Journal of Topology and Analysis
Vol. 10, No. 2 (2018) 447–469

DOI: 10.1142/S1793525318500140

Crossed products and minimal dynamical systems

Huaxin Lin

The Research Center for Operator Algebras
East China Normal University, Shanghai, China

Department of Mathematics, University of Oregon
Eugene, OR 97403-1232, USA
hlin@uoregon.edu

Received 2 June 2016
Accepted 2 October 2016
Published 29 December 2016

Let X be an infinite compact metric space with finite covering dimension and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. We prove that the crossed product C^*-algebras $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\beta \mathbb{Z}$ are isomorphic if and only if they have isomorphic Elliott invariant. In a more general setting, we show that if X is an infinite compact metric space and if $\alpha : X \to X$ is a minimal homeomorphism such that (X, α) has mean dimension zero, then the tensor product of the crossed product with a UHF-algebra of infinite type has generalized tracial rank at most one. This implies that the crossed product is in a classifiable class of amenable simple C^*-algebras.

Keyword: Minimal dynamical systems.

1. Introduction

Let X be an infinite compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism. It is well known that the crossed product C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$ is a unital simple separable amenable C^*-algebra which satisfies the Universal Coefficient Theorem (UCT). There has been a great deal of interaction between dynamical systems and C^*-algebra theory. Let $\beta : X \to X$ be another minimal homeomorphism. A result of Tomiyama [33] stated that two dynamical systems (X, α) and (X, β) are flip conjugate if and only if the crossed products $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\beta Z$ are isomorphic preserving $C(X)$. When X is the Cantor set, Giordano, Putnam and Skau [11] proved that two such systems are strong orbit equivalent if and only if the crossed products are isomorphic as C^*-algebras. In this case, the crossed products are isomorphic to a unital simple $A\mathbb{T}$-algebra of real rank zero. The Elliott program of classification of amenable C^*-algebras plays an important role in this case. In fact, K-theory can be used to determine when two minimal Cantor systems are strong orbit equivalent.

The irrational rotation algebras may be viewed as crossed products from the minimal dynamical systems on the circle by irrational rotations. Elliott and Evans [5] proved a structure theorem which states that every irrational rotation C^*-algebra is isomorphic to a unital simple $A\mathbb{T}$-algebra of real rank zero (there are also some earlier results such as [3]. With the rapid development in the Elliott program, it becomes increasingly important to answer the question when crossed product C^*-algebras from minimal dynamical systems are classifiable. If A is a unital separable simple C^*-algebra of stable rank one with the tracial state space $T(A)$, then the image of $K_0(A)$ under ρ_A in $\mathrm{Aff}(T(A))$ is dense if A has real rank zero [1], where $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ is defined by $\rho_A([p]) = \tau(p)$ for all projection $p \in A$ and for all $\tau \in T(A)$ (see Definition 2.3 below). Let X be an infinite compact metric space with finite covering dimension and let $\alpha : X \to X$ be a minimal homeomorphism. In [23], it is shown that $C = C(X) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero if and only if the image of $K_0(C)$ under ρ_C is dense in $\mathrm{Aff}(T(C))$. If X is connected and α is uniquely ergodic, and if the rotation number of (X, α) has an irrational value, then the image of $K_0(C)$ is dense in $\mathrm{Aff}(T(C))$. This recovers the earlier result of Elliott and Evans [5]. This is because of unital separable simple amenable C^*-algebras of tracial rank zero in the UCT class can be classified by the Elliott invariant up to isomorphism [16]. The Elliott program recently moved beyond the C^*-algebras of finite tracial rank. With Winter's method (see [36]), simple C^*-algebras with finite rational tracial rank in the UCT class have been classified by their Elliott invariant (see [18], [21] and [17]. Toms and Winter [34] improved the result in [23] by showing that, if projections in $C(X) \rtimes_\alpha \mathbb{Z}$ separate the tracial state space (the set of α-invariant Borel probability measures), then the crossed products have rational tracial rank zero and therefore are classifiable by Elliott invariant. Suppose that α, $\beta : X \to X$ are two minimal homeomorphisms and both are uniquely ergodic, then $C(X) \rtimes_\alpha \mathbb{Z}$ is isomorphic to $C(X) \rtimes_\beta \mathbb{Z}$ if and only if their K-groups are isomorphic in a way which also preserves the order and order unit of their K_0-groups. However, as early as the 60s, Furstenburg [10] presented minimal homeomorphisms on the 2-torus which are not Lipschitz. The set of projections of the crossed product C^*-algebras associated with the minimal dynamical systems in this case could not separate the tracial state space of the crossed products. The author was asked whether crossed products of Furstenburg transformation on 2-torus which are not Lipschitz are $A\mathbb{T}$-algebras. These crossed products have the same Elliott invariant as those of unital simple C^*-algebras with tracial rank one (but not zero). They do not have rational tracial rank zero. On the other hand, further developments were made by Strung [30] who showed that the crossed product C^*-algebras from certain minimal homeomorphisms on odd dimensional spheres have rational tracial rank one and therefore are classifiable. In these cases, projections in the crossed products may not separate the tracial state space. It has lately been shown that the crossed product C^*-algebras from any minimal homeomorphisms on $2d+1$ spheres (with $d \geq 1$) and other odd dimensional spaces have rational tracial rank at most one [20]. These results, however, do not cover the cases of the 2-torus given by Furstenburg. In fact, there are many minimal dynamical systems on connected spaces with complicated simplexes of invariant probability Borel measures. In this note we show the following:

Theorem 1.1. *Let X be an infinite compact metric space with finite covering dimension and let $\alpha : X \to X$ be a minimal homeomorphism. Then $C(X) \rtimes_\alpha \mathbb{Z}$ belongs to $\mathcal{N}_1^{\mathcal{Z}}$, a classifiable class of C^*-algebras* (*see Definition* 2.9 *below*). *If Y is another compact metric space and let $\beta : Y \to Y$ be another minimal homeomorphism. Then $C(X) \rtimes_\alpha \mathbb{Z} \cong C(Y) \rtimes_\beta \mathbb{Z}$ if and only if* $\mathrm{Ell}(C(X) \rtimes_\alpha \mathbb{Z}) \cong \mathrm{Ell}(C(Y) \rtimes_\beta \mathbb{Z})$.

(see Definition 2.4 for the definition $\mathrm{Ell}(\cdot)$).

We actually prove the following:

Theorem 1.2. *Let X be an infinite compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism such that (X, α) has mean dimension zero. Then $C(X) \rtimes_\alpha \mathbb{Z}$ belongs to $\mathcal{N}_1^{\mathcal{Z}}$.*

It should be noted that if X has finite dimension, then the minimal dynamical system (X, α) always has mean dimension zero. So Theorem 1.1 follows from theorem 1.2. Moreover, if the minimal dynamical system (X, α) has countably many extreme α-invariant Borel probability measures, then (X, α) has mean dimension zero. In fact, we show that $(C(X) \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{Z}$ is always classifiable, where $\mathcal{Z}$ is the Jiang–Su algebra. It is proved in [8] that crossed product C^*-algebras from minimal dynamical systems of mean dimension zero are $\mathcal{Z}$-stable.

Let X be a compact manifold and $\alpha : X \to X$ be a minimal diffeomorphism. In a longer paper, Lin and Phillips [25] showed that $C(X) \rtimes_\alpha \mathbb{Z}$ is an inductive limit of recursive sub-homogenous C^*-algebras (with bounded dimension in their spectrum). With the classification result in [12], we offer the following:

Corollary 1.3. *Let X be an infinite compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism such that (X, α) has mean dimension zero. Then $C(X) \rtimes_\alpha \mathbb{Z}$ is an inductive limit of sub-homogeneous C^*-algebras described in Remark* 2.10 *with dimension of the spectrum at most three.*

The proof of these results is based on recent advances in the Elliott program. In [12], it is shown that the class $\mathcal{N}_1^{\mathcal{Z}}$ of unital separable simple $\mathcal{Z}$-stable C^*-algebras which have rational generalized tracial rank at most one in the UCT class can be classified by the Elliott invariant. It should be noted that unital infinite dimensional simple C^*-algebras that have generalized tracial rank at most one are all $\mathcal{Z}$-stable. The range theorem in [12] shows that the class $\mathcal{N}_1^{\mathcal{Z}}$ exhausts the Elliott invariants of all possible unital simple amenable $\mathcal{Z}$-stable C^*-algebras and the range of the Elliott invariants is characterized. Moreover, it also shows that a C^*-algebra $A \in \mathcal{N}_1^{\mathcal{Z}}$ is isomorphic to a unital simple inductive limit of subhomogeneous algebras of a certain special form (see Remark 2.10 below) such that the spectrum has dimension at most three. This implies that $C(X) \rtimes_\alpha \mathbb{Z}$ has the same Elliott invariant as one of the model C^*-algebras in [12] as long as it is $\mathcal{Z}$-stable. This immediately provides an opportunity for $C(X) \rtimes_\alpha \mathbb{Z}$ to be embedded into a model C^*-algebras as presented in [12]. Indeed an earlier proof of the main result of this note did just that. However, this is unnecessarily difficult since it involves exact embedding which requires usage of an asymptotic unitary equivalence theorem and repeated usage of a version of Basic Homotopy Lemma. It also uses a new characterization of TA$\mathcal{C}$ for some special class $\mathcal{C}$ of weakly semiprojective C^*-algebras. We realized later that it is not necessary to prove the actual embedding. An approximate version of it would be sufficient. This greatly simplifies the proof.

It is worth pointing out that, without referring to the more recent long paper [12], but referring to [17] and [22], the proof we used in this paper still gives the following result:

Theorem 1.4. *Let X be an infinite compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism Then $C(X) \rtimes_\alpha \mathbb{Z}$ has rationally tracial rank at most one if and only if $K_0(C(X) \rtimes \mathbb{Z}) \otimes \mathbb{Q}$ is a dimension group and each extremal trace gives an extremal state on $K_0(C(X) \rtimes \mathbb{Z})$.*

This result, of course, is a corollary of Theorem 1.2. But it has already covered significant large class of crossed products from minimal action on compact metric spaces. For example, this result implies that the crossed products from non-Lipschitz Furstenburg transformations on 2-torus are unital simple $A\mathbb{T}$-algebras. Theorem 1.4 also covers all minimal actions on odd dimensional spaces studied in [20] (see Corolarry 5.4 below and its remark).

