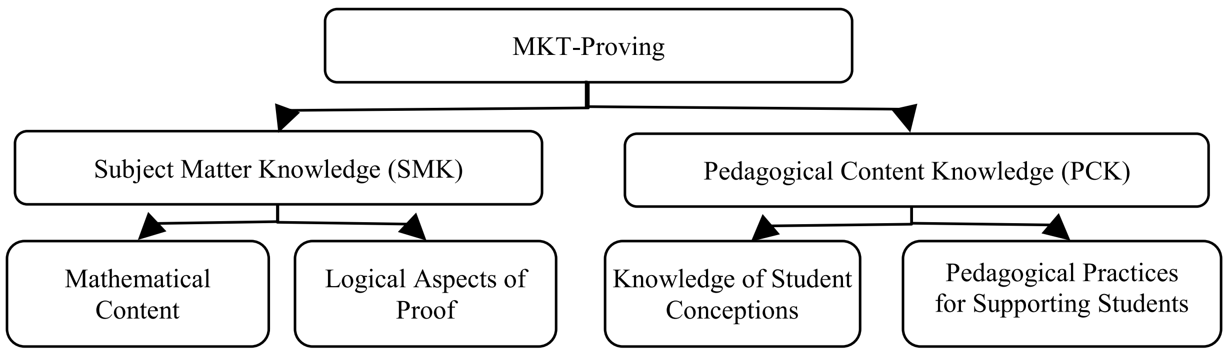


Figure 1. Mathematical Knowledge for Teaching of Proving (Buchbinder & Cook, 2018)

As suggested in Figure 1, to enact reasoning and proving in their classrooms, we hypothesize that teachers must have robust Subject Matter Knowledge (SMK) of mathematical concepts and principles, and also knowledge of the logical aspects of proof, which includes knowledge of different types of arguments, proof techniques, knowledge of logical connections, valid and invalid modes of reasoning, the functions of proof, and the role of examples and counterexamples in proving (Hanna & deVillers, 2012). Teachers also need strong Pedagogical Content Knowledge (PCK) specific to proving, such as knowledge of students' proof-related conceptions and common mistakes, and knowledge of pedagogical strategies for supporting students' proof activities. Thus, we built into the course structure opportunities for PSTs to develop and practice these four types of knowledge (Fig. 1).

The course consists of four modules, each three-weeks long, corresponding to four proof-themes: quantified statements, conditional statements, direct proof and indirect reasoning. These themes were identified in the literature as challenging for students and PSTs (Antonini & Mariotti, 2008; Weber, 2010). Thus, each module was designed with activities to enhance PSTs' knowledge of the logical aspects of proof related to the proof themes, followed by developing and teaching lessons at a local school integrating that proof-theme with current mathematical topics. This combination of the activities was intended to address both subject matter knowledge and pedagogical knowledge specific to proving, as mentioned above. Figure 2 shows the structure of the course (top) and the general structure of a single course module (bottom).

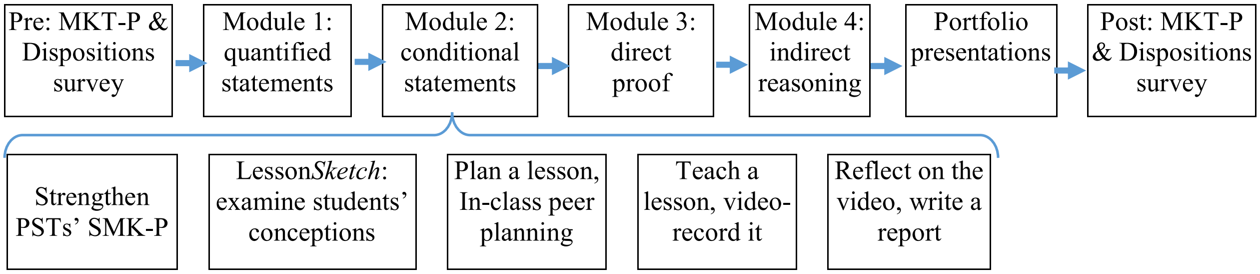


Figure 2. Design of the course *Mathematical Reasoning and Proving for Secondary Teachers*.

