

## INVOLUTIVE BORDERED FLOER HOMOLOGY

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ABSTRACT. We give a bordered extension of involutive  $\widehat{HF}$  and use it to give an algorithm to compute involutive  $\widehat{HF}$  for general 3-manifolds. We also explain how the mapping class group action on  $\widehat{HF}$  can be computed using bordered Floer homology. As applications, we prove that involutive  $\widehat{HF}$  satisfies a surgery exact triangle and compute  $\widehat{HFI}(\Sigma(K))$  for all 10-crossing knots  $K$ .

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### 1. INTRODUCTION

In 2013, Manolescu introduced a  $Pin(2)$ -equivariant version of Seiberg-Witten Floer homology and used it to resolve the Triangulation Conjecture [Man16]. Since then, several authors have given applications of these new invariants, particularly to the homology cobordism group [Man14, Lin15, Sto15b, Sto15a, Sto17]. F. Lin also gave a reformulation of  $Pin(2)$ -equivariant Seiberg-Witten Floer homology and deduced a number of formal properties, such as a surgery exact triangle, in addition to various applications [Lin14, Lin17b, Lin17a, Lin16b, Lin16a].

Two years later, Manolescu and the first author introduced a shadow of  $Pin(2)$ -equivariant Seiberg-Witten Floer homology, called *involutive Heegaard Floer homology* [HM17], in Ozsváth-Szabó’s Heegaard Floer homology [OSz04b]. Involutive Heegaard Floer homology has also had a number of applications, again mainly to the homology cobordism group [HMZ17, BH16, DM17, Zem16].

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As described below, a key step in the definition of involutive Heegaard Floer homology is naturality of the Heegaard Floer invariants [OSz06, JT12]. Another implication of naturality is that the mapping class group of a 3-manifold  $Y$  acts on the Heegaard Floer invariants of  $Y$ ; this action has been studied relatively little.

Bordered Heegaard Floer homology, introduced by Ozsváth, Thurston, and the second author, extends the Heegaard Floer invariant  $\widehat{HF}(Y)$  to 3-manifolds with boundary [LOT08, LOT15], leading to a practical algorithm for computing  $\widehat{HF}(Y)$  [LOT14b]. In this paper we extend that algorithm to compute both the hat variant of involutive Heegaard Floer homology and the mapping class group action on  $\widehat{HF}(Y)$ . Although the two actions are different, their descriptions in terms of bordered Floer homology are quite similar. We also prove a (hitherto unknown) surgery exact triangle for the hat variant of involutive Heegaard Floer homology. In the rest of the introduction we recall some of the definitions and sketch how these algorithms work.

Given a 3-manifold  $Y$ , the minus (respectively hat) involutive Heegaard Floer complex of  $Y$  is defined as follows [HM17]. Fix a pointed Heegaard diagram  $\mathcal{H}$  for  $Y$ . Recall that  $CF^-(\mathcal{H})$  (respectively  $\widehat{CF}(\mathcal{H})$ ) is a chain complex of free  $\mathbb{F}_2[U]$ -modules (respectively  $\mathbb{F}_2$ -vector spaces). Consider the modules

$$CFI^-(\mathcal{H}) = CF^-(\mathcal{H})[-1] \otimes_{\mathbb{F}_2[U]} \mathbb{F}_2[U, Q]/(Q^2),$$

$$\widehat{CFI}(\mathcal{H}) = \widehat{CF}(\mathcal{H})[-1] \otimes_{\mathbb{F}_2} \mathbb{F}_2[Q]/(Q^2)$$

over  $\mathbb{F}_2[U, Q]/(Q^2)$  (respectively  $\mathbb{F}_2[Q]/(Q^2)$ ), where  $Q$  has degree  $-1$  and  $U$  has degree  $-2$ . Define a differential on  $CFI^-(\mathcal{H})$  and  $\widehat{CFI}(\mathcal{H})$  by

$$(1.1) \quad \partial_{CFI}(x) = \partial_{CF}(x) + [x + \iota(x)]Q,$$

where  $\partial_{CF}(x)$  is the usual differential on  $CF^-(\mathcal{H})$  or  $\widehat{CF}(\mathcal{H})$  and  $\iota$  is an endomorphism of  $CF^-(\mathcal{H})$  or  $\widehat{CF}(\mathcal{H})$  defined as follows. Let  $\overline{\overline{\mathcal{H}}}$  be the result of exchanging the roles of the  $\alpha$ - and  $\beta$ -circles and reversing the orientation of the Heegaard surface; i.e., if

$$\mathcal{H} = (\Sigma, \alpha, \beta, z),$$

then

$$\overline{\overline{\mathcal{H}}} = (-\Sigma, \beta, \alpha, z).$$

Given a generator  $\mathbf{x} = \{x_i \in \alpha_i \cap \beta_{\sigma(i)}\} \subset \Sigma$  for  $CF^-(\mathcal{H})$  (respectively  $\widehat{CF}(\mathcal{H})$ ), exactly the same set of points gives a generator  $\eta(\mathbf{x})$  for  $CF^-(\overline{\overline{\mathcal{H}}})$ . For suitable choices of almost complex structures on  $\text{Sym}^g(\Sigma)$  and  $\text{Sym}^g(-\Sigma)$ , the map  $\eta$  is a chain isomorphism. Next, since  $\mathcal{H}$  and  $\overline{\overline{\mathcal{H}}}$  both represent  $Y$ , there is a sequence of Heegaard moves from  $\overline{\overline{\mathcal{H}}}$  to  $\mathcal{H}$ . There is then a corresponding chain homotopy equivalence  $\Phi: CF^-(\overline{\overline{\mathcal{H}}}) \rightarrow CF^-(\mathcal{H})$  (respectively  $\Phi: \widehat{CF}(\overline{\overline{\mathcal{H}}}) \rightarrow \widehat{CF}(\mathcal{H})$ ) associated to this sequence of Heegaard moves (together with changes of almost complex structures) [OSz04b]; the map  $\Phi$  is well-defined up to chain homotopy [OSz06, JT12, HM17]. Then

$$\iota = \Phi \circ \eta.$$

Formula (1.1) makes  $CFI^-(\mathcal{H})$  (respectively  $\widehat{CFI}(\mathcal{H})$ ) into a differential  $\mathbb{F}_2[U, Q]/(Q^2)$ -module (respectively  $\mathbb{F}_2[Q]/(Q^2)$ -module), and hence the homology

$HFI^-(\mathcal{H})$  (respectively  $\widehat{HFI}(\mathcal{H})$ ) is also a module over  $\mathbb{F}_2[U, Q]/(Q^2)$  (respectively  $\mathbb{F}_2[Q]/(Q^2)$ ).

In this paper, we will focus mainly on  $\widehat{CFI}(\mathcal{H})$  and its homology  $\widehat{HFI}(\mathcal{H})$ . Up to isomorphism, the homology groups  $\widehat{HFI}(\mathcal{H})$  are determined by the induced map  $\iota_*: \widehat{HF}(\mathcal{H}) \rightarrow \widehat{HF}(\mathcal{H})$  on homology:

$$\widehat{HFI}(\mathcal{H}) \cong (\ker(\text{Id} + \iota_*) \oplus Q \text{coker}(\text{Id} + \iota_*))[-1]$$

with the obvious  $\mathbb{F}_2[Q]/(Q^2)$ -module structure (e.g., if  $x \in \ker(\text{Id} + \iota_*) \subset \widehat{HF}(\mathcal{H})$ , then  $Qx$  is the image of  $x$  in  $\text{coker}(\text{Id} + \iota_*)$ ).

Before explaining how to compute involutive Heegaard Floer homology, we review the bordered algorithm to compute  $\widehat{HF}(Y)$  [LOT14b]. (This was not the first algorithm to compute  $\widehat{HF}(Y)$ , which was discovered by Sarkar-Wang [SW10].) Choosing a Heegaard splitting of  $Y$  allows us to write  $Y$  as a union of two (standard) handlebodies  $H_g$  of genus  $g$ , glued by a diffeomorphism  $\psi: \Sigma_g \rightarrow \Sigma_g$  of their boundaries. Let  $\mathcal{Z}$  be the split, genus  $g$  pointed matched circle [LOT14b, Figure 4] and let  $F(\mathcal{Z})$  be the corresponding surface. Let  $\phi_0: F(\mathcal{Z}) \rightarrow \partial H_g$  be the 0-framed parametrization [LOT14b, Section 1.4.1]. Then

$$(1.2) \quad \widehat{CF}(Y) \simeq \widehat{CFA}(H_g, \phi_0) \boxtimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(H_g, \phi_0 \circ \psi).$$

The bordered modules  $\widehat{CFA}(H_g, \phi_0)$  and  $\widehat{CFD}(H_g, \phi_0)$  can be described explicitly; see Section 2.2. Further, if  $\widehat{CFDA}(\psi)$  is the type  $DA$  bordered bimodule associated to the mapping cylinder of  $\psi$ , then

$$\widehat{CFD}(H_g, \phi_0 \circ \psi) \simeq \widehat{CFDA}(\psi) \boxtimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(H_g, \phi_0).$$

One factors  $\psi$  as a composition  $\psi = \psi_1 \circ \dots \circ \psi_n$  where each  $\psi_i$  is an arc-slide [LOT14b, Section 2.1]. Then

$$\widehat{CFDA}(\psi) \simeq \widehat{CFDA}(\psi_1) \boxtimes \dots \boxtimes \widehat{CFDA}(\psi_n).$$

The type  $DD$  bimodule  $\widehat{CFDD}(\psi_i)$  associated to each arcslide can be described explicitly [LOT14b, Section 4]. The type  $DA$  bimodule  $\widehat{CFDA}(\psi_i)$  can be computed as

$$\begin{aligned} \widehat{CFDA}(\psi_i) &\simeq \widehat{CFAA}(\mathbb{I}) \boxtimes_{\mathcal{A}(-\mathcal{Z})} \widehat{CFDD}(\psi_i), \\ \widehat{CFAA}(\mathbb{I}) &\simeq \widehat{CFAA}(\text{AZ} \cup \overline{\text{AZ}}), \end{aligned}$$

and  $\text{AZ} \cup \overline{\text{AZ}}$  is a particularly nice bordered Heegaard diagram introduced by Auroux and Zarev [Aur10, Zar10, LOT11] (see Section 2.4) whose type  $AA$  bimodule is, consequently, easy to describe.

Combining these steps gives an algorithm to compute  $\widehat{CF}(Y)$ . This algorithm is practical, at least for manifolds with small Heegaard genus and not-too-complicated gluing maps [LOT14b, Section 9.5]. Further improvements have been made by Zhan [Zha14].

The other key tools for computing involutive Heegaard Floer homology come from earlier work on dualities in bordered Heegaard Floer homology [LOT11]. Recall that a bordered Heegaard diagram consists of an oriented surface-with-boundary  $\Sigma$ , a collection  $\alpha$  of arcs and circles in  $\Sigma$ , a collection  $\beta$  of circles in  $\Sigma$ , and a basepoint  $z$  in  $\partial\Sigma$  satisfying certain conditions [LOT08, Section 4.1]. We can also consider a  $\beta$ -bordered Heegaard diagram in which  $\alpha$  consists only of circles

and  $\beta$  consists of arcs and circles [LOT11, Section 3.1]. Given a bordered Heegaard diagram  $\mathcal{H}$ , there is an associated  $\beta$ -bordered Heegaard diagram  $\mathcal{H}^\beta$ , obtained by exchanging the roles of the  $\alpha$ - and  $\beta$ -curves in  $\mathcal{H}$ . The boundary of a  $\beta$ -bordered Heegaard diagram is a  $\beta$ -pointed matched circle. Given a pointed matched circle  $\mathcal{Z}$ , let  $\mathcal{Z}^\beta$  be the corresponding  $\beta$ -pointed matched circle. Another operation on bordered Heegaard diagrams (respectively pointed matched circles) is reversal of the orientation of the Heegaard surface (respectively circle); we will denote this with a minus sign. Given a Heegaard diagram  $\mathcal{H}$  with boundary  $\mathcal{Z}$ , the invariants of these objects are related as follows:

$$\begin{aligned} \mathcal{A}(\mathcal{Z}^\beta) &= \mathcal{A}(\mathcal{Z})^{\text{op}} = \mathcal{A}(-\mathcal{Z}), \\ \mathcal{A}(-\mathcal{Z}^\beta)\widehat{CFD}(\mathcal{H}^\beta) &= \mathcal{A}(\mathcal{Z})\widehat{CFD}(\mathcal{H}^\beta) \cong \overline{\mathcal{A}(-\mathcal{Z})\widehat{CFD}(\mathcal{H})}, \\ \mathcal{A}(\mathcal{Z})\widehat{CFD}(-\mathcal{H}) &= \widehat{CFD}(-\mathcal{H})^{\mathcal{A}(\mathcal{Z})} \cong \overline{\mathcal{A}(-\mathcal{Z})\widehat{CFD}(\mathcal{H})}, \\ \widehat{CFA}(\mathcal{H}^\beta)_{\mathcal{A}(\mathcal{Z}^\beta)} &= \widehat{CFA}(\mathcal{H}^\beta)_{\mathcal{A}(-\mathcal{Z})} \cong \overline{\widehat{CFA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}}, \\ \widehat{CFA}(-\mathcal{H})_{\mathcal{A}(-\mathcal{Z})} &= \mathcal{A}(\mathcal{Z})\widehat{CFA}(-\mathcal{H}) \cong \overline{\widehat{CFA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}}, \end{aligned}$$

where the overline denotes the dual  $A_\infty$ -module or type  $D$  structure [LOT11]. As usual in the bordered Floer literature, we are using superscripts to denote type  $D$  structures and subscripts for  $A_\infty$  actions.

Given a bordered Heegaard diagram  $\mathcal{H}$  with boundary  $\mathcal{Z}$ , let  $\overline{\mathcal{H}} = -\mathcal{H}^\beta$ , so  $\overline{\mathcal{H}}$  is a  $\beta$ -bordered Heegaard diagram with boundary  $\overline{\mathcal{Z}} = -\mathcal{Z}^\beta$ . From the isomorphisms above, it follows that

$$\mathcal{A}(-\mathcal{Z})\widehat{CFD}(\overline{\mathcal{H}}) \cong \mathcal{A}(-\mathcal{Z})\widehat{CFD}(\mathcal{H}), \quad \widehat{CFA}(\overline{\mathcal{H}})_{\mathcal{A}(\mathcal{Z})} \cong \widehat{CFA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}.$$

These are the analogues of the isomorphism  $\eta$  in the definition of  $CFI$ , and we will denote these isomorphisms by  $\eta$  as well. In particular, it is immediate from the proofs of the isomorphisms (see [LOT11]) that the isomorphism  $\eta$  takes a generator  $\mathbf{x} \subset \alpha \cap \beta \subset \Sigma$  to the same subset of  $\Sigma$ .

The second ingredient in the definition of  $CFI$  is relating  $\overline{\mathcal{H}}$  and  $\mathcal{H}$  by a sequence of Heegaard moves. In the bordered setting this is not possible:  $\overline{\mathcal{H}}$  is  $\beta$ -bordered while  $\mathcal{H}$  is  $\alpha$ -bordered. The Auroux-Zarev piece  $\mathbf{AZ}$  comes to the rescue. Specifically, if we glue  $\mathbf{AZ}$  (respectively  $\overline{\mathbf{AZ}}$ ) to  $\overline{\mathcal{H}}$  along the  $\beta$ -boundary of  $\mathbf{AZ}$  or  $\overline{\mathbf{AZ}}$ , then we have

$$\overline{\mathcal{H}} \cup_{\partial} \mathbf{AZ} \sim \mathcal{H} \sim \overline{\mathcal{H}} \cup_{\partial} \overline{\mathbf{AZ}}$$

[LOT11, Lemma 4.6] (where  $\sim$  means the diagrams are related by a sequence of bordered Heegaard moves or, equivalently, represent the same bordered 3-manifold).

Now, fix bordered Heegaard diagrams  $\mathcal{H}_0, \mathcal{H}_1$  with  $\partial\mathcal{H}_0 = \mathcal{Z} = -\partial\mathcal{H}_1$ . Let  $Y = Y(\mathcal{H}_0 \cup_{\partial} \mathcal{H}_1)$  be the closed 3-manifold represented by  $\mathcal{H}_0 \cup_{\partial} \mathcal{H}_1$ . We show in Theorem 5.1 that, up to homotopy, the involution  $\iota$  on  $\widehat{CF}(Y)$  is the composition

of the following maps:

$$\begin{aligned}
(1.3) \quad & \widehat{CF}(Y) \simeq \widehat{CFA}(\mathcal{H}_0)_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1) \\
& \xrightarrow{\eta} \widehat{CFA}(\overline{\mathcal{H}_0})_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\overline{\mathcal{H}_1}) \\
& = \widehat{CFA}(\overline{\mathcal{H}_0})_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} [\text{Id}_{\mathcal{A}(\mathcal{Z})}]_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\overline{\mathcal{H}_1}) \\
& \xrightarrow{\Omega_1} \widehat{CFA}(\overline{\mathcal{H}_0})_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\overline{\mathcal{H}_1}) \\
& \xrightarrow{\Omega_2} \widehat{CFA}(\overline{\mathcal{H}_0})_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ})_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(AZ)_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\overline{\mathcal{H}_1}) \\
& \xrightarrow{\Psi = \Psi_0 \boxtimes \Psi_1} \widehat{CFA}(\mathcal{H}_0)_{\mathcal{A}(\mathcal{Z})} \boxtimes^{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1) \\
& \simeq \widehat{CF}(Y).
\end{aligned}$$

Here,  ${}^{\mathcal{A}(\mathcal{Z})}[\text{Id}_{\mathcal{A}(\mathcal{Z})}]_{\mathcal{A}(\mathcal{Z})}$  is the (type *DA*) *identity bimodule* of  $\mathcal{A}(\mathcal{Z})$ , i.e., the identity for the operation  $\boxtimes$  [LOT15, Definition 2.2.48], while  $\mathbb{I}_{\mathcal{Z}}$  is the standard bordered Heegaard diagram for the identity map of  $F(\mathcal{Z})$ . The map  $\Omega_1$  is induced by a homotopy equivalence between  $[\text{Id}_{\mathcal{A}(\mathcal{Z})}]$  and  $\widehat{CFDA}(\mathbb{I}_{\mathcal{Z}})$ , while  $\Omega_2$  is induced by a sequence of Heegaard moves from  $\mathbb{I}_{\mathcal{Z}}$  to the bordered Heegaard diagram  $\overline{AZ} \cup AZ$ . The map  $\Psi_0$  is induced by a sequence of Heegaard moves from  $\overline{\mathcal{H}_0} \cup \overline{AZ}$  to  $\mathcal{H}_0$ , and the map  $\Psi_1$  is induced by a sequence of Heegaard moves from  $AZ \cup \overline{\mathcal{H}_1}$  to  $\mathcal{H}_1$ .

To give an algorithm to compute  $\widehat{HFI}(Y)$  we restrict to the case that the  $\mathcal{H}_i$  come from a Heegaard splitting of  $Y$ . As discussed above, we can compute  $\widehat{CFA}(\mathcal{H}_0)$  and  $\widehat{CFD}(\mathcal{H}_1)$  in this case. Further, the diagrams  $AZ$  and  $\overline{AZ}$  are nice (both in the technical and colloquial sense), and so it is routine to compute  $\widehat{CFDA}(AZ)$  and  $\widehat{CFDA}(\overline{AZ})$ . We write these bimodules explicitly in Section 2.4. To compute  $\widehat{HFI}(Y)$  it remains to compute the maps  $\Omega = \Omega_2 \circ \Omega_1$  and  $\Psi = \Psi_0 \boxtimes \Psi_1$ . It turns out that both are determined by being the unique graded homotopy equivalences of the desired form; this is explained in Section 4 (Lemmas 4.2 and 4.5). In particular, one never needs to compute  $\widehat{CFDA}(\text{Id}_{\mathcal{Z}})$ . (These rigidity results were first observed in unpublished work of Ozsváth, Thurston, and the second author, and parallel results in Khovanov homology [Kho06].)

