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Exact interpolation, spurious poles, and uniform convergence of multipoint Padé approximants

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Abstract. We introduce the concept of an exact interpolation index n associated with a function f and open set $\mathscr{L}$: all rational interpolants $R=p/q$ of type (n,n) to f, with interpolation points in $\mathscr{L}$, interpolate exactly in the sense that $fq-p$ has exactly $2n+1$ zeros in $\mathscr{L}$. We show that in the absence of exact interpolation, there are interpolants with interpolation points in $\mathscr{L}$ and spurious poles. Conversely, for sequences of integers that are associated with exact interpolation to an entire function, there is at least a subsequence with no spurious poles, and consequently, there is uniform convergence.

Bibliography: 22 titles.

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§ 1. Introduction

Let $\mathscr{D}$ be an open connected subset of $\mathbb{C}$, and let $f\colon \mathscr{D}\to\mathbb{C}$ be analytic. Given $n\geqslant 1$ and not necessarily distinct points $\Lambda_n=\{z_{jn}\}_{j=1}^{2n+1}$ in $\mathscr{D}$, and

$$\omega_n(z)=\omega_n(\Lambda_n,z)=\prod_{j=1}^{2n+1}(z-z_{jn}),$$

the multipoint Padé approximant to f with interpolation set Λ_n is a rational function

$$R_n(\Lambda_n,z)=\frac{p_n(\Lambda_n,z)}{q_n(\Lambda_n,z)},$$

or more simply,

$$R_n(z)=\frac{p_n(z)}{q_n(z)},$$

where p_n and q_n are polynomials of degree $\leqslant n$ with q_n not identically zero, and

$$\frac{e_n(z)}{\omega_n(z)}=\frac{e_n(\Lambda_n,z)}{\omega_n(\Lambda_n,z)}=\frac{f(z)q_n(z)-p_n(z)}{\omega_n(z)}$$

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is analytic in $\mathscr{D}$. The special case where all $z_{jn} = 0$, gives the Padé approximant $[n/n](z)$. It is well known that R_n exists and is unique, though p_n and q_n are not separately unique. Moreover, it is possible that, in order to satisfy the interpolation conditions, p_n and q_n may need to include some common factors $z - z_{jn}$ with zeros at the interpolation points $\{z_{jn}\}$.

The convergence of sequences of rational interpolants, and especially Padé approximants, is a complex and much studied subject. Many of the beautiful results from the Russian school headed by Gonchar have appeared in this journal. One of the unfortunate properties of such interpolants is the appearance of *spurious poles*: R_n may have poles that bear no relation to singularities of the underlying function f. These are typically close to spurious zeros, that also bear little relation to the zeros of f. See [1], [3]–[6], [8]–[10], [13]–[17], [19], [20] and [22] for some references and surveys of the convergence theory relating to the issue of spurious poles. Of course this is not a precisely defined concept, at least for just one rational interpolant. It is best considered for sequences of interpolants, whose limit points of poles do not approach singularities of the underlying function.

Spurious poles are also known to be related in some sense to the appearance of extra zeros of $e_n(z)$, that is, zeros other than $\{z_{jn}\}_{j=1}^{2n+1}$ (see [2] and [20]). This has been established in a fairly precise sense, especially for algebraic and elliptic functions, in particular for diagonal Padé approximants $\{[n/n]\}_{n\geqslant 1}$. For polynomial interpolation, 'overinterpolation' has been investigated in [7]. The goal of this paper is to explore this relationship further, by considering all interpolants with interpolation points in a given set. This is a new idea to the best of our knowledge and as we shall see has several advantages.

Definition 1.1. Let $\mathscr{D} \subset \mathbb{C}$ be a connected open set, and let $f\colon \mathscr{D} \to \mathbb{C}$ be analytic. Let $\mathscr{L} \subset \mathscr{D}$ be open and $n \geqslant 1$. We say n is an *exact interpolation index for f and $\mathscr{L}$* if for every set of $2n+1$ not necessarily distinct interpolation points $\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1}$ in $\mathscr{L}$, and every corresponding interpolant $R_n(\Lambda_n, z) = p_n(z)/q_n(z)$,

$$\frac{e_n(\Lambda_n, z)}{\omega_n(z)} = \frac{f(z)q_n(z) - p_n(z)}{\omega_n(z)}$$

has no zeros in $\mathscr{L}$.

Note that the condition forces at least one of p_n and q_n to have degree n. Otherwise we can add an extra zero c at any point in $\mathscr{L}$, since $p_n(z)(z-c)$ and $q_n(z)(z-c)$ will have degree at most n, while

$$\frac{f(z)q_n(z)(z-c) - p_n(z)(z-c)}{\omega_n(z)}$$

will have the extra zero c. The property that at least one of p_n and q_n has full degree is typically described as '$R_n(\Lambda_n, z)$ having defect 0'.

The relevance of exact interpolation to spurious poles is clear from the following simple statement.

Proposition 1.2. *Let $\mathscr{D} \subset \mathbb{C}$ be a connected open set, and let $f\colon \mathscr{D} \to \mathbb{C}$ be analytic. Let $n \geqslant 1$, and let $\mathscr{L}$ and $\mathscr{B}$ be open subsets of $\mathscr{D}$. Assume that whenever we are given a set of $2n+3$ not necessarily distinct points $\Lambda_{n+1} \subset \mathscr{L} \cup \mathscr{B}$, $R_{n+1}(\Lambda_{n+1}, z)$ does not have poles in $\mathscr{B}$. Then n is an exact interpolation index for f and $\mathscr{L}$.*

We shall prove this simple proposition in § 2. Note that the pole free interpolant $R_{n+1}(\Lambda_{n+1}, z)$ has type $(n+1, n+1)$, not (n, n). We shall also prove a much deeper partial converse of Proposition 1.2, that exact interpolation to entire functions forces the absence of spurious poles, at least for a subsequence.

