THE LENGTH, WIDTH, AND INRADIUS OF SPACE CURVES

MOHAMMAD GHOMI

Abstract. The width w of a curve γ in Euclidean space $\mathbf{R}^n$ is the infimum of the distances between all pairs of parallel hyperplanes which bound γ, while its inradius r is the supremum of the radii of all spheres which are contained in the convex hull of γ and are disjoint from γ. We use a mixture of topological and integral geometric techniques, including an application of Borsuk Ulam theorem due to Wienholtz and Crofton's formulas, to obtain lower bounds on the length of γ subject to constraints on r and w. The special case of closed curves is also considered in each category. Our estimates confirm some conjectures of Zalgaller up to 99% of their stated value, while we also disprove one of them.

Contents

Date: August 25, 2018 (Last Typeset).

2000 *Mathematics Subject Classification.* Primary: 53A04, 52A40; Secondary: 52A38, 52A15.

Key words and phrases. Width, inradius, convex hull, second hull, Crofton's formulas, sphere inspection, escape path, asteroid survey, optimal search pattern, Bellman's lost in a forest problem.

Research of the author was supported in part by NSF Grant DMS–1308777.

1. Introduction

What is the smallest length of wire which can be bent into a shape that never falls through the gap behind a desk? What is the shortest orbit which allows a satellite to survey a spherical asteroid? These are well-known open problems [31, 21, 12, 17] in classical geometry of space curves $\gamma\colon [a,b] \to \mathbf{R}^3$, which are concerned with minimizing the length L of γ subject to constraints on its *width* w and *inradius* r respectively. Here w is the infimum of the distances between all pairs of parallel planes which bound γ, while r is the supremum of the radii of all spheres which are contained in the convex hull of γ and are disjoint from γ. In 1994–1996 Zalgaller [30, 31] conjectured four explicit solutions to these problems, including the cases where γ is restricted to be closed, i.e., $\gamma(a) = \gamma(b)$. In this work we confirm Zalgaller's conjectures between 83% and 99% of their stated value, while we also find a counterexample to one of them. Our estimates for the width problem are as follows:

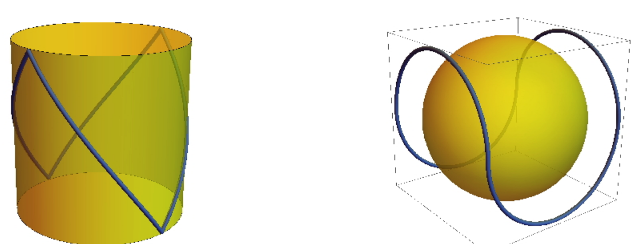
Theorem 1.1. *For any rectifiable curve* $\gamma\colon [a,b] \to \mathbf{R}^3$*,*

$$\frac{L}{w} \geq 3.7669. \tag{1}$$

Furthermore if γ *is closed,*

$$\frac{L}{w} \geq \sqrt{\pi^2 + 16} > 5.0862. \tag{2}$$

In [30] Zalgaller constructs a curve, "L_3", with $L/w \leq 3.9215$. Thus (1) is better than 96% sharp (since $3.7669/3.9215 \geq 0.9605$). Further, in Section 5 we will construct a closed cylindrical curve (Figure 1(a)) with $L/w < 5.1151$, which shows that (2) is at least 99.43% sharp. In particular, the length of the shortest closed curve of width 1 is approximately 5.1. It has been known since Barbier in 1860 [5], and follows from the Cauchy-Crofton formula (Lemma 2.3), that for *closed* planar curves $L/w \geq \pi$, where equality holds only for curves of constant width. The corresponding question for *general* planar curves, however, was answered only in 1961 when Zalgaller [29] produced a caliper shaped curve (Figure 2(a)) with $L/w \approx 2.2782$, which has been subsequently rediscovered several times [1, 19]; see [2, 16]. In 1994, Zalgaller [30] studied the width problem for curves in $\mathbf{R}^3$, and produced a closed curve, "L_5", which he claimed to be minimal; however, our cylindrical example in Section 5 improves upon Zalgaller's curve.



(a) (b)

Figure 1.

Next we describe our estimates for the inradius problem. Obviously $w \geq 2r$, and thus Theorem 1.1 immediately yields $L/r \geq 7.5338$ for general curves, and $L/r \geq 10.1724$ for closed curves. Using different techniques, however, we will improve these estimates as follows:

Theorem 1.2. *For any rectifiable curve* $\gamma\colon [a,b] \to \mathbf{R}^3$,

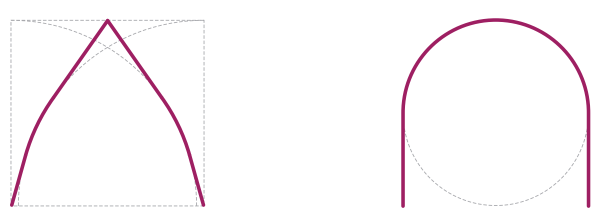
$$\frac{L}{r} \geq \sqrt{(\pi+2)^2+36} > 7.9104. \tag{3}$$

Furthermore if γ *is closed,*

$$\frac{L}{r} \geq 6\sqrt{3} > 10.3923. \tag{4}$$

In [32, Sec. 2.12], Zalgaller constructs a spiral curve with $L/r \leq 9.5767$, which shows that (3) is better than 82.6% optimal. Further, in [31], he produces a curve composed of four semicircles (Figure 1(b)) with $L/r = 4\pi$; see also [21] where this "basebal stitches" is rediscovered in 2011. Thus we may say that (4) is better than 82.69% optimal. For planar curves, the inradius problem is nontrivial only for open arcs, and the answer, which is a horseshoe shaped curve (Figure 2(b)) with $L/r = \pi + 2$, was obtained in 1980 by Joris [18], see also [12, Sec. A30], [14, 16, 13]. Zalgaller studied the inradius problem for space arcs in 1994 [30] and for closed space curves in 1996 [31]. The latter problem also appears in Hiriart-Urruty [17].

Both the width and inradius problems may be traced back to a 1956 question of Bellman [6]: how long is the shortest escape path for a random point (lost hiker) inside an infinite parallel strip (forest) of known width? See [16] for more on these types of problems. Our width problem is the analogue of Bellman's question in $\mathbf{R}^3$. The inradius problem also has an intuitive reformulation known as the "sphere inspection" [32, 21] or the "asteroid surveying" problem [10]. To describe this variation, let us say that a space curve γ *inspects* the sphere $\mathbf{S}^2$, or is an *inspection curve*, provided that γ lies outside $\mathbf{S}^2$ and for each point x of $\mathbf{S}^2$ there exists a point y of γ such that the line segment xy does not enter $\mathbf{S}^2$ (in other words, x is "visible" from y). It is easy to see that γ inspects $\mathbf{S}^2$, after a translation, if and only if its inradius is 1 [31, p. 369]. Thus finding the shortest inspection curve is equivalent to the inradius problem for $r = 1$.



(a) (b)

Figure 2.

The proof of Theorem 1.1 is based on an unpublished result of Daniel Wienholtz [28], which we include in Section 3. This remarkable observation, which follows from Borsuk-Ulam theorem, states that any closed space curve has a pair of parallel support planes H_0, H_1 which intersect the curve at least twice each in alternating fashion (Theorem 3.1). Further, it is easy to see that the same result holds for general curves (Corollary 3.2). Consequently the length L_1 of the projection of γ into the line orthogonal to H_i must be at least $3w$ for general curves, and $4w$ for closed curves. On the other hand we can also bound from below the length L_2 of the projection of γ into H_i by known results in $\mathbf{R}^2$. These estimates yield (1) and (2) due to the basic inequality $L \geq \sqrt{L_1^2 + L_2^2}$ (Lemma 2.2). The proof of (3) also follows from Wienholtz's theorem, once we utilize the theorem of Joris on the inradius problem for planar curves. To prove (4), on the other hand, we develop the notion of *horizon* of a curve, $H(\gamma)$, which is the measure of the tangent planes of $\mathbf{S}^2$, counted with multiplicity, that intersect γ. In Section 7, we derive upper and lower bounds for the horizon, which lead to the proof of (4). Finally in Section 8 we discuss some generalizations of our estimates, including an extension of the inradius estimate to the notion of n^{th} *hull* in geometric knot theory [9].

2. Preliminaries: Projections of Length

Here we will record some basic lemmas on length of projections of curves which will be useful throughout the paper. In this work a *curve* is a continuous mapping $\gamma\colon [a,b] \to \mathbf{R}^n$. By abuse of notation, we also use γ to denote its image $\gamma([a,b])$, and say that γ is *closed* if $\gamma(a) = \gamma(b)$. The *length* of γ is defined as

$$L = L[\gamma] := \sup \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|,$$

where the supremum is taken over all partitions $a := t_0 \leq t_1 \leq \cdots \leq t_n := b$ of $[a,b]$, $n \in \mathbf{N}$. We say that γ is *rectifiable* provided that L is finite. Further γ is *parametrized by arclength* or has *unit speed* if $L[\gamma|_{[t_1,t_2]}] = t_2 - t_1$ for all t_1, $t_2 \in [a,b]$. A curve $\widetilde{\gamma}\colon [c,d] \to \mathbf{R}^n$ is a *reparametrization* of γ provided that there exists a nondecreasing continuous map $\phi\colon [a,b] \to [c,d]$ such that $\gamma = \widetilde{\gamma} \circ \phi$. It is easy to see that $L[\gamma] = L[\widetilde{\gamma}]$. If $\widetilde{\gamma}$ has unit speed, then we say that it is a reparametrization of γ by arclength.

