INF-SUP STABLE FINITE ELEMENTS ON BARYCENTRIC REFINEMENTS PRODUCING DIVERGENCE–FREE APPROXIMATIONS IN ARBITRARY DIMENSIONS

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Abstract. We construct several stable finite element pairs for the Stokes problem on barycentric refinements in arbitrary dimensions. A key feature of the spaces is that the divergence maps the discrete velocity space onto the discrete pressure space; thus, when applied to models of incompressible flows, the pairs yield divergence-free velocity approximations. The key result is a local inf-sup stability that holds for any dimension and for any polynomial degree. With this result, we construct global divergence-free and stable pairs in arbitrary dimension and for any polynomial degree.

1. Introduction. In the papers [2, 25] it was shown that $\boldsymbol{\mathcal{P}}_k^c - \mathcal{P}_{k-1}$ is an inf-sup stable and divergence-free pair on barycentric refined meshes in two and three dimensions if the polynomial size k is sufficiently large. Here, $\mathcal{P}_{k-1}$ denotes the space of piecewise polynomials of degree $\leq (k-1)$, and $\boldsymbol{\mathcal{P}}_k^c$ denotes the space of globally continuous, vector-valued polynomials of degree $\leq k$. The strategy in the analysis, as shown by Zhang [25], is Stenberg's macro-element technique [24], where the crucial step is a local inf-sup stability estimate on each macro tetrahedra/triangle for any $k \geq 1$. Then, Bernardi-Raugel [4] finite elements are implicitly used to control piecewise constants to prove global inf-sup stability. The use of the Bernardi-Raugel finite elements is the reason one needs a restriction on k to ensure global inf-sup stability: For dimension $d = 2$, $k \geq 2$ and for dimension $d = 3$, $k \geq 3$.

One of our contributions in this paper is to extend the results in [2, 25] to arbitrary space dimension $d \geq 2$. The key step, as in [25], is to prove a local inf-sup stability result. Here, adopting the convention in e.g., [20, 6, 16], a barycentric refinement takes a given mesh (which we call the macro mesh) and adds the barycenter of each d-dimensional simplex of the macro mesh to the set of vertices. This is also known as an Alfeld split [17]. We slightly generalize this construction by showing that one can use any arbitrary point in the interior of each simplex (not just the barycenter), as long as the resulting mesh is shape regular.

We then derive several applications of the local inf-sup stability result. First, with the help of the Bernardi-Raugel element, we show that $\boldsymbol{\mathcal{P}}_k^c - \mathcal{P}_{k-1}$ is inf-sup stable on the refined mesh for $k \geq d$ (as was shown in [2, 25] for $d = 2, 3$). For lower order approximations $1 \leq k < d$, we use an idea introduced in [13] and supplement the velocity space to obtain an inf-sup stable pair. To this end, we construct vector-valued, piecewise polynomial functions with respect to the refined mesh that have the same trace as the Bernardi-Raugel face bubbles on the skeleton of the macro mesh. The key difference, compared to the Bernardi-Raugel face bubbles, is that the divergence of these functions are piecewise constant. The existence of such finite element functions, which we call modified face bubbles, is guaranteed by the local inf-sup stability result. Thus, we supplement $\boldsymbol{\mathcal{P}}_k^c$ (for $1 \leq k < d$) locally with these modified face bubbles to get an inf-sup stable pair on the refined mesh.

We also consider finite elements on the (unrefined) macro mesh. We show that $\boldsymbol{\mathcal{P}}_1^c - \mathcal{P}_0$ can be made stable by supplementing the velocity space $\boldsymbol{\mathcal{P}}_1^c$ with the modified face bubbles. Since the divergence of the modified face bubbles are piecewise constant, and thus contained in the pressure space, these finite element will produce divergence-free approximations for the Stokes and NSE problems. These finite elements are developed in arbitrary dimension. The two-dimensional case seems to coincide with a pair of finite elements considered in [7].

A final application is inspired by a finite element introduced in [1]. There, an inf-sup stable and divergence-free macro element pair is constructed in two dimensions with a piecewise linear, continuous pressure space. Again, with the help of the modified face bubbles, we extend these results to arbitrary

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dimension $d \geq 3$.

Advantages of divergence-free pairs for the Stokes/NSE problems include, e.g., better stability and error estimates, and the enforcement of several conservation laws and invariant properties. We refer the reader to the survey article [15] which highlights the benefits of divergence-free pairs. In addition to the above references, several other inf-sup stable pair of spaces that produce divergence-free approximations have been constructed. These include high-order finite elements ($k \geq 2d$) in two and three dimensions [22, 10, 18, 26], as well as lower order pairs supplemented with rational functions [12, 13]. Advantages of the proposed elements given here are its relative simplicity and flexibility with respect to dimension and polynomial degree. The shape functions are piecewise polynomials and therefore quadrature rules are immediately available. We mention that the degrees of freedom of our lowest-order element agree with those given in [7], where divergence-free Stokes elements with respect to Powell-Sabin partitions are considered (e.g., in three dimensions, every tetrahedron is split into 12 sub-elements). However, our elements are defined on a less stringent barycenter partition, which makes the implementation simpler.

The paper is organized as follows. In the next section we introduce some notation used throughout the paper. Then, in Section 3 a local inf-sup stability result is proved. In Section 4 the Bernardi-Raguel face bubble functions and its modification are introduced. Then in Section 5 a low-order, divergence-free, and inf-sup stable pair on the macro mesh is constructed. In Section 6, inf-sup stable and divergence-free finite elements are given on the refined mesh. In Section 7, we discuss implementation aspects of the methods and provide some numerical experiments. Finally, in Section 8 we summarize our results and discuss future directions.

2. Notation and Preliminaries. We consider a family $\{\mathcal{T}_h\}$ of shape regular conforming simplicial triangulations of a polytope domain $\Omega \subset \mathbb{R}^d$. For each $K \in \mathcal{T}_h$ let $x_K \in K$ be an interior point, and consider the refined triangulation to $\mathcal{T}_h$ that subdivides each simplex K into $(d+1)$ simplices by adjoining the vertices of K with the new vertex x_K. The resulting refined triangulation is denoted by $\mathcal{T}_h^r$. We assume that the points $\{x_K\}_{K\in\mathcal{T}_h}$ are chosen so that the family $\{\mathcal{T}_h^r\}$ is also shape regular. If x_K is the barycenter of K, which is the most practical choice, then $\mathcal{T}_h^r$ is the barycentric refinement of $\mathcal{T}_h$. For any simplex K we let $\mathcal{P}_k(K)$ be the space of polynomials of degree at most k defined on K. In the case that k is negative, $\mathcal{P}_k(K)$ is taken to be the trivial set. The vector-valued polynomials on a simplex are given by $\boldsymbol{\mathcal{P}}_k(K) = [\mathcal{P}_k(K)]^d$. We define

$$
\begin{aligned}
\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{S}) &:= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \ \boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}_k(K), \ \forall K \in \mathcal{S}\}, \\
\mathring{\mathcal{P}}_k(\mathcal{S}) &:= \{q \in L_0^2(\Omega) : \ q|_K \in \mathcal{P}_k(K), \ \forall K \in \mathcal{S}\},
\end{aligned}
$$

with either $\mathcal{S} = \mathcal{T}_h$ or $\mathcal{S} = \mathcal{T}_h^r$.

We denote by K^r the triangulation of K:

$$K^r = \{T \in \mathcal{T}_h^r : T \subset K\},$$

and will use the notation

$$\mathring{\boldsymbol{\mathcal{P}}}_k^c(K^r) := \{\boldsymbol{v} \in \boldsymbol{H}_0^1(K) : \ \boldsymbol{v}|_T \in \boldsymbol{\mathcal{P}}_k(T), \ \forall T \in K^r\}, \tag{2.1}$$

$$\mathring{\mathcal{P}}_k(K^r) := \{q \in L_0^2(K) : \ q|_T \in \mathcal{P}_k(T), \ \forall T \in K^r\}. \tag{2.2}$$

3. Local inf-sup stability. In this section we will prove that $\mathring{\boldsymbol{\mathcal{P}}}_k^c(K^r) - \mathring{\mathcal{P}}_{k-1}(K^r)$ is inf-sup stable for each $K \in \mathcal{T}_h$. The result can be stated as follows.

THEOREM 3.1. *Let $k \geq 1$. For any $K \in \mathcal{T}_h$ and for any $p \in \mathring{\mathcal{P}}_{k-1}(K^r)$, there exists $\boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(K^r)$ such that*

$$\operatorname{div} \boldsymbol{v} = p \qquad \text{on } K, \tag{3.1}$$

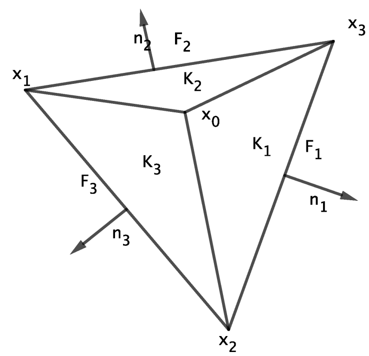

FIG. 3.1. *Figure of macro triangle*

with the bound

$$\|\boldsymbol{v}\|_{H^1(K)} \leq C\|p\|_{L^2(K)}, \tag{3.2}$$

where the constant $C > 0$ only depends on k and the shape regularity of K^r, but is independent of K and p.

The proof of Theorem 3.1 requires some notation and a few preliminary results. For $K \in \mathcal{T}_h$, denote by $S = \{x_1, \ldots, x_{d+1}\}$ the set of vertices of K, and let $x_0 = x_K$. Then the refinement of K is given by $K^r = \{K_i\}_{1 \leq i \leq d+1}$, where K_i is the simplex with vertices $\{x_0\} \cup S \backslash \{x_i\}$.

