

DETECTING FULLY IRREDUCIBLE AUTOMORPHISMS: A POLYNOMIAL TIME ALGORITHM. *WITH AN APPENDIX BY MARK C. BELL.*

ILYA KAPOVICH

ABSTRACT. In [30] we produced an algorithm for deciding whether or not an element $\varphi \in \text{Out}(F_N)$ is an iwip (“fully irreducible”) automorphism. At several points that algorithm was rather inefficient as it involved some general enumeration procedures as well as running several abstract processes in parallel. In this paper we refine the algorithm from [30] by eliminating these inefficient features, and also by eliminating any use of mapping class groups algorithms.

Our main result is to produce, for any fixed $N \geq 3$, an algorithm which, given a topological representative f of an element φ of $\text{Out}(F_N)$, decides in polynomial time in terms of the “size” of f , whether or not φ is fully irreducible.

In addition, we provide a train track criterion of being fully irreducible which covers all fully irreducible elements of $\text{Out}(F_N)$, including both atoroidal and non-atoroidal ones.

We also give an algorithm, alternative to that of Turner, for finding all the indivisible Nielsen paths in an expanding train track map, and estimate the complexity of this algorithm.

An appendix by Mark Bell provides a polynomial upper bound, in terms of the size of the topological representative, on the complexity of the Bestvina-Handel algorithm[3] for finding either an irreducible train track representative or a topological reduction.

1. INTRODUCTION

For an integer $N \geq 2$ an outer automorphism $\varphi \in \text{Out}(F_N)$ is called *fully irreducible* if there no positive power of φ preserves the conjugacy class of any proper free factor of F_N . The notion of being fully irreducible serves as the main $\text{Out}(F_N)$ counterpart of the notion of a pseudo-Anosov mapping class of finite type surface. Fully irreducible automorphisms are crucial in the study of dynamics and geometry of $\text{Out}(F_N)$, and understanding the structural properties of its elements and of its subgroups. If $\varphi \in \text{Out}(F_N)$ is fully irreducible, then φ acts with North-South dynamics on the compactified Outer space $\overline{\text{CV}}_N$ [34], and, under the extra assumption of φ being atoroidal, φ also acts with North-South dynamics on the projectivized space of geodesic currents $\mathbb{P}\text{Curr}(F_N)$ [37] (see also [45]). By a result of Bestvina and Feighn [5], $\varphi \in \text{Out}(F_N)$ acts as a loxodromic isometry of the free factor complex of F_N if and only if φ is fully irreducible. The case where a subgroup $H \leq \text{Out}(F_N)$ contains some fully irreducible is the first case to be treated in the solution of the Tits Alternative for $\text{Out}(F_N)$, see [6, 7, 8]. The modern subgroup structure theory of $\text{Out}(F_N)$ also involves this setting. Thus a result of Handel and Mosher [27] shows that if $H \leq \text{Out}(F_N)$ is a finitely generated subgroup then either H contains a fully irreducible element or H contains a subgroup H_0 of finite index in H , such that H_0 leaves invariant a proper free factor of F_N . Subsequently, Horbez [28] proved, by a very different argument, that this result remains true with the assumption that H be finitely generated dropped. See [16, 26, 31] for other examples illustrating the important role that fully irreducibles play in the study of $\text{Out}(F_N)$.

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In [30] we produced an algorithm that, given $\varphi \in \text{Out}(F_N)$, decided whether or not φ is fully irreducible. That algorithm did not include a complexity estimate and, as written, was computationally quite inefficient. There were several places in the algorithm where two general enumeration procedures were run in parallel, and in one subcase the algorithm also invoked an algorithm from surface theory as a “black box”.

In the present paper we remedy this situation and produce a refined algorithm for deciding whether or not an element φ of $\text{Out}(F_N)$ is fully irreducible. This improved algorithm runs in polynomial time in terms of the “size” of φ or of a topological representative of φ . The algorithm also does not use any surface theory algorithms as subroutines. The main result is stated in Theorem 1.1 below. The term “standard” for a topological representative f' of φ is defined in Section 2, but basically it just means that we view f' as a combinatorial object for algorithmic purposes, as is traditionally done in train track literature.

Given a finite graph Δ , and graph map $g : \Delta \rightarrow \Delta$, we denote $\|g\| := \max_{e \in E\Delta} |g(e)|$, where $|g(e)|$ is the combinatorial length of the edge-path $g(e)$. Let $N \geq 2$ and let $A = \{a_1, \dots, a_N\}$ be a free basis of F_N . Let $\Phi \in \text{Aut}(F_N)$. Denote by $|\Phi|_A := \max_{i=1}^N |\Phi(a_i)|_A$.

The main result of this paper is:

Theorem 1.1. *Let $N \geq 2$ be fixed.*

- (1) *There exists an algorithm that, given a standard topological representative $f' : \Gamma' \rightarrow \Gamma'$ of some $\varphi \in \text{Out}(F_N)$, such that every vertex in Γ has degree ≥ 3 , decides whether or not φ is fully irreducible. The algorithm terminates in polynomial time in terms of $\|f'\|$.*
- (2) *Let $A = \{a_1, \dots, a_N\}$ be a fixed free basis of F_N . There exists an algorithm, that given $\Phi \in \text{Aut}(F_N)$ (where Φ is given as an N -tuple of freely reduced words over $A^{\pm 1}$, $(\Phi(a_1), \dots, \Phi(a_N))$), decides, in polynomial time in $|\Phi|_A$, whether or not Φ is fully irreducible.*
- (3) *Let S be a finite generating set for $\text{Out}(F_N)$. There exists an algorithm that, given a word w of length n over $S^{\pm 1}$, decides whether or not the element φ of $\text{Out}(F_N)$ represented by w is fully irreducible. This algorithm terminates in at most exponential time in terms of the length of the word w .*

After our paper [30], Clay, Mangahas and Pettet [12] produced a different algorithm for deciding whether an element of $\varphi \in \text{Out}(F_N)$ is fully irreducible. Their algorithm was more efficient than that in [30], but did not include an explicit complexity estimate. A key step in their approach was to reduce to a subcase where one can bound the size of the Stallings subgroup graph of a periodic free factor B of Φ by a polynomial in terms of $|\Phi|_A$. They then enumerate *all* Stallings subgroup graphs of possible free factors with that size bound and check if any of them are φ -periodic. This approach produces, at best an exponential time algorithm in terms of the size of φ or of a topological representative of φ (in the setting of parts (1) and (2) of Theorem 1.1 above), and at best a double exponential algorithm in terms of the word length of φ in the generators of $\text{Out}(F_N)$ (in the setting of part (3) Theorem 1.1). Feighn and Handel also gave an alternate algorithm in [23] for deciding if an element of $\text{Out}(F_N)$ is fully irreducible. They use their general extremely powerful machinery of CT train tracks and algorithmic computability of CT train tracks for rotationless elements of $\text{Out}(F_N)$. Their proof does not come with a complexity estimate and it remains unclear if a reasonable complexity estimate can be extracted from that proof.

A key fact that goes into the proof of Theorem 1.1 is Theorem 4.8 providing a train track characterization of being fully irreducible that covers all elements of $\text{Out}(F_N)$ (both atoroidal and non-atoroidal ones). See [39, Lemma 9.9], [40, Proposition 4.1], [29, Proposition 5.1], [30, Proposition 4.4] for related earlier train track criteria of being fully irreducible, in more restricted settings. We include the statement of Theorem 4.8 here because of its potential further utility:

Theorem 1.2. *Let $N \geq 2$ and let $\varphi \in \text{Out}(F_N)$ be arbitrary. Then the following are equivalent:*

- (1) *The automorphism φ is fully irreducible.*
- (2) *The automorphism φ is primitively atoroidal and there exists a weakly clean train track representative $f : \Gamma \rightarrow \Gamma$ of φ .*
- (3) *The automorphism φ is primitively atoroidal, there exists a weakly clean train track representative of φ , and every train track representative of φ is weakly clean.*

The term *weakly clean train track representative* is formally defined in Section 4. An element $\varphi \in \text{Out}(F_N)$ is called *primitively atoroidal* if there do not exist $n \neq 0$ and a primitive element $g \in F_N$ such that $\varphi^n[g] = [g]$. This notion, or rather its negation, first appeared in [12], where non primitively atoroidal elements of $\text{Out}(F_N)$ are called *cyclically reducible*. Primitively atoroidal elements of $\text{Out}(F_N)$ are also specifically studied in [23].

As a step in the proof of Theorem 1.1 we produce an algorithm, with a polynomial time complexity estimate, that given an expanding standard train track map f , finds all the INPs (if any) for f ; see Theorem 5.1. We then refine this algorithm to produce a corresponding statement for finding all periodic INPs, see Theorem 5.13 below. The first (and until now the only) known algorithm for finding INPs in an expanding train track map was due to Turner [44], and did not contain a complexity estimate.¹ Our algorithm provided here is based on fairly different considerations from those of Turner. The main idea is to use precise quantitative estimates of bounded cancellation constants to get efficient control of the length of the two “legs” of an INP. This approach turns out to be sufficient to obtain a polynomial time algorithm for finding INPs and periodic INPs.

