

A_∞ IMPLIES NTA FOR A CLASS OF VARIABLE COEFFICIENT ELLIPTIC OPERATORS

STEVE HOFMANN, JOSÉ MARÍA MARTELL, AND TATIANA TORO

ABSTRACT. We consider a certain class of second order, variable coefficient divergence form elliptic operators, in a uniform domain Ω with Ahlfors regular boundary, and we show that the A_∞ property of the elliptic measure associated to any such operator and its transpose imply that the domain is in fact NTA (and hence chord-arc). The converse was already known, and follows from work of Kenig and Pipher [KP].

CONTENTS

1. Main result	1
2. Preliminaries	4
2.1. Notation and conventions	4
2.2. Some definitions	5
2.3. Dyadic grids and sawtooths	6
2.4. PDE estimates	9
3. Auxiliary results	10
4. Proof of the main result	14
4.1. The symmetric case	19
4.2. The non-symmetric case	21
4.3. Proofs of Lemmas 4.39 and 4.44	25
Appendix A. The A_∞ property in Lipschitz domains: the Kenig-Pipher argument	28
References	31

1. MAIN RESULT

There has been considerable recent activity seeking necessary and sufficient geometric criteria for the absolute continuity of harmonic measure with respect to surface measure, on the boundary of a domain Ω . This paper is concerned with a quantitative version of this problem, in which both the geometric conditions on Ω and its boundary, and the absolute continuity properties of harmonic measure, are expressed in a quantitative, scale invariant fashion. For example, under

Date: January 4, 2017.

2010 Mathematics Subject Classification. 31B05, 35J08, 35J25, 42B99, 42B25, 42B37.

Key words and phrases. Elliptic measure, Poisson kernel, uniform rectifiability, Carleson measures, A_∞ Muckenhoupt weights.

The first author was supported by NSF grant DMS-1361701. The second author was supported in part by ICMAT Severo Ochoa project SEV-2015-0554. He also acknowledges that the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC agreement no. 615112 HAPDEGMT. The last author was partially supported by the Robert R. & Elaine F. Phelps Professorship in Mathematics and DMS grant 1361823.

the background hypotheses that $\partial\Omega$ is Ahlfors regular (Definition 2.1 below), and that Ω satisfies an interior Corkscrew condition (Definition 2.3), in work of the first two authors [HM2]¹, it is shown that $\partial\Omega$ is uniformly rectifiable (a quantitative scale invariant version of rectifiability; see [DS1, DS2]), provided that the Poisson kernel satisfies a certain uniform scale invariant L^q estimate (which is in turn equivalent to the property that harmonic measure satisfies a quantitative, scale-invariant version of absolute continuity, namely, the weak- A_∞ condition; see Definition 2.8).

However, the converse to the result in [HM2] is false (see [BiJ] for a counter-example), and it remains an open problem to find a geometric characterization of quantitative absolute continuity of harmonic measure².

On the other hand, under strengthened background hypotheses, necessary and sufficient conditions for absolute continuity are known. Building on the fundamental result of Dahlberg [Dah] for Lipschitz domains, David and Jerison [DJ], and independently Semmes [Se], showed that for a chord-arc domain Ω , harmonic measure satisfies an A_∞ condition (see Definition 2.8) with respect to surface measure σ on $\partial\Omega$, i.e., harmonic measure is absolutely continuous with respect to σ in a quantitative, scale invariant way. The term “chord-arc” refers to an NTA domain with an Ahlfors regular boundary. In turn, an NTA domain is one which satisfies the Harnack Chain condition (Definition 2.4), as well as both interior and exterior Corkscrew conditions. A domain which satisfies the Harnack Chain condition and interior Corkscrew condition is known in the literature as a *uniform* domain, or a 1-sided NTA domain. Thus, an NTA domain is a uniform domain for which, in addition, the exterior Corkscrew condition holds.

In this work, we take as our background hypotheses that Ω is a uniform (i.e., 1-sided NTA) domain, and that $\partial\Omega$ is Ahlfors regular. We call such domains *1-sided chord-arc domains*. In this context, a necessary and sufficient condition for quantitative absolute continuity (in the form of the A_∞ property) of harmonic measure with respect to surface measure is known. Indeed, one has the following.