This note is organized as follows. The next section serves as a preliminary for the later sections. In Sec. 3, using a recent characterization of $TA\mathcal{C}$ for unital simple C^*-algebras with finite nuclear dimension [37], we present another convenient characterization for a unital separable simple amenable C^*-algebra to have rational generalized tracial rank at most one. In Sec 4, we present a few refinements of results in [12] to be used in this note. In Sec. 5, we present the proof for the main results presented earlier in the Introduction.

The main results of this note were reported at "Dynamics and C^*-algebras: amenability and soficity", a workshop at BIRS Banff Research Station in October 2014.

2. Preliminaries

Definition 2.1. Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A and $U_0(A)$ the normal subgroup of $U(A)$ consisting of the path connected component containing 1_A. Denote by $CU(A)$ the closure of the commutator subgroup of $U_0(A)$. The map $u \mapsto \mathrm{diag}(u, 1_A)$ gives a homomorphism from $U(M_n(A))$ to $U(M_{n+1}(A))$ for each integer $n \geq 1$. We write $U(M_\infty(A))$ for $\cup_{n=1}^\infty U(M_n(A))$ by using the above inclusion. Note we also use the notation $U_0(M_\infty(A)) = \cup_{n=1}^\infty U_0(M_n(A))$ and $CU(M_\infty(A)) = \cup_{n=1}^\infty CU(M_n(A))$.

Definition 2.2. Let A be a unital C^*-algebra and let $T(A)$ be the tracial state space. Let $\tau \in T(A)$. We say that τ is faithful if $\tau(a) > 0$ for all $a \in A_+ \setminus \{0\}$. Denote by $T_{\mathbf{f}}(A)$ the set of all faithful tracial states.

Denote by $\mathrm{Aff}(T(A))$ the space of all real continuous affine functions on $T(A)$ and denote by $\mathrm{LAff}_b(T(A))$ the set of all bounded lower-semi-continuous real affine functions on $T(A)$.

Suppose that $T(A) \neq \emptyset$. There is an affine map $r_{aff} : A_{s.a.} \to \mathrm{Aff}(T(A))$ by

$$r_{aff}(a)(\tau) = \hat{a}(\tau) = \tau(a) \quad \text{for all } \tau \in T(A)$$

and for all $a \in A_{s.a.}$. Denote by $A^q_{s.a.}$ the image $r_{aff}(A_{s.a.})$ and $A^q_+ = r_{aff}(A_+)$.

For each integer $n \geq 1$ and $a \in M_n(A)$, write $\tau(a) = (\tau \otimes Tr)(a)$, where Tr is the (non-normalized) trace on M_n.

Definition 2.3. Let A be a stably finite C^*-algebra with $T(A) \neq \emptyset$. Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the order preserving homomorphism defined by $\rho_A([p]) = \tau(p)$ for any projection $p \in M_n(A)$ (see the above convention), $n = 1, 2, \ldots$

Let A be a unital stably finite C^*-algebra. A map $s : K_0(A) \to \mathbb{R}$ is said to be a state if s is an order preserving homomorphism such that $s([1_A]) = 1$. The set of states on $K_0(A)$ is denoted by $S_{[1_A]}(K_0(A))$.

Denote by $r_A : T(A) \to S_{[1_A]}(K_0(A))$ the map defined by $r_A(\tau)([p]) = \tau(p)$ for all projection $p \in M_n(A)$ (for any integer n) and for all $\tau \in T(A)$.

Definition 2.4. Let A and C be two unital separable stably finite C^*-algebras is with $T(C) \neq \emptyset$ and $T(A) \neq \emptyset$. Let $\kappa_i : K_i(C) \to K_i(A)$ $(i = 0, 1)$ be homomorphism and let $\lambda : T(A) \to T(C)$ be an affine continuous map. We say that (κ_0, λ) is compatible, if

$$r_C(\lambda(t))(x) = r_A(t)(\kappa_0(x)) \quad \text{for all } x \in K_0(C) \quad \text{and} \quad \text{for all } t \in T(A).$$

Denote by $\lambda_\sharp : \mathrm{Aff}(T(C)) \to \mathrm{Aff}(T(A))$ the induced affine continuous map defined by $\lambda_\sharp(f)(\tau) = f \circ \lambda(\tau)$ for all $f \in \mathrm{Aff}(T(A))$ and $\tau \in T(C)$. When (κ_0, λ) is compatible, denote by $\overline{\gamma_\sharp} : \mathrm{Aff}(T(C))/\overline{\rho_A(K_0(C))} \to \mathrm{Aff}(T(A))\overline{\rho_A(K_0(A))}$ the induced continuous map.

Let A be a unital simple C^*-algebra. The Elliott invariant of A, denote by $\mathrm{Ell}(A)$ is the following six tuples

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A).$$

Suppose that B is another unital simple C^*-algebra. We write $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, if there is an order isomorphism $\kappa_0 : K_0(A) \to K_0(B)$ such that $\kappa_0([1_A]) = [1_B]$, an isomorphism $\kappa_1 : K_1(A) \to K_1(B)$ and an affine homeomorphism $\kappa_\rho : T(B) \to T(A)$ such that (κ_0, κ_ρ) is compatible.

Definition 2.5. Let A be a C^*-algebra. Let $a, b \in M_n(A)_+$. Following Cuntz, we write $a \lesssim b$ if there exists a sequence $(x_n) \subset M_n(A)$ such that $\lim_{n\to\infty} x_n^* b x_n = a$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$. The relation "$\sim$" is an equivalence relation.

Denote by $W(A)$ the Cuntz semigroup of the equivalence classes of positive elements in $\cup_{m=1}^{\infty} M_m(A)$ with orthogonal addition and order "$\lesssim$". In particular, when A has stable rank one, if $p, q \in M_n(A)$ are two projections then $p \sim q$ if and only if they are von Neumann equivalent. Denote by $\mathrm{Cu}(A)$ the Cuntz semigroup $W(A \otimes \mathcal{K})$ with order "$\lesssim$".

Definition 2.6. Let A be a C^*-algebra. Denote by $A^{\mathbf{1}}$ the unit ball of A. $A_+^{q,\mathbf{1}}$ is the image of the intersection of $A_+ \cap A^{\mathbf{1}}$ in A_+^q.

Definition 2.7. Let A and B be two unital C^*-algebras and let $\varphi : A \to B$ be a homomorphism. We write $\varphi_{*i} : K_i(A) \to K_i(B)$ for the induced homomorphisms on K-theory and $[\varphi]$ for the element in $KK(A, B)$ as well as $KL(A, B)$ if there is no confusion. We also use $\varphi^{\ddagger} : U(M_\infty(A))/CU(M_\infty(A)) \to U(M_\infty(B))/CU(M_\infty(B))$ for the induced homomorphism. Suppose that $T(A)$ and $T(B)$ are both non-empty. Then $\varphi_T : T(B) \to T(A)$ is the affine continuous map induced by $\varphi_T(\tau)(a) = \tau(\varphi(a))$ for all $a \in A$ and $\tau \in T(A)$. Denote by $\varphi^{cu} : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ the semigroup homomorphisms which preserve of the order.

We use $KL_e(A, B)^{++}$ for the subset of elements $\kappa \in KL(A, B)$ such that $\kappa(K_0(A) \setminus \{0\}) \subset K_0(B) \setminus \{0\}$ and $(\kappa([1_A]) = [1_B]))$.

Definition 2.8. Denote by $\mathcal{C}$ the class of those unital C^*-algebras C which are finite dimensional C^*-algebras or those C which are the pull-back:

$$\begin{array}{ccc} C & \dashrightarrow & C([0,1], F_2) \\ \Big\downarrow{\scriptstyle \pi_e} & & \Big\downarrow{\scriptstyle (\pi_0, \pi_1)} \\ F_1 & \xrightarrow{(\varphi_0, \varphi_1)} & F_2 \oplus F_2, \end{array} \tag{1}$$

where F_1 and F_2 are finite dimensional C^*-algebras and $\varphi_i : F_1 \to F_2$ are homomorphisms. These C^*-algebras are also called one dimensional non-commutative CW complexes (NCCW). C^*-algebra C can also be written as

$$C = \{(f, a) \in C([0,1], F_2) \oplus F_1 : \varphi_0(a) = f(0) \quad \text{and} \quad \varphi_1(a) = f(1)\} \tag{2}$$

and is also called Elliott-Thomsen building block. Denote by $\mathcal{C}_0$ those C^*-algebras C in $\mathcal{C}$ with $K_1(C) = \{0\}$.

C^*-algebras in $\mathcal{C}$ are semiprojective (proved in [4].

Definition 2.9. Let A be a unital simple C^*-algebra and let $\mathcal{S}$ be a class of unital C^*-algebras. We say A is $TA\mathcal{S}$, if for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $a \in A_+ \setminus \{0\}$, there exists a projection $p \in A$ and C^*-subalgebra $C \in \mathcal{S}$ with $1_C = p$ such that

$$\|px - xp\| < \epsilon \quad \text{and} \quad \mathrm{dist}(pxp, C) < \epsilon \quad \text{for all } x \in \mathcal{F} \quad \text{and} \tag{3}$$

$$1 - p \lesssim a. \tag{4}$$

In the case that $\mathcal{S} = \mathcal{C}$, then we say that A has generalized tracial rank at most one and write $gTR(A) \leq 1$. If $gTR(A) \leq 1$, we may say A is TA$\mathcal{C}$. In the above definition, if $C \in \mathcal{C}_0$, then we say A is in TA$\mathcal{C}_0$.

It is proved in [12] that, if $gTR(A \otimes Q) \leq 1$, where Q is the UHF-algebra with $K_0(Q) = \mathbb{Q}$, then $A \otimes Q \in \mathrm{TA}\mathcal{C}_0$(Cor. 29.3 of [12]. By a result in [24], this implies that $A \otimes U$ is $TA\mathcal{C}_0$ for all UHF-algebras U of infinite type.

Denote by $\mathcal{N}_1$ the class of unital simple amenable C^*-algebras in the UCT class such that $gTR(A \otimes U) \leq 1$ for some UHF-algebra U of infinite type.

