CHARACTERS AND SYLOW 2-SUBGROUPS OF MAXIMAL CLASS REVISITED

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ABSTRACT. We give two ways to distinguish from the character table of a finite group G if a Sylow 2-subgroup of G has maximal class. We also characterize finite groups with Sylow 3-subgroups of order 3 in terms of their principal 3-block.

1. INTRODUCTION

Dihedral, semi-dihedral and generalized quaternion 2-groups play an essential role in finite group theory. They have been characterized in many ways:

- as the 2-groups of maximal class,
- as the non-abelian 2-groups whose commutator subgroup has index 4 (O. Taussky-Todd),
- as the non-cyclic 2-groups whose number of involutions is 1 modulo 4 (Alperin-Feit-Thompson),
- as the 2-groups with five rational-valued irreducible characters ([INS]),
- as the non-abelian 2-groups whose group algebra over an infinite field of characteristic 2 has tame representation type.

The representation theory of the groups with a Sylow 2-subgroup of maximal class is a classical theme that has been extensively studied by R. Brauer, G. Glauberman, K. Erdmann, among many others. Here, we wish to go into the other direction: from the character table of a finite group G, how do we distinguish if G possesses a Sylow 2-subgroup of maximal class? We provide with two ways to show that.

Theorem A. *Let G be a finite group, and let $P \in \mathrm{Syl}_2(G)$. Then the following conditions are equivalent.*

(a) $|P/P'| = 4$.

(b) *Either $|P| = 4$, or there exists $g \in P$ such that $|G : \mathbf{C}_G(g)|$ is even and*
$$\mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G)) = \mathbb{Q}(\zeta \pm \zeta^{-1})$$
where $\zeta \in \mathbb{C}$ is a primitive $|P|/2$-th root of unity.

(c) *The principal 2-block B has exactly four irreducible complex characters of odd degree.*

The equivalence of (a) and (b) in Theorem A is quite elementary and will be shown in Section 2. On the other hand, we make use of the classification of finite simple groups to show that (c) implies (a). This is a particular case of the Alperin-McKay conjecture. Unfortunately, despite the recent success [MS] in proving the McKay conjecture for $p = 2$, it seems that there is still a long way to reach the Alperin-McKay conjecture for $p = 2$. For instance, we are not able to prove that if B is an arbitrary block of a finite group with exactly four height zero characters then the defect group of B is of maximal class.

Of course, we are not only interested in $p = 2$. Brauer asked in [Br] what properties of the Sylow p-subgroup of a finite group G can be detected in its character table $X(G)$. After giving a method in [NT2, NST] to check from $X(G)$ if a Sylow p-subgroup P of G is abelian, it is natural to concentrate on the group P/P' now. For instance, does $X(G)$ know if $|P/P'| = p^2$? Again, for $p = 3$, the following is a not so well-known consequence of the Alperin-McKay conjecture.

Conjecture B. *Let G be a finite group, let B be the principal 3-block of G, and let $P \in \mathrm{Syl}_3(G)$. Then $|P/P'| = 9$ if and only if B has either 6 or 9 irreducible complex characters of degree not divisible by 3.*

Before even attempting to prove Conjecture B, one needs first to establish the apparently innocent Theorem C below, which has remained unproven until now. Indeed, the proof of Theorem C is highly non-trivial.

Theorem C. *Let G be a finite group and let B be the principal 3-block of G. Then B has exactly three irreducible characters of degree not divisible by 3 if and only if $|G|$ is divisible by 3 but not by 9.*

Even with Theorem C at our disposal, it remains a challenge for us to prove Conjecture B, and new ideas are welcome. Despite the fact that the Alperin-McKay conjecture was reduced to a question on finite simple groups by B. Späth in [Spa2], we notice that the hypotheses of Conjecture B (or of Theorem A) are not inherited by normal subgroups. Our hope is that, as Brauer did in the case $|P/P'| = 4$, the *small cases* can be handled by different techniques which might have interest on their own.

For primes $p \geq 5$, we have not yet found a way to characterize when $|P/P'| = p^2$ from the character table of G.

2. PROOFS

We denote the n-th cyclotomic field by $\mathbb{Q}_n$. For a finite group G and $g \in G$ let $\mathbb{Q}(g) := \mathbb{Q}\big(\chi(g) : \chi \in \mathrm{Irr}(G)\big) \subseteq \mathbb{Q}_{|\langle g \rangle|}$. The following proves the equivalence of (a) and (b) in Theorem A:

Proposition 2.1. *Let G be a finite group with Sylow 2-subgroup P of order $2^n \geq 8$. Then $|P/P'| = 4$ if and only if there exists a $g \in P$ such that $|G : \mathbf{C}_G(g)|$ is even and $\mathbb{Q}(g) = \mathbb{Q}(\zeta \pm \zeta^{-1})$ where $\zeta \in \mathbb{C}$ is a primitive 2^{n-1}-th root of unity.*

Proof. Assume first that $n = 3$. Then $|P/P'| = 4$ if and only if P is non-abelian. This happens if and only if there exists a $g \in P$ such that $|G : \mathbf{C}_G(g)|$ is even (this is an elementary case of a well-known general result by Camina and Herzog). In this case, the condition $\mathbb{Q}(g) = \mathbb{Q}(\zeta \pm \zeta^{-1})$ is always fulfilled, since $\mathbb{Q}(\zeta \pm \zeta^{-1})$ is either $\mathbb{Q}$ or $\mathbb{Q}(i)$. Therefore, we may assume in the following that $n \geq 4$.

Recall that $|P/P'| = 4$ if and only if there exists some $g \in P$ of order 2^{n-1} such that g is conjugate to g^{-1} or to $g^{-1+2^{n-2}}$ in P. Assume first that such an element exists. Then $|G : \mathbf{C}_G(g)|$ is even and $\mathbb{Q}(g)$ has index 2 in $\mathbb{Q}_{2^{n-1}}$ by [N2, Theorem 4]. Moreover, $\mathbb{Q}(g)$ is contained in the fixed field of the Galois automorphism sending ζ to $\pm\zeta^{-1}$. This shows $\mathbb{Q}(g) = \mathbb{Q}(\zeta \pm \zeta^{-1})$.

Assume conversely that $\mathbb{Q}(g) = \mathbb{Q}(\zeta \pm \zeta^{-1})$ is fulfilled for some $g \in P$. Then the Galois group of $\mathbb{Q}(g)$ over $\mathbb{Q}$ is cyclic of order 2^{n-3}. Since $n \geq 4$, it is easy to see that $\mathbb{Q}(g)$ is not a cyclotomic field. Hence, g has order at least 2^{n-1}. Suppose that P does not have maximal class. Then P is isomorphic either to C_{2^n}, to $C_{2^{n-1}} \times C_2$, or to the modular group M_{2^n} (which is the group $\langle r, s \mid r^{2^{n-1}} = s^2 = 1, r^s = r^{2^{n-2}+1} \rangle$). It follows from Wong [W] that G has a normal 2-complement. This yields the contradiction $\mathbb{Q}_{2^{n-2}} \subseteq \mathbb{Q}(g)$. $\qquad\square$

If B is a p-block, then $\mathrm{Irr}_{p'}(B)$ denotes the set of irreducible characters in B of degree not divisible by p. In general, our notation for blocks follows [N1]. If G is a finite group, $B_0(G)$ denotes the principal block of G.