An arguably even nicer description of  $\iota$ , in terms of morphism complexes, is given in Section 8.

Changing topics slightly, given a closed 3-manifold  $Y$ , the based mapping class group of  $Y$  acts on  $\widehat{HF}(Y)$  [OSz06, JT12]. One can use bordered Floer homology to compute the mapping class group action in a similar way to  $\widehat{HFI}$ , so we explain that algorithm here as well. (We are interested in this action partly because it sometimes allows one to compute the concordance invariant  $q_\tau$  [HLS16].)

So, fix a closed 3-manifold  $Y$ , a basepoint  $p \in Y$ , and a mapping class  $[\chi] \in MCG(Y, p)$ . We can choose a Heegaard splitting  $Y = H_0 \cup_F H_1$  for  $Y$  and a representative  $\chi$  for  $[\chi]$  so that  $\chi$  respects the Heegaard splitting, i.e.,  $\chi(H_i) = H_i$  (Lemma 6.1). Let  $\psi$  denote the gluing map for the Heegaard splitting, so  $\widehat{CF}(Y)$  is computed by equation (1.2), and we know how to compute  $\widehat{CFA}(H_g, \phi_0)$  and  $\widehat{CFD}(H_g, \phi_0 \circ \psi)$ . Let  $\chi|_F$  denote the restriction of  $\chi$  to  $F$ . As described above, we can also compute  $\widehat{CFDA}(\chi|_F)$ . Since  $\chi|_F$  extends over  $H_i$ , the bordered

manifolds  $(H_g, \phi_0)$  and  $(H_g, \phi_0 \circ \chi|_F^{-1})$  are equivalent, as are the bordered manifolds  $(H_g, \phi_0 \circ \psi)$  and  $(H_g, \phi_0 \circ \psi \circ \chi|_F^{-1})$ . Thus, there are (grading-preserving) chain homotopy equivalences

$$\begin{aligned} \widehat{CFA}(H_g, \phi_0) \boxtimes \widehat{CFDA}(\chi|_F) &\xrightarrow{\Theta_0} \widehat{CFA}(H_g, \phi_0), \\ \widehat{CFDA}(\chi|_F^{-1}) \boxtimes \widehat{CFD}(H_g, \phi_0 \circ \psi) &\xrightarrow{\Theta_1} \widehat{CFD}(H_g, \phi_0 \circ \psi). \end{aligned}$$

In fact, we show in Section 4 that there are unique graded homotopy equivalences  $\Theta_0$  and  $\Theta_1$  between these modules (up to homotopy), so  $\Theta_0$  and  $\Theta_1$  are algorithmically computable (cf. Section 3). We show in Theorem 6.2 that the action of  $\chi$  on  $\widehat{HF}(Y)$  is given by the composition

$$\begin{aligned} \widehat{CF}(Y) &\simeq \widehat{CFA}(H_g, \phi_0) \boxtimes \widehat{CFD}(H_g, \phi_0 \circ \psi) \\ &= \widehat{CFA}(H_g, \phi_0) \boxtimes [\text{Id}_{\mathcal{A}(\mathcal{Z})}] \boxtimes \widehat{CFD}(H_g, \phi_0 \circ \psi) \\ (1.4) \quad &\longrightarrow \widehat{CFA}(H_g, \phi_0) \boxtimes \widehat{CFDA}(\chi|_F) \boxtimes \widehat{CFDA}(\chi|_F^{-1}) \boxtimes \widehat{CFD}(H_g, \phi_0 \circ \psi) \\ &\xrightarrow{\Theta_0 \boxtimes \Theta_1} \widehat{CFA}(H_g, \phi_0) \boxtimes \widehat{CFD}(H_g, \phi_0 \circ \psi) \\ &\simeq \widehat{CF}(Y) \end{aligned}$$

for an appropriate homotopy equivalence  $[\text{Id}_{\mathcal{A}(\mathcal{Z})}] \rightarrow \widehat{CFDA}(\chi|_F) \boxtimes \widehat{CFDA}(\chi|_F^{-1})$ . Again, there is a unique such homotopy equivalence, so this map is computable.

The paper has two more contents. In Section 5 we give a definition of *involutive bordered Floer homology*, which describes succinctly what information one needs to compute about a bordered 3-manifold in order to recover  $\widehat{HFI}$  of gluings. In Section 7 we use this description to prove a surgery exact triangle for involutive Heegaard Floer homology. (Previously, Lin proved that  $Pin(2)$ -equivariant monopole Floer homology admits a surgery exact triangle [Lin17b, Theorem 1], but surgery triangles for involutive Heegaard Floer homology have so far been elusive.)

This paper is organized as follows. In Section 2 we collect the results we need from the bordered Floer literature. Section 3 notes that, given two explicit, finitely generated type  $D$ ,  $A$ , or  $DA$  bimodules over the bordered algebras, computing the set of homotopy equivalences between them can be done algorithmically. The rigidity results—that there is a unique isomorphism between type  $D$  or  $A$  modules for the same bordered handlebody, and between type  $DD$ ,  $DA$ , or  $AA$  modules for the same mapping cylinder—are proved in Section 4. The fact that formula (1.3) computes the map  $\iota$  is proved in Section 5, which also proposes a general definition of involutive bordered Floer homology. Section 6 shows that formula (1.4) computes the mapping class group action on  $\widehat{HF}$ . The proof of the surgery triangle is in Section 7. Another computation of  $\iota$ , entirely in terms of type  $D$  modules, is given in Section 8. We conclude with computer computations for the branched double covers of 10-crossing knots in Section 9.

## 2. BACKGROUND

We assume the reader has a passing familiarity with bordered Heegaard Floer homology. The review in this section is focused on fixing notation and recalling some of the less well-known aspects of the theory such as gradings and the Auroux-Zarev diagram.

**2.1. The split pointed matched circle and its algebra.** Let  $\mathcal{Z}_k$  denote the *split pointed matched circle* for a surface of genus  $k$ . That is,  $\mathcal{Z}_k = (Z, \{a_1, \dots, a_{4k}\}, M, z)$ , where  $M$  matches  $a_{4i+1} \leftrightarrow a_{4i+3}$  and  $a_{4i+2} \leftrightarrow a_{4i+4}$ , for  $i = 0, \dots, k-1$ . Note that the matched pairs in  $\mathcal{Z}_k$  are in canonical bijection with  $\{1, \dots, 2k\}$  by identifying  $\{a_{4i+1}, a_{4i+3}\} \mapsto 2i+1$  and  $\{a_{4i+2}, a_{4i+4}\} \mapsto 2i+2$ .

The algebra  $\mathcal{A}(\mathcal{Z}_k)$  has a canonical  $\mathbb{F}_2$ -basis of *strand diagrams* and decomposes as a direct sum

$$\mathcal{A}(\mathcal{Z}_k) = \bigoplus_{i=-k}^k \mathcal{A}(\mathcal{Z}, i).$$

The integer  $i$  denotes the *weight* or *spin<sup>c</sup>-structure* of a strand diagram, which is the number of non-horizontal strands plus half the number of horizontal strands minus  $k$  [LOT08, Definition 3.23]. Only the summand  $\mathcal{A}(\mathcal{Z}_k, 0)$  will be relevant in this paper, and we will often abuse notation and let  $\mathcal{A}(\mathcal{Z}_k)$  denote  $\mathcal{A}(\mathcal{Z}_k, 0)$ .

It will be convenient to have names for certain elements of  $\mathcal{A}(\mathcal{Z}_k)$ . Given a subset  $\mathbf{s} \subset \{1, \dots, 2k\}$  with cardinality  $k$  there is a corresponding basic idempotent  $I(\mathbf{s}) \in \mathcal{A}(\mathcal{Z}_k, 0)$ . Next, for  $1 \leq i < j \leq 4k$  let  $\rho_{i,j}$  be the chord from  $a_i$  to  $a_j$ . There is a corresponding algebra element  $a(\rho_{i,j}) \in \mathcal{A}(\mathcal{Z}_k, 0)$ , the sum of all strand diagrams obtained by adding  $2k-2$  horizontal strands to  $\rho_{i,j}$  in any allowed way. To keep notation simple, we will often denote  $a(\rho_{i,j})$  by  $\rho_{i,j}$ .

In the special case that  $k=1$ ,  $\mathcal{A}(\mathcal{Z}_1, 0)$  has 8 generators:  $I(1)$ ,  $I(2)$ ,  $\rho_{1,2}$ ,  $\rho_{2,3}$ ,  $\rho_{3,4}$ ,  $\rho_{1,3}$ ,  $\rho_{2,4}$ , and  $\rho_{1,4}$ . The multiplication satisfies, for instance,  $\rho_{1,2}\rho_{2,3} = \rho_{1,3}$  and  $I(1)\rho_{1,2}I(2) = \rho_{1,2}$ .

Note that  $\mathcal{Z}_k$  is symmetric under reflection:  $-\mathcal{Z}_k \cong \mathcal{Z}_k$ .

There is an inclusion map

$$\iota: \overbrace{\mathcal{A}(\mathcal{Z}_1) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_1)}^k \hookrightarrow \mathcal{A}(\mathcal{Z}_k)$$

which sends  $\rho_{i,j}$  in the  $\ell$ th copy of  $\mathcal{A}(\mathcal{Z}_1)$  to  $\rho_{4(\ell-1)+i, 4(\ell-1)+j}$ . There is also a projection map

$$\pi: \mathcal{A}(\mathcal{Z}_k) \rightarrow \overbrace{\mathcal{A}(\mathcal{Z}_1) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_1)}^k$$

satisfying  $\pi \circ \iota = \text{Id}_{\mathcal{A}(\mathcal{Z}_1)^{\otimes k}}$  and  $\pi(\rho) = 0$  if  $\rho$  is a strand diagram not in the image of  $\iota$ . (These are special cases of the maps in [LOT15, Section 3.4].)

**2.2. Explicit descriptions of some bordered handlebodies.** Let  $Y_0$  be the *0-framed solid torus*. The type  $D$  structure  $\widehat{CFD}(Y_0)$  has a single generator  $n$  with

$$\delta^1(n) = \rho_{1,3}n.$$

The  $A_\infty$ -module  $\widehat{CFA}(Y_0)$  also has a description with a single generator, but more convenient for us will be the model with three generators  $t, u, v$ :

$$\begin{aligned} m_1(u) &= v, & m_2(u, \rho_{1,2}) &= t, \\ m_2(u, \rho_{1,3}) &= v, & m_2(t, \rho_{2,3}) &= v, \end{aligned}$$

and all other  $A_\infty$  operations vanish. In particular, this model for  $\widehat{CFA}(Y_0)$  is an ordinary *dg* module. (The conventions are chosen so that  $\widehat{CFA}(Y_0) \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} \widehat{CFD}(Y_0) \cong \widehat{HF}(S^2 \times S^1) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ .)

More generally, let  $Y_{0^k}$  be the 0-framed handlebody of genus  $k$ . Then the *standard type D structure for  $Y_{0^k}$* , denoted  $\widehat{CFD}(Y_{0^k})$ , is the image of  $\widehat{CFD}(Y_0)^{\otimes k}$  under the induction map  ${}^{\mathcal{A}(-\mathcal{Z}_1)^{\otimes k}}\text{Mod} \rightarrow {}^{\mathcal{A}(-\mathcal{Z}_k)}\text{Mod}$  associated to  $\iota$  [LOT15, Definition 2.2.48]. Equivalently, if  ${}^{\mathcal{A}(-\mathcal{Z}_k)}[\iota]_{\mathcal{A}(-\mathcal{Z}_1)^{\otimes k}}$  denotes the rank 1 DA bimodule associated to  $\iota$ , then

$$\widehat{CFD}(Y_{0^k}) = {}^{\mathcal{A}(-\mathcal{Z}_k)}[\iota]_{\mathcal{A}(-\mathcal{Z}_1)^{\otimes k}} \boxtimes \left( {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFD}(Y_0) \right)^{\otimes k}.$$

The module  $\widehat{CFA}(Y_{0^k})$  is the image of  $\widehat{CFA}(Y_0)^{\otimes k}$  under the restriction map  $\text{Mod}_{\mathcal{A}(\mathcal{Z}_1)^{\otimes k}} \rightarrow \text{Mod}_{\mathcal{A}(\mathcal{Z}_k)}$  associated to  $\pi$ . Equivalently,

$$\widehat{CFA}(Y_{0^k})_{\mathcal{A}(\mathcal{Z}_k)} = \left( \widehat{CFA}(Y_0)_{\mathcal{A}(\mathcal{Z}_1)} \right)^{\otimes k} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_1)^{\otimes k}}[\pi]_{\mathcal{A}(\mathcal{Z}_k)}.$$

Explicitly, the type D structure  $\widehat{CFD}(Y_{0^k})$  has a single generator  $n$  with

$$\delta^1(n) = (\rho_{1,3} + \rho_{5,7} + \cdots + \rho_{4k-3,4k-1})n.$$

The module  $\widehat{CFA}(Y_{0^k})$  has basis  $\{t, u, v\}^k$ . The module structure is determined as follows. First, operations  $m_i$ ,  $i > 2$ , vanish:  $\widehat{CFA}(Y_{0^k})$  is an honest  $dg$  module. Second,  $m_2(\cdot, \rho_{4i,4i+1})$  and  $m_2(\cdot, \rho_{4i+3,4i+4})$  vanish identically. Third, given a basis element  $(x_1, \dots, x_k) \in \widehat{CFA}(Y_{0^k})$ ,

$$\begin{aligned} m_1((x_1, \dots, x_k)) &= \sum_{x_i=u} (x_1, \dots, x_{i-1}, v, x_{i+1}, \dots, x_k), \\ m_2((x_1, \dots, x_k), \rho_{4i+1,4i+2}) &= \begin{cases} (x_1, \dots, x_i, t, x_{i+2}, \dots, x_k) & x_{i+1} = u, \\ 0 & \text{otherwise,} \end{cases} \\ m_2((x_1, \dots, x_k), \rho_{4i+2,4i+3}) &= \begin{cases} (x_1, \dots, x_i, v, x_{i+2}, \dots, x_k) & x_{i+1} = t, \\ 0 & \text{otherwise,} \end{cases} \\ m_2((x_1, \dots, x_k), \rho_{4i+1,4i+3}) &= \begin{cases} (x_1, \dots, x_i, v, x_{i+2}, \dots, x_k) & x_{i+1} = u, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

**2.3. The type DD identity bimodule.** Fix a pointed matched circle  $\mathcal{Z}$  with orientation-reverse  $-\mathcal{Z}$ . Let  $Y_{\text{Id}}$  be the identity cobordism of  $F(\mathcal{Z})$ . Given a subset  $\mathbf{s} \subset \{1, \dots, 2k\}$ , let  $\mathbf{s}^c$  denote the complement of  $\mathbf{s}$ . Then  $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}) := \widehat{CFDD}(Y_{\text{Id}})$  is generated by  $\{(I(\mathbf{s}) \otimes I(\mathbf{s}^c))\} \subset \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$ . The differential is defined by

$$\delta^1(I(\mathbf{s}) \otimes I(\mathbf{s}^c)) = \sum_{\mathbf{t} \subset \{1, \dots, 2k\}} \sum_{\rho \in \text{chord}(\mathcal{Z})} (I(\mathbf{s}) \otimes I(\mathbf{s}^c))(a(\rho) \otimes a(-\rho)) \otimes (I(\mathbf{t}) \otimes I(\mathbf{t}^c)),$$

where  $\text{chord}(\mathcal{Z})$  denotes the set of chords in the pointed matched circle  $\mathcal{Z}$ , and  $-\rho$  is the chord in the orientation-reverse  $-\mathcal{Z}$  associated to  $\rho$ .

**2.4. The Auroux-Zarev piece.** The *Auroux-Zarev interpolating piece* [Aur10, Zar10] is the  $\alpha$ - $\beta$ -bordered Heegaard diagram  $\text{AZ}(\mathcal{Z})$  defined as follows. For fixed  $k$ , let  $T$  be the triangle defined by the  $y$ -axis, the  $x$ -axis, and the line  $x+y=4k+1$ . Let  $e_y$  be the edge of  $T$  along the  $y$ -axis, let  $e_x$  be the edge along the  $x$ -axis, and let  $e_D$  be the diagonal edge. Produce a genus  $k$  surface  $\Sigma'$  from  $T$  by identifying small neighborhoods of the points  $(i, 4k+1-i)$  and  $(j, 4k+1-j)$  on  $e_D$  whenever  $i$  and  $j$  are matched in  $\mathcal{Z}$ . If  $i$  and  $j$  are matched in  $\mathcal{Z}$ , the two vertical segments  $T \cap \{x=i\}$  and  $T \cap \{x=j\}$  descend to a single arc; declare this to be a  $\beta$ -arc.

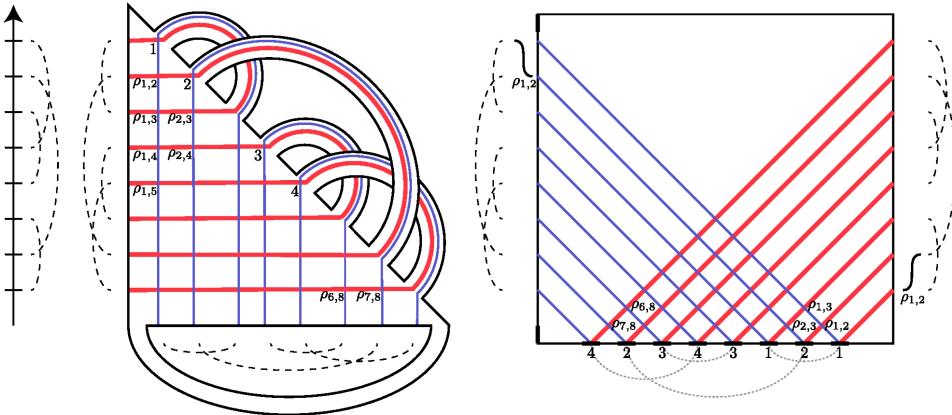


FIGURE 1. The diagram  $\text{AZ}$ . Left: a pointed matched circle  $\mathcal{Z}$ . Center: the diagram  $\text{AZ}(\mathcal{Z})$ , with  $\alpha$ -arcs thick and  $\beta$ -arcs thin. Some of the intersection points are labeled by the corresponding  $\alpha$ -arc if they correspond to a pair of horizontal strands or the chord  $\rho_{i,j}$  otherwise. Labels of generators are to the lower-left of the corresponding intersection point. Right: the same diagram, drawn to show the  $\mathcal{A}(\mathcal{Z})$ -actions on the left and right; the chord  $\rho_{1,2}$  in each algebra is indicated. The thick segments are identified in pairs to give an orientable surface of genus 2 with two boundary components; along the bottom, this identification is indicated by the dotted arcs.

Similarly, the two horizontal segments  $T \cap \{y = 4k + 1 - i\}$  and  $T \cap \{4k + 1 - j\}$  descend to a single arc; declare this to be an  $\alpha$ -arc. Finally, attach a one-handle connecting small neighborhoods of  $(0, 0)$  and  $(4k + 1, 0)$ , giving a surface  $\Sigma$ . Place the basepoint  $z$  at  $(0, 4k + 1)$ . Then  $\text{AZ}(\mathcal{Z}) = (\Sigma, \alpha, \beta, z)$ , and the boundary of  $\text{AZ}(\mathcal{Z})$  is  $\mathcal{Z} \cup \mathcal{Z}^\beta$ . See Figure 1.

