BMO SOLVABILITY AND ABSOLUTE CONTINUITY OF HARMONIC MEASURE

STEVE HOFMANN AND PHI LE

ABSTRACT. We show that for a uniformly elliptic divergence form operator L, defined in an open set Ω with Ahlfors-David regular boundary, BMO-solvability implies scale invariant quantitative absolute continuity (the weak-A_∞ property) of elliptic-harmonic measure with respect to surface measure on $\partial\Omega$. We do not impose any connectivity hypothesis, qualitative or quantitative; in particular, we do not assume the Harnack Chain condition, even within individual connected components of Ω. In this generality, our results are new even for the Laplacian. Moreover, we obtain a partial converse, assuming in addition that Ω satisfies an interior Corkscrew condition, in the special case that L is the Laplacian.

CONTENTS

1. Introduction 1
1.1. Notation and Definitions 4
2. Preliminaries 7
3. Proof of Theorem 1.4: BMO-solvability implies $\omega \in$ weak-A_∞ 7
4. Proof of Theorem 1.6: $\omega \in$ weak-A_∞ implies L^p and BMO-solvability 12
5. The case that Ω is an unbounded domain with bounded boundary 17
References 19

1. INTRODUCTION

The connection between solvability of the Dirichlet problem with L^p data, and scale-invariant absolute continuity properties of harmonic measure (specifically, that harmonic measure belongs to the Muckenhoupt weight class A_∞ with respect to surface measure on the boundary), is well documented, see the monograph of Kenig [Ke], and the references cited there. Specifically, one obtains that the Dirichlet problem is solvable with data in $L^p(\partial\Omega)$ for some $1 < p < \infty$, if and only if harmonic measure ω with some fixed pole is absolutely continuous with respect to surface measure σ on the boundary, and the Poisson kernel $d\omega/d\sigma$ satisfies a reverse Hölder condition with exponent $p' = p/(p-1)$. The most general class of

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domains for which such results had previously been known to hold is that of the so-called "1-sided Chord-arc domains" (see Definition 1.13 below).

The connection between solvability of the Dirichlet problem and scale invariant absolute continuity of harmonic measure was sharpened significantly in work of Dindos, Kenig and Pipher [DKP], who showed that harmonic measure satisfies an A_∞ condition with respect to surface measure, if and only if a natural Carleson measure/BMO estimate (to be described in more detail momentarily) holds for solutions of the Dirichlet problem with continuous data. Their proof was nominally carried out in the setting of a Lipschitz domain, but more generally, their arguments apply, essentially verbatim, to Chord-arc domains. The results of [DKP] were recently extended to the setting of a 1-sided Chord-arc domain by Zihui Zhao [Z].

More precisely, consider a divergence form elliptic operator

$$L := -\operatorname{div} A(X)\nabla, \tag{1.1}$$

defined in an open set $\Omega \subset \mathbb{R}^{n+1}$, where A is $(n+1)\times(n+1)$, real, L^∞, and satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \langle A(X)\xi, \xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(X)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}, \tag{1.2}$$

for some $\lambda > 0$, and for all $\xi \in \mathbb{R}^{n+1}$, and a.e. $X \in \Omega$.

Given an open set $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is everywhere regular in the sense of Weiner, and a divergence form operator L as above, we shall say that the Dirichlet problem is *BMO-solvable*[1] for L in Ω if for all continuous f with compact support on $\partial\Omega$, the solution u of the classical Dirichlet problem with data f satisfies the Carleson measure estimate

$$\sup_{x\in\partial\Omega,\, 0<r<r_0} \frac{1}{\sigma(\Delta(x,r))} \iint_{\Omega\cap B(x,r)} |\nabla u(Y)|^2\, \delta(Y)\, dY \leq C\|f\|^2_{BMO(\partial\Omega)}\,. \tag{1.3}$$

Here, $r_0 := 10\,\operatorname{diam}(\partial\Omega)$, σ is surface measure on $\partial\Omega$, $\delta(Y) := \operatorname{dist}(Y, \partial\Omega)$, and as usual $B(x,r)$ and $\Delta(x,r) := B(x,r)\cap\partial\Omega$ denote, respectively, the Euclidean ball in $\mathbb{R}^{n+1}$, and the surface ball on $\partial\Omega$, with center x and radius r.

For $X \in \Omega$, we let ω_L^X denote elliptic-harmonic measure for L with pole at X, and if the dependence on L is clear in context, we shall simply write ω^X.

The main result of this paper is the following. All terminology used in the statement of the theorem and not discussed already, will be defined precisely in the sequel.

Theorem 1.4. *Suppose that $\Omega \subset \mathbb{R}^{n+1}, n \geq 2$, is an open set, not necessarily connected, with Ahlfors-David Regular boundary. Let L be a divergence form elliptic operator defined on Ω. If the Dirichlet problem for L is BMO-solvable in Ω, then harmonic measure belongs to weak-A_∞ in the following sense: for every ball*

[1]It might be more accurate to refer to this property as "VMO-solvability", but BMO-solvability seems to be the established terminology in the literature. Under less austere circumstances, e.g., in a Lipschitz or (more generally) a Chord-arc domain, it can be seen that the two notions are ultimately equivalent; see [DKP] for a discussion of this point, although for the reader's convenience, we shall show below (see Remark 4.20) that in fact the equivalence holds for a domain with an ADR boundary, in the presence of "$S < N$ estimates" in L^p (thus, in particular, in the special case of the Laplacian, in a domain satisfying an interior Corkscrew condition). In the more general setting of our Theorem 1.4 this matter is not settled.

$B = B(x, r)$, with $x \in \partial\Omega$, and $0 < r < \operatorname{diam}(\partial\Omega)$, and for all $Y \in \Omega \setminus 4B$, harmonic measure $\omega_L^Y \in$ weak-$A_\infty(\Delta)$, where $\Delta := B \cap \partial\Omega$, and where the parameters in the weak-A_∞ condition are uniform in Δ, and in $Y \in \Omega \setminus 4B$.

As mentioned above, this result was established in [DKP], and in [Z], under the more restrictive assumption that Ω is Chord-arc, or 1-sided Chord-arc, respectively. The arguments of [DKP] and [Z] rely both explictly and implicitly on quantitative connectivity of the domain, more precisely, on the Harnack Chain condition (see Definition 1.11 below). The new contribution of the present paper is to dispense with all connectivity assumptions, both qualitative and quantitative. In particular, we do not assume the Harnack Chain condition, even within individual connected components of Ω. In this generality, our results are new even for the Laplacian.

We observe that we draw a slightly weaker conclusion than that of [DKP] (or [Z]), namely, weak-A_∞, as opposed to A_∞, but this is the best that can be hoped for in the absence of connectivity: indeed, clearly, the doubling property of harmonic measure may fail without connectivity. Moreover, even in a connected domain enjoying an "interior big pieces of Lipschitz domains" condition, and having an ADR boundary (and thus, for which harmonic measure belongs to weak-A_∞, by the main result of [BL]), the doubling property may fail in the absence of Harnack Chains; see [BL, Section 4] for a counter-example.

In the particular case that L is the Laplacian, we also obtain the following.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^{n+1}, n \geq 2$, be an open set, not necessarily connected, with Ahlfors-David Regular boundary, and in addition, suppose that Ω satisfies an interior Corkscrew condition (Definition 1.10), and that the Dirichlet problem is BMO-solvable for Laplace's equation in Ω. Then $\partial\Omega$ is uniformly rectifiable (Definition 1.9).*

The proof of the corollary is almost immediate: by Theorem 1.4, harmonic measure belongs to weak-A_∞ (even without the Corkscrew condition), so by the result of [HM][2], in the presence of the interior Corkscrew condition, $\partial\Omega$ is uniformly rectifiable.

We remark that the Corkscrew hypothesis is fairly mild, in the sense that if $\Omega = \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR set, then the Corkscrew condition holds automatically. We also remark that in the absence of the Corkscrew condition, the result of [HM], i.e., the conclusion of uniform rectifiability, may fail; a counter-example will appear in forthcoming work of the first author and J. M. Martell.