The theoretical background underlying the course design also draws on a situated perspective of learning (Peressini, Borko, Romagnano, Knuth, & Willis, 2004) which views learning as patterned participation in social contexts. In particular, Borko et al. (2000) assert that learning to teach occurs across multiple settings through active participation in the social contexts embedded in them, and should not be confined to a university classroom. Building on the literature on practice-based teacher education (e.g., Ball & Forzani, 2011; Grossman, Hammerness & McDonald, 2009) we designed opportunities for PSTs to enhance their proof-related pedagogical knowledge in an environment of reduced complexity and risk, through the virtual learning platform Lesson*Sketch* (Herbst, Chazan, Chieu, Milewski, Kosko, & Aaron, 2016). The learning experiences created and administered through this platform allowed the PSTs to engage in practices such as interpreting sample student work, identifying students' conceptions of proof, evaluating students' mathematical arguments, and envisioning responding to them in ways that challenge and advance students' thinking. We also engaged PSTs in planning and implementing, in local schools, lessons that combine the proof themes with the regular mathematical content. We see this as a critical component and the unique feature of our course design, that aims to bring together the aspects of proof and secondary curriculum.

As PSTs enacted their lesson, they recorded it using 360° video cameras, which captured simultaneously the PSTs' teaching performance and the school students' engagement with proof-oriented lessons. PSTs then watched and analyzed their lesson, wrote a reflection report and received feedback from the course instructor to inform future lesson planning.

The design-based research format of this project requires that we take a careful look at the various course components and the participants' interaction with them. Thus, in the methods and results sections below, we describe one module, Conditional Statements (CS), in order to demonstrate how we address the first goal of our project: to develop and study the capstone course. We focus on the following research questions:

1. *How did the PSTs interact with the CS module?*
2. *What mathematical and pedagogical ideas addressed in the CS module were implemented in the PSTs' lessons?*

At the same time, a second main goal of our project was to improve the preservice teachers' knowledge of and disposition toward teaching proof and reasoning through this course. Thus, we explored the research question:

3. *How did the PSTs' MKT-P and dispositions toward proof develop throughout the course?*

Methods

Participants in the first iteration of the course were 15 PSTs in their senior year (4 middle-school, and 11 high-school track; 6 males and 9 females). The PSTs had completed the majority of their extensive mathematical coursework, and two educational methods courses, one focused on general mathematics education topics that are common across grade levels, and one specific to teaching secondary mathematics.

Multiple measures were used to collect data on PSTs interaction with the CS module. We collected and analyzed PSTs' responses to home- and in-class assignments, and video-recordings of all in-class sessions to answer the first research question. To answer the second question, we analyzed the PSTs' cumulative teaching portfolios containing four lesson plans, video-records of the lessons taught, reflection reports, and sample school students' work. We used a modification of Schoenfeld's (2013) TRU Math rubric to analyze 360° video-records of the PSTs' lessons.

In addition, we investigated how PSTs' knowledge of content and pedagogy, and their dispositions towards proof evolved throughout the course. For that, we used two instruments: pre- and post- measures of mathematical knowledge for teaching proof - the MKT-P questionnaire, and dispositions towards proving survey. Prior to the study, we identified four existing instruments from the literature (Corleis et al., 2008; Kotelawala, 2016; Lesseig, 2016; Nyaumwe & Buzuzi, 2007), that partially matched our research focus. We combined elements of those instruments and supplemented with our own items to create an MKT-P instrument containing four sets of questions corresponding to the four types of MKT-P (Fig. 1). The dispositions towards proof survey included six sets of questions, some open and some closed, addressing PSTs' notions of proof, the purpose and usefulness of proof, suitability of proof for secondary students, PSTs' own confidence and comfort with proving, as well as confidence in teaching proof to students. The survey questions were a combination of items from related instruments (e.g., McCrone & Martin, 2004; Nyaumwe & Buzuzi, 2007). Both the MKT-P instrument and the dispositions survey were reviewed and analyzed by experts in educational assessment. The data collected with the MKT-P questionnaire and dispositions towards proof survey were analyzed quantitatively to answer the third research question.

Results

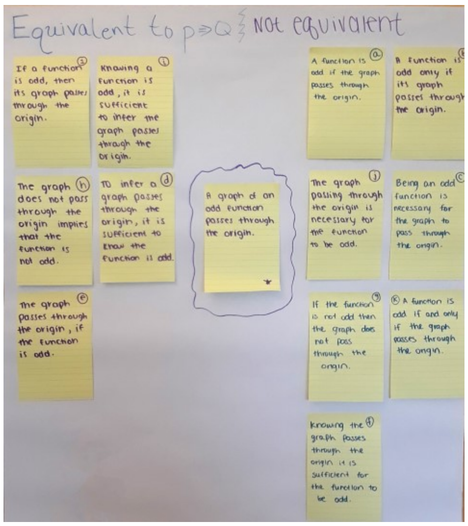
PSTs' Interaction with the Conditional Statements Module

In this section we take a closer look at the Conditional Statements (CS) module and the PSTs interaction with the module components. The Conditional Statements module comprised the following activities: (1) sorting conditional statements, (2) Lesson*Sketch* experience *Who is right?*, (3) analysis of conditional statements in the secondary curriculum, (4) planning and implementing a lesson that incorporates some ideas about conditional statements, and (5) implementation reflection.