Theorem A. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a uniform (aka 1-sided NTA) domain, whose boundary is Ahlfors regular. Then the following are equivalent:*

- (1) $\partial\Omega$ is uniformly rectifiable.
- (2) Ω is an NTA domain, and hence, a chord-arc domain.
- (3) $\omega \in A_\infty$.

As mentioned above, the implication (2) $\implies$ (3) was proved independently in [DJ] and in [Se], while (3) $\implies$ (1) appears in joint work of the first two authors of the present paper, together with I. Uriarte-Tuero [HMU], and (1) $\implies$ (2) was proved by the present authors jointly with J. Azzam and K. Nyström [AHMNT].

In the present work, we give a direct (and more efficient) proof of the fact that (3) $\implies$ (2), without passing through the results of [AHMNT]. More importantly, our approach here allows us to extend these results well beyond the case of the Laplacian, and to treat a much broader class of divergence form elliptic operators, with variable coefficients. For the class of operators that we consider here, the converse direction (2) $\implies$ (3) was known, and follows readily from work of Kenig and Pipher [KP]; we shall return to this point momentarily (in particular, see Corollary 1.9).

More precisely, we shall consider divergence form elliptic operators $Lu = -\operatorname{div}(A\nabla u)$, whose coefficients satisfy the following assumptions.

¹See also [MT] for an alternative proof of the main result of [HM2], and [HLMN] for an extension of these results to the setting of the p -Laplacian.

²On the other hand, we mention that certain other boundary estimates for harmonic functions may be characterized in terms of uniform (i.e. quantitative) rectifiability, by combining the recent results of [HMM] and [GMT].

Hypothesis 1.1. Let $A(X) = (a_{i,j}(X))_{1 \leq i,j \leq n+1}$ be a **real** $(n+1) \times (n+1)$ matrix such that $a_{i,j} \in L^\infty(\Omega)$ for $1 \leq i, j \leq n+1$, and A is uniformly elliptic, that is, there exists $1 \leq \Lambda < \infty$ such that

$$\Lambda^{-1} |\xi|^2 \leq A(X) \xi \cdot \xi, \quad |A(X) \xi \cdot \eta| \leq \Lambda |\xi| |\eta|, \quad \text{for all } \xi, \eta \in \mathbb{R}^{n+1}, \text{ and a.e. } X \in \Omega.$$

Suppose further that A satisfies the following conditions:

(a) $A \in \text{Lip}_{\text{loc}}(\Omega)$.

(b) $\| |\nabla A| \delta \|_{L^\infty(\Omega)} < \infty$, where $\delta(X) = \text{dist}(X, \partial\Omega)$.

(c) ∇A satisfies the Carleson measure estimate:

$$(1.2) \quad \|\nabla A\|_{C(\Omega)} := \sup_{\substack{x \in \partial\Omega \\ 0 < r < \text{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x, r) \cap \partial\Omega)} \iint_{B(x, r) \cap \Omega} |\nabla A(X)| dX < \infty,$$

where $\sigma := H^n|_{\partial\Omega}$ denotes the surface measure and H^n is the n -dimensional Hausdorff measure.

We shall also assume that $\{\omega_L^X\}_{X \in \Omega}$, the elliptic measure associated with L , belongs to $A_\infty(\partial\Omega)$. More precisely, we shall consider elliptic operators whose associated elliptic measure satisfies the following scale invariant higher integrability estimate.

Hypothesis 1.3. There exists $q > 1$ and $C \geq 1$, such that for every surface ball $\Delta = \Delta(x, r) := B(x, r) \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, one has that $\omega_L^{X_\Delta} \ll \sigma$, and the Poisson kernel $k_L^{X_\Delta} := d\omega_L^{X_\Delta}/d\sigma$ satisfies the scale invariant estimate

$$(1.4) \quad \int_{\Delta} k_L^{X_\Delta}(y)^q d\sigma(y) \leq C \sigma(\Delta)^{1-q}.$$

Here, X_Δ is a fixed (or any) Corkscrew point in Ω , relative to Δ (see Definition 2.3).

