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Analysis of a singular Boussinesq model

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Abstract

Recently, a new singularity formation scenario for the 3D axi-symmetric Euler equation and the 2D inviscid Boussinesq system has been proposed by Hou and Luo (Multiscale Model Simul 12(4):1722–1776, 2014, PNAS 111(36):12968–12973, 2014) based on extensive numerical simulations. As the first step to understand the scenario, models with simplified sign-definite Biot–Savart law and forcing have recently been studied in Choi et al. (Commun Pure Appl Math 70(11):2218–2243, 2017, Commun Math Phys 334:1667–1679, 2015), Do et al. (J Nonlinear Sci, 2016. arXiv:1604.07118), Hoang et al. (J Differ Equ 264:7328–7356, 2018), Hou and Liu (Res Math Sci 2, 2015), Kiselev and Tan (Adv Math 325:34–55, 2018). In this paper, we aim to bring back one of the complications encountered in the original equation—the sign changing kernel in the Biot–Savart law. This makes analysis harder, as there are two competing terms in the fluid velocity integral whose balance determines the regularity properties of the solution. The equation we study here is based on the CKY model introduced in Choi et al. (2015). We prove that finite time blow up persists in a certain range of parameters.

1 Introduction

The 2D inviscid Boussinesq system in vorticity form is given by

$$\partial_t \omega + (u \cdot \nabla)\omega = \rho_{x_1}, \tag{1}$$

$$\partial_t \rho + (u \cdot \nabla)\rho = 0, \tag{2}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\omega. \tag{3}$$

The system models ideal fluid driven by buoyancy force [10,19]. Solutions to the 2D Boussinesq system are globally regular if the dissipative terms $\Delta\omega$, $\Delta\rho$ are present in at least one of Eqs. (1), (2), respectively [4,12]. Models with fractional and/or partial diffusion have also been considered in [1,2,8,9,21,22], where the authors show global regularity under various conditions and constraints. In the inviscid case, the finite time blow up vs global regularity question is open; in particular, it appears on the "Eleven Great Problems of Mathematical Hydrodynamics" list by Yudovich [23]. Also, the 2D inviscid Boussinesq system is very similar to the 3D axi-symmetric Euler equation away from the symmetry axis [18]. In particular, the presence of ρ_{x_1} on the right-hand side of (1) enacts vortex stretching which is a common trait among the hardest problems of mathematical fluid mechanics, e.g. 3D Euler equations and 3D Navier–Stokes equations.

A few years ago, Hou and Luo [14] investigated numerically a new possible blow up scenario for the 3D axi-symmetric Euler equation. Their set-up involves an infinite height cylinder with no penetration boundary conditions on the cylinder boundary and periodic

boundary conditions in the vertical direction. The initial data ω^θ is zero, and u^θ is odd with respect to the z variable. Rapid growth of vorticity ω^θ is observed at a ring of hyperbolic points of the flow along the boundary in the $z = 0$ plane [14,15]. For the 2D Boussinesq system, the scenario involves (after $\pi/2$ rotation) an infinite horizontal strip with solutions periodic in x_1 and satisfying no penetration condition on the strip boundary. In the scenario, ω is odd and ρ is even with respect to x_1. Very fast growth of ω is observed at a hyperbolic point of the flow located at $x_1 = 0$ on the strip boundary. It should be noted that there is evidence that hyperbolic points of the flow play an important role in a number of important fluid mechanics phenomena. In particular, a recent experimental paper [20] shows that most instances of extreme dissipation in a turbulent flow happen in regions featuring hyperbolic point-/front-type local geometry of the flow.

Motivated by the Hou–Luo scenario, Kiselev and Sverak [16] considered 2D Euler equation—obtained by setting $\rho = 0$ in (2)—in a similar geometry. They constructed an example of a smooth solution with double exponential in time growth of the gradient of vorticity, showing that the upper bounds on growth of the derivatives of ω available since 1930s are qualitatively sharp.

A 1D model of the Hou–Luo scenario has been proposed already in [14]. Several works have analyzed this and a few other related models, in all cases proving finite time singularity formation [5–7,13]. All these models feature Biot–Savart laws $u(x,t) = -\int_0^\infty K(x,y)\omega(y,t)\,dy$ with non-negative kernels K. This helps prove transport of vorticity and density towards the origin, accompanied by growth in ρ_{x_1} leading to growth of vorticity and thus to nonlinear feedback loop driving blow up.

The first two-dimensional models of the Hou–Luo scenario have been considered in [11,17]. Both models are set in the first quadrant of the plane (implicitly assuming odd symmetry of the solution) and are given by

$$\partial_t \omega + (u\cdot\nabla)\omega = \frac{\rho}{x_1}, \tag{4}$$

$$\partial_t \rho + (u\cdot\nabla)\rho = 0, \tag{5}$$

$$u(x,t) = (-x_1\Omega(x,t), x_2\Omega(x,t)). \tag{6}$$

The models differ in the choice of Ω: in [11]

$$\Omega(x,t) = \int_{S_\alpha} \frac{\omega(y,t)}{|y|^2}dy,$$

where $S_\alpha = \{(x_1,x_2) : 0 < x_1, 0 < x_2 < \alpha x_1\}$ is a sector in the first quadrant with arbitrary large α as a parameter. In [17] a slightly different integration domain, $D = \{(y_1,y_2) : y_1y_2 \geq x_1x_2\}$ is chosen in the definition of Ω. The choice of the Ω in [17] leads to incompressible fluid velocity, while the velocity in [11] is not incompressible but is closer in form to the velocity representation for the 2D Euler solutions established in [16]. Also, both models use simplified mean field forcing term ρ/x_1, which ensures that vorticity has fixed sign. The initial data are taken smooth, and supported away from x_1 axis. In both works, finite time blow up is established for a fairly broad class of initial data.

Both of the above-mentioned modifications as well as all 1D models considered so far share the same feature that particle trajectories for positive vorticity solutions always point to one direction: towards the $x_1 = 0$ axis. However, in the true 2D Boussinesq system, the kernel in the Biot–Savart law is not sign definite. The fluid velocity is given, in a half plane $x_2 \geq 0$ and under the odd in x_1 symmetry assumption on ω, by

$$u_1(x,t) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty \left(\frac{x_2 - y_2}{(x_1-y_1)^2 + (x_2-y_2)^2} - \frac{x_2 - y_2}{(x_1+y_1)^2 + (x_2-y_2)^2} \right.$$
$$\left. - \frac{x_2 + y_2}{(x_1-y_1)^2 + (x_2+y_2)^2} + \frac{x_2 + y_2}{(x_1+y_1)^2 + (x_2+y_2)^2} \right) \omega(y,t) \mathrm{d}y_1\, \mathrm{d}y_2. \quad (7)$$

The second component u_2 is given by a similar formula. It is not hard to see that in (7) the kernel is positive on the part of integration region, and positive vorticity in these regions works against blow up.