There is a canonical identification between the set of generators  $\mathfrak{S}(\text{AZ}(\mathcal{Z}))$  and the strand diagram basis for  $\mathcal{A}(\mathcal{Z})$  as follows [Aur10, LOT11]. Numbering the  $\alpha$ -arcs from the top and the  $\beta$ -arcs from the left, the number of points in  $\alpha_s \cap \beta_t$  is two if  $s = t$  and otherwise is equal to the number of chords in  $\mathcal{Z}$  starting at an endpoint of  $\alpha_t$  and ending at an endpoint of  $\alpha_s$ . If the endpoints of  $\beta_s$  are  $(i, 0)$  and  $(j, 0)$ , the intersection point in  $\alpha_s \cap \beta_s$  which lies on  $e_D$  corresponds to the smeared horizontal strand  $\{i, j\}$ . Other intersection points correspond to upward-sloping chords as follows: if  $z$  lies at coordinates  $(x, 4k + 1 - y)$ , then  $z$  corresponds to the strand  $\rho_{x,y}$  in  $\mathcal{A}(\mathcal{Z})$ . Figure 1 indicates the identifications between intersection points in  $\text{AZ}(\mathcal{Z})$  and chords in  $\mathcal{A}(\mathcal{Z})$ . An arbitrary element of  $\mathfrak{S}(\text{AZ}(\mathcal{Z}))$  is a set of such intersection points and corresponds to a strand diagram in  $\mathcal{A}(\mathcal{Z})$ .

Using the fact that  $\text{AZ}(\mathcal{Z})$  is nice, it is easy to see that the differential on  $\widehat{\text{CFAA}}(\text{AZ}(\mathcal{Z}))$  corresponds to the differential on  $\mathcal{A}(\mathcal{Z})$ . Furthermore,  $m_2$  multiplications correspond to  $k$ -tuples of half-strips on the appropriate boundary [LOT11, Proposition 8.4]. If we treat the  $\alpha$  boundary as the right action and the  $\beta$  boundary as the left action, we have

$$\widehat{\text{CFAA}}(\beta \text{AZ}(\mathcal{Z})^\alpha) \simeq {}_{\mathcal{A}(\mathcal{Z})}\mathcal{A}(\mathcal{Z})_{\mathcal{A}(\mathcal{Z})},$$

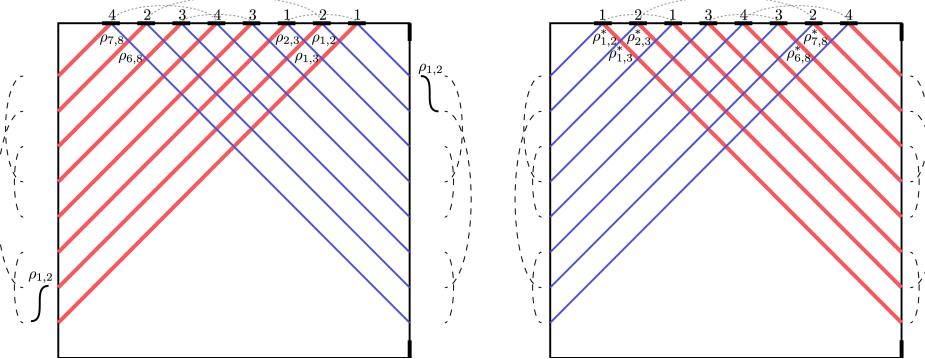


FIGURE 2. The diagrams  $\mathbf{AZ}(-\mathcal{Z})$  and  $\overline{\mathbf{AZ}}(\mathcal{Z})$ . Left: the diagram  $\mathbf{AZ}(-\mathcal{Z})$  for  $\mathcal{Z}$  the same pointed matched circle as in Figure 1, labeled compatibly with a left action by  $\mathcal{A}(\mathcal{Z})$  corresponding to the  $\alpha$ -boundary and a right-action by  $\mathcal{A}(\mathcal{Z})$  corresponding to the  $\beta$ -boundary. Right: the diagram  $\overline{\mathbf{AZ}}(\mathcal{Z})$ . Viewing the  $\alpha$ -boundary as the right action and the  $\beta$ -boundary as the left action, this is a bimodule over  $\mathcal{A}(\mathcal{Z})$ . Labels are to the left of the corresponding intersection point.

whereas if we treat the  $\alpha$  boundary as the left action and the  $\beta$  boundary as the right action, we have

$$\widehat{\mathbf{CFAA}}(\alpha \mathbf{AZ}(\mathcal{Z})^\beta) \simeq_{\mathcal{A}(-\mathcal{Z})} \mathcal{A}(-\mathcal{Z})_{\mathcal{A}(-\mathcal{Z})}.$$

In our computations in Section 4, we will use  $\mathbf{AZ}(-\mathcal{Z})$  (for  $\mathcal{Z}$  the split pointed matched circle) and treat the  $\alpha$ -boundary as the left action and the  $\beta$ -boundary as the right action. Then,

$$\widehat{\mathbf{CFAA}}(\alpha \mathbf{AZ}(-\mathcal{Z})^\beta) \simeq_{\mathcal{A}(\mathcal{Z})} \mathcal{A}(\mathcal{Z})_{\mathcal{A}(\mathcal{Z})}.$$

The corresponding labeling of generators is shown in Figure 2.

We are also interested in a related diagram  $\overline{\mathbf{AZ}}(\mathcal{Z})$  obtained from  $\mathbf{AZ}(\mathcal{Z})$  by switching the  $\alpha$  and  $\beta$  curves. (Equivalently, one could reflect  $\mathbf{AZ}(-\mathcal{Z})$  across the  $x$ -axis, obtaining  $\overline{\mathbf{AZ}}(\mathcal{Z}) = -\mathbf{AZ}(-\mathcal{Z})$ .) Let  $\overline{\mathcal{A}(\mathcal{Z})}$  be the dual, over  $\mathbb{F}_2$ , of  $\mathcal{A}(\mathcal{Z})$ . Since  $\mathcal{A}(\mathcal{Z})$  comes with a preferred basis, the strand diagrams, there is a preferred basis  $\{a^* \mid a \text{ is a strand diagram for } \mathcal{A}(\mathcal{Z})\}$  for  $\overline{\mathcal{A}(\mathcal{Z})}$ . The differential  $\bar{d}$  on  $\overline{\mathcal{A}(\mathcal{Z})}$  is the transpose of the differential  $d$  on  $\mathcal{A}(\mathcal{Z})$ . Moreover,  $\overline{\mathcal{A}(\mathcal{Z})}$  has left and right multiplications by  $\mathcal{A}(\mathcal{Z})$ : on the right,  $a_1^* \cdot a_2$  is the element of  $\overline{\mathcal{A}(\mathcal{Z})}$  which sends an element  $a_3$  to  $a_1^*(a_2 a_3)$ , and on the left  $a_2 \cdot a_1^*$  is the element of  $\overline{\mathcal{A}(\mathcal{Z})}$  which sends an element  $a_3$  to  $a_1^*(a_3 a_2)$ .

By the same computation as above one obtains

$$(2.1) \quad \widehat{\mathbf{CFAA}}(\beta \overline{\mathbf{AZ}}(\mathcal{Z})^\alpha) \simeq_{\mathcal{A}(\mathcal{Z})} \overline{\mathcal{A}(\mathcal{Z})}_{\mathcal{A}(\mathcal{Z})}$$

if the  $\alpha$ -action is on the right [LOT11, Appendix A]. See also Figure 2.

Next we describe  $\widehat{\mathbf{CFDA}}(\alpha \mathbf{AZ}(-\mathcal{Z})^\beta)$  in the case that the  $\alpha$  boundary gives the left type  $D$  structure and the  $\beta$  boundary gives the right type  $A$  structure. From

the pairing theorem,

$$\begin{aligned} {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}({}^\alpha\mathbf{AZ}(-\mathcal{Z})^\beta)_{\mathcal{A}(\mathcal{Z})} &\simeq {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{-\mathcal{Z}})^{\mathcal{A}(\mathcal{Z})} \boxtimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFAA}({}^\beta\mathbf{AZ}(-\mathcal{Z})^\alpha)_{\mathcal{A}(\mathcal{Z})} \\ &= {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{-\mathcal{Z}^\beta})^{\mathcal{A}(\mathcal{Z})} \boxtimes_{\mathcal{A}(\mathcal{Z})} \mathcal{A}(\mathcal{Z})_{\mathcal{A}(\mathcal{Z})}. \end{aligned}$$

Thus, a generator of  $\widehat{CFDA}({}^\beta\mathbf{AZ}(-\mathcal{Z})^\alpha)$  corresponds to  $J \otimes a$ , where  $a$  is a strand diagram in  $\mathcal{A}(\mathcal{Z})$  and  $J$  is the complementary idempotent to the left idempotent  $I$  of  $a$ . The map  $\delta_2^1: \widehat{CFDA}(\mathbf{AZ}(-\mathcal{Z})) \otimes \mathcal{A}(\mathcal{Z}) \rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{CFDA}(\mathbf{AZ}(-\mathcal{Z}))$  is given by multiplication on the right; the image of  $\delta_2^1$  is contained in the subspace  $\widehat{CFDA}(\mathbf{AZ}(-\mathcal{Z})) = 1 \otimes \widehat{CFDA}(\mathbf{AZ}(-\mathcal{Z}))$ . The map  $\delta_1^1: \widehat{CFDA}(\mathbf{AZ}(-\mathcal{Z})) \rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{CFDA}(\mathbf{AZ}(-\mathcal{Z}))$  is given by

$$\delta_1^1(J \otimes a) = J \otimes (J \otimes d(a)) + \sum_{\substack{\rho \in \text{chord}(\mathcal{Z}) \\ (J', I') \text{ complementary}}} J a(\rho) J' \otimes (J' \otimes I' a(\rho) \cdot a).$$

All higher operations  $\delta_k^1$ ,  $k \geq 3$ , vanish.

The same argument, but using equation (2.1), leads to the following description of  $\widehat{CFDA}({}^\beta\overline{\mathbf{AZ}}(\mathcal{Z})^\alpha)$ . As needed by our application, we will treat the  $\beta$  boundary as the left action and the  $\alpha$  boundary as the right action. Generators of  $\widehat{CFDA}({}^\beta\overline{\mathbf{AZ}}(\mathcal{Z})^\alpha)$  correspond to  $J \otimes a^*$ , where  $a$  is a strand diagram in  $\mathcal{Z}$ ,  $a^*$  is the corresponding basis element of  $\overline{\mathcal{A}(\mathcal{Z})}$ , and  $J$  is the complementary idempotent to the left idempotent  $I$  of  $a^*$  (or, equivalently, the right idempotent  $I$  of  $a$ ). The map  $\delta_2^1$  is given by  $\delta_2^1(J \otimes a_1^*, a_2) = J \otimes (J \otimes (a_1^* \cdot a_2))$ . The map  $\delta_1^1: \widehat{CFDA}(\overline{\mathbf{AZ}}(\mathcal{Z})) \rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{CFDA}(\overline{\mathbf{AZ}}(\mathcal{Z}))$  is given by

$$\delta_1^1(J \otimes a^*) = J \otimes (J \otimes \bar{d}(a^*)) + \sum_{\substack{\rho \in \text{chord}(\mathcal{Z}) \\ (J', I') \text{ complementary}}} J a(\rho) J' \otimes (J' \otimes I' a(\rho) \cdot a^*).$$

All higher operations  $\delta_k^1$ ,  $k \geq 3$ , vanish.

To conclude this section, we recall some gluing properties of the diagrams  $\mathbf{AZ}$  and  $\overline{\mathbf{AZ}}$  from [LOT11].

**Lemma 2.2** ([LOT11, Corollary 4.5]). *The Heegaard diagram  ${}^\alpha\mathbf{AZ}(-\mathcal{Z})^\beta \cup {}^\beta\overline{\mathbf{AZ}}(\mathcal{Z})^\alpha$  represents the identity map of  $F(\mathcal{Z}^\alpha)$ , and the diagram  ${}^\beta\overline{\mathbf{AZ}}(-\mathcal{Z})^\alpha \cup {}^\alpha\mathbf{AZ}(\mathcal{Z})^\beta$  represents the identity map of  $F(\mathcal{Z}^\beta)$ .*

**Lemma 2.3** ([LOT11, Corollary 4.6]). *Let  $\mathcal{H}$  be an  $\alpha$ -bordered Heegaard diagram for  $(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$ . Then the Heegaard diagram  $\mathcal{H}^\beta \cup {}^\beta\mathbf{AZ}(-\mathcal{Z})^\alpha$  represents the 3-manifold  $(-Y, \phi: F(-\mathcal{Z}) \rightarrow -\partial Y)$ . In particular,  $\mathcal{H}^\beta \cup {}^\beta\mathbf{AZ}(-\mathcal{Z})^\alpha$  and  $-\mathcal{H}$  represent the same bordered 3-manifold.*

**Convention 2.4.** In the rest of the paper, we will typically drop  $\mathcal{Z}$  from the Auroux-Zarev piece, writing  $\mathbf{AZ}$  (respectively  $\overline{\mathbf{AZ}}$ ) to denote  $\mathbf{AZ}(\mathcal{Z})$  or  $\mathbf{AZ}(-\mathcal{Z})$  (respectively  $\overline{\mathbf{AZ}}(\mathcal{Z})$  or  $\overline{\mathbf{AZ}}(-\mathcal{Z})$ ) as appropriate. Whether  $\mathcal{Z}$  or  $-\mathcal{Z}$  is required is determined by the boundary of the diagram.

**2.5. Gradings on bordered Floer modules.** A key step in our computations is knowing that there are unique graded homotopy equivalences between certain modules and bimodules (as formulated in Section 4). Here we review enough of the gradings in bordered Floer homology to make this statement precise. More details

can be found in the original papers [LOT08, Chapter 10], [LOT15, Sections 2.5, 3.2, 6.5].

Fix a pointed matched circle  $\mathcal{Z}$  representing a surface  $F(\mathcal{Z})$ . The algebra  $\mathcal{A}(\mathcal{Z})$  is graded by a group  $G(\mathcal{Z})$  which is a central extension

$$\mathbb{Z} \rightarrow G(\mathcal{Z}) \rightarrow H_1(F(\mathcal{Z})).$$

Let  $\lambda$  be a generator for the central  $\mathbb{Z}$ . For homogeneous elements  $a, b \in \mathcal{A}(\mathcal{Z})$ , differential satisfies  $\text{gr}(\partial(a)) = \lambda^{-1} \text{gr}(a)$ , and the multiplication satisfies  $\text{gr}(ab) = \text{gr}(a) \text{gr}(b)$ .

Given a bordered 3-manifold  $Y$  with boundary parameterized by  $F(\mathcal{Z})$ ,  $\widehat{\text{CFA}}(Y)$  is graded by a right  $G(\mathcal{Z})$ -set  $S_A(Y)$ , and  $\widehat{\text{CFD}}(Y)$  is graded by a left  $G(-\mathcal{Z})$ -set  $S_D(Y)$ . The  $G$ -orbits in these sets correspond to the  $\text{spin}^c$ -structures on  $Y$ . Similarly, if  $Y$  is a cobordism from  $F(\mathcal{Z}_1)$  to  $F(\mathcal{Z}_2)$ , then  $\widehat{\text{CFDA}}(Y)$  is graded by a set  $S_{DA}(Y)$  with a left action by  $G(-\mathcal{Z}_1)$  and a right action by  $G(\mathcal{Z}_2)$ ;  $\widehat{\text{CFDD}}(Y)$  is graded by a set  $S_{DD}(Y)$  equipped with commuting left actions by  $G(-\mathcal{Z}_1)$  and  $G(-\mathcal{Z}_2)$ ; and  $\widehat{\text{CFAA}}(Y)$  is graded by a set  $S_{AA}(Y)$  equipped with commuting right actions by  $G(\mathcal{Z}_1)$  and  $G(\mathcal{Z}_2)$ . The group  $G(-\mathcal{Z})$  is the opposite group to  $G(\mathcal{Z})$ , so a left  $G(-\mathcal{Z})$ -set is the same data as a right  $G(\mathcal{Z})$ -set;  $S_A(Y)$  and  $S_D(Y)$  are related in this way. (Of course, all groups are isomorphic to their opposites, but here it is convenient to maintain the distinction.)

The  $G(\mathcal{Z})$ -grading on the bordered (bi)modules depends on a choice of *grading refinement data* [LOT08, Section 10.5]. However, up to homotopy equivalence, the bordered invariants are independent of this choice [LOT15, Proposition 6.32].

The special cases of interest to us are:

- (1) Handlebodies. Suppose  $Y$  is a handlebody of genus  $g$ . Then there is a unique  $\text{spin}^c$ -structure on  $Y$ . The corresponding  $G(\mathcal{Z})$ -set  $S_D(Y)$  is the quotient of  $G(-\mathcal{Z})$  by a subgroup isomorphic to  $\mathbb{Z}^g$  which projects isomorphically to  $\ker[H_1(F(\mathcal{Z})) \rightarrow H_1(Y)] \subset H_1(F(\mathcal{Z}))$ . In particular, the grading element  $\lambda$  acts freely on  $S_D(Y)$ .
- (2) Mapping cylinders of diffeomorphisms. If  $\phi: F(\mathcal{Z}_1) \rightarrow F(\mathcal{Z}_2)$  is a strongly based diffeomorphism and  $Y_\phi$  is the associated arced cobordism, then  $S_{DA}(Y)$  is a free, transitive  $G(-\mathcal{Z}_1)$ -set and also a free, transitive  $G(\mathcal{Z}_2)$ -set. Similar statements hold for  $S_{DD}(Y)$  and  $S_{AA}(Y)$ .

Given type  $D$  structures  ${}^{\mathcal{A}(-\mathcal{Z})}P$  and  ${}^{\mathcal{A}(-\mathcal{Z})}Q$ , graded by  $G(-\mathcal{Z})$ -sets  $S$  and  $T$ , respectively, the chain complex of type  $D$  structure morphisms  $\text{Mor}^{\mathcal{A}(-\mathcal{Z})}(P, Q)$  inherits a grading by the  $\mathbb{Z}$ -set  $S^* \times_{G(-\mathcal{Z})} T$  [LOT15, Section 2.5.3], where  $S^*$  is the right  $G(-\mathcal{Z})$ -set with elements  $s^*$  in bijection with  $S$  and action  $s^* \cdot g = (g^{-1} \cdot s)^*$  [LOT15, Definition 2.5.19]. The  $\mathbb{Z}$ -action persists because  $\lambda$  is central in  $G(-\mathcal{Z})$ . The situation for  $A_\infty$ -modules and the various types of bimodules is similar. A morphism is *homogeneous* if it lies in a single grading.

So, if  $G(-\mathcal{Z})$  acts transitively on the grading set  $S$  for  ${}^{\mathcal{A}(-\mathcal{Z})}P$ , then the complex  $\text{Mor}^{\mathcal{A}(-\mathcal{Z})}(P, P)$  is graded by  $S^* \times_{G(-\mathcal{Z})} S \cong (S \times S)/G$  as  $\mathbb{Z}$ -sets. A morphism has *grading 0* if it lands in the summand corresponding to  $(s, s) \in (S \times S)/G$  for some (or equivalently, any)  $s \in S$ .

**Example 2.5.** Let  $Y$  be a 0-framed solid torus, and consider  $\widehat{\text{CFD}}(Y)$ . Since  $\delta^1(n) = \rho_{1,3}n$ , the gradings satisfy  $\text{gr}(\rho_{1,3}x) = \lambda^{-1} \text{gr}(x)$ . Thus, the homomorphism  $\widehat{\text{CFD}}(Y) \rightarrow \widehat{\text{CFD}}(Y)$ ,  $x \mapsto \rho_{1,3}x$  has degree  $\lambda^{-1} \neq 0$ .