Sorting Conditional Statements.

For this in-class activity, PSTs broke up into three groups and each group received one conditional statement. Their task was to identify the statements' hypothesis P and conclusion Q, and use the P and Q to write statements in 11 logical forms such as: $P\ if\ Q$, $P\ only\ if\ Q$, $If\ {\sim}P\ then\ {\sim}Q$, and others. The PSTs were to write each statement on an index card using different colors for true and false statements, and then sort the cards into statements equivalent to $P \Rightarrow Q$, and non-equivalent to it. The original statements given to the groups were: "A graph of an odd function, defined at zero, passes through the origin" (group 1), "A number that is divisible by six is divisible by three" (group 2), and "Diagonals of a rectangle are congruent to each other" (group 3).

Each group created a poster showing how they sorted the cards, and presented their work to others. Figure 3 shows a poster by group 1. The statements on the cards (Fig. 3-a) are written in terms of the original statement, which is located in the middle of the poster. Figure 3-b shows which logical forms PSTs identified as equivalent or non-equivalent to $P \Rightarrow Q$, the incorrect answers are marked with *. Note that the PSTs correctly identified all equivalent forms and all non-equivalent forms. However, they also wrongly identified two equivalent forms (j) and (b) as non-equivalent.



Types of statements as sorted on the poster.
P: f *is an odd function, defined at zero*
Q: *the graph of* f *passes through the origin*

Equivalent to $P \Rightarrow Q$:	Nonequivalent:
If P then Q.	a) P if Q
d) To infer Q, it is sufficient to know P	c) P is necessary for Q
e) Q if P	f) Q is sufficient for P
h) Not Q implies not P	g) If not P then not Q
i) P is sufficient to infer Q	k) P if and only if Q
	j) Q is necessary for P*
	b) P only if Q*

(a) (b)

Figure 3. (a) Group 1 poster, sorting equivalent and non-equivalent statements; the original statement is in the middle; (b) logical forms represented on the poster.

The form that the PSTs in all three groups found most challenging to interpret was P *only if* Q. Some PSTs realized that it is equivalent to a contrapositive, $\sim Q \Longrightarrow \sim P$, and therefore equivalent to the original statement. But other PSTs rejected this idea by arguing that P can be true "not only when Q is true", concluding that P *only if* Q is not equivalent to $P \Longrightarrow Q$. Group 2 argued that a statement "A number is divisible by 6 only if it is divisible by 3" is both untrue and non-equivalent to the original statement, because a number is divisible by 6 *not only* when it is divisible by 3, but also when it is divisible by 2. Since the PSTs found the arguments for equivalence and non-equivalence of the two statements to be equally appealing, it required facilitator intervention to clarify the equivalence using the contrapositive argument. In this explanation, the abstract logical notation appeared to be more useful than the contextualized one.

After all groups presented their posters and the discrepancies were discussed and resolved, PSTs received additional prompts to grapple with, such as: What is the relationship between truth-value of a statement and equivalence of statements? Identify inverse and converse among the given forms, and rewrite the language of necessary and sufficient conditions symbolically. PSTs discussed these prompts in their groups first, and then as a whole class. The follow-up homework assignment (discussed below) introduced the PSTs to students' conceptions of conditional statements.

LessonSketch experience *Who is Right*?

Lesson*Sketch*.org is an interactive-media web-based platform for teacher education, which allows a teacher educator to represent classroom interactions as cartoon sketches, which PSTs can analyze. Such representations preserve much of the authenticity of the real classroom, but allow PSTs more time to interpret student thinking and plan a response (Herbst et al., 2016). The interactive tools of Lesson*Sketch* allow teacher educators to create rich learning experiences for PSTs, that provide PSTs with opportunities to experience situations that resemble classroom interactions, and envision themselves participating in them as teachers. These experiences can involve analyzing samples of student work; watching classroom scenarios in the form of video, animation or story board; responding to prompts about these scenarios; creating their own depictions of classroom interactions; and participating in discussion forums.