We remark that in the setting of a uniform domain with Ahlfors regular boundary, it can be shown that estimate (1.4) is in turn equivalent to the property that the Poisson kernel satisfies an L^q reverse Hölder inequality, equivalently, that elliptic measure satisfies an A_∞ condition (a quantitative, scale-invariant version of absolute continuity; see Definition 2.8). We further remark that Hypothesis 1.3 is equivalent to the solvability of the Dirichlet problem for L with L^p data, and with nontangential maximal function estimates in L^p , for $p = q/(q-1)$ (see, e.g., [Ke]).

Our main result is the following.

Theorem 1.5. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD (see Definition 2.7). Let $Lu = -\text{div}(A \nabla u)$ be a second order divergence form operator in Ω , whose coefficients satisfy the assumptions of Hypothesis 1.1, and suppose that the elliptic measures for both L and its transpose $L^\top$ satisfy Hypothesis 1.3. Then Ω is a chord-arc domain (CAD).

Remark 1.6. In particular, the conclusion that Ω is a CAD implies that $\partial\Omega$ is uniformly rectifiable (UR) (see [DS2]). The UR constants depends on dimension, the ellipticity constants, the 1-sided CAD constants, $\| |\nabla A| \delta \|_\infty$, $\|\nabla A\|_{C(\Omega)}$ and finally q and C in (1.4).

Remark 1.7. We point out that we do not assume symmetry of the coefficient matrix A ; on the other hand, Theorem 1.5 is new even in the symmetric case (and in that case, of course, one requires that Hypothesis 1.3 hold only for elliptic measure associated to L).

Let us observe that given property (b) in Hypothesis 1.1, we have that property (c) immediately implies the more familiar Carleson measure condition

$$(1.8) \quad \sup_{\substack{x \in \partial\Omega \\ 0 < r < \text{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x, r) \cap \partial\Omega)} \iint_{B(x, r) \cap \Omega} |\nabla A(X)|^2 \delta(X) dX < \infty$$

(on the other hand, the converse is not true). Here $\delta(X) := \text{dist}(X, \partial\Omega)$. It has essentially been shown by Kenig and Pipher [KP], that the latter condition, in conjunction with property (b) of Hypothesis 1.1, is sufficient to deduce the A_∞ property of elliptic measure in an arbitrary Lipschitz domain, and thus, by a well-known maximum principle argument, in a chord-arc domain Ω as well, using the method of David and Jerison [DJ]. We refer the reader to [DJ] for the details, which are stated there in the case that L is the Laplacian, but extend immediately to the case of operators satisfying property (b) and (1.8), since these conditions clearly hold uniformly in every Lipschitz subdomain of Ω . In particular, combining our Theorem 1.5 with the result of [KP] and the method of [DJ], we obtain the following necessary and sufficient criterion.

Corollary 1.9. *Let Ω be a 1-sided CAD (cf. Definition 2.7), and let $Lu = -\text{div}(A \nabla u)$ be a variable coefficient second order divergence form operator whose coefficients satisfy Hypothesis 1.1. Then the elliptic measures $\{\omega_L^X\}_{X \in \Omega}$ and $\{\omega_{L^\top}^X\}_{X \in \Omega}$ belong to $A_\infty(\partial\Omega)$, if and only if $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \bar{\Omega}$ satisfies the Corkscrew condition (and thus, Ω is a CAD).*

As noted above, Corollary 1.9 was already known in the special case that L is the Laplacian.

The plan of the paper is as follows. In Section 2 we present some background and preliminaries that are used throughout the paper. Section 3 contains some auxiliary results. The proof of Theorem 1.5 is in Section 4. There, after some reductions, we show that it suffices to establish Proposition 4.25 in the symmetric case and Proposition 4.36 in general. These in turn follow from an integration by parts argument. In Section 4.1 we single out the case of symmetric operators and we pay particular attention to the Laplacian since it is rather simple and models the general case which is treated in Section 4.2. Finally, we observe that the main theorem of [KKiPT] allows for a modest shortcut in the proof of the result of [KP] that we have invoked in Corollary 1.9. For the reader's convenience, in an appendix we briefly sketch the argument of [KP], using the result of [KKiPT] to shorten their proof.

