Exploring Kindergarten Students' Early Understandings of the Equal Sign[1]

*Maria Blanton, Ph.D., TERC, 2067 Massachusetts Avenue, Cambridge, MA 02140; Maria_Blanton@terc.edu, 617-873-9640

Yenny F Otalora Sevilla, Ph.D., Center for Research in Psychology, Cognition and Culture, Universidad del Valle; yenny.otalora@correounivalle.edu.co

Barbara M. Brizuela, Ph.D., Department of Education, Paige Hall, Tufts University, Medford, MA 02155; barbara.brizuela@tufts.edu

Angela Murphy Gardiner, M.A., TERC, 2067 Massachusetts Avenue, Cambridge, MA 02140; Angela_Gardiner@terc.edu

Katie Sawrey, Ph.D., Department of Education, Paige Hall, Tufts University, Medford, MA 02155; Katharine.Sawrey@tufts.edu

Aliska Gibbins, Department of Education, Paige Hall, Tufts University, Medford, MA 02155; Aliska.Gibbins@tufts.edu

Yangsook, Kim, Department of Education, Paige Hall, Tufts University, Medford, MA 02155; Yangsook.Kim@tufts.edu

*Corresponding Author

Introduction

Understanding the equal sign is foundational to algebraic thinking, yet an abundance of studies throughout school mathematics have shown that students hold deep and persistent misconceptions about the equal sign (Stephens, Ellis, Blanton, & Brizuela, 2017). Mathematically speaking, the equal sign is an equivalence relation indicating that two mathematical objects are equivalent (Jones, Inglis, Gilore, & Dowens, 2012). In the parlance of elementary school mathematics, this typically involves equations where the objects are numerical expressions and where "equivalence" means the expressions in an equation have the same value or are the same amount.[2] Yet, rather than interpreting the equal sign in this way—that is, *relationally*—students often view it *operationally*, as an operator symbol prompting them to perform the computation indicated in the expression to the left of the equal sign (Baroody & Ginsburg, 1983; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Kieran, 1981).

Most of the research on students' knowledge of the equal sign has occurred largely in elementary and middle grades, particularly in Grades 1–6 (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012), after the symbol has been introduced through formal instruction. Studies have found that in these settings, an operational view of the equal sign is predominant and leads to students' difficulties with correctly interpreting equations in non-standard formats for which there are operations only to the right of the equal sign, operations on both sides of the equal sign, or no operations at all (e.g., Alibali, 1999; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Carpenter, Franke, & Levi, 2003; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Matthews et al., 2012; McNeil & Alibali, 2005; Powell & Fuchs, 2010; Rittle-Johnson, 2006; Stephens et al., 2013).

Studies also show that a significant consequence of an operational view of the equal sign is that it can interfere with important algebra learning in later grades (e.g., Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Carpenter et al., 2003; Jacobs et al., 2007; Matthews et al., 2012). For instance, Knuth, Stephens, McNeil, and Alibali (2006) found that middle grades students who thought operationally about the equal sign were less successful solving equations than their peers who held a relational view. Elsewhere, Strachota et al. (2017) found that Grade 3 students' understandings of the equal sign significantly predicted their performance on items that assessed their ability to represent a function rule using variables. Students who held a relational view were more likely to recognize a functional relationship in data and correctly represent it with variable notation than students who held an operational view. Studies such as these suggest that students' ability to think relationally about the equal sign is a core prerequisite for success in algebra and, thus, raise for us the importance of creating instructional settings in the elementary grades that foster students' relational thinking.

The difficulties students have with the equal sign, coupled with its importance in school mathematics and its role at the nexus of arithmetic and algebraic thinking (e.g., Carpenter et al., 2003; Kieran, 1981; Knuth et al., 2006), require that we understand why such deep-rooted and widespread misconceptions of this symbol occur at all. There are several perspectives offered to account for students' operational interpretation of the equal sign. One earlier perspective, for example, held that students in elementary grades might not be cognitively ready to understand the equal sign relationally. Baroody and Ginsburg (1983) cite studies (e.g., Collis, 1974) that suggest students' ability to think relationally is a function of the developmental stage they exhibit, and it is possibly not until age 13—after the presumed onset of more formal abstract thinking—that they can see the relational nature of the equal sign and flexibly interpret equations in standard and non-standard formats. Collis (1974) further suggests that students ages 6–10 have a need for closure that prevents them from accepting an indicated operation in unexecuted form as a "result." In other words, such students would find equations with operations on both sides of the equal sign (e.g., 4 + 7 = 2 + 9) problematic because the sum 4 + 7 is equated to 2 + 9 rather than the single value 11. However, more recent research documents the ability of students in Grades 1–3 (that is, ages 6–10) to develop a relational view of the equal sign after instructional interventions have occurred (e.g., Blanton et al., 2015; Matthews et al., 2012), lending support to the doubts Baroody and Ginsburg (1983) raise as to whether students' ability to think relationally is determined by their level of biological maturation.

Another explanation—and one that we ourselves have held—is that an operational understanding of the equal sign is a reflection of the kind of content and instruction students experience in formal schooling relative to this symbol. In other words, the repeated and almost exclusive reliance in mathematics curricula on equations written in standard format (that is, equations where indicated operations appear only to the left of the equal sign), coupled with instruction that historically has not focused deeply on relational thinking, forges an operational view of the equal sign (e.g., Alibali et al., 2007; Baroody & Ginsburg, 1983; Carpenter et al., 2003; Matthews et al., 2012; McNeil et al., 2006; Rittle-Johnson, Matthews, Taylor, McEldoon, 2011).

While particular characteristics of curriculum and instruction or even biological maturation may be contributing factors in students' thinking about the equal sign, it is fair to ask whether other issues play a role. Scholars (e.g., Baroody & Ginsburg, 1983; Seo & Ginsburg, 2003) point out that the informal experiences with number and operation that students bring to formal schooling focus on counting actions to solve addition and subtraction tasks and that the form of these activities may give students a predilection towards a view of the equal sign as an operator symbol. In other words, from very early ages and prior to the use of written symbols, students often have extensive informal experiences with counting strategies (Ginsburg, 1982) based on, for instance, counting the objects in two sets and joining the sets to find the "total" number of objects. This form of activity can be seen as a physical action that has its cognitive counterpart in operational thinking. In contrast, rarely do students have informal experiences decomposing a set of objects and counting the number of objects in the resulting smaller sets (Baroody & Ginsburg, 1983). This predilection towards a "unidirectional action (counting) schema for adding" (Baroody & Ginsburg, 1983, p. 209) is potentially exacerbated by instruction that provides little explicit or sustained attention to the equal sign's relational meaning (Rittle-Johnson et al., 2011) and by curricular materials that offer little variation on equation structure and rely predominantly on equations in standard format (Denmark, Barco, & Voran, 1976; Jones et al., 2012).

As this suggests, what is still not well understood is the nature of the pre-symbolic knowledge about the equal sign that young students bring to formal schooling and whether, as Baroody and Ginsburg (1983) speculate, factors such as the type of informal counting experiences students have had, with its emphasis on combining sets of objects and finding total amounts, have already oriented their thinking towards an operational view. As noted earlier, studies to date have occurred largely within Grades 1–6 (e.g., Matthews et al. 2012), grade levels for which formal instruction about the equal sign has already occurred and for which students have had extensive exposure (in some cases, spanning multiple years) to curricula and instruction that is thought to inadvertently foster operational thinking. While Falkner et al. (1999) provide some evidence that students as early as kindergarten already exhibit operational thinking, studies are needed that systematically unpack the nature of students' thinking about the equal sign at this early, transitional stage.

We are specifically interested in this time frame (that is, kindergarten) because it represents the cognitive space prior to formal instruction on the equal sign. Current practice in the United States (U.S.) advocates that the equal sign, along with the use of written symbols to model mathematical actions and represent relationships in the form of equations, be introduced formally in first grade (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010). The claim by Baroody and Ginsburg (1983) that students might bring a predisposition towards operational thinking to formal instruction raises questions for us regarding what proclivity, if any, students hold prior to such instruction. For example, are they predisposed to think operationally as a result of informal counting actions whose form mimics an operational perspective? Or, given the predominant view that operational thinking is more likely a result of the kinds of instruction and curricular tasks students experience as a result of formal instruction, might kindergarten students be equally predisposed towards relational thinking prior to formal instruction?

The goal of our study, then, is to examine kindergarten students' thinking about the equal sign prior to the formal school instruction that typically occurs in Grade 1 and how their thinking develops through instruction that focuses on a relational view of the equal sign. By this, our aim is to understand any tensions between operational and relational thinking students might exhibit as they begin and progress through formal instruction; how robust a particular perspective might be; and, as a result, what perspective about the equal sign we might expect students to bring into first grade. While we focus here on students' learning about the equal sign as it occurs within a formal school setting, we acknowledge that informal experiences—prior to any usage of this symbol and during the implementation of this study—might have played a role as well. However, such informal experiences are beyond the scope of this study and our focus instead is on what concepts students held at the commencement of, and as they progressed through, instruction on the equal sign. We framed our study around the following research question:

> What knowledge about the equal sign do kindergarten students (ages 5–6) exhibit at the start of, during, and at the completion of instruction designed to foster relational thinking?

Research Design and Methods

The qualitative study reported here is part of a larger research project for which our goals are to understand the cognitive foundations of Grades K–1 students' thinking about core algebraic concepts. Such insights can inform curricular and instructional approaches that support the early introduction of algebraic concepts and, ultimately, help us implement with more fidelity calls in the U.S. for a K–12 approach to teaching and learning algebra (NGA Center & CCSSO, 2010). In the current study, we focus on students' understandings of the equal sign in kindergarten because of the deep significance these understandings hold for both arithmetic and algebra learning and because the equal sign is generally not formally introduced until first grade. Understanding the knowledge of the equal sign that students bring to their first-grade experiences can potentially inform the design of instruction that can build a more robust understanding of this symbol.

Participants

The study took place in two kindergarten classrooms at two elementary schools (Schools A and B) in the Northeastern U.S. For School A, about 20% of the district's school population is categorized as low socioeconomic status (SES) and 6% as English language learners (ELL). For School B, about 45% of the district's school population is categorized as low SES and 27% as

ELL. We selected the participating classrooms by coordinating with the schools' principals to identify a classroom for which the teacher expressed an interest in her or his class participating. All students—approximately 20 students in each classroom, or 40 total—participated in the study.

Data Collection

CTE Lessons. We conducted classroom teaching experiments [CTEs] (Steffe & Thompson, 2000) in the two participating classrooms. Each CTE consisted of approximately two 30-minute lessons per week (14 lessons total) taught over a period of 8 weeks by a member of our project team. The teacher-researcher model employed here is based on the perspective that a phenomenon is best understood when the participatory role of observer enlarges to that of teacher (e.g., Ball, 1999; Cobb & Steffe, 1983; Cobb et al., 2003). Such action research describes a "type of applied research in which the researcher is actively involved in the cause for which the research is conducted" (Bogdan & Biklen, 1992, p. 223). In the case of CTEs, the teacher-researcher model offers a critical perspective for understanding student learning.

Each lesson was videotaped. The full project team met weekly to discuss challenges or opportunities for learning and instruction and to revise subsequent lessons to reflect this. The CTEs addressed the following core algebraic concepts and practices in the 14 lessons: (1) a relational understanding of the equal sign; (2) generalizing, representing, and justifying generalizations in arithmetic (including properties of arithmetic); and (3) generalizing and representing varying, unknown quantities in algebraic expressions. We report here on the first four lessons of the CTE, designed to develop students' relational thinking about the equal sign.

Interviews. Eight students (four from each participating classroom) representing low, medium, and high performance groups were selected to participate in interviews. Because of the lack of formal assessment data at kindergarten, we worked closely with the classroom teachers to select an academically diverse group using classroom-level data. At least one student of the four selected in each classroom was identified for high, medium, and low ability groupings. Criteria included students' facility with number, counting, and simple addition. In addition, all interviewees needed strong verbal skills, regardless of math ability, in order to reasonably communicate their ideas in think aloud interview protocols. Because the CTEs occurred over halfway through the academic year, teachers had a reasonable knowledge of students' ability grouping which could inform our interview selections and allow us to eliminate formal testing of students as a means of selection.

We conducted five individual interviews with each participant (40 interviews total) using both clinical and teaching experiment formats. Clinical interviews (diSessa, 2007; Ginsburg, 1997) were administered as a pre/post assessment so that we could assess students' equal sign knowledge prior to and at the completion of the CTEs. Teaching experiment interviews (Steffe & Thompson, 2000) were conducted as pre-, mid-, and post-interviews throughout the CTEs. The goal of the teaching experiment interviews was to capture the nature of students' thinking over time through interview questions designed to scaffold students' thinking. Each interview lasted no more than 30 minutes and was videotaped. Interviews were conducted by one member of the project team and, when possible, observed by another member of the project team.