We also obtain a partial converse to Theorem 1.4.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^{n+1}, n \geq 2$ be an open set, not necessarily connected, with Ahlfors-David Regular (ADR) boundary. Let L be a divergence form elliptic operator defined on Ω, and suppose that elliptic-harmonic measure for L belongs to weak-A_∞ in the sense of the conclusion of Theorem 1.4. Then the Dirichlet problem for L is L^p-solvable[3] in Ω, for $p < \infty$ sufficiently large. In the special case that L is the Laplacian, the Dirichlet problem is BMO-solvable, provided also that Ω satisfies an interior Corkscrew condition, and that* $\operatorname{diam}(\partial\Omega) = \operatorname{diam}(\Omega)$[4].

[2] See also [HLMN] and [MT] for more general versions of the result of [HM].

[3] We shall say precisely what this means in the sequel; see Proposition 4.5, and Remark 4.8.

[4] I.e., either Ω is a bounded domain, or both Ω and $\partial\Omega$ are unbounded. We discuss a variant of our results, valid in the case that $\partial\Omega$ is bounded but Ω is unbounded, in Section 5.

As noted above, our main new contribution is Theorem 1.4, which establishes the direction BMO-solvability implies $\omega \in$ weak-A_∞; it is in that direction that the lack of connectivity is most problematic. By contrast, our proof of the opposite implication (i.e., Theorem 1.6) is a fairly routine adaptation of the corresponding arguments of [DKP] and of [FN]. On the other hand, let us point out that in Theorem 1.6, we have imposed an extra assumption, namely the Corkscrew condition. At present, we do not know whether the latter hypothesis is necessary to obtain the conclusion of Theorem 1.6 (although as remarked above, in its absence uniform rectifiability of $\partial\Omega$ may fail), nor do we know whether the conclusion of BMO solvability extends to the case of a general divergence form elliptic operator L.

To provide some further context for our results here, let us mention that recently, Kenig, Kirchheim, Pipher and Toro have shown in [KKiPT] that for a Lipschitz domain Ω, a weaker Carleson measure estimate, namely, a version of (1.3) in which the BMO norm of the boundary data is replaced by $\|u\|_{L^\infty(\Omega)}$, still suffices to establish that ω_L satisfies an A_∞ condition with respect to surface measure on $\partial\Omega$. Moreover, the argument of [KKiPT] carries over with minor changes to the more general setting of a uniform (i.e., 1-sided NTA) domain with Ahlfors-David regular boundary [HMT]. However, in contrast to our Theorem 1.4, to deduce absolute continuity of harmonic measure under the weaker L^∞ Carleson measure condition seems necessarily to require some sort of connectivity (such as the Harnack Chain condition enjoyed by uniform domains). Indeed, specializing to the case that L is the Laplacian, an example of Bishop and Jones [BiJ] shows that harmonic measure ω need *not* be absolutely continuous with respect to surface measure, even for domains with uniformly rectifiable boundaries, whereas the first named author of this paper, along with J. M. Martell and S. Mayboroda, have shown in [HMM] that uniform rectifiability of $\partial\Omega$ alone suffices to deduce the L^∞ version of (1.3) in the harmonic case[5] (and indeed, for solutions of certain other elliptic equations as well).

The paper is organized as follows. In the remainder of this section, we present some basic notations and definitions. In Section 2, we recall some known results from the theory of elliptic PDE. In Sections 3 and 4, we give the proofs of Theorems 1.4 and 1.6, respectively. Finally, in Section 5, we discuss the modifications that are needed in the case that Ω is an unbounded domain with bounded boundary.

1.1. **Notation and Definitions.**

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary.
- Given a closed set $E \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on E, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus E$).

[5] We remark that in fact, the L^∞ version of (1.3) actually characterizes uniform rectifiability: a converse to the result of [HMM] has recently been obtained in [GMT].

- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x,r)$ when the center x lies on E, or $B(X,r)$ when the center $X \in \mathbb{R}^{n+1} \setminus E$. A "surface ball" is denoted $\Delta(x,r) := B(x,r) \cap \partial\Omega$.
- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.
- Given a Euclidean or surface ball $B = B(X,r)$ or $\Delta = \Delta(x,r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa B := B(X, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.
- Given a (fixed) closed set $E \subset \mathbb{R}^{n+1}$, for $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \operatorname{dist}(X,E)$.
- We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\lfloor_E$ denote the "surface measure" on a closed set E of co-dimension 1.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\operatorname{int}(A)$ denote the interior of A.
- Given a Borel measure μ, and a Borel set A, with positive and finite μ measure, we set $\fint_A f d\mu := \mu(A)^{-1} \int_A f d\mu$.
- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean dyadic cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. If $\ell(I) = 2^{-k}$, then we set $k_I := k$.

Definition 1.7. (**ADR**) (aka *Ahlfors-David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is ADR if it is closed, and if there is some uniform constant C such that

$$\frac{1}{C}\, r^n \le \sigma(\Delta(x,r)) \le C\, r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{1.8}$$

where $\operatorname{diam}(E)$ may be infinite.

Definition 1.9. (**UR**) (aka *uniformly rectifiable*). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "Big Pieces of Lipschitz Images" of $\mathbb{R}^n$ ("BPLI"). This means that there are positive constants θ and M_0, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0, such that

$$H^n\Big(E \cap B(x,r) \cap \rho\left(\{z \in \mathbb{R}^n : |z| < r\}\right)\Big) \ge \theta\, r^n .$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [DS1]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [MMV] for the case $n = 1$, and [NToV] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [DS2, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions); see [DS1, DS2].

Definition 1.10. (**Corkscrew condition**). Following [JK], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* (more precisely, the *interior* Corkscrew condition) if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x,r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a "Corkscrew point" relative to Δ.

Definition 1.11. (**Harnack Chain condition**). Again following [JK], we say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Lambda\rho$, there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C \operatorname{diam}(B_k)$. The chain of balls is called a "Harnack Chain".

Definition 1.12. (**NTA and uniform domains**). Again following [JK], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA ("Non-tangentially accessible") if it satisfies the Harnack Chain condition, and if both Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the Corkscrew condition. If Ω merely satisfies the Harnack Chain condition and the interior (but not exterior) Corkscrew condition, then it is said to be a *uniform* (aka *1-sided NTA*) domain.

Definition 1.13. (**Chord-arc and 1-sided Chord-arc**). A domain $\Omega \subset \mathbb{R}^{n+1}$ is *Chord-arc* if it is an NTA domain with an ADR boundary; it is *1-sided Chord-arc* if it is a uniform (i.e., 1-sided NTA) domain with ADR boundary.

Definition 1.14. (A_∞, weak-A_∞, and weak-RH_q). Given an ADR set $E \subset \mathbb{R}^{n+1}$, and a surface ball $\Delta_0 := B_0 \cap E$, we say that a Borel measure μ defined on E belongs to $A_\infty(\Delta_0)$ if there are positive constants C and θ such that for each surface ball $\Delta = B \cap E$, with $B \subseteq B_0$, we have

$$\mu(F) \leq C \left(\frac{\sigma(F)}{\sigma(\Delta)} \right)^\theta \mu(\Delta), \qquad \text{for every Borel set } F \subset \Delta. \tag{1.15}$$

Similarly, we say that $\mu \in$ weak-$A_\infty(\Delta_0)$ if for each surface ball $\Delta = B \cap E$, with $2B \subseteq B_0$,

$$\mu(F) \leq C \left(\frac{\sigma(F)}{\sigma(\Delta)} \right)^\theta \mu(2\Delta), \qquad \text{for every Borel set } F \subset \Delta. \tag{1.16}$$

We recall that, as is well known, the condition $\mu \in$ weak-$A_\infty(\Delta_0)$ is equivalent to the property that $\mu \ll \sigma$ in Δ_0, and that for some $q > 1$, the Radon-Nikodym derivative $k := d\mu/d\sigma$ satisfies the weak reverse Hölder estimate

$$\left(\fint_\Delta k^q d\sigma \right)^{1/q} \lesssim \fint_{2\Delta} k \, d\sigma \approx \frac{\mu(2\Delta)}{\sigma(\Delta)}, \quad \forall\, \Delta = B \cap E, \text{ with } 2B \subseteq B_0. \tag{1.17}$$

We shall refer to the inequality in (1.17) as an "RH_q" estimate, and we shall say that $k \in RH_q(\Delta_0)$ if k satisfies (1.17).