In order to prove the equivalence of (a) and (c) in Theorem A, we start with the following elementary result which even holds for non-principal blocks (see [L]).

Lemma 2.2. *Let G be a finite group, let p be a prime, and let $B = B_0(G)$.*

  (i) *Assume that $p = 2$. If 4 divides $|G|$, then 4 divides $|\mathrm{Irr}_{2'}(B)|$.*
  (ii) *Assume that $p = 3$. If 3 divides $|G|$, then 3 divides $|\mathrm{Irr}_{3'}(B)|$.*

Proof. For the convenience of the reader we provide a proof which is slightly easier than the one given in [L]. Let $P \in \mathrm{Syl}_p(G)$. By the weak block orthogonality Corollary 3.7 of [N1], we have that $\sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi(x) = 0$ for every p-singular $x \in G$.

In particular,

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi_P = f\rho_P,$$

where $f \in \mathbb{N}$ and ρ_P is the regular character of P. Suppose that $\mathrm{Irr}_{p'}(B) = \{\chi_1, \ldots, \chi_k\}$ have degrees $\chi_i(1) = b_i$ and that p divides $|G|$. If $p = 2$ and 4 divides $|G|$, we obtain

$$k \equiv b_1^2 + \cdots + b_k^2 \equiv \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)^2 = f\rho_P(1) \equiv 0 \pmod 4.$$

Similarly, if $p = 3$, we have

$$k \equiv b_1^2 + \cdots + b_k^2 \equiv 0 \pmod 3. \quad \square$$

Let us point out that Lemma 2.2(b) is not longer true for $p = 5$. In fact, the dihedral group $G = D_{10}$ is already a counterexample.

We will need the following theorem on simple groups whose proof is deferred until the next section.

Theorem 2.3. *Suppose that S is a finite non-abelian simple group. Then the following statements hold.*

  (i) *Suppose $S \lhd G \leq \mathrm{Aut}(S)$ and $|G/S| \leq 2$. If $|\mathrm{Irr}_{2'}(B_0(G))| = 4$, then $|P/P'| = 4$ for $P \in \mathrm{Syl}_2(G)$.*
 (ii) *At least one of the following two conditions holds for S.*
    (a) *Each $\theta \in \mathrm{Irr}_{2'}(S)$ belongs to the principal 2-block $B_0(S)$ of S.*
    (b) *$\mathrm{Irr}_{2'}(B_0(S))$ contains at least three characters that are each $\mathrm{Aut}(S)$-invariant.*

We will take advantage of [MS] to take care of many almost simple groups, particularly the ones with self-normalizing Sylow 2-subgroups, even though our techniques can also handle them directly:

Proposition 2.4. *Let G be a finite group such that $|\mathrm{Irr}_{2'}(G)| = 4$. Then $|P/P'| = 4$.*

Proof. By the main result of [MS] we have

$$|\mathrm{Irr}_{2'}(\mathbf{N}_G(P))| = |\mathrm{Irr}_{2'}(G)| = 4.$$

It is well-known that the finite groups with 4 conjugacy classes are C_4, $C_2 \times C_2$, A_4 and D_{10}. (This was already known to Burnside, see Note A in [B].) Hence $\mathbf{N}_G(P)/P'$ is one of these four groups, and has a normal Sylow 2-subgroup. Thus we discard D_{10}, and P/P' has order 4. $\quad \square$

Now we are ready to prove Theorem A which we restate for the reader's convenience.

Theorem 2.5. *Let G be a finite group, let B be the principal 2-block of G, and let $P \in \mathrm{Syl}_2(G)$. Then $|P/P'| = 4$ if and only if B has exactly four irreducible characters of odd degree.*

Proof. If $|P/P'| = 4$, then it is well known that B has exactly four irreducible characters of odd degree, by celebrated work of R. Brauer and G. Glauberman.

For the converse, we argue by induction on $|G|$. We may assume that G is not a 2-group, and by Theorem 2.3 that G is not simple. Since $\mathbf{O}_{2'}(G) \subseteq \ker(B)$ by Theorem 6.10 of [N1], we may assume that $\mathbf{O}_{2'}(G) = 1$. In particular, we may assume that 4 divides $|G|$ (for otherwise, $G = P$).

Let $1 < N$ be a normal subgroup of G and recall that $B_0(G/N) \subseteq B_0(G) = B$. Assume that 4 divides $|G/N|$. Therefore, using Lemma 2.2, we have that $\mathrm{Irr}_{2'}(B_0(G/N)) = \mathrm{Irr}_{2'}(B)$. Now, let $\gamma \in \mathrm{Irr}(B_0(PN))$ of odd degree. Now, $\gamma_N = \theta \in \mathrm{Irr}(N)$ is in the principal block of N (using Corollary 11.29 of [I], and Theorem 9.2 of [N1]). By Lemma 4.3 of [Mu], there exists χ in the principal block of G of odd-degree over θ. Thus $\theta = 1$ and γ is linear. By Corollary 3 of [IS], we have that NP has a normal 2-complement, and therefore so does N. Thus N is a 2-group (since $\mathbf{O}_{2'}(G) = 1$) and $N \leq P'$ (since N is in the kernel of every linear character of P). In this case, we are done by induction.

Hence, we may assume that if $1 < N$ is a normal subgroup of G, then G/N has odd order or $|G/N|_2 = 2$. In particular, we have that G/N has a normal 2-complement. Since G does not have a normal 2-complement, we deduce that G has a unique minimal normal subgroup N, which is proper in G (since we are assuming that G is not simple) and that G/N has a Sylow 2-subgroup PN/N of order at most 2.

Again by Corollary 3 of [IS], let $\theta \in \mathrm{Irr}(B)$ be non-linear of odd degree.

Suppose that G has a normal subgroup L such that G/L has prime odd order. If $G = L\mathbf{C}_G(P)$, then by Lemma 1 of [A], we have that restriction defines a bijection $\mathrm{Irr}(B) \to \mathrm{Irr}(B_0(L))$, and then we are done by hypothesis. Hence, we may assume that $\mathbf{C}_G(P) \leq L$. Then, by Lemma 3.1 of [NT1], it follows that $\mathrm{Irr}(G/L) \subseteq \mathrm{Irr}(B)$. Hence, we have that G/L has order 3, and thus $\mathrm{Irr}_{2'}(B) = \{1, \lambda, \bar{\lambda}, \theta\}$, where $\lambda \in \mathrm{Irr}(G/L)$ has order 3. Now, let $\tau \in \mathrm{Irr}(B_0(L))$ be non-trivial of odd-degree (by Lemma 2.2), and let $\psi \in \mathrm{Irr}(B)$ be over τ (by Theorem 9.4 of [N1]). Since G/L has odd order, $\psi(1)$ is odd, and therefore $\psi = \theta$. Since θ_L has at most three irreducible constituents, using Lemma 2.2 we see that L satisfies the hypothesis. Hence, we may assume that $\mathbf{O}^{2'}(G) = G$.