The first lemma below collects some basic facts from measure theory, which allow us to extend some well-known analytic arguments to all rectifiable curves.

Lemma 2.1. *Let $\gamma\colon [a,b] \to \mathbf{R}^n$ be a rectifiable curve.*

1. *If γ is Lipschitz, then $\gamma'(t)$ exists for almost all $t \in [a,b]$ (with respect to the Lebesgue measure), and $L = \int_a^b \|\gamma'(t)\|\, dt$.*
2. *If γ is parametrized by arclength, then $\|\gamma'(t)\| = 1$, for almost all $t \in [a,b]$.*
3. *There exists a (Lipschitz) reparametrization of γ by arclength.*

Proof. The first statement is just Rademacher's theorem. The second statement is proved in [22, Thm. 2], see also [8, Thm. 2.7.6] or [3, Thm. 4.1.6]. For the third statement see [15, 2.5.16] or [8, Prop. 2.5.9]. □

The following lemma is a quick generalization of an observation of Wienholtz [27], which had been obtained by polygonal approximation, see also [26, Lem. 8.2]. Here we offer a quick analytic proof which utilizes the above lemma.

Lemma 2.2 (Length Decomposition). *Let $\gamma\colon [a,b] \to \mathbf{R}^n$ be a rectifiable curve, V_i be a collection of pairwise orthogonal subspaces which span $\mathbf{R}^n$, and L_i be the length of the orthogonal projection of γ into V_i. Then*

$$L \geq \sqrt{\sum_i L_i^2}.$$

Proof. By Lemma 2.1 we may assume that γ is parametrized by arclength. Then γ' exists and $\|\gamma'\| = 1$ almost everywhere. Let γ_i denote the orthogonal projection of γ into V_i. Then γ_i are also Lipschitz, so γ_i' exist almost everywhere as well. Since V_i are orthogonal and span $\mathbf{R}^n$, $\gamma = \sum_i \gamma_i$, which yields $\gamma' = \sum_i \gamma_i'$. Further, $\langle \gamma_i', \gamma_j' \rangle = 0$ for $i \neq j$, since $\gamma_i' \in V_i$. So

$$\sum_i \|\gamma_i'\|^2 = \|\gamma'\|^2 = 1.$$

Now the Cauchy-Schwartz inequality yields

$$\sum_i (L_i)^2 = \sum_i \left(\int_a^b \|\gamma_i'\| \right)^2 \leq (b-a) \sum_i \int_a^b \|\gamma_i'\|^2 = (b-a)^2 = L^2,$$

which completes the proof. □

The next observation we need is a general version of a classical result which goes back to Cauchy. Originally this result was proved for smooth curves; however, it is well-known that it holds for all rectifiable curves [4]. Here we simply record that the original analytic proof may be extended almost verbatim to the general case in light of Lemma 2.1.

Lemma 2.3 (Cauchy-Crofton). *Let $\gamma\colon [a,b] \to \mathbf{R}^2$ be a rectifiable curve, $u \in \mathbf{S}^1$, and γ_u be the projection of γ into the line spanned by u. Then*

$$L = \frac{1}{4} \int_{\mathbf{S}^1} L[\gamma_u]\, du.$$

Proof. Again, by Lemma 2.1, we may assume that γ is Lipschitz (after a reparametrization by arclength), which, since $\gamma_u = \langle \gamma, u \rangle u$, yields that γ_u is Lipschitz as well. Thus

$$\begin{aligned} \int_{\mathbf{S}^1} L[\gamma_u]\, du &= \int_{\mathbf{S}^1} \int_a^b \|\gamma_u'(t)\|\, dt = \int_{\mathbf{S}^1} \int_a^b |\langle \gamma'(t), u \rangle|\, dt \\ &= \int_0^{2\pi} \int_a^b \|\gamma'(t)\|\, |\cos(\theta(t))|\, dt d\theta \\ &= 4L, \end{aligned}$$

as desired. □

The last lemma immediately yields another classical fact, which had been originally proved for smooth curves:

Corollary 2.4 (Barbier)**.** *Let* $\gamma\colon [a,b] \to \mathbf{R}^2$ *be a closed rectifiable curve of width* w*. Then*

$$L \geq \pi w.$$

Proof. Let γ_u be as in Lemma 2.3. Then

$$L = \frac{1}{4}\int_{\mathbf{S}^1} L[\gamma_u] \geq \frac{1}{4}\int_0^{2\pi} 2w\, d\theta = \pi w.$$

□

3. The Theorem of Wienholtz

Motivated by a 1998 conjecture of Kusner and Sullivan [20] on diameter of space curves, Wienholtz made the following fundamental observation in an unpublished work in 2000 [28]; see also [26, Sec. 8] where the argument is outlined. Here we include a complete proof which is significantly shorter than the original, although it is based on the same essential idea.

Theorem 3.1 (Wienholtz)**.** *For any continuous map* $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^n$ *there exists a pair of parallel hyperplanes* H_0*,* $H_1 \subset \mathbf{R}^n$*, and* 4 *points* t_0*,* t_1*,* t_2*,* t_3 *cyclically arranged in* $\mathbf{S}^1$ *such that* γ *lies in between* H_0*,* H_1*, while* $\gamma(t_0)$*,* $\gamma(t_2) \in H_0$*, and* $\gamma(t_1)$*,* $\gamma(t_3) \in H_1$*.*

Proof. For every direction $u \in \mathbf{S}^{n-1}$, let H_u be the support hyperplane of γ with outward normal u, and set $X_u := \gamma^{-1}(H_u)$. We may assume that $H_u \neq H_{-u}$ for all u, for otherwise there is nothing to prove. Let S_u be the open slab bounded by $H_{\pm u}$, and $I_u \subset \mathbf{S}^1$ be a connected component of $\gamma^{-1}(S_u)$ such that the initial point of I_u (with respect to some fixed orientation of $\mathbf{S}^1$) lies in X_{-u} and its final end point lies in X_u. If I_u is not unique for some u, then we are done. So suppose, towards a contradiction, that I_u is unique for every u. Then we will construct a map $f\colon \mathbf{S}^{n-1} \to \mathbf{R}^2 \subset \mathbf{R}^{n-1}$ such that $f(u) \neq f(-u)$ for all u. This violates the Borsuk Ulam theorem, and completes the proof.

To construct f, pick a point $p(u) \in I_u$ for each u. We claim that there exists an open neighborhood $V(u)$ of u in $\mathbf{S}^{n-1}$ such that

$$p(u) \in I_{u'} \text{ for all } u' \in V(u). \tag{5}$$

Indeed, since I_u is unique, the compact sets X_u and X_{-u} lie in the interiors of the (oriented) segments $p(u)p(-u)$ and $p(-u)p(u)$ of $\mathbf{S}^1$ respectively (Figure 3(a)). Thus, since $u \mapsto H_u$ is continuous, we may choose $V(u)$ so small that $X_{u'}$ lies in the interior of $p(u)p(-u)$ and $X_{-u'}$ lies in the in interior of $p(-u)p(u)$. Let $J_{u'}$ be the component of $\gamma^{-1}(A_{u'})$ which contains $p(u)$. Then the final boundary point of $J_{u'}$ must be in $X_{u'}$ and its initial boundary point must be in $X_{-u'}$. Thus $J_{u'} = I_{u'}$ by the uniqueness property, which establishes (5).

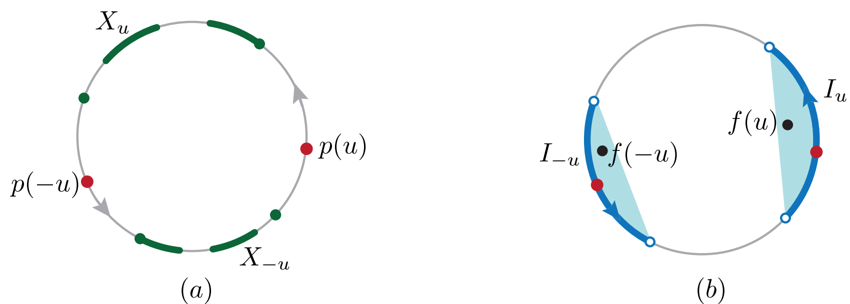


FIGURE 3.