Denote by $\mathcal{Z}$ the Jiang–Su algebra of unital simple C^*-algebra with $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$ which is also an inductive limit sub-homogeneous C^*-algebras [13]. Recall that a C^*-algebra A is $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$. Denote by $\mathcal{N}_1^{\mathcal{Z}}$ those C^*-algebras in $\mathcal{N}_1$ which are $\mathcal{Z}$-stable.

Remark 2.10. Recall [6] that the finite CW complexes $T_{II,k}$ (or $T_{III,k}$) are defined to be a 2-dimensional connected finite CW complex with $H^2(T_{II,k}) = \mathbb{Z}/k$ and $H^1(T_{II,k}) = 0$ (or 3-dimensional finite CW complex with $H^3(T_{III,k}) = \mathbb{Z}/k$ and $H^1(T_{III,k}) = 0 = H^2(T_{III,k})$). For each n, there is a space X_n which is of the form

$$X_n = [0,1] \vee S^1 \vee S^1 \vee \cdots \vee S^1 \vee T_{II,k_1} \vee T_{II,k_2} \vee \\ \cdots \vee T_{II,k_i} \vee T_{III,m_1} \vee T_{III,m_2} \vee \cdots \vee T_{III,m_j}.$$

Let $P_n \in M_\infty(C(X_n))$ be a projection with rank $r(n,1)$. Let

$$A_n = P_n M_\infty(C(X_n))P_n \bigoplus \bigoplus_{i=2}^{p_n} M_{r(n,i)}(\mathbb{C})$$

and $F_n = \bigoplus_{i=1}^{p_n} M_{r(n,i)}(\mathbb{C})$. Fix a point $\xi \in X_n$ and denote by $\pi_\xi : A_n \to F_n$ by $\pi_\xi(f,a) = f(\xi) \oplus a$, where $f \in P_n M_\infty(C(X_n))P_n$ and $a \in \oplus_{i=2}^{p_n} M_{r(n,i)}(\mathbb{C})$. Let E_n be a finite dimensional C^*-algebra and let $\varphi_0, \varphi_1 : F_n \to E_n$ be two unital homomorphisms. Define

$$C_n = \{(c,a) \in C([0,1], E_n) \oplus A_n : c(0) = \varphi_0 \circ \pi_\xi(a) \quad \text{and} \quad c(1) = \varphi_1 \circ \pi_\xi(a)\}.$$

It is proved in [12] (see Theorems of 13.41 and 29.4 of [12]) that every C^*-algebra in $\mathcal{N}_1^{\mathcal{Z}}$ can be written as simple inductive limit of C^*-algebras of the form C_n above (up to isomorphism).

3. Approximate Embeddings

Definition 3.1. Denote by $\mathcal{S}$ a class of unital C^*-algebras which has the following properties: (a) $\mathcal{S}$ contains all finite dimensional C^*-algebras, (b) tensor products of finite dimensional C^*-algebras with C^*-algebras in $\mathcal{S}$ are in $\mathcal{S}$, (c) $\mathcal{S}$ is closed under direct sums, (d) every C^*-algebra in $\mathcal{C}$ are weakly semiprojective, and (e) if $S \in \mathcal{S}$, $J \subset \mathcal{S}$ is a closed two-sided ideal of S, $\epsilon > 0$ and $\mathcal{F} \subset S/I$ is a finite subset, then there exists C^*-subalgebra $C \subset A/I$ such that $C \in \mathcal{S}$ and $\mathrm{dist}(x, C) < \epsilon$ for all $x \in \mathcal{F}$.

It is proved in [12] (see Lemma 3.20 in Definition [12] that the class $\mathcal{C}$ in 2.8 satisfies (a)–(e).

The following is proved in [31].

Proposition 3.2. (cf. Lemma 2.1 of [31]) *Let $\mathcal{S}$ be a class of unital C^*-algebras in Definition* 3.1 *and let U be a UHF-algebra of infinite type. Let A be a unital separable simple stably finite exact C^*-algebra. Then $A \otimes U$ is $TA\mathcal{S}$ if and only if there is $\eta > 0$ such that, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A \otimes U$, there exists a projection p and a C^*-subalgebra $B \subset A \otimes U$ with $1_B = p$ and $B \in \mathcal{S}$ such that*

$$\tau(p) > \eta \quad \text{for all } \tau \in T(A \otimes U), \tag{5}$$

$$\|px - xp\| < \epsilon \quad \text{and} \quad \mathrm{dist}(pap, B) < \epsilon \quad \text{for all } x \in \mathcal{F}. \tag{6}$$

Proof. This is a slight refinement of Lemma 2.1 of [31]. It should be noted, however, that finitely generated assumption on C^*-algebras in $\mathcal{S}$ is not really needed. As in the proof of Lemma 3.2 of [35], let $\gamma < \epsilon_{i+1}/14$. Choose a subset $\mathcal{F}' \subset B_i$ such that, for each $x \in \mathcal{F}$, there exists $x' \in \mathcal{F}'$ such that $\|x - x'\| < \epsilon_i$. For any $\gamma > 0$, there exist of $\vartheta > 0$ and a finite subset $\mathcal{G} \subset B_i$ with the following property: if E is another C^*-algebra $p \in E$ is a projection and $\varphi : B_i \to E$ is a homomorphism satisfying $\|p\varphi(b) - \varphi(b)p\| < \vartheta$ for all $b \in \mathcal{G}$, then there exists a unital homomorphism $\bar{\varphi} : B_i \to pEp$ such that $\|\bar{\varphi}(x') - p\varphi(x')p\| < \gamma$ for all $x' \in \mathcal{F}'$.

Then, as in the proof of Lemma 2.1 of [31], one choose $B'_{i+1} := \varrho(B_i) \oplus F$. By property (e) in Definition 3.1, there exists a unital C^*-subalgebra $B''_{i+1} \in \varrho(B_i)$ with $B''_{i+1} \in \mathcal{S}$ such that

$$\mathrm{dist}(\varrho(x'), B''_{i+1}) < \gamma/2 \quad \text{for all } x' \in \mathcal{F}'. \tag{7}$$

Put $B_{i+1} = B''_{i+1} \oplus F$. Then the rest of the proof remains the same. □

We will also use the following characterization for C^*-algebras in $TA\mathcal{S}$.

Theorem 3.3. (Theorem 2.2 of [37]) *Let $\mathcal{S}$ be a class of unital C^*-algebras in Definition* 3.1 *and let A be a unital separable simple C^*-algebra with $T(A) \neq \emptyset$ and with finite nuclear dimension. Suppose that there are two sequences of contractive completely positive linear maps*: $\varphi_n : A \to B_n$ *and* $h_n : B_n \to A$ *satisfying the following*:

(i) $B_n \in \mathcal{S}$ *for each* $n \in \mathbb{N}$,
(ii) h_n *is an embedding for each* $n \in \mathbb{N}$,
(iii) $\lim_{n\to\infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0$ *for all* $a, b \in A$ *and*
(iv) $\lim_{n\to\infty} \sup_{\tau\in T(A)} |\tau \circ h_n \circ \varphi_n(a) - \tau(a)| = 0$ *for each* $a \in A$. *Then* $A \otimes Q$ *is* $TA\mathcal{S}$.

Lemma 3.4. *Let C and A be two unital separable simple amenable C^*-algebras. Suppose that $C \otimes U$ has finite nuclear dimension and $gTR(A \otimes U) \leq 1$ for some UHF-alegbra U of infinite type. Suppose also that there exists an order isomorphism*

$$\kappa_0 : (K_0(C \otimes U), K_0(C \otimes U)_+, [1_{C\otimes U}]) \to (K_0(A \otimes U), K_0(A \otimes U)_+, [1_{A\otimes U}]),$$

an affine homeomorphism $\lambda : T(A \otimes U) \to T(C \otimes U)$ such that (κ_0, λ) is compatible and there exist a sequence of unital contractive completely positive linear maps

$\psi_n : C \otimes U \to A \otimes U$ *such that*

$$\lim_{n\to\infty} \sup_{\tau \in T(A)} |\tau(\psi_n(a)) - \lambda(\tau)(a)| = 0 \quad \text{for all } a \in C \otimes U \text{ and} \tag{8}$$

$$\lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = 0 \quad \text{for all } a, b \in C \otimes U. \tag{9}$$

Then $gTR(C \otimes U) \le 1$.

Proof. By [24], it suffices to show that $gTR(C \otimes Q) \le 1$. Since $U \otimes Q \cong Q$ and the assumption holds when we replace U by Q, we may assume that $U = Q$. Let $C_1 = C \otimes U$ and $A_1 = A \otimes U$. For unital separable simple C^*-algebras with finite nuclear dimension, we use a characterization for $TA\mathcal{C}_0$ in [37]. We will show that there exist a sequence of unital C^*-algebras $B_n \in \mathcal{C}_0$ and a sequence of monomorphisms $h_n : B_n \to C_1$ such that

$$\lim_{n\to\infty} \sup_{\tau \in T(C_1)} |\tau(h_n \circ \psi_n(c)) - \tau(c)| = 0 \quad \text{for all } c \in C_1. \tag{10}$$

Then, by Theorem 3.3 (Theorem 2.2 in [37], $gTR(C_1) \le 1$.