2. Preliminaries

In this section, we record some known estimates for elliptic harmonic measure ω_L associated to a divergence form operator L as in (1.1) and (1.2), and for solutions of the equation $Lu = 0$, in an open set $\Omega \subset \mathbb{R}^{n+1}$ with an ADR boundary. In the sequel, we shall always assume that the ambient dimension $n + 1 \geq 3$. We recall that, as a consequence of the ADR property, every point on $\partial\Omega$ is regular in the sense of Wiener (see, e.g., [HLMN, Remark 3.26, Lemma 3.27]).

Lemma 2.1 (Bourgain [Bo]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and suppose that $\partial\Omega$ is n-dimensional ADR. Then there are uniform constants $c \in (0,1)$ and $C \in (1,\infty)$, depending only on n, ADR, and the ellipticity parameter λ, such that for every $x \in \partial\Omega$, and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$, then*

$$\omega_L^Y(\Delta(x,r)) \geq 1/C > 0\,. \tag{2.2}$$

We refer the reader to [Bo, Lemma 1] for the proof in the case that L is the Laplacian, but the proof is the same for a general uniformly elliptic divergence form operator.

We note for future reference that in particular, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$, and $\Delta_X := \partial\Omega \cap B(\hat{x}, 10\delta(X))$, then for a slightly different uniform constant $C > 0$,

$$\omega_L^X(\Delta_X) \geq 1/C\,. \tag{2.3}$$

Indeed, the latter bound follows immediately from (2.2), and the fact that we can form a Harnack Chain connecting X to a point Y that lies on the line segment from X to $\hat{x}$, and satisfies $|Y - \hat{x}| = c\delta(X)$.

As a consequence of Lemma 2.1, we have the following (see, e.g., [HKM, Ch. 6]).

Corollary 2.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and suppose that $\partial\Omega$ is n-dimensional ADR. For $x \in \partial\Omega$, and $0 < r < \operatorname{diam}\partial\Omega$, let u be a non-negative solution of $Lu = 0$ in $\Omega \cap B(x,2r)$, which vanishes continuously on $\Delta(x,2r) = B(x,2r) \cap \partial\Omega$. Then for some $\alpha > 0$,*

$$u(Y) \leq C\left(\frac{\delta(Y)}{r}\right)^{\alpha} \frac{1}{|B(x,2r)|} \iint_{B(x,2r)\cap\Omega} u\,, \qquad \forall\, Y \in B(x,r) \cap \Omega\,, \tag{2.5}$$

where the constants C and α depend only on n, ADR and λ.

3. Proof of Theorem 1.4: BMO-solvability implies $\omega \in$ weak-A_∞

The basic outline of the proof follows that of [DKP], but the lack of Harnack Chains requires in addition some slightly delicate geometric arguments inspired in part by the work of Bennewitz and Lewis [BL].

We begin by recalling the following deep fact, established in [BL]. Given a point $X \in \Omega$, let $\hat{x} \in \partial\Omega$ be a "touching point" for the ball $B(X, \delta(X))$, i.e., $|X - \hat{x}| = \delta(X)$ (if there is more than one such point, we just pick one). Set

$$\Delta_X := \Delta(\hat{x}, 10\delta(X))\,. \tag{3.1}$$

Lemma 3.2. *Let $\partial\Omega$ be ADR, and suppose that there are constants $c_0, \eta \in (0,1)$, such that for each $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, and for every Borel set $F \subset \Delta_X$,*

$$\sigma(F) \geq (1-\eta)\sigma(\Delta_X) \implies \omega^X(F) \geq c_0\,. \tag{3.3}$$

Then $\omega^Y \in$ weak-$A_\infty(\Delta)$, where $\Delta = B \cap \partial\Omega$, for every ball $B = B(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and for all $Y \in \Omega \setminus 4B$. Moreover, the parameters in the weak-A_∞ condition depend only on n, ADR, η, c_0, and the ellipticity parameter λ of the divergence form operator L.

Remark 3.4. Lemma 3.2 is not stated explicitly in this form in [BL], but may be gleaned readily from the combination of [BL, Lemma 2.2] and its proof, and [BL, Lemma 3.1]. We mention also that the paper [BL] treats explictly only the case that L is the Laplacian, but the proof of [BL, Lemma 2.2] carries over verbatim to the case of a general uniformly elliptic divergence form operator with real coefficients, while [BL, Lemma 3.1] is a purely real variable result.

Given the BMO-solvability estimate (1.3), it suffices to verify the hypotheses of Lemma 3.2, with η and c_0 depending only on n, ADR, λ, and the constant C in (1.3). To this end, let $X \in \Omega$, $\delta(X) < \operatorname{diam}(\partial\Omega)$, and for notational convenience, set

$$r := \delta(X)\,.$$

We choose $\hat{x} \in \partial\Omega$ so that $|X - \hat{x}| = r$, and let $a \in (0, \pi/10000)$ be a sufficiently small number to be chosen depending only on n and ADR. We then define Δ_X as in (3.1), and set

$$B_X := B(\hat{x}, 10r)\,, \quad B'_X := B(\hat{x}, ar)\,, \quad \Delta'_X := \Delta(\hat{x}, ar)\,. \tag{3.5}$$

We make the following pair of claims.

Claim 1. For a small enough, depending only on n and ADR, there is a constant $\beta > 0$ depending only on n, a, ADR and λ, and a ball $B_1 := B(x_1, ar) \subset B_X$, with $x_1 \in \partial\Omega$, such that $\operatorname{dist}(B'_X, B_1) \geq 5ar$, and

$$\omega_L^X(\Delta_1) \geq \beta\, \omega_L^X(\Delta_X)\,, \tag{3.6}$$

where $\Delta_1 := B_1 \cap \partial\Omega$.

Claim 2. Suppose that u is a non-negative solution of $Lu = 0$ in Ω, vanishing continuously on $2\Delta'_X$, with $\|u\|_{L^\infty(\Omega)} \leq 1$. Then for every $\varepsilon > 0$,

$$u(X) \leq C_\varepsilon \left(\frac{1}{\sigma(\Delta_X)} \iint_{B_X \cap \Omega} |\nabla u(Y)|^2 \delta(Y)\, dY \right)^{1/2} + C\varepsilon^\alpha\,, \tag{3.7}$$

where $\alpha > 0$ is the Hölder exponent in Corollary 2.4.

Momentarily taking these two claims for granted, we now follow the argument in [DKP], with some minor modifications, in order to establish the hypotheses of Lemma 3.2. Let B_1 and Δ_1 be as in Claim 1. Let $F \subset \Delta_X$ be a Borel set satisfying the first inequality in (3.3), for some small $\eta > 0$. If we choose η small enough, depending only on n, ADR, and the constant a in the definition of B'_X, then by inner regularity of σ, there is a closed set $F_1 \subset F \cap \Delta_1$ such that

$$\sigma(F_1) \geq (1 - \sqrt{\eta})\, \sigma(\Delta_1)\,.$$

Set $A_1 := \Delta_1 \setminus F_1$ (so that A_1 is relatively open in $\partial\Omega$), and define

$$f := \max\left(0, 1 + \gamma \log \mathcal{M}(1_{A_1})\right),$$

where $\gamma > 0$ is a small number to be chosen, and $\mathcal{M}$ is the usual Hardy-Littlewood maximal operator on $\partial\Omega$. Note that[6]

$$0 \le f \le 1\,, \qquad \|f\|_{BMO(\partial\Omega)} \le C\gamma\,, \qquad 1_{A_1}(y) \le f(y)\,,\ \forall\, y \in \partial\Omega. \tag{3.8}$$

Note also that if $z \in \partial\Omega \setminus 2B_1$, then

$$\mathcal{M}(1_{A_1})(z) \lesssim \frac{\sigma(A_1)}{\sigma(\Delta_1)} \lesssim \sqrt{\eta}\,,$$

where the implicit constants depend only on n and ADR. Thus, if η is chosen small enough depending on γ, then $1 + \gamma \log \mathcal{M}(1_{A_1})$ will be negative, hence $f \equiv 0$, on $\partial\Omega \setminus 2B_1$.

In order to work with continuous data, we shall require the following.