Let $u_i \in \mathbf{S}^{n-1}$ be a finite number of directions such that $V(u_i)$ cover $\mathbf{S}^{n-1}$, $\phi_i \colon \mathbf{S}^{n-1} \to \mathbf{R}$ be a partition of unity subordinate to $\{V(u_i)\}$, and set

$$f(u) := \sum_i \phi_i(u) p(u_i).$$

If $\phi_i(u) \neq 0$, for some i, then $u \in V(u_i)$, and so (5) yields that $p(u_i) \in I_u$. Then, $f(u) \in \mathrm{conv}(I_u)$, the smallest convex set containing I_u. But $\mathrm{conv}(I_u) \cap \mathrm{conv}(I_{-u}) = \emptyset$, because $I_u \cap I_{-u} = \emptyset$ (Figure 3(b)). Thus $f(u) \neq f(-u)$ as claimed. □

Although Wienholtz stated his theorem only for closed curves, it is easy to see that it holds for all curves:

Corollary 3.2. *For any curve $\gamma \colon [a,b] \to \mathbf{R}^n$ there exist four points $t_0 < t_1 < t_2 < t_3$ in $[a,b]$ and a pair of parallel hyperplanes H_0, H_1 in $\mathbf{R}^n$ such that $\gamma(t_0)$, $\gamma(t_2) \in H_0$ and $\gamma(t_1)$, $\gamma(t_3) \in H_1$.*

Proof. If $\gamma(a) = \gamma(b)$, we may identify $[a,b]$ with $\mathbf{S}^1$ and we are done by Theorem 3.1. Further note that in this case we may assume that $t_i \in [a,b)$. If $\gamma(a) \neq \gamma(b)$, let ℓ be the line segment connecting $\gamma(a)$ and $\gamma(b)$. Then we may extend γ to a closed curve $\widetilde{\gamma} \colon [a,b'] \to \mathbf{R}^n$, for some $b' > b$ such that $\widetilde{\gamma}|_{[b,b']}$ traces ℓ. By Theorem 3.1, there are points $t_0 < t_1 < t_2 < t_3$ in $[a,b')$ such that $\gamma(t_0)$, $\gamma(t_2) \in H_0$ and $\gamma(t_1)$, $\gamma(t_3) \in H_1$. If interior of ℓ is disjoint from H_0 and H_1 then $t_i \notin (b,b')$ and we are done. If interior of ℓ intersects H_j, then ℓ lies entirely in H_j. In this case suppose that $t_i \in [b,b']$. Then $\gamma([t_i,b'])$ lies in H_j. So it would follow that $i = 3$, for otherwise $\gamma(t_i)$ and $\gamma(t_{i+1})$ would lie in the same hyperplane which is not possible. Now that $\gamma(t_3)$ lies in H_j, it follows that $t_2 < b$; because $\gamma([b,t_3])$ lies in H_j and $\gamma(t_2)$ and $\gamma(t_3)$ cannot lie in the same hyperplane. Thus we may replace t_3 with b which concludes the proof. □

4. Estimates for Width: Proof of Theorem 1.1

Using the generalized Wienholtz Theorem (Corollary 3.2) together with the length decomposition lemma (Lemma 2.2), we will now prove our main inequalities for the width:

Proof of Theorem 1.1. Let H_0, H_1 be a pair of parallel bounding planes of γ as in the generalization of Wienholtz's theorem, Corollary 3.2. Further let γ_1 be the projection of γ into a line orthogonal to H_0, γ_2 be the projection of γ into H_0, and L_i, w_i denote the length and width of γ_i respectively.Then

$$L_1 \geq L[\gamma_1|_{[t_0,t_1]}] + L[\gamma_1|_{[t_1,t_2]}] + L[\gamma_1|_{[t_2,t_3]}] \geq 3w.$$

Further recall that, for planar curves, L/w is minimized for Zalgaller's caliper curve, where this quantity is bigger than 2.2782 [1]. Thus

$$L_2 \geq 2.2782\, w_2 \geq 2.2782\, w,$$

So, by Lemma 2.2,

$$L \geq \sqrt{L_1^2 + L_2^2} \geq \sqrt{(3w)^2 + (2.2782w)^2} \geq 3.7669\, w,$$

which establishes (1). Next, to prove (2), suppose that γ is closed. Then $L_1 \geq 4w$. Further, it follows from Corollary 2.4 that $L_2 \geq \pi w_2 \geq \pi w$. Thus, again by Lemma 2.2,

$$L \geq \sqrt{L_1^2 + L_2^2} \geq \sqrt{(4w)^2 + (\pi w)^2},$$

which completes the proof. □

Note 4.1 (The case of equality in (2))**.** As we discussed above, when γ is closed, $L_1 \geq 4w$ and $L_2 \geq \pi w$. Thus the last displayed expression in the above proof shows that equality holds in (2) only if $L_1 = 4w$ and $L_2 = \pi w$. It follows then that γ_2 is a curve of constant width, and γ is composed of four geodesic segments in the cylindrical surface over γ_2. Since optimal objects in nature are usually symmetric, and γ has four segments running between H_0 and H_1, it would be reasonable to expect that the minimal curve γ would be symmetric with respect to a pair of orthogonal planes parallel to u. This would in turn imply that γ_2 is centrally symmetric. The only centrally symmetric curves of constant width, however, are circles. Thus if the equality in (2) is achieved by a symmetric curve, then γ_2 should be a circle. As we show in the next section, however, the equality in (2) never holds for curves which project onto a circles. Thus either (2) is not quite sharp or else the minimal curve is not so symmetric. On the other hand, to add to the mystery, we will produce a symmetric curve in the next section which very nearly realizes the case of equality in (2).

5. A Near Minimizer for L/w

Here we construct a symmetric piecewise geodesic closed curve on a circular cylinder which shows that (2) is very nearly sharp. To this end let C_h be the cylinder of radius 1 and height h in $\mathbf{R}^3$ given by

$$x^2 + y^2 = 1, \qquad -h/2 \leq z \leq h/2.$$

Consider the 4 consecutive points

$$p_1 = (1, 0, h/2),\ p_2 = (0, 1, -h/2),\ p_3 = (-1, 0, h/2),\ p_4 = (0, -1, -h/2)$$

on the boundary of C_h. Let Γ_h be the simple closed curve consisting of 4 geodesic or helical segments connecting these points cyclically, see Figure 4. Note that, as the figure shows, Γ_h may also be constructed by rolling a planar polygonal curve onto the cylinder. Let $L(h)$ and $w(h)$ denote the length and width of Γ_h respectively. We will show that $L(h)/w(h)$ is minimized when the height of C_h is slightly smaller than its diameter, or more precisely $h = h_0 \approx 1.97079$. Then $L(h_0)/w(h_0) < 5.1151$, and thus Γ_{h_0} yields the curve which we mentioned in the introduction.

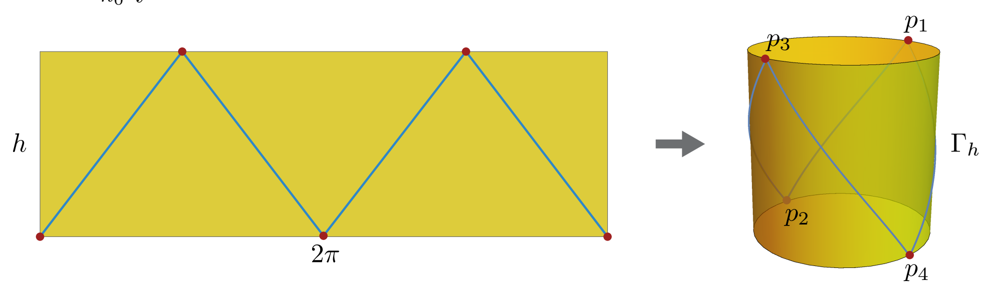


FIGURE 4.

To compute h_0, first note that, by the Pythagorean theorem,

$$L(h) = \sqrt{(4h)^2 + (2\pi)^2}.$$

Next, to find $w(h)$, let $\overline{\Gamma}_h$ be the projection of Γ_h into the xz-plane, see Figure 5, and let $\overline{w}(h)$ denote the width of $\overline{\Gamma}_h$, i.e., the infimum of the distance between all pairs of parallel lines in the plane of $\overline{\Gamma}_h$ which contain $\overline{\Gamma}_h$. Now we record the following lemma, whose proof we will postpone to the end of this section.

Lemma 5.1. *The width of Γ_h is equal to the width of $\overline{\Gamma}_h$:*

$$w(h) = \overline{w}(h).$$

To compute $\overline{w}(h)$, note that there are two possibilities: (i) $\overline{w}(h) = h$, or (ii) $\overline{w}(h)$ is given by the distance $d(h)$ between one of the end points of $\overline{\Gamma}_h$ and the opposite branch of $\overline{\Gamma}_h$. So we have

$$\overline{w}(h) = \min\{h, d(h)\}.$$

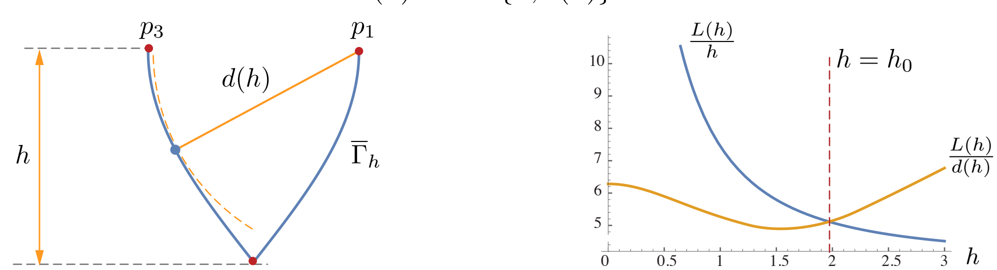


FIGURE 5.