Acknowledgements. The main result of this paper was proved in the Fall of 2014 while the first two authors were visiting the last author at the Mathematics Department of the University of Washington. In addition, these results were presented, in condensed form, by the first two authors at the MSRI Summer Graduate School “Harmonic Analysis and Elliptic Equations on real Euclidean Spaces and on Rough Sets”, June 13-24, 2016. We are grateful to both of these institutions for their kind hospitality.

We mention also that some of the main ideas of this work were used subsequently (but have appeared already) in [ABHM] to obtain a qualitative version of the main result of the present paper. In particular, Proposition 4.25, although first proved in the present work, was previously announced in [ABHM, Proposition 3.3], with a reasonably detailed sketch of the proof³.

2. PRELIMINARIES

2.1. Notation and conventions.

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the “allowable parameters”). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. Unless otherwise specified upper case constants are greater than 1 and lower case constants are smaller than 1. In some occasions it is

³Quite recently, Proposition 4.25, as gleaned from [ABHM], has also been used as one ingredient in work of Azzam, Garnett, Mourougolou, and Tolsa [AGMT], to extend the results of [GMT] to the class of operators with coefficients satisfying Hypothesis 1.1.

important to keep track of the dependence on a given parameter γ , in that case we write $a \lesssim_\gamma b$ or $a \approx_\gamma b$ to emphasize that the implicit constants in the inequalities depend on γ .

- Our ambient space is $\mathbb{R}^{n+1}$, $n \geq 2$.
- Given $E \subset \mathbb{R}^{n+1}$ we write $\text{diam}(E) = \sup_{x,y \in E} |x - y|$ to denote its diameter.
- Given a domain $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z , etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z , etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus \partial\Omega$).
- The open $(n+1)$ -dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \mathbb{R}^{n+1} \setminus \partial\Omega$. A *surface ball* is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$, and unless otherwise specified it is implicitly assumed that $x \in \partial\Omega$.
- If $\partial\Omega$ is bounded, it is always understood (unless otherwise specified) that all surface balls have radii controlled by the diameter of $\partial\Omega$: that is if $\Delta = \Delta(x, r)$ then $r \lesssim \text{diam}(\partial\Omega)$. Note that in this way $\Delta = \partial\Omega$ if $\text{diam}(\partial\Omega) < r \lesssim \text{diam}(\partial\Omega)$.
- For $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \text{dist}(X, \partial\Omega)$.
- We let H^n denote n -dimensional Hausdorff measure, and let $\sigma := H^n|_{\partial\Omega}$ denote the surface measure on $\partial\Omega$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A , i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.
- For a Borel subset $A \subset \partial\Omega$, with $0 < \sigma(A) < \infty$, we set $\int_A f d\sigma := \sigma(A)^{-1} \int_A f d\sigma$.
- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$ -dimensional Euclidean cube with sides parallel to the coordinate axes, and we let $\ell(I)$ denote the side length of I . We use Q to denote dyadic “cubes” on $\partial\Omega$. The latter exist, given that $\partial\Omega$ is AR (cf. [DS1], [Chr]), and enjoy certain properties which we enumerate in Lemma 2.12 below.
- Given a domain Ω , and an elliptic operator L , we let ω_L^X denote L -elliptic measure for Ω with pole at X , and if $\omega_L^X \ll \sigma$, we let $k_L^X := d\omega_L^X/d\sigma$ be the corresponding Poisson kernel. When the operator L is understood, we will at times suppress its appearance in the notation, and write simply ω^X, k^X in place of ω_L^X and k_L^X .