The algorithm given in Theorem 1.1 needs another component, namely the classic Bestvina-Handel algorithm [3] for trying to find a train track representative of a free group automorphism. The Bestvina-Handel algorithm is one of the most useful results in the study of $\text{Out}(F_N)$, both as a theoretical and as an experimental tool. Yet, amazingly enough, until now the computational complexity of this algorithm has not been analyzed. The appendix to the present paper, written by Mark Bell, fills that gap. Theorem A.1 of the appendix provides a polynomial time estimate, in terms of the size $\|f\|$ of a given topological representative of an element of $\text{Out}(F_N)$ (where $N \geq 2$ is fixed), on the running time of the Bestvina-Handel algorithm.

The most complicated part of the algorithm given in Theorem 1.1 consists in finding all the pINPs in a expanding irreducible train-track representative $f : \Gamma \rightarrow \Gamma$ for φ produced by the Bestvina-Handel algorithm, and then processing this data to decide whether a) φ is not primitively atoroidal (in which case we conclude that φ is not fully irreducible and the algorithm stops), or b) φ is not fully irreducible for some other reason, or c) φ is primitively atoroidal (in which case we then use Theorem 1.2 above). It is now known, for geometric reasons, that under fairly mild assumptions on the support of a measure defining a random walk, long random walks on $\text{Out}(F_N)$ produce elements that are atoroidal and fully irreducible, with probability tending to 1 as the length of the walk tends to ∞ . See [36] as the main general result about random walks on groups acting on hyperbolic spaces, together with [22] for the relevant action of $\text{Out}(F_N)$ on the “co-surface graph”. It is also known [33] that a special type of a random walk on $\text{Out}(F_N)$ (where $N \geq 3$), with asymptotically positive probability produces an element of $\text{Out}(F_N)$ which is atoroidal, fully irreducible and is represented by an expanding irreducible train track map with no pINPs. Moreover, experimental data using Thierry Coulbois’ train track package [14] suggests that a simple random walk on $\text{Out}(F_N)$ generically produces an automorphisms that is represented by an

¹There are some issues with Turner’s result requiring further clarification. The “expanding” assumption is missing from the statement of his theorem, and the category of graph maps in which he works is not sufficiently precisely defined, including the precise meaning of a “train track map” of what exactly it means to “compute” and INP and its endpoints algorithmically.

expanded irreducible train track map with no pINPs. Thus it is likely that for processing “random” inputs, the more complicated parts of the algorithm in Theorem 1.1 dealing with manipulating pINPs are rarely needed. It would be extremely interesting to obtain an actual proof of the above suggested absence of pINPs for automorphisms generically produced by a simple random walk on $\text{Out}(F_N)$. In particular, such a result would shed light on the index properties of random elements of $\text{Out}(F_N)$ (see [16] for a survey of the $\text{Out}(F_N)$ index theory).

We thank the referees of this paper for helpful comments. We are particularly grateful to one of the referees for pointing out a simpler and more efficient proof of Proposition 5.15 below than our original argument. Moreover, our original proof of that proposition did not include a complexity estimate, while the argument suggested by the referee produced a polynomial time complexity estimate in the conclusion of Proposition 5.15. This change also resulted in a simplification of the proof of our main result, Theorem 1.1.

2. BACKGROUND

We adopt the same definitions, terminology, and notations regarding graph maps and train track maps as in [30] and [19]. Although this point is not made explicit in [30] (because it is not directly relevant there), we now explicitly assume that all the graphs under considerations are equipped with a PL structure (that is, in the terminology of [19] they are *linear graphs*) and that all graph maps and train track maps under considerations are PL-maps, or, more precisely, that in the terminology of [19] they are *linear graph maps*. Section 2 of [19] discusses in detail the notion of a linear graph map and a more general notion of a “topological graph map”, where for the latter one does not impose any restrictions on the nature of homeomorphisms by which subdivision intervals in an edge get mapped to the edges of the graph. For many combinatorial considerations in train track theory the distinction between topological graph maps and linear graph maps is immaterial. This distinction is almost never discussed in the train track literature. The various papers on this subject almost never explain what they actually mean by saying that a graph map maps an edge to an edge-path (although they typically implicitly assume that they are working in the PL category, that is, they are working with linear graph maps). The distinction between topological graph maps and linear graph maps does matter when Nielsen paths and periodic points of train track maps are discussed. That is why we adopt the convention of working only with linear graph maps here. (Indeed, in the category of topological graph maps an expanding irreducible train track map may have uncountably many periodic and fixed points and may exhibit other pathologies). We refer the reader to Section 2 of [19] for an extended discussion on the topic.

Definition 2.1 (Standard graph map). We say that a graph map $f : \Gamma \rightarrow \Gamma$ is *standard* if Γ is equipped with a linear structure identifying every open edge $e \in E\Gamma$ with an interval $(0, 1)$ via a map $\xi_e : e \rightarrow (0, 1)$ with the following property. If $f(e) = e_1 \dots e_n$, then for each open subinterval J_i of e that f maps to e_i (where $i = 1, \dots, n$) we have $\xi_e(J_i) = (\frac{i-1}{n}, \frac{i}{n})$ and, moreover, $\xi_{e_i} \circ f \circ \xi_e^{-1}$ maps $(\frac{i-1}{n}, \frac{i}{n})$ by an affine orientation-preserving bijection to the interval $(0, 1)$.

Convention 2.2. We will always assume (unless explicitly stated otherwise) that various graph maps are linear graph maps. Most abstract definitions and results will be stated in that context.

For algorithmic results we will assume that our graph maps are standard graph maps, and this assumption will always be mentioned explicitly. This assumption is not burdensome since every graph map $f : \Gamma \rightarrow \Gamma$ is isotopic relative the vertices of Γ to a standard graph map $g : \Gamma \rightarrow \Gamma$. Note however, that in this situation it need not be the case that $f^n : \Gamma \rightarrow \Gamma$ is standard (where $n \geq 1$). Therefore some care will need to be taken when working with powers of standard graph maps, but for most combinatorial considerations these issues will be immaterial.

We also adopt the definitions of [19] regarding the notion of a *path* in a graph Γ . We assume that all paths under considerations are PL-paths, in the terminology of Section 2 of [19]. In particular, every path γ in a graph Γ has an initial point $o(\gamma)$ and a terminal point $t(\gamma)$. The points $o(\gamma)$, $t(\gamma)$ are not required to be vertices of Γ . For a path γ in Γ we denote by γ^{-1} the inverse path of γ , so that $o(\gamma^{-1}) = t(\gamma)$ and $t(\gamma^{-1}) = o(\gamma)$. A path γ is *nontrivial* if $o(\gamma) \neq t(\gamma)$. Note that an essential closed path in Γ is necessarily nontrivial.

For an edge-path γ in a graph Γ we denote by $|\gamma|$ the combinatorial length, that is the number of edges, in γ .

For any path α in a graph Γ there exists a unique smallest edge-path $\hat{\alpha}$ containing α as a subpath. We call $\hat{\alpha}$ the *simplicial support* of α and denote $|\alpha| := |\hat{\alpha}|$.

All of the train track maps in this paper are assumed to be homotopy equivalences (although it turns out that dropping this requirement sometimes does lead to useful results, see [1, 20, 21, 41]).

Recall that a train track map $f : \Gamma \rightarrow \Gamma$ is called *irreducible* if the transition matrix $M(f)$ is irreducible, that is, if for every position i, j there exists $n \geq 1$ such that $(M(f)^n)_{ij} > 0$. Equivalently, a train track map $f : \Gamma \rightarrow \Gamma$ is irreducible if and only if for every (oriented) edges e, e' of Γ there exists $n \geq 1$ such that the path $f^n(e)$ contains an occurrence of e' or of $(e')^{-1}$. Recall also that a graph map $f : \Gamma \rightarrow \Gamma$ is called *expanding* if for every edge e of Γ the combinatorial length $|f^n(e)|$ of the path $f^n(e)$ goes to infinity as $n \rightarrow \infty$. For an irreducible train-track map $f : \Gamma \rightarrow \Gamma$ being expanding is equivalent to the condition $\lambda(f) > 1$, where $\lambda(f)$ is the Perron-Frobenius eigenvalue of the matrix $M(f)$.

We recall the following key notions from train track theory:

Definition 2.3 (Nielsen paths). Let $f : \Gamma \rightarrow \Gamma$ be an expanding train map.

- (1) A point $x \in \Gamma$ is *f-periodic* or just *periodic* if there exists $n \geq 1$ such that $f^n(x) = x$. For a periodic point $x \in \Gamma$, the smallest $n \geq 1$ such that $f^n(x) = x$ is called the *period* of x .
- (2) A nontrivial tight path γ in Γ is called a *Nielsen path* for f if $f(\gamma)$ is homotopic to γ relative the endpoints of γ (which implies that the endpoints of γ are fixed by f).
- (3) A nontrivial tight path γ in Γ is called a *periodic Nielsen path* for f if there exists $n \geq 1$ such that γ is a Nielsen path for f^n ; in this case the smallest such $n \geq 1$ is called the *period* of γ . (Thus if γ is a periodic Nielsen path of period n then each of the endpoints of γ is periodic of period $\leq n$ and is fixed by f^n).
- (4) A *periodic indivisible Nielsen path* (or pINP) is a periodic Nielsen path which cannot be written as a concatenation of two nontrivial periodic Nielsen paths.
- (5) An *indivisible Nielsen path* (or INP) is a Nielsen path which cannot be written as a concatenation of two nontrivial Nielsen paths.