We let F_i be the $(d-1)$ dimensional face of K opposite to x_i, and let $\boldsymbol{n}_i$ be the unit normal to F_i pointing out of K; see Figure 3.1. We let $\lambda_0 \in \mathcal{P}_1^c(K^r)$ be the continuous, piecewise linear function satisfying $\lambda_0(x_j) = \delta_{0j}$ for $0 \leq j \leq d+1$. We note that λ_0 vanishes on ∂K. We let h_K be the diameter of K, and let ρ_K be the diameter of the largest ball inscribed in K. The shape regularity constant of K is defined by

$$C_K = \frac{h_K}{\rho_K}.$$

Analogously we let C_{K_i} be the shape regularity constant of K_i (for $i = 1, 2, \ldots, d+1$), and let $\mathcal{C}_K = \max_{1 \leq i \leq d+1} C_{K_i}$ denote the shape regularity constant of K^r. We see that $\mathcal{C}_K$ is comparable to C_K provided x_0 is sufficiently far from ∂K. A simple calculation shows that

$$\nabla \lambda_0|_{K_i} = -\frac{1}{h_i}\boldsymbol{n}_i, \tag{3.3}$$

where h_i is the distance of x_0 to the $(d-1)$ dimensional hyperplane that contains F_i. We note that

$$h_i \leq h_K, \quad h_K \leq C h_i \quad \text{for } 1 \leq i \leq d+1, \tag{3.4}$$

where $C > 0$ depends only on $\mathcal{C}_K$.

We introduce the local Nédélec space of index $\ell \geq 0$ (see [19, 3]):

$$\boldsymbol{N}_\ell(K) := \boldsymbol{\mathcal{P}}_{\ell-1}(K) + \{\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_\ell(K) : \ \boldsymbol{w} \cdot \boldsymbol{x} = 0\}.$$

Note that $\boldsymbol{N}_{-1}(K) = \boldsymbol{N}_0(K) = \{0\}$. We then define for $\ell \geq 0$

$$\boldsymbol{\mathcal{P}}_\ell^\perp(K) := \{\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_\ell(K) : \int_K \boldsymbol{v} \cdot \boldsymbol{\kappa}\, dx = 0, \text{ for all } \boldsymbol{\kappa} \in \boldsymbol{N}_{\ell-1}(K)\}.$$

Note that $\boldsymbol{\mathcal{P}}_0^\perp(K) = \boldsymbol{\mathcal{P}}_0(K)$ and $\boldsymbol{\mathcal{P}}_1^\perp(K) = \boldsymbol{\mathcal{P}}_1(K)$.

Using the degrees of freedom due to Nédélec we have the following lemma; see [19, 3] for $\ell \geq 1$ and [8] for the case $\ell = 0$.

LEMMA 3.2. *$\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_\ell^\perp(K)$ is uniquely determined by*

$$\begin{aligned}
&\int_{F_i} (\boldsymbol{v} \cdot \boldsymbol{n}_i)\kappa \qquad \forall \kappa \in \mathcal{P}_\ell(F_i), \qquad 1 \leq i \leq d+1, \quad \ell \geq 1,\\
&\int_{F_i} (\boldsymbol{v} \cdot \boldsymbol{n}_i)\kappa \qquad \forall \kappa \in \mathcal{P}_\ell(F_i), \qquad 1 \leq i \leq d, \qquad \ell = 0.
\end{aligned}$$

In addition, the following bounds hold

$$\frac{1}{h_K^2}\|\boldsymbol{v}\|_{L^2(K)}^2 + \|\nabla \boldsymbol{v}\|_{L^2(K)}^2 \leq C\Big[\frac{1}{h_K}\sum_{i=1}^{d+1} \|\boldsymbol{v} \cdot \boldsymbol{n}_i\|_{L^2(F_i)}^2\Big], \qquad \ell \geq 1, \tag{3.5}$$

$$\|\boldsymbol{v}\|_{L^2(K)}^2 \leq C\Big[h_K \sum_{i=1}^{d} \|\boldsymbol{v} \cdot \boldsymbol{n}_i\|_{L^2(F_i)}^2\Big], \qquad \ell = 0, \tag{3.6}$$

where the constant $C > 0$ depends on C_K and ℓ.

We consider the space

$$\boldsymbol{M}_k(K^r) := \{\boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(K^r) : \boldsymbol{v} = \sum_{j=1}^{k} \lambda_0^j \boldsymbol{w}_{k-j}, \text{ with } \boldsymbol{w}_{k-j} \in \boldsymbol{\mathcal{P}}_{k-j}^\perp(K)\}.$$

We can now prove Theorem 3.1. In fact, we prove a slightly stronger result.

THEOREM 3.3. *Let $k \geq 1$. For any $K \in \mathcal{T}_h$ and for any $p \in \mathring{\mathcal{P}}_{k-1}(K^r)$, there exists a* unique *$\boldsymbol{v} \in \boldsymbol{M}_k(K^r)$ such that*

$$\operatorname{div} \boldsymbol{v} = p \qquad on\ K, \tag{3.7}$$

with the bound

$$\|\boldsymbol{v}\|_{H^1(K)} \leq C\|p\|_{L^2(K)}, \tag{3.8}$$

where the constant $C > 0$ only depends on k and the shape regularity of K^r, but is independent of K and p.

Proof. Let $p \in \mathring{\mathcal{P}}_{k-1}(K^r)$. For simplicity of notation we let $\lambda = \lambda_0$. We take $\boldsymbol{v} \in \boldsymbol{M}_k(K^r)$ of the form $\boldsymbol{v} = \sum_{j=1}^{k} \lambda^j \boldsymbol{w}_{k-j}$ with $\boldsymbol{w}_{k-j} \in \boldsymbol{\mathcal{P}}_{k-j}^\perp(K)$.

Step 1: We first show that there exists unique $\boldsymbol{w}_{k-1}, \dots, \boldsymbol{w}_1$ and unique $p_0 \in \mathcal{P}_0(K^r)$ such that

$$\begin{aligned}
&\sum_{j=1}^{k-1} \operatorname{div}(\lambda^j \boldsymbol{w}_{k-j}) = p - \lambda^{k-1} p_0, \\
&\frac{1}{h_K^2}\|\boldsymbol{w}_{k-j}\|_{L^2(K)}^2 + \|\nabla \boldsymbol{w}_{k-j}\|_{L^2(K)}^2 + \|p_0\|_{L^2(K)}^2 \leq C\|p\|_{L^2(K)}^2.
\end{aligned} \tag{3.9}$$

Note that the case $k = 1$ follows by setting $p_0 = -p$; thus, we assume that $k \geq 2$ in this step. We show (3.9) by induction.

We let $p_{k-1} = p$ and we suppose that there exists *unique* $\boldsymbol{w}_{k-1}, \ldots, \boldsymbol{w}_{k-\ell+1}$ and $p_{k-1}, \ldots, p_{k-\ell}$ with $p_{k-j} \in \mathcal{P}_{k-j}(K^r)$ satisfying

$$\operatorname{div}(\lambda^j \boldsymbol{w}_{k-j}) = \lambda^{j-1} p_{k-j} - \lambda^j p_{k-(j+1)}, \tag{3.10}$$

for $1 \le j \le \ell - 1$, and with the bounds

$$\frac{1}{h_K} \|\boldsymbol{w}_{k-j}\|_{L^2(K)} + \|\nabla \boldsymbol{w}_{k-j}\|_{H^1(K)} \le C \|p\|_{L^2(K)} \quad \text{and} \quad \|p_{k-(j+1)}\|_{L^2(K)} \le C \|p\|_{L^2(K)}. \tag{3.11}$$

Using Lemma 3.2 we define $\boldsymbol{w}_{k-\ell} \in \boldsymbol{\mathcal{P}}^{\perp}_{k-\ell}(K)$ satisfying

$$\frac{-\ell}{h_i} \int_{F_i} (\boldsymbol{w}_{k-\ell} \cdot \boldsymbol{n}_i) \kappa \, ds = \int_{F_i} p_{k-\ell} \, \kappa \quad \text{for all } \kappa \in \mathcal{P}_{k-\ell}(F_i), \quad 1 \le i \le d+1, \tag{3.12}$$

and the bound

$$\frac{1}{h_K^2} \|\boldsymbol{w}_{k-\ell}\|^2_{L^2(K)} + \|\nabla \boldsymbol{w}_{k-\ell}\|^2_{L^2(K)} \le C \frac{h_i^2}{h_K \ell^2} \sum_{i=1}^{d+1} \|p_{k-\ell}\|^2_{L^2(F_i)} \le C \|p_{k-\ell}\|^2_{L^2(K)} \le C \|p\|^2_{L^2(K)}. \tag{3.13}$$

Here we used an inverse estimate and (3.11).