Now we have that PN/N has order 2, and let K/N be a normal 2-complement of G/N. Let λ be the non-trivial character of G/K. Then $\mathrm{Irr}_{2'}(B_0(G)) = \{1, \lambda, \theta, \lambda\theta\}$, by using Gallagher's Corollary 6.17 of [I]. Let $C/N = \mathbf{N}_{K/N}(PN/N)$. Now, by the relative Glauberman correspondence (Theorem E of [NTV]), there is a natural bijection $^* : \mathrm{Irr}_P(K) \to \mathrm{Irr}_P(C)$, where $\mathrm{Irr}_P(K)$ denotes the P-invariant irreducible characters in K. Also, χ^* is an irreducible constituent of χ_C, and if $\chi \in \mathrm{Irr}_P(K)$ has

odd-degree, then χ is in $B_0(K)$ if and only if χ^* is in $B_0(C)$. Since χ^* is an irreducible constituent of χ_C and K/N has odd order, it follows that χ has odd degree if and only if χ^* has odd degree. Since G/K has order 2, notice that every P-invariant $\tau \in \mathrm{Irr}(B_0(K))$ extends to some character of G which necessarily lies in the principal 2-block of G (because G/K is a 2-group, using Corollary 9.6 of [N1]). It follows that K has exactly two irreducible P-invariant characters of odd degree in $B_0(K)$ which are 1 and θ_K. Hence C has exactly two irreducible P-invariant characters of odd degree in $B_0(C)$ which are 1 and $\eta = (\theta_K)^*$. Since C has index 2 in $\mathbf{N}_G(P)N$, we conclude that $\mathbf{N}_G(P)N$ satisfies the hypothesis of the theorem. If $\mathbf{N}_G(P)N < G$, then the theorem follows by induction. So we may assume that $PN \lhd G$. Since G/PN has odd order, we conclude that G/N has order 2; in particular, B is the only block that covers the principal 2-block $B_0(N)$ of N. Also, we have that $\mathbf{C}_G(N) = 1$. (Notice that N is non-abelian, since otherwise G would be a 2-group, using that $\mathbf{O}_{2'}(G) = 1$ and that G/N has order 2.) Assume that N is not simple. Then $N = S^n$, where S is a non-abelian simple group. Now we apply Theorem 2.3(ii) to S. In the case of (a), every $\chi \in \mathrm{Irr}_{2'}(G)$ lies above some $\rho \in B_0(N)$ and so belongs to B. Hence we are done by Proposition 2.4. Next we consider the case of (b) and let $\theta_{1,2,3} \in \mathrm{Irr}_{2'}(B_0(S))$ be three distinct characters that are $\mathrm{Aut}(S)$-extendible. It is then easy to see

$$\gamma_i = \theta_i \otimes \theta_i \otimes \ldots \otimes \theta_i \in \mathrm{Irr}_{2'}(B_0(N))$$

is G-invariant, and so extends to G (in two different ways) for each $i = 1, 2, 3$. As B is the only block that covers $B_0(N)$, we see that $B_0(G)$ contains at least six odd-degree characters, a contradiction.

Now we may assume that N is simple, and apply Theorem 2.3(i) to G to complete the proof. $\qquad\square$

3. Odd-degree characters of almost simple groups

In this section we will prove Theorem 2.3. We begin with the following simple observation:

Lemma 3.1. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be a real-valued character of odd degree. Then χ belongs to $B_0(G)$.*

Proof. Consider any irreducible constituent φ of the restriction χ° of χ to $2'$-elements of G. If $\varphi \neq \bar\varphi$, then $\bar\varphi$ is also a constituent of χ°. On the other hand, if $\varphi = \bar\varphi \neq 1_G$, then $2|\varphi(1)$ by Fong's theorem. Since $2 \nmid \chi(1)$, 1_G must be a constituent of χ°. In particular, $\chi \in B_0(G)$. $\qquad\square$

Proposition 3.2. *Theorem 2.3 holds true if at least one of the following conditions is satisfied:*
  (i) *S has self-normalizing Sylow 2-subgroups.*
  (ii) *S is a simple group of Lie type in characteristic 2.*

Proof. First we consider the case of (i): $\mathbf{N}_S(Q) = Q$ for $Q \in \mathrm{Syl}_2(S)$. In this case, $B_0(S)$ is the only 2-block of maximal defect of G, see e.g. [NT1, Lemma 3.1], whence the conclusion (a) of Theorem 2.3(ii) holds for S. Since $|G/S| \leq 2$, we see that $P \in \mathrm{Syl}_2(G)$ is also self-normalizing, and again $B_0(G)$ is the only 2-block of maximal defect of G. Now we can apply Proposition 2.4 to G.

Assume now that (ii) holds. By the main result of [Hu], all characters of positive defect of S lie in $B_0(S)$, including all of odd degree. Since $|G/S| \leq 2$, $B_0(G)$ is the only 2-block of G that covers $B_0(S)$, see e.g. [NTV, Lemma 5.1]. It follows that $\mathrm{Irr}_{2'}(G) = \mathrm{Irr}_{2'}(B_0(G))$. Now we can again apply Proposition 2.4 to G. $\qquad\square$

In what follows, we use the notation $\mathrm{PSL}_n^\epsilon(q)$ to denote $\mathrm{PSL}_n(q)$ if $\epsilon = +$ and $\mathrm{PSU}_n(q)$ if $\epsilon = -$. Similarly, $E_6^\epsilon(q)$ denotes the simple group $E_6(q)$ if $\epsilon = +$ and the simple group ${}^2E_6(q)$ if $\epsilon = -$.

Proposition 3.3. *Theorem 2.3 holds true if $S = \mathrm{PSL}_n^\epsilon(q)$ with $\epsilon = \pm$, $n \geq 3$, and $2 \nmid q$.*

Proof. (i) We aim to show that in the cases under consideration $|\mathrm{Irr}_{2'}(B_0(G))| > 4$, unless $G = S = \mathrm{PSL}_3^\epsilon(q)$ with $4|(q + \epsilon)$. Note that unipotent characters of simple classical groups are uniquely determined by their multiplicities in the Deligne-Lusztig characters and hence in particular they are rational-valued. Furthermore, unipotent characters of S extend to $\mathrm{Aut}(S) \geq G$ by [M, Theorems 2.4, 2.5]. Hence, by Lemma 3.1, it suffices to find 3 odd-degree unipotent characters of S in the case $|G/S| = 2$, and 5 odd-degree unipotent characters of S in the case $G = S$. We can view S as $L/\mathbf{Z}(L)$ for $L := \mathrm{SL}_n^\epsilon(q)$.