Throughout this paper,

$$B_r = \{z\colon |z| < r\}, \qquad r > 0.$$

Theorem 1.3. *Let f be entire. Let $\{n_k\}_{k\geqslant 1}$ be an increasing sequence of positive integers such that for $k \geqslant 1$, and for some integer $L > 1$,*

$$\frac{n_{k+1}}{n_k} \leqslant L. \tag{1.1}$$

Assume that there is an increasing sequence $\{r_k\}_{k\geqslant 1}$ of positive numbers with

$$\lim_{k\to\infty} r_k = \infty, \tag{1.2}$$

and that, for $k \geqslant 1$, $n_k - 1$ is an exact interpolation index for f and the ball B_{r_k}. Then there exists a subsequence $\{n_{k_j}\}_{j\geqslant 1}$ of $\{n_k\}_{k\geqslant 1}$ with the following property: let $r, s > 0$, and for $j \geqslant 1$, choose interpolation sets $\Lambda_{n_{k_j}}$ in B_r. Then for large enough j, $R_{n_{k_j}}(\Lambda_{n_{k_j}}, z)$ is analytic in B_s. Consequently, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{j\to\infty} R_{n_{k_j}}(\Lambda_{n_{k_j}}, z) = f(z).$$

We emphasize that the same subsequence $\{n_{k_j}\}_{j\geqslant 1}$ works for all sets of interpolation points in B_r, and for all r.

When we have mild regularity of errors of best rational approximation, we can establish uniform convergence of full sequences. Let K be a compact set and $f\colon K \to \mathbb{C}$ a continuous function. We let

$$E_n(f, K) = \inf\left\{\left\|f - \frac{p}{q}\right\|_{L_\infty(K)} : p, q \text{ have degree } \leqslant n \text{ and } q \neq 0 \text{ in } K\right\}.$$

A *best approximant of type* (n, n), $R_n^*(f, K) = p_n^*/q_n^*$, is a rational function of type (n, n) satisfying

$$\|f - R_n^*(f, K)\|_{L_\infty(K)} = E_n(f, K).$$

We also let

$$\eta_n(f, K) = E_n(f, K)^{1/n}, \qquad n \geqslant 1.$$

Theorem 1.4. *Let f be entire. Let $\{n_k\}_{k\geqslant 1}$ be a strictly increasing sequence of positive integers. Assume that there is an increasing sequence $\{r_k\}_{k\geqslant 1}$ of positive numbers satisfying (1.2), such that for $k \geqslant 1$, $n_k - 1$ is an exact interpolation index for f and the ball B_{r_k}. Assume in addition that either*

(a) *for some $\tau > 0$, $\delta \in (0, 1)$, integer $M > 1$ and large enough k,*

$$E_{n_k}(f, \overline{B_\tau}) > E_{Mn_k}(f, \overline{B_\tau})^{1-\delta}, \tag{1.3}$$

or

(b) *for some* $T > 1$ *and an unbounded sequence of values of* r,

$$\limsup_{k\to\infty}\left(\frac{E_{n_k}(f,\overline{B_r})}{E_{n_k}(f,\overline{B_{r/4}})}\right)^{1/n_k} < T. \tag{1.4}$$

Then given any $r, s > 0$, *and interpolation sets* Λ_{n_k} *in* B_r *for* $k \geqslant 1$, *for large enough* k *the function* $R_{n_k}(\Lambda_{n_k}, z)$ *is analytic in* B_s. *Consequently, uniformly for* z *in compact subsets of* $\mathbb{C}$,

$$\lim_{k\to\infty} R_{n_k}(\Lambda_{n_k}, z) = f(z).$$

Remark 1.5. (a) We note that the regularity condition (1.3) is a weak one. Indeed, we can reformulate it as

$$\eta_{n_k}(f,\overline{B_\tau}) > \eta_{Mn_k}(f,\overline{B_\tau})^{(1-\delta)M}.$$

and since $(1-\delta)M$ may be very large, while

$$\lim_{k\to\infty} \eta_{n_k}(f,\overline{B_\tau}) = 0,$$

certainly this is true for regularly behaved errors of approximation. For example, if for some ℓ, $\{\eta_n(f,\overline{B_\tau})\}_{n\geqslant\ell}$ is decreasing, then (1.3) is true for $n_k = k$, $k \geqslant 1$. Note too that $\overline{B_\tau}$ can be replaced in (1.3) by any set of positive logarithmic capacity.

(b) Similarly, for regularly behaved functions and for all $r > 0$

$$\lim_{n\to\infty}\left(\frac{E_n(f,\overline{B_r})}{E_n(f,\overline{B_{r/4}})}\right)^{1/(2n)} = 4$$

(see [15]), so (1.4) is not a severe condition. On the other hand, it is easy to construct entire functions with lacunary Maclaurin series for which (1.4) fails for a subsequence of integers.

(c) This circle of ideas can be extended to non-diagonal sequences of interpolants, and probably to functions meromorphic in the plane.

(d) The biggest question that arises from this paper is the existence of sequences of exact indices of interpolation. If for example, f has a normal Padé approximant at 0, so $[n/n] = p_n/q_n$ where p_n and q_n have full degree n, and

$$(fq_n - p_n)(z) = cz^{2n+1} + \cdots$$

with $c \neq 0$, then from classical continuity results for interpolation, there exists $\varepsilon > 0$ such that n is an exact index for f and B_ε. This is an easy consequence of the explicit formulae for rational interpolants in terms of divided differences (see [1], p. 338 and the following pages), which show that the interpolants depend continuously (and even analytically) on the interpolation points. However, the ε of course depends on n. To be useful, we need a sequence of indices exact on balls that are independent of n. Results of this type follow for e^z from Proposition 1.2 and the fact that diagonal multipoint Padé approximants with interpolation points in any compact set have been shown to converge [21], but are worth exploring in

a more general setting. Certainly Proposition 1.2 shows that in the absence of exact interpolation indices, we cannot have uniform convergence of every sequence of interpolants with interpolation points in a (compact) ball.

(e) For rational interpolation to be regarded as stable or 'robust', it would ideally be preferable that when the interpolation points are shifted slightly, new spurious poles do not suddenly arise. Proposition 1.2, Theorems 1.3 and 1.4 suggest that such stability is associated with sequences of exact interpolation indices.

(f) The main tool in proving Theorems 1.3 and 1.4 is Theorem 3.1, which establishes a certain dichotomy. Roughly speaking, this asserts that when there are spurious poles for a sequence of interpolants, then either the preceding indices are not exact interpolation indices, or we have smaller than expected errors of best rational approximation.