To find $d(h)$ note that the left branch of $\overline{\Gamma}_h$ may be parametrized by

$$(-\cos(t), 0, (1/2 + 2t/\pi)h), \qquad -\pi/2 \leq t \leq 0,$$

and the tip of the right branch is $p_1 = (1, 0, h/2)$. Thus

$$d(h) = \min_{-\pi/2 \leq t \leq 0} \sqrt{(\cos(t) + 1)^2 + ((2t/\pi)h)^2}.$$

Finally we have

$$\frac{L(h)}{w(h)} = \frac{L(h)}{\overline{w}(h)} = \max\left\{\frac{L(h)}{h}, \frac{L(h)}{d(h)}\right\}.$$

Graphing these functions shows that $L(h)/w(h)$ is minimized when $\frac{L(h)}{h} = \frac{L(h)}{d(h)}$ or $h = d(h)$; see Figure 5.

Now let h_0 denote the solution to $h = d(h)$. Via a computer algebra system, we can find that $1.97078 < h_0 < 1.97080$. Consequently

$$\min\left(\frac{L(h)}{w(h)}\right) = \frac{L(h_0)}{w(h_0)} = \frac{\sqrt{(4h_0)^2 + (2\pi)^2}}{h_0} < \frac{\sqrt{(4 \times 1.97080)^2 + (2\pi)^2}}{1.97078} < 5.1151.$$

as desired.

It only remains now to prove the last lemma. To this end we need to consider the boundary structure of the convex hull C of Γ_h. Note that Γ_h lies on the boundary ∂C of C, and divides ∂C into a pair of regions by the Jordan curve theorem; see Figure 6. Further, each of these regions is a ruled surface. In one of the regions all the rulings are parallel to p_1p_3, or the x-axis, while in the other region the rulings are parallel to p_2p_4 or the y-axis. Each of these regions can be subdivided into a pair of triangular regions by the lines p_1p_3 and p_2p_4. Thus we may say that ∂C carries a tetrahedral structure, and call these subregions the faces of ∂C. For instance, the face of ∂C which is opposite to p_1 is $p_2p_3p_4$.

Proof of Lemma 5.1. By definition of width, we have $w(h) \leq \overline{w}(h)$. So we just need to establish the reverse inequality. To this end, let H, H' be a pair of parallel planes, with separation distance $w(h)$, which contain Γ_h in between them. First suppose that H (or H') intersects the convex hull C of Γ_h at more than one point. Then, since C is convex, and H is a support plane of C, H must contain a line segment in ∂C, the boundary of C. All line segments in ∂C are parallel to either the x-axis or the y-axis, as we discussed above. Thus H, and consequently H' must be parallel to, say, the y-axis. Consequently, if we let ℓ, ℓ' be the intersections of H, H' with the xz-plane, it follows that $\overline{\Gamma}_h$ is contained between ℓ and ℓ', which are separated by the distance $w(h)$. Thus $\overline{w}(h) \leq w(h)$, as desired.

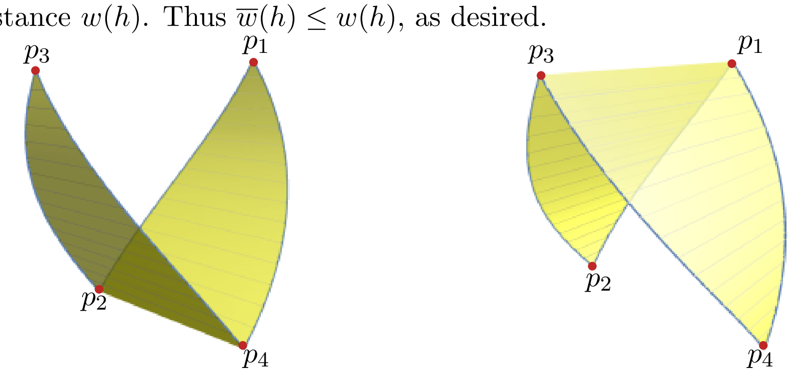


Figure 6.

We may suppose then that H and H' intersect C at precisely one point each, which we call p and p' respectively. Then the line segment pp' must be orthogonal to H and H' (e.g., see [25, p. 86]). Further, if $H \cap C$ and $H' \cap C$ are singletons, then H and H' can intersect C only along Γ_h, because $\partial C - \Gamma_h$ is fibrated by line segments. Now suppose that both p and p' belong to the interior of branches of Γ_h, i.e., the complement of p_i. Then pp' must be orthogonal to Γ_h at both ends. This may happen only when p and p' belong to a pair of opposite branches of Γ_h, and pp' is parallel to the xy plane. It follows then that $\|pp'\| = 2$, which yields $w(h) = 2$. On the other hand $\overline{w}(h) \leq 2$, since the distance between the end points of $\overline{\Gamma}_h$ is 2. Thus again we obtain $\overline{w}(h) \leq w(h)$.

It only remains then to consider the case where p is an end point of a branch of Γ_h, say $p = p_1$. In this case p' must belong to one of the branches of Γ_h which is not adjacent to p_1, i.e., either p_2p_3 or p_3p_4. So p' must belong to face or the triangular region $p_2p_3p_4$ in ∂C. Consequently

$$w(h) \geq \operatorname{dist}(p_1, p_2p_3p_4) = d(h) \geq \overline{w}(h),$$

which completes the proof. Here $d(h)$ is the distance between p_1 and the opposite branch of $\overline{\Gamma}_h$, as we had discussed above. □

Note 5.2. Although the curve Γ_{h_0} we constructed above is the minimizer for the width problem among curves on a circular cylinder, it is not the minimizer for the width problem among all closed curves. Indeed we may replace small segments of Γ_{h_0} which have an end point at p_i with straight line segments without decreasing the width.

6. Zalgaller's L_5 Curve

In [30] Zalgaller describes a closed space curve, "L_5", which he claims minimizes the ratio L/w. Here we show that this conjecture is not true. Indeed, the ratio L/w for Zalgaller's curve, which here we call Z, is bigger than that of the cylindrical curve Γ_{h_0} which we constructed in Section 5. Since Zalgaller does not calculate L/w for this example, we include this calculation below. We will begin by describing the construction of Z, since Zalgaller's paper is not available in English.

The curve Z is modeled on a regular tetrahedron. Note that the width of a regular tetrahedron is the distance between any pairs of its opposite edges. In particular this distance is 1 when the side lengths are $\sqrt{2}$. The basic idea for constructing Z is to take a simple closed curve, which traces 4 consecutive edges of a tetrahedron, say of edge length $\sqrt{2}$, and reduce its length without reducing its width. The error in Zalgaller's construction, however, is that the width does go down below 1, as we will show below.

6.1. **Construction.** Take a regular tetrahedron T with vertices A, B, C, D, as shown in Figure 7. Assume that the edge lengths are $\sqrt{2}$ so that the width of T is 1, i.e., the distance between the edges AB and CD. Let A' be the point on AC whose distance from the face BCD is 1. A simple computation shows that A' is the point on AC whose distance from C is $\sqrt{6}/2$. Similarly, let B' be the point on BD whose distance from D is $\sqrt{6}/2$. Let X be the cylinder of radius 1 with axis CD. Now connect A' and B' with the shortest arc which lies outside X. Note that this

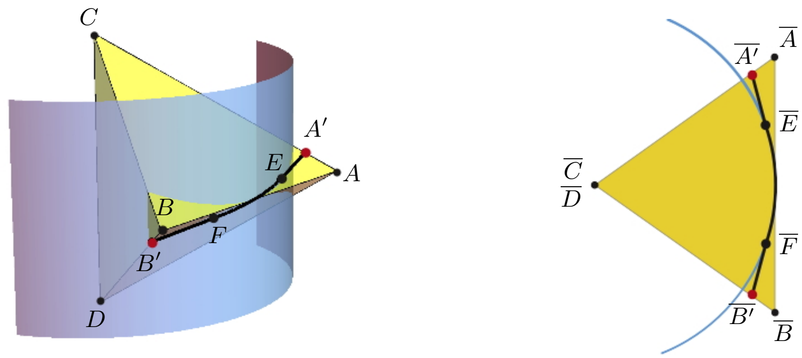


FIGURE 7.

arc is composed of a pair of straight line segments, plus a helical segment which lies on X. This forms the side $A'B'$ of Z. Similarly, we can form the sides $B'C'$, $C'D'$ and $D'A'$ which will yield the whole curve as shown in Figure 8.