Consider the 2-adic decomposition of n:

$$n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_r}, \ n_1 > n_2 > \ldots > n_r \geq 0.$$

By [GKNT, Theorem 4.3], there are exactly $N(n) := 2^{n_1+n_2+\ldots+n_r}$ partitions $\lambda \vdash n$ such that the corresponding irreducible character S^λ of S_n has odd degree. Arguing as in the proof of [NT3, Lemma 4.4] and using the hook formula [FS, (1.15)], one can show that $\chi^\lambda(1) \equiv S^\lambda(1) (\mathrm{mod}\ 2)$, if χ^λ is the unipotent character of S labeled by λ. In particular, if $n \geq 6$, then $N(n) \geq 8$ and S has at least 8 odd-degree unipotent characters. On the other hand, if $r = 1$, i.e. n is a 2-power, then $Q \in \mathrm{Syl}_2(S)$ is self-normalizing by [Ko, Corollary] and so we are done by Proposition 3.2(i).

(ii) It remains to consider the case $n = 3, 5$. Fix $\xi \in \overline{\mathbb{F}}_q^\times$ of order $q - \epsilon$ and $\tilde{\xi} \in \mathbb{C}^\times$ also of order $q - \epsilon$. Also, let $V = \overline{\mathbb{F}}_q^n$ denote natural module for L. If $\epsilon = +$, then L has $q - 1$ irreducible *Weil* characters

$$(3.1) \qquad \zeta_{n,q}^i(g) = \frac{1}{q-1} \sum_{k=0}^{q-2} \tilde{\xi}^{ik} q^{\dim \mathrm{Ker}(g - \xi^k \cdot 1_V)} - 2\delta_{0,i}, \ 0 \leq i \leq q - 2$$

for $g \in L$, see [TZ, T]. Likewise, if $\epsilon = -$, then L has $q+1$ irreducible *Weil* characters

$$(3.2) \qquad \zeta_{n,q}^i(g) = \frac{(-1)^n}{q+1} \sum_{k=0}^q \tilde{\xi}^{ik}(-q)^{\dim \operatorname{Ker}(g - \xi^k \cdot 1_V)}, \ 0 \le i \le q.$$

In particular, if we let $(q-\epsilon)_{2'}$ denote the $2'$-part of $q-\epsilon$ and choose i to be divisible by $(q - \epsilon)_{2'}$, then $\zeta_{n,q}^i$ is trivial at $\mathbf{Z}(L)$ and so can be viewed as an irreducible character of S.

Assume now that $n = 5$. As $N(5) = 4$, S has 4 odd-degree unipotent characters. In particular, we are done if $G > S$. Next, the non-unipotent Weil character $\zeta_{5,q}^{(q-\epsilon)/2}$ of S has odd-degree $(q^5 - \epsilon)/(q - \epsilon)$ and it is real-valued (which can be seen using (3.1)–(3.2)), so we are also done in the case $G = S$ by Lemma 3.1.

(iii) Now we consider the case $n = 3$. It is shown in the proof of [DT, Theorem 7.2] that

$$\zeta_{n,q}^i(g) = \zeta_{n,q}^0(g) + 1$$

for all $2'$-elements $g \in L$, provided that $0 < i = j(q - \epsilon)_{2'} < q - \epsilon$; in particular, $\zeta_{n,q}^i \in B_0(S)$ for such i. It is easy to check that $\zeta_{3,q}^{(q-\epsilon)/2}$ (of degree $(q^3 - \epsilon)/(q - \epsilon)$) is $\operatorname{Aut}(S)$-invariant, as well as the principal character 1_S and the Steinberg character St of S. Thus if $G > S$, then G has at least 6 odd-degree characters in its principal 2-block.

We may now assume that $G = S$. If $4|(q - \epsilon)$, then $B_0(S)$ contains at least 5 odd-degree characters: 1_S, St, and $\zeta_{3,q}^{j(q-\epsilon)/4}$, $j = 1, 2, 3$. Finally, assume that $4|(q+\epsilon)$. Then $P \in \operatorname{Syl}_2(S)$ can be viewed as a Sylow 2-subgroup of $\operatorname{GL}_2^\epsilon(q)$, and it was checked in part (a) of the proof of [GKNT, Lemma 5.1] that $|P/P'| = 4$. $\qquad \square$

Proposition 3.4. *Theorem 2.3 holds true if $S = \operatorname{PSp}_{2n}(q)$ with $n \ge 2$ and $2 \nmid q$.*

Proof. We view $S = L/\mathbf{Z}(L)$, with $L = \operatorname{Sp}_{2n}(q)$. As mentioned in the proof of Proposition 3.3, unipotent characters of S and L are rational-valued; furthermore, as $2 \nmid q$, they are $\operatorname{Aut}(S)$-invariant by [M, Theorem 2.5]. First we exhibit 4 odd-degree unipotent characters of S. Two of them are the principal character and the Steinberg character. Next, consider 4 unipotent characters: $\alpha_{1,2}$ and $\beta_{1,2}$, where

$$\alpha_1(1) = \frac{(q^n + 1)(q^n + q)}{2(q + 1)}, \ \alpha_2(1) = \frac{(q^n - 1)(q^n - q)}{2(q + 1)},$$

$$\beta_1(1) = \frac{(q^n - 1)(q^n + q)}{2(q - 1)}, \ \beta_2(1) = \frac{(q^n + 1)(q^n - q)}{2(q - 1)}.$$

As $\alpha_1(1) - \alpha_2(1) = q^n$, exactly one of $\alpha_{1,2}$ has odd degree. Likewise, exactly one of $\beta_{1,2}$ has odd degree. So we are done if $G > S$. In the case $G = S$, it is well known, see eg. [TZ], that L has two pairs of irreducible *Weil* characters: $\xi_{1,2}$ of degree $(q^n+1)/2$, and $\eta_{1,2}$ of degree $(q^n - 1)/2$, and

$$\xi_1(g) = \eta_1(g) + 1, \ \xi_2(g) = \eta_2(g) + 1$$

for all $2'$-elements $g \in L$. It follows that the two odd-degree characters among these four characters can be viewed as irreducible characters of S and then belong to $B_0(S)$. $\qquad\square$

Proof of Theorem 2.3. By Proposition 3.2 we may assume that $Q \in \mathrm{Syl}_2(S)$ is *not* self-normalizing and furthermore S is not a group of Lie type in characteristic 2. Moreover, if $S = \mathrm{PSL}_n^\epsilon(q)$ with $n \geq 3$ and $2 \nmid q$, or if $S = \mathrm{PSp}_{2n}(q)$ with $n \geq 2$ and $2 \nmid q$, then we are done by Propositions 3.3 and 3.4. Hence, by [Ko, Corollary], it remains to consider the following cases.

(a) $S = E_6^\epsilon(q)$ with $\epsilon = \pm$ and $2 \nmid q$.