The paper is organized as follows: we prove Proposition 1.2 in § 2. We establish a basic alternative in § 3, and prove Theorems 1.3 and 1.4 in § 4.

§ 2. Nonexact interpolation implies spurious poles

We begin by showing the very simple result that if n is not an exact interpolation index, then there are rational interpolants with interpolation points close to a given set of interpolation points, having spurious poles close to any other given point.

Proposition 2.1. *Let $\mathscr{D} \subset \mathbb{C}$ be open, and let $f: \mathscr{D} \to \mathbb{C}$ be analytic. Let $n \geqslant 1$ and suppose that $\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1} \subset \mathscr{D}$ are $2n+1$ given interpolation points which are not necessarily distinct. Assume that*

$$\frac{e_n(\Lambda_n, z)}{\omega_n(\Lambda_n, z)} = \frac{f(z)q_n(z) - p_n(z)}{\omega_n(z)}$$

has a zero b in $\mathscr{D}$. Let $\varepsilon > 0$ and $c \in \mathscr{D}$. Then we can find an interpolation set of $2n+3$ points

$$\Lambda_{n+1} = \{z'_{j,n+1}\}_{j=1}^{2n+1} \cup \{b', c'\}, \tag{2.1}$$

with

$$\max_j |z_{jn} - z'_{j,n+1}| < \varepsilon, \quad 1 \leqslant j \leqslant 2n+1, \qquad |b - b'| < \varepsilon \quad \text{and} \quad |c - c'| < \varepsilon, \tag{2.2}$$

such that

$$R_{n+1}(\Lambda_{n+1}, z) = \frac{p_{n+1}(\Lambda_{n+1}, z)}{q_{n+1}(\Lambda_{n+1}, z)}$$

has a pole and a zero less than an ε distance from c.

Proof. Choose sequences $\{a_m\}$ and $\{b_m\}$ with $a_m \neq b_m$ for all $m \geqslant 1$, and

$$\lim_{m\to\infty} a_m = c = \lim_{m\to\infty} b_m.$$

Assume in addition that $q_n(b_m) \neq 0$ and $p_n(a_m) \neq 0$ for all $m \geqslant 1$. Consider the functions

$$g_m(z) = f(z)q_n(z)(z - a_m) - p_n(z)(z - b_m),$$

$m \geqslant 1$. We see that uniformly for z in compact subsets of D,

$$\lim_{m\to\infty} g_m(z) = (f(z)q_n(z) - p_n(z))(z - c).$$

The right-hand side has zeros at the $2n+3$ zeros of $\omega_n(z)(z-b)(z-c)$ (counting multiplicity), by our hypothesis. By Hurwitz's Theorem, for large enough m, g_m has zeros of total multiplicity $2n+3$ which approach $\{z_j\}_{j=1}^{2n+1} \cup \{b,c\}$ as $m \to \infty$. It follows that for large enough m, we can choose a set Λ_{n+1} satisfying (2.1) and (2.2), and such that

$$R_{n+1}(\Lambda_{n+1}, z) = \frac{p_{n+1}(\Lambda_{n+1}, z)}{q_{n+1}(\Lambda_{n+1}, z)} = \frac{p_n(z)(z-b_m)}{q_n(z)(z-a_m)}$$

and in particular, this rational interpolant has a pole at a_m and a zero at b_m, arbitrarily close to c.

Proof of Proposition 1.2. If n is not exact for f and $\mathscr{L}$, we can find Λ_n in $\mathscr{L}$ for which

$$\frac{f(z)q_n(\Lambda_n, z) - p_n(\Lambda_n, z)}{\omega_n(z)}$$

has a zero in $\mathscr{L}$. Then the construction of Proposition 2.1 shows that we can find $R_{n+1}(\Lambda_{n+1}, z)$ with $2n+2$ of its $2n+3$ interpolation points in $\mathscr{L}$ and one in $\mathscr{B}$ such that $R_{n+1}(\Lambda_{n+1}, z)$ has poles in $\mathscr{B}$, giving a contradiction.

§ 3. The basic alternative

Recall the definition of the Gonchar-Walsh class $\mathscr{R}_0(K)$. Let K be a compact set, and f be analytic at each point of K. We write $f \in \mathscr{R}_0(K)$ if

$$\lim_{n\to\infty} E_n(f, K)^{1/n} = 0.$$

The main result of this section shows that spurious poles either lead to nonexact interpolation, or to smaller than expected errors of best rational approximation for functions in the Gonchar-Walsh class.

We let

$$\|f\|_r = \sup\{|f(z)| : |z| = r\}.$$

Theorem 3.1. *Let f be analytic in a neighbourhood of $\overline{B_1}$ and suppose that f belongs to the Gonchar-Walsh class there. Let $L > 1$ be an integer. Assume that $\delta \in (0,1)$, and that $\varepsilon \in (0, 1/8)$ is so small that*

$$3(8\varepsilon)^{\frac{4}{\pi}\arcsin(\frac{1}{3})} 21^L < 1. \tag{3.1}$$

Assume that an infinite sequence $\mathscr{S}$ of positive integers is given and, for each $n \in \mathscr{S}$, a number $m = m(n)$ (not necessarily in $\mathscr{S}$) such that

$$1 \leqslant \frac{m}{n} \leqslant L.$$

Suppose also that for each $n \in \mathscr{S}$, there exist $2n+1$ not necessarily distinct interpolation points Λ_n in B_ε, such that $R_n(\Lambda_n, z)$ has a pole in B_ε.

Then for large enough $n \in \mathscr{S}$, either

(I) *there is a set Λ_{n-1}, of $2n-1$ interpolation points in B_1, such that if $R_{n-1}(\Lambda_{n-1}, z) = p_{n-1}(z)/q_{n-1}(z)$, then $e_{n-1} = fq_{n-1} - p_{n-1}$ has at least $2n$ zeros in B_1, counting multiplicity,*

or

(II)

$$E_n(f, \overline{B_1}) \leqslant E_m(f, \overline{B_1})^{1-\delta}. \tag{3.2}$$

We begin with a more 'technical' form of Theorem 3.1. Then we present a series of lemmas, and finally prove Theorem 3.1.