The experience *Who is right?* is based on real student data, and was field-tested in prior studies (Buchbinder, 2018). In this experience the PSTs were presented with a false mathematical statement: "If n is a natural number, then $n^2 + n + 17$ is prime," and asked whether it is true, false, sometimes true, or cannot be determined. It is important to note that although a mathematical statement can only be true or false, we discovered in pilot implementations of the experience that PSTs struggled to commit to a dichotomous answer. Instead, they were trying to hedge their responses in the comment box. This tendency to avoid a dichotomous response is reminiscent of "fuzzy logic" (Zazkis, 1995), the form of reasoning that allows one to assign to a statement a value that qualifies one's confidence in its correctness. By providing PSTs four, rather than two, options to choose from when evaluating the truth-value of the statement, we allowed greater flexibility for PSTs to respond to the prompt. As researchers and teacher educators, we were able to assess who among our PSTs had not yet developed bivalent logical reasoning, which aligns with conventional mathematical logic.

After PSTs evaluated the statement on their own, they viewed a set of slides depicting arguments of five pairs of students evaluating the truth-value of that given statement. The arguments were developed to reflect common student misconceptions about conditional statements. For example, "proving" the statement by testing a set of strategically chosen examples, requesting more than one counterexample to disprove a statement, asserting that the truth-value of a statement cannot be determined when both supportive examples and counterexamples exist (Buchbinder & Zaslavsky, 2013). The PSTs' task was to evaluate the correctness of these five students' arguments, identify instances of expertise and gaps in student reasoning and pose questions to advance or challenge that reasoning. In a similar vein, we provided four, rather than two, response options for evaluating students' arguments: correct, more correct than incorrect, more incorrect than correct and incorrect. The design decision to allow PSTs to assign partial correctness to student arguments stemmed from pilot implementations of this Lesson*Sketch* experience (Buchbinder, 2018).

Figure 4 shows the distribution of PSTs' ratings of each of the five arguments. Each row corresponds to one of the five arguments, and the numbers in the rows correspond to the number of PSTs choosing a particular rating (incorrect, more incorrect than correct, more correct than incorrect, or correct). To validate the numerical data, we examined PSTs' justifications of their ratings, which revealed that that PSTs scores are not solely dependent on mathematical correctness of the evaluated argument, but are strongly influenced by pedagogical considerations.

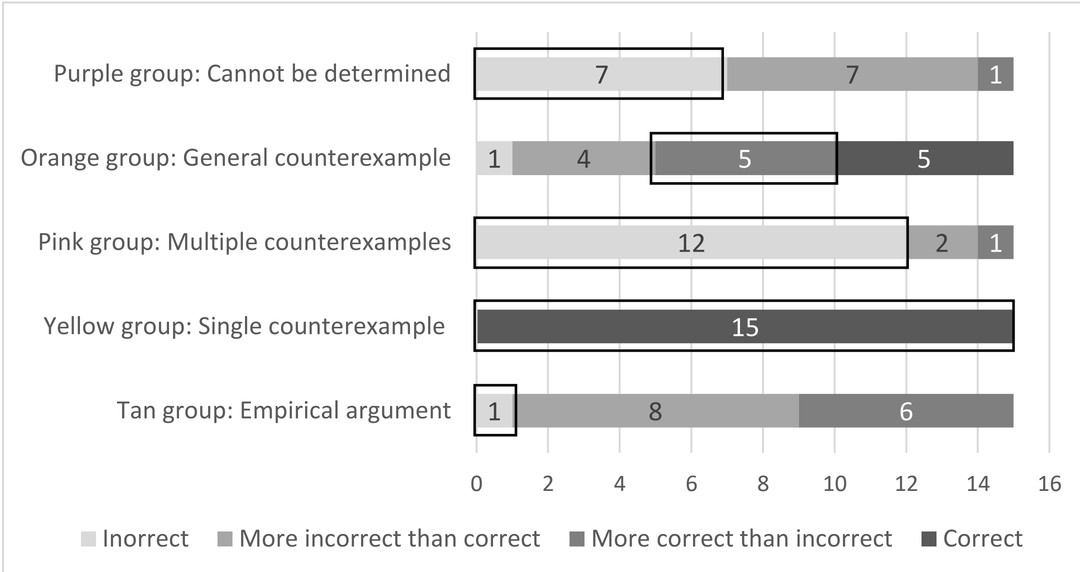


Figure 4: Distribution of PSTs' ratings of five student arguments in the LessonSketch experience *Who is right?* The dark frame indicates expected correct answer.