By the assumption, we have

$$r_{A_1}(\tau)(\kappa_0(x)) = r_{C_1}(\lambda(\tau))(x) \quad \text{for all } x \in K_0(C_1). \tag{11}$$

Let $\{\mathcal{G}_n\}$ be an increasing sequence of finite subsets of A_1 such that $\cup_{n=1}^{\infty} \mathcal{G}_n$ is dense in A_1. For each n, there exists a C^*-subalgebra $B_n \subset A_1$ with $1_{B_n} = p_n$ such that $B_n \in \mathcal{C}_0$ and a contractive completely positive linear map $L_n : A_1 \to B_n$ such that

$$\|p_n x - x p_n\| < 1/2^{n+3} \quad \text{for all } x \in \mathcal{G}_n, \tag{12}$$

$$\|L_n(x) - p_n x p_n\| < 1/2^{n+3} \quad \text{for all } x \in \mathcal{G}_n \quad \text{and} \tag{13}$$

$$\tau(1 - p_n) < \frac{1}{2^{n+3} \max\{\|g\| + 1 : g \in \mathcal{G}_n\}} \quad \text{for all } \tau \in T(A_1). \tag{14}$$

In particular,

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \quad \text{for all } a, b \in A_1. \tag{15}$$

It follows from Theorem 2.5 of [2] that $\mathrm{Cu}(A_1) = V(A_1) \sqcup \mathrm{LAff}_+(T(A_1))$. Since both A_1 and C_1 have stable rank one, the map $\Gamma : \mathrm{Cu}(A_1) \to \mathrm{Cu}(C_1)$ defined by

$$\Gamma([p]) = \kappa_0^{-1}([p]) \quad \text{and} \quad \Gamma(f)(\lambda(\tau)) = f(\tau) \tag{16}$$

for all projections $p \in A_1 \otimes \mathcal{K}$, $f \in \mathrm{LAff}_+(T(A_1))$ and $\tau \in T(A_1)$ is a semigroup isomorphism. Moreover, Γ is order preserving, preserves the suprema and preserves the relation of compact containment. Denote by $j_n : B_n \to A_1$ the embedding and denote by $j_m^{cu} : \mathrm{Cu}(B_n) \to \mathrm{Cu}(A_1)$ the morphism induced by j_m. It follows from the existence part of a result in [29] that there is, for each n, a homomorphism $h_n : B_n \to C_1$ such that $h_n^{cu} = \Gamma \circ j_m^{cu}$. Since j_m is the embedding and Γ is an isomorphism, h_n^{cu} does not vanish. It follows that h_n is an embedding.

Put $\psi_n = L_n \circ \psi'_n$. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of C_1 whose union is dense in C_1. By passing to a subsequence, without loss of generality we may assume that, for each n,

$$\sup_{\tau\in T(A_1)} |\tau \circ \psi_n(c) - \lambda(\tau)(c)| < \frac{1}{2^{n+1}(\max\{\|c\|+1 : c \in \mathcal{F}_n\})} \quad \text{for all } c \in \mathcal{F}_n \quad \text{and} \tag{17}$$

$$\mathrm{dist}(\psi'_n(c), \mathcal{G}_n) < 1/2^{n+2} \quad \text{for all } c \in \mathcal{F}_n. \tag{18}$$

Let $g_c \in \mathcal{G}_n$ such that

$$\|\psi'_n(c) - g_c\| < 1/2^{n+3} \quad \text{for all } c \in \mathcal{F}_n. \tag{19}$$

We then estimate that

$$\begin{aligned} \lambda(\tau)(h_n \circ \psi_n(c)) &= \Gamma \circ j_m^{cu}(\psi_n(c))(\tau) = j_m\widehat{(\psi_n(c))}(\tau) \\ &= \tau(\psi_n(c)) \approx_{1/2^{n+3}} \tau(L_n(g_c)) \approx_{1/2^{n+3}} \tau(p_n g_c p_n) \\ &\approx_{1/2^{n+3}} \tau(g_c) \approx_{1/2^{n+3}} \tau(\psi'_n(c)) \\ &\approx_{1/2^{n+3}} \lambda(\tau)(c) \end{aligned} \tag{20}$$

for all $\tau \in T(A_1)$, where the first approximation follows from (19), the second follows from (13), the third follows from (14), the fourth follows from (19) and the last one follows (17). In other words,

$$\sup_{t\in T(C_1)} |t(h_n \circ \psi_n(c)) - t(c)| < 1/2^n \quad \text{for all } c \in C_1. \tag{21}$$

Then, by applying Theorem 3.3, $gTR(C_1) \le 1$. □

4. Uniqueness and Existence Theorems

The following follows from a result in [12].

Theorem 4.1. *Let X be a compact metric space, let $C = C(X)$, let $A_1 \in \mathcal{A}_1$, let U be a UHF-algebra of infinite type and let $A = A_1 \otimes U$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Suppose also that*

$$[\varphi_1] = [\varphi_2] \text{ in } KL(C, A) \tag{22}$$

$$(\varphi_1)_T = (\varphi_2)_T \text{ and } \varphi_1^\ddagger = \varphi_2^\ddagger. \tag{23}$$

Then φ_1 and φ_2 are approximately unitarily equivalent.

Proof. This follows immediately from Theorem 12.7 of [12]. Define, for each $a \in A_+^{\mathbf{1}} \setminus \{0\}$,

$$\Delta(\hat{a}) = \inf\{\tau(\varphi_1(a)) : \tau \in T(A)\}/2. \tag{24}$$

It is clear that $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ is non-decreasing map. We have

$$\tau(\varphi_2(a)) = \tau(\varphi_1(a)) \ge \Delta(\hat{a}) \quad \text{for all } a \in A_+^{\mathbf{1}} \setminus \{0\}. \tag{25}$$

Then the theorem follows Theorem 12.7 of [12]. □

Definition 4.2. Let A be a unital separable stably finite C^*-algebra with $T(A) \neq \emptyset$. There is a splitting exact sequence with the splitting map J_c^A :

$$0 \to \operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to U_\infty(A)/CU_\infty(A) \overset{\pi_{K_1}^A}{\to} K_1(A) \to 0 \tag{26}$$

(see [32]). In particular, $\pi_{K_1}^A \circ J_c^A = \mathrm{id}_{K_1(A)}$. In what follows, *for each such unital C^*-algebra A, we fix one J_c^A.*

For a fixed pair of unital stable finite C^*-algebras C and A with $T(C) \neq \emptyset$ and $T(A) \neq \emptyset$, we fix J_c^C and J_c^A. Suppose that $\varphi : C \to A$ is a unital homomorphism. Define $\varphi^\rho : K_1(C) \to \operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ by

$$\varphi^\rho = (\mathrm{id} - J_c^A \circ \pi_{K_1}^A) \circ \varphi^\ddagger \circ J_c^C. \tag{27}$$

Note that $\pi_{K_1}^A \circ \varphi^\rho = 0$. Moreover,

$$\pi_{K_1}^A \circ \varphi^\ddagger \circ J_c^C = \varphi_{*1}. \tag{28}$$

Suppose that $\psi : C \to A$ is another unital homomorphism such that $\psi_{*1} = \varphi_{*1}$ and $\varphi^\rho = \psi^\rho$. Then, by (28),

$$\varphi^\ddagger \circ J_c^C = \psi^\ddagger \circ J_c^C. \tag{29}$$

Thus, if $\psi_T = \varphi_T$, then $\varphi^\ddagger = \psi^\ddagger$.

Suppose that $\kappa \in KL_e(C, A)^{++}$, $\lambda : T(A) \to T(C)$ is an affine continuous map and $\gamma : U(M_\infty(C))/CU(M_\infty(C)) \to U(M_\infty(A))/CU(M_\infty(A))$ is a continuous homomorphism. We say $(\kappa, \lambda, \gamma)$ is compatible, if $(\kappa|_{K_0(C)}, \lambda)$ is compatible, $\gamma|_{\operatorname{Aff}(T(C))/\overline{\rho_A(K_0(C))}} = \overline{\lambda_\sharp}$, where $\overline{\lambda_\sharp}$ is defined in Definition 2.4, and $\kappa|_{K_1(C)} = \pi_{K_1}^A \circ \gamma \circ J_c^C$ (which is independent of choice of J_c^C).

Therefore we also have the following:

Corollary 4.3. *Let X be a compact metric space, let $C = C(X)$, let $A_1 \in \mathcal{A}_1$, let U be a UHF-algebra of infinite type and let $A = A_1 \otimes U$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Suppose also that*

$$[\varphi_1] = [\varphi_2] \ in \ KL(C, A) \tag{30}$$

$$(\varphi_1)_T = (\varphi_2)_T \ and \ \varphi_1^\rho = \varphi_2^\rho. \tag{31}$$

Then φ_1 and φ_2 are approximately unitarily equivalent.

The following is well known.

Proposition 4.4. *Let X be a compact metric space and let X_n be a sequence of polyhedrons such that $C(X) = \lim_{n\to\infty}(C(X_n), s_n)$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists an integer $k \geq 1$ such that, for any $n \geq k$ there is a unital ϵ-$\mathcal{F}$-multiplicative contractive completely positive linear maps $L_n : C(X) \to C(X_n)$ such that*

$$\|s_{n,\infty} \circ L_n(f) - f\| < \epsilon \quad for \ all \ f \in \mathcal{F}. \tag{32}$$

Proof. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there is an integer $k_1 \geq 1$ such that $\mathrm{dist}(f, s_{n,\infty}(C(X_n)) < \epsilon/2$ for all $n \geq k_1$. To simplify the notation, without loss of generality, we may assume that $\|f\| \leq 1$ for all $f \in \mathcal{F}$. Put $B_n = s_{n,\infty}(C(X_n))$. Since $C(X)$ is amenable, there exists a unital contractive completely positive linear map $\Phi_n : C(X) \to B_n$ such that

$$\|\Phi_n(f) - f\| < \epsilon/4 \quad \text{for all } f \in \mathcal{F}. \tag{33}$$

Since $s_{n,\infty}(C(X_n))$ is amenable, there exists a unital contractive completely positive linear map $\Psi_n : s_{n,\infty}(C(X_n)) \to C(X_n)$ such that

$$s_{n,\infty} \circ \Psi_n(f) = f \tag{34}$$

for all $f \in s_{n,\infty}(C(X_n))$. For all $f, g \in \mathcal{F}$, since

$$s_{n,\infty} \circ \Psi_n(fg) = fg = s_{n,\infty} \circ \Psi_n(f) \cdot s_{n,\infty} \circ \Psi_n(g) = s_{n,\infty}(\Psi_n(f)\Psi_n(g)),$$

there is an integer $k_2 > n$ such that

$$\|s_{n,m} \circ \Psi_n(fg) - s_{n,m} \circ \Psi_n(f) s_{n,m} \circ \Psi_n(g)\| < \epsilon/2 \tag{35}$$

for all $m \geq k_2$ and $f, g \in \mathcal{F}$. Put $k = k_1 k_2$. For $m \geq k$, define $L_m = s_{k_1+1,m} \circ \Psi_{k_1+1} \circ \Phi_{k_1+1} : C(X) \to C(X_m)$. $m = 1, 2, \ldots$. It follows from (33) and (34) that L_m is ϵ-$\mathcal{F}$-multiplicative. Moreover, by (33) and (34),

$$\|s_{m,\infty} \circ L_m(f) - f\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \qquad \square$$