We recall the following key facts (see [3, 6] for background info):

Proposition 2.4. Let $f : \Gamma \rightarrow \Gamma$ be an expanding train track map. Then the following holds:

- (1) There are only finitely many (possibly none) periodic indivisible Nielsen paths in Γ for f .
- (2) If η is a pINP, then η has the form $\eta = \alpha\beta^{-1}$, where α, β are nontrivial legal paths with $v = t(\alpha) = t(\beta) \in V\Gamma$ (but where $o(\alpha), o(\beta)$ need not be vertices of Γ) such that the turn at v between α and β is illegal.
- (3) A path η is a pINP of period 1 for f if and only if η is an INP for f .
- (4) For two pINPs η_1, η_2 with $t(\eta_2) = o(\eta_1)$ the following conditions are equivalent:
 - (a) the path $\eta_1\eta_2$ is tight;
 - (b) the path $\eta_1\eta_2$ is legal.

- (5) If $\eta_1 = \alpha_1\beta_1^{-1}$ and $\eta_2 = \alpha_2\beta_2^{-1}$ are two pINPs written in the form given by part (2) of this lemma and such that $o(\alpha_1) = o(\alpha_2) = v \in V\Gamma$ and that directions at v given by α_1 and α_2 are the same, then α_1 is an initial segment of α_2 or α_2 is an initial segment of α_1 .
- (6) For a nontrivial cyclically tight circuit γ in Γ , $[\gamma]$ represents an f -periodic conjugacy class in $\pi_1(\Gamma)$ if and only if γ has the form $\gamma = \eta_1 \dots \eta_k$, where $k \geq 1$, where each η_i is a pINP and where the concatenation $\eta_i\eta_{i+1}$ is tight for $i = 1, \dots, k$ (with η_{k+1} interpreted as η_1).

We will also need the following key algorithmic result essentially due to Turner [44]:

Proposition 2.5. *There exists an algorithm that, given an expanding standard train track map $f : \Gamma \rightarrow \Gamma$, finds all the periodic indivisible Nielsen paths for f .*

Later we will show (see Theorem 5.13 below) that one can provide a reasonably satisfactory complexity estimate for the above statement.

Proof. By a result of Feighn and Handel [23], there exists a computable universal bound $n = n(b_1(\Gamma)) \geq 1$ such that η is a periodic indivisible Nielsen path for f if and only if η is an INP for f^n .

We then apply the result of Turner [44] to algorithmically find all the INPs for f^n . $\square$

Remark 2.6. In fact, Feighn and Handel show in [23, Corollary 3.14] that one can take $n(r) = 3^{r^2-1} (g(15r-15))!$, where $r = b_1(\Gamma)$. Here $g(k)$ is Landau's function, which computes the maximal order of an element in the symmetric group S_k . It is known that $g(k) \leq e^{k/e}$, so that $g(k)$ grows at most exponentially in k .

3. GRAPH OF PERIODIC NIELSEN PATHS

We need the following useful definition, which is a slightly adapted version of the graph $S(f)$ defined in Section 8.2 of [23] in the context of CT maps.

In this section we will assume that Γ is a finite connected graph with all vertices of degree ≥ 3 and with $b_1(\Gamma) = N \geq 2$, and equipped with a fixed marking identifying $\pi_1(\Gamma)$ with F_N .

Definition 3.1 (Graph of periodic Nielsen paths). Let $f : \Gamma \rightarrow \Gamma$ be an expanding irreducible train track map.

We define the *graph of periodic Nielsen paths* $S(f)$ for f as follows.

If there are no pINPs for f , define the graph $S(f)$ to be empty.

Otherwise, enumerate all the distinct periodic indivisible Nielsen paths $\eta_1, \dots, \eta_k$ for f , where we eliminate duplication due to inversion (that is, for a pINP η , we include only one of the pINPs η, η^{-1} in the list). Let Y_f be the set of all $x \in \Gamma$ such that x occurs as an endpoint of some η_i . Put $V(S(f)) := Y_f$.

For each $i = 1, \dots, k$ we put a topological edge (i.e. a 1-cell) in $S(f)$ joining vertices $o(\eta_i)$ and $t(\eta_i)$ of $S(f)$. We orient this edge from $o(\eta_i)$ to $t(\eta_i)$ and label it by η_i .

Thus $S(f)$ is a 1-complex as well as an oriented labelled graph in the sense of [30].

The edge-labels in $S(f)$ naturally define, by using concatenation, the *labelling map* μ , associating to every edge-path (respectively, every circuit) in $S(f)$ an edge-path (respectively a circuit) in Γ .

If $x \in Y_f$, then the labelling map μ defines a natural homomorphism $\mu_{*,x} : \pi_1(S_f, x) \rightarrow \pi_1(\Gamma, x)$. Note that this homomorphism, a priori, need not be injective, since it is possible that two distinct pINPs start with the same vertex and the same direction in Γ . So the image of a tight circuit at x in S_f under the labelling map μ may not be tight as a circuit in Γ .

Note that any cyclic concatenation of pINPs that tightens to a homotopically nontrivial closed path in Γ defines a nontrivial $f_{\#}$ -periodic conjugacy class in $\pi_1(\Gamma)$. Proposition 2.4 says that every $f_{\#}$ -periodic conjugacy class arises in this way. Therefore Proposition 2.4 implies:

Proposition 3.2. *Let $f : \Gamma \rightarrow \Gamma$ be an expanding irreducible train track map.*

Then a conjugacy class $[g]$ in $\pi_1(\Gamma)$ is $f_{\#}$ -periodic if and only if there is $x \in Y_f$ such that $[g]$ is represented by an element of $\mu_{,x}(\pi_1(S(f), x))$.*

In particular, $f_{\#}$ is atoroidal if and only if for every $x \in Y_f$ we have $\mu_{,x}(\pi_1(S(f), x)) = \{1\}$.*

We also record the following useful property of the graph $S(f)$ which follows directly from Proposition 2.4.

Proposition 3.3. *Let f be as in Proposition 2.4.*

Let Q be a connected component of $S(f)$, let x be a vertex of Q and let $U \leq F_N$ be the image of $\pi_1(Q, x)$ in F_N under $\mu_{,x}$. Then there is some $n \geq 1$ such that for the element $\varphi \in \text{Out}(F_N)$ represented by f we have $\varphi^n[U] = [U]$, and, moreover, there is a representative of φ^n in $\text{Aut}(F_N)$ which restricts to the identity map on U .*

4. CLEAN TRAIN TRACKS AND FULL IRREDUCIBILITY

Definition 4.1 (Whitehead graphs and stable laminations). Let $f : \Gamma \rightarrow \Gamma$ be an expanding irreducible train track map. Recall for a vertex $v \in V\Gamma$, the *Whitehead graph* $Wh_{\Gamma}(v, f)$ is a simple graph that has as its vertices the set of all $e \in E\Gamma$ with $o(e) = v$, and where two distinct such edges $e, e' \in E\Gamma$ are adjacent in $Wh_{\Gamma}(v, f)$ if and only if there exist $e'' \in E\Gamma$ and $n \geq 1$ such that $e^{-1}e'$ is a subpath of $f^n(e'')$ (note that in this case $(e')^{-1}e$ is a subpath of $f^n((e'')^{-1})$, so that this adjacency condition is symmetric). Recall also that the *stable lamination* $\Lambda(f)$ of f consists of all bi-infinite tight paths (called *leaves* of $\Lambda(f)$) $\gamma = \dots, e_{-1}, e_0, e_1, e_2, \dots$ in Γ such that for all $-\infty < i < j < \infty$ the path $e_i \dots e_j$ occurs as a subpath of $f^n(e)$ for some $e \in E\Gamma$ and some $n \geq 1$. See [30] for more details.

We also recall the following key notion from [30]:

Definition 4.2 (Clean train track map). Let $f : \Gamma \rightarrow \Gamma$ be a train track map.

- (1) We say that f is *clean* if the matrix $M(f)$ is primitive (that is, there exists $t \geq 1$ such that every entry of $M(f^t) = (M(f))^t$ is positive) and if for every vertex $v \in V\Gamma$ the Whitehead graph $Wh_{\Gamma}(v, f)$ is connected.
- (2) We say that f is *weakly clean* if f is expanding and irreducible (that is, the matrix $M(f)$ is irreducible and has $\lambda(f) > 1$) and if for every vertex $v \in V\Gamma$ the Whitehead graph $Wh_{\Gamma}(v, f)$ is connected.

Although this fact is not obvious from the definition, [19] showed:

Proposition 4.3. *Let $f : \Gamma \rightarrow \Gamma$ be an expanding irreducible train track map. Then f is clean if and only if f is weakly clean.*

We recall the following key result from [30]:

Proposition 4.4. *Let $f : \Gamma \rightarrow \Gamma$ be a clean expanding irreducible train track map.*

Let $1 \neq H \leq \pi_1(\Gamma)$ be a finitely generated subgroup of infinite index in $\pi_1(\Gamma)$ and let Δ_H be the Stallings core of the cover of Γ corresponding to H . Then Δ_H does not carry any leaf of $\Lambda(f)$, that is, no leaf of $\Lambda(f)$ lifts to a path in Δ_H .