Lemma 3.2. *Let $\sigma \geqslant 1$ and let f be analytic in $\overline{B_\sigma}$. Let $m, n \geqslant 1$ and $\varepsilon \in (0,1)$. Assume that*

(i) *$2n+1$ not necessarily distinct interpolation points Λ_n in B_ε are given;*

(ii) *$e_n = fq_n - p_n$ has zeros of total multiplicity N $(\geqslant 2n+1)$ in B_ε;*

(iii) *$R_n(\Lambda_n, z) = p_n(z)/q_n(z)$ has a pole $a \in B_\varepsilon$.*

Then either

(I) *there is a set of $2n-1$ interpolation points in B_1, Λ_{n-1}, such that if $R_{n-1}(\Lambda_{n-1}, z) = p_{n-1}(z)/q_{n-1}(z)$, then $e_{n-1} = fq_{n-1} - p_{n-1}$ has at least $N - 1 \geqslant 2n$ zeros in B_1, counting multiplicity,*

or

(II) *if $\varepsilon < r < \rho < 1$, and*

$$\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho} |\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)} \left(\frac{\rho}{r}\right)^{m+n} \left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m \leqslant \frac{1}{2}, \tag{3.3}$$

then

$$E_n(f, \overline{B_\rho}) \leqslant 28n^2 \left(\frac{\rho}{r}\right)^{m+n} \left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m \frac{\|q_n\|_\rho}{\min_{|t|=\rho} |q_n(t)|} E_m(f, \overline{B_\sigma}). \tag{3.4}$$

Proof. First observe that since $q_n(a) = 0$ and a is a pole of $R_n(\Lambda_n, z)$,

$$e_n(a) = -p_n(a) \neq 0,$$

and so

$$e_n(z) - e_n(a) = f(z)q_n(z) - (p_n(z) - p_n(a)) = (z-a)(fq_{n-1}(z) - p_{n-1}(z)),$$

where p_{n-1} and q_{n-1} have degree at most $n-1$.

(I) Suppose that for some $s \in [\varepsilon, 1]$,

$$\min_{|z|=s} |e_n(z)| > |e_n(a)|.$$

Then by Rouché's Theorem, $e_n(z) - e_n(a)$ has the same number of zeros in B_s as e_n counting multiplicity, and in particular, it has at least N. Then $e_{n-1} = fq_{n-1} - p_{n-1}$ has at least $N - 1 \geqslant 2n$ zeros inside $\{z\colon |z| = s\}$, and this gives us Λ_{n-1}. In fact, as we can omit one zero of e_{n-1} from Λ_{n-1}, there might be multiple choices for Λ_{n-1}. So we have (I).

If the hypothesis of (I) fails, then

(II) For all $s \in (\varepsilon, 1]$,

$$\min_{|z|=s} |e_n(z)| \leqslant |e_n(a)| \leqslant \|e_n\|_\varepsilon. \tag{3.5}$$

We apply the Beurling-Nevanlinna Theorem (see [18], p. 120, Theorem 4.5.6). Let $\varepsilon < \rho \leqslant 1$, and

$$u(z) = \frac{\log(|e_n(\rho z)|/\|e_n\|_\rho)}{|\log(\|e_n\|_\varepsilon/\|e_n\|_\rho)|}, \qquad |z| < 1.$$

Then u is subharmonic in $|z| < 1$, and clearly $u \leqslant 0$ in $|z| < 1$, while for $\varepsilon/\rho \leqslant r < 1$, our hypothesis (3.5) shows that

$$\inf_{|z|=r} u(z) \leqslant \frac{\log(\|e_n\|_\varepsilon/\|e_n\|_\rho)}{|\log(\|e_n\|_\varepsilon/\|e_n\|_\rho)|} = -1.$$

On the other hand, for $0 \leqslant r \leqslant \varepsilon/\rho$, the maximum modulus principle shows that even

$$\sup_{|z|=r} u(z) \leqslant \frac{\log(\|e_n\|_\varepsilon/\|e_n\|_\rho)}{|\log(\|e_n\|_\varepsilon/\|e_n\|_\rho)|} = -1.$$

In summary, we have shown that u is subharmonic in $|z| < 1$, that $u(z) \leqslant 0$ there, and for all r, $0 \leqslant r < 1$,

$$\inf_{|z|=r} u(z) \leqslant -1.$$

Then the Beurling-Nevanlinna Theorem (see [18], p. 120, Theorem 4.5.6) shows that for all $|z| \leqslant 1$,

$$u(z) \leqslant -\frac{2}{\pi} \arcsin\left(\frac{1-|z|}{1+|z|}\right),$$

which can be reformulated as

$$\frac{|e_n(\rho z)|}{\|e_n\|_\rho} \leqslant \left(\frac{\|e_n\|_\varepsilon}{\|e_n\|_\rho}\right)^{\frac{2}{\pi}\arcsin\left(\frac{1-|z|}{1+|z|}\right)}.$$

Now considering $\rho|z| = r$ gives

$$\frac{\|e_n\|_r}{\|e_n\|_\rho} \leqslant \left(\frac{\|e_n\|_\varepsilon}{\|e_n\|_\rho}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)}, \qquad 0 < r < \rho < 1.$$

Next, the maximum modulus principle shows that

$$\left\|\frac{e_n}{\omega_n}\right\|_\varepsilon \leqslant \left\|\frac{e_n}{\omega_n}\right\|_\rho,$$

so

$$\frac{\|e_n\|_\varepsilon}{\|e_n\|_\rho} \leqslant \frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}.$$

Thus also

$$\frac{\|e_n\|_r}{\|e_n\|_\rho} \leqslant \left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)}, \qquad 0 < r < \rho < 1. \tag{3.6}$$