Lemma 4.5. *Let X be a compact metric space, let $A \in \mathcal{N}_1$, let U be a UHF-algebra of infinite type and let $B = A \otimes U$. Suppose that $\kappa \in KL_e(C(X), B)^{++}$, $\lambda : T(B) \to T_{\mathbf{f}}(C(X))$ is a continuous affine map and suppose that $\gamma : U_\infty(C(X))/CU_\infty(C(X)) \to U_\infty(A)/CU_\infty(B)$ is a continuous homomorphism such that $(\kappa, \lambda, \gamma)$ is compatible. Then there exists a sequence of unital contractive completely positive linear maps $\varphi_n : C(X) \to B$ such that, for any finite subset $\mathcal{P} \subset \underline{K}(C(X))$,*

$$[\varphi_n]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \quad \textit{for all sufficiently large } n, \tag{36}$$

$$\lim_{n\to\infty} \|\varphi_n(fg) - \varphi_n(f)\varphi_n(g)\| = 0 \quad \textit{for all } f, g \in C(X), \tag{37}$$

$$\lim_{n\to\infty} \max_{\tau \in T(B)} |\tau \circ \varphi_n(f) - \lambda(\tau)(f)| = 0 \quad \textit{for all } f \in C(X)_{s.a.} \textit{ and} \tag{38}$$

$$\lim_{n\to\infty} \mathrm{dist}(\overline{\langle \varphi_n(v) \rangle}, \gamma(\bar{v})) = 0 \quad \textit{for all unitaries } v \in M_m(C(X)),\ m = 1, 2, \ldots. \tag{39}$$

(See 14.5 and 2.20 of [19] for notations in (39).)

Proof. We first note that, for any compact metric space Y, $T_{\mathbf{f}}(C(Y)) \neq \emptyset$. This is well known, but to see this, let $\{y_n\}$ be a dense sequence in Y. Define

$$t(f) = \sum_{n=1}^{\infty} f(y_n)/2^n \quad \text{for all } f \in C(Y). \tag{40}$$

It is clear that t gives a faithful tracial state on $C(Y)$.

We will write $C(X) = \lim_{n\to\infty}(C(X_n), \iota_n)$, where each X_n is a polyhedron and $\iota_{n,\infty} : C(X_n) \to C(X)$ is injective (see Satz 1, p. 229 of [9] and see also [26]). Put $C = C(X)$ and $C_n = C(X_n)$, $n = 1, 2, \ldots$.

Put $\kappa_n = \kappa \circ [\iota_{n,\infty}]$ and $\gamma_n = \gamma \circ \iota^{\ddagger}$. Define $\lambda_n : T(A) \to T_{\mathbf{f}}(C_n)$ by $\lambda_n(\tau)(f) = \lambda(\tau)(\iota_{n,\infty}(f))$ for all $f \in C_n$ and $\tau \in T(A)$, $n = 1, 2, \ldots$. Note that $(\kappa_n, \lambda_n, \gamma_n)$ is compatible since $(\kappa, \lambda, \gamma)$ is compatible. Note also that since $\kappa \in KL_e(C, A)^{++}$ and $\iota_{n,\infty}$ is injective, $\kappa_n \in KL_e(C_n, A)^{++}$. It follows from Theorem 21.14 of [12] that there exists a unital monomorphism $h_n : C_n \to A$ such that

$$[h_n] = \kappa_n,\ (h_n)_T = \lambda_n \quad \text{and} \quad h_n^{\ddagger} = \gamma_n,\ \ n = 1, 2, \ldots. \tag{41}$$

By applying Proposition 4.4, we may assume that there are contractive completely positive linear maps $L_n : C(X) \to C(X_n)$ such that

$$\lim_{n\to\infty} \|L_n(f)L_n(g) - L_n(fg)\| = 0 \quad \text{for all } f, g \in C \quad \text{and} \tag{42}$$

$$\lim_{n\to\infty} \|\iota_{n,\infty} \circ L_n(f) - f\| = 0 \quad \text{for all } f \in C. \tag{43}$$

Define $\varphi_n = h_n \circ L_n$, $n = 1, 2, \ldots$. One easily checks that $\{\varphi_n\}$ meets the requirements. □

4.6. Let A be a unital separable stably finite simple C^*-algebra with $T(A) \neq \emptyset$ and let X be a compact metric space. Suppose that $\lambda : T(A) \to T_{\mathbf{f}}(C(X))$ is an affine continuous map. Define $\lambda_\sharp : C(X)_{s.a.} \to \mathrm{Aff}(T(A))$ by $\lambda_\sharp(f)(\tau) = \lambda(\tau)(f)$ for all $f \in C(X)_{s.a.}$. Since $\lambda(\tau) \in T_{\mathbf{f}}(C(X))$, $\lambda_\sharp$ is strictly positive in the sense that $\lambda_\sharp(f) > 0$ for all $f \in C(X)_+ \setminus \{0\}$. Put

$$\Delta_0(\hat{f}) = \inf\{\lambda_\sharp(f)(\tau) : \tau \in T(A)\} \tag{44}$$

for all $f \in C(X)_+^{\mathbf{1}}$. Note that, since $T(A)$ is compact, $1 \geq \Delta_0(\hat{f}) > 0$ for all $f \in C(X)_+^{\mathbf{1}}$. Thus $\Delta = 3\Delta_0/4$ gives a non-decreasing function from $C(X)_+^{q,\mathbf{1}} \setminus \{0\}$ to $(0, 1)$.

For any finite subset $\mathcal{H}_0 \subset C(X)_+^{q,\mathbf{1}}$, there exist $\sigma > 0$ and a finite subset $\mathcal{H} \subset C(X)_{s.a.}$ satisfying the following: for any unital contractive completely positive linear map $\Phi : C(X) \to A$

$$\tau \circ \Phi(g) \geq \Delta(\hat{g}) \quad \text{for all } g \in \mathcal{H}_0, \tag{45}$$

provided that

$$\max_{\tau\in T(A)} |\tau\circ\Phi(f)-\lambda(\tau)(f)|<\sigma \quad \text{for all } f\in\mathcal{H}. \tag{46}$$

Theorem 4.7. *Let X be a compact metric space, let $A\in\mathcal{N}_1$ and let $B=A\otimes U$, where U is a UHF-algebra of infinite type. Suppose that $\kappa\in KL_e(C(X),B)^{++}$, $\lambda: T(A)\to T_{\mathbf{f}}(C(X))$ is a continuous affine map and suppose that $\chi: K_1(C(X))\to \mathrm{Aff}(T(B))/\overline{\rho_A(K_0(B))}$ is ahomomorphism such that (κ,λ) is compatible. Then there exists a unital monomorphism $h: C(X)\to B$ such that*

$$[h]=\kappa,\ h^{\rho}=\chi \text{ and } h_T=\lambda. \tag{47}$$

(Here $J_c^{C(X)}$ and J_c^B are fixed.)

Proof. Let Δ be induced by λ as defined in 4.6. Define $\gamma: U(M_\infty(C(X))/CU(M_\infty (C(X))\to U(M_\infty(B))/CU(M_\infty(B)))$ as follows:

$$\gamma|_{\mathrm{Aff}(T(C(X)))/\overline{\rho_{C(X)}(K_0(C(X)))}}=\overline{\lambda_\sharp} \quad \text{and} \tag{48}$$

$$\gamma|_{J_c^{C(X)}(K_0(C(X)))}=\chi\circ\pi_{K_1}^{C(X)}+J_c^A\circ\kappa\circ\pi_{K_1}^{C(X)}. \tag{49}$$

Then (κ,λ,γ) is compatible.

Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of $C(X)$ such that its union is dense in $C(X)$. Let $\{\epsilon_n\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty}\epsilon_n<\infty$. We will apply 12.7 of [12] with $C=C(X)$. Let $\mathcal{H}_{1,n}\subset C(X)_+^{q,\mathbf{1}}\setminus\{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\sigma_{1,n}>0$ (in place of γ_1) , $\sigma_{2,n}>0$, $\delta_n>0$ (in place of δ), $\mathcal{G}_n\subset C(X)$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_n\subset \underline{K}(C(X))$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{H}_{2,n}\subset C(X)_{s.a.}$ (in place of $\mathcal{H}_2$) is a finite subset, $\mathcal{U}_n\subset U_\infty(C(X))/CU_\infty(C(X))$ (in place of $\mathcal{U}$) for which $[\mathcal{U}_n]\subset\mathcal{P}_n$ is a finite subset required by 12.7 of [12] for ϵ_n and $\mathcal{F}_n$, $n=1,2,\ldots$. We may assume that $\{\mathcal{H}_{1,n}\}$ and $\{\mathcal{P}_n\}$ are increasing. By applying Lemma 4.5, one obtains a sequence of unital contractive completely positive linear maps $\varphi_n: C(X)\to B$ such that φ_n is δ_n-$\mathcal{G}_n$-multiplicative,

$$[\varphi_n]|_{\mathcal{P}_n}=[\kappa]|_{\mathcal{P}_n}, \tag{50}$$

$$\max_{\tau\in T(A)}|\tau\circ\varphi_n(f)-\lambda(\tau)(f)|<\sigma_{1,n} \quad \text{for all } f\in\mathcal{H}_{2,n} \quad \text{and} \tag{51}$$

$$\mathrm{dist}(\overline{\langle\varphi_n(v)\rangle},\gamma(\bar v))<\sigma_{2,n} \quad \text{for all } v\in\mathcal{U}_n, \tag{52}$$

$n=1,2,\ldots$. By 4.6, we may also assume that

$$\tau\circ\psi_n(g)\ge\Delta(\hat g) \quad \text{for all } g\in\mathcal{H}_{1,n}. \tag{53}$$

It follows from 12.7 of [12] that there exists a sequence of unitaries $u_n\in B$ such that

$$\|\mathrm{Ad}\, u_n\circ\varphi_{n+1}(f)-\varphi_n(f)\|<\epsilon_n \quad \text{for all } f\in\mathcal{F}_n\ n=1,2,\ldots. \tag{54}$$

Define $\psi_1 = \varphi_1$, $\psi_2 = \mathrm{Ad}\, u_1 \circ \varphi_1$, $\psi_3 = \mathrm{Ad}\, u_2 \circ \psi_2$, and $\psi_{n+1} = \mathrm{Ad}\, u_n \circ \psi_n$, $n = 1, 2, \ldots$. Then, by (54),

$$\|\psi_{n+1}(f) - \psi_n(f)\| < \epsilon_n \quad \text{for all } f \in \mathcal{F}_n \ n = 1, 2, \ldots. \tag{55}$$

Since $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $\{\mathcal{F}_n\}$ is an increasing sequence and $\cup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $C(X)$, it is easy to see that $\{\psi_n(f)\}$ is Cauchy for every $f \in C(X)$. Let $h(f) = \lim_{n\to\infty} \psi_n(f)$ for $f \in C(X)$. It is clear that $h : C(X) \to B$ is a unital homomorphism. Moreover,

$$[h] = \kappa \text{ in } KL(C(X), B) \ h_T = \lambda \quad \text{and} \quad h^{\ddagger} = \gamma. \tag{56}$$

Since $\gamma(\tau) \in T_{\mathbf{f}}(C(X))$ for each $\tau \in T(B)$, h is also injective. □

5. The Main Results

We begin with the following lemma.