From (3.12) and (3.3) we get

$$\ell \boldsymbol{w}_{k-\ell} \cdot \nabla \lambda|_{F_i} = \frac{-\ell}{h_i} \boldsymbol{w}_{k-\ell} \cdot \boldsymbol{n}_i|_{F_i} = p_{k-\ell}|_{F_i}, \quad 1 \le i \le d+1,$$

and therefore

$$\ell \boldsymbol{w}_{k-\ell} \cdot \nabla \lambda = p_{k-\ell} + \lambda r_{k-(\ell+1)}$$

for a *unique* $r_{k-(\ell+1)} \in \mathcal{P}_{k-(\ell+1)}(K^r)$. Moreover, by (3.3), (3.4), and (3.13), there holds

$$\|r_{k-(\ell+1)}\|_{L^2(K)} \le C \|\lambda r_{k-(\ell+1)}\|_{L^2(K)} \le \frac{C}{h_K} \|\boldsymbol{w}_{k-\ell}\|_{L^2(K)} + \|p_{k-\ell}\|_{L^2(K)} \le C \|p\|_{L^2(K)}. \tag{3.14}$$

Finally, by setting $p_{k-(\ell+1)} := -(r_{k-(\ell+1)} + \operatorname{div} \boldsymbol{w}_{k-\ell})$, we have

$$\operatorname{div}(\lambda^\ell \boldsymbol{w}_{k-\ell}) = \ell \lambda^{\ell-1} \boldsymbol{w}_{k-\ell} \cdot \nabla \lambda + \lambda^\ell \operatorname{div} \boldsymbol{w}_{k-\ell} = \lambda^{\ell-1} p_{k-\ell} - \lambda^\ell p_{k-(\ell+1)}.$$

From (3.13) and (3.14) we get

$$\|p_{k-(\ell+1)}\|_{L^2(K)} \le C \|p\|_{L^2(K)}.$$

The same arguments show that (3.10) holds for the base case $\ell = 2$. Thus, by induction we have (3.10) and (3.11) for $1 \le j \le k-1$. Hence, statement (3.9) holds by summing (3.10) over j. Also, as a consequence of $\int_K p dx = 0$ we have $\int_K \lambda^{k-1} p_0 = 0$.

Step 2: Applying Lemma 3.2 we choose $\boldsymbol{w}_0 \in \boldsymbol{\mathcal{P}}_0(K)$ satisfying

$$-\frac{k}{h_i} \int_{F_i} \boldsymbol{w}_0 \cdot \boldsymbol{n}_i = \int_{F_i} p_0 \quad \text{for all } \kappa \in \mathcal{P}_{k-\ell}(F_i), \quad 1 \le i \le d, \tag{3.15}$$

with

$$\frac{1}{h_K^2} \|\boldsymbol{w}_0\|^2_{L^2(K)} \le C \frac{h_i^2}{h_K \, k^2} \sum_{i=1}^{d} \|p_0\|^2_{L^2(F_i)} \le C \|p_0\|^2_{L^2(K)} \le C \|p\|^2_{L^2(K)}. \tag{3.16}$$

Hence, $k\nabla\cdot \boldsymbol{w}_0|_{F_i} = p_0|_{F_i}$ for $1 \le i \le d$, and so we have

$$\operatorname{div}(\lambda^k \boldsymbol{w}_0) = k\lambda^{k-1}\nabla\lambda\cdot\boldsymbol{w}_0 = \lambda^{k-1}p_0 \quad \text{on } K\backslash K_{d+1}.$$

Moreover,

$$\int_{K_{d+1}} \lambda^{k-1}(k\nabla\lambda\cdot\boldsymbol{w}_0 - p_0)dx = \int_{K_{d+1}} (\operatorname{div}(\lambda^k\boldsymbol{w}_0) - \lambda^{k-1}p_0)dx = \int_K (\operatorname{div}(\lambda^k\boldsymbol{w}_0) - \lambda^{k-1}p_0)dx = 0.$$

Hence, since $\lambda > 0$ in the interior of K we have $k\nabla\lambda\cdot\boldsymbol{w}_0 = p_0$ on K_{d+1} and thus $\operatorname{div}(\lambda^k\boldsymbol{w}_0) = \lambda^{k-1}p_0$ on K_{d+1}. Therefore, $\operatorname{div}(\lambda^k\boldsymbol{w}_0) = \lambda^{k-1}p_0$ on K. We conclude that (3.7) holds. For the bound (3.8), we use (3.3) and get

$$\|\nabla\boldsymbol{v}\|^2_{L^2(K)} = C\sum_{j=1}^{k}\|\nabla(\lambda^j\boldsymbol{w}_{k-j})\|^2_{L^2(K)} \le C\sum_{j=1}^{k}(\frac{1}{h_K^2}\|\boldsymbol{w}_{k-j}\|^2_{L^2(K)} + \|\nabla\boldsymbol{w}_{k-j}\|^2_{L^2(K)})$$

The bound (3.8) now follows if we use (3.11) and (3.16). ▯

4. The Bernardi-Raugel bubble and its modification. In this section we recall the Bernardi-Raugel face bubbles (cf. [4]) and summarize their stability properties. Then, using Theorem 3.1, we propose a modification of these bubble functions such that the resulting vector fields have constant divergence.

Recall, that for a simplex $K \in \mathcal{T}_h$, the vertices are denoted by $\{x_1, x_2, \ldots, x_{d+1}\}$, and that F_i is the $(d-1)$-dimensional face of K opposite to x_i with outward unit normal $\boldsymbol{n}_i$. We denote by $\{\mu_1, \mu_2, \ldots, \mu_{d+1}\} \subset \mathcal{P}_1(K)$ the barycentric coordinates of K, i.e., $\mu_i(x_j) = \delta_{ij}$. We define scalar face bubbles as

$$B_i := \prod_{\substack{j=1\\ j\neq i}}^{d+1}\mu_j, \qquad \text{for } 1 \le i \le d+1.$$

The Bernardi-Raugel face bubbles are given as

$$\boldsymbol{b}_i := B_i\boldsymbol{n}_i \qquad \text{for } 1 \le i \le d+1. \tag{4.1}$$

We note that $\boldsymbol{b}_i \in \boldsymbol{\mathcal{P}}_d(K)$.

We define the local Bernardi-Raugel bubble space as follows:

$$\boldsymbol{V}^{\mathrm{BR}}(K) := \operatorname{span}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{d+1}\} + \boldsymbol{\mathcal{P}}_1(K).$$

The corresponding global space is given by

$$\boldsymbol{V}_h^{\mathrm{BR}} := \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}^{\mathrm{BR}}(K), \text{ for all } K \in \mathcal{T}_h\}.$$

The following result is well known; see [4].

Proposition 4.1. *For any $p \in \mathring{\mathcal{P}}_0(\mathcal{T}_h)$, there exists a $\boldsymbol{v} \in \boldsymbol{V}_h^{\mathrm{BR}}$ so that*

$$\int_K \operatorname{div}\boldsymbol{v} = \int_K p \qquad \textit{for all } K \in \mathcal{T}_h,$$

with the bound

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|p\|_{L^2(\Omega)}.$$

In the next result, we modify the function $\boldsymbol{b}_i$ so that the resulting vector-valued function has constant divergence.

PROPOSITION 4.2. *There exists $\boldsymbol{\beta}_i \in \boldsymbol{\mathcal{P}}_d^c(K^r)$ such that*

$$\boldsymbol{\beta}_i|_{\partial K} = \boldsymbol{b}_i|_{\partial K}, \quad \operatorname{div} \boldsymbol{\beta}_i \in \mathcal{P}_0(K), \tag{4.2}$$

with the bound

$$\|\boldsymbol{\beta}_i\|_{H^1(K)} \le C\|\boldsymbol{b}_i\|_{H^1(K)}.$$

Proof. Set

$$g_i = \operatorname{div} \boldsymbol{b}_i - \frac{1}{|K|}\int_K \operatorname{div} \boldsymbol{b}_i \in \mathring{\mathcal{P}}_{d-1}(K) \subset \mathring{\mathcal{P}}_{d-1}(K^r).$$

By Theorem 3.3, there exists a unique $\boldsymbol{w}_i \in \boldsymbol{M}_d(K^r)$ such that

$$\operatorname{div} \boldsymbol{w}_i = g_i \quad \text{on } K, \qquad \|\boldsymbol{w}_i\|_{H^1(K)} \le C\|g_i\|_{L^2(K)}. \tag{4.3}$$

The function $\boldsymbol{\beta}_i := \boldsymbol{b}_i - \boldsymbol{w}_i$ then satisfies (4.2) since $\boldsymbol{\beta}_i|_{\partial K} = \boldsymbol{b}_i|_{\partial K}$ and $\operatorname{div}\boldsymbol{\beta}_i = \operatorname{div}\boldsymbol{b}_i - \operatorname{div}\boldsymbol{w}_i = |K|^{-1}\int_K \operatorname{div}\boldsymbol{b}_i$. The stability estimate follows from (4.3) and the bound $\|g_i\|_{L^2(K)} \le C\|\boldsymbol{b}_i\|_{H^1(K)}$. ◻

We let $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_{d+1}\} \subset \boldsymbol{\mathcal{P}}_d^c(K^r)$ be functions from Proposition 4.2. We call these functions the modified face bubbles. We define the local finite element space of these functions as follows:

$$\boldsymbol{V}^{\mathrm{MB}}(K) := \operatorname{span}\{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{d+1}\} + \boldsymbol{\mathcal{P}}_1(K).$$

LEMMA 4.3. *A function $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{MB}}(K)$ is uniquely determined by*

$$\int_{F_i} \boldsymbol{v}\cdot\boldsymbol{n}_i \qquad \textit{for } 1 \le i \le d+1, \tag{4.4}$$

$$\boldsymbol{v}(x_i) \qquad \textit{for all vertices } x_i \textit{ of } K. \tag{4.5}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{MB}}(K)$, and write $\boldsymbol{v} = \boldsymbol{w} + \sum_{i=1}^{d+1} a_i\boldsymbol{\beta}_i$ for some $a_i \in \mathbb{R}$ and $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_1(K)$. Suppose that $\boldsymbol{v}$ vanishes on the degrees of freedom (4.4) and (4.5). Then, since $\boldsymbol{\beta}_i$ vanish on the vertices, $\boldsymbol{w}$ vanishes on the vertices of K and hence $\boldsymbol{w} \equiv 0$. Moreover, $\boldsymbol{\beta}_i|_{\partial K\setminus F_i} = \boldsymbol{b}_i|_{\partial K\setminus F_i} = 0$, we have

$$0 = \int_{F_i} \boldsymbol{v}\cdot\boldsymbol{n}_i = a_i\int_{F_i} \boldsymbol{b}_i\cdot\boldsymbol{n}_i = a_i\int_{F_i} B_i.$$

Since $B_i > 0$ on F_i, we conclude that $a_i = 0$ and $\boldsymbol{v} \equiv 0$. The dimension of $\boldsymbol{V}^{\mathrm{MB}}(K)$ is clearly $2(d+1)$, and therefore we conclude that (4.4) and (4.5) uniquely determines a function in $\boldsymbol{V}^{\mathrm{MB}}$. ◻

5. A low order inf-sup stable pair on $\mathcal{T}_h$.

With the modified face bubble spaces, we now provide several inf-sup stable and divergence-free pairs applicable to incompressible flows. First, let us recall the definitions of an inf-sup stable pair and a divergence-free pair.