Next, write $R_m^*(f, \overline{B_\sigma}) = p_m^*/q_m^*$, and observe that if $e_m^* = fq_m^* - p_m^*$, then

$$e_n q_m^* - e_m^* q_n = p_m^* q_n - p_n q_m^*.$$

By Bernstein's growth inequality (see [18], p. 156) applied to the right-hand side, which is a polynomial of degree $\leqslant m+n$,

$$\|e_n q_m^* - e_m^* q_n\|_\rho \leqslant \left(\frac{\rho}{r}\right)^{m+n} \|e_n q_m^* - e_m^* q_n\|_r$$

$$\Longrightarrow \|e_n q_m^*\|_\rho - \|e_m^* q_n\|_\rho \leqslant \left(\frac{\rho}{r}\right)^{m+n} (\|e_n q_m^*\|_r + \|e_m^* q_n\|_r)$$

$$\Longrightarrow \|e_n\|_\rho \min_{|t|=\rho} |q_m^*(t)| - \|e_m^*\|_\rho \|q_n\|_\rho \leqslant \left(\frac{\rho}{r}\right)^{m+n} (\|e_n\|_r \|q_m^*\|_r + \|e_m^*\|_\rho \|q_n\|_\rho)$$

$$\Longrightarrow \|e_n\|_\rho \left\{1 - \frac{\|e_n\|_r}{\|e_n\|_\rho}\left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_m^*\|_r}{\min_{|t|=\rho} |q_m^*(t)|}\right\}$$

$$\leqslant 2\|e_m^*\|_\rho \left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_n\|_\rho}{\min_{|t|=\rho} |q_m^*(t)|}; \tag{3.7}$$

recall that $r < \rho$. Next, as q_m^* has no zeros in $\overline{B_\sigma}$,

$$\frac{\|q_m^*\|_r}{\min_{|t|=\rho} |q_m^*(t)|} \leqslant \frac{\|q_m^*\|_\rho}{\min_{|t|=\rho} |q_m^*(t)|} \leqslant \left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m. \tag{3.8}$$

Then using (3.6),

$$\frac{\|e_n\|_r}{\|e_n\|_\rho}\left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_m^*\|_r}{\min_{|t|=\rho} |q_m^*(t)|}$$

$$\leqslant \left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho} |\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)} \left(\frac{\rho}{r}\right)^{m+n} \left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m \leqslant \frac{1}{2}$$

by our hypothesis (3.3). So (3.7) gives

$$\|e_n\|_\rho \leqslant 4\|e_m^*\|_\rho \left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_n\|_\rho}{\min_{|t|=\rho} |q_m^*(t)|}. \tag{3.9}$$

Here, provided that q_n has no zeros on the circle $|t| = \rho$,

$$\|e_n\|_\rho \geqslant \min_{|t|=\rho} |q_n(t)| \left\|f - \frac{p_n}{q_n}\right\|_\rho$$

$$\geqslant \min_{|t|=\rho} |q_n(t)| E_n(f, \{t\colon |t| = \rho\}) \geqslant \min_{|t|=\rho} |q_n(t)| \frac{1}{7n^2} E_n(f, \overline{B_\rho})$$

by a classical inequality of Gonchar and Grigorjan's for the analytic parts of meromorphic functions, for the simply connected domain B_ρ (see [12], p. 145, Corollary 1, and [11], p. 571, Theorem 1). Moreover,

$$\|e_m^*\|_\rho \leqslant E_m(f, \overline{B_\sigma})\|q_m^*\|_\rho.$$

Combining the last two inequalities and (3.9) gives

$$\min_{|t|=\rho} |q_n(t)| \frac{1}{7n^2} E_n(f, \overline{B_\rho}) \leqslant 4 \frac{\|q_m^*\|_\rho}{\min_{|t|=\rho} |q_m^*(t)|} \left(\frac{\rho}{r}\right)^{m+n} \|q_n\|_\rho E_m(f, \overline{B_\sigma}).$$

Applying (3.8) once more, we obtain (3.4).

We also give an alternative form, which involves errors of the same approximant on balls of different radii.

Lemma 3.3. *Assume the hypotheses of Lemma 3.2 hold. Then either we have* (I) *there, or*

(II′) *for* $\varepsilon < r < \rho < 1$ *and* $\rho < s < \sigma$,

$$\frac{E_n(f,\overline{B_r})}{E_n(f,\overline{B_s})} \leqslant \frac{7n^2}{1-\rho/s}\left(\frac{2}{1-\rho/s}\right)^n \left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin(\frac{\rho-r}{\rho+r})} \times \frac{\|\omega_n\|_\rho}{\min_{|t|=s}|\omega_n(t)|}\,\frac{\|q_n\|_s}{\min_{|t|=r}|q_n(t)|}.$$

Proof. (II′) We start with (3.6). As in the previous proof,

$$\|e_n\|_r \geqslant \frac{1}{7n^2}\min_{|t|=r}|q_n(t)|E_n(f,\overline{B_r}). \tag{3.10}$$

Also, if $\rho < s \leqslant \sigma$ and $R_n^*(f,\overline{B_s}) = p_n^*/q_n^*$, Cauchy's integral formula gives for $|z| < s$,

$$\begin{aligned}\frac{(fq_n-p_n)(z)q_n^*(z)}{\omega_n(z)} &= \frac{1}{2\pi i}\int_{|t|=s}\frac{(fq_n-p_n)(t)q_n^*(t)}{\omega_n(t)}\,\frac{dt}{t-z}\\ &= \frac{1}{2\pi i}\int_{|t|=s}\frac{(fq_n^*-p_n^*)(t)q_n(t)}{\omega_n(t)}\,\frac{dt}{t-z}\end{aligned}$$

(since $(p_n^*q_n - p_nq_n^*)(t)/(\omega_n(t)(t-z))$ is analytic outside this circle and $O(t^{-2})$ at ∞). We deduce that

$$\|e_n\|_\rho \leqslant \frac{\|\omega_n\|_\rho}{\min_{|t|=s}|\omega_n(t)|}E_n(f,\overline{B_s})\frac{\|q_nq_n^*\|_s}{\min_{|t|=\rho}|q_n^*|(t)}\,\frac{1}{1-\rho/s}.$$

Combining this, (3.10) and (3.6) gives

$$\begin{aligned}\frac{E_n(f,\overline{B_r})}{E_n(f,\overline{B_s})} &\leqslant \frac{7n^2}{\min_{|t|=r}|q_n(t)|}\,\frac{\|e_n\|_r}{\|e_n\|_\rho}\,\frac{\|e_n\|_\rho}{E_n(f,\overline{B_s})}\\ &\leqslant \frac{7n^2}{1-\rho/s}\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin(\frac{\rho-r}{\rho+r})}\\ &\quad\times\frac{\|\omega_n\|_\rho}{\min_{|t|=s}|\omega_n(t)|}\,\frac{\|q_nq_n^*\|_s}{\min_{|t|=r}|q_n(t)|\min_{|t|=\rho}|q_n^*|(t)},\end{aligned}$$

provided q_n has no zeros on the circle $|t| = r$. Finally, as q_n^* has no zeros in $\overline{B_s}$,

$$\frac{\|q_n^*\|_s}{\min_{|t|=\rho}|q_n^*|(t)} \leqslant \left(\frac{2}{1-\rho/s}\right)^n.$$

The lemma is proved.