Lemma 5.1. *Let C be a unital separable amenable C^*-algebra and let $\alpha \in Aut(C)$ be such that $\tau(c) = \tau(E(c))$ for all $c \in C \rtimes_\alpha \mathbb{Z}$ and for all $\tau \in T(C \rtimes_\alpha \mathbb{Z})$, where E is the canonical conditional expectation. Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. Suppose that $\lambda : T(A) \to T(C \rtimes_\alpha \mathbb{Z})$ is a surjective continuous affine map. Suppose $\varphi : C \to A$ is a unital monomrophism and $\psi_n : C \rtimes_\alpha \mathbb{Z} \to A$ is a contractive completely positive linear map, $n = 1, 2, \ldots,$ such that*

$$\lim_{n\to\infty} \|\psi_n(a)\psi_n(b) - \psi_n(ab)\| = 0 \quad \textit{for all } a, b \in C \rtimes_\alpha \mathbb{Z}, \tag{57}$$

$$\lim_{n\to\infty} \|\psi_n(c) - \varphi(c)\| = 0 \quad \textit{for all } c \in C \textit{ and} \tag{58}$$

$$\tau \circ \varphi(a) = \lambda(\tau)(a) \quad \textit{for all } a \in C \textit{ and } \tau \in T(A). \tag{59}$$

Then, for any $\epsilon > 0$ and any finite subset set $\mathcal{F} \subset C \rtimes_\alpha \mathbb{Z}$, there exists $N \geq 1$ such that

$$\sup_{\tau \in T(A)} |\tau \circ \psi_n(a) - \lambda(\tau)(a)| < \epsilon \quad \textit{for all } a \in \mathcal{F} \tag{60}$$

and for all $n \geq N$.

Proof. We first show that, for any $\tau \in T(A)$,

$$\lim_{n\to\infty} |\tau \circ \psi_n(c) - \lambda(\tau)(c)| = 0 \quad \text{for all } c \in C \rtimes_\alpha \mathbb{Z}. \tag{61}$$

Fix $\tau \in T(A)$. If (61) fails, then there exist at least one $c \neq 0$ in $C \rtimes_\alpha \mathbb{Z}$, one $\tau \in T(C \rtimes_\alpha \mathbb{Z})$ and a subsequence $\{n_k\}$ such that

$$\liminf_k |\tau \circ \psi_{n_k}(c) - \lambda(\tau)(c)| = \eta > 0. \tag{62}$$

Since the state space of $C \rtimes_\alpha \mathbb{Z}$ is weak*-compact, one can choose a limit point t of $\{\tau \circ \psi_{n_k}\}$. Then there exists a sequence $\{n'_k\} \subset \{n_k\}$ such that

$$\lim_{k\to\infty} |\tau \circ \psi_{n'_k}(c) - t(c)| = 0 \quad \text{for all } c \in C. \tag{63}$$

By (57), t is a tracial state. Let $E : C \to C(X)$ be the canonical conditional expectation. Then, by the assumption, $t(c) = t(E(c))$ for all $c \in C$. By combining with (63),

$$\lim_{n\to\infty} |\tau \circ \psi_{n'_k}(c) - \tau \circ \psi_{n'_k}(E(c))| = 0 \quad \text{for all } c \in C. \tag{64}$$

However, by (58) and by (64)

$$\begin{aligned}\lim_{n\to\infty} |\tau \circ \psi_{n'_k}(c) - \lambda(\tau)(c)| &= \lim_{n\to\infty} |\tau \circ \psi_{n'_k}(c) - \lambda(\tau)(E(c))| \\ &= \lim_{n\to\infty} |\tau \circ \psi_{n'_k}(c) - \tau \circ \varphi(E(c))| \\ &= \lim_{n\to\infty} |\tau \circ \psi_{n'_k}(E(c)) - \tau \circ \varphi(E(c))| = 0\end{aligned} \tag{65}$$

for all $c \in C$. This contradicts with (62). So the claim is proved.

Suppose that the lemma is not true. There exist $\epsilon_0 > 0$, a finite subset $\mathcal{F}$, a sequence of tracial states $\{\tau_k\} \subset T(B)$, and an increasing sequence $\{n_k\}$ of integers such that

$$\max_{a\in\mathcal{F}} |\tau_k \circ \psi_{n_k}(a) - \lambda(\tau_k)(a)| \geq \epsilon_0 \tag{66}$$

for all $k \geq 1$. We will again use the fact that the state space of $C \rtimes_\alpha \mathbb{Z}$ is weak*-compact. Let t_0 be a limit point of $\{\tau_k \circ \psi_{n_k}\}$ in $S(C \rtimes_\alpha \mathbb{Z})$. It follows from (57) that $t_0 \in T(C \rtimes_\alpha \mathbb{Z})$. For each $c \in C \rtimes_\alpha \mathbb{Z}$,

$$\lim_{k\to\infty} |\tau_k \circ \psi_{n_k}(c) - t_0(c)| = 0 \quad \text{for all } c \in C \rtimes_\alpha \mathbb{Z}. \tag{67}$$

However, $t_0(c) = t_0(E(c))$ for any $c \in C \rtimes_\alpha \mathbb{Z}$. This implies that

$$\lim_{k\to\infty} \tau_k \circ \psi_{n_k}(E(c)) = t_0(E(c)) = t_0(c) \quad \text{for all } c \in C \rtimes_\alpha \mathbb{Z}. \tag{68}$$

It follows that

$$\lim_{k\to\infty} |\tau_k \circ \psi_{n_k}(c) - t_k \circ \psi_{n_k}(E(c))| = 0 \quad \text{for all } c \in C \rtimes_\alpha \mathbb{Z}. \tag{69}$$

On the other hand, by (58),

$$\lim_{k\to\infty} |\tau_k \circ \psi_{n_k}(E(c)) - \lambda(\tau_k)(c)| = \lim_{k\to\infty} |\tau_k \circ \varphi(E(c)) - \lambda(\tau_k)(E(c))| = 0 \tag{70}$$

for all $c \in C \rtimes_\alpha \mathbb{Z}$. In particular, for any $a \in \mathcal{F}$,

$$\lim_{k\to\infty} |\tau_k \circ \psi_{n_k}(a) - \lambda(\tau_k)(a)| = 0. \tag{71}$$

This contradicts with (66). □

We now prove the following:

Theorem 5.2. *Let X be an infinite compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism. Then $gTR((C(X) \rtimes_\alpha \mathbb{Z}) \otimes U) \leq 1$ for any UHF-alegbra U of infinite type.*

Proof. By [24], it suffices to show that $gTR((C(X) \rtimes_\alpha \mathbb{Z}) \otimes Q) \le 1$. Let $C = C(X) \rtimes_\alpha \mathbb{Z}$ and $C_1 = C \otimes Q$. We will use the result in [7] that C_1 has finite nuclear dimension (when X has finite covering dimension, it was proved in [34]).

Note that C_1 is a unital separable simple amenable $\mathcal{Z}$-stable C^*-algebra. By the Range Theorem (Theorem 13.41) in [12], there is a unital separable simple C^*-algebra A in UCT class with $gTR(A) \le 1$ such that

$$\begin{aligned}(K_0(C_1), K_0(C_1)_+, [1_{C_1}], K_1(C_1), T(C_1), r_{C_1})\\ = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A).\end{aligned} \tag{72}$$

Since $C_1 \cong C_1 \otimes Q$, we may assume that $A \cong A \otimes Q$.

Let $\kappa \in KL_e(C_1, A)^{++}$ which gives the part of the above identification:

$$(K_0(C_1), K_0(C_1)_+, [1_{C_1}], K_1(C_1)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Let $\lambda : T(A) \to T(C_1)$ be an affine homeomorphism given by (72) which is compatible with κ. Put $\kappa_0 = \kappa|_{K_0(C_1)}$ and $\kappa_1 = \kappa|_{K_1(C_1)}$. Let $\imath_T : T(C_1) \to T_{\mathbf{f}}(C(X))$ be defined by $\imath_T(\tau)(f) = \tau(\imath(f))$ for all $\tau \in T(C_1)$ and for all $f \in C(X)$, where $\imath : C(X) \to C$ is the embedding. By theorem 4.7, there exists a unital monomorphism $\varphi' : C(X) \to A$ such that

$$[\varphi'] = \kappa \circ [\imath], \ \ \varphi'_T = \imath_T \circ \lambda \quad \text{and} \quad (\varphi')^\rho = 0. \tag{73}$$

Consider $\psi = \varphi' \circ \alpha : C(X) \to A$. Then $[\imath \circ \alpha] = [\imath]$, $(\imath \circ \alpha)_T = \imath_T$. It follows that

$$[\psi] = [\varphi'], \psi_T = \varphi'_T. \tag{74}$$

Note that since $\psi^\rho = \varphi^\rho \circ \alpha_{*1}$,

$$\psi^\rho = (\varphi')^\rho. \tag{75}$$

It follows from corollary 4.3 that there exists a sequence of unitaries $\{u_n\} \subset A$ such that

$$\lim_{n\to\infty} \|u_n^* \varphi'(f) u_n - \varphi' \circ \alpha(f)\| = 0 \quad \text{for all } f \in C(X). \tag{76}$$