In this paper, we propose a 1D model set on $\mathbb{R}$ given by

$$\partial_t \omega + u \cdot \partial_x \omega = \frac{\rho}{x}, \quad (8)$$

$$\partial_t \rho + u \cdot \partial_x \rho = 0, \quad (9)$$

$$u(x,t) = x \int_{\min(\beta_2 x, 1)}^{\min(\beta_1 x, 1)} \frac{\omega(y,t)}{y} \mathrm{d}y - x \int_{\min(\beta_1 x, 1)}^{1} \frac{\omega(y,t)}{y} \mathrm{d}y, \quad (10)$$

$$\omega(x,0) = \omega_0(x), \quad \rho(x,0) = \rho_0(x), \quad (11)$$

where $0 < \beta_2 \le 1 \le \beta_1 < \infty$ are two prescribed parameters. Note that we effectively limit the meaningful evolution to (0, 1) interval since we are interested in dynamics near zero. Extending integration in the Biot–Savart law beyond 1 does not add anything essential to the model since the kernel is regular in the added region, but leads to some technicalities associated with estimating growth of support of ω, ρ. In what follows below, for the sake of notational simplicity, we will omit the min condition in the limits of Biot–Savart integral. It should be always understood that if $\beta_{1,2} x \ge 1$, the integral limits are cut off at 1.

The model is close to the CKY model of [5], which can be obtained by setting $\beta_1 = \beta_2$ in (10) and replacing ρ/x_1 with ρ_{x_1}. The "anti-blow up" region is $(\beta_1 x, \beta_2 x)$, and it is the part of the integration region closest to $x = 0$, a feature that is also shared by (7). This region also tends to include the largest values of vorticity, making the overall balance highly nontrivial. The main purpose of this paper is to begin to assemble the technical tools needed for analysis of models with more complex Biot–Savart relationships, with the eventual goal of getting insight into the workings of the true Biot–Savart laws appearing in the key equations of fluid mechanics such as the 2D Boussinesq system or the SQG equation.

To set up local well-posedness theory, we will follow [17] and use the space K^n of compactly supported in (0, 1) functions. We say that $f \in K_n$ if

$$\|f\|_{K^n} := \|f\|_{C^n} + (\min_x \{\mathrm{supp}(f)\})^{-1} < \infty.$$

Here n is an integer. Note that $\|\cdot\|_{K_n}$ is not a norm, but this will not affect our arguments. This space is well adapted to the mean field forcing term in (8). We also denote $K^\infty = \bigcap_{n\ge 1} K^n$.

Theorem 1 *Given non-negative initial data* $(\omega_0, \rho_0) \in K^n((0,1)) \times K^n((0,1))$, $n \ge 1$, *there exists* $T = T(\omega_0, \rho_0)$ *such that the system* (8)–(11) *has a local-in-time unique solution* $(\omega, \rho) \in C([0,T], K^n((0,1)) \times C([0,T), K^n((0,1)))$.

Remark It is not difficult to remove the non-negative assumption on the initial data. We do not pursue the most general case here since proving finite time singularity formation is our main objective.

Theorem 2 *Assume that $\beta_2 \le 1 \le \beta_1 < 2\beta_2$. There exist compactly supported $(\omega_0, \rho_0) \in K^\infty((0,1)) \times K^\infty((0,1))$ such that the corresponding solution of (8)–(11) blows up in finite time in the sense that*

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty} \mathrm{d}t = \infty, \quad \int_0^T \|\partial_x \rho(\cdot,t)\|_{L^\infty} \mathrm{d}t = \infty, \quad \int_0^T \|\partial_x u(\cdot,t)\|_{L^\infty} \mathrm{d}t = \infty$$

for some $T \in (0,\infty)$.

Remark The assumption $\beta_2 \le 1 \le \beta_1 < 2\beta_2$ is necessary for the current argument to yield finite time singularity formation. It seems likely that this condition is not sharp, but new ideas are needed to improve the blow up parameter range.

2 Local well-posedness and continuation criteria

The proof of the local existence Theorem 1 for the model (8)–(11) can be carried out with essentially the same argument as in [17], so we will provide just a sketch of the proof for the sake of brevity. The key is to control the distance from the support of the solution to the origin. It is not hard to see that while this distance remains positive, the system (8)–(11) has well-controlled forcing and Biot–Savart law, and the solutions retain original regularity.

Denote this distance by

$$\delta(t) := \min_x \{\mathrm{supp}(\omega) \cup \mathrm{supp}(\rho)\}.$$

The next lemma explains how we can bound $\delta(t)$ away from zero for at least a short period of time. This is an a priori estimate; to properly show local existence of solutions and the associated bounds, one needs to use an iterative approximation scheme similar to [17]. Let $\Phi(x,t)$ be particle trajectories defined as usual by

$$\partial_t \Phi(x,t) = u(\Phi(x,t),t), \quad \Phi(x,0) = x. \tag{12}$$

Lemma 1 *Suppose that ω_0, ρ_0 are as in the assumption of Theorem 1, and let $\omega, \rho \in C(K_n, [0,T])$ solve (8)–(11). Write*

$$\Psi(x,t) = \sup_{s \le t} \log(1/\Phi(x,s)).$$

Then, $\Psi(x,t)$ satisfies

$$\partial_t \Psi(x,t) \le C\Psi(x,t)(1 + te^{\Psi(x,t)}), \quad \Psi(x,0) = \log x^{-1}. \tag{13}$$

Therefore, there exists $T > 0$ such that for $0 \le t \le T$, $\Psi(x,t)$ remains finite for all $x \in \mathrm{supp}(\omega_0, \rho_0)$.

Proof Solving the equations along trajectories Φ defined in (12) we obtain

$$\rho(x,t) = \rho_0(\Phi^{-1}(x,t)), \quad \omega(x,t) = \omega_0(\Phi^{-1}(x,t)) + \rho_0(\Phi^{-1}(x,t)) \int_0^t \frac{1}{\Phi(\Phi^{-1}(x,t),s)} \mathrm{d}s. \tag{14}$$

This in particular indicates preservation of non-negativity of ρ and ω by the evolution.

Due to positivity of ω and $\beta_1 \ge 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(x,t) \geq -\Phi(x,t)\int_{\Phi(x,t)}^{1}\frac{\omega(y,t)}{y}\mathrm{d}y \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\log(1/\Phi(x,t)) \leq \int_{\Phi(x,t)}^{1}\frac{\omega(y,t)}{y}\mathrm{d}y.$$

Now by (14)

$$\omega(y,t) \leq \|\omega_0\|_{L^\infty} + \|\rho_0\|_{L^\infty}\int_0^t \frac{1}{\Phi(\Phi^{-1}(y,t),s)}\mathrm{d}s \leq C\left(1+\int_0^t \frac{1}{\Phi(\Phi^{-1}(y,t),s)}\mathrm{d}s\right). \tag{15}$$

Also, if $y \in [\Phi(x,t),1]$, then $\Phi^{-1}(y,t) \in [x,1]$. Since the trajectories cannot cross while solution remains regular, we have

$$\frac{1}{\Phi(\Phi^{-1}(y,t),s)} \leq \frac{1}{\Phi(x,s)} \leq e^{\Psi(x,t)}.$$

Therefore,

$$\partial_t\Psi(x,t) \leq C\int_{\Phi(x,t)}^{1}\frac{1}{y}\left(1+\int_0^t e^{\Psi(x,t)}\mathrm{d}s\right)\mathrm{d}y \leq C\Psi(x,t)(1+te^{\Psi(x,t)})$$

yielding (13). □

Note that $\delta(t) = e^{-\Psi(\delta(0),t)}$, so Lemma 1 allows control of $\delta(t)$ for $t \leq T$ (and so implies regularity of the solution).