Figure 4 shows that all PSTs accepted a disproof by a single counterexample, and the vast majority of PSTs rated as incorrect the request for multiple counterexamples. Also, 14 out of 15 PSTs rated negatively the assertion that the truth-value cannot be determined when both supportive and counterexamples exist. The one PST who rated this student response as more correct than incorrect justified it with a pedagogical consideration. She wrote:

> This statement should be written with a universal quantifier to make it easier for students to understand it, but since it is not I can see where the students are confused and are not sure if they can prove the statement true or false. However, finding one counterexample is enough to prove this statement false, which they don't understand yet.

This kind of response from the PSTs was not an isolated instance. Overall, across all PSTs' responses to the variety of student arguments we observed that mathematical correctness of the argument was not the ultimate evaluation criterion; PSTs' evaluations were strongly affected by pedagogical considerations, reflecting the PSTs' desire to award partial credit for correct students' work or computation. In particular, the eight PSTs who rated Tan group's empirical argument as more incorrect than correct, noted that it is not an appropriate way to prove a conditional statement, but acknowledged the correctness of student calculations and validated their efforts.

We note that that while valuing student contributions is an important pedagogical practice, it is crucial for PSTs to recognize which arguments are mathematically valid, and help students to progress towards more mathematically accepted logical arguments. This is, however, heavily dependent on the PSTs' own mathematical reasoning. In our sample, there were five PSTs who believed the given statement to be true because they, themselves, could not find a counterexample. These PSTs also rated the students' empirical argument as more correct than incorrect, mimicking their own invalid reasoning.

PSTs completed the *Who is right?* activity at home, and then shared and discussed their responses as a whole class. The main focus of the discussion was on the students' conceptions of proof and the type of evidence needed to determine the truth-value of conditional statements.

PSTs also discussed the ways to acknowledge student effort, while pointing students towards more valid modes of argumentation.

Conditional statements in the secondary curriculum.

The final in-class activity in the Conditional Statements module focused on analyzing where conditional statements appear in the secondary curriculum. Working in groups, PSTs analyzed excerpts from glossary sections in a few textbooks. Their task was to write some of the rules or theorems stated there in the form of conditional statements or in some of the equivalent forms. The goals of this activity were (1) to demonstrate the prevalence of conditional statements in high school mathematics curricula aside from geometry, and (2) to anticipate student difficulties in reasoning about such statements. The discussion questions for this activity addressed the importance of understanding conditional statements, potential student difficulties related to conditional statements and brainstorming ways to support student thinking.

Planning and implementing a lesson on conditional statements.

The lesson planning process included several steps. About a week prior to the lesson, PSTs contacted their cooperating teacher to find out the mathematical topic for their lesson. Based on this information, the PSTs developed a lesson incorporating that topic with some ideas about conditional statements. During an in-class session, the PSTs worked in small groups sharing the lesson plans, testing out ideas, giving and receiving feedback with their peers and the course instructor. After improving the lesson plans through this process, the PSTs implemented their lessons in middle school and high school classrooms participating in the study.

PSTs in the middle school worked with one teacher and class, throughout the semester. These PSTs could develop closer relationships with the students, but all their lessons were tied to one mathematical unit, namely exponents. The high school track PSTs rotated among different classrooms and teachers. This allowed exposure to a variety of mathematical topics, but limited the PSTs ability to establish long-term connections with students. This also complicated their lesson planning, since the PSTs struggled to envision the students' mathematical background. The high school track PSTs taught lessons in Pre-algebra, Algebra 1, and Geometry on a range of topics such as order of operations, variable expressions, linear equations, classifying triangles, and parallel lines.

The analysis of PSTs' lesson plans showed that PSTs came up with a variety of creative ways to integrate conditional statements in their lessons while appropriately adjusting them to the students' level. Almost all PSTs used real-world examples, such as, "If I do my homework, I will get good grades" to introduce students to the general structure of a conditional statement, and to identify the hypothesis (P) and the conclusion (Q). One of the PSTs, Sam, came up with a particularly creative way to introduce conditional statements: She showed students a few product advertisements, asked them to turn the slogans into conditional statements and analyze their structure in terms of P and Q.