Let ϕ_m denote the mth cyclotomic polynomial in q. Using [C, §13.9] we will exhibit 6 odd degrees each of which is the degree of exactly one unipotent character (and so it is $\mathrm{Aut}(S)$-invariant and rational-valued, since $\mathrm{Aut}(S)$ and $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ preserve the set of unipotent characters):

- $\epsilon = +$: 1, q^{36}, $q^6\phi_3^3\phi_6^2\phi_9\phi_{12}$, $q^{10}\phi_3^3\phi_6^2\phi_9\phi_{12}$, $\frac{1}{2}q^3\phi_5\phi_6^2\phi_8\phi_9$, $\frac{1}{2}q^3\phi_5\phi_8\phi_9\phi_{12}$
- $\epsilon = -$: 1, q^{36}, $q^6\phi_3^2\phi_6^3\phi_{12}\phi_{18}$, $q^{10}\phi_3^2\phi_6^3\phi_{12}\phi_{18}$, $\frac{1}{2}q^3\phi_3^2\phi_8\phi_{10}\phi_{18}$, $\frac{1}{2}q^3\phi_8\phi_{10}\phi_{12}\phi_{18}$

(b) $S = {}^2G_2(q)$ with $q = 3^{2a+1} \geq 27$.

Here, since $|\mathrm{Out}(S)| = 2a + 1$, we have $G = S$. As shown in [F], $B_0(S)$ contains 8 odd-degree characters, all being $\mathrm{Aut}(S)$-invariant.

(c) $S = \mathrm{PSL}_2(q)$ with $5 \leq q \equiv \pm 3 \,(\mathrm{mod}\, 8)$.

Here, $Q \cong C_2^2$ and all (four) odd-degree irreducible characters of S belong to $B_0(S)$. Finally, if $G > S$, then $G \cong \mathrm{PGL}_2(q)$ and $P \cong D_8$.

(d) $S = J_1, J_2, J_3, Suz$, or HN.

These five cases can be checked directly using the available decomposition matrices [ModAt]. In all cases, $|\mathrm{Irr}_{2'}(B_0(G))| > 4$ and moreover $\mathrm{Irr}_{2'}(B_0(S))$ contains at least three $\mathrm{Aut}(S)$-invariant characters. $\qquad\square$

4. Proof of Theorem C

Theorem 4.1. *It suffices to prove Theorem C for non-abelian simple groups G.*

Proof. Since the proof is quite similar to Theorem 2.5, we only sketch the argument. Let $P \in \mathrm{Syl}_3(G)$. We argue by induction on $|G|$. Arguing as in Theorem 2.5, we may assume that $\mathbf{O}_{3'}(G) = 1$, and that if $1 < N$ is a normal subgroup of G, then G/N has $3'$-order. In particular, G has a unique minimal normal subgroup L of order divisible by 3, and G/L is a $3'$-group. Suppose that $N \triangleleft G$ has $3'$-index with G/N simple. Notice that $N\mathbf{C}_G(P) \triangleleft G$ by the Frattini argument. If $G = N\mathbf{C}_G(P)$, then we apply [A] to conclude that N satisfies the hypothesis. If $\mathbf{C}_G(P) \leq N$, then all the irreducible characters of G/N lie in the principal block of G (and all of them have $3'$-degree). Hence G/N has at most three conjugacy classes. Therefore $G/N = C_2$, since G/N is a $3'$-group. Thus $\mathrm{Irr}_{3'}(B) = \{1, \lambda, \theta\}$, where $N \leq \ker(\lambda)$ and θ is non-linear. Now, if $1 \neq \gamma \in \mathrm{Irr}_{3'}(B_0(N))$ then γ should lie below θ, and therefore

$|\operatorname{Irr}_{3'}(B_0(N))| \leq 3$, and we are done by induction. We conclude that $\mathbf{O}^{3'}(G) = G$. Thus we have that $G = L$ is a direct product of non-abelian simple groups. Since $|\operatorname{Irr}_{p'}(B)| = 3$, necessarily G is simple. $\square$

We will spend the rest of this section to prove Theorem C for simple groups G. In what follows, we will omit the trivial check whether $|G|_3 = 3$.

Lemma 4.2. *Theorem C holds if G is a sporadic simple group or if $G \cong {}^2F_4(2)'$.*

Proof. This is easily checked with [GAP]. $\square$

Lemma 4.3. *Theorem C holds in the case $G = \mathsf{A}_n$ with $n \geq 5$. Furthermore, $|\operatorname{Irr}_{3'}(\mathsf{S}_n)| \geq 6$ if $n \geq 5$.*

Proof. The cases $5 \leq n \leq 8$ can be checked directly using [GAP], so we will assume $n \geq 9$. It now suffices to prove that $B_0(\mathsf{S}_n)$ contains at least 8 characters of $3'$-degree. Let $P_n \in \operatorname{Syl}_3(\mathsf{S}_n)$. Since the Alperin-McKay conjecture holds for S_n [O], it suffices to show that $\mathbf{N}_{\mathsf{S}_n}(P_n)$ contains at least 9 characters of $3'$-degree in its principal 3-block.

Write n 3-adically: $n = \sum_{i=0}^t a_i 3^i$ with $0 \leq a_i \leq 2$ and $a_t > 0$. Since $n \geq 9$, we have that $t \geq 2$. We can choose

$$P = P_{n_0} \times P_{n_1} \times \ldots \times P_{n_t},$$

with $P_{n_i} \in \operatorname{Syl}_3(\mathsf{S}_{n_i})$ and $n_i := a_i 3^i$. Then

$$\mathbf{N}_{\mathsf{S}_n}(P) = \mathbf{N}_{\mathsf{S}_{n_0}}(P_{n_0}) \times \mathbf{N}_{\mathsf{S}_{n_1}}(P_{n_1}) \times \ldots \times \mathbf{N}_{\mathsf{S}_{n_t}}(P_{n_t}),$$

and so it suffices to show that $B_0(\mathbf{N}_{\mathsf{S}_{n_t}}(P_{n_t}))$ contains at least 9 characters of $3'$-degree. Applying the Alperin-McKay conjecture to S_{n_t}, it suffices to show that $B_0(\mathsf{S}_{n_t})$ contains at least 9 characters of $3'$-degree. Now observe that any hook $\lambda \vdash n_t = a_t 3^t$ has empty 3-core and so the irreducible character χ^λ labeled by λ belongs to $B_0(\mathsf{S}_{n_t})$; also $3 \nmid \chi^\lambda(1)$. Thus $|\operatorname{Irr}_{3'}(B_0(\mathsf{S}_{n_t}))| \geq n_t \geq 9$, and we are done. $\square$

Henceforth we will assume that G is a non-abelian simple group, not isomorphic to any of the groups considered in Lemmas 4.2, 4.3. Thus we can find a simple algebraic group $\mathcal{H}$ of adjoint type, defined over a field of characteristic $r > 0$, and a Frobenius endomorphism $F : \mathcal{H} \to \mathcal{H}$, such that $G \cong [H, H]$, where $H := \mathcal{H}^F$. Let the pair $(\mathcal{H}^*, F^*)$ be dual to $(\mathcal{H}, F)$ and let $H^* := \mathcal{H}^{*F^*}$. Also let St_H denote the Steinberg character of H.