Next, we apply Cartan's Lemma in the standard way.

Lemma 3.4. *Let Q be a polynomial of degree $\leqslant n$ and $s \geqslant 1$. Let $\eta \in (0,1)$. There exists a set $\mathscr{E} \subset [0,s]$ of linear measure $\leqslant s\eta$ such that for $r \in [0,s] \setminus \mathscr{E}$,*

$$\frac{\|Q\|_s}{\min_{|t|=r}|Q(t)|} \leqslant \left(\frac{12e}{\eta}\right)^n.$$

Proof. We can factorize Q as

$$Q(z) = \left(\prod_{|z_j|<2s}(z-z_j)\right)\left(\prod_{|z_j|\geqslant 2s}\left(1-\frac{z}{z_j}\right)\right).$$

Let k be the number of terms in the first product and ℓ the number in the second. Then for $r \leqslant s$,

$$\frac{\|Q\|_s}{\min_{|t|=r}|Q(t)|} \leqslant \frac{(3s)^k 3^\ell}{\min_{|t|=r}\left|\prod_{|z_j|<2s}(t-z_j)\right|}.$$

By Cartan's Lemma (see [1], p. 325, Theorem 6.6.7),

$$\left|\prod_{|z_j|<2s}(t-z_j)\right| \geqslant \varepsilon^k$$

outside a union of at most k circles, the sum of whose diameters is at most $4e\varepsilon$. Let $\varepsilon = s\eta/(4e)$ and let $\mathscr{E}$ be the set of all $r \in [0,\infty)$ for which some z with $|z| = r$ lies in one of these circles. Then it is clear that $\mathscr{E}$ has linear measure at most $4e\varepsilon = s\eta$, and for $r \notin \mathscr{E}$,

$$\min_{|t|=r}\left|\prod_{|z_j|<2s}(t-z_j)\right| \geqslant \left(\frac{s\eta}{4e}\right)^k,$$

so

$$\frac{\|Q\|_s}{\min_{|t|=r}|Q(t)|} \leqslant \frac{(3s)^k 3^\ell}{(\frac{s\eta}{4e})^k} \leqslant \left(\frac{12e}{\eta}\right)^n.$$

The lemma is proved.

The next lemma appears in [14], p. 514, Lemma 3.3, as a consequence of a more general result. However, for completeness, we give a simpler proof of this special case.

Lemma 3.5. *Let $\mathscr{D}$ be a bounded simply connected open set and let $f \in \mathscr{R}_0(\overline{\mathscr{D}})$. Let T and K be compact subsets of $\mathscr{D}$ with T having positive logarithmic capacity. Let $\delta \in (0,1)$. Then for large enough n,*

$$E_n(f,K) \leqslant E_n(f,T)^{1-\delta}.$$

Proof. Write $R_n^*(f,T) = p_n^*/q_n^*$. Let $\theta \in (0,1)$ and for k so large that $E_n(f,T) < 1$,

$$k = k(n) = \text{ least integer } \geqslant \frac{\log E_n(f,T)}{\log\theta}. \tag{3.11}$$

We shall choose θ small enough later. Observe that $k \to \infty$ as $n \to \infty$, and

$$\theta^k \leqslant E_n(f,T). \tag{3.12}$$

Since $f \in R_0(\overline{\mathscr{D}})$, we can find for large enough n and $k = k(n)$, a rational function $p_k^{\#}/q_k^{\#}$ of type (k,k) such that

$$\left\| f - \frac{p_k^{\#}}{q_k^{\#}} \right\|_{L_\infty(\overline{\mathscr{D}})} \leqslant \theta^k. \tag{3.13}$$

Then

$$\left\| \frac{p_n^*}{q_n^*} - \frac{p_k^{\#}}{q_k^{\#}} \right\|_{L_\infty(T)} \leqslant E_n(f,T) + \theta^k \leqslant 2E_n(f,T),$$

so that

$$\|p_n^* q_k^{\#} - p_k^{\#} q_n^*\|_{L_\infty(T)} \leqslant 2E_n(f,T)\|q_n^* q_k^{\#}\|_{L_\infty(T)}.$$

Next, as T has positive logarithmic capacity, the Bernstein-Walsh inequality (see [18], p. 156) shows that there is a constant C_0, depending only on T and $\overline{\mathscr{D}}$, such that

$$\|p_n^* q_k^{\#} - p_k^{\#} q_n^*\|_{L_\infty(\overline{\mathscr{D}})} \leqslant C_0^{n+k}\|p_n^* q_k^{\#} - p_k^{\#} q_n^*\|_{L_\infty(T)},$$

and hence

$$\|p_n^* q_k^{\#} - p_k^{\#} q_n^*\|_{L_\infty(\overline{\mathscr{D}})} \leqslant 2C_0^{n+k} E_n(f,T)\|q_n^* q_k^{\#}\|_{L_\infty(T)}. \tag{3.14}$$

Then for $z \in \overline{\mathscr{D}}$,

$$\begin{aligned} \left| f - \frac{p_n^*}{q_n^*} \right|(z) &\leqslant \left| f - \frac{p_k^{\#}}{q_k^{\#}} \right|(z) + \frac{|p_k^{\#} q_n^* - p_n^* q_k^{\#}|(z)}{|q_n^* q_k^{\#}|(z)} \\ &\leqslant E_{nn}(f,T)\left\{ 1 + 2C_0^{n+k} \frac{\|q_n^* q_k^{\#}\|_{L_\infty(T)}}{|q_n^* q_k^{\#}|(z)} \right\}. \end{aligned}$$