Let C_0 be the subalgebra of C whose elements have the form $\sum_{i=-k}^k f_i u^i$, where $f_i \in C(X)$ and $u \in C$ is a unitary which implement the action α, i.e. $u^* f u = f \circ \alpha$ for all $f \in C(X)$. Define a linear map $L_n : C_0 \to A$ by

$$L_n\left(\sum_{i=-k}^k f_i u^i\right) = \sum_{i=-k}^k f_i u_n^i. \tag{77}$$

Let $L : C_0 \to l^\infty(A)$ be defined by $L(c) = \{L_n(c)\}$ and let $\pi : l^\infty(A) \to l^\infty(A)/c_0(A)$ be the quotient map. Then $\pi \circ L : C_0 \to l^\infty(A)/c_0(A)$ is a unital $*$-homomorphism. In particular, it is a covariant representation of $(C(X), \alpha)$. Thus $\pi \circ L$ gives a unital homomorphism $\Phi : C \to l^\infty(A)/c_0(A)$ such that $\Phi|_{C_0} = \pi \circ L$. Since C is amenable, there exists a contractive completely positive linear map $\Lambda : C \to l^\infty(A)$ such that $\pi \circ \Lambda = \Phi$. Let $\pi_n : l^\infty(A) \to A$ be the projection to the nth coordinate. Put $\varphi'_n = \pi_n \circ \Lambda$. Then $\varphi'_n : C \to A$ is a contractive completely positive linear map. Moreover, since $\pi \circ \Lambda = \Phi$, we have

$$\lim_{n\to\infty} \|L_n(c) - \varphi'_n(c)\| = 0 \quad \text{for all } c \in C. \tag{78}$$

In particular,

$$\lim_{n\to\infty} \|\varphi'_n(a)\varphi'_n(b) - \varphi'_n(ab)\| = 0 \quad \text{for all } a, b \in C \quad \text{and} \tag{79}$$

$$\lim_{n\to\infty} \|\varphi'_n(a) - \varphi(a)\| = 0 \quad \text{for all } a \in C(X). \tag{80}$$

Note that $\varphi_T = \imath_T \circ \lambda$, i.e.

$$\tau \circ \varphi(a) = \lambda(\tau)(a) \quad \text{for all } a \in C. \tag{81}$$

It follows from Lemma 5.1 that

$$\lim_{n\to\infty} \sup_{\tau \in T(A)} |\tau \circ \varphi'_n(c) - \lambda(\tau)(c)| = 0 \quad \text{for all } c \in C. \tag{82}$$

We then define $\varphi_n : C_1 \to A \otimes U \cong A$ by $\varphi_n(c \otimes a) = \varphi'_n(c) \otimes a$ for all $c \in C$ and $a \in U$. Then

$$\lim_{n\to\infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \quad \text{for all } a, b \in C_1, \tag{83}$$

$$\lim_{n\to\infty} \sup_{\tau \in T(A)} |\tau \circ \varphi_n(a) - \lambda(\tau)(a)| = 0 \quad \text{for all } a \in C_1. \tag{84}$$

Since C_1 has finite nuclear dimension, by Lemma 3.4, the above implies that $gTR(C_1) \le 1$. □

Corollary 5.3. *Let X be a compact metric space, let $\alpha : X \to X$ be a minimal homeomorphism and let $C = C(X) \rtimes_\alpha \mathbb{Z}$. Then $C \otimes \mathcal{Z} \in \mathcal{N}_1^{\mathcal{Z}}$.*

The proof of Theorems 1.2 and 1.1. It is proved in [8] that, when (X, α) is a minimal dynamical system with mean dimension zero, $C(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable. Thus Theorem 1.2 follows immediately from Corollary 5.3. When X has finite dimension, every minimal dynamical system (X, α) has mean dimension zero. So Theorem 1.1 follows.

The proof of Corollary 1.3. Let $C = C(X) \rtimes_\alpha \mathbb{Z}$. By 1.2, $C \in \mathcal{N}_1^{\mathcal{Z}}$. It follows from the Range Theorem (Theorem 13.41) in [12] that there exists a unital simple inductive limit A of sub-homogeneous C^*-algebras described in Remark 2.10 such that $\mathrm{Ell}(A) \cong \mathrm{Ell}(C)$. By the isomorphism theorem (Theorem 29.4) in [12], $C \cong A$.

The proof of Theorem 1.4. As mentioned earlier, this can be proved without using [12]. The "only if" part follows from [22]. For "if" part, let $C = C(X) \rtimes_\alpha \mathbb{Z}$. By [22] again, there is a unital simple amenable C^*-algebra A which satisfies the UCT and $A \otimes Q$ has tracial rank at most one such that $\mathrm{Ell}(A) \cong \mathrm{Ell}(C)$. Using this A, exactly the same proof of theorem 5.2 shows that $TR(C \otimes Q) \le 1$.

The following is of course a consequence of Theorem 5.2. But it is also a corollary of Theorem 1.4:

Corollary 5.4. *Let X be an infinite compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $K_0(C(X) \rtimes_\alpha \mathbb{Z})$ has a unique state. Then* $TR((C(X) \rtimes_\alpha \mathbb{Z}) \otimes Q) \le 1$.

Proof. Put $C = C(X) \rtimes_\alpha \mathbb{Z}$ and $C_1 = C \otimes Q$. Then $S_{[1_{C_1}]}(K_0(C_1))$ has a single point. Therefore it is a Choquet simplex. In particular, by 5.10 of [22], $K_0(C_1)$ is a dimension group. It is also clear that all extremal points of $T(C_1)$ maps to the extremal point of $S_{[1_{C_1}]}(K_0(C_1))$. So Theorem 1.4 applies. ◻

The above certainly applies to all cases that were studied in [20] (see Theorem 6.1 and 6.2 of [20]). It also applies to all connected X with torsion $K_0(C(X))$.

Theorem 5.5. *Let $C \in \mathcal{N}_1$ be a unital separable simple C^*-algebra and let $\alpha \in Aut(C)$ such that $\tau(a) = \tau(E(a))$ for all $a \in C \rtimes_\alpha \mathbb{Z}$ and $\tau \in T(C \rtimes_\alpha \mathbb{Z})$. Suppose that $(C \rtimes_\alpha \mathbb{Z}) \otimes U$ has finite nuclear dimension for some UHF-algebra U of infinite type. Then $C \rtimes_\alpha \mathbb{Z} \in \mathcal{N}_1$.*

Proof. The proof is almost identical to that of Theorem 5.2 except that we have to use a different existence and uniqueness theorems. Let $C_1 = C \rtimes_\alpha \mathbb{Z}$ and $C_2 = C_1 \otimes Q$. We will show that $gTR(C_2) \le 1$.

It follows from 4.4 of [15] that α^k is strongly outer for all integer $k \ne 0$. Therefore C_1 is simple (see Theorem 3.1 of [14]). Hence C_2 is a unital separable simple amenable $\mathcal{Z}$-stable C^*-algebra. By the Range Theorem (Theorem 13.41) in [12], there is a unital separable simple C^*-algebra A in UCT class with $gTR(A) \le 1$ such that

$$\begin{aligned}&(K_0(C_2), K_0(C_2)_+, [1_{C_2}], K_1(C_2), T(C_2), r_{C_2})\\ &\quad = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A). \qquad (85)\end{aligned}$$

Since $C_2 \cong C_2 \otimes Q$, we may assume that $A \cong A \otimes Q$.

Let $\kappa \in KL_e(C_2, A)^{++}$ which gives the part of the above identification:

$$(K_0(C_2), K_0(C_2)_+, [1_{C_2}], K_1(C_2)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Let $\lambda : T(A) \to T(C_2)$ be an affine homeomorphism above which is compatible with κ. Put $\kappa_0 = \kappa|_{K_0(C_2)}$ and $\kappa_1 = \kappa|_{K_1(C_2)}$. Let $\imath_T : T(C_2) \to T(C \otimes Q)$ be defined by $\imath_T(\tau)(c) = \tau(\imath(c))$ for all $\tau \in T(C_2)$ and for all $c \in C \otimes Q$, where $\imath : C \otimes Q \to C_1 \otimes Q$ is the embedding. Since (κ_0, λ) is compatible, λ induces an isomorphism

$$\bar{\lambda}_\sharp : \mathrm{Aff}(T(C_2))/\overline{\rho_{C_2}(K_0(C_2))} \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

Fix $J_c^{C_2}$ and J_c^A as in Definition 4.2. Define $\gamma : U(C\otimes Q)/CU(C\otimes Q) \to U(A)/CU(A)$ as follows:

$$\gamma|_{\mathrm{Aff}(T(C\otimes Q))/\overline{\rho_{C\otimes Q}(K_0(C\otimes Q))}} = \overline{(\imath_T\circ\lambda)_\sharp} \quad \text{and} \tag{86}$$

$$\gamma|_{J_c^{C\otimes Q}(K_1(C\otimes Q)} = J_c^A\circ\kappa_1\circ\imath_{*1}\circ(\pi_{K_1(C\otimes Q)})|_{J_c^{C\otimes Q}(K_1(C\otimes Q))}. \tag{87}$$

By Theorem 21.9 of [12], there exists a unital monomorphism $\varphi : C\otimes Q \to A$ such that

$$[\varphi] = \kappa\circ[\imath],\ \varphi_T = \imath_T\circ\lambda \quad \text{and} \quad (\varphi)^\ddagger = \gamma. \tag{88}$$

Define $\beta = \alpha\otimes \mathrm{id}_U : C\otimes Q\to C\otimes Q$ and consider $\psi = \varphi\circ\beta : C\otimes Q\to A$. Then $[\imath\circ\alpha] = [\imath]$, $(\imath\circ\alpha)_T = \imath_T$. It follows that

$$[\psi] = [\varphi], \psi_T = \varphi_T. \tag{89}$$

Note, since $\imath_{*1}\circ\beta_{*1} = \imath_{*1}$,

$$\begin{aligned}
\psi^\ddagger|_{\mathrm{Aff}(T(C\otimes Q))/\overline{\rho_{C\otimes Q}(K_0(C\otimes Q))}} &= \bar\lambda_\sharp = \varphi^\ddagger|_{\mathrm{Aff}(T(C\otimes Q))/\overline{\rho_{C\otimes Q}(K_0(C\otimes Q))}} \quad \text{and}\\
\psi^\ddagger|_{J_c^{C\otimes Q}(K_1(C\otimes Q))} &= \varphi^\ddagger\circ\beta^\ddagger|_{J_c^{C\otimes Q}(K_1(C\otimes Q))}\\
&= J_c^A\circ\kappa_1\circ\imath_{*1}\circ\beta_{*1}\circ\pi_{K_1}^{C\otimes Q}|_{J_c^{C\otimes Q}(K_1(C\otimes Q))}\\
&= J_c^A\circ\kappa_1\circ\imath_{*1}\circ\pi_{K_1}^{C\otimes Q}|_{J_c^{C\otimes Q}(K_1(C\otimes Q))}\\
&= \varphi^\ddagger|_{J_c^{C\otimes Q}(K_1(C\otimes Q))}.
\end{aligned} \tag{90}$$

It follows that $\varphi^\ddagger = \psi^\ddagger$.