6.2. **Length.** To compute the length of Z, we are going to assume that $A = (1, \sqrt{2}/2, 0)$, $B = (1, -\sqrt{2}/2, 0)$, $C = (0, 0, \sqrt{2}/2)$, and $D = (0, 0, -\sqrt{2}/2)$. For any set $S \subset \mathbf{R}^3$, let $\overline{S}$ denote its projection into the xy-plane, and note that $\overline{C}$ and $\overline{D}$ coincide with the origin o of the xy-plane. So the cylinder X will intersect the xy-plane in a circle of radius 1 centered at o; see the right diagram in Figure 7. Let $\overline{A'B'}$ denote the projection of $A'B'$ into the xy-plane, and h be the distance of A' or B' from the xy-plane. Then

$$L(A'B') = \sqrt{L(\overline{A'B'})^2 + (2h)^2}.$$

The reason behind the above equality is that $A'B'$ lies on the cylindrical surface over $\overline{A'B'}$, and is a geodesic in that surface (which has zero curvature); thus, the Pythagorean theorem applies. Next, note that

$$A' = \left(1 - \frac{\sqrt{3}}{2}\right) C + \frac{\sqrt{3}}{2} A, = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{2-\sqrt{3}}{2\sqrt{2}}\right).$$

Thus

$$\overline{A'} = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2\sqrt{2}}\right), \quad \text{and} \quad h = \frac{2-\sqrt{3}}{2\sqrt{2}}.$$

Next, to compute $L(\overline{A'B'})$, note that $\overline{A'B'} = \overline{A'E} \cup \overline{EF} \cup \overline{FB'}$, where $A'E$ and FB' are line segment, and EF is a circular arc. To find $L(\overline{EF})$, write $\overline{E} = (\cos(\theta), \sin(\theta))$, and note that $\langle \overline{E} - \overline{A}', \overline{E} \rangle = 0$, which yields that $\theta = \tan^{-1}(\sqrt{5}/2)$. So $L(\overline{EF}) = 2\arctan(\sqrt{5}/2)$. Further, it follows that $\overline{E} = (5, \sqrt{2})/(3\sqrt{3})$, which in turn allows us to compute that $L(\overline{A'E}) = \sqrt{2}/4$. So we conclude that

$$L(\overline{A'B'}) = 2L(\overline{A'E}) + L(\overline{EF}) = \frac{1}{\sqrt{2}} + 2\tan^{-1}\left(\frac{\sqrt{5}}{2}\right),$$

which in turn yields

$$L(Z) = 4L(A'B') = 4\sqrt{\left(\frac{1}{\sqrt{2}} + 2\tan^{-1}\left(\frac{\sqrt{2}}{5}\right)\right)^2 + \left(\frac{2-\sqrt{3}}{\sqrt{2}}\right)^2} \approx 5.0903$$

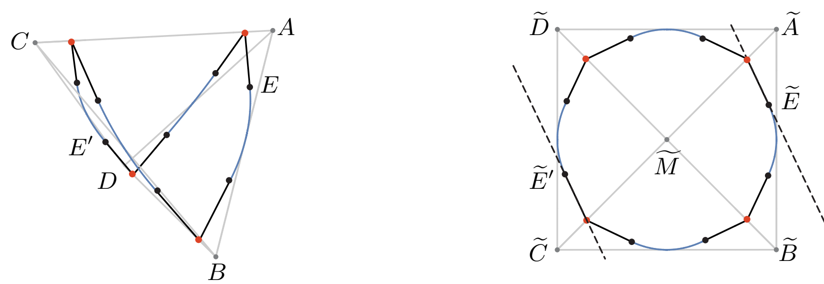


FIGURE 8.

6.3. **Width.** To estimate the width of Z we are going to project it into a plane Π orthogonal to

$$u := \frac{A+C}{2} - \frac{B+D}{2} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right);$$

see the right diagram in Figure 8. For any set $S \subset \mathbf{R}^3$, we let $\widetilde{S}$ denote its projection into Π. Let E' be the point on the segment DC of Z which lies at the end of the line segment in DC starting at D. Further let M denote the center of mass of the tetrahedron T. Note that

$$w(Z) \leq w(\widetilde{Z}) \leq \|\widetilde{EE'}\| = 2\|\widetilde{EM}\|.$$

The first inequality above is obvious from the definition of w; the second inequality follows from the fact that $\widetilde{Z}$ is contained in between the lines spanned by $\widetilde{A'E}$ and $\widetilde{C'E'}$; and the last equality of course is due to the fact that $\widetilde{M}$ is the midpoint of $\widetilde{EE'}$. It only remains then to compute $\|\widetilde{EM}\|$. To this end first note that

$$M = (A+B+C+D)/4 = (1/2, 0, 0).$$

Next, to compute E, recall that we already computed its first two components given by $\overline{E} = (5, \sqrt{2})/(3\sqrt{3})$. To find the third component of E recall that $A'B'$ is a linear graph over its projection $\overline{A'B'}$. More specifically, the height of this graph is given by $z(t) = \frac{h}{L/8}\, t$, where t measures the distance from the center of $\overline{A'B'}$. Thus

$$E = \left(\frac{5}{3\sqrt{3}}, \frac{\sqrt{2}}{3\sqrt{3}}, \frac{h}{L/8}\tan^{-1}(\frac{\sqrt{5}}{2})\right).$$

Finally recall that $\widetilde{E} = E - \langle E, u\rangle u$ and $\widetilde{M} = M - \langle M, u\rangle u$. So we now have all the information to compute that

$$w(Z) \leq 2\|\widetilde{EM}\| = 2\|\widetilde{E} - \widetilde{M}\| \approx 0.980582.$$

In particular, the computation contradicts Zalgaller's conjecture that Z has width 1. Using these computations, we now have

$$\frac{L(Z)}{w(Z)} \geq 5.1911,$$

which is bigger than 5.1151, the ratio L/w for the cylindrical curve we constructed in the last section. Thus Z does not minimize L/w, contrary to Zalgaller's conjecture.

Note 6.1 (Original Statement of the L_5 Conjecture)**.** Since Zalgaller's paper [30] is not available in English, here we include a translation of the conjecture on the shortest closed curve of width 1, which we disproved above.

"16. A similar problem can be posed for closed curves. On the plane any curve of constant width 1 is the shortest closed curve of width 1. In space consider the regular tetrahedron with edge $\sqrt{2}$ (fig 9). The 4-segment polygonal curve $L_4 = ABCDA$ is an example of a closed curve of width 1. Mark the middle points O_1, O_2, O_3, O_4 on the edges of L_4. On the edge AC which is not in L_4 mark the point A' which is at distance 1 from the plane BCD and also mark a point C' which is at distance 1 from the plane ABD. Similarly, on the edge BD which is also not in L_4 mark points B', D' which are at distance 1 from the planes ACD, ABC, respectively. Form the

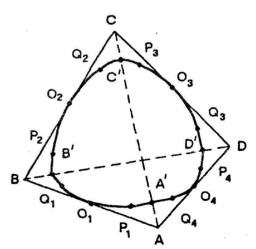


FIGURE 9.

curve L_5 from four congruent C^1-smooth portions $A'B'$, $C'D'$, $D'A'$. It is enough to describe the portion $A'B'$. We construct it as the shortest curve $A'P_1O_1Q_1B'$ that joins A' and B' and goes around outside the circular cylinder Z of radius 1 with axis CD. This portion is of the form $A'B' = A'P_1 + P_1O_1Q_1 + Q_1B'$ where $A'P_1$ and Q_1B' are straight line segments and $P_1O_1Q_1$ is a screw-rotational arc on the cylinder Z. Similarly, one constructs the portions $B'C' = B'P_2O_2Q_2C'$, $C'D' = C'P_3O_3Q_3D'$, $D'A' = D'P_4O_4Q_4A'$.

17. Conjecture 2. The curve L_5 has width 1 and is the shortest closed space curve of width 1."

Note 6.2. If Zalgaller had been correct in his conjecture that the width of L_5 is 1, then the ratio L/w for this curve would have been approximately 5.0903 according to our computation of length in Section 6.2. In this sense, the L_5 conjecture predicted that $L/w \geq 5.0903$, which interestingly enough is within 0.01% of the lower bound (2) in Theorem 1.1.

7. Estimates for Inradius: Proof of Theorem 1.2

The general estimate in Theorem 1.2 follows quickly from the Wienholtz theorem as was the case for the estimates for the width in the proof of Theorem 1.1. For the case of closed curves, however, we will work harder to obtain a better estimate via the notion of horizon developed below.

7.1. **The general case.** Here we prove (3). Let γ_i be as in the proof of Theorem 1.1, and L_i, r_i denote the length, and inradius of the convex hull of γ_i respectively. Then

$$L_1 \geq 3w \geq 6r.$$

Further note that r_2 is not smaller than the inradius of the convex hull of γ, which in turn is not smaller than r. Thus

$$L_2 \geq (2+\pi)r_2 \geq (2+\pi)r$$

by the theorem of Joris [18]. Consequently

$$L \geq \sqrt{L_1^2 + L_2^2} \geq \sqrt{(6r)^2 + ((2+\pi)r)^2} > 7.90164\, r,$$

which establishes (3).