Lemma 4.4. *Theorem C holds in the case G is a simple group of Lie type in characteristic $r = 3$.*

Proof. According to [Hu], $B_0(G) = \operatorname{Irr}(G) \smallsetminus \{\mathsf{St}_G\}$, where St_G denotes the Steinberg character of G. Note that H/G is an abelian $3'$-group and St_G extends to St_H. It follows that every irreducible constituent of χ_G, where $\chi \in \operatorname{Irr}_{3'}(H)$, belongs to $\operatorname{Irr}_{3'}(B_0(G))$. Note that each $\alpha \in \operatorname{Irr}(G)$ can lie under at most $[H : G]$ characters $\chi \in \operatorname{Irr}(H)$. Hence, it suffices to show that H has more than $3[H : G]$ irreducible

characters of $3'$-degree. It is easy to check this assertion for $G = {}^2G_2(q^2)$ (with $q^2 \geq 27$), so we will assume $G \not\cong {}^2G_2(q^2)$.

Recall [C, §8.4] that every semisimple class s^{H^*} in H^* yields a character $\chi_s \in \mathrm{Irr}_{3'}(H)$, and morever $\chi_s = \chi_t$ implies that the two semisimple elements s and t are conjugate in H^*. As $\mathcal{H}^*$ is simply connected, the number of semisimple classes in H^* is q^l, where $l = \mathrm{rank}(\mathcal{H})$ and q is the common absolute value of eigenvalues of F acting on the character group $X(\mathcal{T})$ of an F-stable maximal torus $\mathcal{T}$ of $\mathcal{H}^*$, cf. [C, Theorem 3.7.6(ii)]. If $l \geq 3$ or if $l = 2$ but $q \geq 9$, then

$$q^l > 3(q+1) \geq 3[G : H],$$

and we are done. In the remaining cases, that is where $G = \mathrm{PSL}_2(q)$ with $q \geq 9$ or $G \in \{\mathrm{SL}_3(3), \mathrm{SU}_3(3), \mathrm{PSp}_4(3), G_2(3)\}$, it is easy to verify Theorem C directly. $\qquad\square$

From now on we may assume that $r \neq 3$. Recall an element $g \in G$ is *p-central* if $p \nmid |g^G|$. We will rely on the following construction:

Proposition 4.5. *Let $r \neq 3$ and suppose the following two conditions hold for H:*
(a) *The Steinberg character St_H of H belongs to $B_0(H)$;*
(b) *There exists a 3-central 3-element $t \in H^* \smallsetminus \mathbf{Z}(H^*)$ such that $\mathbf{C}_{\mathcal{H}^*}(t)$ is not a torus.*
Then $|\mathrm{Irr}_{3'}(B_0(G))| \geq 4$.

Proof. First we construct 4 characters in $\mathrm{Irr}_{3'}(B_0(H))$. Two of them are 1_H and St_H. Next, we consider the semisimple character χ_t and the regular character χ_t^* labeled by the conjugacy class of $t \in H^*$. Since t is 3-central, but not central, $\chi_t(1)$ and $\chi_t^*(1)$ are coprime to 3, and

(4.1) $$\chi_t(1) = \chi_t(1)_{r'} = \chi_t^*(1)_{r'} = [H^* : \mathbf{C}_{H^*}(t)]_{r'} > 1.$$

As $\mathcal{H}^*$ is simply connected, $\mathbf{C}_{\mathcal{H}^*}(t)$ is connected. Since $\mathbf{C}_{\mathcal{H}^*}(t)$ is not a torus, it follows that

(4.2) $$\chi_t^*(1)_r = |\mathbf{C}_{H^*}(t)|_r > 1.$$

Now, since $\mathbf{Z}(\mathcal{H}) = 1$, we can apply [H1, Corollary 3.4] to see that χ_t belongs to $B_0(H)$. Similarly, [H1, Corollary 3.3] implies that χ_t^* and St_H belong to the same block, and so $\chi_t^* \in B_0(H)$.

Recall that $B_0(H)$ covers only $B_0(G)$, and that St_H restricts to St_G. So we get at least four characters in $\mathrm{Irr}_{3'}(B_0(G))$: 1_G, St_G, and α lying below χ_t and β lying below χ_t^*. It remains to show that these four characters are pairwise distinct. Recall that H/G is an abelian r'-group and St_G extends to H. Hence any irreducible character of H lying above 1_G or St_G has degree being an r-power. It follows by (4.1) that $\alpha, \beta \notin \{1_G, \mathsf{St}_G\}$. Next, $\chi_t^*(1)/\beta(1)$ divides $|H/G|$ and so $r|\beta(1)$ by (4.2), whereas $\alpha(1)$ divides $\chi_t(1)$ and so it is coprime to r by (4.1). Hence $\beta \neq \alpha$. $\qquad\square$

Next we show

Proposition 4.6. *Assume $r \neq 3$. Then $\mathsf{St}_G \in B_0(G)$ and $\mathsf{St}_H \in B_0(H)$.*

Proof. (i) Note that the statements follow from the main result of [H2] if G is of type $^2B_2$, G_2, $^3D_4$, $^2F_4$, or F_4, in which cases we also have $G = H$.

Next we note that the first statement implies the second. Indeed, it is well known that St_G extends to St_H, and so

$$\mathrm{Irr}(H|\mathsf{St}_G) = \{\mathsf{St}_H \cdot \lambda \mid \lambda \in \mathrm{Irr}(H/G)\}.$$

As $B_0(H)$ covers $B_0(G) \ni \mathsf{St}_G$, by [N1, Theorem 9.4] at least one of these extensions $\mathsf{St}_H \cdot \lambda$ belongs to $B_0(H)$. Let χ_s denote the semisimple character labeled by a semisimple element $s \in H^*$ and $\mathcal{E}(H, (s))$ denote the Lusztig series labeled by s. Since $\mathsf{St}_H \in \mathcal{E}(H, (1))$, we have that $\mathrm{Irr}(H/G) = \{\chi_z \mid z \in \mathbf{Z}(H^*)\}$ and $\mathsf{St}_H \cdot \chi_z \in \mathcal{E}(H, (z))$ by [DM, Proposition 13.30]. Moreover, if z is not a 3-element, then $\mathcal{E}(H, (z))$ cannot contain any character from $B_0(H)$ by the main result of [BM]. On the other hand, if z is a 3-element, then St_H and $\mathsf{St}_H \cdot \chi_z$ have the same restriction to $3'$-elements of H. It follows that $\mathsf{St}_H \in B_0(H)$.

(ii) Now we prove that $\mathsf{St}_G \in B_0(G)$ for the remaining cases. We will view G as $\mathcal{G}^F/\mathbf{Z}(\mathcal{G}^F)$ for a suitable simple simply connected algebraic group $\mathcal{G}$ in characteristic r and a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$. Then St_G can be viewed (by inflation) as the Steinberg character of $\mathcal{G}^F$, and it suffices to show that $\mathsf{St}_G \in B_0(\mathcal{G}^F)$. Let e denote the order of q modulo 3, where $\mathcal{G}^F$ is defined over $\mathbb{F}_q$; in particular $e \in \{1, 2\}$. As shown in the proof of [NTV, Proposition 5.4], see also the proof of [BLP, Theorem 4.2], if e is a *regular number* for $(\mathcal{G}, F)$ in the sense of [Spr, §5] (equivalently, there is a Sylow e-torus $\mathcal{S}$ such that $\mathbf{C}_\mathcal{G}(\mathcal{S})$ is a maximal torus, see Definition 2.5 and Remark 2.6 of [Spa1]), then $\mathsf{St}_G \in B_0(\mathcal{G}^F)$.