Recall that an automorphism $\varphi \in \text{Out}(F_N)$ is called *atoroidal* if there do not exist $n \geq 1$ and $1 \neq g \in F_N$ such that $\varphi^n([g]) = [g]$, where $[g]$ is the conjugacy class of g in F_N . We need the following slightly more general notion (see Definition 13.1 of Feighn and Handel [23]):

Definition 4.5. We say that $\varphi \in \text{Out}(F_N)$ is *primitively atoroidal* if there do not exist $n \geq 1$ and a primitive element $g \in F_N$ such that $\varphi^n([g]) = [g]$.

Recall the following well-known result of Bestvina-Handel [3]:

Proposition 4.6. *Let $\varphi \in \text{Out}(F_N)$ be non-atoroidal, where $N \geq 2$.*

Then φ is fully irreducible if and only if there exist a compact connected (possibly non-orientable) surface Σ with one boundary component and with $\pi_1(\Sigma) = F_N$ and a pseudo-Anosov homeomorphism $g : \Sigma \rightarrow \Sigma$ such that $\varphi = g_\#$. In this case φ is primitively atoroidal, and $[w]$ is a nontrivial φ -periodic conjugacy class in F_N if and only if $w = [u^k]$ for some $k \neq 0$, where u is the peripheral curve of Σ .

Remark 4.7. Thus all fully irreducible elements of $\text{Out}(F_N)$ are primitively atoroidal. Moreover, if $\varphi \in \text{Out}(F_N)$ admits periodic conjugacy classes $[w], [z]$ such that for all nonzero t, s we have $[w^t] \neq [z^s]$, then φ is not fully irreducible.

The following result provides a unified train track characterization of all fully irreducibles, both atoroidal and non-atoroidal ones, following our approach in [30].

Theorem 4.8. *Let $N \geq 2$ and let $\varphi \in \text{Out}(F_N)$ be arbitrary. Then the following are equivalent:*

- (1) *The automorphism φ is fully irreducible.*
- (2) *The automorphism φ is primitively atoroidal and there exists a weakly clean train track representative $f : \Gamma \rightarrow \Gamma$ of φ .*
- (3) *The automorphism φ is primitively atoroidal, there exists a weakly clean train track representative of φ , and every train track representative of φ is weakly clean.*

Proof. We first show that (1) implies (3). Thus suppose that φ is fully irreducible. Note that by Remark 4.7 φ is primitively atoroidal. Also, by a result of Bestvina-Handel [3], there exists an expanding irreducible train track representative of φ . Let $f : \Gamma \rightarrow \Gamma$ be any train track representative of φ . Since φ is fully irreducible, exactly the same argument as in the proof of implication (1) $\Rightarrow$ (2) of Proposition 4.4 of [30] shows that f is weakly clean. Thus (1) indeed implies (3).

Part (3) directly implies part (2).

Thus suppose that (2) holds and that $f : \Gamma \rightarrow \Gamma$ is a weakly clean train track representative of a primitively atoroidal automorphism $\varphi \in \text{Out}(F_N)$. Then f is clean by Proposition 4.3. Suppose that φ is not fully irreducible. Let $H \leq F_N$ be a proper free factor of minimal rank such that the conjugacy class $[H]$ of H is φ -periodic. Then $\text{rank}(H) \geq 2$ since the case $\text{rank}(H) = 1$ is ruled out by the assumption that φ is primitively atoroidal. The minimality assumption on H implies that there exists a representative $\Psi \in \text{Aut}(F_N)$ of φ^k for some $k \geq 1$ such that $\Psi(H) = H$ and such that $\Psi|_H$ is a fully irreducible automorphism of H . Hence there exists $1 \neq h \in H$ such that the conjugacy class $[h]$ of h is not φ -periodic. Let Δ_H be the Γ -Stallings core for H . Let γ be an immersed circuit in Γ representing the conjugacy class of h . Since $[h]$ is not φ -periodic, the cyclically tightened length of $f^n(\gamma)$ tends to ∞ as $n \rightarrow \infty$. Let s be the simplicial length of γ , so that $\gamma = e_1 \dots e_s$. Let γ_n be the immersed circuit in Γ given by the cyclically tightened form of $f^n(\gamma)$. We can obtain γ_n by cyclic tightening of the path $f^n(e_1) \dots f^n(e_s)$. Thus γ_n is a concatenation of $\leq s$ segments, each of which is a subsegment of $f^n(e)$ for some $e \in E\Gamma$. Since the simplicial length of γ_n goes to infinity as $n \rightarrow \infty$, the length of at least one of these segments tends

to infinity as $n \rightarrow \infty$. Since $\varphi[H] = [H]$, the circuit γ_n lifts to a circuit in Δ_H . Hence there exists a sequence of segments α_n in Γ such that each α_n lifts to a path in Δ_H , such that the simplicial length of α_n goes to infinity as $n \rightarrow \infty$ and such that there are $e_n \in E\Gamma$ and $t_n \geq 1$ with the property that α_n is a subpath of $f^{t_n}(e_n)$. Moreover, since $E\Gamma$ is finite, after passing to a subsequence we can even assume that $e_n = e \in E\Gamma$ for all $n \geq 1$. By a standard compactness argument, it follows that H carries a leaf of $\Lambda(f)$, contrary to the conclusion of Proposition 4.4. $\square$

5. FINDING INPs

In this section we give an alternative algorithm (to that of Turner [44]) for finding all INPs in an expanding train track map. Our algorithm is based on more direct and elementary considerations than that of Turner, although it is possible that Turner's algorithm is computationally more efficient.

The algorithm presented here is known in the folklore. Essential ideas for this algorithm, particularly regarding bounding the length of the “legs” of an INP using the bounded cancellation constant, are already present in the 1992 paper of Bestvina and Handel [3]. The structure of the algorithm is similar to that of the algorithm for finding INPs in a CT train track obtained by Feighn and Handel in [23]. However, the computational complexity analysis that we provide here, is new, and this analysis plays an essential role in the proof of our main result, Theorem 1.1.

For a graph map $g : \Delta \rightarrow \Delta$ (where Δ is a finite graph), we denote $\|g\| := \max_{e \in E\Delta} |g(e)|$. Also, for a finite graph Δ we denote by $m(\Delta)$ the number of topological edges of Γ .

Note that if Δ is a finite connected graph with all vertices of degree ≥ 3 and if r is the rank of the free group $\pi_1(\Delta)$ then $r \leq m(\Delta) \leq 3r - 3$.

Theorem 5.1. *The exists a deterministic algorithm $\mathfrak{A}$ with the following property.*

Given an expanding standard train track map $f : \Gamma \rightarrow \Gamma$ (where Γ is a finite connected graph with all vertices of degree ≥ 3), the algorithm $\mathfrak{A}$ determines whether or not f has any INPs and if yes, finds all the INPs for f . The algorithm terminates in time $O(m^5 \|f\|^{3m+6} \log m \log \|f\|)$, where $m = m(\Gamma)$. In particular, if $m(\Gamma)$ is fixed, the algorithm runs in polynomial time in $\|f\|$.

Convention 5.2. For the remainder of this section we will assume that $f : \Gamma \rightarrow \Gamma$ is an expanding standard train track map, where Γ is a finite graph where every vertex has degree ≥ 3 . We denote $m := m(\Gamma)$.

Remark 5.3. In the setting of Theorem 5.1 the number m of topological edges of Γ is not fixed, but rather is a variable parameter of the problem. Therefore, in various algorithmic computations below, we first need to index the edges of Γ by binary integers, which introduces a factor of $\log m$ in various computations. In fact, describing the graph map $f : \Gamma \rightarrow \Gamma$ itself requires time $O(\|f\| m \log m)$ for that reason. Similar considerations apply to the subdivision Γ' of Γ discussed below.

Recall that for a path (not necessarily an edge-path) γ in some graph, we denote by $|\gamma|$ the combinatorial length of γ , that is the combinatorial length of the minimal edge-path $\hat{\gamma}$ containing γ as a subpath.

We need the following quantitative version of the statement known as “Bounded Cancellation Lemma” that was originally proved in [15].

Proposition-Definition 5.4 (Bounded Cancellation Constant). [6, Lemma 3.1]

Let $f : \Gamma \rightarrow \Gamma$ and m be as Convention 5.2. Put $C_f := m\|f\|$.

Consider any tight path $\gamma = \alpha\beta$ in Γ (where the endpoints of α, β are not assumed to be vertices). Let α' be the tightened form of $f(\alpha)$, and let β' be the tightened form of $f(\beta)$.

Then for the maximal terminal segment γ of α' , that cancels with an initial segment of β' when the path $\alpha'\beta'$ is tightened, we have $|\gamma| \leq C_f$.