2.2. Some definitions.

Definition 2.1 (Ahlfors regular). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is *n-dimensional Ahlfors regular* (AR for shortness) if there is some uniform constant C such that

$$(2.2) \quad C^{-1} r^n \leq H^n(E \cap B(x, r)) \leq C r^n, \quad x \in E, \quad 0 < r < \text{diam}(E).$$

Definition 2.3 (Corkscrew condition). Following [JK], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a *corkscrew point relative to Δ* , (or, relative to B). We note that we may allow $r < C \text{diam}(\partial\Omega)$ for any fixed C , simply by adjusting the constant c .

Definition 2.4 (Harnack Chain condition). Again following [JK], we say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Lambda\rho$, there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial\Omega) \leq C \text{diam}(B_k)$. The chain of balls is called a *Harnack Chain*.

Definition 2.5 (1-sided NTA and NTA). We say that a domain Ω is a *1-sided NTA domain* if it satisfies both the Corkscrew and Harnack Chain conditions. Furthermore, we say that Ω is an

NTA domain if it is 1-sided NTA and if, in addition, $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ also satisfies the Corkscrew condition.

Remark 2.6. The abbreviation NTA stands for non-tangentially accessible. In the literature, 1-sided NTA domains are also called *uniform domains*. We remark that the 1-sided NTA condition is a quantitative form of path connectedness.

Definition 2.7 (1-sided CAD and CAD). A *1-sided chord-arc domain* (1-sided CAD) is a 1-sided NTA domain with AR boundary. A *chord-arc domain* (CAD) is an NTA domain with AR boundary.

Definition 2.8. (A_∞ , weak- A_∞ , and RH_q). Given an n -dimensional Ahlfors regular set $E \subset \mathbb{R}^{n+1}$, and a surface ball $\Delta_0 := B_0 \cap E$, we say that a Borel measure μ defined on E belongs to $A_\infty(\Delta_0)$ if there are positive constants C and s such that for each surface ball $\Delta = B \cap E$, with $B \subseteq B_0$, we have

$$(2.9) \quad \mu(A) \leq C \left(\frac{\sigma(A)}{\sigma(\Delta)} \right)^s \mu(\Delta), \quad \text{for every Borel set } A \subset \Delta.$$

Similarly, we say that $\mu \in \text{weak-}A_\infty(\Delta_0)$ if for each surface ball $\Delta = B \cap E$, with $2B \subseteq B_0$,

$$(2.10) \quad \mu(A) \leq C \left(\frac{\sigma(A)}{\sigma(\Delta)} \right)^s \mu(2\Delta), \quad \text{for every Borel set } A \subset \Delta.$$

We recall that, as is well known, the condition $\mu \in A_\infty(\Delta_0)$ is equivalent to the property that $\mu \ll \sigma$ in Δ_0 , and that for some $q > 1$, the Radon-Nikodym derivative $k := d\mu/d\sigma$ satisfies the reverse Hölder estimate

$$(2.11) \quad \left(\int_\Delta k^q d\sigma \right)^{1/q} \lesssim \int_\Delta k d\sigma \approx \frac{\mu(\Delta)}{\sigma(\Delta)}, \quad \forall \Delta = B \cap E, \text{ with } B \subseteq B_0.$$

The inequality in (2.11) is often referred to as an L^q Reverse Hölder (“ RH_q ”) estimate.

2.3. Dyadic grids and sawtooths. We first give a lemma concerning the existence of a “dyadic grid” which can be found in [DS1, DS2, Chr].

Lemma 2.12 (Existence and properties of the “dyadic grid”). [DS1, DS2], [Chr]. *Suppose that $E \subset \mathbb{R}^{n+1}$ satisfies the AR condition (2.2). Then there exist constants $a_0 > 0$, $\eta > 0$ and $C_1 < \infty$, depending only on dimension and the AR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets (“cubes”)*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{J}_k\},$$

where $\mathfrak{J}_k$ denotes some (possibly finite) index set depending on k , satisfying

- (i) $E = \cup_j Q_j^k$ for each $k \in \mathbb{Z}$.
- (ii) If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.
- (iii) For each (j, k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.
- (iv) $\text{diam}(Q_j^k) \leq C_1 2^{-k}$.
- (v) Each Q_j^k contains some surface ball $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$.
- (vi) $H^n(\{x \in Q_j^k : \text{dist}(x, E \setminus Q_j^k) \leq \tau 2^{-k}\}) \leq C_1 \tau^\eta H^n(Q_j^k)$, for all k, j and for all $\tau \in (0, a_0)$.