DEFINITION 5.1. *A pair of spaces $\boldsymbol{V}_h - M_h$, with $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$ and $M_h \subset L_0^2(\Omega)$ is* inf-sup stable *if there exists a constant $\gamma > 0$, independent of h, such that*

$$0 < \gamma \le \inf_{0\ne q\in M_h}\ \sup_{0\ne\boldsymbol{v}\in\boldsymbol{V}_h} \frac{\int_\Omega q \operatorname{div}\boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}\|p\|_{L^2(\Omega)}}.$$

The pair is said to be a divergence-free pair *if* $\operatorname{div}\boldsymbol{V}_h \subset M_h$.

In this section, we give an example of a pair defined on the macro mesh $\mathcal{T}_h$ satisfying the two conditions in Definition 5.1. To this end, we define the global space:

$$\boldsymbol{V}_h^{\mathrm{MB}} := \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}^{\mathrm{MB}}(K), \text{ for all } K \in \mathcal{T}_h\}.$$

Lemma 4.3 shows that the degrees of freedom for $\boldsymbol{v} \in \boldsymbol{V}_h^{\mathrm{MB}}$ are

$$\int_F \boldsymbol{v} \cdot \boldsymbol{n} \quad \text{for all interior } (d-1)\text{-dimensional faces } F \text{ of } \mathcal{T}_h, \tag{5.1}$$

$$\boldsymbol{v}(x) \quad \text{for all interior vertices } x \text{ of } \mathcal{T}_h. \tag{5.2}$$

Theorem 5.2. *It holds,* $\operatorname{div} \boldsymbol{V}_h^{\mathrm{MB}} \subset \mathring{\mathcal{P}}_0(\mathcal{T}_h)$. *Moreover, for any* $p \in \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ *there exists a* $\boldsymbol{v} \in \boldsymbol{V}_h^{\mathrm{MB}}$ *so that*

$$\operatorname{div} \boldsymbol{v} = p \qquad \text{in } \Omega, \tag{5.3}$$

with the bound

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|p\|_{L^2(\Omega)}.$$

Proof. First, by (4.2) we see that $\operatorname{div} \boldsymbol{V}_h^{\mathrm{MB}} \subset \mathring{\mathcal{P}}_0(\mathcal{T}_h)$.

For a fixed $p \in \mathring{\mathcal{P}}_0(\mathcal{T}_h)$, there exists $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ such that (see for example [9]).

$$\operatorname{div} \boldsymbol{w} = p \quad \text{in } \Omega,$$

with the bound

$$\|\boldsymbol{w}\|_{H^1(\Omega)} \le C\|p\|_{L^2(\Omega)}.$$

Let $\boldsymbol{I}_h^{SZ}\boldsymbol{w}$ be the Scott-Zhang interpolant into $\mathring{\boldsymbol{\mathcal{P}}}_1^c(\mathcal{T}_h) \subset \boldsymbol{V}_h^{\mathrm{MB}}$ [23]. We then define $\boldsymbol{w}_1 \in \boldsymbol{V}_h^{\mathrm{MB}}$ by

$$\int_F \boldsymbol{w}_1 \cdot \boldsymbol{n} = \int_F (\boldsymbol{w} - \boldsymbol{I}_h^{SZ}\boldsymbol{w}) \cdot \boldsymbol{n} \quad \text{for all interior } (d-1)\text{-dimensional faces } F \text{ of } \mathcal{T}_h.$$

$$\boldsymbol{w}_1(x) = 0 \quad \text{for all interior vertices } x \text{ of } \mathcal{T}_h.$$

We set $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{I}_h^{SZ}\boldsymbol{w} \in \boldsymbol{V}_h^{\mathrm{MB}}$. A simple scaling argument gives

$$\|\nabla \boldsymbol{w}_1\|_{H^1(K)} \le C(\frac{1}{h_K}\|\boldsymbol{w} - \boldsymbol{I}_h^{SZ}\boldsymbol{w}\|_{L^2(K)} + \|\nabla(\boldsymbol{w} - \boldsymbol{I}_h^{SZ}\boldsymbol{w})\|_{L^2(K)}) \text{ for all } K \in \mathcal{T}_h.$$

Hence,

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \le \|\boldsymbol{w}_1\|_{H^1(\Omega)} + \|\boldsymbol{I}_h^{SZ}\boldsymbol{w}\|_{H^1(\Omega)} \le C\|\boldsymbol{w}\|_{H^1(\Omega)} \le C\|p\|_{L^2(\Omega)}.$$

Moreover, an application of the divergence theorem shows

$$\int_K \operatorname{div} \boldsymbol{v} = \int_K \operatorname{div} \boldsymbol{w} = \int_K p \text{ for all } K \in \mathcal{T}_h.$$

Since $\operatorname{div} \boldsymbol{v}, p \in \mathring{\mathcal{P}}_0(\mathcal{T}_h)$, this proves (5.3). □

Remark 5.3. *The two-dimensional case of Theorem 5.2 although not written in this exact form, has appeared in [7].*

Remark 5.4. *The degrees of freedom of* $\boldsymbol{V}_h^{\mathrm{MB}}$ *are the same as the Bernardi-Raugel finite element space [4]; see Figure 5.1.*

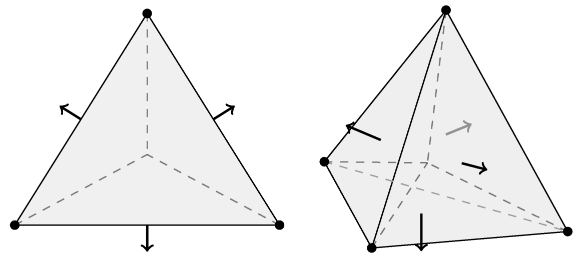

FIG. 5.1. *Degrees of freedom for the velocity space in Lemma 4.3 in two dimensions (left) and three dimensions (right). Solid circles indicate function evaluation and arrows indicate normal component evaluation.*

6. Inf-sup stable pair of spaces on $\mathcal{T}_h^r$. In this section, we apply Theorem 5.2 to construct divergence-free and inf-sup stable pairs on the refined mesh $\mathcal{T}_h^r$. To do so, we will use the following result repeatedly.

PROPOSITION 6.1. *Let $k \geq 1$, and suppose that $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$ satisfies $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) \subset \boldsymbol{V}_h$ and $\boldsymbol{V}_h - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable. Then $\boldsymbol{V}_h - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ is inf-sup stable.*

Proof. Let $q \in \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$, and define $\bar{q}$ to be its L^2-projection onto $\mathring{\mathcal{P}}_0(\mathcal{T}_h)$, i.e.,

$$\bar{q}|_K = \frac{1}{|K|}\int_K q\,dx \qquad \forall K \in \mathcal{T}_h.$$

By Theorem 3.1 there exists $\boldsymbol{w} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) \subset \boldsymbol{V}_h$ so that

$$\operatorname{div} w = q - \bar{q} \quad \text{ on } \Omega,$$

$$\|w\|_{H^1(\Omega)} \leq C\|q - \bar{q}\|_{L^2(\Omega)}. \tag{6.1}$$

Hence,

$$\|q - \bar{q}\|_{L^2(\Omega)}^2 = \int_\Omega (q - \bar{q})(q - \bar{q}) = \int_\Omega q(q - \bar{q}) = \int_\Omega q \operatorname{div} \boldsymbol{w}.$$

Applying (6.1), we find

$$\|q - \bar{q}\|_{L^2(\Omega)}^2 = \|w\|_{H^1(\Omega)} \frac{\int_\Omega q \operatorname{div} \boldsymbol{w}}{\|w\|_{H^1(\Omega)}} \leq C\|q - \bar{q}\|_{L^2(\Omega)} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}},$$

and therefore,

$$\|q - \bar{q}\|_{L^2(\Omega)} \leq C \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}}.$$

By the hypothesis, there is a constant $\gamma_0 > 0$ such that

$$\gamma_0 \|\bar{q}\|_{L^2(\Omega)} \leq \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega \bar{q} \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \leq \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}} + \|q - \bar{q}\|_{L^2(\Omega)}.$$

Hence, using the last two inequalities we get

$$\|q\|_{L^2(\Omega)} \leq (\|q - \bar{q}\|_{L^2(\Omega)} + \|\bar{q}\|_{L^2(\Omega)}) \leq (C(\frac{1}{\gamma_0} + 1) + \frac{1}{\gamma_0}) \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}}.$$

This proves the result. ∎

6.1. Higher-order Elements with discontinuous pressures. Using Proposition 6.1, we now show that the pair $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ for $k \geq d$ is inf-sup stable.