Lemma 3.9. *There exists a collection of continuous functions $\{f_s\}_{0<s<ar/1000}$, defined on $\partial\Omega$, with the following properties.*

1. $0 \le f_s \le 1$, *for each s.*
2. $\operatorname{supp}(f_s) \subset 3B_1 \cap \partial\Omega$.
3. $1_{A_1}(z) \le \liminf_{s\to 0} f_s(z)$, *for every $z \in \partial\Omega$.*
4. $\sup_s \|f_s\|_{BMO(\partial\Omega)} \le C\|f\|_{BMO(\partial\Omega)} \lesssim \gamma$, *where $C = C(n, ADR)$.*

The proof is based on a standard mollification of the function f constructed above. We defer the routine proof to the end of this section.

Let u_s be the solution of the Dirichlet problem for the equation $Lu_s = 0$ in Ω, with data f_s. Note that f_s vanishes on $2\Delta'_X$, by the distance condition in Claim 1 and Lemma 3.9-(2). Then, for a small $\varepsilon > 0$ to be chosen momentarily, by Lemma 3.9, Fatou's lemma, and Claim 2, we have

$$\omega_L^X(A_1) \le \int_{\partial\Omega} \liminf_{s\to 0} f_s\, d\omega^X \le \liminf_{s\to 0} u_s(X) \le C_\varepsilon\gamma + C\varepsilon^\alpha\,, \tag{3.10}$$

where in the last step we have used (3.7), (1.3), and Lemma 3.9-(4). Combining (3.10) with (2.3), we find that

$$\omega_L^X(A_1) \le (C_\varepsilon\gamma + C\varepsilon^\alpha)\,\omega_L^X(\Delta_X)\,. \tag{3.11}$$

Next, we set $A := \Delta_X \setminus F$, and observe that by definition of A and A_1, along with Claim 1, and (3.11),

$$\omega_L^X(A) \le \omega_L^X(\Delta_X \setminus \Delta_1) + \omega_L^X(A_1) \le (1 - \beta + C_\varepsilon\gamma + C\varepsilon^\alpha)\,\omega_L^X(\Delta_X)\,.$$

We now choose first $\varepsilon > 0$, and then $\gamma > 0$, so that $C_\varepsilon\gamma + C\varepsilon^\alpha < \beta/2$, to obtain that

$$\omega_L^X(F) \ge \frac{\beta}{2}\,\omega_L^X(\Delta_X) \ge c\beta\,,$$

where in the last step we have used (2.3). We therefore conclude that (3.3) holds.

It now remains only to establish the two claims, and to prove Lemma 3.9.

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[6]The BMO estimate for f in (3.8) follows from the fact that $\mathcal{M}(1_A)^{1/2}$ is an A_1 weight with A_1 constant depending only on dimension, and that the log of an A_1 weight w belongs to BMO, with BMO norm depending only on the A_1 constant of w; see, e.g., [GR, Ch. 2, Theorems 3.3 and 3.4].

Proof of Claim 1. By translation and rotation, we may suppose without loss of generality that $\hat{x} = 0$, and that the line segment joining $\hat{x}$ to X is purely vertical, thus, $X = re_{n+1}$, where as usual $e_{n+1} := (0, ..., 0, 1)$. Let $\Gamma, \Gamma', \Gamma''$ denote, respectively, the open inverted vertical cones with vertex at X having angular apertures $200a$, $100a$, and $20a$, respectively (recall that $a < \pi/10000$). Then $B'_X \subset \Gamma''$ (where B'_X is defined in (3.5)). Recalling that $r = \delta(X)$, we let $B_0 := B(X, r)$ denote the open "touching ball", so that $B_0 \cap \partial\Omega = \emptyset$, and define a closed annular region $R_0 := \overline{5B_0} \setminus B_0$. We now consider two cases:

Case 1. $\partial\Omega \cap (R_0 \setminus \Gamma)$ is non-empty. In this case, we let x_1 be the point in $\partial\Omega \cap (R_0 \setminus \Gamma)$ that is closest to X (if there is more than one such point, we just pick one). Then by construction $r \leq |X - x_1| \leq 5r$, and the ball $B_1 = B(x_1, ar)$ misses Γ', hence $\operatorname{dist}(B_1, B'_X) \geq \operatorname{dist}(B_1, \Gamma'') > 5ar$. Moreover, since x_1 is the closest point to X, setting $\rho := |X - x_1|$, we have that $\Omega' \cap \partial\Omega = \emptyset$, where

$$\Omega' := (B(X, \rho) \setminus \overline{\Gamma}) \cup B_0 \,.$$

Note that Ω' satisfies the Harnack Chain condition, with constants depending only upon n and a. Consequently, we may construct a Harnack Chain within the subdomain $\Omega' \subset \Omega$, connecting X to a point $Y \in B(x_1, car) \cap \Omega'$, with $\delta(Y) \geq car/2$, where c is the constant in Lemma 2.1. Thus, by Harnack's inequality and Lemma 2.1,

$$\omega_L^X(\Delta_1) \gtrsim \omega_L^Y(\Delta_1) \geq 1/C \,.$$

Since $\omega_L^X(\Delta_X) \leq 1$, we obtain (3.6), and thus Claim 1 holds in the present case.

Case 2. $\partial\Omega \cap (R_0 \setminus \Gamma) = \emptyset$. By ADR, we have that

$$\sigma(\Delta(0, 10ar)) \leq C(ar)^n \,, \qquad \sigma(B(X, 4r) \cap \partial\Omega) \geq r^n/C \,.$$

Thus, for a chosen small enough, depending only on n and ADR, we see that the set $\partial\Omega \cap (B(X, 4r) \setminus B(0, 10ar))$ is non-empty. Consequently, under the scenario of Case 2,

$$\partial\Omega \cap \left(\overline{B(X, 4r)} \setminus B(0, 10ar)\right) \subset \Gamma \,.$$

Define

$$\theta_0 := \min\{\theta \in [0, 200a) : \partial\Omega \cap (\overline{B(X, 4r)} \setminus B(0, 10ar)) \subset \Gamma_\theta\} \,,$$

where Γ_θ is the inverted cone with vertex at X of angular aperture θ. It is not hard to see that since $n \geq 2$, we necessarily have $\theta_0 > 0$, as a consequence of the ADR property; see, e.g., [DS1, Lemma 5.8]. Then by construction, there is a point

$$x_1 \in \partial\Gamma_{\theta_0} \cap \partial\Omega \cap \left(\overline{B(X, 4r)} \setminus B(0, 10ar)\right).$$

Note that $B_1 = B(x_1, ar)$ misses $B(0, 9ar)$, so that in particular, $\operatorname{dist}(B_1, B'_X) > 5ar$. Moreover, $\Omega' \cap \partial\Omega = \emptyset$, where now

$$\Omega' := \left((B(X, 4r) \setminus \overline{\Gamma_{\theta_0}}) \cup B_0\right) \setminus \overline{B(0, 10ar)} \,.$$

Thus, as in Case 1, there is a point $Y \in B(x_1, car) \cap \Omega'$, with $\delta(Y) > cr/2$, which may be joined to X via a Harnack Chain within the subdomain $\Omega' \subset \Omega$, as follows: starting at Y, we move on a great circle on the sphere $\partial B(X, R)$, where $R = |X - Y|$, and then horizontally until we reach X; since the smallest ball in this Harnack Chain has radius on the order of car, we can see that the number of balls will depend only on a, n, and implicitly on ADR (since c depends on n and ADR).