### 3. COMPUTATION OF HOMOTOPY EQUIVALENCES

Two key steps in our descriptions of involutive Floer homology and the mapping class group involve computing homotopy equivalences between  $A_\infty$ -modules or between type  $DA$  bimodules. We explain in this section that the bordered algebras have finiteness properties which imply that these computations can be carried out to any order desired.

**Lemma 3.1.** *Given a pointed matched circle  $\mathcal{Z}$  there is an integer  $K$  so that any product of  $n > K$  chords in  $\mathcal{A}(\mathcal{Z})$  vanishes.*

*Proof.* This is immediate from the fact that no two strands in a strand diagram can start at the same point in the matched circle. So, if  $\mathcal{Z}$  represents a surface of genus  $k$ ,

$$K = 1 + 2 + \cdots + 4k - 1 = 2k(4k - 1)$$

suffices. (This bound is not optimal.)  $\square$

**Proposition 3.2.** *Fix a dg algebra  $\mathcal{B}$  and let  $M$  and  $N$  be type  $DA$  bimodules over  $\mathcal{B}$  and  $\mathcal{A}(\mathcal{Z})$ , where  $\mathcal{Z}$  is a pointed matched circle. Let  $K$  be as in Lemma 3.1. Suppose  $\ell \geq K$  and  $\{f_{1+n}^1: M \otimes \mathcal{A}(\mathcal{Z})^{\otimes n} \rightarrow \mathcal{B} \otimes N\}_{n=0}^\ell$  satisfy the type  $DA$  homomorphism relations with up to  $\ell + 1$  inputs. Then there is a type  $DA$  module homomorphism  $g: M \rightarrow N$  so that  $g_{1+n}^1 = f_{1+n}^1$  for all  $0 \leq n \leq \ell$ .*

Since  $A_\infty$ -modules are a special case of type  $DA$  bimodules, this proposition covers  $A_\infty$ -modules as well. Roughly, the proposition says that, after building a homomorphism which takes up to  $K$  inputs, one never gets stuck in extending the homomorphism to take one more input.

*Proof of Proposition 3.2.* View  $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$  as a left-right type  $DD$  structure over  $\mathcal{A}(\mathcal{Z})$  and  $\mathcal{A}(\mathcal{Z})$ . The functor  $\cdot \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$  gives an equivalence of categories from the category of type  $DA$  bimodules over  $\mathcal{B}$  and  $\mathcal{A}(\mathcal{Z})$  to the category of (left-right) type  $DD$  bimodules over  $\mathcal{B}$  and  $\mathcal{A}(\mathcal{Z})$ . This functor sends a morphism  $f \in \text{Mor}(M, N)$  to  $f \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$ . As we will see, the key point is that the form of the differential  $\delta^1$  on  $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$  and Lemma 3.1 imply that the map  $f \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$  depends only on the terms  $f_{1+n}^1$  for  $n \leq K$ .

Fix data  $f = \{f_{1+n}^1\}_{n=0}^\ell$  as in the statement of the proposition. Temporarily declare  $f_i^1 = 0$  for  $i > \ell$ , and form  $f \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$ . It follows from Lemma 3.1 and the form of  $\delta^1$  on  $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$  (see also Section 2.3) that  $f \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$  is a type  $DD$  structure homomorphism. Since  $\cdot \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$  is a homotopy equivalence of  $dg$  categories, there is a type  $DA$  structure homomorphism  $g$  so that  $g \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$  is homotopic to  $f \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$ . So,  $(g - f) \boxtimes \text{Id}_{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}$  is nullhomotopic, so  $g - f$  is itself nullhomotopic. Let  $h$  be a nullhomotopy of  $g - f$ , i.e.,  $g - f = d(h)$ . Write  $h = h' + h''$ , where  $h'$  consists of the terms with  $\leq \ell + 1$  inputs and  $h''$  consists of the terms with  $> \ell + 1$  inputs. Let  $\tilde{f} = f + d(h'')$ . Then  $\tilde{f}_{1+n}^1 = f_{1+n}^1$  for all  $n \leq \ell$ . Furthermore,

$$\tilde{f} = g + d(h'),$$

so  $\tilde{f}$  is a type  $DA$  structure homomorphism. This proves the result.  $\square$

Proposition 3.2 implies that if  $M$  and  $N$  are homotopy equivalent, then one can compute a homotopy equivalence. First one finds terms with up to  $K + 1$  inputs satisfying the type  $DA$  structure relations with up to  $K + 1$  inputs, and so that this map has an up-to- $(K + 1)$ -input homotopy inverse. This is a finite (albeit huge) computation. Proposition 3.2 then implies that one can extend any such solution to more inputs by solving the type  $DA$  structure relation inductively; one never gets stuck.

Maybe a final word is in order about the meaning of the word *compute*. We have finitely generated modules  $M$  and  $N$  with only finitely many non-zero operations. A type  $DA$  structure homomorphism from  $M$  to  $N$  is a computer program (Turing machine)  $f$  which takes as input an integer  $\ell$  and inputs  $m \in M$  and  $a_1, \dots, a_\ell \in \mathcal{A}(\mathcal{Z})$  and gives as output an element of  $N$ . Being able to compute  $f$  means we can write a computer program  $\mathcal{F}$  which takes as inputs homotopy equivalent modules  $M$  and  $N$  and outputs a computer program  $f$  representing a type  $DA$  homotopy equivalence from  $M$  to  $N$ .

#### 4. RIGIDITY RESULTS

In this section we prove that, up to homotopy, there are unique homogeneous homotopy equivalences between certain modules. The results in this section were originally observed by P. Ozsváth, D. Thurston, and the second author.

We will call a map (and, in particular, a homotopy equivalence)  $f$  *homogeneous* if  $f$  is homogeneous with respect to the grading on morphism spaces (cf. Section 2.5).

**Lemma 4.1.** *Let  $Y_{0^k}$  be the 0-framed handlebody of genus  $k$  and  $\widehat{CFD}(Y_{0^k})$  the standard type  $D$  module for  $Y_{0^k}$  (as in Section 2.2). Then there is a unique homogeneous homotopy equivalence  $\widehat{CFD}(Y_{0^k}) \rightarrow \widehat{CFD}(Y_{0^k})$ .*

*Proof.* Let  $f^1: \widehat{CFD}(Y_{0^k}) \rightarrow \widehat{CFD}(Y_{0^k})$  be a homogeneous homotopy equivalence. Write

$$f^1(n) = (a_1 + \dots + a_m)n$$

where the  $a_i$  are strand diagrams (basic elements of  $\mathcal{A}(-\mathcal{Z}_k)$ ). Let  $\mathcal{I} \subset \mathcal{A}(-\mathcal{Z}_k)$  denote the ideal spanned by strand diagrams not of the form  $I(\mathbf{s})$  (i.e., in which at least one strand is not horizontal). Then, as algebras,

$$\mathcal{A}(-\mathcal{Z}_k)/\mathcal{I} \cong \bigoplus_{\mathbf{s} \subset \{1, \dots, 2k\}} \mathbb{F}_2.$$

Let  $\widehat{CFD}(Y_{0^k})/\mathcal{I}$  be the result of extending scalars from  $\mathcal{A}(-\mathcal{Z}_k)$  to  $\mathcal{A}(-\mathcal{Z}_k)/\mathcal{I}$ . Then  $\widehat{CFD}(Y_{0^k})/\mathcal{I}$  is isomorphic to  $\mathbb{F}_2$ , with trivial differential. Since  $f^1$  must induce a homotopy equivalence

$$\widehat{CFD}(Y_{0^k})/\mathcal{I} \rightarrow \widehat{CFD}(Y_{0^k})/\mathcal{I},$$

it follows that one of the  $a_i$ , say  $a_1$ , is the idempotent  $I(\{1, 3, 5, 7, \dots\})$ . That is,

$$f^1(n) = n + (a_2 + \dots + a_m)n,$$

where  $a_2, \dots, a_m \in \mathcal{I}$ .

Next we claim that  $a_2 = \dots = a_m = 0$ . Since both the left and right idempotents of  $a_i$  must agree with the left idempotent  $I_n$  of  $n$ , the  $a_i$  are in the algebra generated by

$$\{\rho_{1,3}I_n, \rho_{5,7}I_n, \dots\}.$$

As in Example 2.5,

$$\text{gr}(\rho_{4i+1,4i+3}n) = \lambda^{-1} \text{gr}(n).$$

Since  $f^1$  is homogeneous and  $n$  appears in  $f^1(n)$  so every term in  $f^1(n)$  has the same grading as  $n$ , it follows that  $a_2 = \dots = a_m = 0$  and so  $f^1(n) = n$ .  $\square$

Suppose that  $\mathcal{H}$  is a Heegaard diagram for a bordered handlebody and  $P$  is a  $G$ -set graded type  $D$  structure homotopy equivalent to  $\widehat{CFD}(\mathcal{H})$ . Then

$$H_* \text{Mor}^{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(\mathcal{H}), P) \cong \widehat{HF}((S^1 \times S^2)^{\#k})$$

[LOT11, Theorem 1] is graded by a free  $\mathbb{Z}$ -set (cf. Section 2.5). Further, the homology lies over a single  $\mathbb{Z}$ -orbit in the grading set. This  $\mathbb{Z}$ -orbit inherits a total order by declaring that  $a > b$  if  $a = \lambda^n b$  for some  $n \in \mathbb{Z}$ . Thus, it makes sense to talk about a non-trivial homomorphism (that is, a homomorphism whose image in  $H_* \text{Mor}^{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(\mathcal{H}), P)$  is non-trivial) of maximal grading. The same discussion holds for type  $A$  invariants.

**Lemma 4.2.** *Let  $\mathcal{H}$  be a Heegaard diagram for a bordered handlebody and let  $P$  (respectively  $M$ ) be a  $G$ -set-graded type  $D$  structure (respectively  $A_\infty$ -module) homogeneous homotopy equivalent to  $\widehat{CFD}(\mathcal{H})$ . Then up to chain homotopy there is a unique homogeneous homotopy equivalence  $\widehat{CFD}(\mathcal{H}) \rightarrow P$  (respectively  $\widehat{CFA}(\mathcal{H}) \rightarrow M$ ). Further, this homotopy class is represented by any non-trivial homomorphism of maximal grading.*

So, if  $\mathcal{H}$  and  $\mathcal{H}'$  represent the same bordered handlebody, to find a homotopy equivalence  $\widehat{CFD}(\mathcal{H}) \rightarrow \widehat{CFD}(\mathcal{H}')$ , say, it suffices to find any grading-preserving, non-nullhomotopic homomorphism.

*Proof.* First, if  $P$  and  $Q$  are homotopy equivalent, then the set of homotopy classes of homotopy equivalences from  $P$  to  $Q$  is a torsor for the set of homotopy classes of homotopy equivalences from  $P$  to  $P$ . So, it suffices to prove the lemma in the case that  $P = \widehat{CFD}(\mathcal{H})$  and  $M = \widehat{CFA}(\mathcal{H})$ .

If  $\mathcal{H}_0$  represents the standard 0-framed handlebody, then by Lemma 4.1 there is a unique homogeneous homotopy equivalence  $\widehat{CFD}(\mathcal{H}_0) \rightarrow \widehat{CFD}(\mathcal{H}_0)$ . Next, there is a mapping class  $\phi$  so that  $\mathcal{H}$  represents a handlebody with boundary parameterized by  $\phi$ . Then the pairing theorem gives a homogeneous homotopy equivalence

$$(4.3) \quad \widehat{CFDA}(\phi) \boxtimes \widehat{CFD}(\mathcal{H}_0) \simeq \widehat{CFD}(\mathcal{H}).$$

Tensoring with  $\widehat{CFDA}(\phi)$  is an equivalence of homotopy categories of  $G$ -set-graded type  $D$  structures, with inverse  $\widehat{CFDA}(\phi^{-1}) \boxtimes \cdot$  [LOT15, Corollary 8.1], so the set of homotopy classes of homogeneous homotopy auto-equivalences of  $\widehat{CFDA}(\phi) \boxtimes \widehat{CFD}(\mathcal{H}_0)$  is in bijection with the set of homotopy classes of homogeneous homotopy auto-equivalences of  $\widehat{CFD}(\mathcal{H}_0)$ . Thus, by equation (4.3) there is a unique homotopy class of homogeneous homotopy auto-equivalences of  $\widehat{CFD}(\mathcal{H})$ . Finally,

$$\widehat{CFA}(\mathcal{H}) \simeq \widehat{CFAA}(\mathbb{I}) \boxtimes \widehat{CFD}(\mathcal{H}).$$

Since tensoring with  $\widehat{CFAA}(\mathbb{I})$  is an equivalence of homotopy categories, with inverse given by tensoring with  $\widehat{CFDD}(\mathbb{I})$  [LOT15, Corollary 8.1], there is a unique homotopy class of homogeneous homotopy auto-equivalences of  $\widehat{CFA}(\mathcal{H})$ .

For the second part of the statement, observe that any other non-trivial homogeneous homomorphism  $\widehat{CFD}(\mathcal{H}_0) \rightarrow \widehat{CFD}(\mathcal{H}_0)$  has grading strictly smaller than the identity map. This property, too, is preserved by homotopy equivalences and equivalences of the homotopy category.  $\square$

There is an analogous result for the bimodules associated to mapping classes.

**Lemma 4.4.** *Let  $\widehat{CFDD}(\mathbb{I})$  be the standard type DD bimodule for the trivial cobordism (as in Section 2.3). Then there is a unique homogeneous homotopy equivalence  $\widehat{CFDD}(\mathbb{I}) \rightarrow \widehat{CFDD}(\mathbb{I})$ , which is also the unique non-trivial homomorphism of maximal grading.*

*Proof.* Since different choices of grading refinement data lead to graded chain homotopy equivalent modules  $\widehat{CFDD}(\mathbb{I})$  [LOT15, Proposition 6.32], it suffices to prove the lemma for any choice of grading refinement data. Choose any grading refinement data for  $\mathcal{Z}$ , and work with the induced grading refinement data for  $-\mathcal{Z}$ . With respect to these choices, all of the generators of  $\widehat{CFDD}(\mathbb{I})$  are in the same grading.

Let  $f^1: \widehat{CFDD}(\mathbb{I}) \rightarrow \widehat{CFDD}(\mathbb{I})$  be a homotopy equivalence. Write

$$f^1(I(\mathbf{s}) \otimes I(\mathbf{s}^c)) = \sum_{\mathbf{t} \in \{1, \dots, 2k\}} \sum_i (a_{\mathbf{s}, \mathbf{t}, i} \otimes a'_{\mathbf{s}, \mathbf{t}, i}) \otimes (I(\mathbf{t}) \otimes I(\mathbf{t}^c)),$$

where the  $a_{\mathbf{s}, \mathbf{t}, i}$  and  $a'_{\mathbf{s}, \mathbf{t}, i}$  are strand diagrams. Note that for each  $\mathbf{s}$ ,  $\mathbf{t}$ , and  $i$ ,

$$I(\mathbf{s})a_{\mathbf{s}, \mathbf{t}, i}I(\mathbf{t}) = a_{\mathbf{s}, \mathbf{t}, i}, \quad I(\mathbf{s}^c)a'_{\mathbf{s}, \mathbf{t}, i}I(\mathbf{t}^c) = a'_{\mathbf{s}, \mathbf{t}, i}.$$

Considering  $\mathcal{A}(-\mathcal{Z})/\mathcal{I}$  and  $\mathcal{A}(\mathcal{Z})/\mathcal{I}$  as in the proof of Lemma 4.1 shows that for each generator  $I(\mathbf{s}) \otimes I(\mathbf{s}^c)$ , one of the terms  $a_{\mathbf{s}, \mathbf{s}, i}$  must be  $I(\mathbf{s}) \otimes I(\mathbf{s}^c)$ . We claim that these are the only terms in  $f^1$ .

To see this note that the fact that  $f^1$  is homogeneous implies that the supports of  $a_{\mathbf{s}, \mathbf{t}, i}$  and  $a'_{\mathbf{s}, \mathbf{t}, i}$  (in  $H_1(Z, \mathbf{a})$ ) must be the same. (This statement depends on the fact that we are using corresponding grading refinement data for  $\mathcal{Z}$  and  $-\mathcal{Z}$ .) That is,  $a_{\mathbf{s}, \mathbf{t}, i} \otimes a'_{\mathbf{s}, \mathbf{t}, i}$  lies in the *diagonal subalgebra* [LOT14b, Definition 3.1]. Every basic element in the diagonal subalgebra can be factored as a product of chord-like elements  $a(\rho) \otimes a(-\rho)$  [LOT14b, Lemma 3.5]. Since  $(a(\rho) \otimes a(-\rho)) \otimes (I(\mathbf{t}) \otimes I(\mathbf{t}^c))$  occurs in the differential on  $\widehat{CFDD}(\mathbb{I})$ , it follows that the grading of a product of  $n$  chord-like elements is  $-n$ . Thus, since  $f^1$  is homogeneous, each term  $a_{\mathbf{s}, \mathbf{t}, i} \otimes a'_{\mathbf{s}, \mathbf{t}, i}$  must be a product of 0 chord-like elements, i.e., have the form  $I(\mathbf{s}) \otimes I(\mathbf{s}^c)$ . This proves the result.  $\square$

**Lemma 4.5.** *If  $\phi: F(\mathcal{Z}) \rightarrow F(\mathcal{Z}')$  is a mapping class and  $M$  is a type DA bimodule homogeneous homotopy equivalent to  $\widehat{CFDA}(\phi)$  (respectively  $\widehat{CFAA}(\phi)$ ,  $\widehat{CFDD}(\phi)$ ), then there is a unique homogeneous homotopy equivalence between  $\widehat{CFDA}(\phi)$  (respectively  $\widehat{CFAA}(\phi)$ ,  $\widehat{CFDD}(\phi)$ ) and  $M$ . Further, the homotopy equivalence is the unique non-zero homotopy class of homomorphisms of maximal grading.*

*Proof.* Since tensoring with  $\widehat{CFAA}(\mathbb{I})$  gives an equivalence of homotopy categories, it suffices to prove the statement for  $\widehat{CFDD}(\phi)$ . Further, since tensoring with  $\widehat{CFDA}(\phi)$  gives an equivalence of categories, it suffices to prove the statement for  $\widehat{CFDD}(\mathbb{I})$ . Since the number of homotopy equivalences is preserved by homotopy

equivalences, it suffices to show there is a unique homotopy equivalence  $\widehat{CFDD}(\mathbb{I}) \rightarrow \widehat{CFDD}(\mathbb{I})$  and that this homotopy equivalence is the unique non-nullhomotopic map of maximal grading. So, the result now follows from Lemma 4.4 and its proof.  $\square$

**Corollary 4.6.** *Up to homotopy, there is a unique homogeneous homotopy equivalence*

$$\Omega: {}^{\mathcal{A}(\mathcal{Z})}[\text{Id}_{\mathcal{A}(\mathcal{Z})}]_{\mathcal{A}(\mathcal{Z})} \xrightarrow{\simeq} {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\overline{\mathbf{AZ}})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbf{AZ})_{\mathcal{A}(\mathcal{Z})}.$$

*Proof.* Since  $\overline{\mathbf{AZ}} \cup \mathbf{AZ}$  represents the identity diffeomorphism, this follows from the pairing theorem and Lemma 4.5.  $\square$

## 5. INVOLUTIVE BORDERED FLOER HOMOLOGY

We start by proving that the bordered description of  $\widehat{CFI}$  in the introduction does, in fact, give  $\widehat{CFI}$ .