There was a great variation among PSTs' use of mathematical vocabulary in the lessons. For example, although both Cindy and Audrey developed a lesson on exponents asking 8^{th} grade students to evaluate the truth-value of several conditional statements, Cindy did not introduce any vocabulary in her lesson, while Audrey mentioned explicitly conditional statements, and used P and Q notation. Nate, who taught 10^{th} grade geometry, included in his lesson on parallel lines the definitions of converse, inverse and contrapositive. The majority of PSTs used a softer approach, explaining that statements of the form "if__then__" are called conditional statements, and used language of given and claim, instead of hypothesis and conclusion for P and Q. These kinds of adjustments were discussed by PSTs throughout the in-class sessions of the module as possible ways to support students' engagement with proof, particularly at the lower grade levels.

The most utilized types of tasks implemented by the PSTs were True or False, and Always-Sometimes-Never, in which the PSTs had students identify the hypothesis and the conclusion in each statement, determine whether the statement is true or false, and provide justifications or counterexamples. But there were other types of tasks utilized by the PSTs. For example, Bill created two sets of notecards: one set contained hypotheses (e.g., a triangle is not equilateral) and another set contained conclusions (e.g., a triangle is isosceles). First, Bill asked his students to match hypotheses to conclusions to produce conditional statements about triangles. After determining as a group which statements are true and which are false, Bill asked the students to change the order of cards by physically switching between the hypothesis and the conclusion. Then he had students examine the relationship between the converse and the original statement.

Another PST, Logan, modified a common Algebra 1 task into a proof-related activity on conditional statements. First, he asked his Algebra 1 students to produce an algebraic expression describing a certain sequence of operations: pick a number, quadruple it, subtract 6 and divide the result by 2. Then, he asked students to determine the truth value of several statements about the resulting expression, for example: "If your output is 34, then your input had to have been 8," or "If your input is even, then your output will be odd." Overall, except for four PSTs who only minimally addressed conditional statements, the majority of PSTs successfully and creatively integrated conditional statements in their lessons.

Reflection on lesson implementation.

As mentioned above, the PSTs recorded their lessons using 360° cameras to capture both the teacher and the student interactions. Each PST watched and reflected on their video by (a) annotating it, and (b) writing a report on how the lesson went, according to a given set of prompts. To support their claims, PSTs were required to provide time-stamps in the video. Some reflection prompts were common across all lessons, such as the following questions: In what ways did you engage students in making sense of mathematics? What aspects of students' thinking did you find particularly interesting or surprising? On the scale 1 (low) – 5 (high) evaluate your own performance in the lesson and explain the rating. In addition, each module had questions specific to the proof-theme. In the CS module, the PSTs were asked to reflect on aspects of their lessons that were specific to conditional statements. The following excerpt is taken from Grace's reflection report in which she responded to a prompt: What ideas about conditional statements do you think students understood by the end of your lesson? How do you know? Grace wrote:

> The pair work allowed [the students] to bounce ideas off each other and I could tell that they understood how to write the converse and contrapositive based on their discussions. For example, I overheard things like, "Converse, ok we need Q then P" or "This is the contrapositive, right? … If not Q, then not P?" I was happy to hear these conversations because not only were the students engaged in the activity, but they were working and communicating well together.

Reflecting on the video-recording of the lesson was quite a time consuming process, however when asked to reflect on this aspect of the course at the end of the semester, the majority of PSTs indicated that this contributed to their learning. In the summative course evaluation Ellen wrote:

> I felt that the video recordings were extremely beneficial for my learning. Even when there were parts of the lesson that I thought went very smooth at the time, I later found when watching the videos that things did not always go as smooth as I had thought.

This idea of occasional mismatch between one's feeling of their teaching performance and the lesson as recorded on camera was a recurring theme in our data. For example, Audrey, who was initially happy with her lesson, wrote after watching the video: "I realized that I am so quick to answer students' questions, that I am not stepping back and asking for other students' ideas." And concluded: "I want to make sure I am reaching all my students." Overall, the 360° video capturing allowed the PSTs to see how students reacted to their teaching in general, and to the specific proof-theme.

Evidence of PSTs' Learning

In this section we provide data from the MKT-P and disposition questionnaires to examine research question three: *How did the PSTs' MKT-P and dispositions toward proof develop throughout the course?* To trace changes in PSTs' knowledge and dispositions as a result of the course, we administered pre- and post- measures of MKT-P and pre- and post-surveys on dispositions towards proof (Fig. 2). The MKT-P contained 12 questions, 3 in each of the four areas of MKT-P (Fig. 1). In line with Hill and Ball (2004) all MKT-P items were embedded in pedagogical contexts, that is, as representing student mathematical work. The items in our measures, therefore, called for analyzing, interpreting and responding to students' conceptions of proof, similar to activities described above in the CS module.