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Chr], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the Ahlfors regularity property (2.2), the result already appears in [DS1, DS2].

- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \text{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k , i.e.,

$$\mathbb{D} := \cup_k \mathbb{D}_k,$$

where, if $\text{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \text{diam}(E)$.

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the “length” of Q . Evidently, $\ell(Q) \approx \text{diam}(Q)$.
- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r_Q)$ and a surface ball $\Delta(x_Q, r_Q) := B(x_Q, r_Q) \cap E$ such that

$$(2.13) \quad c\ell(Q) \leq r_Q \leq \ell(Q) \quad \text{and} \quad \Delta(x_Q, 2r_Q) \subset Q \subset \Delta(x_Q, Cr_Q),$$

for some uniform constants C, c . We shall denote this ball and surface ball by

$$(2.14) \quad B_Q := B(x_Q, r_Q), \quad \Delta_Q := \Delta(x_Q, r_Q),$$

and we shall refer to the point x_Q as the “center” of Q .

It will be useful to dyadicize the Corkscrew condition, and to specify precise Corkscrew constants. Let us now specialize to the case that $E = \partial\Omega$ is AR, with Ω satisfying the Corkscrew condition. Given $Q \in \mathbb{D}(\partial\Omega)$, we shall sometimes refer to a “Corkscrew point relative to Q ”, which we denote by X_Q , and which we define to be a corkscrew point X_{Δ_Q} relative to the surface ball Δ_Q (see (2.13), (2.14) and Definition 2.3). We note that

$$(2.15) \quad \delta(X_Q) \approx \text{dist}(X_Q, Q) \approx \text{diam}(Q).$$

Definition 2.16 (*c_0 -exterior Corkscrew condition*). Fix a constant $c_0 \in (0, 1)$, and a domain $\Omega \subset \mathbb{R}^{n+1}$, with AR boundary. We say that a cube $Q \in \mathbb{D}(\partial\Omega)$ satisfies the c_0 -exterior Corkscrew condition, if there is a point $z_Q \in \Delta_Q$, and a point $X_Q^- \in B(z_Q, r_Q/4) \setminus \overline{\Omega}$, such that $B(X_Q^-, c_0 \ell(Q)) \subset B(z_Q, r_Q/4) \setminus \overline{\Omega}$, where $\Delta_Q = \Delta(x_Q, r_Q)$ is the surface ball defined above in (2.13)–(2.14).

Following [HM1, Section 3] we next introduce the notion of **Carleson region** and **discretized sawtooth**. Given a cube $Q \in \mathbb{D}(\partial\Omega)$, the **discretized Carleson region** $\mathbb{D}_Q$ relative to Q is defined by

$$\mathbb{D}_Q = \{Q' \in \mathbb{D}(\partial\Omega) : Q' \subset Q\}.$$

Let $\mathcal{F}$ be family of disjoint cubes $\{Q_j\} \subset \mathbb{D}(\partial\Omega)$. The **global discretized sawtooth region** relative to $\mathcal{F}$ is the collection of cubes $Q \in \mathbb{D}$ that are not contained in any $Q_j \in \mathcal{F}$;

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j}.$$

For a given $Q \in \mathbb{D}$ the **local discretized sawtooth region** relative to $\mathcal{F}$ is the collection of cubes in $\mathbb{D}_Q$ that are not in contained in any $Q_j \in \mathcal{F}$;

$$\mathbb{D}_{\mathcal{F}, Q} := \mathbb{D}_Q \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q.$$