**Theorem 5.1.** *Fix bordered Heegaard diagrams  $\mathcal{H}_0, \mathcal{H}_1$  with  $\partial\mathcal{H}_0 = \mathcal{Z} = -\partial\mathcal{H}_1$ . Let  $Y = Y(\mathcal{H}_0 \cup_{\partial} \mathcal{H}_1)$  be the closed 3-manifold represented by  $\mathcal{H}_0 \cup_{\partial} \mathcal{H}_1$ . Then, under the identification  $\widehat{CF}(Y) \simeq \widehat{CFA}(\mathcal{H}_0)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\mathcal{H}_1)$  from the pairing theorem [LOT08, Theorem 3], the map*

$$\Psi \circ \Omega \circ \eta: \widehat{CFA}(\mathcal{H}_0)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\mathcal{H}_1) \rightarrow \widehat{CFA}(\mathcal{H}_0)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\mathcal{H}_1)$$

*from formula (1.3) is homotopic to the map  $\iota: \widehat{CF}(Y) \rightarrow \widehat{CF}(Y)$ .*

*Proof.* In outline, the proof is that, up to homotopy, the map  $\eta$  in formula (1.3) agrees with the map  $\eta$  in the definition of  $\widehat{HFI}$ , while the composition  $\Psi \circ \Omega$  agrees with the map  $\Phi$  in the definition of  $\widehat{HFI}$ . To check this we need to verify that:

(1) Up to homotopy, the following diagram commutes:

$$(5.2) \quad \begin{array}{ccc} \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1) & \xrightarrow{\eta} & \widehat{CFA}(\overline{\mathcal{H}_0}) \boxtimes \widehat{CFD}(\overline{\mathcal{H}_1}) \\ \downarrow & & \downarrow \\ \widehat{CF}(\mathcal{H}_0 \cup \mathcal{H}_1) & \xrightarrow{\quad \eta \quad} & \widehat{CF}(\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_1}), \end{array}$$

where the vertical arrows come from the pairing theorem for bordered Floer homology.

(2) Up to homotopy, the following diagrams commute, where in each case the bottom arrow is the chain homotopy equivalence on  $\widehat{CF}$  (from [OSz06, JT12]) induced by a sequence of Heegaard moves and the vertical arrows come from the pairing theorem:

$$(5.3) \quad \begin{array}{ccc} \widehat{CFA}(\overline{\mathcal{H}_0}) \boxtimes [\text{Id}] \boxtimes \widehat{CFD}(\overline{\mathcal{H}_1}) & \xrightarrow{\Omega_1} & \widehat{CFA}(\overline{\mathcal{H}_0}) \boxtimes \widehat{CFDA}(\mathbb{I}) \boxtimes \widehat{CFD}(\overline{\mathcal{H}_1}) \\ \downarrow & & \downarrow \\ \widehat{CF}(\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_1}) & \xrightarrow{\quad \quad} & \widehat{CF}(\overline{\mathcal{H}_0} \cup \mathbb{I} \cup \overline{\mathcal{H}_1}) \end{array}$$

$$(5.4) \quad \widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFDA}(\mathbb{I}) \boxtimes \widehat{CFD}(\overline{\mathcal{H}}_1) \xrightarrow{\Omega_2} \begin{matrix} \widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFDA}(\overline{AZ}) \\ \boxtimes \widehat{CFDA}(AZ) \boxtimes \widehat{CFD}(\overline{\mathcal{H}}_1) \end{matrix} \downarrow \quad \downarrow \\ \widehat{CF}(\overline{\mathcal{H}}_0 \cup \mathbb{I} \cup \overline{\mathcal{H}}_1) \longrightarrow \widehat{CF}(\overline{\mathcal{H}}_0 \cup \overline{AZ} \cup AZ \cup \overline{\mathcal{H}}_1)$$

and

$$(5.5) \quad \begin{matrix} \widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFDA}(\overline{AZ}) \boxtimes \widehat{CFDA}(AZ) \boxtimes \widehat{CFD}(\overline{\mathcal{H}}_1) \xrightarrow{\Psi} \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1) \\ \downarrow \quad \downarrow \\ \widehat{CF}(\overline{\mathcal{H}}_0 \cup \overline{AZ} \cup AZ \cup \overline{\mathcal{H}}_1) \longrightarrow \widehat{CF}(\mathcal{H}_0 \cup \mathcal{H}_1). \end{matrix}$$

(Note that the top-left square of diagram (5.3) is canonically isomorphic to  $\widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFD}(\overline{\mathcal{H}}_1)$ .)

The fact that diagram (5.2) commutes is straightforward from either proof of the pairing theorem. For example, the time-dilation proof [LOT08, Chapter 9] has two steps. In the first, one chooses complex structures  $j_n$  on  $\mathcal{H}_0 \cup \mathcal{H}_1$  with increasingly long necks around  $\partial \mathcal{H}_0 = \partial \mathcal{H}_1$ . For  $n$  sufficiently large, the differential on  $\widehat{CF}(\mathcal{H}_0 \cup \mathcal{H}_1)$  agrees with a count of pairs of holomorphic curves in  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , subject to a matching condition. We may as well assume that  $\widehat{CF}(\mathcal{H}_0 \cup \mathcal{H}_1)$  is computed with respect to one of these sufficiently large  $j_n$ . One then deforms the matching condition and observes that after a sufficiently large deformation the resulting differential agrees with  $\widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1)$ . Complexes with different deformation parameters are chain homotopy equivalent. Now, if one chooses the conjugate complex structure to  $j_n$  on  $\overline{\mathcal{H}}_0 \cup \overline{\mathcal{H}}_1$  and then performs exactly the same deformation, at every stage the moduli spaces of holomorphic curves for  $(\mathcal{H}_0, \mathcal{H}_1)$  and  $(\overline{\mathcal{H}}_0, \overline{\mathcal{H}}_1)$  are identified. Thus, diagram (5.2) can be chosen to commute on the nose. (The argument via the nice diagrams proof [LOT08, Chapter 8] is even simpler and is left as an exercise.)

Consider next diagram (5.5). By a similar argument to the one just given, it suffices to show that the corresponding diagram

$$\begin{matrix} \widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFD}(AZ \cup \overline{\mathcal{H}}_1) \xrightarrow{\Psi} \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1) \\ \downarrow \quad \downarrow \\ \widehat{CF}(\overline{\mathcal{H}}_0 \cup \overline{AZ} \cup AZ \cup \overline{\mathcal{H}}_1) \longrightarrow \widehat{CF}(\mathcal{H}_0 \cup \mathcal{H}_1) \end{matrix}$$

homotopy commutes. Recall that  $\Psi$  is the box product of maps  $\Psi_0$  and  $\Psi_1$ , induced by Heegaard moves from  $\overline{\mathcal{H}}_0 \cup \overline{AZ}$  to  $\mathcal{H}_0$  and from  $AZ \cup \overline{\mathcal{H}}_1$  to  $\mathcal{H}_1$ , respectively. By definition,  $\Psi_0 \boxtimes \Psi_1 = (\Psi_0 \boxtimes \text{Id}) \circ (\text{Id} \boxtimes \Psi_1)$ , but this is canonically homotopic to  $(\text{Id} \boxtimes \Psi_1) \circ (\Psi_0 \boxtimes \text{Id})$  [LOT15, Section 3.2]. Thus, we can break this into two steps

by considering the diagram

$$\begin{array}{ccccc}
 \widehat{CFA}(\overline{\overline{\mathcal{H}_0}} \cup \overline{\mathcal{A}\mathcal{Z}}) \boxtimes \widehat{CFD}(\mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}}) & \xrightarrow{\Psi_0 \boxtimes \text{Id}} & \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}}) & \xrightarrow{\text{Id} \boxtimes \Psi_1} & \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1) \\
 \downarrow & & \downarrow & & \downarrow \\
 \widehat{CF}(\overline{\overline{\mathcal{H}_0}} \cup \overline{\mathcal{A}\mathcal{Z}} \cup \mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}}) & \longrightarrow & \widehat{CF}(\mathcal{H}_0 \cup \mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}}) & \longrightarrow & \widehat{CF}(\mathcal{H}_0 \cup \mathcal{H}_1).
 \end{array}$$

The proofs of commutativity of the two squares are essentially the same, so we will focus on the left square. We can relate  $\overline{\overline{\mathcal{H}_0}} \cup \overline{\mathcal{A}\mathcal{Z}}$  to  $\mathcal{H}_0$  by a sequence of bordered Heegaard moves; let  $\mathcal{H}^1, \mathcal{H}^2, \dots, \mathcal{H}^k$  be the sequence of bordered Heegaard diagrams obtained by doing these moves one at a time, with  $\mathcal{H}^1 = \overline{\overline{\mathcal{H}_0}} \cup \overline{\mathcal{A}\mathcal{Z}}$  and  $\mathcal{H}^k = \mathcal{H}_0$ . There is a corresponding sequence of closed Heegaard diagrams

$$\mathcal{H}^1 \cup \mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}}, \mathcal{H}^2 \cup \mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}}, \dots, \mathcal{H}^k \cup \mathcal{A}\mathcal{Z} \cup \overline{\overline{\mathcal{H}_1}},$$

each successive pair of which is related by a Heegaard move. So, it suffices to check that:

**Lemma 5.6.** *If  $\mathcal{H}^i$  and  $\mathcal{H}^{i+1}$  are bordered Heegaard diagrams related by a bordered Heegaard move and  $\mathcal{H}'$  is another bordered Heegaard diagram with  $\partial\mathcal{H}' = -\partial\mathcal{H}^i$ , then the diagram*

$$\begin{array}{ccc}
 \widehat{CFA}(\mathcal{H}^i) \boxtimes \widehat{CFD}(\mathcal{H}') & \longrightarrow & \widehat{CFA}(\mathcal{H}^{i+1}) \boxtimes \widehat{CFD}(\mathcal{H}') \\
 \downarrow & & \downarrow \\
 \widehat{CF}(\mathcal{H}^i \cup \mathcal{H}') & \longrightarrow & \widehat{CF}(\mathcal{H}^{i+1} \cup \mathcal{H}')
 \end{array}$$

commutes up to homotopy. (Here, the horizontal arrows come from the invariance proofs for bordered and classical Heegaard Floer homology.)

*Proof.* For stabilizations (near the basepoint  $z$ ), this is obvious: if  $y$  is the intersection point between the new  $\alpha$ -circle and the new  $\beta$ -circle, then both horizontal maps send a generator  $\mathbf{x}$  to  $\mathbf{x} \cup \{y\}$ , and none of the moduli spaces used to define the vertical maps are affected. For handleslides, both horizontal maps are defined by counting holomorphic triangles, and the fact that this diagram commutes up to homotopy is a special case of the pairing theorem for triangles [LOT14a]. For isotopies, commutativity follows by imitating the proof of the pairing theorem but with dynamic boundary conditions.  $\square$

Commutativity of diagram (5.4) follows from a similar argument. Here, the horizontal maps come from a sequence of Heegaard moves relating the identity Heegaard diagram to the diagram  $\overline{\mathcal{A}\mathcal{Z}} \cup \mathcal{A}\mathcal{Z}$ . Working one Heegaard move at a time, the result follows from the obvious bimodule analogue of Lemma 5.6.

For diagram (5.3), note that there are two homotopy equivalences,

$$\widehat{CFA}(\overline{\overline{\mathcal{H}_0}}) \cong \widehat{CFA}(\overline{\overline{\mathcal{H}_0}}) \boxtimes [\text{Id}] \rightarrow \widehat{CFA}(\overline{\overline{\mathcal{H}_0}}) \boxtimes \widehat{CFDA}(\mathbb{I}),$$

one given by a sequence of Heegaard moves from  $\overline{\overline{\mathcal{H}_0}}$  to  $\overline{\overline{\mathcal{H}_0}} \cup \mathbb{I}$  and the pairing theorem, and the other given by tensoring with the homotopy equivalence  $[\text{Id}] \simeq \widehat{CFDA}(\mathbb{I})$ . The second of these is the map  $\Omega_1$ , while for the first of these diagram (5.3) clearly commutes. So, it suffices to show these two maps are homotopic. In the case that  $\mathcal{H}_0$  represents a handlebody, this follows from Lemma 4.2.

For the general case, since tensoring with  $\widehat{CFDD}(\text{Id})$  is a quasi-equivalence of  $dg$  categories, it suffices to show that the two maps

$$\widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFDD}(\mathbb{I}) \rightarrow \widehat{CFA}(\overline{\mathcal{H}}_0) \boxtimes \widehat{CFDA}(\mathbb{I}) \boxtimes \widehat{CFDD}(\mathbb{I}),$$

one induced by a sequence of Heegaard moves and the other induced by the equivalence  $[\text{Id}] \simeq \widehat{CFDA}(\mathbb{I})$ , are homotopic. By homotopy associativity of the box tensor product and the pairing theorem (see [LOT15]), it suffices to show that the two maps

$$\widehat{CFDD}(\mathbb{I}) \rightarrow \widehat{CFDA}(\mathbb{I}) \boxtimes \widehat{CFDD}(\mathbb{I}),$$

one given by a sequence of Heegaard moves and the other by the equivalence  $[\text{Id}] \simeq \widehat{CFDA}(\mathbb{I})$ , are homotopic. This last statement follows from rigidity of  $\widehat{CFDD}(\mathbb{I})$ , Lemma 4.4.  $\square$

Next we abstract the bordered information required to compute involutive Heegaard Floer homology.

**Definition 5.7.** Fix a pointed matched circle  $\mathcal{Z}$ . An *involutive type D module* over  $\mathcal{A}(\mathcal{Z})$  consists of a pair  $({}^{\mathcal{A}(\mathcal{Z})}P, \Psi_P)$  where  ${}^{\mathcal{A}(\mathcal{Z})}P$  is a type D structure over  $\mathcal{A}(\mathcal{Z})$  and

$$\Psi_P: {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\text{AZ})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}P \rightarrow {}^{\mathcal{A}(\mathcal{Z})}P$$

is a homotopy equivalence of type D structures. We call two involutive type D structures  $({}^{\mathcal{A}(\mathcal{Z})}P, \Psi_P)$  and  $({}^{\mathcal{A}(\mathcal{Z})}Q, \Psi_Q)$  *equivalent* if there is a type D structure homotopy equivalence  $g: {}^{\mathcal{A}(\mathcal{Z})}P \rightarrow {}^{\mathcal{A}(\mathcal{Z})}Q$  so that  $g \circ \Psi_P$  is homotopic to  $\Psi_Q \circ (\text{Id} \boxtimes g)$ .

Similarly, an *involutive  $A_\infty$ -module* over  $\mathcal{A}(\mathcal{Z})$  consists of a pair  $(M_{\mathcal{A}(\mathcal{Z})}, \Psi_M)$  where  $M_{\mathcal{A}(\mathcal{Z})}$  is an  $A_\infty$ -module over  $\mathcal{A}(\mathcal{Z})$  and

$$\Psi_M: M_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\overline{\text{AZ}})_{\mathcal{A}(\mathcal{Z})} \rightarrow M_{\mathcal{A}(\mathcal{Z})}$$

is a homotopy equivalence of  $A_\infty$ -modules. We call involutive  $A_\infty$ -modules  $(M_{\mathcal{A}(\mathcal{Z})}, \Psi_M)$  and  $(N_{\mathcal{A}(\mathcal{Z})}, \Psi_N)$  *equivalent* if there is an  $A_\infty$ -module homotopy equivalence  $g: M_{\mathcal{A}(\mathcal{Z})} \rightarrow N_{\mathcal{A}(\mathcal{Z})}$  so that  $g \circ \Psi_M$  is homotopic to  $\Psi_N \circ (g \boxtimes \text{Id})$ .

**Definition 5.8.** Given a bordered 3-manifold  $Y$  with boundary  $-F(\mathcal{Z})$  and bordered Heegaard diagram  $\mathcal{H}$  for  $Y$ , let  $\widehat{CFDI}(\mathcal{H}) = (\widehat{CFD}(\mathcal{H}), \Psi_D)$  be the involutive type D module where  $\Psi_D$  is the map

$${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\text{AZ})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\mathcal{H}) \xrightarrow{\sim} {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\text{AZ} \cup \overline{\mathcal{H}}) \xrightarrow{\sim} {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\mathcal{H})$$

in which the first equivalence is given by the pairing theorem and the second is induced by some sequence of Heegaard moves from  $\text{AZ} \cup \overline{\mathcal{H}}$  to  $\mathcal{H}$ .

Similarly, given a bordered 3-manifold  $Y$  with boundary  $F(\mathcal{Z})$  and bordered Heegaard diagram  $\mathcal{H}$  for  $Y$ , let  $\widehat{CFAI}(\mathcal{H}) = (\widehat{CFA}(\mathcal{H}), \Psi_A)$  be the involutive  $A_\infty$ -module where  $\Psi_A$  is the map

$$\widehat{CFA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\overline{\text{AZ}})_{\mathcal{A}(\mathcal{Z})} \xrightarrow{\sim} \widehat{CFA}(\overline{\mathcal{H}} \cup \overline{\text{AZ}})_{\mathcal{A}(\mathcal{Z})} \xrightarrow{\sim} \widehat{CFA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$$

in which the first equivalence is given by the pairing theorem and the second is induced by some sequence of Heegaard moves from  $\overline{\mathcal{H}} \cup \overline{\text{AZ}}$  to  $\mathcal{H}$ .

**Conjecture 5.9.** The involutive type D structure  $\widehat{CFDI}(\mathcal{H})$  and involutive  $A_\infty$ -module  $\widehat{CFAI}(\mathcal{H})$  are invariants of the bordered 3-manifold  $Y$ .

The missing ingredient to prove Conjecture 5.9 is an analogue of Ozsváth-Szabó-Juhász-Thurston-Zemke's naturality theorem. That is, we do not know that the maps  $\Psi_A$  and  $\Psi_D$  are independent of the choice of sequence of Heegaard moves. In the special case that  $Y$  is a handlebody, Conjecture 5.9 follows from Lemma 4.2. In general, it is not even known that  $\widehat{CFDI}(\mathcal{H})$  and  $\widehat{CFAI}(\mathcal{H})$  are invariants of the Heegaard diagram  $\mathcal{H}$ , since as far as we know different sequences of Heegaard moves would give different maps  $\Psi_D$  and  $\Psi_A$ .

The rest of this paper does *not* depend on Conjecture 5.9.

**Definition 5.10.** The tensor product

$$(M_{\mathcal{A}(\mathcal{Z})}, \Psi_M) \boxtimes (\mathcal{A}(\mathcal{Z})P, \Psi_P)$$

of an involutive type  $D$  structure  $(\mathcal{A}(\mathcal{Z})P, \Psi_P)$  and an involutive  $A_\infty$ -module  $(M_{\mathcal{A}(\mathcal{Z})}, \Psi_M)$  is the mapping cone of the map

$$M \boxtimes P \cong M \boxtimes [\text{Id}] \boxtimes P \xrightarrow{\text{Id} + [(\Psi_M \boxtimes \Psi_P) \circ (\text{Id} \boxtimes \Omega \boxtimes \text{Id})]} M \boxtimes P,$$

where  $\Omega: [\text{Id}] \rightarrow \widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ})$  is the homotopy equivalence from Corollary 4.6. This tensor product is a differential module over  $\mathbb{F}_2[Q]/(Q^2)$  in an obvious way.