COROLLARY 6.2. *The pair $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ with $k \geq d$ is a divergence-free, inf-sup stable pair.*

Proof. Since $\boldsymbol{V}_h^{\mathrm{BR}} \subset \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)$ for $k \geq d$ and $\boldsymbol{V}_h^{\mathrm{BR}} - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable (cf. Proposition 4.1), the corollary follows by applying Proposition 6.1. ◻

In order to establish that $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ is inf-sup stable we used that $\boldsymbol{V}_h^{\mathrm{BR}} - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable; in other words, the inclusion $\boldsymbol{V}_h^{\mathrm{BR}} \subset \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h)$ implies that $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h) - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable. An interesting fact is that the converse is true.

THEOREM 6.3. *The pair $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ is inf-sup stable if and only if $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h) - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable.*

Proof. Assume $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h) - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable. Then by the inclusion $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h) \subset \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)$, the pair $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable. Thus, $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ is inf-sup stable by Proposition 6.1.

Now suppose that $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ is inf-sup stable. Let $q \in \mathring{\mathcal{P}}_0(\mathcal{T}_h)$. Due to the inclusion $\mathring{\mathcal{P}}_0(\mathcal{T}_h) \subset \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$, there exist $\gamma > 0$ such that

$$\gamma \|q\|_{L^2(\Omega)} \leq \sup_{0 \neq \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}}.$$

Let $\boldsymbol{I}_h : [C(\bar{\Omega})]^d \to \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h)$ be the canonical (nodal) interpolant. We then have

$$(\boldsymbol{I}_h \boldsymbol{v} - \boldsymbol{v})|_{\partial K} = 0 \quad \text{for all } K \in \mathcal{T}_h, \text{ for all } \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r). \tag{6.2}$$

Moreover,

$$\|\boldsymbol{I}_h \boldsymbol{v}\|_{H^1(\Omega)} \leq C \|\boldsymbol{v}\|_{H^1(\Omega)} \quad \text{for all } \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r). \tag{6.3}$$

By (6.2) and the divergence theorem we get that

$$\int_\Omega q \operatorname{div} \boldsymbol{v} = \int_\Omega q \operatorname{div} \boldsymbol{I}_h \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r).$$

Hence, by (6.3),

$$\gamma \|q\|_{L^2(\Omega)} \leq \sup_{0 \neq \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)} \frac{\int_\Omega q \operatorname{div} \boldsymbol{I}_h \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \leq C^{-1} \sup_{0 \neq \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)} \frac{\int_\Omega q \operatorname{div} \boldsymbol{I}_h \boldsymbol{v}}{\|\boldsymbol{I}_h \boldsymbol{v}\|_{H^1(\Omega)}} \leq C^{-1} \sup_{0 \neq \boldsymbol{w} \in \mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h)} \frac{\int_\Omega q \operatorname{div} \boldsymbol{w}}{\|\boldsymbol{w}\|_{H^1(\Omega)}}.$$

Therefore, $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h) - \mathring{\mathcal{P}}_0(\mathcal{T}_h)$ is inf-sup stable. ◻

6.2. Low-order elements with discontinuous pressures. For $k < d$, we can always augment $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)$ with $\boldsymbol{V}_h^{\mathrm{BR}}$ and it will lead to an inf-sup stable pair. However, the resulting pair will not be a divergence-free pair. Therefore, we instead supplement $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)$ with $\boldsymbol{V}_h^{\mathrm{MB}}$.

COROLLARY 6.4. *Let $1 \leq k < d$. The pair $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) + \boldsymbol{V}_h^{\mathrm{MB}} - \mathring{\mathcal{P}}_{k-1}(\mathcal{T}_h^r)$ is a divergence-free, inf-sup stable pair.*

Proof. The result follows from Proposition 6.1 and Lemma 5.2. ◻

REMARK 6.5. *It follows from Proposition 4.2 and Lemma 4.3 that the degrees of freedom for $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r) + \boldsymbol{V}_h^{\mathrm{MB}}$ $(k < d)$ are the canonical degrees of freedom of $\mathring{\boldsymbol{\mathcal{P}}}_k^c(\mathcal{T}_h^r)$ plus the degrees of freedom* (5.1)*; see Figure 6.1.*

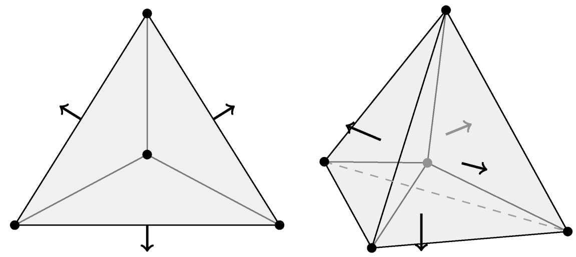

FIG. 6.1. *Degrees of freedom for the lowest order velocity space in Corollary 6.4 in two dimensions (left) and three dimensions (right). Solid circles indicate function evaluation and arrows indicate normal component evaluation.*

6.3. Low-order Stokes pairs with continuous pressure. In this section we, in some sense, generalize the inf-sup stable pair of spaces found in [1] to higher dimensions. In the paper [1], the pressure space is the space of continuous, piecewise linear polynomials with respect to the refined triangulation:

$$W_h^R = \mathring{\mathcal{P}}_1(\mathcal{T}_h^r) \cap H^1(\Omega).$$

Their velocity space is given by $\mathring{\boldsymbol{\mathcal{P}}}_2^c(\mathcal{T}_h^r) \cap \boldsymbol{H}^1(\mathrm{div}\,;\Omega)$, where

$$\boldsymbol{H}^1(\mathrm{div}\,;S) := \{\boldsymbol{v} \in \boldsymbol{H}^1(S) : \mathrm{div}\,\boldsymbol{v} \in H^1(S)\}.$$

It is shown in [1] that this pair of spaces is inf-sup stable in two dimensions. It is clearly a divergence-free pair.

To generalize these results to higher dimensions it seems necessary to supplement the velocity space with the modified face bubbles given in Proposition 4.2. The following result defines the local space and a unisolvent set of degrees of freedom.

THEOREM 6.6. *Define, for $d \geq 3$,*

$$\boldsymbol{V}_R(K) := \boldsymbol{\mathcal{P}}_2(K^r) \cap \boldsymbol{H}^1(\mathrm{div}\,;K) \oplus \mathrm{span}\{\boldsymbol{\beta}_i\}_{i=1}^{d+1}.$$

Then a function $\boldsymbol{v} \in \boldsymbol{V}_R(K)$ is uniquely determined by the values

$$\boldsymbol{v}(x_i),\ \mathrm{div}\,\boldsymbol{v}(x_i) \quad \text{for all vertices } x_i \text{ of } K, \tag{6.4a}$$

$$\int_{e_i} \boldsymbol{v}, \quad \text{for all one-dimensional edges } e_i, \tag{6.4b}$$

$$\int_{F_i} \boldsymbol{v}\cdot\boldsymbol{n}_i, \quad \text{for all } (d-1)\text{-dimensional faces } F_i. \tag{6.4c}$$

Before we prove this result, we note that in the case $d = 2$ (which is not considered in this Theorem), the degrees of freedom (6.4b) would contain the degrees of freedom (6.4c). Therefore, in the case $d = 2$, one simply has to eliminate the functions that give rise to (6.4c), which are $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$; see [1] for details.

Proof of Theorem 6.6., The constraint $\boldsymbol{v} \in \boldsymbol{H}^1(\mathrm{div}\,;K)$ for $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_2^c(K^r)$ represents $(d+1)^2-(d+2)$ equations. Therefore

$$\begin{aligned}\dim \boldsymbol{V}_R(K) &\geq \dim \boldsymbol{\mathcal{P}}_2^c(K^r) - \big((d+1)^2 - (d+2)\big) + (d+1)\\ &= \frac{1}{2}(d+1)(d^2+2d+4).\end{aligned}$$

On the other hand, the number of degrees of freedom given is

$$d(d+1) + (d+1) + \frac{d^2}{2}(d+1) + (d+1) = \frac{1}{2}(d+1)(d^2+2d+4).$$

Now suppose that $\boldsymbol{v} \in \boldsymbol{V}_R(K)$ vanishes on the degrees of freedom (6.4), and write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{s}$, where $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{P}}_2^c(K^r)$ and $\boldsymbol{s} = \sum_{i=1}^{d+1} c_i \boldsymbol{\beta}_i$ for some $c_i \in \mathbb{R}$. Then, since $\boldsymbol{s}|_{F_i} = \boldsymbol{b}_i$ on each $(d-1)$-dimensional face, we conclude that

$$\boldsymbol{v}_0(x_i) = 0, \qquad \int_{e_i} \boldsymbol{v}_0 = 0$$

for all vertices x_i and edges e_i of K. These conditions imply that $\boldsymbol{v}_0 = 0$ on ∂K. Therefore we have

$$0 = \int_{F_i} \boldsymbol{v} \cdot \boldsymbol{n}_i = \int_{F_i} \boldsymbol{s} \cdot \boldsymbol{n}_i = c_i \int_{F_i} \boldsymbol{b}_i \cdot \boldsymbol{n}_i.$$

Since $\boldsymbol{b}_i \cdot \boldsymbol{n}_i > 0$ on F_i, we obtain that $c_i = 0$, and so $\boldsymbol{s} \equiv 0$ and $\boldsymbol{v} = \boldsymbol{v}_0 \in \boldsymbol{\mathcal{P}}_2^c(K^r)$. Moreover, because $\operatorname{div} \boldsymbol{v}$ restricted to a $(d-1)$-dimensional face is a linear polynomial, we conclude from the condition $\operatorname{div} \boldsymbol{v}(x_i) = 0$ that $\operatorname{div} \boldsymbol{v}$ vanishes on ∂K as well.