7.2. **The horizon.** Here we develop some integral formulas needed to prove (4). For a point x outside $\mathbf{S}^2$ consider the cone generated by all rays which emanate from x and are tangent to $\mathbf{S}^2$. This cone touches $\mathbf{S}^2$ along a circle which we call the horizon of x. The *horizon* of a curve γ, which we denote by $H(\gamma)$, is defined as the total area, counted with multiplicity, covered by horizons of all points of γ. Note that a point p of $\mathbf{S}^2$ belongs to $H(\gamma)$ if and only if the tangent plane $T_p\mathbf{S}^2$ intersects γ. Thus

$$H(\gamma) := \int_{p \in \mathbf{S}^2} \#\big(\gamma^{-1}(T_p\mathbf{S}^2)\big)\, dp.$$

The closedness of γ together with a bit of convexity theory, quickly yields the following lower bound for the horizon. Recall that we say a curve $\gamma\colon [a,b] \to \mathbf{R}^3$ inspects the sphere $\mathbf{S}^2$, or is an inspections curve provided that it lies outside $\mathbf{S}^2$ and $\mathbf{S}^2$ lies in its convex hull.

Lemma 7.1. *If γ is a closed inspection curve of $\mathbf{S}^2$, then*

$$8\pi \leq H(\gamma). \tag{6}$$

Proof. We claim that for every $p \in \mathbf{S}^2 \setminus \gamma$, $T_p\mathbf{S}^2$ intersects γ in at least two points. To see this set $C := \mathrm{conv}(\gamma)$. Either $T_p\mathbf{S}^2$ is a support plane of C, or else C has points in the interior of each of the closed half-spaces determined by $T_p\mathbf{S}^2$. In the latter case it is obvious that the claim holds. Suppose then that $T_p\mathbf{S}^2$ is a support plane of C. By Caratheodory's theorem [25, p. 3], p must lie in a line segment or a triangle Δ whose vertices belong to γ. Since, by assumption $p \notin \gamma$, p must belong to the relative interior of Δ. Consequently Δ has to lie in $T_p\mathbf{S}^2$. Then the vertices of Δ yield the desired points. $\square$

Next we develop an analytic formula for computing H, following the basic outline of the proof of Crofton's formula in Chern [11, p. 116]. Suppose that γ is piecewise $\mathcal{C}^1$, and its projection into $\mathbf{S}^2$, $\overline{\gamma} := \gamma/\|\gamma\|$ has non vanishing speed. Let $\overline{T} := \overline{\gamma}'/\|\overline{\gamma}'\|$ and $\overline{\nu} := \overline{\gamma} \times \overline{T}$. Then $(\overline{\gamma}, \overline{T}, \overline{\nu})$ is a moving orthonormal frame along $\overline{\gamma}$. It is easy to check that the derivative of this frame is given by

$$\begin{pmatrix} \overline{\gamma} \\ \overline{T} \\ \overline{\nu} \end{pmatrix}' = \begin{pmatrix} 0 & v & 0 \\ -v & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix} \begin{pmatrix} \overline{\gamma} \\ \overline{T} \\ \overline{\nu} \end{pmatrix},$$

where $v := \|\overline{\gamma}'\|$ and $\lambda \colon [a, b] \to \mathbf{R}$ is some scalar function. Define $F \colon [a, b] \times [0, 2\pi] \to \mathbf{S}^2$ by

$$F(t, \theta) := h(t)\overline{\gamma}(t) + r(t)\big(\cos(\theta)\overline{T}(t) + \sin(\theta)\overline{\nu}(t)\big),$$

where

$$r := \frac{\sqrt{\|\gamma\|^2 - 1}}{\|\gamma\|} \quad \text{and} \quad h := \frac{1}{\|\gamma\|}.$$

For each $t \in [a, b]$, $F(t, \theta)$ parametrizes the horizon of $\gamma(t)$. In particular, for all $p \in \mathbf{S}^2$,

$$F^{-1}(p) = \gamma^{-1}(T_p\mathbf{S}^2).$$

Thus the area formula [15, Thm 3.2.3] yields that

$$H(\gamma) = \int_{p \in \mathbf{S}^2} \#F^{-1}(p)\, dp = \int_a^b \int_0^{2\pi} \mathrm{Jac}(F)\, d\theta dt, \tag{7}$$

where $\mathrm{Jac}(F) := \|\partial F/\partial t \times \partial F/\partial \theta\|$ denotes the Jacobian of F. A computation shows that

$$\mathrm{Jac}(F) = \big| r(t)v(t)\cos(\theta) - h'(t) \big|.$$

Further we have

$$v = \frac{\sqrt{\|\gamma'\|^2\|\gamma\|^2 - \langle \gamma, \gamma' \rangle^2}}{\|\gamma\|^2} \quad \text{and} \quad h' = -\frac{\langle \gamma, \gamma' \rangle}{\|\gamma\|^3}.$$

So we conclude that

$$\mathrm{Jac}(F) = \frac{1}{\|\gamma\|^3} \left| \sqrt{(\|\gamma\|^2 - 1)(\|\gamma'\|^2\|\gamma\|^2 - \langle \gamma, \gamma' \rangle^2)}\, \cos(\theta) + \langle \gamma, \gamma' \rangle \right|.$$

Note that if $\|\gamma'\| = 1$ and $\alpha(t)$ is the angle between $\gamma'(t)$ and $\gamma(t)$, then

$$\cos(\alpha) = \frac{\langle \gamma, \gamma' \rangle}{\|\gamma\|}, \quad \text{and} \quad \sin(\alpha) = \frac{\sqrt{\|\gamma\|^2 - \langle \gamma, \gamma' \rangle^2}}{\|\gamma\|},$$

which yields

$$\mathrm{Jac}(F) = \frac{1}{\|\gamma\|^2} \left| \sqrt{\|\gamma\|^2 - 1}\, \sin(\alpha)\cos(\theta) + \cos(\alpha) \right|.$$

The observations in this section may now be summarized as follows:

Lemma 7.2. *Let $\gamma\colon [a,b] \to \mathbf{R}^3$ be a piecewise $\mathcal{C}^1$ curve, and $\alpha(t)$ be the angle between $\gamma(t)$ and $\gamma'(t)$. Suppose that $\alpha(t) \neq 0, \pi$ except at finitely many points, and $\|\gamma'\| = 1$. Then*

$$H(\gamma) = \int_a^b \int_0^{2\pi} \frac{1}{\|\gamma\|^2} \left| \sqrt{\|\gamma\|^2 - 1}\, \sin(\alpha)\cos(\theta) + \cos(\alpha) \right| d\theta dt. \tag{8}$$

□

Note 7.3. If $\|\gamma\| = c$, then (6) together with Lemma 7.2 yield

$$8\pi \leq H(\gamma) = \frac{\sqrt{c^2-1}}{c^2} 4L \leq 2L,$$

and equality holds only if $c = \sqrt{2}$. Thus, as has been noted by Jean-Marc Schlenker [21], see also [32, Sec. 2.2]: If γ inspects $\mathbf{S}^2$ and $\|\gamma\| = c$, then $L \geq 4\pi$, which is the optimal inequality for closed inspection curves originally conjectured by Zalgaller [31], and also suggested by Gjergji Zaimi [21]. We will improve this observation in Proposition 8.1 below.

7.3. **The closed case.** To prove (4), we begin by recording a pair of lemmas which yield an upper bound for the horizon. Let us say that a piecewise $\mathcal{C}^1$ curve γ inspects the sphere $\mathbf{S}^2$ *efficiently*, provided that that γ inspects $\mathbf{S}^2$ and the tangent lines of γ do not enter $\mathbf{S}^2$.

