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## Special Issue Editorial: Small Scales and Singularity Formation in Fluid Dynamics

Alexander Kiselev<sup>1</sup>

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Fluids are all around us, and attempts to understand fluid motion go back many centuries. Anyone witnessing dramatic phenomena like tornado or hurricane or even an everyday river flow or ocean wave breaking can easily imagine the complexity of the task. There has been tremendous accumulation of knowledge in the field, yet it is remarkable that some of the fundamental properties of key equations of fluid mechanics remain poorly understood. Many equations of fluid mechanics are both nonlinear and nonlocal, meaning that the behavior of solution over the entire region influences what happens at every point. This makes mathematical analysis of these equations quite challenging.

A special role in fluid mechanics is played by the incompressible Euler and Navier–Stokes equations. These equations model a broad spectrum of fluid behavior, and are widely used in science and engineering. One of the key questions that remain open for the three dimensional Euler and Navier–Stokes equations is whether singularities can spontaneously form in initially smooth solutions. This question for the Navier–Stokes equation is one of the celebrated Millennium problems of Clay Institute. To answer this question, completely new insight into nonlinear workings of the equation is likely necessary. More broadly, one can ask a question about formation of small scales: fine structures that appear for example when a laminar solution loses stability, and in fully developed turbulence. The mechanisms behind creation of these structures, as well as their effect on key aspects of evolution such as for instance mixing efficiency, are not fully understood even in two dimensions.

The main goal of this issue is to present an overview of some recent developments in the area. The collection of articles is largely focused on singularity formation in

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✉ Alexander Kiselev  
kiselev@math.duke.edu

<sup>1</sup> Department of Mathematics, Duke University, Durham, NC 27708, USA

equations of fluid mechanics and associated models, questions of stability analysis, mixing, and modeling of turbulence. These articles are not intended to provide a comprehensive review of this exciting and active domain, but rather to demonstrate recent advancements in nonlinear science research in the areas of small-scale creation and stability analysis in fluid mechanics.

Bedrossian et al. consider the problem of stability for shear flows close to Couette flows for the Navier–Stokes equations in two dimensions. The authors consider perturbations that are small in Sobolev spaces. They provide an estimate on the stability neighborhood (in terms of the viscosity coefficient) and show long time convergence to a decaying shear flow. An interesting aspect of their results is an enhanced dissipation estimate for the components of the flow orthogonal to the functions depending on the variable of the shear. This effect exceeds pure heat equation relaxation and is a consequence of mixing imposed by the shear flow.

Cordoba and Pernas-Castano analyze singularity formation in Muskat problem, which models fluid flow in porous medium. They consider an interface between vacuum and fluid in porous medium with two different permeabilities. The authors consider two types of scenario for singularity formation at the interface: splash, where the interface may self-intersect at a single point, and splat, where the intersection is along an arc. They show that starting with a sufficiently regular interface satisfying Raleigh–Taylor condition one can never arrive at a splat singularity, but that there exist initial data leading to a finite-time splash singularity.

Do et al. consider a one-dimensional model of the Hou–Luo blowup scenario for the 3D Euler equation. The model is a generalization of the one-dimensional model suggested by Hou and Luo, for which finite time blow has been proved. The authors first study a model including lower order terms that more closely aligns with the Hou–Luo scenario. Then they introduce a broader class of models and prove that finite-time blowup happens in all of these generalized models. This suggests a certain structural stability of the blowup dynamics, providing an additional argument in favor that the scenario indeed persists in higher dimension.

Doering and Miles analyze mixing by incompressible flow in the context of advection–diffusion equation. Mixing is most commonly measured by decay of the negative Sobolev norms of the solution. The goal is to derive optimal bounds on mixing under reasonable constraints on the flow (such as fixed energy or enstrophy), with or without diffusion. The authors develop a shell model to get insight into these questions. The model leads to results consistent with the full PDE setting in cases where analogous results are available—for instance, when there is no diffusion, perfect mixing in finite time is possible for energy constrained flows and exponential decay applies for the enstrophy constrained case. The model exhibits interesting properties for the case with diffusion, where finite-time perfect mixing is no longer possible for the energy constrained flow.

Feng et al. consider the bipolar compressible Navier–Stokes–Maxwell equations. This system models motion of charged fluids, such as for example plasmas, and consists of coupled equations for fluid, electric, and magnetic fields as well as for the energy transfer. The authors construct small amplitude non-trivial steady state solutions to the system. Then they show that there exists a small neighborhood of these stationary

states, in an appropriate metric, where solutions stay globally regular. Moreover, they converge in time to the corresponding steady solution.

Hou et al. study a family of models of the incompressible Euler and Navier–Stokes equations which are obtained by making pure convective terms in the equations weaker. The models are a generalization of the earlier model proposed by Hou and Lei, and can be seen as interpolating between the Hou–Lei model and the original equations. The authors prove a number of results for the whole spectrum of the models establishing sufficient conditions for global regularity, such as an analog of the Beale–Kato–Majda criterion. The authors also analyze the models numerically, and find that for the case of weak convection term, numerical simulations suggest finite-time blowup in the inviscid setting. The blowup solution appears to have self-similar features, and the authors find approximately the scaling and key exponents characterizing blowup. For stronger advection, the scenario under analysis does not seem to lead to singularity formation.

Hu and Sverak develop a family of models inspired by fluid turbulence. Their starting point is a geometric point of view on fluid motion as a trajectory in the group of volume preserving diffeomorphisms of the domain containing the fluid. Physical observations and conjectures on the nature of turbulent flows include, in particular, nonlinearity-enabled energy flux and propagation from a few randomly forced Fourier modes to other modes in the system. An example of rigorous result in this direction is the well-known Hairer–Mattingly theorem on ergodicity of the stochastically forced 2D Navier–Stokes equation. Mathematical mechanism behind this effect is hypoellipticity of the system, based on the action of nonlinearity and analysis of Hörmander condition. The authors consider simplified models of the process, where the group of diffeomorphisms is replaced by a finite-dimensional Lie group. They establish a sharp algebraic necessary and sufficient condition on the structure of the group and stochastic forcing leading to ergodicity and convergence to equilibrium.

Zlatoš considers solutions of the 2D Euler equation on a disk, and constructs an example where two level sets of vorticity approach each other along a curve at an exponential-in-time rate. Geometry of the scenario considered by Zlatoš involves odd symmetry of the solution, and a hyperbolic point of the flow at the boundary. The vorticity level sets are squeezed by the hyperbolic flow, and the convergence of the level sets happens in the bulk of the fluid, away from the boundary. Zlatoš also shows that under a number of additional assumptions, which are plausible but difficult to establish rigorously, the rate of growth can be improved to super-exponential or even double exponential in time.

This collection demonstrates the richness of the area and the key role nonlinear analysis plays in advancing our understanding of fluid motion. From small-scale creation to stability analysis and models of turbulence, nonlinearity is the driving force behind the observed phenomena. The collection also illustrates some of the very recent advances in the field. These include, for instance, new advances in understanding possible singularity formation in interface problems and in 3D fluids or sharp novel threshold estimates in classical problems on stability. In many cases, these exceedingly difficult problems have been approached by a combination of cutting edge numerical simulations providing critical insight into the nature of dynamics as well as new analytic ideas capable of detailing complex mechanisms involved at least on the level of con-

crete scenarios with additional constraints and symmetries. Many exciting challenges lie ahead, and we hope that this special issue will serve as an inspiration for more researchers to join the effort in this vibrant field.