**Lemma 5.11.** *If  $(\mathcal{A}(\mathcal{Z})P, \Psi_P)$  and  $(\mathcal{A}(\mathcal{Z})Q, \Psi_Q)$  (respectively  $(M_{\mathcal{A}(\mathcal{Z})}, \Psi_M)$  and  $(N_{\mathcal{A}(\mathcal{Z})}, \Psi_N)$ ) are equivalent involutive type  $D$  structures (respectively  $A_\infty$ -modules) over  $\mathcal{A}(\mathcal{Z})$ , then the box tensor products  $(M_{\mathcal{A}(\mathcal{Z})}, \Psi_M) \boxtimes (\mathcal{A}(\mathcal{Z})P, \Psi_P)$  and  $(N_{\mathcal{A}(\mathcal{Z})}, \Psi_N) \boxtimes (\mathcal{A}(\mathcal{Z})Q, \Psi_Q)$  are quasi-isomorphic differential modules over  $\mathbb{F}_2[Q]/(Q^2)$ .*

*Proof.* The proof is straightforward and is left to the reader.  $\square$

The following is the pairing theorem for involutive bordered Floer homology:

**Theorem 5.12.** *Fix bordered Heegaard diagrams  $\mathcal{H}_1$  and  $\mathcal{H}_2$  with  $\partial\mathcal{H}_1 = \mathcal{Z} = -\partial\mathcal{H}_2$ . Then there is a chain homotopy equivalence*

$$\widehat{CFI}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2) \simeq \widehat{CFAI}(\mathcal{H}_1) \boxtimes \widehat{CFDI}(\mathcal{H}_2).$$

*Proof.* This follows from Theorem 5.1.  $\square$

## 6. COMPUTING THE MAPPING CLASS GROUP ACTION

We start by recalling a well-known lemma.

**Lemma 6.1.** *Let  $\phi: (Y, p) \rightarrow (Y, p)$  be an orientation-preserving, based diffeomorphism. Then there is a Heegaard splitting  $Y = H_0 \cup_{\Sigma} H_1$  with  $p \in \Sigma$  and a diffeomorphism  $\chi$  isotopic to  $\phi$  (rel.  $p$ ) so that  $\chi(H_i) = H_i$ .*

*Proof.* Start with any Heegaard splitting  $Y = H_0 \cup_{\Sigma} H_1$  of  $Y$ . Then  $\phi(H_0) \cup_{\phi(\Sigma)} \phi(H_1)$  is another Heegaard splitting of  $Y$ . Since any pair of Heegaard splittings becomes isotopic after sufficiently many stabilizations, after stabilizing enough times we may assume that  $(H_0, H_1)$  is isotopic to  $(\phi(H_0), \phi(H_1))$  by some ambient isotopy  $\psi_t: Y \rightarrow Y$ . Consider the map  $\psi_1^{-1} \circ \phi$ . Since  $\psi_1^{-1}$  is isotopic to the identity,  $\psi_1^{-1} \circ \phi$  is isotopic to  $\phi$ . Clearly  $\psi_1^{-1} \circ \phi$  preserves the Heegaard splitting  $Y = H_0 \cup_{\Sigma} H_1$ .  $\square$

With notation as in the introduction, the main goal of this section is to prove:

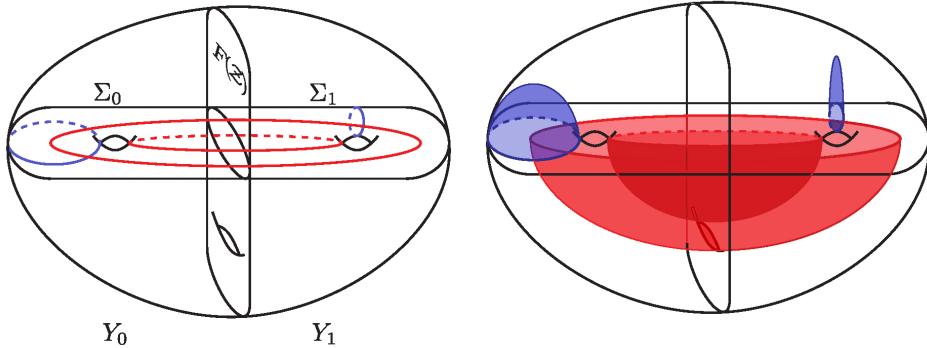


FIGURE 3. Embedded bordered Heegaard surfaces. Left: a schematic of how the bordered Heegaard surfaces  $\Sigma_0$  and  $\Sigma_1$  and the boundary  $F(\mathcal{Z}) = \partial Y_0 = -\partial Y_1$  lie in  $Y$ . Right: a schematic of the descending disks of index 2 critical points and ascending disks of index 1 critical points.

**Theorem 6.2.** *The action of a mapping class  $[\chi]$  on  $\widehat{HF}(Y)$  is given by the composition of the maps in formula (1.4).*

*Proof.* The proof is similar to the proof of Theorem 5.1. Choose a Heegaard splitting as in Lemma 6.1. Let  $F$  denote the Heegaard surface and let  $\psi: F \rightarrow F$  be the gluing diffeomorphism for the Heegaard splitting. Let  $\mathcal{H}_0 = (\Sigma_0, \alpha^0, \beta^0, z^0)$  be a bordered Heegaard diagram representing the 0-framed handlebody and let  $\mathcal{H}_1 = (\Sigma_1, \alpha^1, \beta^1, z^1)$  be a bordered Heegaard diagram representing  $(H_g, \phi_0 \circ \psi)$ , so  $\mathcal{H}_0 \cup_{\partial} \mathcal{H}_1$  is a Heegaard diagram for  $Y$ . Here, we view  $\Sigma_0$  and  $\Sigma_1$  as subsets of  $Y$ ; see Figure 3.

Applying  $\chi$  to  $\Sigma_0$  and  $\Sigma_1$  gives new Heegaard diagrams  $(\chi(\Sigma_i), \chi(\alpha^i), \chi(\beta^i), \chi(z^i))$  for  $H_i$ . (Abstractly, of course, these diagrams are diffeomorphic to the original ones, but they are new subsets of the manifolds  $H_i$ .) Let  $C_\chi$  denote the mapping cylinder of  $\chi|_F$ , and let  $\mathcal{H}_\chi$  be a bordered Heegaard diagram for  $C_\chi$ . Cutting  $Y$  along  $F$  and gluing in  $C_\chi C_{\chi^{-1}}$  does not change the 3-manifold. At the level of Heegaard diagrams, this corresponds to gluing  $\mathcal{H}_\chi$  to  $\chi(\mathcal{H}_0)$  and  $\mathcal{H}_{\chi^{-1}}$  to  $\chi(\mathcal{H}_1)$ . Further, this cutting and regluing can be realized by a path of Heegaard diagrams from the standard Heegaard diagram for the identity map to  $\mathcal{H}_\chi \cup \mathcal{H}_{\chi^{-1}}$ .

Now,  $\chi(\mathcal{H}_0) \cup \mathcal{H}_\chi$  and  $\mathcal{H}_0$  are bordered Heegaard diagrams representing  $H_0$ , and the Heegaard surfaces are embedded so that they have the same boundary. Similarly,  $\mathcal{H}_{\chi^{-1}} \cup \chi(\mathcal{H}_1)$  and  $\mathcal{H}_1$  both represent  $H_1$ . Choose a path of Heegaard diagrams from  $\chi(\mathcal{H}_0) \cup \mathcal{H}_\chi$  to  $\mathcal{H}_0$ , and a path from  $\mathcal{H}_{\chi^{-1}} \cup \chi(\mathcal{H}_1)$  to  $\mathcal{H}_1$ . By definition, the map on  $\widehat{HF}$  induced by  $\chi$  comes from the composition of the Heegaard Floer continuation map associated to the path which introduces  $\mathcal{H}_\chi \cup \mathcal{H}_{\chi^{-1}}$ , and then the Heegaard Floer continuation maps associated to the Heegaard moves from  $\chi(\mathcal{H}_0) \cup \mathcal{H}_\chi$  to  $\mathcal{H}_0$  and  $\mathcal{H}_{\chi^{-1}} \cup \chi(\mathcal{H}_1)$  to  $\mathcal{H}_1$ .

By the pairing theorem for holomorphic triangle maps [LOT14a, Proposition 5.35], these continuation maps agree with the tensor products of the bordered continuation maps associated to the pieces which are changing. So, a similar argument to the proof of commutativity of diagrams (5.3), (5.4), and (5.5) shows that the

action of  $\chi$  on  $\widehat{HF}$  is given by the composition

$$\begin{aligned} \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1) &= \widehat{CFA}(\mathcal{H}_0) \boxtimes [\text{Id}] \boxtimes \widehat{CFD}(\mathcal{H}_1) \\ &\rightarrow \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFDA}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{CFD}(\mathcal{H}_1) \\ &\rightarrow \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFDA}(\chi) \boxtimes \widehat{CFDA}(\chi^{-1}) \boxtimes \widehat{CFD}(\mathcal{H}_1) \\ &\xrightarrow{\Theta_0 \boxtimes \Theta_1} \widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1), \end{aligned}$$

where the first map comes from the homotopy equivalence  $[\text{Id}] \simeq \widehat{CFDA}(\mathbb{I}_{\mathcal{Z}})$ , the second map comes from some homotopy equivalence  $\widehat{CFDA}(\mathbb{I}_{\mathcal{Z}}) \rightarrow \widehat{CFDA}(\chi|_F) \boxtimes \widehat{CFDA}(\chi^{-1}|_F)$ , and the third map comes from some homotopy equivalences  $\widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFDA}(\chi|_F) \rightarrow \widehat{CFA}(\mathcal{H}_0)$  and  $\widehat{CFDA}(\chi^{-1}|_F) \boxtimes \widehat{CFD}(\mathcal{H}_1) \rightarrow \widehat{CFD}(\mathcal{H}_1)$ . By Lemmas 4.2 and 4.5, up to homotopy there is a unique homotopy equivalence in each case.  $\square$

As noted in the introduction, each of the maps in formula (1.4) is the unique homotopy class of homotopy equivalences between the given source and target. So, after computing the modules and bimodules by factoring into mapping classes [LOT14b], computing the homotopy equivalences required to describe the mapping class group action is straightforward (and, in particular, algorithmic).

## 7. THE SURGERY EXACT TRIANGLE

The goal of this section is to prove:

**Theorem 7.1.** *Let  $K$  be a framed knot in a 3-manifold  $Y$ . Then there is a surgery exact triangle*

$$\cdots \rightarrow \widehat{HFI}(Y) \rightarrow \widehat{HFI}(Y_{-1}(K)) \rightarrow \widehat{HFI}(Y_0(K)) \rightarrow \widehat{HFI}(Y) \rightarrow \cdots.$$

Before turning to the proof, to fix notation we recall the modules and maps used in the bordered proof of the surgery exact triangle for  $\widehat{HF}$  [LOT08, Section 11.2]. (The reader is referred to the original paper for a more leisurely account.)

Let  $\mathcal{H}_0$ ,  $\mathcal{H}_1$ , and  $\mathcal{H}_\infty$  be the standard, genus 1 Heegaard diagrams for the 0-framed, 1-framed, and  $\infty$ -framed solid tori, respectively. It is easy to compute that

$$\begin{aligned} \widehat{CFD}(\mathcal{H}_\infty) &= \langle r \mid \delta^1(r) = \rho_{2,4}r \rangle, \\ \widehat{CFD}(\mathcal{H}_{-1}) &= \langle a, b \mid \delta^1(a) = (\rho_{1,2} + \rho_{3,4})b, \delta^1(b) = 0 \rangle, \\ \widehat{CFD}(\mathcal{H}_0) &= \langle n \mid \delta^1(n) = \rho_{1,3}n \rangle. \end{aligned}$$

Further, there is a short exact sequence

$$0 \rightarrow \widehat{CFD}(\mathcal{H}_\infty) \xrightarrow{\phi} \widehat{CFD}(\mathcal{H}_{-1}) \xrightarrow{\psi} \widehat{CFD}(\mathcal{H}_0) \rightarrow 0,$$

where  $\phi$  and  $\psi$  are given by

$$\phi(r) = b + \rho_{2,3}a, \quad \psi(a) = n, \quad \psi(b) = \rho_{2,3}n.$$

Given any bordered 3-manifold  $Y$  with boundary  $T^2$ , tensoring this short exact sequence with  $\widehat{CFA}(Y)$  gives a long exact sequence in homology [LOT08, Proposition 2.36]—the desired surgery exact sequence. (This exact sequence agrees with Ozsváth-Szabó’s original [OSz04a], as proved in [LOT14a, Corollary 5.41].)

For notational convenience, in this section let  $\mathbf{AZ} = \mathbf{AZ}(-\mathcal{Z}_1)$ . The main work in extending these bordered computations to prove Theorem 7.1 is the following lemma.

**Lemma 7.2.** *There are homotopies  $G: \widehat{\mathit{CFDA}}(\mathbf{AZ}) \boxtimes \widehat{\mathit{CFD}}(\mathcal{H}_\infty) \rightarrow \widehat{\mathit{CFD}}(\mathcal{H}_{-1})$  and  $H: \widehat{\mathit{CFDA}}(\mathbf{AZ}) \boxtimes \widehat{\mathit{CFD}}(\mathcal{H}_{-1}) \rightarrow \widehat{\mathit{CFD}}(\mathcal{H}_0)$  making each square of the following diagram homotopy commute:*

$$\begin{array}{ccccc} \widehat{\mathit{CFDA}}(\mathbf{AZ}) \boxtimes \widehat{\mathit{CFD}}(\mathcal{H}_\infty) & \xrightarrow{\text{Id} \boxtimes \phi} & \widehat{\mathit{CFDA}}(\mathbf{AZ}) \boxtimes \widehat{\mathit{CFD}}(\mathcal{H}_{-1}) & \xrightarrow{\text{Id} \boxtimes \psi} & \widehat{\mathit{CFDA}}(\mathbf{AZ}) \boxtimes \widehat{\mathit{CFD}}(\mathcal{H}_0) \\ \Psi \downarrow & \dashrightarrow \xrightarrow{G} & \Psi \downarrow & \dashrightarrow \xrightarrow{H} & \Psi \downarrow \\ \widehat{\mathit{CFD}}(\mathcal{H}_\infty) & \xrightarrow{\phi} & \widehat{\mathit{CFD}}(\mathcal{H}_{-1}) & \xrightarrow{\psi} & \widehat{\mathit{CFD}}(\mathcal{H}_0). \end{array}$$

Further,  $\psi \circ G = H \circ (\text{Id} \boxtimes \phi)$ .

*Proof.* This is a direct computation.

Recall from Section 2.4 that  $\widehat{\mathit{CFDA}}(\mathbf{AZ})$  is the type  $DA$  bimodule with generators

$$\begin{aligned} \iota_1 \otimes \iota_0, \quad & \iota_1 \otimes \rho_{1,2}, \quad \iota_1 \otimes \rho_{1,3}, \quad \iota_1 \otimes \rho_{1,4}, \\ \iota_1 \otimes \rho_{3,4}, \quad & \iota_0 \otimes \iota_1, \quad \iota_0 \otimes \rho_{2,3}, \quad \iota_0 \otimes \rho_{2,4}. \end{aligned}$$

The operation  $\delta_2^1: \widehat{\mathit{CFDA}}(\mathbf{AZ}) \otimes \mathcal{A}(T^2) \rightarrow \mathcal{A}(T^2) \otimes \widehat{\mathit{CFDA}}(\mathbf{AZ})$  is the obvious right action of  $\mathcal{A}(T^2)$ , and  $\delta_1^1: \widehat{\mathit{CFDA}}(\mathbf{AZ}) \rightarrow \mathcal{A}(T^2) \otimes \widehat{\mathit{CFDA}}(\mathbf{AZ})$  is induced by

$$\begin{aligned} \delta_1^1(\iota_1 \otimes \iota_0) &= \rho_{2,3} \otimes (\iota_0 \otimes \rho_{2,3}), \\ \delta_1^1(\iota_0 \otimes \iota_1) &= \rho_{1,2} \otimes (\iota_1 \otimes \rho_{1,2}) + \rho_{3,4} \otimes (\iota_1 \otimes \rho_{3,4}) + \rho_{1,4} \otimes (\iota_1 \otimes \rho_{1,4}), \end{aligned}$$

and the Leibniz rule with  $\delta_2^1$ . All higher  $\delta_k^1$ ,  $k \geq 3$ , vanish.

Thus, the type  $D$  structure  $\widehat{\mathit{CFDA}}(\mathbf{AZ}) \boxtimes \widehat{\mathit{CFD}}(\mathcal{H}_\infty)$  has generators

$$\iota_1 \otimes \rho_{1,2} \otimes r, \quad \iota_1 \otimes \rho_{1,4} \otimes r, \quad \iota_1 \otimes \rho_{3,4} \otimes r, \quad \iota_0 \otimes \iota_1 \otimes r, \quad \iota_0 \otimes \rho_{2,4} \otimes r$$

(as a type  $D$  structure) with differential given by

$$\begin{aligned} \delta^1(\iota_1 \otimes \rho_{1,2} \otimes r) &= \iota_1 \otimes (\iota_1 \otimes \rho_{1,4} \otimes r), \\ \delta^1(\iota_1 \otimes \rho_{1,4} \otimes r) &= 0, \\ \delta^1(\iota_1 \otimes \rho_{3,4} \otimes r) &= \rho_{2,3} \otimes (\iota_0 \otimes \rho_{2,4} \otimes r), \\ \delta^1(\iota_0 \otimes \iota_1 \otimes r) &= \iota_0 \otimes (\iota_0 \otimes \rho_{2,4} \otimes r) + \rho_{1,2} \otimes (\iota_1 \otimes \rho_{1,2} \otimes r) \\ &\quad + \rho_{3,4} \otimes (\iota_1 \otimes \rho_{3,4} \otimes r) + \rho_{1,4} \otimes (\iota_1 \otimes \rho_{1,4} \otimes r), \\ \delta^1(\iota_0 \otimes \rho_{2,4} \otimes r) &= \rho_{1,2} \otimes (\iota_1 \otimes \rho_{1,4} \otimes r). \end{aligned}$$

Here, some terms come from the operation  $\delta^1$  on  $\widehat{\mathit{CFD}}(\mathcal{H}_\infty)$  (together with the operation  $\delta_2^1$  on  $\widehat{\mathit{CFDA}}(\mathbf{AZ})$ ), while other terms come from the operation  $\delta_1^1$  on  $\widehat{\mathit{CFDA}}(\mathbf{AZ})$ . The quasi-isomorphism  $\Psi$  is given by

$$\Psi(\iota_1 \otimes \rho_{3,4} \otimes r) = \iota_1 \otimes r, \quad \Psi(\iota_0 \otimes \rho_{2,4} \otimes r) = \rho_{3,4} \otimes r,$$

$$\Psi(\iota_1 \otimes \rho_{1,2} \otimes r) = \Psi(\iota_1 \otimes \rho_{1,4} \otimes r) = \Psi(\iota_0 \otimes \iota_1 \otimes r) = 0.$$

These formulas are perhaps easier to absorb, and check, graphically:

$$\widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) \quad \widehat{CFD}(\mathcal{H}_\infty)$$

Here, we have replaced tensor signs with vertical bars. Unlabeled arrows are implicitly labeled by idempotents. Dashed arrows represent the map  $\Psi$ , while solid arrows represent  $\delta^1$ . Labels are always above the corresponding arrows. The check that  $\Psi$  is a homomorphism reduces to examining all length-two paths from a vertex on the left to  $r$ . The map is clearly a quasi-isomorphism.

After this warm-up, the complexes  $\widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(\mathcal{H}_{-1})$  and  $\widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(\mathcal{H}_0)$ , the maps  $\Psi$  on them, the morphisms  $\phi$  and  $\psi$  and induced maps  $\text{Id} \boxtimes \phi$  and  $\text{Id} \boxtimes \psi$ , and the homotopies are shown in Figure 4.