By the uniqueness theorem (Theorem 12.11) of [12], there exists a sequence of unitaries $\{u_n\}\subset A$ such that

$$\lim_{n\to\infty}\|u_n^*\varphi'(f)u_n - \varphi'\circ\alpha(f)\| = 0 \quad \text{for all } f\in C\otimes Q. \tag{91}$$

Let C_0 be a subalgebra of C whose elements have the form $\sum_{i=-k}^k f_iu^i$, where $f_k\in C\otimes Q$ and $u\in C\otimes Q\rtimes_\beta\mathbb{Z}$ is a unitary which implement the action β, i.e. $u^*fu = f\circ\beta$ for all $f\in C\otimes Q$. Define a linear map $L_n : C_0\to A$ by

$$L_n\left(\sum_{i=-k}^k f_iu^i\right) = \sum_{i=-k}^k f_iu_n^i. \tag{92}$$

Since $C\in\mathcal{N}_1$, $C\otimes Q$ is in the so-called bootstrap class. Therefore so Theorem is $C_2\cong(C\otimes Q)\rtimes_\beta\mathbb{Z}$. In particular, C_2 is amenable. The same argument used in the proof of Theorem 5.2 shows that there exists a sequence of contractive completely positive linear map $\varphi_n : C_2\to A$ such that

$$\lim_{n\to\infty}\|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \quad \text{for all } a,b\in C_2 \quad \text{and} \tag{93}$$

$$\lim_{n\to\infty}\|\varphi_n(a) - \varphi(a)\| = 0 \quad \text{for all } a\in C\otimes Q. \tag{94}$$

Note that $\varphi_T = \imath_T \circ \lambda$, i.e.

$$\tau \circ \varphi(a) = \lambda(\tau)(a) \quad \text{for all } a \in C \otimes Q. \tag{95}$$

It follows from Lemma 5.1 that

$$\lim_{n\to\infty} \sup_{\tau \in T(A)} |\tau \circ \varphi_n(c) - \lambda(\tau)(c)| = 0 \quad \text{for all } c \in C. \tag{96}$$

Since C_2 has finite nuclear dimension, by Lemma 3.4, $gTR(C_2) \le 1$. In other words, $C \rtimes_\alpha \mathbb{Z} \in \mathcal{N}_1$. □

Remark 5.6. Let C be a unital simple separable C^*-algebra and $\alpha \in Aut(C)$. By a refined argument of Kishimoto, as in Remark 2.8 of [27], if α has weak Rokhlin property, then $\tau(au^k) = 0$ for any $a \in C$ and any integer $k \neq 0$. Thus $\tau(a) = \tau(E(a))$ for all $a \in C \rtimes_\alpha \mathbb{Z}$ and all $\tau \in T(C \rtimes_\alpha \mathbb{Z})$, where $E : C \rtimes_\alpha \mathbb{Z} \to C$ is the canonical conditional expectation. On the other hand, if $\tau(a) = \tau(E(a))$ for all $a \in C \rtimes_\alpha \mathbb{Z}$ and all $\tau \in T(C \rtimes_\alpha \mathbb{Z})$, by the proof of 4.4 of [15], the $\mathbb{Z}$ action induced by α is strongly outer (see also Remark 2.8 of [27]). Thus, in Theorem 5.5, if in addition, $C \in \mathcal{N}_1^{\mathcal{Z}}$, then by Corollary 4.10 of [28], $C \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable. In light of this, it seems that the condition $\tau(a) = \tau(E(a))$ for all $a \in C \rtimes_\alpha \mathbb{Z}$ is a reasonable replacement for some version of weak Rokhlin property.

Acknowledgment

This work was partially supported by The Research Center for Operator Algebras at East China Normal University which is partially funded by a Chinese NSF grant (No. 11531003) and by the Science and Technology Commission of Shanghai Municipality (grant No. 13dz2260400) and by an NSF grant (DMS-1361431).

References

1. B. Blackadar and D. Handelman, Dimension functions and traces on C^*-algebras, *J. Funct. Anal.* **45** (1982) 297–340.
2. N. Brown and A. Toms, Three applications of the Cuntz semigroup, *Int. Math. Res. Not.* **2007** (2007) rnm068.
3. M-D. Choi, G. A. Elliott and N. Yui, Gauss polynomials and the rotation algebra, *Invent. Math.* **99** (1990) 225–246.
4. S. Eilers, T. A. Loring and G. K. Pedersen, Stability of anticommutation relations: an application of noncommutative CW complexes, *J. Reine Angew. Math.* **499** (1998) 101–143.
5. G. A. Elliott and D. E. Evans, The structure of the irrational rotation C^*-algebra, *Ann. Math.* **138** (1993) 477–501.
6. G. A. Elliott and G. Gong, On the classification of C^*-algebras of real rank zero. II, *Ann. Math.* **144** (1996) 497–610.
7. G. A. Elliott, Z. Niu, L. Santiago, A. P. Tikuisis and W. Winter, Decomposition rank of minimal crossed products, draft preprint, 2015.
8. G. A. Elliott and Z. Niu, The C^*-algebra of a minimal homeomorphism of zero mean dimension, arXiv:1406.2382.

9. H. Freudenthal, Entwicklungen yon Rdumen und ihren Gruppen, *Compositio Math.* **4** (1937) 145–234.
10. H. Furstenberg, Strict ergodicity and transformation of the torus, *Amer. J. Math.* **83** (1961) 573–601.
11. T. Giordano, I. Putnam and C. Skau, Topological orbit equivalence and C^*-crossed products, *J. Reine Angew. Math.* **469** (1995) 51–111.
12. G. Gong, H. Lin and Z. Niu, Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, arXiv:1501.00135.
13. X. Jiang and H. Su, On a simple unital projectionless C^*-algebra, *Amer. J. Math.* **121** (1999) 359–413.
14. A. Kishimoto, Outer automorphisms and reduced crossed products of simple C^*-algebras, *Comm. Math. Phys.* **81** (1981) 429–435.
15. A. Kishimoto, The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras, *J. Funct. Anal.* **140** (1996) 100–123.
16. H. Lin, Classification of simple C^*-algebras of tracial topological rank zero, *Duke Math. J.* **125** (2004) 91–119.
17. H. Lin, Asymptotic unitary equivalence and classification of simple amenable C^*-algebras, *Invent. Math.* **183** (2011) 385–450.
18. H. Lin, Localizing the Elliott conjecture at strongly self-absorbing C^*-algebras, II, *J. Reine Angew. Math.* **692** (2014) 233–243.
19. H. Lin, Locally AH-algebras, *Mem. Amer. Math. Soc.* **235** (2015) No. 1107.
20. H. Lin, Minimal dynamical systems on connected odd dimensional spaces, *Can. J. Math.* **67** (2015) 870–892.
21. H. Lin and Z. Niu, Lifting KK-elements, asymptotic unitary equivalence and classification of simple C^*-algebras, *Adv. Math.* **219** (2008) 1729–1769.
22. H. Lin and Z. Niu, The range of a class of classifiable separable simple amenable C^*-algebras, *J. Funct. Anal.* **260** (2011) 1–29.
23. H. Lin and N. C. Phillips, Crossed products by minimal homeomorphisms, *J. Reine Angew. Math.* **641** (2010) 95–122.
24. H. Lin and W. Sun, Tensor products of classifiable C^*-algebras, arXiv:1203.3737.
25. Q. Lin and N. C. Phillips, The structure of C^*-algebras of minimal diffeomorphisms, preprint.
26. S. Mardesic, On covering dimension and inverse limits of compact spaces, *Illinois J. Math.* **4** (1960) 278–291.
27. H. Matui and Y. Sato, $\mathcal{Z}$-stability of crossed products by strongly outer actions, *Comm. Math. Phys.* **314** (2012) 193–228.
28. H. Matui and Y. Sato, $\mathcal{Z}$-stability of crossed products by strongly outer actions II, *Amer. J. Math.* **136** (2014) 1441–1496.
29. L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, *Adv. Math.* **231** (2012) 2802–2836.
30. K. Strung, C^*-algebras of minimal dynamical systems of the product of a Cantor set and an odd dimensional sphere, *J. Funct. Anal.* to appear, arXiv:1403.3136.
31. K. R. Strung and W. Winter, UHF-slicing and classification of nuclear C^*-algebras, *J. Topol. Anal.* **6** (2014) 465–540.
32. K. Thomsen, Traces, unitary characters and crossed products by Z, *Publ. Res. Inst. Math. Sci.* **31** (1995) 1011–1029.
33. J. Tomiyama, Topological full groups and structure of normalizers in transformation group C^*-algebras, *Pacific J. Math.* **173** (1996) 571–583.
34. A. Toms and W. Winter, Minimal dynamics and the classification of C^*-algebras, *Proc. Natl. Acad. Sci. USA* **106** (2009) 16942–16943.

35. W. Winter, On the classification of simple $\mathcal{Z}$-stable C^*-algebras with real rank zero and finite decomposition rank, *J. London Math. Soc.* **74** (2006) 167–183.
36. W. Winter, Localizing the Elliott conjecture at strongly self-absorbing C^*-algebras, *J. Reine Angew. Math.* **692** (2014) 193–231.
37. W. Winter, Classifying crossed product C^*-algebras, arXiv:1308.5084.