Out of the three items measuring PSTs' Knowledge of the Logical Aspects of Proof (or Logical Knowledge, LK, for a shorthand), two items addressed knowledge related to conditional statements. In one item, a geometrical statement about quadrilaterals, and its converse were given, accompanied by a set of four claims about these statements, for example, "to prove statement (1) is false it is sufficient to prove statement (2) is false", or "to prove statement (2) is true, it is sufficient to prove statement (2) is false". The task for PSTs was to identify the correct claim about the given pair of statements. Almost all PSTs performed well on this item in both pre- and post- questionnaires, indicating that the item was too easy for PSTs in our sample, and did not sufficiently discriminate among them. We plan to modify this item in the future.

The second item dealing with conditional statements had several parts. It first introduced four statements about real numbers: (a) If $x < 1$, then $x^2 < x$, (b) If $x^2 < x$, then $x < 1$, (c) If $x^2 > x$, then $x > 1$, and (d) If $x > 1$ then $x^2 > x$. PSTs were asked for each statement to determine whether it is true or false, and if false, provide a counterexample. Next, we referred to statement (a) as "If P then Q", and asked the PSTs to identify the logical form of the rest of the statements, accordingly. Lastly, we asked the PSTs to identify an equivalent statement to (a) from a given list of distractors. Across all parts of the item, we documented a 14% score increase from pre- to post-test, with most gains occurring in identifying equivalent statements.

Overall, the data analysis revealed that PSTs had relatively high initial scores on three out of four types of MKT-P, that is Knowledge of Logical Aspects of Proof, Knowledge of Students' Conception of proof, and Knowledge of Pedagogical Practices for supporting students (Fig. 5). Because of the high initial scores we calculated the calculated the percentage of possible growth in lieu of examining the point increase. That is, we calculated what percent of the possible growth (difference of maximum score and pre-test score average) constitutes the observed growth (difference in post-test and pre-test averages). For example, although the increase in the Knowledge of Logical Aspects of Proof was only 1.15 points, it constitutes 48% of possible 2.4

points needed to obtain the maximum score. The highest gain of 66% occurred in the Knowledge of Pedagogical Practices for supporting students' learning of proof. This was reflected in the items, among others, that called for interpreting students' conceptions of proof related to activities from the CS module.

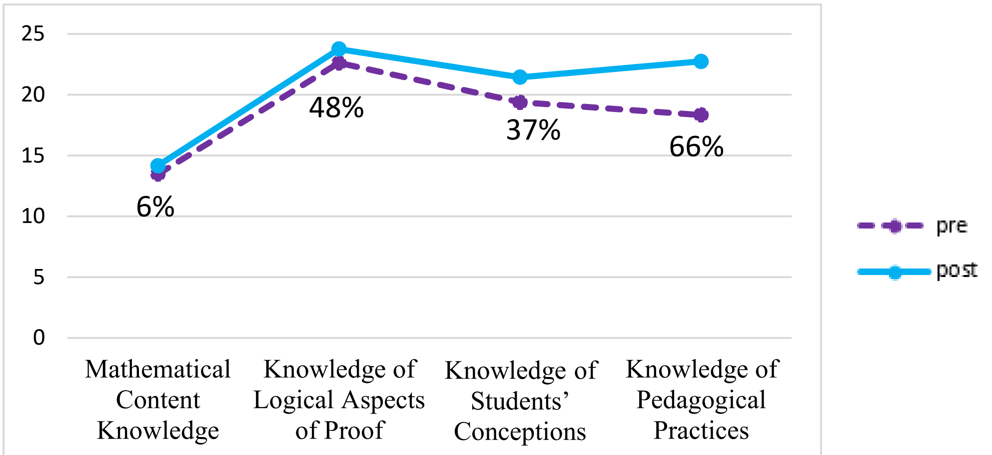


Figure 5: Sub-score averages across all participants (out of 25 points) for the four components of the MKT-P survey, with percent increase.