Design for CTE Lessons and Interviews

Certain task types are commonly used in both instruction and research to build students' relational thinking about the equal sign as well as to elicit their understanding of this symbol (e.g., Carpenter et al., 2003; McNeil & Alibali, 2005; Powell & Fuchs, 2010). Three prominent types include tasks that (1) elicit definitions of the equal sign; (2) ask students to identify whether an equation is true or false; and (3) require students to find an unknown value in an equation (e.g., Matthews et al., 2012; Rittle-Johnson et al., 2011). A critical aspect of task types (2) and (3) is that equations be represented in both non-standard and standard formats. Matthews et al. (2012) further identified item difficulty relative to these formats for task types (2) and (3) and found that order of difficulty could generally be characterized in terms of where the operations occurred. Equations with operations only on the left of the equal sign (i.e., $a + b = c$) were less difficult than equations with operations only on the right of the equal sign (i.e., $a = b + c$), while both of these formats were generally less difficult than equations with operations on both sides (i.e., $a + b = c + d$) or equations with no operations at all (i.e., $a = a$).

These task types and the order of difficulty for equation structures were important in the design of our equal sign tasks for CTEs and interviews. However, because these tasks are framed around the use of written symbols (i.e., numerals, operations, and the equal sign) to represent mathematical relationships (equations), their use requires students to already have some facility with these symbolic forms in order to interpret their meaning. Because kindergarten students typically have not had these experiences, our CTE and interview designs needed to account for this. As such, we included additional task types that were rooted in students' physical actions with objects: (1) decomposing numbers by breaking apart sets of objects and using this as the basis for modeling actions with equations in the non-standard format $a = b + c$; (2) finding the number of missing objects in a set to obtain equivalent amounts within two sets; and (3) a card game in which students found equivalent quantities on matching cards, where quantities were depicted through numerals and/or pictorial representations of numerals.

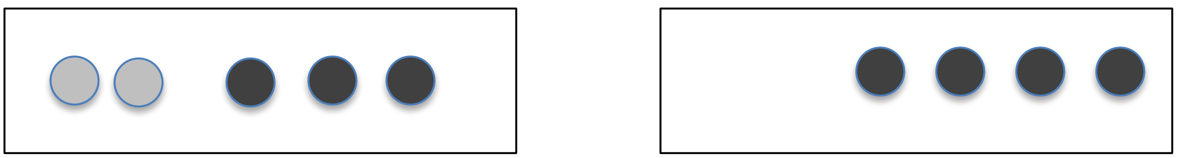
Baroody and Ginsburg (1983) argue that, historically, students at this age have had little experience with separating sets of objects and counting the resulting objects in each set, an action that could be mathematically framed as "decomposing." More recently, though, tasks that involve decomposing numbers have been advocated for use in kindergarten (NGA Center & CCSSO, 2010). In our view, such tasks can serve as a rich context for beginning to symbolize the mathematical action of decomposing using the non-standard equation format $a = b + c$. However, Falkner et al. (1999) note that kindergarten-age students may understand mathematical equivalence with collections of objects, yet have difficulty symbolizing their understandings with equations. Thus, supporting the transition of actions on objects to symbolic representations of equivalence was important to us.

With this in mind, we designed tasks for CTE lessons[3] that would extend the action of decomposing sets to include modeling this action with an equation in the non-standard format $a = b + c$. For example, students were given a set of 10 cubes and asked to explore how many ways they could break apart 10 cubes into 2 groups of cubes. They were then prompted to record the different ways they "made 10" by writing an equation of the form 10 = ___ + ___. Classroom discussions about students' findings focused on the nature of students' representations and their validity. Our goal was to promote students' ability to symbolize their action of breaking a total number of objects into smaller sets with an equation whose non-standard format would promote a relational view of the equal sign. In this sense, we hypothesized that decomposition tasks could help bridge students' informal understandings of equivalence—rooted in physical actions on objects—to symbolic representations of equivalent quantities.

Additionally, we designed tasks for CTE lessons and interviews in which numerical expressions were represented with colored circles and where students were asked to find a missing number of circles for a given color and represent their action with expressions and equations. For example, Figure 1 shows a figural representation of '2 + 3' (i.e., 2 grey circles and

3 black circles) and '___ + 4', where the task is to find the number of grey circles needed to combine with the 4 black circles so that the cards have the same number of circles:

Figure 1. Task for finding two equivalent expressions.



How many grey circles are needed in the second card so that the cards have the same number of circles?

The teacher-researcher explored with students through whole class discussion in CTE lessons how they might find the missing number of grey circles, then represent each expression in symbolic form (e.g., 2 + 3, 1 + 4) and their equivalence with an equation (e.g., 2 + 3 = 1 + 4). We hypothesized that reasoning with the pre-symbolic forms represented on the cards could support students in equation-solving tasks where they were to find an unknown value in an equation.

Finally, to strengthen students' understandings of representing equivalent amounts or expressions with equations, we designed a card game in which students were asked to identify equivalent amounts on different cards and represent the equivalency with an equation. In CTE lessons, students found a student "partner" in class based on identifying equivalent amounts on the cards they held. The cards had values represented as numerical expressions, including single numerical values. Once a matching card pair was found, the student dyad was expected to represent their “matching cards” with an equation. For example, two students might have the matching cards containing the expressions ‘3 + 5’ and ‘2 + 6’, and the resulting equation could be depicted as 3 + 5 = 2 + 6. The card game was used in the pre-interview as well, with individual students finding all possible “matches” in the set of cards. Depending on what was depicted on the cards, the resulting equation might necessarily be given in non-standard format, thereby offering students more opportunities to construct equations and engage in discussions that could promote relational thinking.

We coordinated the above task types and activities to design four CTE equal sign lessons and the interview protocols. Overall, our design goal was for ideas about equivalence to build from informal experiences rooted in either physical actions on sets (e.g., decomposing) or examining equivalent amounts prior to the representation of their equivalence with equations (e.g., finding the number of missing objects in a set, finding “matching” cards by identifying equivalent quantities), towards tasks that focused on interpreting written equations (e.g., determining if an equation is true or false, finding the missing value in an equation through open equation tasks).

We introduced non-standard equation structures early on and continued this practice throughout the study as a way to promote students’ relational thinking (e.g., Baroody & Ginsburg, 1983; Matthews et al., 2012). Moreover, different equation structures were introduced in a sequence generally corresponding with their level of difficulty (Matthews et al., 2012). For example, equations that students were to identify as true or false were generally sequenced in interview protocols in the following way: (1) an equation in standard format (e.g., 2 + 3 = 5); (2) an equation with an operation only on the right of the equal sign (e.g., 6 = 3 + 2); (3) an equation with no operations (e.g., 5 = 5); and (4) an equation with operations on both sides of the equal sign (e.g., 4 + 1 = 5 + 3).

Open equations (i.e., equations with an unknown or missing value) were designed to follow a similar sequence of difficulty and to vary the position of the unknown to elicit different ways of thinking. For example, open equations with the missing value directly after the equal sign (e.g., 4 + 2 = ___ + 1) reveal operational thinking if a child adds 4 and 2 and determines the missing value to be 6. Open equations of the form $a = b + c$ with a missing value to the left of the equal sign (e.g., ___ = 5 + 2) reveal operational thinking if a child rejects the equation as valid because the expression to the left of the equal sign does not contain an operation.

Selected tasks used in CTE lessons and interview protocols were piloted prior to the CTE, reviewed by the project's advisory board, and revised prior to implementation of the study. Table 1 contains the chronology for the interview/CTE lesson sequence, as well as the task types used in each event. Appendix A contains an overview of Lessons 1–4 and interview protocols relative to the equal sign.

Table 1. Chronology of the CTE Equal Sign Lessons and Interviews.

Event Sequence	**Task Type and Sequence**
Pre-Assessment	• Provide a Definition of the Equal Sign • Identify an Equation as True or False • Find the Missing Value in an Open Equation
Pre-Interview	• Provide a Definition of the Equal Sign • Find Equivalent Amounts and Represent with an Equation (card game) • Identify an Equation as True or False • Find the Missing Set of Objects to Make Equivalent Amounts • Find the Missing Value in an Open Equation
CTE Lesson 1	• Decompose Quantities and Model with Equations
CTE Lesson 2	• Find Equivalent Amounts and Represent with an Equation (card game) • Identify an Equation as True or False
CTE Lesson 3	• Find the Missing Set of Objects to Make Equivalent Amounts • Find the Missing Value in an Open Equation

CTE Lesson 4	• Identify an Equation as True or False
Mid-Interview	• Provide a Definition of the Equal Sign • Identify an Equation as True or False • Find the Missing Value in an Open Equation
Remaining CTE lessons on additional algebraic concepts and practices (10 lessons, 5 weeks). These lessons are not included in the study reported here.	
Post-Interview	• Provide a Definition of the Equal Sign • Identify an Equation as True or False • Find the Missing Value in an Open Equation
Post-Assessment (same as pre-assessment)	• Provide a Definition of the Equal Sign • Identify an Equation as True or False • Find the Missing Value in an Open Equation

Treatment of the Equal Sign in CTE Instruction

Prior to the CTE lessons, students' exposure to the equal sign had involved limited and informal use of equations and symbols associated with equations (e.g., +, =). They had not received formal instruction on the equal sign, its meaning, or its use in equations.[4] As such, the four equals sign lessons (Lessons 1–4) implemented in our CTEs were designed to be familiar to students in terms of numbers and operations used (i.e., we used only small numbers and the operation of addition), yet designed around tasks that would foster a relational view of the equal sign. In this sense, we needed equations that would help us probe the boundary points of students' thinking. Thus, while equations of the form $a + b = c$ were potentially more familiar to students, we intentionally designed equations involving non-standard formats that we hypothesized would be within students' zones of proximal development (Vygotsky, 1962/1934), yet which research (e.g., Carpenter et al., 2003; Rittle-Johnson et al., 2011) suggests would challenge their thinking about the meaning of the equal sign.

Overall, lesson activities were introduced through whole class discussion, after which students were asked to explore tasks in small groups or individually. Throughout the lessons, students were asked to justify their reasoning and listen to the reasoning of others. Lessons concluded with a discussion of student understandings and new ideas gleaned through lesson activities.

The equal sign was initially introduced in Lesson 1 through the activity of decomposing quantities and representing the equivalence of decomposed quantities. Students were given the opportunity to explore breaking apart sets of cubes and representing their observations using a non-standard equation (e.g., 11 = 10 + 1). Starting with decomposition gave students the opportunity to have whole group discussions about equivalent quantities. Some of their observations included: "11 is the same as 10 + 1"; "If I start with eleven blocks and break them apart to a group of 10 and a group of 1, I still have the same amount. I can show this with 11 blocks on one side of the equal sign and 10 + 1 blocks on the other side of the equal sign, so the same amount is on both sides."

In Lesson 2, the class started with a review of decomposition, after which students played the card game to explore different expressions and find expressions on the cards that represented the same amounts. The lesson concluded with creating true equations. Lessons 3 and 4 continued a focus on relational thinking by solving open number sentences (e.g., 10 + ___ = 11 + 5) and deciding if equations were true or false. Students were encouraged to use drawings or manipulatives to support their thinking.

Data Analysis

Data for the study reported here consisted of videotaped recordings of student interviews and students' written work produced in interviews. Our goal was to analyze these data in terms of students' equal sign knowledge prior to, during, and after formal instruction on the equal sign. In particular, we analyzed the nature of kindergarten students' thinking about the equal sign across the following distinct task types: (1) providing a definition of the equal sign; (2) representing equivalent amounts through an equation; (3) evaluating an equation as true or false; and (4) finding a missing value in an equation. All task types involved equations in standard and non-standard formats, that is, either an operation only on the left of the equal sign ($a + b = c$), an operation only on the right of the equal sign ($a = b + c$), an operation on both sides of the equal sign ($a + b = c + d$), or no operation at all ($a = a$).[5]

One member of the project team initiated the analysis by transcribing and reviewing all equal sign data from the pre- and post-test clinical interviews and the pre-, mid-, and post-teaching experiment interviews for the four interview participants in the participating classroom from School A (20 interviews total). During this initial review, student interview responses were coded regarding students' equal sign knowledge from a broad perspective of either operational or relational thinking. The individual response was the unit of analysis. Theoretical memos (Glaser, 1998) were made that included transcript evidence supporting a designation of operational or relational thinking.

Because students' understanding of the equal sign has been so extensively researched, we then drew from the research base to identify an existing framework that might allow us to conduct a more fine-grained analysis of students' equal sign knowledge than simply an operational/relational lens. Jones et al. (2012) note that over the last several decades, researchers have developed increasingly nuanced models of development organized around the constructs of operational and relational thinking. We incorporated one of these models, a recent framework "Construct Map for Mathematical Equivalence Knowledge" developed by Rittle-Johnson et al. (2011) in a study of students' equal sign knowledge across Grades 2–6, in a second tier of analysis of student interview data from School A.