Lemma 5.5. *Let $g : \Delta \rightarrow \Delta$ be an expanding graph map, where Δ is a finite graph with $s \geq 1$ topological edges. Then for every $e \in E\Delta$ we have $|g^s(e)| \geq 2$.*

Proof. Let $e \in E\Delta$. If $|g(e)| \geq 2$ then $|g^s(e)| \geq |g(e)| \geq 2$, and we are done. Thus assume that $|g(e)| = 1$. Since by assumption g is expanding, $|g^n(e)| \rightarrow \infty$ as $n \rightarrow \infty$. Let $k \geq 1$ be the maximal integer such that $|g^k(e)| = 1$. Thus for $i = 1, \dots, k$ $e_i := g^i(e)$ is a single edge, and $|g^{k+1}(e)| \geq 2$. Also put $e_0 := e$. The underlying topological edges of the edges $e_0, e_1, \dots, e_k$ must be distinct since otherwise $|g^n(e)| = 1$ for all $n \geq 1$, yielding a contradiction. Therefore $k + 1 \leq s$. Hence $|g^s(e)| \geq |g^{k+1}(e)| \geq 2$, as required. $\square$

Corollary 5.6. *Let η be an INP for f , so that by Proposition 2.4, η has the form $\eta = \alpha\beta^{-1}$, where α, β are nontrivial legal paths with $v = t(\alpha) = t(\beta) \in V\Gamma$ such that the turn at v between α and β is illegal.*

Then $|\alpha|, |\beta| \leq m||f||^m + 4$.

Proof. Suppose that $|\alpha| > m||f||^m + 4$, that is $|\alpha| \geq m||f||^m + 5$. Thus α contains as a subpath an edge-path of combinatorial length $|\alpha| - 2 \geq m||f||^m + 3$. Then by Lemma 5.5, $|f^m(\alpha)| \geq 2(|\alpha| - 2)$.

Note that $f^m : \Gamma \rightarrow \Gamma$ is a train-track map with $||f^m|| \leq ||f||^m$. Hence, by Proposition-Definition 5.4, $C_{f^m} \leq m||f||^m$. The paths $f^m(\alpha)$ and $f^m(\beta)$ are already tight and legal. Therefore $f^m(\alpha)$ contains an edge-subpath τ that survives after tightening $f^m(\alpha)f^m(\beta^{-1})$ and such that

$$|\tau| \geq 2(|\alpha| - 2) - C_{f^m} \geq 2|\alpha| - 4 - m||f||^m \geq |\alpha| + 1$$

where the last inequality holds because we assumed that $|\alpha| \geq m||f||^m + 5$.

However, $\eta = \alpha\beta^{-1}$ is a Nielsen path for f^m , with α, β legal and the turn between α and β illegal. Therefore, after tightening $f^m(\alpha)f^m(\beta^{-1})$ the portion of $f^m(\alpha)$ that does not cancel must be exactly its initial segment α which has combinatorial length $|\alpha|$, yielding a contradiction. Thus $|\alpha| \leq m||f||^m + 4$. A symmetric argument shows that $|\beta| \leq m||f||^m + 4$. $\square$

Convention 5.7. Note that the expanding assumption on f implies that if $e \in E\Gamma$ and $k \geq 1$ then $f^k(e) \neq e^{\pm 1}$. Let $Fix(f) := \{x \in \Gamma \mid f(x) = x\}$ be the fixed set of f .

If $f(e) = e_1 \dots e_n$ and $e_i = e^{\pm 1}$ for some $1 < i < n$ or if $f(e_i) = e^{-1}$ for $i \in \{1, n\}$ then we get a fixed point of f in the interior of the subinterval of e that gets mapped to e_i . Moreover, every fixed point of f , contained in the interior of some edge, arises in this fashion. (Here we are using our PL assumption on f). Thus for every $e \in E\Gamma$ there are at most $|f(e)|$ fixed points of f in the interior of e and at most $|f(e)| + 1$ fixed points in the closure of e . Therefore $\#(Fix(f)) \leq m(||f|| + 1) \leq 2m||f||$.

Let Γ' be the graph obtained from Γ by subdividing along all elements of $Fix(f)$ which are not already vertices of Γ . We denote by $f' : \Gamma' \rightarrow \Gamma'$ the map f considered as a graph map $\Gamma' \rightarrow \Gamma'$.

Every path in Γ is a path in Γ' . Sometimes, in order to avoid confusion, for a path α in Γ , we will denote its combinatorial length (in the sense defined in the beginning of Section 2) in Γ by $|\alpha|_\Gamma$ and we will denote its combinatorial length in Γ' by $|\alpha|_{\Gamma'}$.

The following is a direct consequence of the definition of f' and Γ' :

Lemma 5.8. *The following hold:*

- (1) *Each edge of Γ got subdivided into $\leq |f(e)|_\Gamma$ edges in Γ' . Hence the graph Γ' and the map $f' : \Gamma' \rightarrow \Gamma'$ can be computed in time $O(||f||m(\log m + \log ||f||))$. [The $\log m + \log ||f||$ -factor arises since Γ' has $\leq m||f||$ topological edges and, when describing f' , we first need to index these edges by binary integers.]*

- (2) For every path α in Γ we have $|\alpha|_\Gamma \leq |\alpha|_{\Gamma'} \leq \|f\| \cdot |\alpha|_\Gamma$.
- (3) We have $\|f'\| \leq \|f\|^2$.

Remark 5.9. If the number m of topological edges of Γ is considered a constant, and α is an edge-path in Γ then for $t \geq 1$ we have $|f^t(\alpha)|_\Gamma \leq \|f\|^t |\alpha|_\Gamma$ and the path $f^t(\alpha)$ can be computed in $O(\|f\|^t |\alpha|_\Gamma)$ steps. However, it is often more efficient to use a different computational bound in this situation, since $|f^t(\alpha)|_\Gamma$ may be much smaller than $\|f\|^t |\alpha|_\Gamma$. Note that $f(\alpha)$ can be computed in $O(|f(\alpha)|_\Gamma)$ steps and hence $f^t(\alpha)$ can be computed in $O(t|f^t(\alpha)|_\Gamma)$ steps (if m is viewed as a constant). If m is a variable parameter, then in these computations we first need to index the edges of Γ as $e_1, \dots, e_m$, with the subscript i of e_i written in binary form. Therefore computing $f(\alpha)$ can be done in $O(|f(\alpha)|_\Gamma \log m)$ steps and hence $f^t(\alpha)$ can be computed in $O(t|f^t(\alpha)|_\Gamma \log m)$ steps. The graph Γ' has $m' \leq m\|f\|$ topological edges. Therefore, by similar reasoning, given an edge-path α in Γ' , we can compute $(f')^t(\alpha)$ in $O(t|(f')^t(\alpha)|_{\Gamma'}(\log m + \log \|f\|))$ steps. We will use these bounds in the computations below.

Definition 5.10 (Eigenrays). Let d be a fixed direction in Γ given by a nondegenerate segment τ in Γ starting at a fixed point $x \in \Gamma$ and such that $f(\tau)$ has τ as an initial segment. Since f is expanding, the PL assumption on f implies that τ is a proper initial segment of $f(\tau)$ and that $|f^k(\tau)| \rightarrow \infty$ as $k \rightarrow \infty$. Thus we can form a reduced legal path ρ_d in Γ such that for every $k \geq 1$ $f^k(\tau)$ is an initial segment of ρ_d .

This path ρ_d is called the *eigenray* of f determined by d . (It is not hard to check that this definition of ρ_d does not depend on the choice of τ as above).

Note that in the above situation $f(\rho_d) = \rho_d$. Since $f = f'$ as functions, and since the graphs Γ and Γ' have the same set of fixed points and the same set of fixed directions at fixed points, in the above definition it does not matter whether we use $f : \Gamma \rightarrow \Gamma$ or $f' : \Gamma' \rightarrow \Gamma'$. Also, the maps f and f' have exactly the same collection of INPs.

Now part (2) of Proposition 2.4 implies:

Lemma 5.11. Let $\eta = \alpha\beta^{-1}$ be an INP in Γ , where α, β are as in part (2) of Proposition 2.4. Let d be the direction at $o(\alpha)$ given by α and let d' be the direction at $o(\beta)$ given by β .

Then:

- (1) There is a legal path γ in Γ such that $f(\alpha) = \alpha\gamma$ and $f(\beta) = \beta\gamma$.
- (2) The directions d and d' are fixed by f .
- (3) The path α is an initial segment of the eigenray ρ_d , and the path β is an initial segment of the eigenray $\rho_{d'}$.

Remark 5.12. Let $f : \Gamma \rightarrow \Gamma$ be a standard expanding irreducible train track map. We should explain now what we mean by “computing” an INP or a pINP for f . Recall that every pINP η for f has the form $\eta = \alpha\beta^{-1}$, where α, β are as in part (2) of Proposition 2.4. Let $k \geq 1$ be some integer such that η is an INP for f^k . In particular α and β begin at f -periodic points fixed by f^k and meet at nondegenerate illegal turn.

Then either α is an edge-path in Γ or α begins at an interior point x of an edge e of Γ . In the first case by “computing” α we just mean recording the corresponding edge-path.