We also introduce the “geometric” Carleson regions and sawtooths. In the sequel, $\Omega \subset \mathbb{R}^{n+1}$ ($n \geq 2$) will be a 1-sided CAD domain. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω (see [St, Chapter VI]), so that the cubes in $\mathcal{W}$ form a covering of Ω with non-overlapping interiors, and which satisfy

$$(2.17) \quad 4 \text{diam}(I) \leq \text{dist}(4I, \partial\Omega) \leq \text{dist}(I, \partial\Omega) \leq 40 \text{diam}(I)$$

and

$$(2.18) \quad \text{diam}(I_1) \approx \text{diam}(I_2), \text{ whenever } I_1 \text{ and } I_2 \text{ touch.}$$

Let $X(I)$ denote the center of I , let $\ell(I)$ denote the side length of I , and write $k = k_I$ if $\ell(I) = 2^{-k}$.

Given $0 < \lambda < 1$ and $I \in \mathcal{W}$ we write $I^* = (1 + \lambda)I$ for the “fattening” of I . By taking λ small enough, we can arrange matters so that, first, $\text{dist}(I^*, J^*) \approx \text{dist}(I, J)$ for every $I, J \in \mathcal{W}$, and secondly, I^* meets J^* if and only if ∂I meets ∂J . (Fattening ensures I^* and J^* overlap for any pair $I, J \in \mathcal{W}$ whose boundaries touch. Thus, the Harnack Chain property holds locally in $I^* \cup J^*$ with constants depending on λ .) By picking λ sufficiently small, say $0 < \lambda < \lambda_0$, we may also suppose that there is $\tau \in (1/2, 1)$ such that for distinct $I, J \in \mathcal{W}$, $\tau J \cap I^* = \emptyset$. In what follows we will need to work with dilations $I^{**} = (1 + 2\lambda)I$ or $I^{***} = (1 + 4\lambda)I$ and in order to ensure that the same properties hold we further assume that $0 < \lambda < \lambda_0/4$.

For every Q we can construct a non-empty family $\mathcal{W}_Q^* \subset \mathcal{W}$ and define

$$(2.19) \quad U_Q := \bigcup_{I \in \mathcal{W}_Q^*} I^*,$$

satisfying the following properties: $X_Q \in U_Q$ and there are uniform constants k^* and K_0 such that

$$(2.20) \quad \begin{aligned} k(Q) - k^* &\leq k_I \leq k(Q) + k^* & \forall I \in \mathcal{W}_Q^*, \\ X(I) &\rightarrow_{U_Q} X_Q & \forall I \in \mathcal{W}_Q^*, \\ \text{dist}(I, Q) &\leq K_0 2^{-k(Q)} & \forall I \in \mathcal{W}_Q^*. \end{aligned}$$

Here $X(I) \rightarrow_{U_Q} X_Q$ means that the interior of U_Q contains all the balls in a Harnack Chain (in Ω) connecting $X(I)$ to X_Q , and moreover, for any point Z contained in any ball in the Harnack Chain, we have $\text{dist}(Z, \partial\Omega) \approx \text{dist}(Z, \Omega \setminus U_Q)$ with uniform control of the implicit constants. The constants k^* , K_0 and the implicit constants in the condition $X(I) \rightarrow_{U_Q} X_Q$ in (2.20) depend on at most allowable parameters and on λ . For later use, it will be convenient to associate to Whitney boxes a particular nearest dyadic cube. Let $I \in \mathcal{W}$ with $\ell(I) \lesssim \text{diam}(\partial\Omega)$ and pick $z \in \partial\Omega$ (there could be more than one z with this property but we just pick one) such that $\text{dist}(I, \partial\Omega) = \text{dist}(I, z)$. We define $Q_I^* \in \mathbb{D}$ as the unique dyadic cube such that $z \in Q_I^*$ with $\ell(Q_I^*) = \ell(I)$. We note that the construction in [HM1] guarantees that $I \in \mathcal{W}_{Q_I^*}^*$ (indeed, this property holds for any other nearest dyadic cube with side length $\ell(I)$). The reader is referred to [HM1] for full details.