All the regular numbers e *greater than* 1 are listed in [Spr, §§5, 6], showing in particular that 2 is a regular number. Next we observe that 1 is also regular. Indeed, let W denote the Weyl group of $\mathcal{G}$ and let $(\mathcal{G}, F)$ correspond to the coset $W\phi$. We need to show that some element $h \in W\phi$ fixes a (nonzero) regular vector in $V = \mathbb{C}^l$, the defining module for W. In the split case, i.e. $\phi = 1_V$, we can certainly take $h = 1_V$. Consider the non-split case of $^2E_6$. According to the line $d = 2$ of [Spr, Table 8], $W\phi$ contains -1_V, whence

$$(4.3) \qquad\qquad W\phi = \{-1_V g \mid g \in W\}.$$

Next, the line $d = 2$ of [Spr, Table 1] shows that there is $g \in W$ such that $g(v) = -v$ for some regular vector $v \in V$. Now taking $h = -1_V g \in W$ we have that $h(v) = v$ as stated. The same argument applies to the case of ^2A_n, where we again have (4.3). In the case of ^2D_n, see [Spa1, Table 1]; alternatively, observe that $\mathcal{G}$ admits a Sylow Φ_1-torus $\mathcal{S}$ (so that $|\mathcal{S}^F| = (q-1)^{n-1}$) with $\mathbf{C}_\mathcal{G}(\mathcal{S})$ being a maximal torus.

Thus both 1 and 2 are regular numbers, and so we are done. $\qquad\square$

Lemma 4.7. *Theorem C holds in the case where $G = E_6^\epsilon(q)$, $\epsilon = \pm$, and $3|(q - \epsilon)$.*

Proof. In this case $H/G \cong C_3$. According to [Lu2], there is a semisimple element $s \in H^*$ of order 3 such that $|\mathbf{C}_{H^*}(s)| = |\mathrm{SL}_3^\epsilon(q)|^3$. Using [H1, Corollary 3.4] as in the proof of Proposition 4.5, we see that the semisimple character χ_s labeled by s belongs to $B_0(H)$. We also view $G = \mathcal{G}^F/\mathbf{Z}(\mathcal{G}^F)$ for a simple simply connected algebraic group of type E_6 and a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$. Inspecting the tables in [Lu1], one can check that $\mathcal{G}^F$ has no irreducible character of degree $\chi_s(1)$ and exactly three irreducible characters of degree $\chi_s(1)/3$. It follows that G has no irreducible character of degree $\chi_s(1)$ and three irreducible characters $\alpha_{1,2,3}$ of degree $\chi_s(1)/3$ with $(\chi_s)|_G = \alpha_1 + \alpha_2 + \alpha_3$. Now $1_G, \alpha_{1,2,3} \in B_0(G)$, and $3 \nmid \alpha_i(1)$, and so we are done. $\square$

Lemma 4.8. *Theorem C holds in the case where $G = \mathrm{PSL}_n^\epsilon(q)$, $3 \nmid q$, $\epsilon = \pm$, $n \geq 2$, and furthermore $3|(q - \epsilon)$ if $n \geq 4$.*

Proof. (i) Assume $n = 2$. Using the results in [Bu] one can check that

$$|\mathrm{Irr}_{3'}(B_0(G))| = (|G|_3 + 3)/2,$$

and so we are done in this case.

Next assume that $n = 3$ but $3|(q + \epsilon)$. In this case $G \cong H \cong H^* \cong \mathrm{SL}_3^\epsilon(q)$ and $P \in \mathrm{Syl}_3(G)$ is cyclic of order $3^a = (q + \epsilon)_3$. If $\epsilon = -$, then [Ge, Theorem 4.1] implies that $|\mathrm{Irr}_{3'}(B_0(G))| = (3^a + 3)/2$, and so we are again done. Consider the case $\epsilon = +$. By the main result of [BM], a character from the Lusztig series $\mathcal{E}(G, (s))$ can belong to $B_0(G)$ only when $s \in H^*$ is a 3-element. Moreover, if $s \in H^*$ is a 3-element, then the semisimple character χ_s labeled by s belongs to $B_0(G)$ by [H1, Corollary 3.4]. Now if $a \geq 2$, then we can choose s to be of order 3 or 9, leading to at least 4 characters in $B_0(G)$ (together with 1_G and St_G). If $a = 1$, then $H^* \cong \mathrm{SL}_3(q)$ has a unique conjugacy class of non-trivial 3-elements – any such element s is conjugate over $\overline{\mathbb{F}}_3$ to $\mathrm{diag}(1, \omega, \omega^2)$ where $\omega \in \mathbb{F}_{q^2}^\times$ has order 3. Note that $\mathbf{C}_{H^*}(s) \cong \mathrm{GL}_1(q^2)$. We have therefore shown that $\mathrm{Irr}_{3'}(B_0(G))$ consists of exactly three characters: 1_G, St_G, and χ_s (of degree $q^3 - 1$).

(ii) We may now assume that $n \geq 3$ and $3|(q - \epsilon)$. By [CE, Theorem 13], all irreducible characters in $\mathcal{E}(H, (t))$ for any 3-element $t \in H^*$ belong to $B_0(H)$. In particular, all unipotent characters of H belong to $B_0(H)$. Since they all restrict irreducibly to G, the same arguments as in p. (i) of the proof of Proposition 4.6 show that their restrictions to G are pairwise distinct and all belong to $B_0(G)$. The unipotent characters χ^λ of H are labeled by $\lambda \vdash n$. Let S^λ denote the irreducible character of S_n labeled by the partition $\lambda \vdash n$. Since $3|(q - \epsilon)$, it follows from the hook formula for the degree of unipotent characters of G, see [FS, (1.15)] that $3 \nmid \chi^\lambda(1)$ if and only if $3 \nmid S^\lambda(1)$. Now if $n \geq 5$, then $|\mathrm{Irr}_{3'}(\mathsf{S}_n)| \geq 6$ by Lemma 4.3, and so we have shown that $|\mathrm{Irr}_{3'}(B_0(G))| \geq 6$.