Suppose now the latter happens. Then $\alpha = e'\alpha'$ where e' is a proper terminal segment of e starting with x , and where α' is a (possibly empty) edge-path in Γ . Recall that f^k is not necessarily a standard graph map. Still, $f^k(e)$ is, combinatorially, an edge-path in Γ of the form $e_1, e_2, \dots, e_r$. The edge e is subdivided into r consecutive closed intervals $J_1, J_2, \dots, J_r$, with f^k taking the interior of J_i to the open edge e_i for $i = 1, \dots, r$. There is a unique index q such that x belongs to the interior of J_q ; moreover in this case $e_q = e^{\pm 1}$. Recall that in our graph Γ the edge e is equipped

with the characteristic map $\xi_e : e \rightarrow (0, 1)$. Applying ξ_e to the subdivision $J_1, J_2, \dots, J_r$ of e , we get a subdivision

$$0 = s_0 < s_1 < \dots < s_r = 1$$

of $[0, 1]$, with $\xi_e(x) \in (s_{q-1}, s_q)$.

In this case, by “computing” e' we mean finding k , the path $e_1, e_2, \dots, e_r$, the subdivision $0 = s_0 < s_1 < \dots < s_r = 1$ and the (rational) number $\xi_e(x) \in (s_{q-1}, s_q)$. Then “computing” α means computing e' together with computing the edge-path α' in Γ .

The meaning of “computing” β is defined similarly.

Finally, by computing an INP or a pINP η for f we mean finding the decomposition of η in the form $\eta = \alpha\beta^{-1}$ as above and computing each of α and β in the above sense.

Note also that for practical computations with INPs and pINPs knowing the combinatorial structure of α and β is often sufficient. Thus, when describing e' above, it is often sufficient to specify k , the combinatorial path $e_1, \dots, e_k$ and the index q such that x occurs in the interior of the interval J_q being mapped by f^k to $e_q = e^{\pm 1}$. This point of view is taken in the train track computational software package developed by Thierry Coulbois [14].

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. First we compute the set $Fix(f)$ (using the PL definition of f), and construct the graph Γ' and compute the map $f' : \Gamma' \rightarrow \Gamma'$. By Lemma 5.8, this task can be done in $O(m||f||)$ steps. We then find all the fixed directions at vertices of Γ' by checking which edges $e' \in E\Gamma'$ have the property that $f'(e')$ starts with e' . Note that the number of vertices of Γ' and of directions at vertices of Γ' is $O(m||f||)$, and therefore this task can also be done in $O(m||f||)$ steps. Put m' to be the number of non-oriented edges in Γ' , so that $m' = O(m||f||)$.

For each edge e' of Γ' defining a fixed direction of f' we start computing the eigenray $\rho_{e'}$ by iterating f on e' . We iterate until we have found $k \geq 1$ such that $|f^{km'}(e')|_{\Gamma'} \geq ||f||((m||f||^m + 5))$, so that $|f^{km'}(e')|_{\Gamma} \geq (m||f||^m + 5)$. By Lemma 5.5 applied to the map f' , we know that $f^{m'}$ sends every edge of Γ' to an edge-path of length ≥ 2 in Γ' . Thus $|f^{tm'}(e')|_{\Gamma} \geq 2^t$ for $t \geq 1$. Therefore we can find $k \geq 1$ such that $|f^{km'}(e')|_{\Gamma'} \geq ||f||((m||f||^m + 5))$ with $k = O(m \log ||f||)$. Since $m' = O(m||f||)$, we have $km' = O(m^2||f|| \log ||f||)$. Therefore, by Remark 5.9, for a given e' computing the path $\tau(e') := f^{km'}(e')$ in Γ' can be done in $O(m^2||f|| \log ||f|| (m||f||^m + 5)(\log m + \log ||f||)) = O(m^3||f||^{m+2} \log^2 ||f|| \log m)$ steps.

As there are at most $O(m||f||)$ edges in Γ' giving fixed directions for f' , we find all such edges $e'_1, \dots, e'_q$ (with $q = O(m||f||)$), and for each of them compute the path $\tau(e'_i)$ as above. This requires at most $O(m^4||f||^{m+3} \log^2 ||f|| \log m)$ steps.

Now enumerate pairs (α, β) where α is an initial segment of some $\tau(e'_i)$ and where β is an initial segment of some $\tau(e'_j)$. Since $q = O(m||f||)$ and $|\tau(e')|_{\Gamma'} = O(m||f||^{m+1})$, the number of such pairs (α, β) is at most $O(m^4||f||^{2m+4})$ and they can be enumerated in time $O(m^4||f||^{2m+4} \log m \log ||f||)$.

For each such pair (α, β) we check whether $\eta = \alpha\beta^{-1}$ is a path in Γ' and if this path is tight. If yes, we compute $f(\eta)$ and then compute the tightened form of $f(\eta)$. For a given pair (α, β) , this check can be done in $O(m||f||^{m+2} \log m \log ||f||)$ steps, and doing this for all pairs (α, β) requires at most $O(m^5||f||^{3m+6} \log m \log ||f||)$ steps. If the tightened form of $f(\eta)$ is η , then by Proposition 2.4 η is an INP for f (note that in this case the turn between α and β in η is automatically illegal since f is expanding), and, also by Proposition 2.4, every INP for f arises in this way. Thus running this check for every pair (α, β) as above computes all the INPs for f .

Summing up the above computations, the total running time of this process can be bounded above by $O(m^5||f||^{3m+6} \log m \log ||f||)$ steps. $\square$

Theorem 5.1 can be used to establish the following more general result:

Theorem 5.13. *Let $N \geq 2$ be fixed.*

The exists a deterministic algorithm $\mathfrak{A}'$ with the following property. Given an expanding standard train track map $f : \Gamma \rightarrow \Gamma$ (where Γ is a finite connected graph with all vertices of degree ≥ 3 and with $\pi_1(\Gamma)$ free of rank N), the algorithm $\mathfrak{A}'$ determines whether or not f has any periodic INPs and if yes, computes all the periodic INPs for f .

The algorithm then constructs the graph $S(f)$.

The algorithm terminates in polynomial time in $\|f\|$.

Proof. As noted in Remark 2.6, Feighn and Handel showed in [23, Corollary 3.14] that there exists a computable power $t = t(N)$ such that for $f^t : \Gamma \rightarrow \Gamma$ a path η is a periodic indivisible Nielsen path for f if and only if η is an INP for f^t . Note that $\|f^t\| \leq \|f\|^t$.

Thus we want to first replace f by f^t and then apply Theorem 5.1 to f^t . However, a technical complication arises in that $f^t : \Gamma \rightarrow \Gamma$ is not necessarily a standard graph map and thus Theorem 5.1 technically does not directly apply to f^t .

However, there is a unique standard graph map $g : \Gamma \rightarrow \Gamma$ which is isotopic to f^t relative VG .

There is a also polynomial time computable “reparameterization” map $h : G \rightarrow G$, such that $h|_{VG} = ID_{VG}$, such that the restriction of h to each edge e of G is a piecewise-linear orientation-preserving homeomorphism from e to e , and such that h conjugates f^t to g . More precisely, we have $f^t = h \circ g \circ h^{-1}$. Then whenever η' is an INP for g then $h\eta'$ is an INP for f^t , and, moreover, all INPs for f^t arise in this way.

Thus to find pINPs for f , we use Theorem 5.1 to compute all INPs (if any) $\eta'_1, \dots, \eta'_k$ for g . Then $h\eta'_1, \dots, h\eta'_k$ are all the pINPs for f . $\square$

Remark 5.14. Let $N \geq 3$ and $f : \Gamma \rightarrow \Gamma$ be as in Theorem 5.13. Then Γ has $\leq 3N$ topological edges, $\leq 6N$ oriented edges and $\leq 36N^2$ illegal turns at vertices. Therefore there are at most $36N^2$ pINPs distinct in Γ (up to inversion). Let $\eta_1, \eta_2, \dots, \eta_s$ be these pINPs, so that $0 \leq s \leq 36N^2$.

Recall that, as noted in Remark 2.6, there exists a uniform power $b = b(N)$ such that every pINP of f is an INP of f^b . Corollary 5.6 implies that if η is an INP for f^b then $|\eta| \leq 6N\|f^b\|^{3N} + 8$. Thus for each $0 \leq i \leq s$ we have $|\eta_i| \leq 6N\|f\|^{3bN} + 8$.

Proposition 5.15. *Let an integer $N \geq 2$ be fixed. There exists an algorithm that, given an expanding irreducible standard train track map $f : \Gamma \rightarrow \Gamma$ with $\pi_1(\Gamma) \cong F_N$, decides whether or not $\varphi = f_\#$ is primitively atoroidal. This algorithm terminates in at most polynomial time in $\|f\|$.*

Proof. We first algorithmically construct the graph $S(f)$ using Theorem 5.13; this can be done in polynomial time in $\|f\|$. If $b_1(S(f)) = 0$, then φ is atoroidal and, in particular, primitively atoroidal. Thus assume that $b_1(S(f)) \geq 1$.