In their framework, Rittle-Johnson et al. (2011) distinguish two levels of operational thinking (“rigid” and “flexible”) and two levels of relational thinking (“basic” and “advanced”). Our goal in the second tier of analysis was to ascertain whether and how this more nuanced model might help us further flesh out what we found in the first tier relative to students’ operational and relational thinking. In particular, the first coder compared the data coded as operational or relational from the first tier of analysis with the levels of the framework developed by Rittle-Johnson et al. (2011), looking for places of agreement as well as gaps or inconsistencies between what we observed in younger students’ thinking and what was reported for older students.

School A data were then independently coded by a second member of the project team. During this round of analysis, code types were either confirmed or alternative codes were suggested to account for students’ thinking about the equal sign for which questions remained in code assignments for School A data. Discrepancies between the code assignments of the two coders were discussed among the full project team and codes were revised. The first coder then recoded all School A data to reflect the modified framework; the second coder subsequently reviewed all coded School A data for any discrepancies and codes were finalized when full agreement was reached. Once our codes were stable, the second coder coded all of school B data using the modified framework. The coding scheme was further refined as any questions arose with School B data, and School A data were re-coded as needed. Data that could not be clearly linked to a specific code, including data where students’ words were inaudible or students did not provide sufficient information, were labeled “inconclusive.”

All data for a given code were then analyzed to develop an overall profile of the type of thinking characteristic of that code across task types and. Given the lack of data on kindergarten students' thinking about the equal sign in the research literature, our goal was to identify rich descriptions of students' thinking at this age that we might characterize relative to types of thinking within operational or relational perspectives. Additionally, we analyzed data chronologically by looking across participants' responses for patterns in thinking relative to equation format. Our goal in this was to quantify the types of thinking that occurred and examine whether different formats elicited particular forms of thinking and whether there were trends towards a particular type of thinking (e.g., relational) across time. We hypothesized that making sense of students' types of thinking relative to equation formats might provide insights into designing instructional experiences that could better support students' understandings of the equal sign.

Results

In what follows, we detail the nature of kindergarten students' thinking about the equal sign as they interacted with this symbol in different task types and equation formats through individual interviews. We present our findings in terms of three levels of sophistication we observed in students' knowledge of the equal sign. We frame our discussion around these three levels in terms of how students talked about the meaning of the equal sign, how they used the equal sign in equation construction, and how they interpreted its use in equations.

We see the levels as an ordering of the mathematical sophistication in the nature of thinking we observed. Levels should be viewed as related to students' responses on a given task and are not intended to be descriptions of students' overall thinking on an entire interview. Moreover, the levels should not be viewed as stages and, thus, are not intended to convey a mandatory sequence by which they occur in students' thinking. That is, in general students' thinking may reflect movement bi-directionally between levels as situations in their learning environment, such as task types, change (Clements & Sarama, 2014). In this sense, we do not describe children as being at a "level," but rather that the thinking they exhibit on a particular task reflects a certain level and that different tasks can elicit different types of thinking for the same student.

Level 1: Operational Thinking

Students' definitions of the equal sign were categorized as *operational* if they defined the equal sign as an operator symbol that connoted an operation was to be performed or an intended action of *totalizing* (Behr, Erlwanger, & Nichols, 1980; Kieran, 1981). In this case, students spoke about the equal sign as meaning, for example, "what you have all together," "the number is going to come after it," or "combining two numbers together." Their views were consistent with those exhibited by older students—even as late as middle grades—whose operational characterizations of the equal sign include descriptions such as "the total" or "the answer" (Knuth et al., 2006; McNeil & Alibali, 2005).

The other task types used in interviews involved students' analyses of given equations or their representation of equivalent quantities with an equation. In their responses to these tasks, we again observed thinking that we would characterize as operational, based on the use of this term in the literature. In particular, students sometimes exhibited an equal-sign-as-operator view by which they interpreted equations—regardless of the equation structure—through a "standard format" lens. In this, they conceptualized equations as an active process of operating on two numbers to find a total—rather than comparing two quantities—in order to determine a solution for a given task such as finding a missing value or determining if an equation was true or false.

Students who exhibited what we describe here as operational thinking could correctly find missing values, represent equivalent quantities with an equation, or interpret equations as true or false *when equations were written in standard format*. As Seo and Ginsburg (2003) note, an operational view of the equal sign is a legitimate view with such equations. For the task in which students were asked to write an equation to represent equivalent quantities, if one of the quantities was in the form $a + b$ and the other c (that is, a single value), then students' (correct) representation of the equivalence of these quantities as $a + b = c$ was characterized as operational. For example, for cards containing the value 6 and the expression 2 + 4, responses in which students (correctly) represented the equivalence of these quantities as 2 + 4 = 6 was coded as operational.

Furthermore, student responses to equations in standard format were characterized as operational if students reasoned in a way that indicated they performed the operation to the left of the equal sign and checked if the "answer" was the value immediately to the right of the equal sign. For example, one student explained that the equation 3 + 1 = 4 was true because "you're just adding another number up to make 4." Moreover, student responses to equations in standard format were characterized as operational when students argued, for example, that equations were false if the result of the operation on the left of the equal sign was not equal to the value to its right. As one child explained about the equation 1 + 1 = 7, "[It is] false because one plus one equals two and there's not enough numbers to make it seven." Similarly, students' thinking was characterized as operational if students reasoned that the missing value in equations such as 2 + 3 = ___ was '5' because "if you count off from 3 and you add two more, you get 5." What this seems to suggests is that equations in standard format prompted the use of verbal explanations based on operating on two quantities to produce a single value. While this view is legitimate for equations in standard format (Seo and Ginsburg (2003), as we discuss next, it is problematic for equations that are not.

Unlike equations in standard format, equations in non-standard formats revealed the misconceptions in students' operational thinking. For these equation structures (i.e., $a = b + c$, $a = a$, and $a + b = c + d$), students' responses were categorized as operational if their responses indicated that they rejected such equations as valid and either abandoned solving the task or incorrectly re-interpreted the equation using a "standard format" lens. In this, we found that students either inverted both the place of the equal sign and the operation in an equation, replaced the equal sign with an operation, or disregarded parts of an equation altogether.

For instance, students' responses were characterized as operational if they rewrote equations of the form $a = b + c$ as $a – b = c$, almost subconsciously replacing the '=' with the similar symbol '–' and '+' with '='. This suggests that they already held a particular view of the form an equation *should* take—standard form, where the indicated operation is to the left of the equal sign—without paying close attention to how the equation was actually configured.

For equations with no operations (i.e., of the form $a = a$), responses in which students replaced the equal sign with an operation that resulted in a new equation or expression that was not equivalent to the original equation were characterized as operational because such actions demonstrated a view that an equation required an action—such as adding—even though the resulting equation and the original were not equivalent. In particular, such students interpreted the equation as either an expression (for example, 5 = 5 was interpreted as 5 + 5, with no apparent recognition that this new object was no longer an equation) or as a completely new equation (for example, 5 = 5 was interpreted as 5 + 5 = 10). This response of rewriting equations given in non-standard format in standard format has been reported with older students as well (e.g., Behr et al., 1980).

For equations with operations on both sides, responses were coded as operational when students either had no strategy for finding missing values in such equations or found a missing value by either ignoring indicated operations or values that fell outside of the $a + b = c$ format or imposing a standard format structure on the equation and attempting to make sense of operations or values that fell outside of that format. For example, students with an operational view of the equal sign stated that the missing value in the equation 4 + 2 = __ + 1 was 6 “because 4 + 2 is equal to 6”, not referencing or acknowledging “+1” in their explanations. Students also explained that the equation 2 + 3 = ___ + 1 is false “because the answer is 5 (sum of 2 and 3) and not 1,” suggesting that, while they did not ignore the “1” (although we note that they did ignore the operation preceding it), they seemed to impose a standard format structure on the equation, interpreting it as (2 + 3 = ___) + 1. In this scenario, they attempted to make sense of “1,” but from the operational perspective of whether the sum of 2 and 3 could be the single value 5 or the single value 1.

Finally, responses in which students were unable to represent equivalent quantities with an equation in the matching card game and instead used an operation to relate equivalent quantities were also categorized as operational. In this case, students represented equivalent amounts as addends in an expression rather than as equivalent quantities in an equation. For example, students denoted the equivalency of 5 and 5 as ‘5 + 5’ rather than ‘5 = 5’.

Overall, these scenarios across nonstandard equation types underscored for us that students who held an operational view of the equal sign needed for the equation to include an operation in order for the equation to be viewed as valid, and they needed the operation to be located in a certain position in the equation consistent with standard format. Operational thinking was characterized by students’ inability to successfully interpret, solve, evaluate, or represent equations in non-standard format, as evidenced by an active stance in which they viewed such equations as invalid or incorrectly interpreted relationships between equivalent quantities using an equal-sign-as-operator view. This is consistent with Rittle-Johnson et al's (2011) notion of *rigid operational thinking*, not only in how students defined the equal sign (operationally, or as an operator symbol), but also in that they were only successful in solving, evaluating, and encoding equations in standard format.

Table 2 details the nature of students' operational thinking observed in interview data. It presents the criteria to determine if a child's response is operational and illustrates these criteria according to task types and equation structures.

Level 2: Emergent Relational Thinking

Some of the thinking exhibited by students had hybrid characteristics with roots in operational thinking, yet also emerging characteristics consistent with relational thinking. We characterized students' responses as *emergent relational* when they could represent equivalent quantities by writing equations in non-standard format and/or if they viewed these non-standard structures as valid representations—characteristics consistent with relational thinking—yet still interpreted the equal sign's role within an equation as an operator by talking about equations in terms of operations on numbers rather than as a comparison of equivalent quantities.

In particular, students whose responses were characterized as emergent relational interpreted true/false equations of the form $a = b + c$ as valid, albeit "backwards." For example, one student correctly described the equation 6 = 3 + 2 as false because "if I put [the equation 6 = 3 + 2] around to make it this (writes '2 + 3 = 6'), it won't equal 6." His explanation entails what we characterize as an emergent relational view because he did not reject the non-standard format as valid, as students whose thinking was characterized as operational did. Still, his reasoning has

Table 2. Characteristics of kindergarten students' operational thinking across task types.

Task Type	**Characteristics of Operational Thinking with Examples/Evidence**
Providing a Definition of the Equal Sign	Defines equal sign as an operator symbol denoting an action to be performed (e.g., Kieran, 1981), with the result of that action being a single value. *The equal sign means:* *"that it equals a number, like one plus five equals six"* *"what you have all together"* *"the number is going to come after it"* *"combining two numbers together"* *"you get two numbers and that (pointing to the equal sign) combines them"* *"it means how many are left"*
Representing Equivalent Amounts with an Equation	Given equivalent amounts in the form of a numeral on one card and an expression on another, can write an equation in standard format that shows equivalence. *Represents the equivalence of 2 + 4 and 6 as 2 + 4 = 6.* Given equivalent amounts in the form of numerals or expressions that cannot be represented in a standard format, cannot write an equation that shows equivalence. *Represents the equivalence of 3 and 3 as the expression 3 + 3 or the equation 3 + 3 = 6.* *Claims the equation 3 = 3 to be invalid because "the equal sign goes after (the second three in 3 + 3)" or because "they forgot the plus and they forgot what it was all together."* *Represents equivalence of 6 + 7 and 6 + 7 as 6 + 7 = 13 and claims 6 + 7 = 6 + 7 does not "make sense" because it involves "just writing 6 + 7 over and over again."*
Determining if an Equation is True or False	Equation structure: $a + b = c$ Correctly interprets equations as true or false using an operations-equals-answer lens.

Sees 2 + 3 = 5 as true because "2 and 3 really equals 5."

Sees 3 + 1 = 4 as true because "you're just adding another number up to make 4."

Sees 1 + 1 = 7 as false because "1 plus 1 equals 2 and there's not enough numbers to make it 7."

Equation structure: $a = b + c$

Rejects the $a = b + c$ format as a valid structure and/or interprets equations of the form $a = b + c$ in standard format.

Equation of the form a = b + c is not valid because "it's backwards" or because "the equal sign doesn't come first."

Equation 6 = 3 + 2 is not valid "because I see this one over here (pointing to the '=') is not supposed to be here, it's supposed to be here (pointing to where the '+' symbol appears) and this one (pointing to the '+') is supposed to be here (pointing to the '=').

Reads 6 = 3 + 2 as "six minus three equals two."

Equation structure: $a = a$

Rejects the $a = a$ format as a valid structure and interprets equations of the form $a = a$ as either the expressions "$a + a$" or "$a – a$" or as a new equation of the form $a + a = b$.