The proposition that we prove next is an analogue of the well-known result due to Beale et al. [3]. It will provide continuation criteria for solutions.

Proposition 1 *Let $n \geq 1$ be an integer. The following are equivalent:*

(a) *The solution $(\omega,\rho) \in C([0,T),K^n)\times C([0,T),K^n)$ can be continued past T,*
(b) $\int_0^T \|\partial_x u(\cdot,t)\|_{L^\infty}\mathrm{d}t < \infty$,
(c) $\int_0^T \|\partial_x \rho(\cdot,t)\|_{L^\infty}\mathrm{d}t < \infty$,
(d) $\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\mathrm{d}t < \infty$,
(e) $\liminf_{t\to T}\delta(t) > 0$.

Proof The equivalence of (a) and (e) follows from the definition of the norm K^n, and the above discussion on how positive $\delta(t)$ ensures local existence of solution in K^n on a time interval depending only on the size of δ. In fact, (e) implies all other conditions in the lemma by the argument mentioned above: the solution supported away from $x = 0$ uniformly in a given time interval maintains regularity by straightforward estimates.

Equivalence between (a) and (b) can be obtained through a standard argument based on the Lagrangian formulation of the system. Note that we only need to show (b) implies (a). A standard estimate on the trajectories, using the fact that the origin is a fixed point of the flow, yields

$$\delta'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\delta(0),t) \geq -\|\partial_x u(\cdot,t)\|_{L^\infty}\delta(t). \tag{16}$$

Thus by Gronwall,

$$\delta(t) \geq \delta(0)\exp\left(-\int_0^T \|\partial_x u(\cdot,t)\|_{L^\infty}\,\mathrm{d}t\right).$$

To prove the implication (b) $\Rightarrow$ (c) , differentiate (9) and compose with Φ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_x\rho(\Phi(x,t),t) = -\partial_x u(\Phi(x,t),t)\partial_x\rho(\Phi(x,t),t)$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x\rho\|_{L^\infty} \le \|\partial_x u\|_{L^\infty}\|\partial_x\rho\|_{L^\infty}$$

to which we can again apply Grönwall's inequality and establish (b) ⇒ (c).

The implication (c) ⇒ (d) follows from integrating (8) in Lagrangian coordinates and estimating (using $\rho(0,t)=0$, before blow up)

$$\begin{aligned}|\omega(\Phi(x,t),t)| &= \left|\omega_0(x) + \int_0^t \frac{\rho(\Phi(x,s),s)}{\Phi(x,s)}\mathrm{d}s\right| \\ &\le \|\omega_0\|_{L^\infty} + \int_0^t \frac{\|\partial_x\rho(\cdot,s)\|_{L^\infty}\cdot|\Phi(x,s)|}{|\Phi(x,s)|}\mathrm{d}s\end{aligned}$$

for all x.

To show (d) ⇒ (e), assume solution exists up to T and $\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\mathrm{d}t = M$. Observe that differentiating (10) we obtain

$$|\partial_x u(\Phi(x,t),t)| \le \left|\int_{\beta_2\Phi(x,t)}^1 \frac{\omega(y,t)}{y}\mathrm{d}y\right| + C\|\omega\|_{L^\infty} \le C\|\omega(\cdot,t)\|_{L^\infty}(1+|\log\Phi(x,t)|), \tag{17}$$

where C depends only on $\beta_{1,2}$. Taking $x=\delta(0)$ and combining (16) and (17), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta(t) \ge -C\|\omega(\cdot,t)\|_{L^\infty}\delta(t)(1+\log(1/\delta(t)))$$

So

$$\log\delta(t)^{-1} \le \log\delta(0)^{-1}e^{C\int_0^t\|\omega(\cdot,s)\|_{L^\infty}\mathrm{d}s} + \left(e^{C\int_0^t\|\omega(\cdot,s)\|_{L^\infty}\mathrm{d}s}-1\right),$$

finishing the proof. □

3 Warming-up: a special case with sign-definite Biot–Savart law

As a warm-up, let us first take a look at a special case of the model (8)–(11) by further simplifying the Biot–Savart law. Take $\beta_1=\beta_2=1$ and consider the following model on unit interval $[0,1]$:

$$\partial_t\omega + u\cdot\partial_x\omega = \frac{\rho}{x}, \tag{18}$$

$$\partial_t\rho + u\cdot\partial_x\rho = 0, \tag{19}$$

$$u(x,t) = -x\int_x^1 \frac{\omega(y,t)}{y}\mathrm{d}y, \tag{20}$$

$$\omega(x,0)=\omega_0(x), \quad \rho(x,0)=\rho_0(x). \tag{21}$$

The model then becomes close to the CKY model, but even easier due to simpler forcing term. The proof of blow up is very transparent.

Theorem 3 *There exists* $(\omega_0,\rho_0)\in K^\infty((0,1))\times K^\infty((0,1))$ *such that the corresponding solution of* (18)–(21) *blows up in finite time in the sense that*

$$\int_0^T\|\omega(\cdot,t)\|_{L^\infty}\mathrm{d}t=\infty, \quad \int_0^T\|\partial_x\rho(\cdot,t)\|_{L^\infty}\mathrm{d}t=\infty, \quad \int_0^T\|\partial_x u(\cdot,t)\|_{L^\infty}\mathrm{d}t=\infty$$

for some $T\in(0,\infty)$.

Proof Denote $I = (0, 1)$. Consider $\rho_0 \in C_0^\infty(I)$ such that $0 \le \rho_0 \le 1$ and $\rho_0 \equiv 1$ on $[\frac{1}{3}, \frac{2}{3}]$. For simplicity, choose $\omega_0 = 0$.

The idea is to control how the support of ρ_0 moves towards the origin. We assume that the solution stays regular and show that the characteristics originating at the points with nonzero ρ_0 arrive at the origin in finite time, thus implying that $\delta(t)$ becomes zero in finite time and then all other blow up characterizations of Proposition 1 hold.