Thus, by Harnack's inequality, Lemma 2.1, and the fact that $\omega_L^X(\Delta_X) \le 1$, we again obtain (3.6). Claim 1 therefore holds in all cases. □

Proof of Claim 2. As in the proof of Claim 1, we may assume by translation and rotation that $\hat{x} = 0$, and that $X = re_{n+1}$, with $r = \delta(X)$. Let Γ denote the *upward* open vertical cone with vertex at 0, of angular aperture $\pi/100$. We let S denote the spherical cap inside Γ, i.e., $S := S^n \cap \Gamma$ (recall that our ambient dimension is $n+1$). Then by Harnack's inequality, letting μ denote surface measure on the unit sphere, we have

$$u(X) \lesssim \int_S u(r\xi)\, d\mu(\xi) = \int_S \Big(u(r\xi) - u(\varepsilon r\xi)\Big)\, d\mu(\xi) \; + \; O(\varepsilon^\alpha) \; =: \; I + O(\varepsilon^\alpha)\,,$$

where we have used Corollary 2.4 to estimate the "big-O" term. In turn,

$$|I| = \left| \int_S \int_{\varepsilon r}^r \frac{\partial}{\partial t}(u(t\xi))\, dt\, d\mu(\xi) \right| \le (\varepsilon r)^{-n} \iint_{\Gamma\cap R_\varepsilon} |\nabla u(Y)|\, dY\,,$$

where $R_\varepsilon := B(0,r) \setminus B(0,\varepsilon r)$, and we have used polar co-ordinates in $n+1$ dimensions. We then have

$$|I| \lesssim (\varepsilon r)^{-n} r^{(n+1)/2} \left(\iint_{\Gamma\cap R_\varepsilon} |\nabla u(Y)|^2\, dY \right)^{1/2}$$
$$\lesssim (\varepsilon)^{-n-1/2} r^{-n/2} \left(\iint_{B(0,r)\cap\Omega} |\nabla u(Y)|^2\, \delta(Y)\, dY \right)^{1/2},$$

where we have used that by construction, $\Gamma\cap R_\varepsilon \subset B(0,r)\cap\Omega$, with $\delta(Y) \approx |Y| \ge \varepsilon r$ in $\Gamma \cap R_\varepsilon$. Estimate (3.7) now follows, by ADR and the definition of B_X. □

Proof of Lemma 3.9. Let $\zeta \in C_0^\infty(\mathbb{R}^{n+1})$, with

$$\operatorname{supp}(\zeta) \subset B(0,1)\,, \quad \zeta \equiv 1 \text{ on } B(0,1/2)\,, \quad 0 \le \zeta \le 1\,.$$

Given $s \in (0, ar/1000)$, and $z, y \in \partial\Omega$, set

$$\Lambda_s(z,y) := b(z,s)^{-1} \zeta(s^{-1}(z-y))\,,$$

where

$$b(z,s) := \int_{\partial\Omega} \zeta(s^{-1}(z-y))\, d\sigma(y) \; \approx s^n\,, \tag{3.12}$$

uniformly in $z \in \partial\Omega$, by the ADR property. Furthermore,

$$\int_{\partial\Omega} \Lambda_s(z,y)\, d\sigma(y) \equiv 1\,, \qquad \forall\, z \in \partial\Omega\,.$$

We now define

$$f_s(z) := \int_{\partial\Omega} \Lambda_s(z,y)\, f(y)\, d\sigma(y)\,,$$

so that f_s is continuous, by construction. Let us now verify (1)-(4) of Lemma 3.9. We obtain (1) immediately, by (3.8), and the properties of Λ_s, while (2) follows directly from the smallness of s and the fact that $\operatorname{supp}(f) \subset 2B_1 \cap \partial\Omega$. Next, observe that since A_1 is a relatively open set in $\partial\Omega$, we have that for every $z \in \partial\Omega$,

$$1_{A_1}(z) \le \liminf_{s\to 0} \int_{\partial\Omega} \Lambda_s(z,y)\, 1_{A_1}(y)\, d\sigma(y) \le \liminf_{s\to 0} f_s(z)\,,$$

by the last inequality in (3.8). Hence (3) holds.

To prove (4), we observe that the second inequality is simply a re-statement of the second inequality in (3.8), so it suffices to show that

$$\|f_s\|_{BMO(\partial\Omega)} \lesssim \|f\|_{BMO(\partial\Omega)}\,, \qquad \text{uniformly in } s\,. \tag{3.13}$$

To this end, we fix a surface ball $\Delta = \Delta(x, r)$, and we consider two cases.

Case 1: $s \geq r$. In this case, set $c := \fint_{\Delta(x,2s)} f$, so that by ADR, (3.12) and the construction of Λ_s,

$$\fint_\Delta |f_s - c|\, d\sigma \,\lesssim\, \fint_\Delta \fint_{\Delta(x,2s)} |f - c|\, d\sigma d\sigma \,\lesssim\, \|f\|_{BMO(\partial\Omega)}\,.$$

Case 2: $s < r$. In this case, set $c := \fint_{2\Delta} f$. Then by Fubini's Theorem,

$$\fint_\Delta |f_s(z) - c|\, d\sigma(z) \,\lesssim\, \fint_{2\Delta} |f(y) - c| \int_{\partial\Omega} \Lambda_s(z, y)\, d\sigma(z)\, d\sigma(y) \,\lesssim\, \|f\|_{BMO(\partial\Omega)}\,,$$

where again we have used ADR, (3.12) and the compact support property of $\Lambda_s(z, y)$.

Since these bounds are uniform over all $x \in \partial\Omega$, and $r \in (0, \mathrm{diam}(\partial\Omega))$, we obtain (3.13). □

4. Proof of Theorem 1.6: $\omega \in$ weak-A_∞ implies L^p and BMO-solvability

In this section, we suppose that Ω is an open set with ADR boundary $\partial\Omega$, and that for every ball $B_0 = B(x_0, r)$, with $x_0 \in \partial\Omega$, and $0 < r < \mathrm{diam}(\partial\Omega)$, and for all $Y \in \Omega \setminus 4B_0$, elliptic-harmonic measure $\omega_L^Y \in$ weak-$A_\infty(\Delta_0)$, where $\Delta_0 := B_0 \cap \partial\Omega$. Thus, $\omega_L^Y \ll \sigma$ in Δ, and the Poisson kernel $k^Y := d\omega_L/d\sigma$ satisfies the weak reverse Hölder condition (1.17), for some uniform $q > 1$. In our proof of BMO-solvability (but not for L^p solvability), we shall also require, at precisely one point in the argument, that the Corkscrew condition (Definition 1.10) is satisfied in Ω. Even in the absence of the Corkscrew condition, it may happen that there is a Corkscrew point X_Δ relative to some particular Δ (e.g., for every $X \in \Omega$, this is true for the surface ball Δ_X as in (3.1), with X itself serving as a Corkscrew point), and in this case, we have the following consequence of the weak-RH_q estimate:

$$\left(\fint_\Delta \left(k^{X_\Delta}\right)^q d\sigma\right)^{1/q} \,\leq\, C\,\sigma(\Delta)^{-1}\,. \tag{4.1}$$

Indeed, one may cover Δ by a collection of surface balls $\{\Delta' = B' \cap \partial\Omega\}$, in such a way that $X_\Delta \in \Omega \setminus 4B'$, but each Δ' has radius comparable to that of Δ (hence $\sigma(\Delta') \approx \sigma(\Delta)$, by the ADR property), depending on the constant in the Corkscrew condition, and such that the cardinality of the collection $\{\Delta'\}$ is uniformly bounded; one may then readily derive (4.1) by applying (1.17) in each Δ', and using the crude estimate that $\omega^{X_\Delta}(2\Delta')/\sigma(\Delta') \leq \sigma(\Delta')^{-1} \approx \sigma(\Delta)^{-1}$.

Our first step is to establish an L^p solvability result. To this end, we define non-tangential "cones" and maximal functions, as follows. First, we fix a collection of standard Whitney cubes covering Ω, and we denote this collection by $\mathcal{W}$. Given $x \in \partial\Omega$, set

$$\mathcal{W}(x) := \{I \in \mathcal{W} :\ \mathrm{dist}(x, I) \leq 100\,\mathrm{diam}(I) < 1000\,\mathrm{diam}(\partial\Omega)\}\,, \tag{4.2}$$

and define the (possibly disconnected) non-tangential "cone" with vertex at x by

$$\Upsilon(x) \,:=\, \cup_{I \in \mathcal{W}(x)}\, I\,. \tag{4.3}$$

For a continuous u defined on Ω, the non-tangential maximal function of u is defined by

$$N_*u(x) := \sup_{Y\in\Upsilon(x)} |u(Y)|\,. \tag{4.4}$$

Recall that $\mathcal{M}$ denotes the (non-centered) Hardy-Littlewood maximal operator on $\partial\Omega$. We have the following.