Equation of the form a = a is not valid because "there's no plus sign," "it needs a plus and a number," "it does not make any sense"

5 = 5 is interpreted as 5 + 5, 5 – 5, or 5 + 5 = 10

Equation structure: $a + b = c + d$

Interprets equations with operations on both sides in standard format, ignoring the operation and value (+ d) to the right of the equal sign.

The equation 8 + 2 = 10 + 3 is seen as true because 8 + 2 = 10; does not know how to interpret "+3"
The equation 1 + 6 = 3 + 4 is seen as false because 1 + 6 ≠ 3; ignores "+4".

Does not accept equations with operations on both sides as valid.

	Equations of the form $a + b = c + d$ are not valid because "there's two plus sentences," "you can't make doubles in a number sentence," "(the equation) has too much numbers," or "(an equation) can't have two plus signs."
Finding a Missing Value in an Equation	Equation structure: $a + b = c$ Correctly finds a missing value in an equation using an operations-equals-answer lens. *Explains that the missing value in 2 + 3 = ___ is 5 because "two plus three equals five because I know that three and then count up two more"; "5…because that's what it's all together"* *Explains that the missing value in 8 + 2 = ___ is 10 because "I counted up (from 8) two more" or because student uses cubes to make an 8-tower and 2-tower and counts the total.* Equation structure: $a = b + c$ Cannot find a missing value for equation of the form ___ = $b + c$. *Student has "never seen a problem where the blank came first" and no reason can be given for incorrect solutions.* Interprets equations of the form $a = b +$ ___ in standard format by finding the missing value to be a + b *States that missing value in 5 = 2 + ___ is 7 (that is, 5 + 2).* Equation structure: $a = a$ *No examples of operational thinking for $a = a$ structure of this task type.* Equation structure: $a + b = c + d$ Interprets equations with a missing value directly to the right of the equal sign (i.e., $a + b =$ ___ $+ d$) operationally by claiming the missing value to be the sum of a and b. *States that the missing value in the equation 4 + 2 = __ + 1 is 6 (that is, 4 + 2).*

Interprets equations with a missing value directly to the right of the equal sign (i.e., $a + b = \_\_\_ + d$) operationally by comparing the sum $a + b$ to d.

States that the equation 2 + 3 = ___ + 1 is false because the "answer" is 5 (sum of 2 and 3) and not 1.

Reasons about equations of the form $a + \_\_\_ = c + d$ by ignoring one of the operations or performing all operations to find the missing value.

Rewrites the 2 + __ = 3 + 1 as 1 + 3 = ___ and finds the missing value to be 4.
States that the missing value in 2 + __ = 3 + 1 is 1 because "2 plus 1 is 3."
States that the missing value in 6 + __ = 4 + 2 is 12 after adding the values 6, 4 and 2.

roots in an operational view because it is based on starting with performing an operation (adding 2 and 3) to see if the result is 6. In particular, rather than starting with 6 and comparing this to the quantity 3 + 2, this student needed to start with the side containing the operation (3 + 2), even though to do so required him to contort the equation. In other words, he needed to reverse the equation so that it was in a format whereby he could then first “do” something in order to get to (or, not get to) 6.

We suggest that a child who views the equal sign relationally would not need to “put [the equation] around,” but could start with ‘6’ and might reason that “the equation is false because 6 is not the same as 3 + 2, since 3 + 2 is 5.” While an operation is performed in both cases, we see the interpretations of the meanings of the quantities relative to each other as different. On the one hand, a child sees one quantity as a way to “get to” a second quantity by operating on the first (operational), while another sees two quantities as mutually co-existent and comparable (relational).

For open equations of the type $a = b + c$, students exhibiting emergent relational thinking accepted the structure with no operation to the left of the equal sign and could successfully find the missing value. However, the descriptions of their thinking reflected an operational perspective in which they performed an operation to find the missing value. As one student explained about the equation ___ = 8 + 1, the missing value is 9 because “if you flip it around, one plus eight equals nine.” As with the previous example, we see that this student accepted the “backwards” format (that is, he did not reject the equation or task outright as not “making sense,” as a student reasoning from a fully operational perspective would). At the same time, he needed to contort the equation into a standard format in order to “do” something to 1 and 8 to get to 9 (the missing value) as a unique product. In our view, an attribute of this thinking is that there is a linear movement that characterizes the child's left-to-right thinking about an equation, as opposed to a holistic view of an equation by which the child sees the quantities as existing together, in equal relationship to each other.

Students with an emergent relational view accepted the format for equations of the form $a = a$ as valid, but their reasoning was based on revising the equation to an equivalent form that included an indicated operation. For example, students maintained that 5 = 5 is true because, as one student explained, "I think it's saying like five plus nothing equals five but it just doesn't show the plus sign and the zero." For open equations of the form $a = a$, students saw the structure as valid and could successfully find a missing value. However, they represented or viewed this "value" as an expression with an indicated operation. For example, one student reasoned that the missing value in 8 = __ was '5 + 3', while another explained that the missing value in 18 = ___ was '18' because "you can write 18 equals 18 plus zero." An attribute of this thinking is that students were able to look for strategies to make equations valid (such as decomposing the number '37' into '30 + 7' in order to argue that 37 = 37 is a true equation). This was in contrast to a child whose operational reasoning about such equations resulted in either rejecting the equation outright as not valid or revising the equation to a nonequivalent form that included an operation.

As the previous discussion suggests, we do not see emergent relational thinking as fully operational because students who exhibited emergent relational thinking seemed comfortable with non-standard formats of the form $a = b + c$ and $a = a$ and could correctly reason about equations in these formats. In contrast, students who viewed the equal sign operationally rejected outright that equations of the form $a = b + c$ and $a = a$ could be valid. At the same time, we do not view this type of thinking as fully relational because students' descriptions of their reasoning were not based on comparing equivalent quantities, but on performing an operation or interpreting an equation without operations in a (correct) equivalent form that included operations. We view this interpretation as consistent with that characterized in Matthews et al. (2012) and Rittle-Johnson et al. (2011) as "flexible operational" thinking. In particular, Matthews et al. (2012) describe flexible operational thinking as a transitional level in which children "maintain an operational view of the equal sign, but become somewhat more flexible with respect to the types of equation formats that they correctly solve and accept as valid" (p. 320) and are "comfortable with equations that are atypical but remain consistent with an operational view of the equal sign, such as equations that are 'backwards'" (pp. 320-321).

Table 3 details the nature of students' emergent relational thinking observed in interview data. As Table 3 indicates, students did not provide any definitions of the equal sign that we would characterize as emergent relational. Similar to Table 2, Table 3 presents the criteria by which a child's response is characterized as emergent relational and illustrates these criteria according to task types and equation structures.

Level 3: Relational Thinking

In spite of kindergarten students' lack of exposure to a formal study of the equal sign prior to the CTEs, we did find evidence of relational thinking in students' responses to interview tasks. We discuss here the nature of relational thinking exhibited by participants in how they defined the equal sign, used the equal sign to represent equivalent quantities, and interpreted the equal sign in equations for tasks that required them to determine if an equation was true or false or to find a missing value in an equation.

Table 3. Characteristics of kindergarten students' emergent relational thinking across task types.

Task Type	**Characteristics of Emergent Relational Thinking with Examples/Evidence**
Providing a Definition of the Equal Sign	*No examples of emergent relational thinking*
Representing Equivalent Amounts with an Equation	Given equivalent amounts in the form of numerals or expressions, represents equivalence through an equation in non-standard format or may represent an equivalence correctly using standard format, but indicates a non-standard format is an equally valid representation *Represents the equivalence of 4 + 1 and 5 as 5 = 4 + 1* *Represents the equivalence of 1 + 1 and 2 as 2 = 1 + 1* *Represents the equivalence of 2 + 4 and 6 as 2 + 4 = 6 and indicates 6 = 2 + 4 is also valid*
Determining if an Equation is True or False	Equation structure: $a + b = c$ *No examples of emergent relational thinking for structure a + b = c.*[6] *(This structure was not included on the post-test.)* Equation structure: $a = b + c$ Successfully evaluates equations of the form $a = b + c$ using an operational lens; views the equation structure as "backwards" but valid. *Sees 6 = 3 + 2 as false because "three plus two equals five", "it's a number sentence backwards – two plus three doesn't really equal six because it has to be five instead of six," "if I put this around to make it this (writes '2 + 3 = 6'), it won't equal 6,", or "if I use 2 and 3 it equals 5 not 6 (indicating he accepts format as written)"* *Sees 12 = 10 + 2 as true because "it's a number sentence that's a true equation but backwards", "10 plus 2 is 12."*

	Equation structure: $a = a$ Accepts the structure $a = a$ as valid and successfully evaluates an equation as true or false when it can be revised to an equivalent equation that contains an indicated operation. *Sees 5 = 5 as true because "I think it's saying like five plus nothing equals five but it just doesn't show the plus sign and the zero…it can make five, five plus, five equal five."* *Sees 37 = 37 as true when rewritten as 37 = 30 + 7.* Equation structure: $a + b = c + d$ *No examples of emergent relational thinking for this equation structure.*
Finding a Missing Value in an Equation	Equation structure: $a + b = c$ *No examples of emergent relational thinking for structure a + b = c. (This structure was not included in the post-test.)* Equation structure: $a = b + c$ Accepts structure with no operation to the left of the equal sign as valid and successfully finds missing value for equations of the form ___ = $a + b$ and $a = b +$ ___ using an operational lens. Strategies for finding the missing value are computational in nature and include counting cubes that represent values in the equation, recalling math facts, or counting on. *For ___ = 5 + 2, gives answer of '7' because "5 adding 2 more is 6, 7", "3, 4, 5 (counting on 3 more from 2), it equals 5."* *For 6 = 3 + ___, says missing value is '3' because "3 and 3 makes 6 and I practice on 6 a lot in the classroom", "3 plus 3 equals 6."* *For __ = 10 + 1:* *Uses cubes to build a 10-tower and 1-tower, then counts the total: "That must be 11!"* *Writes '11' in blank because "1 plus 10 equals 11" and says it doesn't matter that the "answer" is on the left of the equal sign.*

Writes 11 in the blank and explains: "Because 1 makes 11 with 10."

For ___ = 8 + 1, counts and explains, "Nine. I counted from 8, then I counted 1 more and that makes 9", "because if you flip it around, 1 plus 8 equals 9."

Equation structure: $a = a$

Sees structure $a = a$ as valid and successfully finds missing value for equations of the form $a = a$ using an operational lens in which the missing value is represented as an expression (that is, an indicated operation rather than a single value). Using this, can write an equation that is seen as valid in non-standard format.

For 8 = __, writes 5 + 3 in blank, but also sees '8' as a solution;

For 18 = ___, states that the missing value is '18' because "18 adding nothing is still 18", or "you can write 18 equals 18 plus zero"

Equation structure: $a + b = c + d$

No examples of emergent relational thinking for this equation structure.

Students' definitions of the equal sign were characterized as relational if they described the equal sign as a symbol that indicated equivalence between quantities. Their descriptions took various forms, including that the equal sign means "balance," "the number on one side could be the same number on the other side," or "one side is the same [as] the other side."

Students' responses to tasks requiring them to evaluate equations as true or false or find missing values in equations were characterized as relational when students solved these tasks by comparing the expressions or values in a given equation to determine if they were the same amount or by finding missing values that would result in the same amounts. For example, responses in which students reasoned that equations of the form $a = a$ were true because "they both have the same amount" and "[the equal sign] is a balance sign and if they were different numbers it wouldn't be balanced and it would be false" were characterized as relational. The distinction in this type of thinking and that of the two levels described previously is that students did not reject an equation of the form $a = a$ as valid or attempt to revise it into an inequivalent form that included an operation (as they did with operational thinking), nor did they attempt to revise the equation to an equivalent form that included an operation (as they did with emergent relational thinking).

Students exhibited similar reasoning about the idea of balance for equations with operations on both sides. One student explained that the equation 8 + 2 = 10 + 3 was false because "They aren't balancing. I just know that 8 with 2 more is 10, but there is 10 already and 3 more (indicating the expression to the right of the equal sign)." For the equation 4 + 2 = ___ + 1, another student explained that the missing value was 5 because "it would be balanced, it would be the same (amount on both sides)."

While students' reasoning here involved operating on numbers as did the reasoning of students whose responses were coded as operational or emergent relational, the distinction here lies in part in the purpose of that computational work. In particular, in relational thinking it seemed that students performed computations for the purpose of comparing quantities and for which one quantity was *not* intended to be used to derive the other. That is, both quantities seemed to mutually co-exist and their indicated operations were independently performed for the purpose of comparing amounts. In contrast, students who exhibited operational thinking seemed to assume a linear perspective on equations in which computation was for the purpose of determining if one action (i.e., computation) "totaled" an existing amount. In this action-oriented sense, a quantity served as either a starting point (where an operation was to be performed) or an ending point (representing the "total" to be reached).