Note that since ρ and ω are non-negative, trajectories always move in the negative x direction. Compute

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left(\frac{1}{\Phi(x,t)}\right) &= -\frac{\mathrm{d}\Phi(x,t)}{\mathrm{d}t}\cdot\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)} + \int_{\Phi(x,t)}^1 \frac{-u\partial_x\omega + \frac{\rho}{y}}{y}\mathrm{d}y\\ &= \omega(\Phi(x,t),t)\int_{\Phi(x,t)}^1 \frac{\omega(y,t)}{y}\mathrm{d}y - \frac{u\omega}{y}\Big|_{\Phi(x,t)}^1\\ &\quad + \int_{\Phi(x,t)}^1 \frac{\omega^2(y,t)}{y}\mathrm{d}y + \int_{\Phi(x,t)}^1 \frac{\rho(y,t)}{y^2}\mathrm{d}y\\ &= \int_{\Phi(x,t)}^1 \frac{\omega^2(y,t)}{y}\mathrm{d}y + \int_{\Phi(x,t)}^1 \frac{\rho(y,t)}{y^2}\mathrm{d}y\\ &\ge \int_{\Phi(x,t)}^1 \frac{\rho(y,t)}{y^2}\mathrm{d}y = \int_{\Phi(x,t)}^1 \frac{\rho_0(\Phi^{-1}(y,t))}{y^2}\mathrm{d}y\end{aligned}$$

Also

$$\int_{\Phi(\frac{1}{3},t)}^1 \frac{\rho_0(\Phi^{-1}(y,t))}{y^2}\mathrm{d}y \ge \int_{\Phi(\frac{1}{3},t)}^{\Phi(\frac{2}{3},t)} \frac{1}{y^2} = \frac{1}{\Phi(\frac{1}{3},t)} - \frac{1}{\Phi(\frac{2}{3},t)},$$

where we have used the fact $\rho_0 \equiv 1$ on $[\frac{1}{3}, \frac{2}{3}]$. Moreover, Eqs. (12) and (20) together also imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{1}{\Phi(\frac{1}{3},t)}\right) \ge \frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{1}{\Phi(\frac{2}{3},t)}\right)$$

leading to

$$\log\left(\frac{1}{\Phi(\frac{1}{3},t)}\right) - \log\left(\frac{1}{\Phi(\frac{2}{3},t)}\right) \ge \log 2,$$

or equivalently

$$\frac{1}{\Phi(\frac{1}{3},t)} \ge \frac{2}{\Phi(\frac{2}{3},t)}$$

Combining all of the above, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left(\frac{1}{\Phi(\frac{1}{3},t)}\right) \ge \int_{\Phi(\frac{1}{3},t)}^1 \frac{\rho_0(\Phi^{-1}(y,t))}{y^2}\mathrm{d}y \ge \frac{1}{\Phi(\frac{1}{3},t)} - \frac{1}{\Phi(\frac{2}{3},t)} \ge \frac{1}{2\Phi(\frac{1}{3},t)}. \tag{22}$$

Write $y(t) = 1/\Phi(\frac{1}{3}, t)$. Then based on (22) we have $y(t) \ge G(t)$, where

$$G''(t) = \frac{1}{2}G^2(t), \quad G(0) = 1, \quad G'(0) = 0. \tag{23}$$

The choice of the initial condition for the derivative in (23) follows from $y'(0) = 0$, which is a consequence of our choice $\omega_0(x) \equiv 0$. Finite time blow up for G is not hard to establish. Introduce a new variable $v = G'$ and observe that

$$\frac{1}{6}\frac{\mathrm{d}(G^3)}{\mathrm{d}G} = \frac{1}{2}G^2 = G'' = v' = \frac{\mathrm{d}v}{\mathrm{d}G}G' = \frac{1}{2}\frac{\mathrm{d}(v^2)}{\mathrm{d}G} \quad \Longrightarrow \quad v^2 = \frac{1}{3}(G^3 - 1). \tag{24}$$

Then

$$v' = G'' = \frac{1}{2}G^2 = (3v^2 + 1)^{2/3}, \quad v(0) = G'(0) = 0.$$

Due to $v' \geq 1$ and $v(0) = 0$, we can fix some time $t_0 > 0$ such that $v(t_0) = v_0 > 0$. A change of variable in time $\tilde{t} = t - t_0$ gives

$$\frac{\mathrm{d}v}{\mathrm{d}\tilde{t}} > v(\tilde{t})^{4/3}, \quad v(0) = v_0 > 0$$

or explicitly

$$v(\tilde{t}) > \frac{v_0}{(1 - Cv_0^{1/3}\tilde{t})^3}$$

From this, we can deduce that $v(t)$ and also, according to (24), $G(t)$ blow up in finite time. □

4 The model with non-sign-definite Biot–Savart law

To study the non-sign-definite model, it will be convenient to introduce a change of variable $z = -\log x$. Denote $\tilde{\rho}(z,t) = \rho(x(z),t)$, $\tilde{\omega}(z,t) = \omega(x(z),t)$ and $\tilde{u}(z,t) = -x(z)^{-1}u(x(z),t)$. In the $z-$coordinate, Eqs. (8), (9) and (10) take form

$$\partial_t\tilde{\omega} + \tilde{u}\cdot\partial_z\tilde{\omega} = \tilde{\rho}\cdot e^z, \tag{25}$$

$$\partial_t\tilde{\rho} + \tilde{u}\cdot\partial_z\tilde{\rho} = 0, \tag{26}$$

$$\tilde{u}(z,t) = \int_0^{z-\gamma_1}\tilde{\omega}(y,t)\mathrm{d}y - \int_{z-\gamma_1}^{z+\gamma_2}\tilde{\omega}(y,t)\mathrm{d}y, \tag{27}$$

where $\gamma_1 = \log\beta_1, \gamma_2 = \log\beta_2^{-1}$. Note that $\gamma_{1,2} > 0$ and $2e^{-\gamma_1} - e^{\gamma_2} > 0$ due to our assumptions on $\beta_{1,2}$ in Theorem 2.

We will work with the model (25)–(27) for the rest of the paper and abuse notation to suppress tilde and write (ω, ρ, u) as the solution to (25)–(27) instead of $(\tilde{\omega}, \tilde{\rho}, \tilde{u})$. We will also abuse notation to denote Φ the particle trajectories defined by $\tilde{u}$ via (12). Note that in the z formulation, the blow up condition $\delta(t) \to 0$ becomes $\Phi(Z,t) \to \infty$ for $Z = \sup(\mathrm{supp}(\omega_0, \rho_0))$.

Unlike the method we used previously on the warm-up model, the blow up of the full model becomes more delicate. It is conceivable that the negative contribution in (27) arrests propagation of trajectories to infinity, especially since the negative contribution comes from the largest z in the support of solution where we can expect ω to be largest due to the forcing term (25). We will need to establish a sort of monotonicity structure that allows to prove blow up. The argument will focus on growth of $\partial_z\Phi(z,t)$.

4.1 The choice of initial data and parameters

For the rest of the paper, we fix β_1, β_2 (which in succession fixes γ_1, γ_2, respectively) and ϵ small enough such that

$$2e^{-\gamma_1-\gamma_2-\epsilon} - 1 > 0. \tag{28}$$

Note that due to our assumption that $\beta_1/\beta_2 < 2$, which translates into $\gamma_1 + \gamma_2 < \log 2$, we are able to find $\epsilon > 0$ such that (28) holds. Next, let the parameters $L_{0,1,2,3,4}$ have the ordering $1 < L_0 < L_1 < L_2 < L_3 < L_4$. Fix L_0, L_1 such that $L_0 \le L_1/4$, $\gamma_{1,2} < L_1/4$, and $\epsilon < L_1/10$. The choice of L_2, L_3 will be specified later, and L_4 will be fixed with only one constraint $L_4 > L_3$ once L_3 is chosen. The initial data ω_0, ρ_0 will be constructed as follows: $\omega_0 = 0$ for simplicity; $\rho_0 \in C_0^\infty$ is supported on $[1, L_4]$ and such that $0 \le \rho \le 1$, $\rho(x) > 0$ if $x \in (1, L_4)$, $\rho_0([L_0, L_3]) = 1$ and ρ_0 is monotone decreasing for $z > L_3$.