Since $\boldsymbol{v}$ vanishes on ∂K we can write $\boldsymbol{v} = \lambda_0 \boldsymbol{p}$ (see Section 3 for definition of λ_0) for some $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_1^c(K^r)$. We then find that

$$0 = \operatorname{div} \boldsymbol{v} = \nabla \lambda_0 \cdot \boldsymbol{p} + \lambda_0 \operatorname{div} \boldsymbol{p} = \nabla \lambda_0 \cdot \boldsymbol{p} \quad \text{on } \partial K.$$

The gradient of λ_0 restricted to K_i is parallel to the outward unit normal of the face $\partial K_i \cap \partial K$, and so we conclude that $\boldsymbol{p} \cdot \boldsymbol{n} = 0$ on ∂K. This implies, since $\boldsymbol{p}$ is continuous, that $\boldsymbol{p}$ vanishes at the vertices of K. But since $\boldsymbol{p}$ is piecewise linear, we obtain that $\boldsymbol{p}|_{\partial K} = 0$, and so $\boldsymbol{v} = \boldsymbol{c} \lambda_0^2$ for some $\boldsymbol{c} \in \mathbb{R}^d$. However, it is easy to see that $\operatorname{div} \boldsymbol{v} = 2\lambda_0 \boldsymbol{c} \cdot \nabla \lambda_0$ is only continuous if $\boldsymbol{c} \equiv 0$. Thus, $\boldsymbol{v} \equiv 0$, and so the degrees of freedom are unisolvent on $\boldsymbol{V}_R(K)$. ☐

REMARK 6.7. *Note that* $\operatorname{div} \boldsymbol{\beta}_i \in \mathcal{P}_0(K) \subset \mathcal{P}_1(K^r) \cap H^1(K)$. *Therefore* $\boldsymbol{V}_R(K) \subset \boldsymbol{H}^1(\operatorname{div}; K)$.

REMARK 6.8. *If* $\boldsymbol{v} \in \boldsymbol{V}_R(K^r)$ *vanishes at the degrees of freedom restricted to one face, then we can argue as in the proof of Theorem 6.6 that* $\boldsymbol{v} = 0$ *and* $\operatorname{div} \boldsymbol{v} = 0$ *on that face. Thus, the degrees of freedom induce an* $\boldsymbol{H}^1(\operatorname{div}; \Omega)$*–conforming finite element space.*

The local spaces and degrees of freedom lead to the global finite element space:

$$\boldsymbol{V}_h^R = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^1(\operatorname{div}; \Omega) : \ \boldsymbol{v}|_K \in \boldsymbol{V}_R(K) \ \forall K \in \mathcal{T}_h\}.$$

THEOREM 6.9. *The pair* $\boldsymbol{V}_h^R - W_h^R$ *is inf-sup stable.*

Proof. Let $q \in W_h^R$ and let $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ satisfy $\operatorname{div} \boldsymbol{w} = q$ and $\|\boldsymbol{w}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. We then define $\boldsymbol{v} \in \boldsymbol{V}_h^R$ such that

$$\begin{aligned}
&\boldsymbol{v}(x) = \boldsymbol{I}_h^{SZ}\boldsymbol{w}(x),\ \operatorname{div} \boldsymbol{v}(x) = q(x), && \text{for all vertices } x,\\
&\int_e \boldsymbol{v} = \int_e \boldsymbol{I}_h^{SZ}\boldsymbol{w} && \text{for all one-dimensional edges } e,\\
&\int_F \boldsymbol{v} \cdot \boldsymbol{n} = \int_F \boldsymbol{w} \cdot \boldsymbol{n} && \text{for all } (d-1)\text{-dimensional faces } F,
\end{aligned}$$

where $\boldsymbol{I}_h^{SZ}\boldsymbol{w}$ is the Scott-Zhang interpolant of $\boldsymbol{w}$ [23]. We then have $\operatorname{div} \boldsymbol{v}(x) = q(x)$ at all vertices and

$$\int_K \operatorname{div} \boldsymbol{v} = \int_K \operatorname{div} \boldsymbol{w} = \int_K q.$$

Since $q, \operatorname{div} \boldsymbol{v}|_K \in \mathcal{P}_1^c(K^r)$, we conclude that $\operatorname{div} \boldsymbol{v} = q$. Uniform inf-sup stability then comes from a standard scaling argument. ☐

6.3.1. Reduced velocity space of $\boldsymbol{V}_h^R$. In this section, we give a basis for the local space $\boldsymbol{\mathcal{P}}_2(K^r) \cap \boldsymbol{H}^1(\mathrm{div}\,; K)$, and as a byproduct, construct reduced spaces of $\boldsymbol{V}_h^R$. To this end, recall that, for a simplex $K \in \mathcal{T}_h$, $\lambda_i \in \mathcal{P}_1^c(K^r)$ satisfy $\lambda_i(x_j) = \delta_{ij}$. For each $i \in \{1, \ldots, d+1\}$ we set

$$\boldsymbol{\psi}_i = \boldsymbol{c}_i \lambda_i^2,$$

where the constant $\boldsymbol{c}_i \in \mathbb{R}^d$ is chosen so that

$$2\boldsymbol{c}_i \cdot \nabla \lambda_i|_{K_j} = 1 \text{ for all } 1 \le j \le d+1, j \ne i.$$

This is possible since $\nabla \lambda_i|_{K_j}$ for $1 \le j \le d+1, j \ne i$ are linearly independent. We then see that $\mathrm{div}\, \boldsymbol{\psi}_i = \lambda_i$, and so $\boldsymbol{\psi}_i \in \boldsymbol{\mathcal{P}}_2(K^r) \cap \boldsymbol{H}^1(\mathrm{div}\,; K)$. We note from the proof of Theorem 6.6 that

$$\dim \boldsymbol{\mathcal{P}}_2(K^r) \cap \boldsymbol{H}^1(\mathrm{div}\,; K) = \dim \boldsymbol{\mathcal{P}}_2(K) + d + 1.$$

From this dimension count, we conclude that

$$\boldsymbol{\mathcal{P}}_2(K^r) \cap \boldsymbol{H}^1(\mathrm{div}\,; K) = \boldsymbol{\mathcal{P}}_2(K) + \mathrm{span}\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_{d+1}\}.$$

Next, using this construction, we reduce the dimension of $\boldsymbol{V}_h^R$ while still getting an inf-sup stable pair. Recall from Section 4 that $\{\mu_i\}_{i=1}^{d+1}$ are the barycentric coordinates of K. By the labeling convention, we then see that $\mu_i = \lambda_i$ on ∂K for all $1 \le i \le d+1$. We then define

$$\boldsymbol{\theta}_i = \frac{1}{2}\boldsymbol{c}_i(\lambda_i^2 - \mu_i^2),$$

and choose $\boldsymbol{c}_i$ so that

$$\boldsymbol{c}_i \cdot \nabla(\lambda_i - \mu_i)|_{K_j} = 1 \text{ for all } 1 \le j \le d+1, j \ne i.$$

We then have

$$\mathrm{div}\, \boldsymbol{\theta}_i = (\lambda_i - \mu_i)\boldsymbol{c}_i \cdot \nabla \mu_i + \lambda_i.$$

In particular, there holds $\mathrm{div}\, \boldsymbol{\theta}_i|_{\partial K} = \lambda_i$, and since $(\boldsymbol{c}_i \cdot \nabla \mu_i)$ is constant on K, $\mathrm{div}\, \boldsymbol{\theta}_i$ is continuous. Thus, these functions have the following properties:

$$\begin{aligned} \boldsymbol{\theta}_i \in& \boldsymbol{H}^1(\mathrm{div}\,; K) \cap \boldsymbol{H}_0^1(K), \\ \mathrm{div}\, \boldsymbol{\theta}_i(x_j) =& \delta_{ij}. \end{aligned}$$

We then define the space $\boldsymbol{V}^S(K) = \mathrm{span}\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_{d+1}\}$. We see that this space is a space of bubbles (i.e. vanish on ∂K), and that the degrees of freedom of $\boldsymbol{V}^S(K) + \boldsymbol{V}^{\mathrm{MB}}(K)$ are given by

$$\begin{aligned} &\int_{F_i} \boldsymbol{v} \cdot \boldsymbol{n} \quad && \text{for all } (d-1)\text{-dimensional faces } F_i \text{ of } K, \\ &\boldsymbol{v}(x_i), \mathrm{div}\, \boldsymbol{v}(x_i) && \text{for all vertices } x_i \text{ of } K. \end{aligned}$$

We can then define

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{H}^1(\mathrm{div}\,; \Omega) \cap \boldsymbol{H}_0^1(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}^S(K) + \boldsymbol{V}^{\mathrm{MB}}(K), \text{ for all } K \in \mathcal{T}_h\},$$

and the degrees of freedom of this space are

$$\begin{aligned} &\int_F \boldsymbol{v} \cdot \boldsymbol{n} \quad && \text{for all interior } (d-1)\text{-dimensional faces } F \text{ of } \mathcal{T}_h, \\ &\mathrm{div}\, \boldsymbol{v}(x) && \text{for all vertices } x \text{ of } \mathcal{T}_h, \\ &\boldsymbol{v}(x) && \text{for all interior vertices } x \text{ of } \mathcal{T}_h. \end{aligned}$$

These degrees of freedom give us the following result. Its proof is identical to the proof of Theorem 6.9 and is therefore omitted.