We found that students generally exhibited relational thinking only for true/false equations and open equations of the form $a = a$ or $a + b = c + d$.[7] Moreover, no student who exhibited relational thinking used a *compensation strategy* (Carpenter et al., 2003) in reasoning about equations of these types.[8] A compensation strategy involves looking at the structure of an equation and using relationships between quantities to find a missing value or determine if an equation is true. For example, a student is using a compensation strategy if he reasons that the missing value in the equation 5 + 2 = __ + 3 is 4 because 3 is one more than 2, so the unknown value must be one less than 5. In other words, we found relational thinking consistent with what Rittle-Johnson et al (2011) characterize as *basic relational,* but no evidence of what they describe as *comparative relational* thinking.

Instead, kindergarten students used only arithmetic strategies to reason about missing values in open equations. In their arithmetic strategies, some recognized a math fact, while others used mental math or their fingers or cubes to find a missing addend. One student explained his solution of '4' as the missing value in the equation 5 + 2 = __ + 3 as follows: "I used my fingers and I imagined I covered these (using one hand, he covered three fingers on his other hand) and there was four left out." He further explained that he imagined holding up 7 fingers from which he "subtracted" 3 fingers. He concluded, "Three plus four is seven and five plus two is seven. It's just different ways of making seven." In his reasoning, he implicitly uses the generalization that two quantities equal to the same amount are equal to each other in that both of the quantities '5 + 2' and '4 + 3' are being compared to a third, independent quantity (7). In this, we suggest that he sees the two quantities as mutually co-existent and comparable (that is, relational).

Table 4 details the nature of students' *relational thinking* observed in interview data across different task types and equation formats. As with Tables 2 and 3, Table 4 presents the criteria by which a child's response is characterized as relational and illustrates these criteria according to task types and equation structures.

Characterizing Students' Thinking about the Equal Sign across the CTEs

Although the results given in the previous section help categorize the types of thinking kindergarten students exhibited about the equal sign, they do not convey a sense of how these understandings occurred across the CTEs. In order to provide some context to this, we wanted to capture a picture of how instances of operational, emergent relational, or relational thinking occurred for participants across the interviews and for particular types of tasks. In this section, we discuss the frequency of the types of thinking that occurred across interviews, as well as the case of one student and what interviews reveal about his learning across the CTE.

Table 4. Characteristics of kindergarten students' relational thinking across task types.

Task Type	Characteristics of Relational Thinking with Examples/Evidence
Providing a Definition of the Equal Sign	Defines equal sign as indicating equivalence between quantities. *The equal sign means:* *"balance"* *"the number on one side could be the same number on the other side"* *"Making sure one is balanced with the other (using hands to gesture idea of balance)"* *"Like if you had a 5 over here and a 5 over here, it's a balance sign"* *"One side is the same to the other side"* *"It's equal to each other."*
Representing Equivalent Amounts with an Equation	*No examples of relational thinking for this task type. (This task was used only in pre-interview.)*
Determining if an Equation is True or False	Equation structure: $a + b = c$ *No examples of relational thinking for structure* $a + b = c$. Equation structure: $a = b + c$ *No examples of relational thinking for structure* $a = b + c$. Equation structure: $a = a$ Views equations of the form $a = a$ as true because the quantities are the same amount. *Equation of the form* $a = a$ *is true because "they both have the same amount," "it's saying the same thing over and over again," "[the equal sign] is a balance sign and if they were different numbers it*

	wouldn't be balanced and it would be false," "it's balanced," and "equals means even (gesturing with hands to show balance) so the equation is even." <u>Equation structure:</u> *a + b = c + d* Interprets truth of an equation by examining whether the expressions on either side of the equal sign are equivalent. *The equation 1 + 2 = 3 + 3 is seen as false because "they are both not the same amount because 1 + 2 is 3 and 3 + 2 is 5."* *The equation 8 + 2 = 10 + 3 is seen as false "because they aren't balancing - I just know that 8 with 2 more is 10, but there is 10 already and 3 more (indicating the expression to the right of the equal sign)."* *The equation 4 + 1 = 2 + 3 is seen as true because it "does balance because 4 and 1 make 5 and 2 and 3 make 5."* *The equation 10 + 1 = 11 + 3 is false because "11 plus 3 equals 14 [and the other side equals] 11."*
Finding a Missing Value in an Equation	<u>Equation structure:</u> *a + b = c* *No examples of relational thinking for structure a + b = c.* <u>Equation structure:</u> *a = b + c* Finds the missing value by appealing to a structural argument concerning the relationship between the quantities rather than an operational argument based on operating on two values. *Sees the missing value in ______ = 8 + 1 as '1 + 8': "I'm thinking of putting two numbers (writes 1 + 8 in the blank). I put a switcheroo[9] there."* <u>Equation structure:</u> *a = a* Finds missing value by reasoning that the amounts on either side of the equal sign should be the same; represents the missing value as a single value and not as an expression that requires an operation. *Sees the missing value in 18 = ___ as '18' because "I think it's balanced; That's (pointing to '=') a*

balance symbol, so I think it's balanced because the numbers are the same."

Equation structure: $a + b = c + d$

Using a computational approach, finds the missing value by determining what amount would make the expressions equivalent.

Finds the missing value in 2 + __ = 3 + 1 to be 2 because "2 and 2 equals 4 and 1 and 3 equals 4."

Finds the missing value in 2 + 3 = __ + 1 to be 4 because "4 plus 1 equals 5 and 2 plus 3 equals 5."

Finds the missing value in 5 + 2 = __ + 3 to be 4 because "3 plus 4 is 7 and 5 plus 2 is 7. It's just different ways of making 7."

Finds the missing value in 4 + 2 = ___ + 1 to be 5 because "it would be balanced, it would be the same (amount on both sides)."

Finds the missing value in 5 + 3 = _____ + 4 to be 4 because "5 plus 3 equals 8, and they have to equal the same thing. So I put in a 4, because 4 plus 4 equals 8."

Quantifying Students' Responses across the CTEs

Figure 2 displays the frequency of the types of thinking (e.g., operational) that occurred for the different equation structures (e.g., $a + b = c$), across all five interviews (e.g., pre-interview). The most compelling observation we draw from these data is the predominance

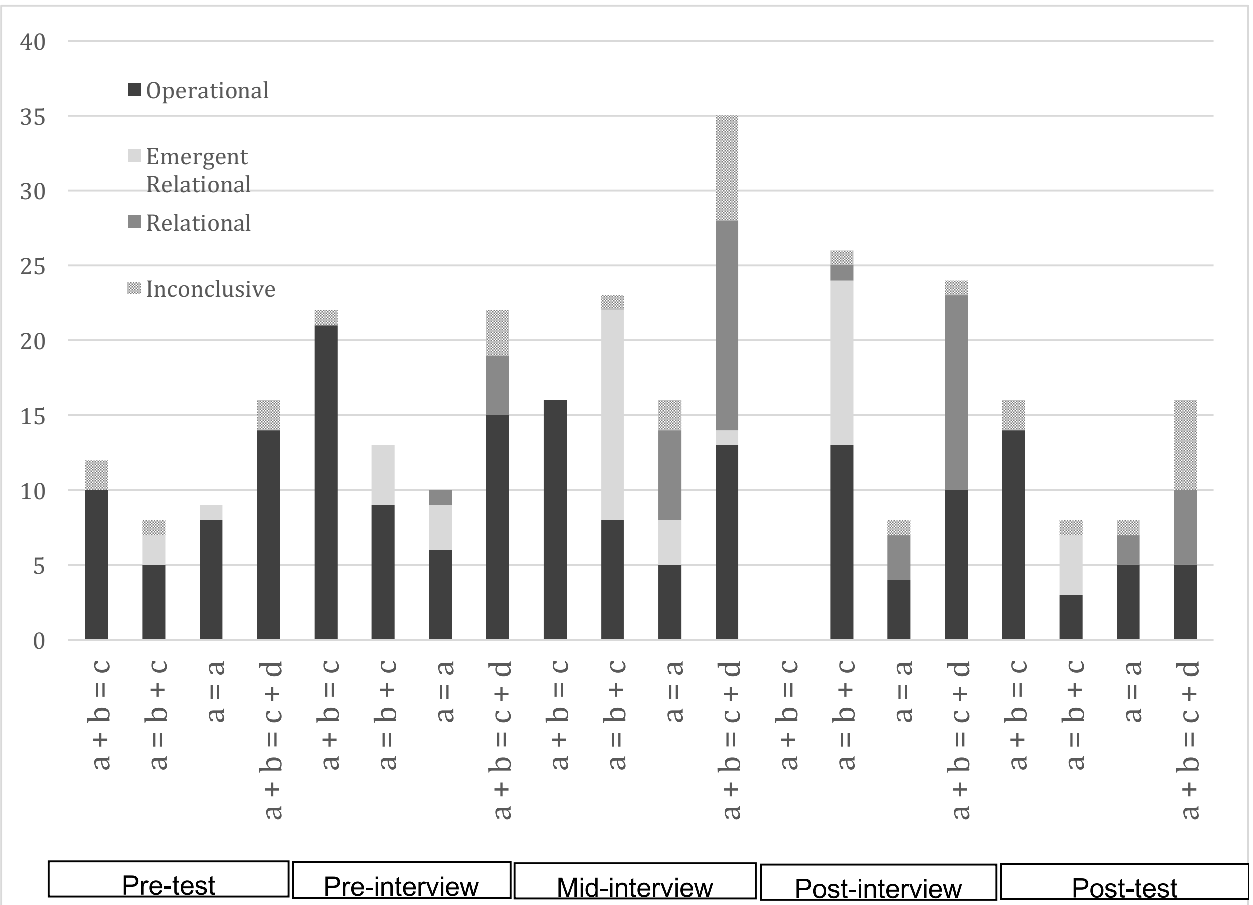


Figure 2. Frequency of levels of thinking for equation structures across interviews[10].

of operational thinking across all equation structures throughout the study, and particularly prior to any instruction in the CTEs (i.e., at pre-test). In fact, kindergarten participants exhibited no relational thinking at pre-test. In our view, this deeply challenges a presumption that children might bring an equal proclivity to either relational or operational thinking to formal instruction. It seems fair to suggest that, prior to formal schooling, factors—such as students' informal, pre-symbolic experiences with arithmetic—were already orienting these students towards an operational view of the equal sign where, by kindergarten, young children were exhibiting the same misconceptions as older students.

Although a predominantly operational view of the equal sign persisted throughout the remaining four interviews, some shifts towards emergent relational and relational thinking did occur that showed a potential for more sophisticated thinking about the equal sign prior to first grade. That these shifts occurred essentially for equation structures $a = b + c$ and $a = a$ (emergent relational) or $a = a$ and $a + b = c + d$ (relational) reinforces the importance of using non-standard formats to promote a relational view of the equal sign. Moreover, the increase in emergent relational or relational thinking occurred largely at the mid-interview and post-interviews, suggesting that even when there is potential for relational thinking, children require sustained support and scaffolding to develop these more sophisticated ways of thinking.

The relatively low occurrence of emergent relational and relational thinking, coupled with the high frequency of operational thinking, underscores that while a relational view of the equal sign is challenging for young children to develop, students are at greater risk of organically building deep misconceptions about this symbol in the absence of curricular and instructional experiences, long before a formal introduction to the equal sign is even considered. This points to the assertion by Baroody and Ginsburg (1983) that the totalizing approach privileged in informal experiences with counting actions that students bring to formal schooling may already predispose them towards an operational view of the equal sign. If this is the case, then students' informal experiences in kindergarten (and even preschool) need to be intentionally re-structured in a way that aligns with a relational understanding of the equal sign.

Dan's[8] Case

To provide more perspective on how students' thinking emerged across the CTEs, we consider Dan's case. While a showcase of all of the profiles that occurred across the different ability groups would be beyond the scope of the study reported here, a close examination of a case where learning did seem to shift over time can be instructive. Dan's case was selected for this reason.

At pre-test, a clinical interview in which no instruction or scaffolding was provided, Dan exhibited operational thinking on 6 of the 8 questions involving the three primary task types (i.e., defining the equal sign, determining whether an equation is true or false, and finding the missing value in an equation). This seems to underscore our finding that, even without formal instruction on the equal sign and any extensive exposure to equations—including those written in standard format—students may already hold a proclivity towards operational thinking.