Let us start with a useful a priori bound on Φ, which is just a z-variant of Lemma 1.

Lemma 2 *Take ρ_0, ω_0 as above. Let $\Gamma(z,t)$ be the solution to*

$$\partial_t \Gamma(z,t) = e^{\Gamma(z,t)} \cdot \Gamma(z,t) \cdot t, \quad \Gamma(z,0) = z. \tag{29}$$

Then, we have $\Phi(z,t) \le \Gamma(z,t)$ for all z for as long time as Γ is defined.

Proof Local existence of Γ follows by Picard's theorem. Write $\Psi(z,t) := \sup_{s\in[0,t]} \Phi(z,s)$. Along the particle trajectories $\Phi(z,t)$, we now have

$$\rho(z,t) = \rho_0(\Phi^{-1}(z,t)), \quad \omega(z,t) = \rho_0(\Phi^{-1}(z,t)) \int_0^t e^{\Phi(\Phi^{-1}(z,t)),s)} \mathrm{d}s. \tag{30}$$

So ω and ρ remain non-negative if they are non-negative initially. Then, given $\omega_0 = 0$, we have

$$\partial_t \Psi(z,t) \le \int_0^{\Phi(z,t)} \omega(y,t)\mathrm{d}y \le \int_0^{\Phi(z,t)} \int_0^t e^{\Phi(\Phi^{-1}(y,t)),s)} \mathrm{d}s\, \mathrm{d}y.$$

Here we have used $0 \le \rho_0 \le 1$ and $\omega \ge 0$. For y in the integration domain $[0, \Phi(z,t)]$, one must have $\Phi(\Phi^{-1}(y,t), s) \le \Phi(z,s) \le \Psi(z,s)$. This is due to non-crossing of trajectories, i.e. $\Phi(z_1,t) \le \Phi(z_2,t)$ for all t if $z_1 \le z_2$ and similarly for the inverse trajectories Φ^{-1}. Thus, we continue to estimate and arrive at

$$\partial_t \Psi(z,t) \le \int_0^{\Psi(z,t)} \int_0^t e^{\Psi(z,s)} \mathrm{d}s\, \mathrm{d}y \le \Psi(z,t) \cdot e^{\Psi(z,t)} \cdot t$$

as Ψ is increasing in t. A simple comparison $\Phi(z,t) \le \Psi(z,t) \le \Gamma(z,t)$ completes the proof. □

A key quantity that we will need to estimate is

$$\frac{\partial_z u(\Phi(z,t),t)}{\partial_z \Phi(z,t)} = 2\omega(\Phi(z,t) - \gamma_1, t) - \omega(\Phi(z,t) + \gamma_2, t).$$

The first step is showing that this quantity becomes positive on most of the support of ρ_0 for a very short initial time. This would imply that in this range, $\partial_z \Phi(z,t)$ is initially growing. One can think of this estimate as a sort of establishment of induction base, to be followed by "induction step".

Lemma 3 *There exists $t_0 = t_0(L_1, L_4)$ such that for all $0 < t \le t_0$ and $L_1 \le z < \Phi^{-1}(\Phi(L_4, t) + \gamma_1, t)$, we have*

$$2\omega(\Phi(z,t) - \gamma_1, t) - \omega(\Phi(z,t) + \gamma_2, t) > 0. \tag{31}$$

Note also that the expression in (31) is zero for any $z \ge \Phi^{-1}(\Phi(L_4, t) + \gamma_1, t)$ and any time t while solution exists.

Proof Note that with the initial data ρ_0 described above, the local existence time is controlled by finiteness of the solution to (29) corresponding to the value $z = L_4$. Then by local existence and continuity of $\Phi(L_1, t)$, as well as the assumption $L_0 \le L_1/4$, there exists a short time T_0 such that the solution stays regular and

$$\Phi(L_0, t) \le L_1/2 \quad \text{and} \quad \Phi(L_1, t) \ge \frac{5L_1}{6}, \quad \forall t \le T_0. \tag{32}$$

Then, we must have

$$\Phi(z,t) - \gamma_1 \ge \Phi(L_1, t) - \gamma_1 \ge L_1/2, \quad \forall z \ge L_1, \quad \forall t \le T_0 \tag{33}$$

because $\gamma_1 < L_1/4$; otherwise, trajectories will cross. Denote

$$z_-(t) = \Phi^{-1}(\Phi(z,t) - \gamma_1, t), \quad z_+(t) = \Phi^{-1}(\Phi(z,t) + \gamma_2, t).$$

Of course $z_\pm$ depend on z, but we will suppress this in notation. Now notice that if $z \in [L_1, \Phi^{-1}(\Phi(L_4, t) + \gamma_1, t))$, then by (32) we have $L_0 \le z_-(t) < L_4$ if $t \le T_0$, and therefore $\rho_0(z_-(t)) > 0$. We utilize (30) and monotonicity of ρ_0 in the region $z > L_0$ to get

$$\begin{aligned} &2\omega(\Phi(z,t) - \gamma_1, t) - \omega(\Phi(z,t) + \gamma_2), t) \\ &\quad = 2\rho_0(z_-(t)) \int_0^t e^{\Phi(z_-(t),s)} \mathrm{d}s - \rho_0(z_+(t)) \int_0^t e^{\Phi(z_+(t),s)} \mathrm{d}s \\ &\quad \ge \rho_0(z_-(t)) \int_0^t \left(2e^{\Phi(z_-(t),s)} - e^{\Phi(z_+(t),s)}\right) \mathrm{d}s, \end{aligned} \tag{34}$$

for all $z \in [L_1, \Phi^{-1}(\Phi(L_4, t) + \gamma_1, t))$. Now fix $t_0 < T_0$ such that for every $z \in [L_1, \Phi^{-1}(\Phi(L_4, t) + \gamma_1, t))$

$$\Phi(z_-(t), s) \ge z - \gamma_1 - \epsilon_1, \quad \Phi(z_+(t), s) \le z + \gamma_2 + \epsilon_2 \tag{35}$$

for all $0 \le s \le t \le t_0$ with $\epsilon_{1,2} > 0$, $\epsilon_1 + \epsilon_2 \le \epsilon$ (see definition of ϵ in (28)). Such t_0 can be found due to local existence, continuity of $\Phi(z,t)$, and finiteness of the domain. Therefore, plugging (35) into (34) yields that for $t \le t_0$ and $z \in [L_1, \Phi^{-1}(\Phi(L_4, t) + \gamma_1, t))$ we have

$$\begin{aligned} 2\omega(\Phi(z,t) - \gamma_1, t) - \omega(\Phi(z,t) + \gamma_2), t) &\ge \rho_0(z_-(t)) \int_0^t \left(2e^{z - \gamma_1 - \epsilon_1} - e^{z + \gamma_2 + \epsilon_2}\right) \mathrm{d}s \\ &\ge \rho_0(z_-(t)) e^{z + \gamma_2 + \epsilon_2} (2e^{-\gamma_1 - \gamma_2 - \epsilon} - 1)t > 0. \end{aligned}$$

□

Let us now outline the plan of the proof of our main result Theorem 2. As we already mentioned, Lemma 3 can be viewed as an "induction base"—we established positivity of a key quantity for a very short time depending on "fast" parameter L_4. In the next proposition, we show that this positivity is preserved, provided that the solution stays regular, for a period of time that depends only on the "slow" parameters L_1, $\gamma_{1,2}$, ϵ. We will then use this positivity to show singularity formation in an arbitrary short time provided that we choose the fast parameter L_3 large enough.