Again, checking that this diagram is correct reduces to looking at length-two paths. Have fun!  $\square$

*Proof of Theorem 7.1.* The framing of  $K$  makes  $X(K) := Y \setminus \text{nbd}(K)$  into a bordered 3-manifold. We claim that the squares in the following diagram commute up to the dashed homotopies shown:

$$\begin{array}{ccccccc}
 \widehat{CFA}(X(K)) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) & \xrightarrow{\text{Id} \boxtimes \phi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) & \xrightarrow{\text{Id} \boxtimes \psi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFD}(\mathcal{H}_0) & & \\
 \Omega \downarrow & & \Omega \downarrow & & \Omega \downarrow & & \\
 \widehat{CFA}(X(K)) \boxtimes \widehat{CFDA}(\overline{\text{AZ}}) & \xrightarrow{\text{Id}^3 \boxtimes \phi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFDA}(\overline{\text{AZ}}) & \xrightarrow{\text{Id}^3 \boxtimes \psi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFDA}(\overline{\text{AZ}}) & & \\
 \boxtimes \widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) & & \boxtimes \widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) & & \boxtimes \widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(\mathcal{H}_0) & & \\
 \Psi_0 \boxtimes \text{Id}^2 \downarrow & & \Psi_0 \boxtimes \text{Id}^2 \downarrow & & \Psi_0 \boxtimes \text{Id}^2 \downarrow & & \\
 \widehat{CFA}(X(K)) \boxtimes \widehat{CFDA}(\text{AZ}) & \xrightarrow{\text{Id}^2 \boxtimes \phi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFDA}(\text{AZ}) & \xrightarrow{\text{Id}^2 \boxtimes \psi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFDA}(\text{AZ}) & & \\
 \boxtimes \widehat{CFD}(\mathcal{H}_\infty) & & \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) & & \boxtimes \widehat{CFD}(\mathcal{H}_0) & & \\
 \text{Id} \boxtimes \Psi_1 \downarrow & \searrow \text{Id} \boxtimes G & \text{Id} \boxtimes \Psi_1 \downarrow & \searrow \text{Id} \boxtimes H & \text{Id} \boxtimes \Psi_1 \downarrow & & \\
 \widehat{CFA}(X(K)) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) & \xrightarrow{\text{Id} \boxtimes \phi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) & \xrightarrow{\text{Id} \boxtimes \psi} & \widehat{CFA}(X(K)) \boxtimes \widehat{CFD}(\mathcal{H}_0). & & 
 \end{array}$$

Indeed, the fact that the top two rows commute on the nose follows from basic properties of the box tensor product [LOT15, Lemma 2.3.3]. For the third row, commutativity up to the homotopies follows from these properties and Lemma 7.2. Further, by Lemma 7.2, the homotopies satisfy

$$(\text{Id} \boxtimes H) \circ (\text{Id}^2 \boxtimes \phi) = (\text{Id} \boxtimes \psi) \circ (\text{Id} \boxtimes G).$$

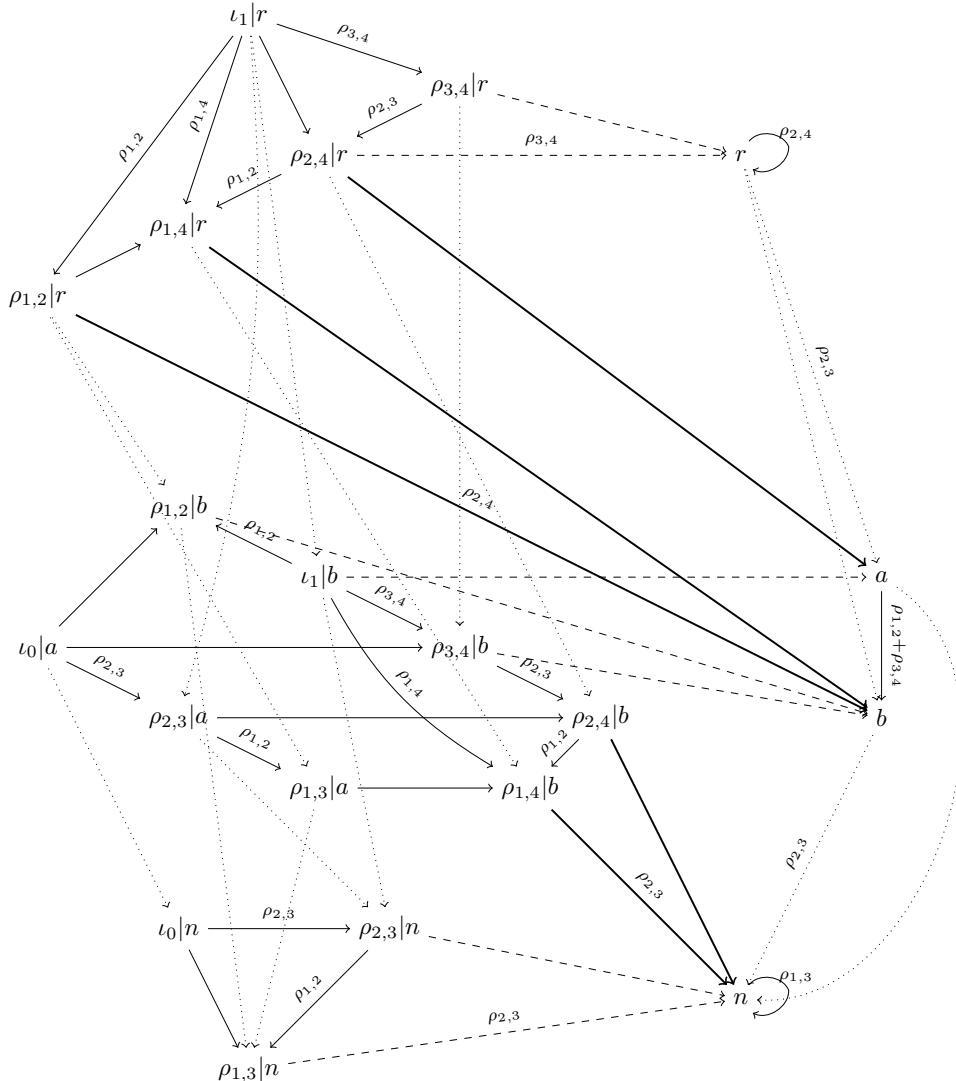


FIGURE 4. Proof of Lemma 7.2. The maps  $\Psi$  are dashed,  $\phi$  and  $\psi$  are dotted, and the homotopies are thick. We have dropped the first idempotent in the label for each generator (since it is determined by the other data), so for instance the generator  $\iota_1 \otimes r$  is denoted  $\rho_{3,4}|r$ . Arrow labels, which indicate type  $D$  outputs, are always above the center of the corresponding arrow (except for the self-arrows of  $n$  and  $r$ ).

Since by Theorem 5.1 the composition of the three vertical arrows in any column is the map  $\iota$ , it follows that there is a homotopy commutative diagram

$$(7.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widehat{CF}(Y) & \xrightarrow{i} & \widehat{CF}(Y_{-1}(K)) & \xrightarrow{p} & \widehat{CF}(Y_0(K)) \longrightarrow 0 \\ & & \downarrow \text{Id}+\iota & \searrow \text{G}' & \downarrow \text{Id}+\iota & \searrow \text{H}' & \downarrow \text{Id}+\iota \\ 0 & \longrightarrow & \widehat{CF}(Y) & \xrightarrow{i} & \widehat{CF}(Y_{-1}(K)) & \xrightarrow{p} & \widehat{CF}(Y_0(K)) \longrightarrow 0 \end{array}$$

where the rows are short exact sequences inducing the surgery exact triangle on homology, and the diagonal arrows are the homotopies  $G' = (\text{Id} \boxtimes G) \circ (\Psi_0 \boxtimes \text{Id}^2) \circ \Omega$  and  $H' = (\text{Id} \boxtimes H) \circ (\Psi_0 \boxtimes \text{Id}^2) \circ \Omega$ .

The theorem now follows from the commutative diagram (7.3) and homological algebra (cf. [HM17, Proof of Proposition 4.1]). That is, by Lemma 7.2, the homotopies in diagram (7.3) satisfy

$$(7.4) \quad p \circ G' = H' \circ i.$$

Take the mapping cone of each vertical map in the diagram to obtain a sequence of chain complexes

$$0 \rightarrow \text{Cone}((\text{Id}+\iota)_{\widehat{CF}(Y)}) \rightarrow \text{Cone}((\text{Id}+\iota)_{\widehat{CF}(Y_{-1}(K))}) \rightarrow \text{Cone}((\text{Id}+\iota)_{\widehat{CF}(Y_0(K))}) \rightarrow 0,$$

where the maps are given by the matrices

$$\begin{bmatrix} i & 0 \\ G & i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p & 0 \\ H & p \end{bmatrix}.$$

Homotopy commutativity of diagram (7.3) implies that these maps are chain maps, and exactness of the rows in diagram (7.3) together with equation (7.4) implies that this sequence is exact. The associated long exact sequence is the statement of the lemma.  $\square$

*Remark 7.5.* The proof of Theorem 7.1 also shows that the map induced by  $\iota$  on homology commutes with the maps in the surgery exact triangle for  $\widehat{HF}$ . Lidman points out that this commutativity can be deduced more directly by an argument that also applies to  $HF^\pm$ . Specifically, the maps in the surgery exact triangle for  $\widehat{HF}$  or  $HF^\pm$  are induced by cobordisms, and cobordism maps commute with the conjugation isomorphism (cf. [OSz06, Theorem 3.6]).

## 8. INVOLUTIVE FLOER HOMOLOGY AS MORPHISM SPACES

In this section we give some formulas purely in terms of  $\widehat{CFD}$  for the map  $\iota: \widehat{CF}(Y) \rightarrow \widehat{CF}(Y)$  and the map associated to a mapping class, which may be helpful in computer implementations.

Given a type  $D$  structure  ${}^A P$  over a  $dg$  algebra  $\mathcal{A}$  over  $\mathbb{F}_2$ , consisting of a finite-dimensional underlying vector space  $X$  and a map  $\delta^1: X \rightarrow \mathcal{A} \otimes X$ , the *dual type D structure*  $\overline{P}^A$  has underlying vector space  $X^*$ , the dual space to  $X$ , and operation

$$\delta_{\overline{P}}^1: X^* \rightarrow X^* \otimes \mathcal{A}$$

induced from  $\delta^1 \in \text{Hom}(X, \mathcal{A} \otimes X)$  via the identifications

$$\text{Hom}(X, \mathcal{A} \otimes X) \cong X^* \otimes \mathcal{A} \otimes X \cong X \otimes X^* \otimes \mathcal{A} \cong \text{Hom}(X^*, X^* \otimes \mathcal{A}).$$

Given a bordered 3-manifold  $Y$  with boundary  $F(\mathcal{Z})$ , recall that

$$\widehat{CFA}(Y) \simeq \overline{\widehat{CFD}(-Y)} \boxtimes \mathcal{A}(\mathcal{Z})$$

[LOT11, Theorem 2], so given bordered 3-manifolds  $Y_1$  and  $Y_2$  with  $\partial Y_1 = F(\mathcal{Z}) = -\partial Y_2$ ,

$$(8.1) \quad \begin{aligned} \widehat{CF}(Y_1 \cup_{\partial} Y_2) &\simeq \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2) \simeq \overline{\widehat{CFD}(-Y_1)} \boxtimes \mathcal{A}(\mathcal{Z}) \boxtimes \widehat{CFD}(Y_2) \\ &= \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2)) \end{aligned}$$

[LOT11, Theorem 1].

Using this, we explain how to compute the map  $\iota$  without mentioning  $\widehat{CFA}$ . Fix a Heegaard splitting  $Y = H \cup_{\psi} H$ . To compute  $\widehat{HF}(Y)$  one first computes  $\widehat{CFD}(H, \psi \circ \phi_0)$  and  $\widehat{CFD}(H, \phi_0)$ , where  $\phi_0: F(\mathcal{Z}_g) \rightarrow \partial H$  is the 0-framing (as in Section 2.2). The computation of  $\widehat{CFD}(H, \phi_0 \circ \psi)$  uses a factorization of  $\psi$  into arcsides and the identity

$$\widehat{CFD}(Y, \phi \circ \psi) \simeq \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFDD}(-\psi), \widehat{CFD}(\phi))$$

(see [LOT14b]). Then one uses formula (8.1). Indeed, this algorithm has already been implemented by Lipshitz-Ozsváth-Thurston [LOT14b] and Zhan [Zha].

Recall that a  $DA$  bimodule  ${}^B P_{\mathcal{A}}$  is called *quasi-invertible* if there is a  $DA$  bimodule  ${}^A Q_{\mathcal{B}}$  so that

$${}^B P_{\mathcal{A}} \boxtimes {}^A Q_{\mathcal{B}} \simeq {}^B [\text{Id}_{\mathcal{B}}]_{\mathcal{B}} \quad \text{and} \quad {}^A Q_{\mathcal{B}} \boxtimes {}^B P_{\mathcal{A}} \simeq {}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}}.$$

Let  $\text{Mor}^B({}^B P_{\mathcal{A}}, {}^B P_{\mathcal{A}})$  denote the complex of left type  $D$  morphisms of  $P$ . This morphism complex is an  $\mathcal{A}$ -bimodule. (The module structure is somewhat intricate; see [LOT15, Section 2.3.4].)

We have the following Yoneda lemma.

**Lemma 8.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be dg algebras and let  ${}^B P_{\mathcal{A}}$  be a quasi-invertible  $DA$  bimodule. Then there is a quasi-isomorphism of  $A_{\infty}$ -bimodules*

$$\Omega: {}_{\mathcal{A}} \mathcal{A}_{\mathcal{A}} \xrightarrow{\sim} \text{Mor}^B({}^B P_{\mathcal{A}}, {}^B P_{\mathcal{A}})$$

which sends the multiplicative identity  $1 \in \mathcal{A}$  to the identity morphism  $\text{Id}_P$ . More generally, the  $A_{\infty}$ -bimodule map  $\Omega$  is given by

(8.3)

$$\Omega_{m,1,n}(a_1, \dots, a_m, a, a'_1, \dots, a'_n)(x) = \delta_{1+m+1+n}^1(x, a_1, \dots, a_m, a, a'_1, \dots, a'_n),$$

where  $\delta_k^1$  is the structure map of  $P$ .

*Proof.* Let  $\text{Mor}^A({}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}}, {}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}})$  be the chain complex of type  $D$  structure morphisms. Then the map  $F: \mathcal{A} \rightarrow \text{Mor}_{\mathcal{A}}({}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}}, {}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}})$  defined by

$$\begin{aligned} F_1(a) &= (1 \mapsto a \otimes 1) \\ F_n &= 0, \quad n > 1, \end{aligned}$$

is a chain homotopy equivalence. Next, since  $P$  is quasi-invertible, the functor  $P \boxtimes \cdot$  is a quasi-equivalence of dg categories. Thus, the map

$$\begin{aligned} G: \text{Mor}^A({}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}}, {}^A [\text{Id}_{\mathcal{A}}]_{\mathcal{A}}) &\rightarrow \text{Mor}^B(P \boxtimes [\text{Id}_{\mathcal{A}}], P \boxtimes [\text{Id}_{\mathcal{A}}]), \\ G(f) &= \text{Id}_P \boxtimes f \end{aligned}$$

is a quasi-isomorphism. (Compare [LOT15, Proposition 2.3.36].) The composition  $G \circ F$  is the desired equivalence. Tracing through the definitions gives the formula (8.3).  $\square$

**Corollary 8.4.** *Under the identification*

$$\widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFAA}}(\text{AZ}) \boxtimes \widehat{\text{CFDA}}(\text{AZ}) \simeq \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFDA}}(\text{AZ}), \widehat{\text{CFDA}}(\text{AZ})),$$

*the unique homogeneous homotopy equivalence (of  $A_\infty$ -bimodules)*

$$\mathcal{A}(\mathcal{Z}) = \widehat{\text{CFAA}}(\text{AZ}) \xrightarrow{\Omega} \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFDA}}(\text{AZ}), \widehat{\text{CFDA}}(\text{AZ}))$$

*is given by*

$$\begin{aligned} \Omega_1(a)(x) &= \delta_2^1(x, a) \\ \Omega_n(a_1, \dots, a_n) &= 0, \quad n > 1. \end{aligned}$$

*Proof.* This is immediate from Lemma 8.2 and the fact that the structure map  $\delta_n^1$  for  $\widehat{\text{CFDA}}(\text{AZ})$  vanishes for  $n > 2$ .  $\square$

**Theorem 8.5.** *Fix a Heegaard splitting  $Y = (-H_0) \cup H_1$  of  $Y$ . Then up to homotopy the map  $\iota: \widehat{\text{CF}}(Y) \rightarrow \widehat{\text{CF}}(Y)$  is given by the composition*

$$\begin{aligned} &\text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFD}}(H_0), \widehat{\text{CFD}}(H_1)) \\ &\xrightarrow{\Omega} \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(H_0), \widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(H_1)) \\ &\xrightarrow{\Psi} \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFD}}(H_0), \widehat{\text{CFD}}(H_1)), \end{aligned}$$

*where*

$$\begin{aligned} \Omega: &\text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFD}}(\mathcal{H}_0), \widehat{\text{CFD}}(\mathcal{H}_1)) \\ &\rightarrow \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_0), \widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_1)) \end{aligned}$$

*sends a morphism  $f$  to  $\text{Id}_{\widehat{\text{CFDA}}(\text{AZ})} \boxtimes f$ , and if  $\Psi_i: \widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_0) \rightarrow \widehat{\text{CFD}}(\mathcal{H}_i)$  is the homogeneous homotopy equivalence, then  $\Psi$  sends a morphism  $g \in \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_0), \widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_1))$  to  $\Psi_1 \circ g \circ \Psi_0^{-1}$ .*

This seems to be a succinct, and computer-friendly, description of the map  $\iota$ .

*Proof.* Choose a Heegaard diagram  $\mathcal{H}_i$  for  $H_i$ . Then the pairing theorem gives

$$\text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFD}}(H_0), \widehat{\text{CFD}}(H_1)) \simeq \widehat{\text{CF}}((-H_0) \cup \text{AZ} \cup \mathcal{H}_1),$$

which is identified, via  $\eta$ , with  $\widehat{\text{CF}}(\mathcal{H}_0^\beta \cup \overline{\text{AZ}}^\beta \cup \overline{\mathcal{H}_1})$ .