LEMMA 6.10. *It holds,* $\operatorname{div} \boldsymbol{V}_h \subset W_h^R$. *Moreover, for any* $p \in W_h^R$ *there exists a* $\boldsymbol{v} \in \boldsymbol{V}_h$ *so that*

$$\operatorname{div} \boldsymbol{v} = p \qquad \text{on } \Omega, \tag{6.5}$$

with the bound

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|p\|_{L^2(\Omega)}.$$

Finally, it is clear that the above space is indeed a subspace $\boldsymbol{V}_h^R$. However, the velocity approximation will converge with one order less.

7. Implementation Aspects and Numerical Experiments. In this section we discuss some implementation aspects of the proposed methods, in particular, how to compute the modified face bubbles $\{\boldsymbol{\beta}_i\}_{i=1}^{d+1}$ stated in Proposition 4.2 (see (4.2)). In the construction we have $\boldsymbol{\beta}_i = \boldsymbol{b}_i - \boldsymbol{w}_i$ where $\boldsymbol{w}_i \in \mathring{\boldsymbol{\mathcal{P}}}_d^c(K^r)$ satisfies (4.3), and $\boldsymbol{b}_i \in \boldsymbol{\mathcal{P}}_d(K)$ is the Bernardi-Raugel face bubble defined by (4.1). Therefore the computation of $\boldsymbol{\beta}_i$ reduces to the computation of $\boldsymbol{w}_i$.

Let $\hat{K} \subset \mathbb{R}^d$ be the reference element, and for $K \in \mathcal{T}_h$, let $G : \hat{K} \to K$ be an affine mapping with $G(\hat{x}) = A\hat{x} + b$, $A \in \mathbb{R}^{d\times d}$, and $b \in \mathbb{R}^d$. Note that $G(\hat{K}_i) = K_i$. For an index $i \in \{1, 2, \dots, d+1\}$, we set $\boldsymbol{s}_i = A^{-1}\boldsymbol{n}_i$, where we recall that $\boldsymbol{n}_i$ is the outward unit normal of the face $F_i = \partial K \cap \partial K_i$. Let $\hat{B}_i \in \mathcal{P}_d(\hat{K})$ denote the scalar face bubble associated with $\hat{F}_i \subset \partial\hat{K}$.

We specify $\hat{\boldsymbol{w}}_i \in \mathring{\boldsymbol{\mathcal{P}}}_d^c(\hat{K}^r)$ such that

$$\widehat{\operatorname{div}}\, \hat{\boldsymbol{w}}_i = \boldsymbol{s}_i \cdot \hat{\nabla}\hat{B}_i - \frac{1}{|\hat{K}|}\int_{\hat{K}} \boldsymbol{s}_i \cdot \hat{\nabla}\hat{B}_i. \tag{7.1}$$

Theorem 3.1 guarantees the existence of such a $\hat{\boldsymbol{w}}_i$. We then set $\boldsymbol{w}_i \in \mathring{\boldsymbol{\mathcal{P}}}_d^c(K^r)$ such that

$$\boldsymbol{w}_i(x) = A\hat{\boldsymbol{w}}_i(\hat{x}), \quad x = G(\hat{x}).$$

An application of the chain rule shows that $\operatorname{div} \boldsymbol{w}_i(x) = \widehat{\operatorname{div}}\, \hat{\boldsymbol{w}}_i(\hat{x})$. The same calculation also shows that

$$\operatorname{div} \boldsymbol{b}_i(x) = \widehat{\operatorname{div}}\,(A^{-1}\boldsymbol{n}_i\hat{B}_i)(\hat{x}) = \widehat{\operatorname{div}}\,(\boldsymbol{s}_i\hat{B}_i)(\hat{x}) = \boldsymbol{s}_i \cdot \hat{\nabla}\hat{B}_i(\hat{x}).$$

Thus, we have

$$\begin{aligned}\operatorname{div} \boldsymbol{w}_i(x) = \widehat{\operatorname{div}}\, \hat{\boldsymbol{w}}_i(\hat{x}) &= \boldsymbol{s}_i \cdot \hat{\nabla}\hat{B}_i - \frac{1}{|\hat{K}|}\int_{\hat{K}} \boldsymbol{s}_i \cdot \hat{\nabla}\hat{B}_i\\ &= \operatorname{div} \boldsymbol{b}_i - \frac{1}{|K|}\int_K \operatorname{div} \boldsymbol{b}_i,\end{aligned}$$

and therefore $\boldsymbol{w}_i$ satisfies (4.3). To summarize, the computation of $\boldsymbol{\beta}_i$ consists of (i) constructing $\hat{\boldsymbol{w}}_i$ satisfying (7.1) (with $\boldsymbol{s}_i = A^{-1}\boldsymbol{n}_i$); (ii) setting $\boldsymbol{\beta}_i(x) = \hat{B}_i(\hat{x})\boldsymbol{n}_i - A\hat{\boldsymbol{w}}_i(\hat{x})$. Closed-form formulas for the function $\hat{\boldsymbol{w}}_i$ in terms of a vector $\boldsymbol{s}_i$ can easily be obtained by symbolic mathematical software and hard-coded into a finite element subroutine. We provide the explicit formulas for $\hat{\boldsymbol{w}}_i$ in two dimension in Figure 7.1.

We perform some numerical experiments for the Stokes problem in three dimensions:

$$\begin{aligned}-\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad &&\text{in } \Omega,\\ \operatorname{div} \boldsymbol{u} &= 0 &&\text{in } \Omega,\\ \boldsymbol{u} &= 0 &&\text{on } \partial\Omega,\end{aligned}$$

$$
\begin{aligned}
\hat{\boldsymbol{w}}_1|_{\hat{K}_1} &= -\frac{1}{18}\hat{\lambda}_0\Big(s_1(3\hat{x}_1+6\hat{x}_2-2)+s_2(6\hat{x}_1+12\hat{x}_2-6), s_1(6\hat{x}_1-6\hat{x}_2-2)+s_2(15\hat{x}_1-6\hat{x}_2-6)\Big),\\
\hat{\boldsymbol{w}}_1|_{\hat{K}_2} &= -\frac{1}{18}\hat{\lambda}_0\Big(s_1(6\hat{x}_1+3\hat{x}_2-2)+s_2(6\hat{x}_1+12\hat{x}_2-6), -2s_1+s_2(6\hat{x}_1+3\hat{x}_2-6)\Big),\\
\hat{\boldsymbol{w}}_1|_{\hat{K}_3} &= -\frac{1}{18}\hat{\lambda}_0\Big(s_1(9\hat{x}_1+9\hat{x}_2-5)+s_2(6\hat{x}_1+12\hat{x}_2-6), -s_1(6\hat{x}_1+12\hat{x}_2-4)-s_2(3\hat{x}_1+15\hat{x}_2-3)\Big).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\boldsymbol{w}}_2|_{\hat{K}_1} &= -\frac{1}{18}\hat{\lambda}_0\Big((3\hat{x}_1+6\hat{x}_2-6)s_1-2s_2, (3\hat{x}_1+6\hat{x}_2-2)s_2+(12\hat{x}_1+6\hat{x}_2-6)s_1\Big),\\
\hat{\boldsymbol{w}}_2|_{\hat{K}_2} &= -\frac{1}{18}\hat{\lambda}_0\Big(-(6\hat{x}_1-6\hat{x}_2+2)s_2-(6\hat{x}_1-15\hat{x}_2+6)s_1, (6\hat{x}_1+3\hat{x}_2-2)s_2+(12\hat{x}_1+6\hat{x}_2-6)s_1\Big),\\
\hat{\boldsymbol{w}}_2|_{\hat{K}_3} &= -\frac{1}{18}\hat{\lambda}_0\Big(-(12\hat{x}_1+6\hat{x}_2-4)s_2-(15\hat{x}_1+3\hat{x}_2-3)s_1, (9\hat{x}_1+9\hat{x}_2-5)s_2+(12\hat{x}_1+6\hat{x}_2-6)s_1\Big).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\boldsymbol{w}}_3|_{\hat{K}_1} &= -\frac{1}{18}\hat{\lambda}_0\Big(-(3\hat{x}_1+6\hat{x}_2-2)s_1+2s_2, -(3\hat{x}_1+6\hat{x}_2-2)s_2-(6\hat{x}_1-6\hat{x}_2-2)s_1\Big),\\
\hat{\boldsymbol{w}}_3|_{\hat{K}_2} &= -\frac{1}{18}\hat{\lambda}_0\Big(-(-6\hat{x}_1+6\hat{x}_2-2)s_2-(6\hat{x}_1+3\hat{x}_2-2)s_1, -(6\hat{x}_1+3\hat{x}_2-2)s_2+2s_1\Big),\\
\hat{\boldsymbol{w}}_3|_{\hat{K}_3} &= -\frac{1}{18}\hat{\lambda}_0\Big(-(-12\hat{x}_1-6\hat{x}_2+4)s_2-(9\hat{x}_1+9\hat{x}_2-5)s_1, (-9\hat{x}_1-9\hat{x}_2+5)s_2+(6\hat{x}_1+12\hat{x}_2-4)s_1\Big).
\end{aligned}
$$