Lemma 7.4. *For every closed polygonal curve γ which inspects $\mathbf{S}^2$, there is a closed polygonal curve $\widetilde{\gamma}$, with $L[\widetilde{\gamma}] \leq L[\gamma]$, which inspects $\mathbf{S}^2$ efficiently.*

Proof. Let E be an edge of γ whose corresponding line enters $\mathbf{S}^2$ (if E does not exist, then there is nothing to prove). Let p be the vertex of E which is farthest from $\mathbf{S}^2$, and C be the cone with vertex p which is tangent to $\mathbf{S}^2$. Then the other vertex of E, say p' belongs to the region X which lies inside C and outside $\mathbf{S}^2$, see Figure 10. Consider the polygonal arc $p'p$ of γ which is different from E. Note that γ cannot lie entirely in X for then $\mathbf{S}^2$ cannot be in the convex hull of γ. So $p'p$ must have a point outside X. In particular, there is a point of $p'p$, other than p which belongs to ∂C. Let q be the first such point, and replace the subsegment $p'q$ of $p'p$ with the line segment joining p and q. This procedure removes E and does not increase the number of edges of γ or its length. Further, the new curve still inspects $\mathbf{S}^2$, because p "sees" all points of $\mathbf{S}^2$ which were visible from any points of pq. Since γ has only finitely many edges, repeating this procedure eventually yields the desired curve $\widetilde{\gamma}$. □

Lemma 7.5. *Suppose that γ is a piecewise $\mathcal{C}^1$ curve which inspects $\mathbf{S}^2$ efficiently, then*

$$H(\gamma) \leq \frac{4\pi}{3\sqrt{3}} L. \tag{9}$$

Proof. By (7), it suffices to show that

$$\int_0^{2\pi} \mathrm{Jac}(F)\, d\theta \leq \frac{4\pi}{3\sqrt{3}}. \tag{10}$$

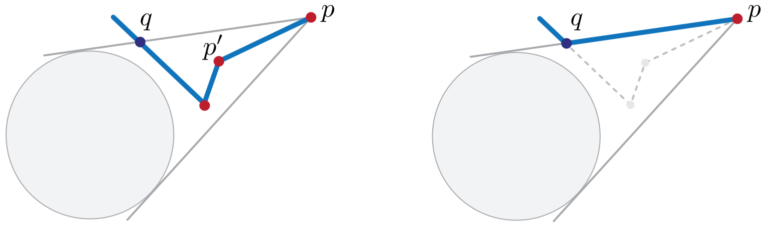


FIGURE 10.

To this end note that, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\mathrm{Jac}(F) &= \frac{1}{\|\gamma\|^2}\left|\Big\langle\big(\sqrt{\|\gamma\|^2-1}\cos(\theta),1\big),\big(\sin(\alpha),\cos(\alpha)\big)\Big\rangle\right|.\\ &\leq \frac{1}{\|\gamma\|^2}\|\big(\sqrt{\|\gamma\|^2-1}\cos(\theta),1\big)\| \leq \frac{1}{\|\gamma\|}.\end{aligned}$$

Thus (10) is satisfied whenever $\|\gamma\| \geq 3\sqrt{3}/2$. So it suffices now to check (10) for $\|\gamma\| < 3\sqrt{3}/2 < 2.6$. To this end, set

$$I(x,y) := \int_0^{2\pi} \frac{1}{x^2}\left|\sqrt{x^2-1}y\cos(\theta)+\sqrt{1-y^2}\right| d\theta.$$

Then by (8), $\int_0^{2\pi}\mathrm{Jac}(F)\,d\theta = I(\|\gamma\|,\sin(\alpha))$, because replacing $\cos(\alpha)$ with $|\cos(\alpha)|$ in (8) amounts at most to switching θ to $-\theta$, which does not affect the value of the integral. Further note that, by elementary trigonometry, the tangent lines of γ avoid the interior of $\mathbf{S}^2$ if and only if

$$\sin(\alpha) \geq \frac{1}{\|\gamma\|}.$$

So we just need to check that $I \leq 4\pi/(3\sqrt{3}) \approx 2.4$ for $1 \leq x \leq 3$ and $1/x \leq y \leq 1$, which may be done with the aid of a computer algebra system. In particular, graphing I shows that the maximum of I over the given region is achieved on the boundary curve $y = 1/x$, see Figure 11. Then it remains to note that

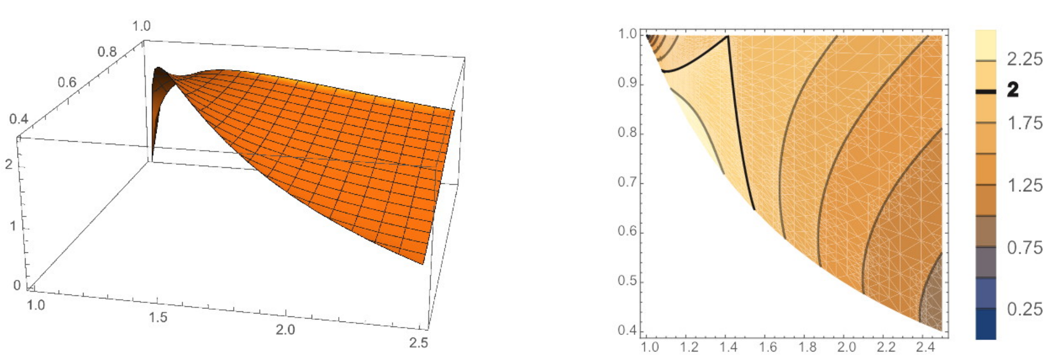


FIGURE 11.

$$I\left(x, \frac{1}{x}\right) = \frac{\sqrt{x^2-1}}{x^3} \int_0^{2\pi} (\cos(\theta)+1)\, d\theta = 2\pi \frac{\sqrt{x^2-1}}{x^3} \leq \frac{4\pi}{3\sqrt{3}},$$

which completes the proof. □

Now we are ready to complete the proof of (4). We may assume, after a translation, that $\mathbf{S}^2$ is a sphere of maximal radius which is contained in the convex hull of γ, and whose interior is disjoint from γ. In particular $r = 1$. We need to show then that $L \geq 6\sqrt{3}$. To this end we may assume that γ is polygonal. Indeed, let γ_i be a sequence of polygonal curves, converging to γ. Then $L_i/r_i \to L$ where L_i and r_i are the length and inradius of γ_i respectively. Thus, if $L_i/r_i \geq 6\sqrt{3}$, it follows that $L/r \geq 6\sqrt{3}$ as desired. Now we may let $\widetilde{\gamma}$ be as in Lemma 7.4. Then by Lemmas 7.1 and 7.5

$$8\pi \leq H(\widetilde{\gamma}) \leq \frac{4\pi}{3\sqrt{3}} L[\widetilde{\gamma}] \leq \frac{4\pi}{3\sqrt{3}} L,$$

which completes the proof of Theorem 1.2.

Note 7.6. We were able to establish the conjectured sharp inequality $L \geq 4\pi$ only for the case of $\|\gamma\| = c$, since in this case $H(\gamma) \leq 2L$. If the same upper bound may be established for the class of all closed curves which inspect $\mathbf{S}^2$ efficiently, then we may replace the right hand side of the last displayed expression by $2L$, and thus obtain $L \geq 4\pi$ for all closed curves inspecting $\mathbf{S}^2$. The contour graph in Figure 11 shows that $H(\gamma) \leq 2L$ if $\min \|\gamma\| \geq 1.6$. Thus, in this case $L \geq 4\pi$.

8. Generalizations

8.1. **More inradius estimates via Crofton.** Here we use Crofton's formulas to generalize the earlier observation in Note 7.3, on inspection curves of constant height:

Proposition 8.1. *Let $\gamma\colon [a,b] \to \mathbf{R}^3$ be a closed rectifiable curve which inspects $\mathbf{S}^2$, and set $M := \max \|\gamma\|$, $m := \min \|\gamma\|$. Then*

$$L \geq \frac{2\pi M m}{\sqrt{M^2-1}}.$$

In particular, when $M = m$, or $M \leq 2/\sqrt{3}$, then $L \geq 4\pi$.

Recall that, as we pointed out in Note 7.6, the conjectured inequality $L \geq 4\pi$ holds when $M \geq 1.6$. This, together with the above proposition shows that if there exists a closed inspection curve with $L < 4\pi$, then $1.15 \leq \|\gamma(t)\| \leq 1.6$ for some $t \in [a,b]$. To establish the above inequality, let us record that:

Lemma 8.2 (Crofton-Blaschke-Santalo)**.** *For every point $p \in \mathbf{S}^2$, and $0 \leq \rho \leq \pi/2$, let $C_\rho(p)$ denote the circle of spherical radius ρ centered at p. Then*

$$L = \frac{1}{4\sin(\rho)} \int_{p \in \mathbf{S}^2} \#\gamma^{-1}(C_\rho(p))$$

Crofton was the first person to obtain integrals of this type for planar curves [24]. According to Santalo [23], Blaschke observed the analogous phenomena on the sphere for regular curves [7], which were then extended to all rectifiable curves by Santalo [23, (37)].