Proposition 4.5. *Suppose that there is a $q > 1$, such that (1.17) holds for the Poisson kernel k^Y, for every surface ball $\Delta = B \cap \partial\Omega$, centered on $\partial\Omega$, provided $Y \in \Omega \setminus 4B$. Given g continuous with compact support on $\partial\Omega$, let u be the elliptic-harmonic measure solution of the Dirichlet problem for L with data g. Then for $p = q/(q-1)$, and for all $x \in \partial\Omega$*

$$N_*u(x) \lesssim \left(\mathcal{M}(|g|^p)(x)\right)^{1/p}\,. \tag{4.6}$$

Thus, for all $s > p$, the Dirichlet problem is L^s-solvable, i.e.,

$$\|N_*u\|_{L^s(\partial\Omega)} \leq C_s\, \|g\|_{L^s(\partial\Omega)}\,. \tag{4.7}$$

Remark 4.8. As is well known, the weak-RH_q estimate (1.17) is self-improving, i.e., weak-RH_q implies weak-$RH_{q+\varepsilon}$, for some $\varepsilon > 0$, thus, in particular, one may self-improve (4.7) to the case $s = p$. We also remark that our definition of L^p-solvability of the Dirichlet problem entails only a non-tangential maximal function estimate, and does not address the issue of non-tangential convergence a.e. to the data. The latter would seem to require that the Whitney boxes in the definition of $\mathcal{W}(x)$ (see (4.2)) exist at infinitely many scales, for a.e. $x \in \partial\Omega$; e.g., the interior Corkscrew condition would be more than enough to guarantee this property.

Proof of Proposition 4.5. Splitting the data g into its positive and negative parts, we may suppose without loss of generality that $g \geq 0$, hence also $u \geq 0$. Let $x \in \partial\Omega$, fix $Y \in \Upsilon(x)$, and let $\hat{y} \in \partial\Omega$ be a touching point, i.e., $|Y - \hat{y}| = \delta(Y)$. Set

$$\Delta_Y^* := \Delta(\hat{y}, 1000\delta(Y))\,, \qquad B_Y^* := B(\hat{y}, 1000\delta(Y))\,,$$

and note that $x \in \Delta_Y^*$. Define a continuous partition of unity $\sum_{k\geq 0} \varphi_k \equiv 1$ on $\partial\Omega$, such that $0 \leq \varphi_k \leq 1$ for all $k \geq 0$, with

$$\operatorname{supp}(\varphi_0) \subset 4\Delta_Y^*\,, \quad \operatorname{supp}(\varphi_k) \subset R_k := 2^{k+2}\Delta_Y^* \setminus 2^k\Delta_Y^*\,,\ k \geq 1\,, \tag{4.9}$$

set $g_k := g\varphi_k$, and let u_k be the solution of the Dirichlet problem with data g_k (it may be that for some k, the boundary annulus R_k is empty; for such k, we have that g_k, and hence u_k, are identically zero). Thus, $u = \sum_{k\geq 0} u_k$ in Ω. By construction, Y is a Corkscrew point for $4\Delta_Y^*$, and $x \in 4\Delta_Y^*$, hence

$$u_0(Y) \leq \int_{\partial\Omega} g_0\, k^Y\, d\sigma \lesssim \left(\fint_{4\Delta_Y^*} g^p\, d\sigma\right)^{1/p} \lesssim \left(\mathcal{M}(g^p)(x)\right)^{1/p}\,,$$

where in the next to last step we have used (4.1).

Next, we claim that for $k \geq 1$,

$$u_k(Y) \lesssim 2^{-k\alpha}\left(\mathcal{M}(g^p)(x)\right)^{1/p}\,. \tag{4.10}$$

Given this claim, we may sum in k to obtain (4.6). Thus, it suffices to verify (4.10). To this end, we set

$$\mathcal{W}_k := \{I \in \mathcal{W} :\ I \text{ meets } 2^{k-1}B_Y^*\}\,,$$

and for each $I \in \mathcal{W}_k$, we fix a point $X_I \in I \cap 2^{k-1}B_Y^*$, and we define

$$\Delta_I := \Delta_{X_I},$$

as in (3.1), with $X = X_I$. We now choose a collection of balls $\{B_i\}_{1\leq i\leq N}$, with N depending only on n and ADR, and corresponding surface ball $\Delta_i := B_i \cap \partial\Omega$, such that $R_k \subset \cup_{i=1}^N \Delta_i$, and such that for each $i = 1, 2, ..., N$, with $r := \delta(Y)$,

$$r_{B_i} \approx 2^k r \quad \text{and} \quad 2^{k-1}B_Y^* \subset \mathbb{R}^{n+1} \setminus 4B_i .$$

Then by definition of R_k (see (4.9)), and the ADR property,

$$\begin{aligned} (4.11) \quad u_k(X_I) \leq \int_{R_k} g\, d\omega^{X_I} &\lesssim (2^k r)^n \left(\fint_{2^{k+2}\Delta_Y^*} g^p\, d\sigma \right)^{1/p} \left(\sum_{i=1}^N \fint_{\Delta_i} \left(k^{X_I}\right)^q d\sigma \right)^{1/q} \\ &\lesssim \left(\fint_{2^{k+2}\Delta_Y^*} g^p\, d\sigma \right)^{1/p} \lesssim \left(\mathcal{M}(g^p)(x)\right)^{1/p}, \end{aligned}$$

where in the next-to-last step we have used the weak-RH_q estimate (1.17) in each Δ_i, along with the crude bound $\omega^{X_I}(2\Delta_i) \leq 1$, and the fact that each Δ_i has radius $r_{\Delta_i} \approx 2^k r$.

Next, by Corollary 2.4,

$$\begin{aligned} (4.12) \quad u_k(Y) &\lesssim 2^{-k\alpha} \frac{1}{|2^{k-1}B_Y^*|} \iint_{2^{k-1}B_Y^* \cap \Omega} u_k(Z)\, dZ \\ &\lesssim 2^{-k\alpha} \frac{1}{(2^k r)^{n+1}} \sum_{I \in \mathcal{W}_k} \iint_I u_k(Z)\, dZ \approx 2^{-k\alpha} \frac{1}{(2^k r)^{n+1}} \sum_{I \in \mathcal{W}_k} |I|\, u_k(X_I) \\ &\lesssim 2^{-k\alpha} \left(\mathcal{M}(g^p)(x)\right)^{1/p}, \end{aligned}$$

where in the last two lines we have used Harnack's inequality in the Whitney box I, and then (4.11), and the fact that the Whitney boxes in $\mathcal{W}_k$ are non-overlapping and are all contained in a Euclidean ball of radius $\approx 2^k r$. □

With Proposition 4.5 in hand, we turn to the proof of BMO-solvability. Our approach here follows that in [DKP], which in turn is based on that of [FN]. We now suppose that the Corkscrew condition holds in Ω, and that L is the Laplacian. In this case, by the result of [HM] (see also [HLMN] and [MT]), the weak-A_∞ condition for harmonic measure implies that $\partial\Omega$ is uniformly rectifiable, and thus, by a result of [HMM2], we have the following square function/non-tangential maximal function estimate: for u harmonic in Ω,

$$(4.13) \qquad \int_{\partial\Omega} (\mathcal{S}u)^p\, d\sigma \leq C_p \int_{\partial\Omega} (N_* u)^p\, d\sigma, \qquad 1 < p < \infty,$$

where C_p depends also on n, and the UR constants for $\partial\Omega$ (and thus on the ADR, Corkscrew and weak-A_∞ constants), and where

$$\mathcal{S}u(x) := \left(\iint_{\Upsilon(x)} |\nabla u(Y)|^2\, \delta(Y)^{1-n}\, dY \right)^{1/2}.$$

We recall that $\Upsilon(x)$ and $N_* u$ were defined in (4.3) and (4.4).

Remark 4.14. In fact, the interested reader may observe that the proof below does not require, *per se*, either the Corkscrew assumption or that L be the Laplacian, but only that the "$S < N$" bounds (4.13) hold.