FIG. 7.1. *The unique $\hat{\boldsymbol{w}}_i \in \mathring{\boldsymbol{\mathcal{P}}}_2^c(\hat{K}^r)$ in two dimensions satisfying* (7.1). *Here, the labeling is chosen such that $\hat{B}_1 = (1-\hat{x}_1-\hat{x}_2)\hat{x}_2$, $\hat{B}_2 = (1-\hat{x}_1-\hat{x}_2)\hat{x}_1$, and $\hat{B}_3 = \hat{x}_1\hat{x}_2$. The modified bubble functions in Proposition 4.2 are given by $\boldsymbol{\beta}_i(x) = \hat{B}_i(\hat{x})\boldsymbol{n}_i - A\hat{\boldsymbol{w}}_i(\hat{x})$ with $\boldsymbol{s}_i = A^{-1}\boldsymbol{n}_i$.*

where $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is a given source function, and $\nu > 0$ is the viscosity. The finite element method seeks $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$
\begin{aligned}
\int_\Omega \nu\nabla\boldsymbol{u}_h : \nabla\boldsymbol{v} - \int_\Omega (\operatorname{div}\boldsymbol{v})p_h &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall \boldsymbol{v}\in\boldsymbol{V}_h,\\
\int_\Omega (\operatorname{div}\boldsymbol{u}_h)q &= 0 \qquad\qquad \forall q\in M_h.
\end{aligned}
$$

We perform two sets of numerical experiments. In the first set, we take $\boldsymbol{V}_h \times M_h$ to be the Bernardi-Raugel pair, i.e., $\boldsymbol{V}_h = \boldsymbol{V}_h^{\mathrm{BR}}$ and $M_h = \mathring{\mathcal{P}}_0(\mathcal{T}_h)$. In the second set, the pair is taken to be the modified Bernardi-Raugel pair defined in Section 5, i.e., $\boldsymbol{V}_h = \boldsymbol{V}_h^{\mathrm{MB}}$ and $M_h = \mathring{\mathcal{P}}_0(\mathcal{T}_h)$. We note that the degrees of freedom of these two pairs are the same, and thus their global dimensions coincide as well.

In the first case, with $\boldsymbol{V}_h = \boldsymbol{V}_h^{\mathrm{BR}}$, the velocity error satisfies [4, 5]

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} \le Ch\big(|\boldsymbol{u}|_{H^2(\Omega)} + \nu^{-1}|p|_{H^1(\Omega)}\big), \tag{7.2}
$$

where the constant $C > 0$ depends on Ω, the shape-regularity of $\mathcal{T}_h$, and the inf-sup constant $\gamma > 0$ stated in Definition 5.1. On the other hand, the second case $\boldsymbol{V}_h = \boldsymbol{V}_h^{\mathrm{MB}}$ satisfies $\operatorname{div}\boldsymbol{V}_h \subseteq M_h$, and therefore the velocity error has the upper bound [15]

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} \le C\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{H^1(\Omega)} \le C\inf_{\boldsymbol{v}\in\mathring{\boldsymbol{\mathcal{P}}}_1^c(\mathcal{T}_h)}\|\boldsymbol{u}-\boldsymbol{v}\|_{H^1(\Omega)} \le Ch|\boldsymbol{u}|_{H^2(\Omega)}. \tag{7.3}
$$

We emphasize that the velocity error of the divergence-free pair is decoupled from the pressure and is robust respect to the viscosity. In both cases, the pressure error satisfies

$$
\|p-p_h\|_{L^2(\Omega)} \le C\big(h|p|_{H^1(\Omega)} + \nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)}\big).
$$

In the numerical experiments, the domain is taken to be the unit cube $\Omega = (0,1)^3$, the viscosity is $\nu = 10^{-5}$, and the source function $\boldsymbol{f}$ is taken such that the exact solution is given by

$$\boldsymbol{u} = \mathbf{curl}\,(\psi,\psi,\psi), \qquad p = x - y, \qquad \psi = x_1^2(1-x_1)^2 x_2^2(1-x_2)^2 x_3(1-x_3)^2.$$

We plot the resulting errors against the number of degrees of freedom (DOFs) of the two sets of experiments in Figures 7.2 and 7.3 (note $h \approx DOFs^{-1/3}$). The plots show the expected rates of convergence $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} = \mathcal{O}(DOFs^{-1/3})$ and $\|p - p_h\|_{L^2(\Omega)} = \mathcal{O}(DOFs^{-1/3})$ in both cases. However, the velocity error of the divergence-free pair is roughly 10^5 times smaller than the Bernardi-Raugel pair; the error estimates (7.2)–(7.3) predict this behavior. In addition, the plots also show that the L^2 velocity error converges with rate $\mathcal{O}(DOFs^{-2/3})$. Finally, the divergence-free pair yields divergence errors comparable to machine epsilon, while the Bernardi-Raugel method satisfies $\|\nabla \cdot \boldsymbol{u}_h\|_{L^2(\Omega)} = \mathcal{O}(DOFs^{-1/3})$.

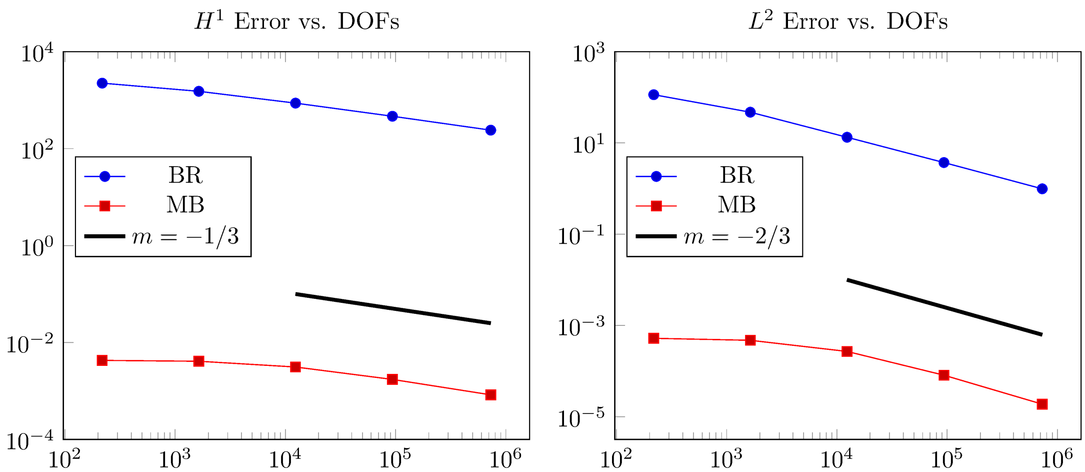

FIG. 7.2. *The H^1 (left) and L^2 (right) velocity errors for the Bernardi-Raugel method (blue) and the modified Bernardi-Raugel divergence-free method (red). The H^1 error converges with rate $\mathcal{O}(DOFs^{-1/3})$, and the L^2 error converges with rate $\mathcal{O}(DOFs^{-2/3})$.*

8. Concluding Remarks. In this paper we have constructed several divergence-free and inf-sup stable Stokes elements in arbitrary dimension and for any polynomial degree. This is achieved by first establishing the result locally on each macro element and then constructing modified Bernardi-Raugel bubble functions with constant divergence. With these two ingredients we have constructed finite element pairs on macro and refined meshes, and pairs with continuous pressure spaces.

As is the case with other divergence-free Stokes pairs, it is expected that the finite element pairs developed here fit within a smooth, discrete de Rham complex (Stokes complex). For example, 2D complexes based on a barycenter refinements have been constructed in [7]. We are currently developing discrete complexes in higher dimensions that incorporate the Stokes pairs proposed in this paper.

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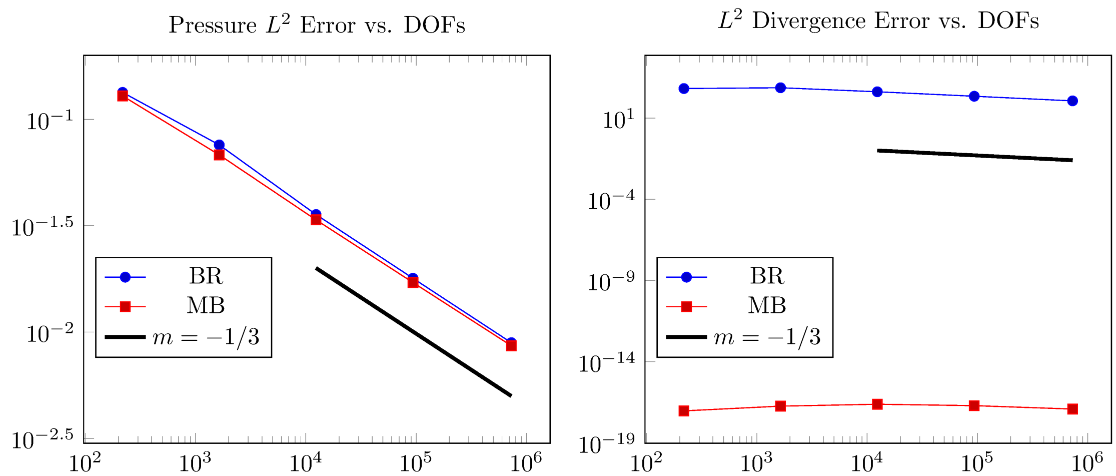

FIG. 7.3. *The L^2 pressure error (left) and the L^2 divergence errors (right) for the Bernardi-Raugel method (blue) and the modified Bernardi-Raugel divergence-free method (red). The pressure errors converge with rates $\mathcal{O}(DOFs^{-1/3})$ for both methods. The divergence error of the Bernardi-Raugel method converges with rate $\mathcal{O}(DOFs^{-1/3})$. The divergence error of the modified Bernardi-Raugel is comparable to machine epsilon.*

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