We list all the non-contractible connected components $Q_1, \dots, Q_k$ of $S(f)$. In view of Remark 5.14, the number k is bounded above by $36N^2$, and the total sum of the lengths of the μ -labels of edges of $S(f)$ is bounded above by a polynomial function in terms of $\|f\|$. Then, using the Stallings folding algorithm, for each Q_i we find a finite generating set of the subgroup $U_i = \mu(\pi_1 Q_i) \leq \pi_1(\Gamma)$. Note that the sum of the lengths of the generators of $U_1, \dots, U_k$, expressed as loops in Γ , is bounded by a polynomial in $\|f\|$. Again, this step requires at most polynomial time in $\|f\|$.

Proposition 3.2 implies that a nontrivial conjugacy class $[g]$ in $\pi_1(\Gamma)$ is φ -periodic if and only if g is conjugate in $\pi_1(\Gamma)$ to an element of U_i for some $1 \leq i \leq k$.

Now Corollary 13.3 in [23] implies that either for $i = 1, \dots, k$ every element of U_i is trivial in $H_1(F_N, \mathbb{Z}/2\mathbb{Z})$, or $\varphi = f_\#$ is not primitively atoroidal (and these two cases are mutually exclusive).

We then compute the images $\overline{U_1}, \dots, \overline{U_k}$ of $U_1, \dots, U_k$ in $H_1(F_N, \mathbb{Z}/2\mathbb{Z})$ and check if we have $\overline{U_i} = \{0\}$ for $i = 1, \dots, k$. [It is clear that this check can also be done in polynomial time.] If yes, then $\varphi = f_\#$ is primitively atoroidal, and if not then $\varphi = f_\#$ is not primitively atoroidal. $\square$

Remark 5.16. Our original proof of Proposition 5.15 was different and used the result of Clifford-Goldstein [13], or alternatively, of Dicks [17], to decide whether any of the subgroups $U_1, \dots, U_k$ of F_N contains a primitive element of F_N . The referee then pointed to us a simpler argument, presented above, using Corollary 13.3 of [23].

Feighn and Handel [23, Corollary 3.14] obtain a general algorithm for deciding if an element $\varphi \in \text{Out}(F_N)$ is primitively atoroidal. However, their approach relies on heavy duty machinery of CT train tracks and needs, as an essential ingredient, the main result of [23] about algorithmically constructing CTs. That result does not come with any complexity estimate and it remains to be seen if reasonable complexity estimates can be obtained there. Therefore the general algorithm given in [23, Corollary 3.14] also does not, for the time being, have a complexity estimate.

6. REFINED FULL IRREDUCIBILITY TESTING

In this section we provide a refinement of the algorithm from [30] for detecting full irreducibility.

Proposition 6.1. *Let $N \geq 2$ be fixed.*

There exists an algorithm that, given a finite connected graph where every vertex has degree ≥ 3 and $\pi_1(\Gamma) \cong F_N$, and given a standard expanding irreducible train track map $f : \Gamma \rightarrow \Gamma$, computes, in polynomial time, in terms of $\|f\|$, the Whitehead graph $Wh_\Gamma(v, f)$ for every vertex $v \in V\Gamma$.

Proof. Under the assumptions made on Γ there are at most $3N$ topological edges and at most $6N$ oriented edges. Thus there are at most $36N^2$ turns in Γ .

Let us enumerate all the edges $e_1, \dots, e_k$ of Γ (so that $k \leq 6N$).

We put $\mathcal{T}_0 = \emptyset$, and $\mathcal{T}_1$ to be the set of turns contained in $f(e_1), \dots, f(e_k)$. Computing $\mathcal{T}_1$ requires at most $O(\|f\|)$ time.

Then, inductively, given a set of turns $\mathcal{T}_i$, we define $\mathcal{T}_{i+1} := \mathcal{T}_i \cup Df(\mathcal{T}_i)$ where Df is the derivative map of f . Thus

$$\emptyset = \mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \dots \subseteq \mathcal{T}$$

where $\mathcal{T}$ is the set of all turns in Γ .

Note that, by construction, once there is some i such that $\mathcal{T}_i = \mathcal{T}_{i+1}$ then $\mathcal{T}_j = \mathcal{T}_i$ for all $j \geq i$.

Since $\#\mathcal{T} \leq 36N^2$, this chain terminates in at most $36N^2$ steps with some $q \leq 36N^2$ such that $\mathcal{T}_q = \mathcal{T}_{q+1}$. Summing up the times, we see that the set $\mathcal{T}_q$ can be computed in polynomial time in $\|f\|$. The set $\mathcal{T}_q$ is the set of all turns “taken” by the train track map f . Using this set we can now construct all the Whitehead graphs $Wh_\Gamma(v, f)$ of the vertices of Γ , using Definition 4.1. Note that the number of vertices of Γ is $\leq 6N$, and that for every vertex of Γ its degree is $\leq 2N$. This gives us constant (in terms of $\|f\|$) bounds on the number of Whitehead graphs we need to construct, and on the sizes of each of these graphs. Therefore the total computational bound remains polynomial in terms of $\|f\|$. $\square$

Theorem 6.2. *Let $N \geq 2$ be fixed.*

- (1) *There exists an algorithm that, given a standard topological representative $f' : \Gamma' \rightarrow \Gamma'$ of some $\varphi \in \text{Out}(F_N)$, such that every vertex in Γ has degree ≥ 3 , decides whether or not φ is fully irreducible. The algorithm terminates in polynomial time in terms of $\|f'\|$.*

- (2) Let $A = \{a_1, \dots, a_N\}$ be a fixed free basis of F_N . There exists an algorithm, that given $\Phi \in \text{Aut}(F_N)$ (where Φ is given as an N -tuple of freely reduced words over $A^{\pm 1}$, $(\Phi(a_1), \dots, \Phi(a_N))$), decides, in polynomial time in $|\Phi|_A$, whether or not Φ is fully irreducible.
- (3) Let S be a finite generating set for $\text{Out}(F_N)$. There exists an algorithm that, given a word w of length n over $S^{\pm 1}$, decides whether or not the element φ of $\text{Out}(F_N)$ represented by w is fully irreducible. This algorithm terminates in at most exponential time in terms of the length of the word w .

Proof. (1) We first run the algorithm of Bestvina-Handel for trying to find an irreducible standard train track representative of φ . By Theorem A.1, this algorithm always terminates in polynomial time in $\|f'\|$, and either finds a reducible topological representative of φ (in which case we conclude that φ is not fully irreducible), or it finds an irreducible standard train track representative $f : \Gamma \rightarrow \Gamma$ of φ . Assume that the latter has occurred. In this case, by Remark A.6, we also have that $\|f\|$ is bounded by a polynomial function in terms of $\|f'\|$. We then check whether $M(f)$ is a permutation matrix. If yes, then φ has finite order in $\text{Out}(F_N)$ and thus is not fully irreducible. If not, then f is an expanding irreducible train track representative of φ .

We then use the algorithm obtained in Proposition 5.15 to decide, in polynomial time in $\|f'\|$, whether or not φ is primitively atoroidal. If φ is not primitively atoroidal, then some positive power of φ preserves the conjugacy class of a rank-1 free factor of F_N , and thus φ is not fully irreducible.

Suppose now that φ is primitively atoroidal. We then construct the Whitehead graphs $Wh_\Gamma(v, f)$ for every vertex $v \in V\Gamma$. Note that by Proposition 6.1 this step can be done in polynomial time in terms of $\|f\|$. Moreover, since there are at most $6N$ vertices in Γ , and for every $v \in V\Gamma$ the Whitehead graphs $Wh_\Gamma(v, f)$ has $\leq 2N$ vertices, we can also verify in polynomial time in $\|f\|$ whether or not the Whitehead graphs $Wh_\Gamma(v, f)$ are connected for all $v \in V\Gamma$.

Theorem 4.8 now implies that φ is fully irreducible if and only if all these Whitehead graphs are connected.

Thus we have produced an algorithm which decides whether or not φ is fully irreducible. Summing up the total time expended, we see that the algorithm does terminate in polynomial time in $\|f'\|$.

(2) The statement of part (2) follows from part (1). We take Γ' to be an N -rose with petals marked by $a_1, \dots, a_N$. Take $u_i = \Phi(a_i)$ for $i = 1, \dots, N$. Now construct a standard graph map $f' : \Gamma' \rightarrow \Gamma'$ where the i -th petal of Γ' gets mapped according to the word u_i . Then $\|f'\| = |\Phi|_A$ and f' is a standard topological representative of (the outer automorphism class of) φ . Applying the result of part (1) to f' we obtain the desired conclusion.

(3) The statement of part (3) follows from part (1).

Again, we take Γ' to be an N -rose. If $S = \{\sigma_1, \dots, \sigma_k\}$ and $F_N = F(a_1, \dots, a_N)$, we put $C = \max_{i=1}^N |\sigma_i(a_i)|_A$. We first compute the freely reduced words $u_1, \dots, u_N$ over $A = \{a_1, \dots, a_N\}$ where $u_i = \varphi(a_i)$. Since φ is represented by a word w over $S^{\pm 1}$, we have $|u_i|_A \leq C^{|w|}$ for $i = 1, \dots, N$.