Remark 4.15. There is a technical point that we wish to address before proceeding further. Recall that in the BMO solvability part of Theorem 1.6, we have imposed the further assumption that $\operatorname{diam}(\partial\Omega) = \operatorname{diam}(\Omega)$. It is in the nature of the space BMO that constant data should not be detected by the Carleson measure expression in (1.3); i.e., constant data $f \equiv c$ should produce a constant elliptic-harmonic measure solution u. If either $\operatorname{diam}(\Omega) < \infty$, or $\operatorname{diam}(\Omega) = \operatorname{diam}(\partial\Omega) = +\infty$, this is trivially true: elliptic-harmonic measure is a probability measure in these cases, so the elliptic-harmonic measure solution with constant data is equal to the same constant in Ω. However, if $\partial\Omega$ is bounded, but Ω is unbounded, then elliptic-harmonic measure is a probability measure only if we consider it to exist on $\partial_\infty\Omega$, that is, the boundary of Ω with point at infinity appended[7]. Thus, in the case of an unbounded domain with bounded boundary, we are forced to make certain modifications, which we discuss in Section 5.

Now consider a ball $B = B(x,r)$, with $x \in \partial\Omega$, and $0 < r < \operatorname{diam}(\Omega)$, and corresponding surface ball $\Delta = B \cap \partial\Omega$. Let f be continuous with compact support on $\partial\Omega$, and set $h := f - c_\Delta$, where $c_\Delta := \fint_{40\Delta} f$. With a slight abuse of notation, we let u denote the elliptic-harmonic measure solution with data h, and observe that this u differs from our original solution u by a constant. We construct a smooth partition of unity $\sum_{k\geq 0} \varphi_k \equiv 1$ on $\partial\Omega$ as before, but now with 10Δ in place of Δ_Y^*. Set $h_k := h\varphi_k$, and let u_k be the solution to the Dirichlet problem with data h_k. Set

$$\mathcal{W}_B := \{I \in \mathcal{W} : I \text{ meets } B\}, \qquad \mathcal{W}_B^j := \{I \in \mathcal{W}_B : \ell(I) = 2^{-j}\}, \tag{4.16}$$

and for each $I \in \mathcal{W}_B$, fix a point $X_I \in I \cap B$. As above, let $\Delta_I := \Delta_{X_I}$ be defined as in (3.1), and note that by construction,

$$z \in \Delta_I \quad \Longrightarrow \quad I \in \mathcal{W}(z),$$

where $\mathcal{W}(z)$ is defined in (4.2). Consequently, given $z \in \partial\Omega$,

$$\sum_{I:\, z\in\Delta_I} \iint_I |\nabla u_0(Y)|^2\, \delta(Y)^{1-n}\, dY \lesssim (\mathcal{S}u_0(z))^2 . \tag{4.17}$$

Let us note also that

$$I \in \mathcal{W}_B \quad \Longrightarrow \quad \Delta_I \subset \Delta(x, Cr) =: \Delta^* , \tag{4.18}$$

[7] Recall that our ambient dimension is $n+1$, with $n \geq 2$, so that the fundamental solution of L decays to 0 at infinity.

for C chosen large enough. Let us fix $p \in [2,\infty)$ such that the Poisson kernel satisfies an L^q reverse Hölder estimate for $q = p/(p-1)$. We then have

$$\begin{aligned}\iint_{B\cap\Omega} |\nabla u_0(Y)|^2\, \delta(Y)\, dY &\lesssim \sum_{I\in\mathcal{W}_B} \iint_I |\nabla u_0(Y)|^2\, \delta(Y)\, dY \\ &\approx \sum_{I\in\mathcal{W}_B} \fint_{\Delta_I} \iint_I |\nabla u_0(Y)|^2\, \delta(Y)\, dY\, d\sigma \\ &\lesssim \int_{\Delta^*} (\mathcal{S}u_0(z))^2\, d\sigma(z) \\ &\lesssim \sigma(\Delta)^{(p-2)/p} \left(\int_{\Delta^*} (\mathcal{S}u_0(z))^p\, d\sigma(z)\right)^{2/p},\end{aligned}$$

where in the last two steps we have used the ADR property and (4.17), and then ADR again. Therefore, by (4.13), and then Proposition 4.5/Remark 4.8, and the definition of u_0,

$$\frac{1}{\sigma(\Delta)} \iint_{B\cap\Omega} |\nabla u_0(Y)|^2\, \delta(Y)\, dY \lesssim \sigma(\Delta)^{-2/p} \left(\int_{40\Delta} |f - c_\Delta|^p\right)^{2/p} \lesssim \|f\|^2_{BMO(\partial\Omega)}.$$

Next, we observe that

(4.19)

$$\begin{aligned}\left(\iint_{B\cap\Omega} |\nabla(u(Y)-u_0(Y))|^2\, \delta(Y)\, dY\right)^{1/2} &\lesssim \left(\sum_{I\in\mathcal{W}_B} \ell(I) \iint_I |\nabla(u(Y)-u_0(Y))|^2\, dY\right)^{1/2} \\ &\lesssim \left(\sum_{I\in\mathcal{W}_B} \ell(I)^{-1} \iint_{I^*} |(u(Y)-u_0(Y))|^2\, dY\right)^{1/2} \\ &\lesssim \sum_{k=1}^{\infty} \left(\sum_{I\in\mathcal{W}_B} \ell(I)^{-1} \iint_{I^*} |u_k(Y)|^2\, dY\right)^{1/2}.\end{aligned}$$

For $k \geq 1$, we set $g_k := |h_k| = |f - c_\Delta|\varphi_k$, and let v_k be the solution of the Dirichlet problem with data g_k. Thus, $|u_k| \leq v_k$. For $k \geq 0$, set

$$\widetilde{B} := 40B = B(x, 40r), \quad B_k := 2^k\, \widetilde{B}, \quad \Delta_k := B_k \cap \partial\Omega,$$

and let Δ_k^* be a sufficiently large concentric fattening of Δ_k. Given $I \in \mathcal{W}$, define $I^* = (1+\tau)I$, with τ chosen small enough that $\operatorname{dist}(I^*, \partial\Omega) \approx \operatorname{dist}(I, \partial\Omega) \approx \operatorname{diam}(I)$. Then for $Y \in I^*$, with $I \in \mathcal{W}_B^j$, by Corollary 2.4,

$$\begin{aligned}v_k(Y) &\lesssim \left(\frac{\ell(I)}{2^k r}\right)^{\alpha} \frac{1}{|B_{k-1}|} \iint_{B_{k-1}\cap\Omega} v_k \lesssim (2^j 2^k r)^{-\alpha} \fint_{\Delta_k^*} N_* v_k\, d\sigma \\ &\lesssim (2^j 2^k r)^{-\alpha} \left(\fint_{\Delta_k^*} (N_* v_k)^p\, d\sigma\right)^{1/p} \lesssim (2^j 2^k r)^{-\alpha} \left(\fint_{\Delta_{k+2}} |f - c_\Delta|^p\, d\sigma\right)^{1/p} \\ &\lesssim k\, (2^j 2^k r)^{-\alpha}\, \|f\|_{BMO(\partial\Omega)},\end{aligned}$$

where in the last two steps we have used Proposition 4.5/Remark 4.8, and then a well known telescoping argument. Consequently, setting $\Delta^* = \Delta(x, Cr)$ as in

(4.18), we find that the squares of the summands in the last line of (4.19) satisfy

$$\sum_{I\in\mathcal{W}_B}\ell(I)^{-1}\iint_{I^*}|u_k(Y)|^2\,dY \lesssim k^2\,2^{-2k\alpha}\,\|f\|^2_{BMO(\partial\Omega)}\sum_{j:\,2^{-j}\lesssim r}(2^j r)^{-2\alpha}\sum_{I\in\mathcal{W}_B^j}\sigma(\Delta_I)$$
$$\lesssim k^2\,2^{-2k\alpha}\,\|f\|^2_{BMO(\partial\Omega)}\,\sigma(\Delta^*)\,,$$

since for each fixed j, the surface balls Δ_I with $I\in\mathcal{W}_B^j$ have bounded overlaps, and are all contained in Δ^*. By ADR, $\sigma(\Delta^*)\approx\sigma(\Delta)$, so we may then plug this last estimate into (4.19) and sum in k to obtain (1.3), thus concluding the proof of Theorem 1.6.