Proof of Proposition 8.1. Let $\rho(t)$ be the (spherical) radius of the "visibility circle", generated by rays which emanate from $\gamma(t)$ and are tangent to $\mathbf{S}^2$. Then $\cos(\rho) = 1/\|\gamma\| \geq 1/M$ by simple trigonometry, which in turn yields that

$$\sin(\rho) \leq \frac{\sqrt{M^2-1}}{M}.$$

Let $\overline{\rho}$ be the supremum of the radii of the visibility circles. Then the union of all spherical disks of radius $\overline{\rho}$ centered at points of $\overline{\gamma}$ cover $\mathbf{S}^2$. Consequently, $\overline{\gamma}$ intersects all circles of radius $\overline{\rho}'$ in $\mathbf{S}^2$ at least twice for all $\overline{\rho}' > \overline{\rho}$. So, by the Crofton formula, Lemma 8.2,

$$\overline{L} \geq \frac{1}{4\sin(\overline{\rho})}\int_{\mathbf{S}^2} 2 \geq \frac{2\pi}{\sin(\overline{\rho})}$$

where $\overline{L}$ denotes the length of $\overline{\gamma}$. Finally note that

$$L \geq m\overline{L},$$

since $m\overline{L}$ is the length of the projection of γ into the sphere of radius m centered at the origin, and $\|\gamma\| \geq m$. Combining the last three displayed expressions completes the proof. □

8.2. **Estimates for the n^{th} inradius.** The convex hull of a closed curve γ in $\mathbf{R}^3$ coincides with the set of all points p such that almost every plane through p intersects γ in at least 2 points. Motivated by this phenomenon, the n^{th} *hull* of γ has been defined [9] as the set of all points p such that every plane through p intersects γ in at least $2n$ points. Accordingly, the n^{th} *inradius* r_n of γ may be defined as the supremum of the radii of all balls which are contained in the n^{th} hull of γ and do not intersect γ, which generalizes the notion of the inradius defined in the introduction. The proof of (4) may now be easily generalized as follows:

Theorem 8.3. *For any closed rectifiable curve $\gamma\colon [a,b] \to \mathbf{R}^3$,*

$$\frac{L}{r_n} \geq 6\sqrt{3}\, n.$$

Proof. As in the proof of Theorem 1.2, we may assume that $r_n = 1$, and $\mathbf{S}^2$ is a sphere of maximal radius contained in the n^{th} hull of γ, and whose interior is disjoint from γ. Then every tangents plane $T_p\mathbf{S}^2$ intersects γ at least $2n$ times, and consequently $H(\gamma) \geq 4\pi \times 2n$ by the definition of the horizon. On the other hand $H(\gamma) \leq 4\pi/(3\sqrt{3})L$ by Lemma 7.5. Thus

$$8n\pi \leq H(\gamma) \leq \frac{4\pi}{3\sqrt{3}}L,$$

which yields $L \geq 6\sqrt{3}\, n$ as desired. □

The notion of n^{th} hull is of interest in geometric knot theory, since it was established in [9] that knotted curves have nonempty second hulls.

8.3. **Estimates for width and inradius in $\mathbf{R}^n$.** The generalized Wienholtz theorem (Corollary 3.2) together with the length decomposition lemma (Lemma 2.2) quickly yield:

Lemma 8.4. *Let $\gamma\colon [a,b] \to \mathbf{R}^n$ be a rectifiable curve. Suppose that for all projections of γ into hyperplanes of $\mathbf{R}^n$ we have $L/w \geq c_1$ and $L/r \geq c_2$. Then*

$$L \geq \sqrt{c_1^2 + 9}\, w \quad \text{and} \quad L \geq \sqrt{c_2^2 + 36}\, r$$

Further, if γ is closed, and for all projections of γ into hyperplanes of $\mathbf{R}^n$ we have $L/w \geq c_3$ and $L/r \geq c_4$, then

$$L \geq \sqrt{c_3^2 + 16}\, w \quad \text{and} \quad L \geq \sqrt{c_4^2 + 64}\, r.$$

□

Thus we may inductively extend our estimates for the width and inradius problems, in Theorems 1.1 and 1.2, to higher dimensions:

Theorem 8.5. *Let $\gamma\colon [a,b] \to \mathbf{R}^{2+k}$ be a rectifiable curve. Then*

$$L \geq \sqrt{2.2782^2 + 9k}\, w \quad \text{and} \quad L \geq \sqrt{(\pi+2)^2 + 36k}\, r.$$

Further, if γ is closed,

$$L \geq \sqrt{\pi^2 + 16k}\, w \quad \text{and} \quad L \geq \sqrt{(6\sqrt{3})^2 + 64(k-1)}\, r.$$

□

Acknowledgements

The author thanks Joseph O'Rourke for posting the sphere inspection problem on MathOverflow [21] which provided the initial stimulus for this work. Thanks also to the commenters on MathOverflow, specially Gjergji Zaimi and Jean-Marc Schlenker, for useful observations on this problem. Finally the author thanks Igor Belegradek for his translation of Zalgaller's L_5-conjecture, included in Note 6.1.

References

[1] A. Adhikari and J. Pitman. The shortest planar arc of width 1. *Amer. Math. Monthly*, 96(4):309–327, 1989.

[2] R. Alexander. The geometry of wide curves in the plane. *J. Geom.*, 93(1-2):1–20, 2009.

[3] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[4] S. Ayari and S. Dubuc. La formule de Cauchy sur la longueur d'une courbe. *Canad. Math. Bull.*, 40(1):3–9, 1997.

[5] E. Barbier. Note sur le problème de l'aiguille et le jeu du joint couvert. *Journal de mathématiques pures et appliquées*, pages 273–286, 1860.

[6] R. Bellman. A minimization problem. *Bulletin of the AMS*, 62:270, 1956.

[7] W. Blaschke. *Vorlesungen über integralgeometrie*. Number 20. BG Teubner, 1937.

[8] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[9] J. Cantarella, G. Kuperberg, R. B. Kusner, and J. M. Sullivan. The second hull of a knotted curve. *Amer. J. Math.*, 125(6):1335–1348, 2003.

[10] T. M. Chan, A. Golynski, A. Lopez-Ortiz, and C.-G. Quimper. The asteroid surveying problem and other puzzles. In *Proceedings of the Nineteenth Annual Symposium on Computational Geometry*, SCG '03, pages 372–373, New York, NY, USA, 2003. ACM.

[11] S. S. Chern. Curves and surfaces in euclidean space. *Studies in Global Geometry and Analysis*, 4(1):967, 1967.

[12] H. T. Croft, K. J. Falconer, and R. K. Guy. *Unsolved problems in geometry.* Springer-Verlag, New York, 1994. Corrected reprint of the 1991 original [MR 92c:52001], Unsolved Problems in Intuitive Mathematics, II.

[13] H. G. Eggleston. The maximal inradius of the convex cover of a plane connected set of given length. *Proc. London Math. Soc. (3)*, 45(3):456–478, 1982.

[14] V. Faber and J. Mycielski. The shortest curve that meets all the lines that meet a convex body. *Amer. Math. Monthly*, 93(10):796–801, 1986.

[15] H. Federer. *Geometric measure theory.* Springer-Verlag New York Inc., New York, 1969. Die Grundlehren der mathematischen Wissenschaften, Band 153.

[16] S. R. Finch and J. E. Wetzel. Lost in a forest. *Amer. Math. Monthly*, 111(8):645–654, 2004.

[17] J.-B. Hiriart-Urruty. Du calcul différentiel au calcul variationnel: un aperçu de l'évolution de p. fermata nos jours. *Quadrature*, (70):8–18, 2008.

[18] H. Joris. Le chasseur perdu dans la forêt. *Elem. Math.*, 35(1):1–14, 1980. Un problème de géométrie plane.

[19] R. Klötzler and S. Pickenhain. Universale rettungskurven ii. *Zeitschrifte für Analysis und ihre Anwendungen*, 6:363–369, 1987.

[20] R. B. Kusner and J. M. Sullivan. On distortion and thickness of knots. In *Topology and geometry in polymer science (Minneapolis, MN, 1996)*, pages 67–78. Springer, New York, 1998.

[21] J. O'Rourke. Shortest closed curve to inspect a sphere (question posted on mathoverflow). *mathoverflow.net/questions/69099/shortest-closed-curve-to-inspect-a-sphere*, June 2011.

[22] M. J. Pelling. Classroom Notes: Formulae for the Arc-Length of a Curve in R^N. *Amer. Math. Monthly*, 84(6):465–467, 1977.

[23] L. A. Santaló. Integral formulas in Crofton's style on the sphere and some inequalities referring to spherical curves. *Duke Math. J.*, 9:707–722, 1942.

[24] L. A. Santaló. *Integral geometry and geometric probability.* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1.

[25] R. Schneider. *Convex bodies: the Brunn-Minkowski theory.* Cambridge University Press, Cambridge, 1993.

[26] J. M. Sullivan. Curves of finite total curvature. In *Discrete differential geometry*, volume 38 of *Oberwolfach Semin.*, pages 137–161. Birkhäuser, Basel, 2008.

[27] D. Wienholtz. The smallest diameter of projections of closed curves into hyperplanes. *Unpublished Manuscript*, 2000.

[28] D. Wienholtz. A special way how two planes can bound a given closed curve. *Unpublished Manuscript*, 2000.

[29] V. A. Zalgaller. How to get out of the woods. *On a problem of Bellman (in Russian), Matematicheskoe Prosveshchenie*, 6:191–195, 1961.

[30] V. A. Zalgaller. The problem of the shortest space curve of unit width. *Mat. Fiz. Anal. Geom.*, 1(3-4):454–461, 1994.

[31] V. A. Zalgaller. Extremal problems on the convex hull of a space curve. *Algebra i Analiz*, 8(3):1–13, 1996.

[32] V. A. Zalgaller. Shortest inspection curves for a sphere. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 299(Geom. i Topol. 8):87–108, 328, 2003.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332
E-mail address: ghomi@math.gatech.edu
URL: www.math.gatech.edu/∼ghomi