Fix $\tau_0 > 0$ to be such that

$$\tau_0 e^{\Gamma(3L_1,\tau_0)} = \frac{\epsilon}{\tau_0(\gamma_1+\gamma_2+\epsilon)}. \tag{36}$$

The existence of τ_0 follows from the local bounds on Γ evident from (29). We will also assume that

$$\Phi(L_0,t) \le \Gamma(L_0,t) < L_0 + \frac{L_1}{6} \le \frac{L_1}{2}, \tag{37}$$

$$\Phi(L_1,t) \le \Gamma(L_1,t) < \frac{3L_1}{2}, \tag{38}$$

for all $0 \le t \le \tau_0$ if necessary by decreasing τ_0.

Proposition 2 *For all $t \in [0,\tau_0]$ and while the regular solution exists, and for all $z \in [L_1, \Phi^{-1}(\Phi(L_4,t)+\gamma_1,t))$, we have*

$$2\omega(\Phi(z,t)-\gamma_1,t) - \omega(\Phi(z,t)+\gamma_2,t) > 0. \tag{39}$$

Proof Suppose not. Observe that for every t while the solution exists, there is an $\eta(t) > 0$ such that for $z \in [\Phi^{-1}(\Phi(L_4,t)+\gamma_1,t)-\eta(t),t), \Phi^{-1}(\Phi(L_4,t)+\gamma_1,t))$ we must have strict inequality in (39). This $\eta(t)$ can be determined by the condition that

$$\Phi(\Phi^{-1}(\Phi(L_4,t)+\gamma_1,t)-\eta(t),t)+\gamma_2 \ge \Phi(L_4,t),$$

so that the second term in (39) vanishes. That $\eta(t) > 0$ follows from non-intersection of the trajectories while solution stays regular.

Now due to Lemma 3, continuity of solution, and compactness of domain

$$\{t_0 \le t \le \tau_0, \quad L_1 \le z \le \Phi^{-1}(\Phi(L_4,t)+\gamma_1,t)-\eta(t)\},$$

the only way (39) can be violated is if there exists τ_1, $t_0 < \tau_1 \le \tau_0$ such that for $t < \tau_1$ (39) holds but

$$2\omega(\Phi(z_1,\tau_1)-\gamma_1,\tau_1) - \omega(\Phi(z_1,\tau_1)+\gamma_2,\tau_1) = 0 \tag{40}$$

for some $z_1 \in [L_1, \Phi^{-1}(\Phi(L_4,\tau_1)+\gamma_1,\tau_1)-\eta(\tau_1)]$. We will need the following lemma to complete the proof.

Lemma 4 *We have*

$$\Phi(L_1,t)-\gamma_1 \ge \Phi(L_0,t), \quad \Phi(2L_1,t) \ge \Phi(L_1,t)+\gamma_2 \tag{41}$$

for $0 \le t \le \tau_1$.

Proof Let us focus on the proof of the first inequality in (41), as the proof of the second one is similar modulo using (38) instead of (37). Integrating (12) in t and differentiating in z gives

$$\partial_z \Phi(z,t) = 1 + \int_0^t \left(2\omega(\Phi(z,s)-\gamma_1,s) - \omega(\Phi(z,s)+\gamma_2,s)\right)\partial_z\Phi(z,s)\mathrm{d}s. \tag{42}$$

Definition of τ_1 implies that $\partial_z\Phi(z,t) \geq 1$ for all $z \geq L_1$, $t \leq \tau_1$. Denote $Z_1(t) := \Phi^{-1}(\Phi(L_1,t)+\gamma_2,t)$ which is valid on $t \leq \tau_1$, then

$$\gamma_2 = \Phi(Z_1(t),t) - \Phi(L_1,t) \geq Z_1(t) - L_1$$

which yields

$$Z_1(t) \leq L_1 + \gamma_2. \tag{43}$$

By (27), (30), $\rho_0 \leq 1$, the definition of τ_0 (36) and $\gamma_2 \leq L_1/4$, we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(L_1,t) &\geq -\int_{\Phi(L_1,t)-\gamma_1}^{\Phi(L_1,t)+\gamma_2} \omega(y,t)\mathrm{d}y \geq -\int_{\Phi(L_1,t)-\gamma_1}^{\Phi(L_1,t)+\gamma_2}\left(\int_0^t e^{\Phi(\Phi^{-1}(y,t),s)}\mathrm{d}s\right)\mathrm{d}y\\
&\geq -\int_{\Phi(L_1,t)-\gamma_1}^{\Phi(L_1,t)+\gamma_2}\left(\int_0^t e^{\Phi(Z_1(t),s)}\mathrm{d}s\right)\mathrm{d}y\\
&\geq -\int_{\Phi(L_1,t)-\gamma_1}^{\Phi(L_1,t)+\gamma_2}\left(\int_0^t e^{\Phi(L_1+\gamma_2,s)}\mathrm{d}s\right)\mathrm{d}y\\
&\geq -(\gamma_2+\gamma_1)te^{\Gamma(L_1+\gamma_2,t)} \geq -(\gamma_2+\gamma_1)\frac{\epsilon}{\tau_0(\gamma_1+\gamma_2+\epsilon)} \geq -\frac{\epsilon}{\tau_0}, \quad \forall t \leq \tau_1,
\end{aligned}$$

where the third inequality follows from definition of Z_1 and the fourth from (43). Therefore, using our assumptions on ϵ, γ_2, L_0 and (37), we have for all $t \leq \tau_1$

$$\Phi(L_1,t) \geq L_1 - \epsilon \geq \gamma_1 + \Phi(L_0,t)$$

which finishes the proof of the lemma. □

Equipped with Lemma 4, let us continue to show Proposition 2. We write $z_-^1(\tau_1) = \Phi^{-1}(\Phi(z_1,\tau_1)-\gamma_1,\tau_1)$, $z_+^1(\tau_1) = \Phi^{-1}(\Phi(z_1,\tau_1)+\gamma_2,\tau_1)$, then naturally $z_-^1(\tau_1) < z_+^1(\tau_1)$ by non-intersection of trajectories. Let $0 < s < \tau_1$ be such that

$$\Phi(z_-^1(\tau_1),s) \leq \Phi(z_+^1(\tau_1),s) - \gamma_1 - \gamma_2 - \epsilon. \tag{44}$$