Here we have used (3.12)–(3.14). Next, as in the proof of Lemma 3.4, given $\eta > 0$,

$$\frac{\|q_n^* q_k^{\#}\|_{L_\infty(T)}}{|q_n^* q_k^{\#}|(z)} \leqslant \left(\frac{C_1}{\eta} \right)^{n+k}$$

for $z \in \mathscr{D} \setminus \mathscr{E}$, where C_1 is a constant that depends only on the diameter of $\mathscr{D}$, and $\mathscr{E}$ is the union of at most $n+k$ open balls the sum of whose diameters is at most η. Choose η to be half the distance from K to the boundary of $\mathscr{D}$. This parameter is independent of f, n, k and θ. Then we can find a simple closed contour Γ in $\mathscr{D}$ that encloses K, but lies inside $\mathscr{D}$, that does not intersect any ball in $\mathscr{E}$. For example, we can take Γ to be $\{t \in \mathscr{D}\colon \operatorname{dist}(t, \partial\mathscr{D}) = \eta/3\}$, but where this level curve intersects $\mathscr{E}$, we deform Γ to run along the boundary of $\mathscr{E}$. Thus

$$\sup_{z \in \Gamma} \left| f - \frac{p_n^*}{q_n^*} \right|(z) \leqslant 4E_n(f,T)\left(\frac{C_0 C_1}{\eta} \right)^{n+k}.$$

Next, as the interior of Γ is simply connected, Gonchar-Grigorjan's classical inequality shows that

$$E_n(f,K) \leqslant 7n^2 E_n(f,\Gamma) \leqslant 7n^2 \sup_{z\in\Gamma}\left|f-\frac{p_n^*}{q_n^*}\right|(z)$$
$$\leqslant 28n^2 E_n(f,T)\left(\frac{C_0C_1}{\eta}\right)^{n+k}.$$

Here, letting $B = C_0C_1/\eta$, our choice (3.11) of k gives

$$\left(\frac{C_0C_1}{\eta}\right)^{k-1} = \exp((k-1)\log B) \leqslant E_n(f,T)^{\log B/\log\theta}.$$

Thus

$$E_n(f,K) \leqslant E_n(f,T)(28n^2B^{n+1}E_n(f,T)^{\log B/\log\theta}) \leqslant E_n(f,T)^{1-\delta}$$

for n large enough, if we choose θ so small that $|\log B/\log\theta| \leqslant \delta/2$, and also use the fact that

$$\lim_{n\to\infty} E_n(f,T)^{1/n} = 0.$$

The lemma is proved.

Proof of Theorem 3.1. We simplify (3.3) and (3.4). We choose $\sigma = 1$ in Lemma 3.2 and $s = 1$ and $\eta = 1/5$ in Lemma 3.4. Lemma 3.4 shows that there exists $\rho \in [1/2, 3/4]$ with

$$\frac{\|q_n\|_\rho}{\min_{|t|=\rho}|q_n(t)|} \leqslant \frac{\|q_n\|_1}{\min_{|t|=\rho}|q_n(t)|} \leqslant (60e)^n.$$

Also, we choose $r = 1/4$. Then $(\rho - r)/(\rho + r) \geqslant 1/3$. Also as $m \leqslant Ln$ and all the zeros of ω_n lie in B_ε,

$$\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin(\frac{\rho-r}{\rho+r})}\left(\frac{\rho}{r}\right)^{m+n}\left(\frac{1+\rho}{1-\rho}\right)^m$$
$$\leqslant \left(\frac{2\varepsilon}{\rho-\varepsilon}\right)^{\frac{2}{\pi}(2n+1)\arcsin(\frac{1}{3})} 3^{n(1+L)}\left(\frac{1+3/4}{1-3/4}\right)^{Ln}$$
$$\leqslant [(8\varepsilon)^{\frac{4}{\pi}\arcsin(\frac{1}{3})}3^{1+L}7^L]^n < \frac{1}{2}$$

for large enough n, by (3.1). So (3.3) is satisfied. Next, we reformulate (3.4) as

$$E_n(f,\overline{B_\rho}) \leqslant 28n^2 3^{m+n}\left(\frac{1+3/4}{1-3/4}\right)^m (60e)^n E_m(f,\overline{B_1}).$$

Since $m \geqslant n$, and $f \in \mathscr{R}_0(\overline{B_s})$, for some $s > 1$, this in turn implies that for large enough n,

$$E_n(f,\overline{B_\rho}) \leqslant E_m(f,\overline{B_1})^{1-\delta/2}.$$

In view of Lemma 3.5, we can replace B_ρ by B_1 for large enough n. The theorem is proved.

As an immediate corollary of Theorem 3.1, we have the following.

Corollary 3.6. *Assume the hypotheses of Theorem 3.1 hold, apart from the hypothesis about the poles of* $\{R_n(\Lambda_n, z)\}$. *Assume also that for* $n \in \mathscr{S}$, $n-1$ *is an exact interpolation index for* f *and* B_1, *and* (3.2) *fails. Then for large enough* $n \in \mathscr{S}$ *and any* $\Lambda_n \subset B_\varepsilon$, $R_n(\Lambda_n, z)$ *has no poles in* B_ε.