The dispositions towards proof survey included five categories of questions: (1) the PSTs' notions of a proof, (2) the purpose and usefulness of proof, (3) confidence and comfort with proving, (4) the suitability of proof in the school curriculum, and (5) confidence in and knowledge about teaching proof to students. In general, the pre- and post-test results did not show much change in the PSTs' thinking around categories 1, 2 and 4, but a few noteworthy results had to do with categories 3 and 5. Prior to the course, about 65% of the PSTs agreed or strongly agreed that they felt confident in their ability to prove mathematical results from the school curriculum, whereas 92% responded positively after completing the course. A slightly greater change was noted in confidence of teaching proof to students, from 50% agreement in the pretest responses to 84% agreement in post-test responses. The fact that PSTs' overall growth of dispositions was relatively modest may be attributed to the not necessarily justifiable feeling of confidence at the start of the semester due to the PSTs' prior mathematical coursework.

Discussion

This paper described the first iteration of a 3-year design-based-research project aimed to enhance PSTs' knowledge and dispositions for teaching proof at the secondary level. We have described the overall structure of the course *Mathematical Reasoning and Proving for Secondary Teachers*, the theoretical underpinnings of its design, and provided details on the Conditional Statements module, one of four course modules. Our first two goals in this paper were to describe how PSTs interacted with the Conditional Statements module, and to examine what mathematical and pedagogical ideas addressed in the Conditional Statements module were implemented in PSTs' lessons. Towards this end we provided descriptions of the various components of the module, such as an activity on sorting conditional statements and the Lesson*Sketch* experience *Who is right?* which required the PSTs to evaluate the logical reasoning of students in relation to determining the validity of a conditional statement. We also presented data on how the PSTs interacted with these components of the module, and identified particular strengths and weaknesses in PSTs' knowledge. Our data show that PSTs interacted with different components of the CS module in productive and meaningful ways. Although the scope of the paper does not allow presenting the full detail of the interactions, the analysis of classroom videos showed that PSTs engaged in rich discussions around the logical aspects of conditional statements actively seeking to clarify their meaning, especially when presented in the alternative forms to "If P then Q". The PSTs also were deeply concerned with the pedagogical side of teaching middle- or high-school students about conditional statements, as their lesson plans and reflections attest.

The data on PSTs' interactions with the CS module serve as a backdrop for understanding the gains in PSTs' content and pedagogical knowledge of proof, following their participation in the capstone course. The comparison of PSTs' performance on pre- and post- measures of MKT-P shows that the areas in which the PSTs' growth of MKT-P was most evident are those that were emphasized in the course, namely, the logical aspects of proof and pedagogical knowledge specific to proving. Due to the small sample size we were unable to test whether the gains were statistically significant, however, we are encouraged by these outcomes, especially since they are based on evidence beyond PSTs' self-report.

One of the critical elements of design-based-research, according to Edelson (2002), is to treat a particular study as an instance of a more general phenomena to develop educational design theories that go beyond the specific research context. This can be achieved by examining the relationship between the design features of the intervention – the capstone course – and the PSTs' learning. In the descriptions above we highlighted the range of practice-based elements in the course design: Analyzing students' conceptions of proof; devoting course time and resources to lesson planning and sharing; teaching in middle school and high school classrooms; and reflecting on one's teaching, supported by video technology. We assert that all these elements contributed to enhancement of PSTs' MKT-P and dispositions towards proof. One particular aspect where this can be seen is the PSTs' pedagogical knowledge for proving. Throughout the course, and in the final course reflection, many PSTs acknowledged that they often found it challenging to integrate the proof-themes with pedagogical practices. However, as we showed in the case of CS module above, the PSTs' lesson plans and classroom implementations clearly reflect the proof-themes addressed in the course (although there was obvious variation among the PSTs). This is also visible in the increased PSTs' scores on the MKT-P portion related to the Pedagogical Aspects of Proof. This suggests to us, that the repeated cycles of lesson development, implementation and video-supported reflection contributed to PSTs' pedagogical knowledge for proving.

Our data analysis is still ongoing, as we examine video of on-campus sessions and of the PSTs' teaching to create a more fine-grained description of how PSTs' content and pedagogical knowledge evolved throughout the course, and to match this growth to the design principles of the course. The results of this analysis will inform future iterations of our project. In particular, we plan to further conceptualize and enhance instructional scaffolding of the course in the subsequent iterations of the study, to better support PSTs' learning.

Through our data analysis we seek to generate an evidence-based instructional model, and four proof-modules that can be adopted by other courses or institutions to improve preparation of secondary mathematics teachers, and, potentially advance the field's understanding of how to support PSTs' development of mathematical knowledge for teaching proof.

Acknowledgments

This research was supported by the National Science Foundation, Award No. 1711163. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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