Similarly,

$$\begin{aligned} &\text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(H_0), \widehat{\text{CFDA}}(\text{AZ}) \boxtimes \widehat{\text{CFD}}(H_1)) \\ &\simeq \widehat{\text{CF}}(\mathcal{H}_0^\beta \cup \text{AZ}^\beta \cup \overline{\text{AZ}}^\beta \cup \overline{\text{AZ}}^\beta \cup \overline{\mathcal{H}_1}). \end{aligned}$$

Consider a sequence of Heegaard moves

$$\begin{aligned} \mathcal{H}_0^\beta \cup \overline{\text{AZ}}^\beta \cup \overline{\mathcal{H}_1} &\rightarrow (\mathcal{H}_0^\beta \cup \text{AZ}^\beta) \cup (\overline{\text{AZ}}^\beta \cup \overline{\text{AZ}}^\beta \cup \overline{\mathcal{H}_1}) \\ &\rightarrow (-H_0) \cup (\text{AZ} \cup \mathcal{H}_1), \end{aligned}$$

where the first arrow does not change the diagrams at the end and the second arrow consists of bordered Heegaard moves changing the diagrams on the two sides of the

union sign. There are two associated maps on  $\widehat{CF}$ . By the pairing theorem for triangles, the first map is induced by a map

$$\begin{aligned} \mathcal{A}(\mathcal{Z}) &= \widehat{CFAA}(\overline{A}\overline{Z}^\beta) \rightarrow \widehat{CFAA}(AZ^\beta \cup \overline{A}\overline{Z}^\beta \cup \overline{A}\overline{Z}^\beta) \\ &\simeq \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFDA}(AZ), \widehat{CFDA}(AZ)). \end{aligned}$$

By uniqueness, this map is the map  $\Omega$  of Corollary 8.4. It follows from the definition of  $\Omega$  and the pairing theorem that the induced map

$$\begin{aligned} \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(\mathcal{H}_0), \widehat{CFD}(\mathcal{H}_1)) \\ \rightarrow \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFDA}(AZ) \boxtimes \widehat{CFD}(\mathcal{H}_0), \widehat{CFDA}(AZ) \boxtimes \widehat{CFD}(\mathcal{H}_1)) \end{aligned}$$

sends  $f$  to  $\text{Id} \boxtimes f$ . Similarly, by the pairing theorem for triangles, the second map is induced by an equivalence on each of the parenthesized pieces and thus agrees with the map  $\Psi$ .  $\square$

The mapping class group action admits a similar description: the action of  $\chi$  is given by

$$\begin{aligned} \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(H_g, \phi_0), \widehat{CFD}(H_g, \phi_0 \circ \psi)) \\ \rightarrow \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFDA}(\chi^{-1}) \boxtimes \widehat{CFD}(H_g, \phi_0), \widehat{CFDA}(\chi^{-1}) \boxtimes \widehat{CFD}(H_g, \phi_0 \circ \psi)) \\ \rightarrow \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(H_g, \phi_0), \widehat{CFD}(H_g, \phi_0 \circ \psi)), \end{aligned}$$

where the first map sends a morphism  $f$  to  $\text{Id} \boxtimes f$  and the second sends  $g$  to  $\Theta_1 \circ g \circ \Theta_0^{-1}$ . We can rewrite this using  $\widehat{CFDD}(\chi)$  instead of  $\widehat{CFDA}(\chi)$  as

$$\begin{aligned} \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(H_g, \phi_0), \widehat{CFD}(H_g, \phi_0 \circ \psi)) \\ \rightarrow \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFDD}(\chi), \widehat{CFD}(H_g, \phi_0)), \\ \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFDD}(\chi), \widehat{CFD}(H_g, \phi_0 \circ \psi))) \\ \rightarrow \text{Mor}^{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(H_g, \phi_0), \widehat{CFD}(H_g, \phi_0 \circ \psi)), \end{aligned}$$

where the first arrow sends a morphism  $f$  to the morphism which sends a morphism  $h$  to  $f \circ h$  and the second arrow is again induced by the unique homotopy equivalences  $\text{Mor}(\widehat{CFDD}(\chi), \widehat{CFD}(H_g, \phi_0)) \simeq \widehat{CFD}(H_g, \phi_0)$  and

$$\text{Mor}(\widehat{CFDD}(\chi), \widehat{CFD}(H_g, \phi_0 \circ \psi)) \simeq \widehat{CFD}(H_g, \phi_0 \circ \psi).$$

The proof that this gives the mapping class group action is similar to the proof of Theorem 8.5 and is left to the reader.

## 9. EXAMPLES

For a knot  $K$  in  $S^3$ , let  $\Sigma(K)$  denote the branched double cover of  $K$ . To illustrate the algorithm for computing  $\iota$ , we finish the computation of  $\widehat{HFI}(\Sigma(K))$  for knots  $K$  through 10 crossings.

If  $\Sigma(K)$  is an  $L$ -space, then, since  $\Sigma(K)$  is a rational homology sphere with a unique spin-structure,  $\widehat{HFI}(\Sigma(K)) \cong \mathbb{F}_2^{\det(K)+1}$ . That is,  $\widehat{HFI}(\Sigma(K))$  has two generators for each conjugacy class of spin<sup>c</sup>-structures. The  $Q$ -action takes one generator corresponding to the spin-structure to the other and vanishes on all other generators. All knots  $K$  with 9 or fewer crossings have  $\Sigma(K)$  an  $L$ -space. Indeed, except

TABLE 1. The 10-crossing knots with  $\Sigma(K)$  not an  $L$ -space. The table lists the dimensions of  $\widehat{HF}(\Sigma(K))$  and  $\widehat{HFI}(\Sigma(K))$ , as computed by Zhan's program and its extension, for these knots as well as  $\det(K) = |H_1(\Sigma(K))|$ . Computations of  $\det(K)$  are taken from The Knot Atlas, [katlas.org](http://katlas.org).

| Knot $K$   | $\det(K)$ | $\dim \widehat{HF}(\Sigma(K))$ | $\dim \widehat{HFI}(\Sigma(K))$ |
|------------|-----------|--------------------------------|---------------------------------|
| $10_{139}$ | 3         | 5                              | 6                               |
| $10_{145}$ | 3         | 5                              | 6                               |
| $10_{152}$ | 11        | 13                             | 14                              |
| $10_{153}$ | 1         | 5                              | 6                               |
| $10_{154}$ | 13        | 15                             | 16                              |
| $10_{161}$ | 5         | 7                              | 8                               |

for  $8_{19} = T(3, 4)$ ,  $9_{42}$ , and  $9_{46}$ , every knot  $K$  with 9 or fewer crossings is quasi-alternating [JS09, Jab14]; for quasi-alternating knots,  $\Sigma(K)$  is an  $L$ -space [OSz05]. It turns out that  $\Sigma(8_{19})$ ,  $\Sigma(9_{42})$ , and  $\Sigma(9_{46})$  are  $L$ -spaces. (This can be checked using Zhan's computer program [Zha].)

The 10-crossing knots  $K$  for which  $\Sigma(K)$  is not an  $L$ -space are listed in Table 1. The computation of which of these spaces are not  $L$ -spaces and the dimensions of their Floer homologies was accomplished by Zhan. Computation of  $\widehat{HFI}$  for these manifolds was carried out by a modest extension of Zhan's program, using the algorithm described above. The first two knots,  $10_{139}$  and  $10_{145}$ , are Montesinos knots; hence our computation is implied by (and agrees with) the computation of  $HFI^-$  for Seifert fibered spaces [DM17]. We make a few further comments about the details of our implementation below.

Both Zhan's code and our extension, which is now included in Zhan's package [Zha], are written in Python (version 2.7). Zhan's code includes classes for chain complexes, type  $D$  structures, and type  $DA$  structures, as well as for morphisms between them. He also, of course, implemented basic operations on these structures, including taking the box tensor product of a type  $D$  structure and a type  $DA$  structure and computing the morphism complex between two type  $D$  structures. His program also automates computation of  $\widehat{HF}(\Sigma(K))$  given a bridge diagram for  $K$ . The algorithms behind Zhan's code use properties of the bordered bimodules which appear only in his thesis [Zha14] to compute tensor products without writing down all of the generators. (He calls this technique *extending by the identity* and the local objects that he extends *local type DA structures*.) The upshot is that his code computes  $\widehat{CFD}(H_0)$  and  $\widehat{CFD}(H_1)$  efficiently.

In our extension, we implemented the bimodule  $\widehat{CFDA}(\text{AZ})$ , mapping cones of maps between type  $D$  structures and chain complexes, composition of morphisms between type  $D$  structures, and the tensor product of a morphism of type  $D$  structures with the identity map of a type  $DA$  structure. Computing mapping cones gives some easy sanity checks: it makes testing whether maps are quasi-isomorphisms trivial by checking whether their mapping cones are acyclic.

Our code computes the rank of  $\widehat{HFI}$  by:

- (1) Computing  $\widehat{CFD}(H_0)$ ,  $\widehat{CFD}(H_1)$ ,  $\widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(H_0)$ , and  $\widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(H_1)$ , as well as various morphism complexes between them.
- (2) Computing a basis  $\{f_1, \dots, f_n\}$  for  $H_* \text{Mor}(\widehat{CFD}(H_0), \widehat{CFD}(H_1))$ , consisting of explicit cycles in  $\text{Mor}(\widehat{CFD}(H_0), \widehat{CFD}(H_1))$ .
- (3) For each basis element  $f_i$ , computing  $\text{Id}_{\widehat{CFDA}(\text{AZ})} \boxtimes f_i$ .
- (4) Computing a basis for  $H_* \text{Mor}(\widehat{CFD}(H_0), \widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(H_0))$  and for  $H_* \text{Mor}(\widehat{CFDA}(\text{AZ}) \boxtimes \widehat{CFD}(H_1), \widehat{CFD}(H_1))$ . Even though we do not implement the grading for  $\widehat{CFDA}(\text{AZ})$ , the way that Zhan's code computes homology automatically gives bases of homogeneous elements. Each of these bases has  $2^k$  elements where  $k$  is the genus of the Heegaard splitting. For the computations in Table 1,  $k = 2$ , so each of these bases has 4 elements.
- (5) Searching through these bases to find the unique homotopy equivalences  $\Psi_0^{-1}$  and  $\Psi_1$ .
- (6) For each  $f_i$ , computing the composition  $\Psi_1 \circ (\text{Id}_{\widehat{CFDA}(\text{AZ})} \boxtimes f_i) \circ \Psi_0^{-1}$ . The map

$$f_i \mapsto \Psi_1 \circ (\text{Id}_{\widehat{CFDA}(\text{AZ})} \boxtimes f_i) \circ \Psi_0^{-1}$$

is a map

$$[H_* \text{Mor}(\widehat{CFD}(H_0), \widehat{CFD}(H_1))] \rightarrow \text{Mor}(\widehat{CFD}(H_0), \widehat{CFD}(H_1))$$

representing  $\iota$ . (Mapping from the homology of the complex to the complex means we do not have to choose a projection from the morphism complex to its homology.) Abusing notation, we call this map  $\iota$ .

- (7) There is also an inclusion

$$\text{Id}: H_* \text{Mor}(\widehat{CFD}(H_0), \widehat{CFD}(H_1)) \rightarrow \text{Mor}(\widehat{CFD}(H_0), \widehat{CFD}(H_1))$$

induced by the choice of cycles  $f_1, \dots, f_n$ . The involutive Floer homology is then the homology of  $\text{Cone}(\iota + \text{Id})$ .

The computations in Table 1 are fairly slow: on a circa 2016 MacBook Pro with 16 GB of RAM the code computes  $\widehat{HF}(\Sigma(K))$  within a few minutes, but each computation of  $\widehat{HFI}(\Sigma(K))$  takes up to several hours. (We have not made a serious attempt to improve the efficiency of our code.)

**9.1. Computing  $HFI^-$  from  $\widehat{HFI}$ .** Sometimes, one can recover  $HFI^-(Y)$  from  $\widehat{HF}(Y)$  and  $\widehat{HFI}(Y)$ . (This is desirable given that most known applications use  $HFI^-(Y)$  or  $HFI^+(Y)$  rather than  $\widehat{HFI}(Y)$ .) We illustrate the process of recovering  $HFI^-$  by computing  $HFI^-(\Sigma(10_{161}))$  up to a grading shift.

Let  $\mathfrak{s}_0$  denote the spin-structure on  $\Sigma(10_{161})$ . If  $\mathfrak{s} \in \text{Spin}^c(\Sigma(10_{161}))$  is any other spin<sup>c</sup>-structure, then, since  $\widehat{HF}(\Sigma(10_{161}), \mathfrak{s}) \cong \mathbb{F}_2$ ,  $HFI^-(\Sigma(10_{161}), [\mathfrak{s}]) \cong \mathbb{F}_2[U] \oplus \mathbb{F}_2[U]$  with trivial  $Q$ -action, where  $[\mathfrak{s}]$  denotes the orbit consisting of the spin<sup>c</sup> structure and its conjugate. So, for the rest of the section we focus on  $HFI^-(\Sigma(10_{161}), \mathfrak{s}_0)$ .

**Lemma 9.1.** *Let  $d = d(\Sigma(10_{161}), \mathfrak{s}_0)$  be the Heegaard Floer correction term of the  $\text{spin}^c$ -structure  $\mathfrak{s}_0$  on  $\Sigma(10_{161})$ . Then*

$$HFI^-(\Sigma(10_{161}), \mathfrak{s}_0) \simeq \mathbb{F}_2[U]_{(d-3)}\langle a \rangle \oplus \mathbb{F}_2[U]_{(d-2)}\langle b \rangle \oplus (\mathbb{F}_2)_{(d-2)}\langle c \rangle$$

with  $Q$ -action given by  $Qa = Ub$  and  $Qb = Qc = 0$ .

In [HM17], C. Manolescu and the first author extract two invariants of  $\mathbb{F}_2$ -homology cobordism from involutive Heegaard Floer homology, called the *involutive correction terms*. Given a rational homology sphere  $Y$  and a conjugation-invariant  $\text{spin}^c$ -structure  $\mathfrak{s}$ , in terms of the minus variant, these invariants are

$$\underline{d}(Y, \mathfrak{s}) = \max\{r \mid \exists x \in HFI_r^-(Y, \mathfrak{s}), \forall n, U^n x \neq 0 \text{ and } U^n x \notin \text{Im}(Q)\} + 1$$

and

$$\bar{d}(Y, \mathfrak{s}) = \max\{r \mid \exists x \in HFI_r^-(Y, \mathfrak{s}), \forall n, U^n x \neq 0; \exists m \geq 0 \text{ s.t. } U^m x \in \text{Im}(Q)\} + 2.$$

We therefore have the following corollary of Lemma 9.1.

**Corollary 9.2.** *The involutive correction terms of  $\Sigma(10_{161})$  in the unique spin structure are related to  $d = d(\Sigma(10_{161}), \mathfrak{s}_0)$  by*

$$\begin{aligned} \underline{d}(\Sigma(10_{161}), \mathfrak{s}_0) &= d - 2, \\ \bar{d}(\Sigma(10_{161}), \mathfrak{s}_0) &= d. \end{aligned}$$

*Proof of Lemma 9.1.* Let  $K = 10_{161}$ . Zhan's code for computing  $\widehat{HF}(\Sigma(K))$  can be used to compute relative gradings and  $\text{spin}^c$ -structures for generators of  $\widehat{HF}(\Sigma(K))$ . Arbitrarily numbering the  $\text{spin}^c$ -structures of the generators by  $0, \dots, 4$ , the code finds that, up to a shift, the gradings of the generators representing the different  $\text{spin}^c$ -structure are:

| $\mathfrak{s}$ | gr  | $\mathfrak{s}$ | gr  |
|----------------|-----|----------------|-----|
| 3              | 6/5 | 0              | 2/5 |
| 3              | 6/5 | 1              | 2/5 |
| 3              | 1/5 | 2              | 0   |
|                |     | 4              | 0   |

Thus, the  $\text{spin}^c$ -structure labeled 3 must be the central  $\text{spin}^c$ -structure. From the computer computation,  $\text{rank}(\widehat{HFI}(\Sigma(10_{161}))) = 8 = \text{rank}(\widehat{HF}(\Sigma(10_{161}))) + 1$ , so  $\iota_*$  must have exactly one fixed point, which must be the generator in relative grading 1/5. The other two elements in this  $\text{spin}^c$ -structure must, up to a change of basis, be interchanged by  $\iota_*$ . We conclude that  $\widehat{HF}(\Sigma(K), \mathfrak{s}_0)$  contains three elements, two in some grading  $q$  and one in grading  $q-1$ , and that up to a change of basis, the two elements in grading  $q$  are interchanged by  $\iota_*$ .

Now, recall that there is a long exact sequence

$$(9.3) \quad \cdots \rightarrow HF^-(\Sigma(K)) \xrightarrow{\cdot U} HF^-(\Sigma(K)) \rightarrow \widehat{HF}(\Sigma(K)) \rightarrow HF^-(\Sigma(K)) \rightarrow \cdots$$

such that the map  $HF^-(\Sigma(K)) \rightarrow \widehat{HF}(\Sigma(K))$  increases the grading by 2 and the map  $\widehat{HF}(\Sigma(K)) \rightarrow HF^-(\Sigma(K))$  decreases the grading by 1 [OSz04a, Proposition 2.1]. This long exact sequence commutes at every step with  $\iota_*$  [HM17, Proof of Proposition 4.1]. (Strictly speaking, this was proved for the analogous sequence for  $HF^+$ , but the proof for  $HF^-$  is identical.) It follows from the existence of this long exact sequence that there is a non-canonical isomorphism  $HF^-(\Sigma(K)) \simeq$

$\mathbb{F}_2[U]\langle\alpha\rangle \oplus \mathbb{F}_2\langle\beta\rangle$ , where both  $\alpha$  and  $\beta$  lie in grading  $q-2$ . In particular, the ordinary Heegaard Floer correction term is  $d(\Sigma(K), \mathfrak{s}_0) = q$ . Further, the grading shifts imply that the summand of  $\widehat{HF}(\Sigma(K))$  in grading  $q$  is precisely the image of the summand of  $HF^-(\Sigma(K))$  in grading  $q-2$ , which is spanned as a vector space by  $\alpha$  and  $\beta$ . Therefore since the long exact sequence (9.3) respects the action  $\iota_*$ , the involution on  $\widehat{HF}(\Sigma(K))$  is determined by the involution on  $HF^-(\Sigma(K))$ . There are exactly two  $U$ -equivariant involutions on  $HF^-(\Sigma(K))$ : the identity and the involution  $\iota_*(\alpha) = \alpha + \beta$ ,  $\iota_*(\beta) = \beta$ . The first of these induces the identity involution on  $\widehat{HF}(\Sigma(K))$ , contradicting the computer computation. Thus,  $\iota_*(\alpha) = \alpha + \beta$ ,  $\iota_*(\beta) = \beta$ .

Recall that there is an exact triangle

$$(9.4) \quad \begin{array}{ccc} HF^-(Y, \mathfrak{s}) & \xrightarrow{Q(1 + \iota_*)} & Q \cdot HF^-(Y, \mathfrak{s})[-1] \\ & \swarrow & \searrow \\ & HFI^-(Y, \mathfrak{s}) & \end{array}$$

[HM17, Proposition 4.6].

Ordinarily, the existence of this triangle is insufficient to determine  $HFI^-(Y, \mathfrak{s})$ . (That is,  $HFI^-$  is in general not a mapping cone of the map  $1 + \iota_*$  on  $HF^-$ , unlike the hat variant.) However, in this case the complex is sufficiently small that given our computation of  $\iota_*$ , the mapping cone of  $(1 + \iota_*)$  is the unique  $\mathbb{F}_2[U, Q]/(Q^2)$ -module that fits into the long exact triangle. The map  $Q(1 + \iota_*)$  takes  $\alpha$  to  $Q\beta$ . So,  $HFI^-(\Sigma(K), \mathfrak{s}_0)$  is generated by  $U\alpha = a$ ,  $Q\alpha = b$ , and  $\beta = c$ , and those elements lie in gradings  $d-3$ ,  $d-2$ , and  $d-2$  respectively.  $\square$

*Remark 9.5.* The reader may have noticed that the complex  $HF^-(\Sigma(10_{161}), \mathfrak{s}_0)$  is (after a change of basis) a symmetric graded root. Indeed, I. Dai and C. Manolescu recently showed that whenever  $(Y, \mathfrak{s})$  is such that  $HF^-(Y, \mathfrak{s})$  is a symmetric graded root with involution given by the canonical symmetry,  $HFI^-(Y, \mathfrak{s})$  is a mapping cone on  $HF^-(Y, \mathfrak{s})$  [DM17, Theorem 1.1].

*Remark 9.6.* It may be interesting to compare these computations with Lin's spectral sequence from a variant of Khovanov homology to involutive monopole Floer homology of the branched double cover [Lin16a].

*Remark 9.7.* One could call a rational homology sphere  $Y$   $\widehat{HFI}$ -trivial if for each spin-structure  $\mathfrak{s}$  on  $Y$ ,  $\widehat{HFI}(Y, \mathfrak{s}) \cong \widehat{HF}(Y, \mathfrak{s}) \oplus \mathbb{F}_2$  where  $Q \cdot \widehat{HF}(Y, \mathfrak{s}) = 0$  and  $Q$  is non-vanishing on the remaining generator. At the time of writing, no  $\widehat{HFI}$ -non-trivial rational homology sphere  $Y$  is known.

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