Then construct a standard graph map $f' : \Gamma' \rightarrow \Gamma'$ where the i -th petal of Γ' gets mapped according to the word u_i . Note that $\|f'\| \leq C^{|w|}$. Now applying the result of part (1) to f' we obtain the desired conclusion for (3). $\square$

APPENDIX A. THE COMPLEXITY OF THE BESTVINA-HANDEL ALGORITHM

*by Mark C. Bell*University of Illinois
mcbell@illinois.edu

Recall that a *topological representative* of an outer automorphism $\varphi \in \text{Out}(F_N)$ is a homotopy equivalence $f : \Gamma \rightarrow \Gamma$ taking vertices to vertices, where Γ is a finite connected graph whose fundamental group has been identified with F_N and where each vertex has degree at least three, such that f induces φ on $\pi_1(\Gamma)$. For ease of notation, we will say that a topological representative f of φ is *good* if it is either a train track map or is reducible.

In a seminal 1992 paper [3] Bestvina and Handel gave an algorithm to improve any topological representative $f' : \Gamma' \rightarrow \Gamma'$ of $\varphi \in \text{Out}(F_N)$ to a good one $f : \Gamma \rightarrow \Gamma$. Hence showing that every irreducible outer automorphism has a train track representative [3, Theorem 1.7]. Here we provide an explicit polynomial bound on the running time of their algorithm:

Theorem A.1 (Complexity of the Bestvina-Handel algorithm). *Fix an integer $N \geq 2$. Given a topological representative f' of $\varphi \in \text{Out}(F_N)$, the Bestvina-Handel algorithm terminates in polynomial time in $\|f'\|$.*

Corollary A.2. *Fix an integer $N \geq 2$ and let S be a finite generating set of $\text{Out}(F_N)$. There is an algorithm that, given a word w over $S^{\pm 1}$ corresponding to $\varphi \in \text{Out}(F_N)$, finds a good topological representative of φ . Moreover, this algorithm terminates in exponential time in $|w|$.*

Proof. Fix a free basis $A = \{a_1, \dots, a_N\}$ of F_N . We first compute the freely reduced words $u_1, \dots, u_N$ over A where $u_i = \varphi(a_i)$. Since φ is given by a word w over $S^{\pm 1}$, we have that $|u_i|_A \leq C^{|w|}$ where

$$C := \max_{\sigma \in S} \max_{a_i \in A} |\sigma(a_i)|_A$$

is a constant that depends only on A and S .

Now let Γ' to be the N -rose with the petals marked by $a_1, \dots, a_N$. Let $f' : \Gamma' \rightarrow \Gamma'$ be the *standard* topological representative of φ where the i -th petal of Γ' gets mapped according to the word u_i . Note that the bounds on $|u_i|_A$ imply that $\|f'\| \leq C^{|w|}$ and so the result now follows by applying Theorem A.1 to f' . $\square$

The Bestvina-Handel algorithm works by producing a sequence of topological representatives $f_0, \dots, f_K$, starting with $f_0 = f'$. The representative f_{i+1} is computed from f_i by applying one of four basic operations [3, Section 1]. These operations ensure each topological representative is better than the last: the stretch factor of f_{i+1} is strictly less than the stretch factor of f_i . Bestvina and Handel showed that when this procedure terminates — when no such operation can improve the topological representative — the map f_K is good [3, Theorem 1.7].

Examination of the different possible transformations shows that f_{i+1} can be computed from f_i in $\text{poly}(\|f_i\|)$ time and that $\|f_{i+1}\| \leq \text{poly}(\|f_i\|)$. However these bounds alone are not sufficient to conclude that the Bestvina-Handel algorithm runs in polynomial time. Thus we prove Theorem A.1 by showing that K and $\|f_i\|$ are bounded by polynomial functions of $\|f'\|$. We achieve this by considering the transition matrices M_i associated to f_i .

Recall that a matrix $M \in \mathbb{N}_0^{n \times n}$ is *irreducible* if for each i and j there exists k such that $(M^k)_{ij} \neq 0$ [38, Page 671]. We may think of a matrix as the adjacency matrix of a directed graph. Doing so we obtain that a matrix is irreducible if and only if its corresponding digraph is strongly

connected [38, Page 671]. From this we deduce that we may strengthen the definition of a matrix being irreducible to also include that $1 \leq k \leq n$.

An irreducible matrix M has an associated *Perron–Frobenious eigenvalue* $\lambda_{PF}(M)$. This is equal to its *spectral radius* $\rho(M) \geq 1$ and is the largest eigenvalue of M in absolute value [38, Page 673].

For a matrix $M = (m_{ij})$, let $\|M\| := \max(m_{ij})$ denote the maximum of the entries of M . By the Gershgorin circle theorem [25], we immediately obtain an upper bound on $\lambda_{PF}(M)$:

Lemma A.3. *If $M \in \mathbb{N}_0^{n \times n}$ is irreducible then*

$$\lambda_{PF}(M) \leq n \cdot \|M\|. \quad \square$$

The main tool powering this section is a corresponding lower bound.

Proposition A.4. *If $(m_{ij}) = M \in \mathbb{N}_0^{n \times n}$ is an irreducible matrix then*

$$\|M\| \leq \sum_{i,j=1}^n m_{ij} \leq n\lambda^{n+1}$$

where $\lambda := \lambda_{PF}(M)$.

Proof. Let $(x_1 \cdots x_n)^T \neq 0$ be a real eigenvector of M corresponding to λ such that $x_i \geq 0$ [38, Equation 8.3.2]. For ease of notation let $m_{ij}^k := (M^k)_{ij}$. Now note that as M is irreducible for each i and j there is a $1 \leq k \leq n$ such that $m_{ij}^k \geq 1$. From this we observe that for each i we have that

$$\begin{aligned} n\lambda^n x_i &\geq (\lambda^n + \cdots + \lambda)x_i \\ &= \sum_{k=1}^n \sum_{j=1}^n m_{ij}^k x_j \\ &\geq x_1 + \cdots + x_n. \end{aligned}$$

Thus

$$\begin{aligned} \lambda(x_1 + \cdots + x_n) &= \sum_{i=1}^n \lambda x_i \\ &= \sum_{i,j=1}^n m_{ij} x_j \\ &\geq \frac{x_1 + \cdots + x_n}{n\lambda^n} \sum_{i,j=1}^n m_{ij}. \end{aligned}$$

By cancelling the $x_1 + \cdots + x_n \neq 0$ term on each side and rearranging this inequality we obtain

$$\sum_{i,j=1}^n m_{ij} \leq n\lambda^{n+1}$$

as required. The other inequality follows trivially. $\square$

Now let $\text{PF}(n, \lambda)$ denote the set of $m \times m$ irreducible matrices, where $m \leq n$, with entries in $\mathbb{N}_0$ and Perron–Frobenious eigenvalue at most λ . As an immediate consequence of Proposition A.4 we obtain the following bound on the cardinality of $\text{PF}(n, \lambda)$:

Corollary A.5. *There are at most $n(n\lambda^{n+1} + 1)^{n^2}$ matrices in $\text{PF}(n, \lambda)$. In particular, when n is fixed this bound is polynomial in λ . $\square$*

We now have the tools needed to prove the main theorem:

Proof of Theorem A.1. Let $f_0, \dots, f_K$ be the sequence of topological representatives produced by the Bestvina–Handel algorithm. Let M_i be the transition matrix of f_i . Note that M_0 is the transition matrix of $f_0 = f'$. Hence if M_0 is reducible then the algorithm terminates immediately and there is nothing to check as $f = f'$.

Otherwise, M_0 is irreducible and so

$$\lambda_{\text{PF}}(M_0) \leq N \|M_0\| \leq N \|f'\|$$

by Lemma A.3.

These matrices have size at most $R := 3N - 3$ and, while it is possible that M_K is reducible, the matrices $M_0, \dots, M_{K-1}$ are irreducible. Now $\lambda_{\text{PF}}(M_i) < \lambda_{\text{PF}}(M_{i-1})$ for each $0 < i < K$. Thus $M_0, \dots, M_{K-1} \in \text{PF}(R, N \|f'\|)$ and these matrices are all pairwise distinct. Hence

$$K \leq |\text{PF}(R, N \|f'\|)| + 1$$

which is at most a polynomial function of $\|f'\|$ by Corollary A.5 as N is fixed.

Furthermore, for each $0 \leq i < K$ the fact that $M_i \in \text{PF}(R, N \|f'\|)$ implies that $\|M_i\| \leq R(N \|f'\|)^{R+1}$ by Proposition A.4. This shows that for the corresponding topological representatives

$$\|f_i\| \leq R^2(N \|f'\|)^{R+1}$$

which again is a polynomial function of $\|f'\|$ as N is fixed.

Finally, as noted above

$$\|f_K\| \leq \text{poly}(\|f_{K-1}\|) \leq \text{poly}(R^2(N \|f'\|)^{R+1}).$$

Hence the Bestvina–Handel algorithm terminates in polynomial time in terms of $\|f'\|$. $\square$

Remark A.6. Suppose that f' is a topological representative of $\varphi \in \text{Out}(F_N)$ and that f is the good topological representative of φ produced by the Bestvina–Handel algorithm starting from f' . We highlight that the proof of Theorem A.1 also shows that, when N is fixed, $\|f\|$ is bounded above by a polynomial function of $\|f'\|$.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA

<http://www.math.uiuc.edu/~kapovich/>

E-mail address: kapovich@math.uiuc.edu