Dan's thinking began to shift to include relational thinking as early as the pre-interview. As described earlier, this interview was in a teaching experiment format and when students gave incorrect responses to tasks, they were questioned in order to challenge and extend their thinking. Moreover, students were given brief instruction in the pre-interview on the meaning of the equal sign as a symbol denoting "is the same value or amount as." Consider the following contrast in Dan's thinking on the pre-test and pre-interview assessment. On the pre-test, Dan's reasoning about the truth of the equation 4 + 1 = 5 + 3 was that it was false because "I think it's saying nothing. It makes like no sense because you can't make doubles in a number sentence. It's like doubles, like three plus two equals one plus four." We characterize his thinking as operational because he rejected this type of equation (with operations on both sides) as a valid form. However, at the pre-interview (and after brief instruction on the meaning of the equal sign), Dan reasoned that the equation 1 + 2 = 3 + 2 was false since “They are both not the same amount. One plus two is three and three plus two is five, so they are not the same amount.” In other words, he accepted the equation type as valid and justified that the equation was false by comparing the amounts of the quantities.

It is worth considering what types of scaffolding in the interview might have led to this shift in Dan’s thinking. The interview was structured so that Dan could first compare quantities represented on small cards with numerals and/or a corresponding set of squares. Dan’s initial strategy was to attend to the squares themselves, counting the number of squares on a given card and comparing the result to the number of squares on a second card. Later in the game, he transitioned to comparing only the numerals on the cards. His initial reasoning with a non-symbolic quantity (that is, a quantity represented by a corresponding number of squares) seemed to scaffold his ability to recognize equivalent quantities represented abstractly (i.e., with numerals). The use of non-symbolic quantities, then, seemed to be a constructive strategy for engaging Dan’s initial thinking about equivalence, as he was able to correctly match all the cards in the set.

However, the next step of writing an equation to represent the equivalence denoted in a pair of cards revealed that symbolizing the equivalence was more difficult. For example, Dan wrote the equation 3 + 3 = 6, rather than 3 = 3, to represent the “match” for two cards that each contained the numeral 3. When a child wrote an incorrect equation such as this, the strategy used by the interviewer was often to introduce another child’s correct solution. Here, Dan was asked whether he viewed another child's equation of 3 = 3 as correctly representing the matching cards. Dan explained why he thought the student was incorrect:

Dan: He did it wrong because the equal sign goes over here (pointing to the right of the '3' on the right of the equal sign).

Interviewer: So you think the equal sign should be moved to the other side of this second 3?

Dan: Uh huh.

Interviewer: Why do you want to do that?

Dan: Because that's (referring to the equal sign) what tells us that it's (referring to 3 and 3) going together.

Dan's operational interpretation of the equal sign entailed that the two quantities (3 and 3) had to go "together" or be added, with the equal sign showing that this had happened.

The interviewer moved to other matching card sets that Dan had correctly identified as equivalent, engaging Dan in extensive discussions regarding the types of equations he used to represent the equivalent quantities on the card sets. Although Dan had found all matching card sets, he sometimes struggled with correctly representing equivalent quantities with an equation. However, the modeling activity introduced him to non-standard ways to write equations, even if they were not in his original thinking. In the following exchange, Dan was asked to write an equation for the matching set he found, where one card contained the expression "8 + 0" along with 8 corresponding squares drawn underneath the numeral 8, and the second card simply contained "8." He wrote the equation 8 + 0 = 8 to correctly represent the equivalence:

Dan: Eight plus zero equals eight.

Interviewer: I saw another child who wrote 8 = 8 + 0. What do you think?

Dan: It's backwards. Like, it goes that way (pointing right to left on the equation). That makes no sense.

Interviewer: The way he's written it here makes no sense?

Dan: Yeah.

Interviewer: So you would rather see it written the way you wrote it?

Dan: Uh huh.

Interviewer: Should he not write it that way?

Dan: It's backwards, but he could write it that way.

By now, Dan seemed to be more open to equations that were not in standard format, even if it was "backwards" to him. Whereas before he viewed the students' equation 3 =3 as incorrect, he later seemed willing to accept 8 = 8 + 0 as a valid equation, although part of his acceptance might be due to the fact that the equation contains an operation, unlike the case of 3 = 3. Our point is that the lengthy discussions around the task of constructing equations to represent the relationship between quantities on card sets that Dan had already found to be equivalent created a context for exploring equations in non-standard formats. In this, we suggest that the focus on equations in non-standard format, along with open-ended discussions with the interviewer, were both important in scaffolding Dan's thinking.

In the next task, Dan was given a series of equations in different formats and asked to find the missing value in each one. The series was structured to begin with an equation of the form $a + b = c$ (2 + 3 = ___), which we expected he could successfully solve because he could interpret the equal sign operationally and still correctly find the missing value. He did, and went on to find the missing value in equations of the form $a = b + c$, where the missing value was placed at different points in the equation. He seemed increasingly comfortable with equations in this non-standard format. In the exchange around the second equation ___ = 5 + 2, Dan correctly wrote 7 in the blank, but noted that the equation was backwards and did not "make sense." However, on the third equation, 6 = 3 + ___, when asked again whether this format made sense, he nodded, "It makes sense." Later, when asked to find the missing value in the equation 2 + ___ = 3 + 1, Dan placed a "1" in the blank because "2 plus 1 equals 3." However, when the interviewer asked him if 2 + 1 was the same as 2 + 1, he said no and changed his answer to 2, explaining that "two plus two equals four and three plus one equals four". We suggest that what prompted him to reconsider his answer was hearing the words "is the same as" rather than interpreting the written symbol "=" In other words, it seemed important that oral language be used in context with written symbolic language. He applied his thinking to the next equation, finding the missing value in 2 + 3 = ___ + 1 to be 4 because "two plus three is five and four plus one is five."

It seemed that Dan's progression through a set of task types that moved from reasoning about equivalence with non-symbolic representations, to modeling equivalence with symbolic forms (i.e., through equations), to finding missing values in an equation and evaluating equations as true or false, all of which were built around discussions that promoted thinking about equations in non-standard formats, served to scaffold his relational thinking about the equal sign. However, we found that Dan's relational thinking—as well as that of other students—was not consistent. Over the four equal sign lessons within the CTE, the teacher-researcher continued to engage Dan (and his peers) with explorations of the meaning of the equal sign through tasks similar to those he encountered in the pre-test and interviews. Although Dan did exhibit trends towards relational thinking across the CTE (see Table 5), there were still clear instances of operational (or emergent relational) thinking in his responses across the interviews. For example, Dan consistently provided an operational definition of the equal sign each time he was questioned about this. His responses included "it means that's what it means all together" (pre-test) and "combining two numbers together" (post-test).

	Pre-test	Pre-interview	Mid-interview	Post-interview	Post-test
Operational	6	3	5	0	2
Emergent Relational	2	4	4	3	1
Relational		4	5	4	3

Table 5. Frequency of type of thinking for Dan across 5 interviews.

As we looked across Dan's interview responses, we found trends in the types of responses relative to equation structures. For equations with an operation only on the right (i.e., of the form $a = b + c$), his thinking was always categorized as emergent relational. This might point to flexibility in thinking at an early age that can be leveraged through decomposition tasks—such as those used in our CTEs—because such tasks lend themselves to modeling the mathematical action of decomposing with equations of this type. We also observed that Dan's relational thinking occurred only on equations for which there were either no operations at all or an operation on both sides of the equation (i.e., those of the form $a = a$ or $a + b = c + d$), suggesting that these types can provide important contexts for promoting relational thinking. Finally, Dan never exhibited emergent relational or relational thinking on equations in standard form (i.e., $a + b = c$), although he correctly reasoned about all equations of this type from an operational perspective.

Overall, while Dan exhibited a capacity to think relationally, he did not consistently use this type of thinking across equation structures. This phenomenon has been described elsewhere (e.g., see McNeil & Alibali, 2005; Seo & Ginsburg, 2003). In a study of seventh-grade students' understandings of the equal sign, McNeil and Alibali (2005) found that students did not discard well-established (operational) views of the equal sign even though this view did not work across all equation structures. They suggest that students may change their interpretation of the equal sign based on the context (e.g., the equation structures) in a way that does not require a reorganization of their knowledge of the equal sign. They conclude that students could "hold on to their original conception of the equal sign in most contexts, while also interpreting the equal sign as a relational symbol of equivalence in contexts that elicit ideas such as equivalent to and same amount as" (p. 302). The fluctuation in Dan's thinking about the equal sign across different contexts (e.g., equation structures) seems consistent with this perspective. Moreover, it underscores the reality that students struggle with relational thinking and current practices do not seem to adequately address these struggles.

Discussion

Comparisons of Younger and Older Students' Thinking about the Equal Sign

We found that kindergarten students think about the equal sign in ways very similar to older students. First, the forms of operational thinking observed in our study's participants are consistent with those identified elsewhere in older students' thinking (e.g., Baroody & Ginsburg, 1983; Behr et al., 1980; Falkner et al., 1999). For example, Rittle-Johnson et al. (2011) characterize *rigid operational thinking* among students in Grades 2–6 as when students are "only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure" and that, additionally, such students "define the equal sign operationally" (p. 87). This aligns with what we observed in students' thinking that we characterize here as operational, where students who held an operational view consistently used a standard-format lens to view all equation structures. This was evidenced by students either explicitly rejecting the validity of equations in non-standard format (for example, noting that such equations did not "make sense") or implicitly rejecting such equations by rewriting them in a nonequivalent standard format and using the nonequivalent forms to perform a given task.

Secondly, as with older students, participants in our study sometimes interpreted the equal sign within a single context in ways that reflected both operational and relational thinking. As described earlier, we see this emergent relational thinking as consistent with the "flexible operational" level identified by Rittle-Johnson et al. (2011), in which students are able to "successfully solve, evaluate, and encode atypical equation structures" (p. 87) of the form $a = b + c$ or $a = a$. It seems noteworthy that, with the exception of one response, emergent relational thinking occurred exclusively with equations of the form $a = b + c$ or $a = a$ for students in our study as well. The characteristic that seemed to allow for this type of flexibility in thinking—that is, thinking that did not reject equations in non-standard formats as valid structures but also did not fully view equivalent quantities as mutually co-existent objects—was the type of equation structure itself. That is, the hybrid aspect seen in emergent relational thinking occurred only with non-standard equations that were similar to standard format in that at least one of the quantities was a single value. These particular formats seemed to allow students to successfully reason with equations operationally to find missing values or determine if an equation was true or false. In other words, these structures allowed students to interpret equations operationally, yet correctly, by flexibly reading equations "backwards" (e.g., reading 6 = 3 + 2 as "2 plus 3 equals 6") or inserting operations and expressions into equations (e.g., viewing 8 = 8 as 8 = 5 + 3) in a way that produced an equivalent form by which students could correctly reason from a "standard format" perspective. Equations of the form $a + b = c + d$ did not allow for this, and results showed that, with one exception, students whose responses were not coded as "inconclusive" exhibited either operational or relational thinking for this equation structure (see Table 4 and Figure 2).

One caveat concerning the operational and emergent relational levels is whether children might reason about an equation for which there is an operation on only one side (i.e., those of the form $a + b = c$ or $a = b + c$) in a way that is consistent with an operational view, but hold an underlying relational view of the equal sign that is not articulated, perhaps because of constraints of early language skills for young children. How do we ascertain this? These particular equation structures are challenging because students can reason correctly about the equation (that is, they can successfully find a missing value or evaluate whether the equation is true or false) while interpreting the equal sign operationally, an approach that we and others (e.g., Seo & Ginsburg, 2003) see as valid, albeit incomplete.

Moreover, this caveat prompts a reasonable question as to what relational thinking about such equations would even resemble. While there is clearly a relationship between students' thinking about the equal sign relative to the type of equation format used, the relationship is not so clear regarding how or even if students might think relationally about equations in standard format—or, perhaps more broadly, equations with only one operation—because such equations seemed to elicit operational thinking. As noted earlier, we found that students did not exhibit relational thinking for equations in either standard format or the form $a = b + c$, with one exception (see Table 4). If this had occurred, we would expect to see language that showed students reasoning about the relationship between equivalent quantities as if the quantities mutually co-existed. For example, in the one case where this occurred, the student reasoned that the missing "value" in the equation ____ = 8 + 1 was '1 + 8' by appealing to a structural argument ("I put a switcheroo there") concerning the relationship between the quantities (1 + 8 and 8 + 1), rather than an operational argument based on operating on two values (8 and 1) to find a missing value (9) (see Table 4). A child might possibly make a similar argument for an equation of the form 8 + 1 = _____, though no student in our study did so. But does the absence of language that indicates relational thinking imply that students cannot think relationally about such equations or that they simply choose not to because there are other more intrinsically meaningful or pragmatic (operational) ways for them to reason about the problem?

We acknowledge that this is possible and suggest this as a further area of research beyond our current data corpus. We took the approach that, in the absence of data indicating that students actively compared the equivalence of quantities, even if operations needed to be performed to create quantities that could be compared, we could not conclusively characterize the child's thinking as fully relational.