Note that such s must exist. Otherwise, Lemma 4 guarantees that $\rho(z_-^1(t)) > 0$, and the breakthrough scenario at τ_1 cannot happen due to (34) and (28). Let us focus on $s < \tau_1$ that is the maximal time for which equality in (44) holds. Now

$$\begin{aligned}
\gamma_1+\gamma_2 &= (\Phi(z_1,\tau_1)+\gamma_2) - (\Phi(z_1,\tau_1)-\gamma_1)\\
&= \Phi[\Phi^{-1}(\Phi(z_1,\tau_1)+\gamma_2,\tau_1),\tau_1] - \Phi[\Phi^{-1}(\Phi(z_1,\tau_1)-\gamma_1,\tau_1),\tau_1]\\
&= \left(\Phi(z_+^1(\tau_1),s) + \int_s^{\tau_1} u(\Phi(z_+^1(\tau_1)),r)\mathrm{d}r\right)\\
&\quad - \left(\Phi(z_-^1(\tau_1),s) + \int_s^{\tau_1} u(\Phi(z_-^1(\tau_1)),r)\mathrm{d}r\right)
\end{aligned}$$

$$= \gamma_1 + \gamma_2 + \epsilon + \int_s^{\tau_1} \mathrm{d}r \int_{\Phi(z_-^1(\tau_1),r)}^{\Phi(z_+^1(\tau_1),r)} \frac{\partial u}{\partial y}(y,r)\mathrm{d}y$$
$$= \gamma_1 + \gamma_2 + \epsilon + \int_s^{\tau_1} \mathrm{d}r \int_{\Phi(z_-^1(\tau_1),r)}^{\Phi(z_+^1(\tau_1),r)} 2\omega(y-\gamma_1,r) - \omega(y+\gamma_2,r)\mathrm{d}y.$$

The choice of s and (44) immediately give $\Phi(z_+^1(\tau_1),r) - \Phi(z_-^1(\tau_1),r) < \gamma_1 + \gamma_2 + \epsilon$ for all $r \in (s,\tau_1)$, which implies that there must exist some $r \in (s,\tau_1)$ and some $y_0 \in [\Phi(z_-^1(\tau_1),r), \Phi(z_+^1(\tau_1),r)]$ such that

$$2\omega(y_0-\gamma_1,r) - \omega(y_0+\gamma_2,r) < -\frac{\epsilon}{(\tau_1-s)(\gamma_1+\gamma_2+\epsilon)} < -\frac{\epsilon}{\tau_0(\gamma_1+\gamma_2+\epsilon)}. \quad (45)$$

From the definition of τ_1, we can infer that the only possibility is that $y_0 = \Phi(z_0,r)$ for some $z_0 \in [0,L_1)$. Once (45) is established, what is left to obtain contradiction is just to estimate $\omega(y_0+\gamma_2,r)$. Note that by the second inequality in (41), we have $\Phi(z_0,r)+\gamma_2 \le \Phi(2L_1,r)$. Using this and (30), we get

$$\omega(y_0+\gamma_2,r) = \rho_0(\Phi^{-1}(y_0+\gamma_2,r)) \int_0^r e^{\Phi(\Phi^{-1}(y_0+\gamma_2,r),r')}\mathrm{d}r'$$
$$\le re^{\Gamma(2L_1,r)} < \tau_0 e^{\Gamma(2L_1,\tau_0)}. \quad (46)$$

Non-negativity of ω together with (45) and (46) jointly contradict the choice of τ_0 (36) and the proof is complete. □

Let us reiterate that as opposed to Lemma 3, Proposition 2 holds for τ_0 independent of L_3. We are now free to choose L_3 large enough and assume (39) for all times while solution exists. The next proposition strengthens the bound in (39) in a narrower range of z.

Proposition 3 *Suppose $L_3 > L_2 \ge L_1 + \gamma_1$. Then for all $z \in [L_2,L_3]$ and $t \in [0,\tau_0]$, and while the regular solution exists, we have*

$$2\omega(\Phi(z,t)-\gamma_1,t) - \omega(\Phi(z,t)+\gamma_2,t) \ge (2e^{-\gamma_1-\gamma_2}-1)\int_0^t e^{\Phi(z,s)}\mathrm{d}s. \quad (47)$$

Proof First notice that for such choice of L_2, due to (42) and Proposition 2 we have $\Phi(L_2,t) \ge \Phi(L_1,t)+\gamma_1$ for all $t \in [0,\tau_0]$. This in turn ensures that $z_-(t) = \Phi^{-1}(\Phi(z,t)-\gamma_1,t) \ge L_1$ for all $z \ge L_2$. Then, based on Proposition 2 we obtain that

$$\gamma_1+\gamma_2 = (\Phi(z,t)+\gamma_2) - (\Phi(z,t)-\gamma_1)$$
$$= \left(\Phi(z_+(t),s) + \int_s^t u(\Phi(z_+(t),r),r)\mathrm{d}r\right)$$
$$- \left(\Phi(z_-(t),s) + \int_s^t u(\Phi(z_-(t),r),r)\mathrm{d}r\right)$$
$$= \Phi(z_+(t),s) - \Phi(z_-(t),s)$$
$$+ \int_s^t \mathrm{d}r \int_{\Phi(z_-(t),r)}^{\Phi(z_+(t),r)} (2\omega(y-\gamma_1,r) - \omega(y+\gamma_2,r))\,\mathrm{d}y$$
$$\ge \Phi(z_+(t),s) - \Phi(z_-(t),s), \quad \forall 0 < s < t. \quad (48)$$

However, observe also that when $z \in [L_2, L_3]$, we always have $\rho_0(z_-(t)) = 1$. So we can recall (34) and combine with (48) to deduce that

$$2\omega(\Phi(z,t)-\gamma_1,t)-\omega(\Phi(z,t)+\gamma_2,t) \geq \int_0^t \left(2e^{\Phi(z_-(t),s)} - e^{\Phi(z_+(t),s)}\right) \mathrm{d}s$$
$$\geq \int_0^t e^{\Phi(z_+(t),s)}(2e^{-\gamma_1-\gamma_2}-1)\mathrm{d}s \geq (2e^{-\gamma_1-\gamma_2}-1)\int_0^t e^{\Phi(z,s)}\mathrm{d}s.$$

□

Let us prove one last lemma before we move on to show finite time singularity formation.

Lemma 5 *Define $f(z,t)$ for $z \in [L_2, L_3]$ by*

$$\partial_t f(z,t) = c\int_0^t e^{\int_{L_2}^z f(y,s)\mathrm{d}y}\mathrm{d}t, \quad f(z,0) \equiv 1/2, \tag{49}$$

where c is a fixed positive constant. Then

(a) *For each $L_2 < L_3 < \infty$, the equation is locally well-posed;*
(b) *Given any $\tau_0 > 0$, $L_3 < \infty$ can be chosen so that $f(L_3, t)$ becomes infinite before τ_0.*

Proof (a) Local existence of solutions can be done via a standard iteration argument. For an a priori bound, set $h(t) := \sup_{z\in[L_2,L_3]} f(z,t)$. Then, differentiating (49) gives

$$\partial_{tt} h \leq ce^{h(t)(L_3-L_2)}, \quad h(0) = 1/2, \quad h'(0) = 0, \tag{50}$$

which clearly controls $h(t)$ locally (we can use equality in (50) to derive an upper bound). Suppose $h(t)$ stays bounded on $[0, T_0]$. Define the iteration scheme by

$$\partial_t f_n(t) = c\int_0^t e^{\int_{L_2}^z f_{n-1}(y,s)\mathrm{d}y}\mathrm{d}s, \quad f_n(z,0) = \frac{1}{2}.$$