Remark 4.20. We observe that a slight refinement of the argument above shows that for a domain Ω with ADR boundary, for which the "$S<N$ estimate" (4.13) holds for every $p\in(1,\infty)$, the notions of VMO-solvability and BMO-solvability are equivalent. Indeed, suppose that the Carleson measure estimate (1.3) holds for all continuous data f. Then harmonic measure is weak-A_∞ in the sense of Theorem 1.4, and therefore, by the first part of Theorem 1.6 (i.e., by Proposition 4.5), we have that (4.7) holds for all sufficiently large (but finite) powers s. We may then repeat the preceding argument essentially verbatim, but now with the continuous, compactly supported data f replaced by an arbitrary $f\in BMO(\partial\Omega)$; as before, we construct elliptic-harmonic measure solutions u and u_k, corresponding to the BMO data $f-c_\Delta$, and to the dyadic pieces $h_k=(f-c_\Delta)\varphi_k$, respectively. In order to carry out the rest of the argument to obtain the Carleson measure estimate (1.3), we need only verify that the solution u is well defined, and that $u(Y)=\sum_k u_k(Y)$ pointwise, for every $Y\in\Omega$. In the case that $\partial\Omega$ is bounded, these facts follow immediately from the John-Nirenberg inequality, the higher integrability of the Poisson kernel, and the fact that only finitely many terms appear in the sum.

If $\partial\Omega$ is unbounded, we proceed as follows. Without loss of generality, we may suppose that $c_\Delta=0$, and for $Y\in\Omega$ fixed, as above we set $\Delta_Y^*:=\partial\Omega\cap B(\hat{y},10000\delta(Y))$, where $\hat{y}\in\partial\Omega$ satisfies $|Y-\hat{y}|=\delta(Y)$. We then define a smooth partition of unity $\sum_{m\geq0}\tilde{\varphi}_m$ relative to Δ_Y^* as in (4.9) (but where we now change φ_k to $\tilde{\varphi}_m$ to avoid confusion with the the partition of unity used to define u_k), set $c_Y:=\fint_{4\Delta_Y^*}f$, and observe that

$$\int_{\partial\Omega}|f|d\omega^Y\leq\int_{\partial\Omega}|f-c_Y|d\omega^Y+|c_Y|\leq\sum_{m=0}^{\infty}\int_{\partial\Omega}|f-c_Y|\tilde{\varphi}_m d\omega^Y+|c_Y|$$
$$=:\sum_{m=0}^{\infty}a_m(Y)\,+\,|c_Y|\,.$$

By slightly modifying the proof of Proposition 4.5 *mutatis mutandi* (see in particular (4.11) and (4.12)), and using a standard telescoping argument, we find that $a_m(Y)\lesssim m2^{-m\alpha}\|f\|_{BMO(\partial\Omega)}$, and therefore that $u(Y)=\int_{\partial\Omega}fd\omega^Y$ is well defined, and that $u(Y)=\sum_k u_k(Y)$, by dominated convergence.

5. The case that Ω is an unbounded domain with bounded boundary

We suppose now that Ω is unbounded, but that $\partial\Omega$ is bounded. In this case, as noted above in Remark 4.15, estimate (1.3) cannot hold in general: indeed, for constant data f, the right hand side of (1.3) is zero, but the left hand side is non-zero, since the solution $u(X) = \int_{\partial\Omega} f d\omega^X$ is not constant. We therefore consider the following variant of (1.3):

$$\sup_{x\in\partial\Omega,\,0<r<\varepsilon R_0} \frac{1}{\sigma(\Delta(x,r))} \iint_{\Omega\cap B(x,r)} |\nabla u(Y)|^2\,\delta(Y)\,dY \leq C\left(\|f\|^2_{BMO(\partial\Omega)} + \kappa(\varepsilon)\,\|f\|^2_{L^\infty(\partial\Omega)}\right), \quad \forall\varepsilon\in(0,1)\,, \tag{5.1}$$

where $R_0 := \operatorname{diam}(\partial\Omega)$, f is any continuous function defined on $\partial\Omega$, and u is the elliptic-harmonic measure solution with data f, and where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let us now observe that suitable variants of Theorems 1.4 and 1.6 hold in this context.

Assuming that (5.1) holds for all continuous f defined on $\partial\Omega$, for some $\kappa(\varepsilon)$ with $\lim_{\varepsilon\to 0}\kappa(\varepsilon) = 0$, we may repeat the proof of Theorem 1.4 with minor modifications, provided that $\delta(X) \leq \varepsilon R_0$ with ε small enough. Note that the function f_s that we have constructed satisfies $\|f_s\|_\infty \leq 1$. Consequently, a version of estimate (3.7) still holds, but with the small error $C\varepsilon^\alpha$ replaced by $C(\varepsilon^\alpha + \sqrt{\kappa(\varepsilon)})$. For ε (and hence also $\kappa(\varepsilon)$) small enough, and now fixed, the rest of the proof of Theorem 1.4 goes through unchanged, and we obtain that (3.3) holds whenever $\delta(X) \leq \varepsilon R_0$. In turn, an examination of the proof of Lemma 3.2 (the result of [BL]) reveals that the conclusion of Lemma 3.2 continues to hold for all surface balls Δ of radius at most εR_0. It then readily follows that the weak-A_∞ property holds for all surface balls of radius up to R_0; the constants depend on ε, but the latter constant has now been fixed depending only on the various parameters in the hypotheses of the theorem. Thus Theorem 1.4 is still valid in this setting. We leave the details to the interested reader.

Conversely, suppose that elliptic-harmonic measure belongs to weak-A_∞ in the sense of Theorem 1.4, and that the "$S < N$" bound (4.13) holds (in particular, as noted above, this is true when L is the Laplacian and Ω satisfies an interior Corkscrew condition). We seek to establish the Carleson measure estimate (5.1). As in the proof of the BMO-solvability part of Theorem 1.6, we let u be the elliptic-harmonic measure solution with given continuous data f defined on $\partial\Omega$, and for $B = B(x,r)$, $\Delta = \Delta(x,r)$, with $r \leq \varepsilon R_0$, we set $h := f - c_\Delta$, where again $c_\Delta = \fint_{40\Delta} f$. Note that

$$|c_\Delta| \leq \|f\|_{L^\infty(\partial\Omega)}\,. \tag{5.2}$$

We then have

$$u(X) = \int_{\partial\Omega} h\,d\omega^X + c_\Delta(v(X)-1) + c_\Delta\,,$$

where $v(X) := \omega^X(\partial\Omega)$, and therefore, setting $\tilde{u}(X) := \int_{\partial\Omega} h\,d\omega^X$, we see in turn that

$$\nabla u(X) = \nabla\tilde{u}(X) + c_\Delta\nabla\tilde{v}(X)\,,$$

where $\tilde{v} := v - 1$. We may handle the contribution of $\tilde{u}$ exactly as in Section 3, making a dyadic annular decomposition of h into a sum of terms h_k, which give rise to elliptic-harmonic measure solutions u_k. The new ingredient is the contribution of $\tilde{v}$. By (5.2), it is enough to show that

$$\frac{1}{\sigma(\Delta(x,r))} \iint_{\Omega\cap B(x,r)} |\nabla\tilde{v}(Y)|^2\,\delta(Y)\,dY \leq C\varepsilon^{2\alpha}\,, \tag{5.3}$$

which yields (5.1) with $\kappa(\varepsilon) = C\varepsilon^{2\alpha}$. With $B = B(x, r)$, define $\mathcal{W}_B$, $\mathcal{W}_B^j$ as in (4.16). Note that $\tilde{v}$ vanishes continuously on $\partial\Omega$, and is bounded in absolute value by 1, so that the left hand side of (5.3) is no larger than a constant times

$$(5.4)\quad r^{-n} \sum_{j:\, 2^{-j} \lesssim r} \sum_{I \in \mathcal{W}_B^j} \ell(I)^{-1} \iint_I |\tilde{v}(Y)|^2 dY$$
$$\leq r^{-n} \sum_{j:\, 2^{-j} \lesssim r} \sum_{I \in \mathcal{W}_B^j} \ell(I)^{n+2\alpha} R_0^{-2\alpha} \leq \left(\frac{r}{R_0}\right)^{2\alpha},$$

where we have used first Caccioppoli's inequality, then Hölder continuity at the boundary (Corollary 2.4), along with the ADR property and the definition of $\mathcal{W}_B^j$. Since $r \leq \varepsilon R_0$, we obtain (5.3).

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Steve Hofmann, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: hofmanns@missouri.edu

Phi Le, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: llc33@mail.missouri.edu