Finally, while the relational thinking described here is generally consistent with that reported elsewhere, we do see one subtle distinction with findings in some of the research literature in that the relational thinking of kindergarten participants was not confined to the equation structure with operations on both sides of the equal sign, but included equations with no operations at all (*cf.* Rittle-Johnson et al., 2011). For example, referring to the open equation 18 = ___ and her choice of 18 as the missing value, one student noted, "That's (pointing to '=') a balance symbol, so I think it's balanced because the numbers are the same." We suggest that some participants in our study were able to think relationally about equation structures other than those with operations on both sides, unlike that reported for older students in Rittle-Johnson et al (2011), where relational thinking (both basic and comparative) was identified only for equations with operations on both sides. Given the computational complexity of equations with operations on both sides for young children, this finding suggests that simpler equation structures such as *a*

= *a* might provide more reasonable starting points for engaging children in developing a relational understanding of the equal sign.

Differences Between Students' Definitions of the Equal Sign and How They Interpreted It

We found that the meanings students gave for the equal sign were not always consistent with how they interpreted this symbol in equations. We again see a distinction here with findings elsewhere in how students' definitions of the equal sign—operational or relational—were coupled with their interpretation of the equal sign in equation-based tasks. For example, Rittle-Johnson et al. (2011) characterize students at the level "basic relational" as generating a relational definition of the equal sign, but students at an operational level (i.e., "rigid operational" or "flexible operational") as producing an operational definition. We found, however, that kindergarten students might give an operational definition of the equal sign, but interpret the equal sign's use in equations relationally. Conversely, students sometimes gave a relational definition of the equal sign, yet interpreted its use in equations operationally. Moreover, this misalignment between definition and interpretation occurred throughout the pre-, mid-, and post-interviews. That is, the misalignment was not a phenomenon that stabilized in a particular direction over the course of the study.

While such inconsistencies between the definitions students produce for the equal sign and their interpretation of the equal sign in equations are known, it is usually the case that students have more difficulty producing a relational definition than interpreting its use relationally in tasks that require them to solve or evaluate equations (e.g., Knuth et al., 2006; Rittle-Johnson et al., 2011). That we see both directions occurring in young students' thinking suggests that more is at play here than simply one direction being more difficult than another (for example, that producing a relational definition is more difficult than interpreting the equal sign in an equation). Instead, it raises the issue of whether young students may not yet connect meanings of a construct with its usage, nor even understand that they can or should make these connections. In other words, at times it seemed that producing a definition and interpreting the equal sign were two unrelated activities for participants in our study. This fluctuation seems consistent with Siegler's (2007) notion of *within-child variability*, a phenomenon in which children often regress in the strategies and approaches they use trial-to-trial, even though the broader trajectory of learning reflects an upward arc. According to Siegler (2007), such variation can provide insights into our understanding of how cognitive change occurs. For us, this raises the question: What are implications for learning, if any, of the fluctuations we observed between students' definitions and interpretations of the equal sign?

Implications for Curriculum and Instruction

Our findings point to implications for how the equal sign could be treated in curriculum and instruction to support students' understanding of this symbol. Arcavi (1994) argues that symbol sense—which we take to include a "sense" about the equal sign symbol as well—is central to what it means to be competent in algebra and should be developed through rich experiences that help students construct meanings for symbols in context. What might such experiences entail for the equal sign at the start of formal schooling?

First, teachers who begin instruction on the equal sign with an explanation of its (relational) definition might assume that students will apply this definition when interpreting the equal sign in an equation and might not provide students with sustained support concerning how its meaning relates to its interpretation in equations. Our data suggest that although our CTE lessons began with a relational definition of the equal sign, students did not necessarily apply this to their interpretations of equations. This raises a design question for us: Instead of providing a formal definition of the equal sign *a priori*, should students instead first be provided with more extensive informal, pre-symbolic experiences in which they explore equal or unequal quantities? From such experiences, they might then construct their own (relational) definition of the equal sign, much in the same way suggested by Dougherty (2008). Such an organic approach might more successfully result in students' construction of a relational meaning for the equal sign and might strengthen the connections students make between the meaning of the symbol and their interpretation of its use in equations.

On a related note, our findings suggest that informal experiences should be more intentionally linked to the construction of formal symbolisms to support students' relational interpretation of the equal sign in use. For example, how would an emphasis on decomposing sets in kindergarten (and even preschool), coupled with increasing support for symbolizing these actions with equations in the non-standard format $a = b + c$ (as opposed to the activity of decomposing in isolation), support a relational interpretation of the equal sign in equations? While our CTEs briefly addressed this, a more sustained approach might yield stronger results.

Consider, too, a scenario where students spend significant time comparing equivalent and non-equivalent quantities among sets and using this as a context for qualifying relationships as "the same amount as," "more than," or "less than" before symbolizing these relationships with "<", ">", and "=". How might such qualitative comparisons better enable students to interpret the equal sign relationally in equations (as well as strengthen their meaning for the equal sign as a relational symbol)?

Some have argued that such an approach might be more effective in developing a relational view of the equal sign (Capraro, Ding, Matteson, Capraro, & Li, 2007; McNeil, Eyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011). Seo and Ginsburg (2003), for example, found that children were more successful in developing a relational view of the equal sign in contexts where no arithmetic operation was performed (e.g., comparing the lengths of Cuisenaire rods) than contexts involving written, symbolic forms (such as those used in our study). They caution, however, that students in their study had difficulty making connections between the relational view they exhibited in problems that did not involve operations and the (operational) view of the equal sign they exhibited in symbolic contexts that involved operations.

They offer several design features for curriculum and instruction, including that curriculum and instruction needs to facilitate discussions about the meaning of the equal sign when it is used in symbolic contexts, use equations in non-standard formats to prompt students' relational thinking, and be sensitive to framing problems in which students need to "make" or "produce" a certain number. However, in spite of our attention to features such as these in our CTE design, we were surprised at the persistent tendency towards operational thinking exhibited by participants in our study, suggesting to us that the role of other factors (such as children's informal experiences) needs to be explored further.

This is consistent with our findings about the types of equation structures that support the development of a relational understanding of the equal sign and the resulting implications for curriculum and instruction. In short, experiences with formal symbols and equations might be more thoughtfully crafted to build a relational understanding. Our findings that, with the exception of one response, emergent relational thinking occurred only with equations of the form $a = b + c$ or $a = a$ suggest that such formats have the potential to promote flexibility in thinking and might serve as an entry point into relational thinking. For instance, if one were to design an instructional sequence that would help students transition from operational to relational thinking, it seems reasonable that equations of these types (that is, $a = b + c$ or $a = a$) might serve as a bridge towards the more complicated structure $a + b = c + d$ and, as such, might be introduced before equations that have operations on both sides but simultaneous to equations in standard format (that is, $a + b = c$). This ordering is consistent with the item difficulty analysis reported by Matthews et al. (2012) relative to older students.

Moreover, our finding that students did not use a compensation strategy even when they correctly performed tasks involving equations of the form $a + b = c + d$ suggests that they had limited facility in thinking about relationships between numbers, the heart of the compensation strategy. Experiences in which students compare relationships between numbers in numerical expressions, prior to the introduction of equations, might build a proclivity towards looking for efficient strategies based on relationships in number rather than rushing to arithmetic procedures. For example, a focus on comparing 3 + 1 and 3 + 2 and noticing that 3 + 2 is one more than 3 + 1 because 2 is one more than 1, rather than finding and comparing their sums (4 and 5), encourages students to think about relationships *before* arithmetic strategies are invoked.

We also found that students did not exhibit relational thinking when modeling equivalent quantities with an equation for the matching card game. This might be explained by the fact that, of all the interviews, the card game was used only in the pre-interview. Another underlying issue is whether students might understand when quantities are equivalent—even when represented numerically or pictorially on a set of cards—but still struggle to represent that equivalence symbolically with an equation, particularly when the representation has the more complex structure $a + b = c + d$. This claim is supported by research that shows students have shown knowledge of numerical equivalence—that is, an ability to match sets of objects based on the number of objects in the sets—as early as age 4 (e.g., Gelman & Gallistel, 1986), yet this knowledge seems to precede their ability to connect it to encoding, interpreting, and solving equations (Rittle-Johnson et al., 2011). Falkner et al. (1999), too, note that students as early as kindergarten understand equality relations but are not able to connect this to equations. For us, this raises the question of whether initial tasks should focus more on informal interpretations of relationships rather than encoding equations.

In summary, our findings point to the need for curriculum and instruction in kindergarten to begin with extensive, pre-symbolic, informal experiences where students compare quantities, before they are measured or counted, in order to construct understandings of what it means for one quantity to be less than, equal to, or greater than another. As students gain an understanding of relationships between quantities in non-symbolic settings, numbers, operations, and relational symbols such as =, <, and > can gradually be introduced in "rich contexts" (Arcavi, 1994) as tools by which students can represent their thinking. Such contexts might include concrete representations (e.g., bins of cubes on balance scales) or visual representations (e.g., drawings that depict equivalence), both of which should be connected to the abstract, symbolic representation of equations and inequalities. They should also be grounded in discussions that engage students in talking about their thinking.

In this, care is needed regarding the type of equations that students encounter. Equations in non-standard formats not only reveal misconceptions in students' thinking, they can also promote relational thinking and, thus, should be as dominant in students' experiences as equations in standard format are. Particularly, non-standard equations with no operations ($a = a$) or with operations only on the right of the equal sign ($a = b + c$) seem to bridge students' thinking in important ways: Equations of the form $a = a$ map well to explorations of relationships in unspecified quantities (e.g., comparing the weights of a set of cubes in two bins), while equations of the form $a = b + c$ are natural extensions for modeling the result of number decompositions. Later, equations with operations on both sides may be considered. Elsewhere, two of the authors are exploring whether such an approach might increase kindergarten students' flexibility in their thinking about the equal sign and the symbol sense needed to reason with equations (Arcavi, 1994).

Conclusion

A widely-held perspective on the genesis of an operational view of the equal sign pinpoints the role of curriculum and instruction in students' formal school experiences. Scholars rightly argue that the repeated use of equations in standard format over time reinforces in students' thinking a view of the equal sign as signifying an action to be performed (e.g., Falkner et al., 1999; McNeil & Alibali, 2005; McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur, & Krill, 2006). However, findings from the study reported here suggest that the development of students' understanding of the equal sign might be more nuanced than this and that an operational view of this symbol is likely being formed long before students' formal school experiences begin.

Like others (e.g., Falkner et al., 1999), prior to the study reported here we hypothesized that students who have had little or no experience with the equal sign—such as kindergarten students—might not yet have formed an operational view. We found instead that participants in our study not only exhibited types of thinking similar to that observed among older students who have had both formal instruction on the equal sign and significant and repeated exposure to tasks based on standard format equations, they also demonstrated predominantly operational thinking.

This naturally raises further questions for us regarding the genesis of operational thinking. While we and others (e.g., Seo & Ginsburg, 2003) have pointed to the potential for the forms of informal counting experiences to influence children's views of the equal sign, even this alone might not account for the early operational conceptions about the equal sign held by children participating in this study, particularly given the research that highlights a disconnect between children's pre-symbolic notions of equivalent sets and their ability to interpret and solve written equations (e.g., Falkner et al., 1999; Rittle-Johnson et al., 2011). Further study is needed to understand both the influence pre-symbolic counting (particularly, combining sets to find a total quantity vis-à-vis decomposing sets and comparing quantities in the original and decomposed sets) has on children's views of the equal sign and how symbols themselves mediate or inhibit children's thinking.

In sum, our findings elevate the importance and urgency of two core research goals for young children: (1) understanding how informal learning experiences in preschool and kindergarten can be crafted, prior to a formal introduction of the equal sign, in ways that orient students' thinking towards a relational understanding of this symbol; and (2) understanding how formal learning experiences with equations in lower elementary grades can build from informal learning to develop a robust, stable understanding of the equal sign. The potential payoffs in increasing students' success in algebra are worth our investment in these goals.

References

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology, 35*, 127–145. doi:10.1037/0012-1649.35.1.127

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*, 221–247. doi:10.1080/10986060701360902

Arcavi, A. (1994) 'Symbol sense: informal sense-making in formal mathematics', For the Learning of Mathematics 14(3), 24-35.

Ball, D. L. (1999). Working on the inside: Using one's own practice as a site for studying teaching and learning. In A. Kelly & R. Lesh (Eds.), Handbook of research in mathematics education (pp. 365-402). Hillsdale, NJ: Lawrence Erlbaum.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *The Elementary School Journal, 84*, 199–212. doi:10.1086/461356

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92*, 13–15.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A., Isler, I., & Kim, J. (2015). The Development of Children's Algebraic Thinking: The Impact of a Comprehensive Early Algebra Intervention in Third Grade. *Journal for Research in Mathematics Education, 46*(1), 39-87. http://dx.doi.org/10.5951/jresematheduc.46.1.0039

Bogdan, R. C., & Biklen, S. K. (1992). Qualitative research for education: An introduction to theory and methods. Boston: Allyn & Bacon.