§ 4. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Fix $A > 1$ and let $g(z) = f(Az)$. This is entire, and for large enough k, our hypothesis on f ensures that $n_k - 1$ is an exact interpolation index for g and B_1. For $k \geqslant 1$, define $k^* = k^*(k)$ by

$$n_{k^*} = \inf\{n_j : n_j \geqslant Ln_k\}.$$

This is well defined as $\{n_j\}$ is increasing and has limit ∞. Moreover,

$$n_{k^*-1} < Ln_k,$$

so using (1.1),

$$Ln_k \leqslant n_{k^*} \leqslant Ln_{k^*-1} < L^2 n_k.$$

Next, we have

$$\lim_{k\to\infty} \eta_{n_k}(g, \overline{B_1}) = 0,$$

so we can choose a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ with the property that

$$\eta_{n_{k_j}}(g, \overline{B_1}) > \eta_{n_\ell}(g, \overline{B_1}) \quad \text{whenever } \ell > k_j.$$

Observe that with k_j^* defined as above, we have

$$Ln_{k_j} \leqslant n_{k_j^*} < L^2 n_{k_j},$$

and by the choice of k_j, for j large enough

$$E_{n_{k_j}}(g, \overline{B_1}) > (E_{n_{k_j^*}}(g, \overline{B_1}))^{n_{k_j}/n_{k_j^*}} \geqslant E_{n_{k_j^*}}(g, \overline{B_1})^{1/L} = E_{n_{k_j^*}}(g, \overline{B_1})^{1-\delta},$$

where $\delta = 1 - 1/L > 0$. Thus with $n = n_{k_j}$ and $m = n_{k_j^*}$, (3.2) in Theorem 3.1 is not true. Assume now that ε satisfies (3.1); it does not depend on A, g and f, but does depend on L. If for infinitely many k, and the corresponding $\Lambda_{n_k} \subset B_{A\varepsilon}$, the interpolant $R_{n_{k_j}}(\Lambda_{n_{k_j}}, \cdot)$ for f has a pole in $B_{A\varepsilon}$, then for the corresponding interpolant for g with points in B_ε, the interpolant has a pole in B_ε. In this case, we are fulfilling the initial hypotheses of Theorem 3.1, but neither of the alternative conclusions (I) and (II) hold, so we have a contradiction. Thus for large enough k, and any $\Lambda_{n_k} \subset B_{A\varepsilon}$, the interpolant $R_{n_{k_j}}(\Lambda_{n_{k_j}}, \cdot)$ for f cannot have poles in $B_{A\varepsilon}$. Since A is arbitrary, and ε is independent of A, the proof is complete.

Proof of Theorem 1.4 assuming (1.3). Let $A > 1$. Choose $\delta' \in (\delta, 1)$. By Lemma 3.5, for large enough k, our hypothesis (1.3) gives

$$E_{n_k}(f, \overline{B_A}) > E_{Mn_k}(f, \overline{B_A})^{1-\delta'}.$$

Applying this to $g(z) = f(Az)$ gives

$$E_{n_k}(g, \overline{B_1}) > E_{Mn_k}(g, \overline{B_1})^{1-\delta'}.$$

Also for large enough k, $n_k - 1$ is an exact interpolation index for g and B_1. We can then apply Theorem 3.1 with $\sigma = 1$, $n = n_k$, $m = Mn_k$, and L replaced by M. Since both alternatives (I), (II) of Theorem 3.1 fail, it follows that for large enough $n = n_k$, the interpolant $R_{n_k}(\Lambda_{n_k}, z)$ for g has no poles in B_ε, where ε satisfies

$$3(8\varepsilon)^{\frac{4}{\pi}\arcsin(\frac{1}{3})} 21^M < 1.$$

Then for large enough k, the interpolant $R_{n_k}(\Lambda_{n_k}, z)$ for f has no poles in $B_{A\varepsilon}$. As ε does not depend on A, the proof is complete.

Proof of Theorem 1.4 assuming (1.4). We apply Lemma 3.3. Let $g(z) = f(2rz)$, where r is one of the sequence of values r with the property (1.4). Assume that for infinitely many $n = n_k$, the interpolant $R_{n_k}(\Lambda_{n_k}, z)$ for g has a pole in B_ε, where $\varepsilon \in (0, 1/8)$. Assume that $\varepsilon < r < \rho < 1$ and $\rho < s < \sigma$. Our hypothesis on exact indices for f shows that alternative (I) in Lemma 3.3 is not possible for g. We now show that this leads to a contradiction in the alternative (II′) in Lemma 3.3. Combining Lemmas 3.3 and 3.4, we have for $n \in \{n_k\}$, such that the interpolant $R_{n_k}(\Lambda_{n_k}, z)$ for g has a pole in B_ε,

$$\frac{E_n(g, \overline{B_r})}{E_n(g, \overline{B_s})} \leqslant \frac{7n^2}{1-\rho/s}\left(\frac{2}{1-\rho/s}\right)^n \left(\frac{2\varepsilon}{\rho-\varepsilon}\right)^{\frac{2}{\pi}(2n+1)\arcsin(\frac{\rho-r}{\rho+r})} \times \left(\frac{\rho+\varepsilon}{s-\varepsilon}\right)^{2n+1}\left(\frac{12e}{\eta}\right)^n.$$

Here also by Lemma 3.4, we need $r \in [0,1] \setminus \mathscr{E}$, where $\mathrm{meas}(\mathscr{E}) < \eta$. Choose $\eta = 1/5$, $s = 1$, $\sigma = 2$, $\rho = 7/8$ and some suitable $r \in [1/2, 3/4]$. Then, using the monotonicity of errors of rational approximation in the set, we obtain

$$\frac{E_n(g, \overline{B_{1/2}})}{E_n(g, \overline{B_2})} \leqslant 56n^2 16^n (4\varepsilon)^{\frac{2}{\pi}(2n+1)\arcsin(\frac{1}{13})}\left(\frac{8}{7}\right)^{2n+1}(60e)^n.$$

For large enough n, our hypothesis (1.4) transferred from f to g, gives

$$\left(\frac{1}{T}\right)^n \leqslant 56n^2 16^n (4\varepsilon)^{\frac{2}{\pi}(2n+1)\arcsin(\frac{1}{13})}\left(\frac{8}{7}\right)^{2n+1}(60e)^n.$$

Let $T' > T$. Taking nth roots, for large enough $n \in \{n_k\}$,

$$\frac{1}{T'} \leqslant 16(4\varepsilon)^{\frac{4}{\pi}\arcsin(\frac{1}{13})}\left(\frac{8}{7}\right)^2 60e.$$

Thus for large enough $n \in \{n_k\}$, any interpolant $R_{n_k}(\Lambda_{n_k}, z)$ for g with interpolation points in B_ε has no poles in B_ε if ε is so small that it violates this last bound. Hence also any interpolant $R_{n_k}(\Lambda_{n_k}, z)$ for f with points in $B_{A\varepsilon}$ has no poles in $B_{A\varepsilon}$. Theorem 1.4 is proved.

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