It can be seen by induction that $f_n(z,t)$ is increasing for every z, t. Note that f_n is bounded by h uniformly for all n, z, and t. Thus, for $(z,t) \in [L_2, L_3] \times [0, T_0]$ we have

$$|\partial_t(f_n - f_{n-1})(z,t)| \leq c\int_0^t e^{\|h\|_{L^\infty}(z-L_2)} \max_{[L_2,L_3]\times[0,t]} |f_n(z,s) - f_{n-1}(z,s)|\mathrm{d}s.$$

Let $F_n(t) = \max_{[L_2,L_3]\times[0,t]} |f_n(z,s) - f_{n-1}(z,s)|$, then $F(t)$ satisfies

$$F_n'(t) \leq C(L_3, T_0)\int_0^t F_{n-1}(s)\mathrm{d}s$$

for some $C(L_3, T_0) < \infty$ which inductively gives

$$F_1(t) \leq \int_0^t \mathrm{d}s\int_0^s e^{L_3/2}\mathrm{d}r = e^{L_3/2}\frac{t^2}{2}, \quad F_n(t) \leq \frac{C(L_3,T_0)^n t^{2n}}{(2n)!}.$$

It is clear that the series F_n converges uniformly in z if $t \leq T_0$.

(b) We will show now that L_3 can always be chosen so that $f(L_3, t)$ will go to infinity before τ_0. First of all, note that from the definition we have $\partial_t f \geq 0$ for all $t \in [0, \tau_0]$ and $z \in [L_2, L_3]$ and hence for $z \in [L_3 - (L_3 - L_2)/2, L_3]$ and $t \geq \tau_0/2$

$$f(z,t) \geq \frac{1}{2} + ce^{\frac{L_3-L_2}{4}}\frac{\tau_0^2}{8}.$$

Denote $G_n := (n+1)2^{n+2}$ and $\Delta = L_3 - L_2$ and choose for $\Delta > 8$ large enough so that $ce^{\Delta/4}\frac{\tau_0^2}{8} \geq G_1 \equiv 16$. We assert that in fact

$$f(z, \tau_0(1-2^{-n})) \geq G_n, \quad \forall z \in [L_3 - \Delta 2^{-n}, L_3]. \tag{51}$$

This assertion can be shown by an inductive argument. The case $n = 1$ is instantaneous from the assumption on Δ. Assume by induction that (51) holds for $n = k$. When $n = k+1$, for each $z \in [L_3 - \Delta 2^{-k-1}, L_3]$ we have

$$\begin{aligned} f(z, \tau_0(1-2^{-k-1})) &\geq \int_{\tau_0(1-2^{-k})}^{\tau_0(1-2^{-k-1})} \mathrm{d}t \int_{\tau_0(1-2^{-k})}^{t} \mathrm{d}s \exp\left(\int_{L_3-\Delta 2^{-k}}^{L_3-\Delta 2^{-k-1}} G_k \mathrm{d}y\right) \\ &\geq c\frac{\tau_0^2}{2^{2k+3}} e^{2(k+1)\Delta} \geq \frac{e^{2k\Delta}}{2^{2k+3}} \geq 2^{10k} \geq (k+1)2^{2k+3} = G_{k+1}. \end{aligned}$$

This shows that $\lim_{t\to\tau} f(L_3, t) = \infty$ for some $\tau \leq \tau_0$ as desired. □

Now, we are well prepared to prove Theorem 2.

Proof Take ρ_0 as described in the beginning of this section. Choose L_0, L_1 as above. Let τ_0 satisfy (36), (37), and (38). Suppose that $L_2 \geq L_1 + \gamma_1$. Let f satisfy (49) with $c = 2e^{-\gamma_1-\gamma_2} - 1$. Choose L_3 so that the blow up time T of $f(L_3, t)$ satisfies $T \leq \tau_0$. Fix $L_4 > L_3$. Consider

$$\frac{\partial^2 \Phi(z,t)}{\partial t \partial z} = \partial_z u(\Phi(z,t), t) = \partial_z \Phi(z,t)(2\omega(\Phi(z,t) - \gamma_1, t) - \omega(\Phi(z,t) + \gamma_2, t)) \tag{52}$$

with $\partial_z \Phi(z, 0) \equiv 1$. By Proposition 2, we see that for all $z \in [L_1, \infty)$ and $t \in [0, \tau_0]$,

$$\frac{\partial^2 \Phi(z,t)}{\partial t \partial z} \geq 0, \tag{53}$$

which indicates that $\partial_z \Phi(z, t) \geq 1$ for all t, z in these ranges. Using (53) and invoking Proposition 3, we can infer that

$$\frac{\partial^2 \Phi(z,t)}{\partial t \partial z} \geq c\partial_z \Phi(z,t) \cdot \int_0^t e^{\Phi(z,s)} \mathrm{d}s > c\partial_z \Phi(z,t) \cdot \int_0^t e^{\int_{L_2}^z \partial_z \Phi(y,s) \mathrm{d}y} \mathrm{d}s \tag{54}$$

for $z \in [L_2, L_3]$, $0 \leq t \leq \tau_0$ while solution exists. However, it is not hard to establish the comparison $\partial_z \Phi(z,t) \geq f(z,t)$ for all $z \in [L_2, L_3]$ all the way until blow up time $T \leq \tau_0$. Indeed, note that $\partial_z \Phi(z, 0) \equiv 1 > f(z, 0)$. Suppose $T_1 < T$ is the first time when there exists $z_1 \in [L_2, L_3]$ such that

$$\partial_z \Phi(z_1, t) = f(z_1, t).$$

But then by (54), for every $s \leq T_1 \leq T$ we have

$$\left.\frac{\partial^2 \Phi(z,t)}{\partial t \partial z}\right|_{(z,t)=(z_1,s)} > c\int_0^s e^{\int_{L_2}^{z_1} \partial_z \Phi(y,r) \mathrm{d}y} \mathrm{d}r \geq c\int_0^s e^{\int_{L_2}^{z_1} f(y,r) \mathrm{d}y} \mathrm{d}r = \left.\frac{\mathrm{d}}{\mathrm{d}t} f(z_1, t)\right|_{t=s},$$

which is a contradiction.

This argument shows that unless singularity develops earlier in some other way required by Proposition 1, $\partial_z \Phi(z,t)$ becomes infinite for some $t \leq \tau_0$. However, this implies that

$\partial_z \tilde{u}(z,t)$ becomes infinite too, where we are returning to the $\tilde{u}$ notation for the velocity in z representation (27) in order to avoid confusion. But, we have

$$\partial_z \tilde{u}(z,t) = -\partial_x u(x(z)) - \frac{u(x(z))}{x(z)}.$$

Therefore, it is not hard to see that blow up in $\partial_z \tilde{u}(z)$ forces blow up in either $\|\partial_x u\|_{L^\infty}$, or $\|\omega\|_{L^\infty}$, or $\delta(t)^{-1}$. At this point, we can invoke Proposition 1 which gives us a set of minimal conditions that must happen when singularity forms. □

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