Capraro, M. M., Ding, M., Matteson, S., Capraro, R. M., & Li, X. (2007). Representational implications for understanding equivalence. *School Science and Mathematics, 107*(3), 86-88.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Clements, D. H., & Sarama, J. (2014). Learning trajectories: Foundations for effective, research-based education. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 1–30). Charlotte, NC: Information Age Publishing.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9-13.

Cobb, P., & Steffe, L. (1983). The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14(2), 83-94.

Collis, K. (1974). *Cognitive development and mathematics learning*. Paper presented at the Psychology of Mathematics Education Workshop, Centre for Science Education, Chelsea College, London.

Denmark, T., Barco, B., & Voran, J. (1976). *Final report: A teaching experiment on equality* (PMDC Technical Report No. 6). Tallahassee: Florida State University.

Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). New York: Lawrence Erlbaum Associates.

diSessa, A. A. (2007). An interactional analysis of clinical interviewing. *Cognition and Instruction*, *25*(4), 523-565.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*, 232–236.

Gelman, R., & Gallistel, C. R. (1986). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Ginsburg, H. (1982). *Children's arithmetic: The learning process* (Rev. ed.). New York: D. Van Nostrand.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Glaser, B. G. (1998). *Doing grounded theory: Issues and discussions*. Mill Valley, CA: Sociology Press.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*, 258–288.

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology, 113*(2012), 166–176.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326. doi:10.1007/BF00311062

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*, 297–312. doi:10.1207/s1532690xci2403_3

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316–350.

McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experi- ence: All contexts are not created equal. *Journal of Cognition and Development, 6*, 285–306. doi: 10.1207/s15327647jcd0602_6

McNeil, N. M., Eyfe, E. R., Petersen, L. A., Dunwiddie, A. E., & Brletic-Shipley, H. (2011). Benefits of practicing 4 = 2 + 2: Nontraditional problem formats facilitate children's understanding of mathematical equivalence. *Child Development, 82*(5), 1620–1633.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction*, *24*, 367–385. doi:10.1207/s1532690xci2403_3

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Council of Chief State School Officers. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Powell, S. R., & Fuchs, L. S. (2010). Contribution of equal-sign instruction beyond word-problem tutoring for third-grade students with mathematics difficulty. *Journal of Educational Psychology, 102*, 381–394. doi:10.1037/a0018447

Rittle-Johnson, B. (2006). Promoting transfer: Effects of self-explanation and direct instruction. *Child Development, 77*, 1–15. doi:10.1111/j.1467-8624.2006.00852.x

Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldoon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology, 103,* 85–104. doi:10.1037/a0021334

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence...":

Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Recent research and theory* (pp. 161-187). Mahwah, NJ: Lawrence Erlbaum Associates.

Siegler, R. S. (2007). Cognitive variability. *Developmental science*, *10*(1), 104-109.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principals and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267 – 306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for Research in Mathematics Education*. Reston, VA: National Council of Teachers of Mathematics.

Stephens, A., Knuth, E., Blanton, M., Isler, I., Gardiner, A., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior 32*(2), 173-182.

Strachota, S., Isler, I., Stephens, A., Blanton, M., Knuth, E., Demers, L., & Gardiner, A. (2017). *The Co-development of Core Concepts in Algebra: Functional Thinking and Equality.* Paper presented at the Research Conference of the National Council of Teachers of Mathematics, San Antonio, Texas.

Vygotsky, L. (1962). *Thought and language.* (E. Hanfmann and G. Vakar, Trans.). Cambridge, MA: Massachusetts Institute of Technology. (Original work published in 1934.)

Appendix A

CTE Equal Sign Lessons – Tasks and Sequence

Equal Sign – Lesson 1

Lesson Objective: Decompose numbers and model decomposition with equations in non-standard format $a = b + c$.

A. How many ways can you make 10?

Explore & Record:

1. In what ways can you break apart 10 cubes into groups?
2. In what ways can you break apart10 cubes into 2 groups of cubes? What numbers of cubes are in your group?
3. Write down the ways you made 10 by writing an equation in the form "10 = __ + ___".

Discuss:

1. *Compose a list of all the ways to make 10, representing each way in the form $a = b + c$.*
2. *Discuss whether each equation is true or false and why.*
3. *Discuss:* Do we have all possible ways to break apart 10? Why?

B. How many ways can you make 11?

Explore & Record:

1. (In partner pairs) In what ways can you break apart 11 cubes into two groups of cubes? Use the chart provided to record your work.

Discuss:

1. *Discuss:* Do we have all possible ways to break apart 11? Why?

Equal Sign – Lesson 2

Lesson Objective: Find equivalent quantities and representing their equivalence with equations. Identify equations as true or false.

A. Equivalent Quantities – Can you find the same amount?

Play "Equivalent Quantities Game"

Game Rules:

1. Pass out cards, where each card has a value or indicated sum (e.g., 3, 2 + 4). Students find their "partner" (person with card containing expression with equivalent amount).
2. The student pair writes an equation that shows the expressions are equivalent.

Discuss:

1. How do you know your cards show the same amount?

3. *On the board, write the equation that shows the relationship you found. The resulting equations will reflect all equation types (i.e., standard and non-standard format).*

Discuss.

B. True or False?

Discuss: Are the following equations true or false? Why?

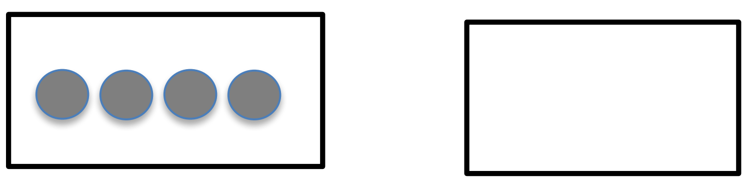
2 + 2 = 10
4 = 4
1 + 2 = 1 + 2
5 = 1 + 3
3 + 3 = 6 + 2
18 + 3 = 18

Equal Sign – Lesson 3

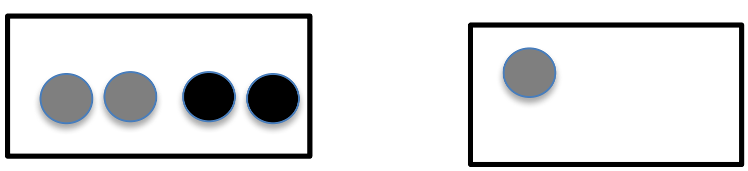
<u>Lesson Objective</u>: – Find the number of objects that yields equivalent quantities. Solve simple open equations by finding missing values.

A. Finding Balance

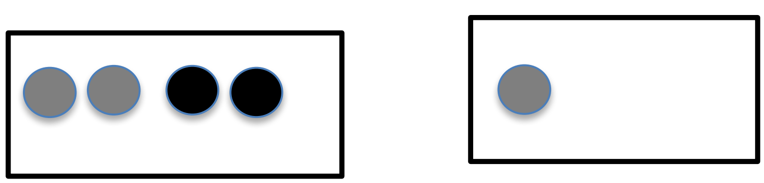
1. How many grey circles should we add to the empty rectangle so that the rectangles have the same number of circles? How do you know? *(Write an equation to model this.)*



2. How many black circles should we add to the rectangle on the right so that the rectangles have the same number of circles? How do you know?



3. How would you complete the expression to show the number of black circles you need to add to the rectangle on the right so that the rectangles have the same number of circles?



2 + 2 1 + ___

Write an equation that shows this relationship.

4. Use the expressions to draw a picture using blue and red circles such that the rectangles have the same total number of circles.

4 + 2	1 + 5

Write an equation that shows what you found.

B. Find the missing value in each equation.

3 + 3 = ___

3 + ___ = 7

__ = 4 + 2

10 = 5 + ___

2 + 2 = _____

5 + 1 = 6 + _____

4 + 2 = ___ + 3

Equal Sign – Lesson 4

<u>Lesson Objective</u>: Determine if equations are true or false.

Are the following true or false? How do you know? How could you explain your answer without adding the numbers on each side?

2 + 3 = 6

1 + 1 = 1 + 1 + 1

3 + 1 + 1 = 3

10 = 0 + 10

16 + 1 = 16 + 3

28 = 28

7 + 0 = 0 + 7

8 = 2 + 6

2 + 3 = 3 + 2

8 + 2 = 10 + 2

Interview Protocols

Pre/Post Test (Clinical Format)

1. Have you seen this symbol ('=') before?
If so:
- What is it?
- What does it mean?
- Can you give (write) me an example of how you use it?

2. Have you heard the word "equation" or "number sentence" before?
If so:
- Tell me what you know about it.
- Can you give an example of how you have heard these used?

3. *Ask the questions below for each of the following equations:*

2 + 3 = 5
6 = 3 + 2
4 + 1 = 5 + 3
5 = 5

- Can you read this to me?
- Explain to me what you think this is saying.
- *If they know what an equation is*: Do you think this equation is true? Why or why not?

4. *Ask the questions below for each of the following equations:*

2 + 3 = ___
4 + 2 = ___ + 1

- Can you read this to me?
- Explain to me what you think this is saying.
- *If they know what an equation is*: Can you tell me what number should go in the blank? How do you know?

Pre-Interview (Teaching Experiment Format)

1. Finding and representing equivalent quantities

a) Play *Finding Equivalent Amounts*
b) Briefly explain what the equal sign means (a symbol that shows two expressions have the same value; model an example of an equation if needed).
c) For each pair found in the game, ask student to write an equation that shows the relationship.

2. Are the following equations true or false? Why?

$1 + 1 = 7$

$3 + 1 = 4$

$3 = 3$

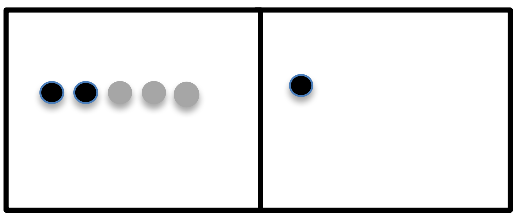
$2 + 3 = 2 + 3$

$1 + 2 = 3 + 2$

$5 + 4 = 5$

3. Complete the picture to make the quantities equivalent

How many grey circles should we put in the rectangle on the right so that the rectangles have the same number of circles? How do you know?



4. Find the missing value that makes the equation true. Explain.

2 + 3 = ______

______ = 5 + 2

6 = 3 + ______

8 = _______

2 + ______ = 3 + 1

2 + 3 = ______ + 1

Mid-Interview (Teaching Experiment Format)

1. Meaning of Equal Sign

- What does this symbol (‘=’) mean?
- Can you give (write) me an example of how you use it?

2. True/False Equations

Ask the questions below for each of the following equations:

$7 + 2 = 9$

6 = 3 + 2
10 + 1 = 11 + 3
37 = 37
5 + 3 = 6 + 2

- Can you read this to me?
- Explain to me what you think this is saying.
- Do you think this equation is true? Why or why not?

3. Open Equations

Find the missing value that makes the equation true. Explain.

8 + 2 = _____

_____ = 10 + 1

4 = 2 + _____

18 = ______

6 + _____ = 4 + 2

5 + 2 = _____ + 3

Post-Interview (Teaching Experiment Format)

1. True/False Equations
Ask the questions below for each of the following equations:

12 = 10 + 2
8 + 2 = 10 + 3
42 = 42
1 + 6 = 3 + 4

- Can you read this to me?
- Do you think this equation is true? Why or why not?

2. Open Equations

Find the missing value that makes the equation true. Explain.

_____ = 8 + 1

5 = 2 + _____

5 + 3 = _____ + 4

---

[1] The research reported here was supported in part by the National Science Foundation under Grant No. DRL-1415509. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[2] Hereafter, this is the mathematical context in which we explore students' thinking about the equal sign.

[3] This particular task type was not used in individual interviews.

[4] In the pre-test, some participants reported having seen the symbol (and recognized the symbol as the equal sign), while others stated they had not previously seen the symbol.

[5] The exception was the task type "providing a definition of the equal sign," which did not explicitly involve the use of equations.

[6] By definition, emergent relational thinking occurred only with equations in non-standard format. As such, there are no examples of this type of thinking for equations in standard format ($a + b = c$) for the task types of determining if an equation is true or false or finding a missing value in an equation.

[7] One exception to this is an open equation with format $a = b + c$, where one student interpreted the given equation relationally to find a missing value (see Table 4).

[8] The compensation strategy was not explicitly taught and did not spontaneously occur in kindergarten.

[9] "Switcheroo" was a term used by the teacher in this student's classroom to denote commutativity.

[10] There were no questions on the post-interview (the 4th interview) for equations of the type $a + b = c$, so there is no corresponding data stack in the graph.
[8] Student names are pseudonyms.