Suppose now that $n = 3$. As mentioned above, the unipotent characters of G of degree 1, $q(q + \epsilon)$, and q^3 all belong to $B_0(G)$. If $\epsilon = -$, then [Ge, Theorem 4.5] shows

that $\mathrm{Irr}_{3'}(B_0(G))$ also contains three characters of degree $(q+\epsilon)(q^2+q\epsilon+1)/3$. Assume now that $\epsilon = +$ and again fix $\omega \in \mathbb{F}_{q^2}^{\times}$ of order 3. Then it follows from [KT, Theorem 1.1] that the irreducible character $\chi := S(1,(1)) \circ S(\omega,(1)) \circ S(\omega^2,(1)) \in \mathrm{Irr}(\mathrm{GL}_3(q))$ splits into a sum of three irreducible characters $\alpha_{1,2,3}$ of $\mathrm{SL}_3(q)$. We again have that $\chi \in B_0(\mathrm{GL}_3(q))$, and $3 \nmid \alpha_i(1) = (q+\epsilon)(q^2+q\epsilon+1)/3$; in particular, α_i is trivial at $\mathbf{Z}(\mathrm{SL}_3(q)) \cong C_3$. It follows that $\alpha_{1,2,3} \in \mathrm{Irr}_{3'}(B_0(G))$.

Finally, assume that $n = 4$. With ω chosen as above, consider $t \in H^* \cong \mathrm{SL}_4^\epsilon(q)$ conjugate to $\mathrm{diag}(\omega,\omega,\omega,1)$ over $\overline{\mathbb{F}}_3$. Note that $\mathbf{C}_{H^*}(t) \cong \mathrm{GL}_3^\epsilon(q)$. It follows that t is 3-central, not central, and $\mathbf{C}_{\mathcal{H}^*}(t)$ is not a torus. Hence we are done by Proposition 4.5. $\qquad\square$

Completion of the proof of Theorem C. (i) It remains to prove Theorem C for non-abelian simple groups G, not isomorphic to any of the groups considered in Lemmas 4.2, 4.3, 4.4, 4.7, and 4.8. Then $\mathsf{St}_G \in B_0(G)$ by Proposition 4.6. By Proposition 4.5, it suffices to show that H^* contains a 3-central $t \notin \mathbf{Z}(H^*)$ of order 3 such that $\mathbf{C}_{\mathcal{H}^*}(t)$ is not a torus. In all the remaining cases, $3 \nmid |\mathbf{Z}(H^*)|$; hence $t \notin \mathbf{Z}(H^*)$ for any element $t \in H^*$ of order 3. We may also assume that $G \not\cong {}^2B_2(q)$.

First we consider exceptional groups of Lie type. Under the assumptions made on G, one can check using [Lu2] that $\mathbf{C}_{\mathcal{H}^*}(v)$ is not a torus for any element $v \in H^*$ of order 3. Hence we are done in this case by choosing $t \in \mathbf{Z}(P)$ of order 3 for some $P \in \mathrm{Syl}_3(H^*)$.

(ii) Now we handle the (remaining) simple classical groups. Suppose that $G = \mathrm{PSL}_n^\epsilon(q)$ with $n \geq 4$ and $3|(q+\epsilon)$. Then $H = \mathrm{PGL}_n^\epsilon(q)$ and $H^* = \mathrm{SL}_n^\epsilon(q)$. Set $m := \lfloor n/2 \rfloor$ and note that $\mathrm{GL}_n^\epsilon(q) \geq \mathrm{GL}_{2m}^\epsilon(q)$ contains a subgroup $X \cong \mathrm{GL}_m(q^2)$. Since $H^* = \mathbf{O}^{3'}(\mathrm{GL}_n^\epsilon(q))$, H^* contains all 3-elements of X as well as $[X,X] \cong \mathrm{SL}_m(q^2)$. In particular, H^* contains $t \in \mathbf{Z}(X) \cong C_{q^2-1}$ of order 3. Furthermore, X contains a Sylow 3-subgroup P of H^*. Hence $P \leq \mathbf{C}_{H^*}(t)$, i.e. t is 3-central. Finally, as $m \geq 2$ and $\mathbf{C}_{H^*}(t) \geq [X,X]$, we conclude that $\mathbf{C}_{\mathcal{H}^*}(t)$ is not a torus.

Next assume that $G = \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$ with $n \geq 2$. Then $H^* = \mathrm{Spin}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$. Choose $\kappa = \pm 1$ such that $3|(q-\kappa)$, and observe that $\mathrm{SO}_{2n+1}(q)$, respectively $\mathrm{Sp}_{2n}(q)$ contains a subgroup $X \cong \mathrm{GL}_n^\kappa(q)$. Arguing as above, we see that $Y = \Omega_{2n+1}(q)$, respectively $\mathrm{Sp}_{2n}(q)$ contains a 3-central (in Y) element $\bar{t} \in \mathbf{Z}(X)$ of order 3, and $\mathbf{C}_Y(\bar{t}) \geq \mathrm{SL}_n^\kappa(q)$. Since $Y = H^*/Z$ for a central 2-subgroup Z, we can then lift $\bar{t}$ to a 3-central element t of order 3 with non-toral $\mathbf{C}_{\mathcal{H}^*}(t)$.

Finally, consider the case $G = P\Omega_{2n}^\epsilon(q)$ with $\epsilon = \pm$ and $n \geq 4$. Then $H^* = \mathrm{Spin}_{2n}^\epsilon(q)$. Again we choose $\kappa = \pm 1$ such that $3|(q-\kappa)$. We will define a certain subgroup X of $3'$-index in $\mathrm{SO}_{2n}^\epsilon(q)$ with $3\|\mathbf{Z}(X)|$ as follows.

(a) Suppose $\epsilon = +$. Then choose $X \cong \mathrm{GL}_n(q) < \mathrm{SO}_{2n}^+(q)$ if $\kappa = +$. When $\kappa = -$, choose $X \cong \mathrm{GU}_{n-1}(q) < \mathrm{SO}_{2n-2}^+(q) < \mathrm{SO}_{2n}^+(q)$ if $2 \nmid n$, and $X \cong \mathrm{GU}_n(q) < \mathrm{SO}_{2n}^+(q)$ if $2|n$.

(b) Suppose $\epsilon = -$. Then choose $X \cong \mathrm{GL}_{n-1}(q) < \mathrm{SO}_{2n-2}^+(q) < \mathrm{SO}_{2n}^-(q)$ if $\kappa = +$. When $\kappa = -$, choose $X \cong \mathrm{GU}_{n-1}(q) < \mathrm{SO}_{2n-2}^-(q) < \mathrm{SO}_{2n}^-(q)$ if $2|n$, and $X \cong \mathrm{GU}_n(q) < \mathrm{SO}_{2n}^-(q)$ for $2 \nmid n$.

Again arguing as above, we see that $Y = \Omega_{2n}^\epsilon(q)$ contains a 3-central (in Y) element $\bar{t} \in \mathbf{Z}(X)$ of order 3, and $\mathbf{C}_Y(\bar{t}) \geq \mathrm{SL}_m^\gamma(q)$ for some $\gamma = \pm$ and some $m \in \{n, n-1\}$. Since $Y = H^*/Z$ for a central 2-subgroup Z, we can again lift $\bar{t}$ to a 3-central element t of order 3 with non-toral $\mathbf{C}_{\mathcal{H}^*}(t)$, finishing the proof of Theorem C. $\qquad\square$

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