For a given $Q \in \mathbb{D}$, the **Carleson box** relative to Q is defined by

$$T_Q := \text{int} \left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \right).$$

For a given family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$ and a given $Q \in \mathbb{D}$ we define the **local sawtooth region** relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F}, Q} := \text{int} \left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F}, Q}} U_{Q'} \right) = \text{int} \left(\bigcup_{I \in \mathcal{W}_{\mathcal{F}, Q}} I^* \right),$$

where $\mathcal{W}_{\mathcal{F}, Q} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F}, Q}} \mathcal{W}_{Q'}^*$. Analogously, we can slightly fatten the Whitney boxes and use I^{**} to define new fattened Whitney regions and sawtooth domains. More precisely,

$$(2.21) \quad T_Q^* := \text{int} \left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}^* \right), \quad \Omega_{\mathcal{F}, Q}^* := \text{int} \left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F}, Q}} U_{Q'}^* \right), \quad U_{Q'}^* := \bigcup_{I \in \mathcal{W}_{Q'}^*} I^{**}.$$

Similarly, we can define T_Q^{**} , $\Omega_{\mathcal{F}, Q}^{**}$ and U_Q^{**} by using I^{***} in place of I^{**} .

One can easily see that there is $\kappa_0 > c^{-1}$ (depending only on the allowable parameters and where c is the constant in (2.13)) so that

$$(2.22) \quad T_Q \subset T_Q^* \subset T_Q^{**} \subset \overline{T_Q^{**}} \subset \kappa_0 B_Q \cap \overline{\Omega} =: B_Q^* \cap \overline{\Omega}, \quad \forall Q \in \mathbb{D}.$$

Given a pairwise disjoint family $\mathcal{F} \subset \mathbb{D}$ (we also allow $\mathcal{F}$ to be the null set) and a constant $\rho > 0$, we derive another family $\mathcal{F}(\rho) \subset \mathbb{D}$ from $\mathcal{F}$ as follows. Augment $\mathcal{F}$ by adding cubes $Q \in \mathbb{D}$ whose side length $\ell(Q) \leq \rho$ and let $\mathcal{F}(\rho)$ denote the corresponding collection of maximal cubes with respect to the inclusion. Note that the corresponding discrete sawtooth region $\mathbb{D}_{\mathcal{F}(\rho)}$ is the union of all cubes $Q \in \mathbb{D}_{\mathcal{F}}$ such that $\ell(Q) > \rho$. For a given constant ρ and a cube $Q \in \mathbb{D}$, let $\mathbb{D}_{\mathcal{F}(\rho), Q}$ denote the local discrete sawtooth region and let $\Omega_{\mathcal{F}(\rho), Q}$ denote the local geometric sawtooth region relative to disjoint family $\mathcal{F}(\rho)$.

Given $Q \in \mathbb{D}$ and $0 < \epsilon < 1$, if we take $\mathcal{F}_0 = \emptyset$, one has that $\mathcal{F}_0(\epsilon \ell(Q))$ is the collection of $Q' \in \mathbb{D}$ such that $\epsilon \ell(Q)/2 < \ell(Q') \leq \epsilon \ell(Q)$. We then introduce $U_{Q, \epsilon} = \Omega_{\mathcal{F}_0(\epsilon \ell(Q)), Q}$, which is a Whitney region relative to Q whose distance to $\partial\Omega$ is of the order of $\epsilon \ell(Q)$. For later use, we observe that given $Q_0 \in \mathbb{D}$, the sets $\{U_{Q, \epsilon}\}_{Q \in \mathbb{D}_{Q_0}}$ have bounded overlap with constant that may depend on ϵ . Indeed, suppose that there is $X \in U_{Q, \epsilon} \cap U_{Q', \epsilon}$ with $Q, Q' \in \mathbb{D}_{Q_0}$. By construction $\ell(Q) \approx_\epsilon \delta(X) \approx_\epsilon \ell(Q')$ and

$$\text{dist}(Q, Q') \leq \text{dist}(X, Q) + \text{dist}(X, Q') \lesssim_\epsilon \ell(Q) + \ell(Q') \approx_\epsilon \ell(Q).$$

The bounded overlap property follows then at once.

2.4. PDE estimates. Next, we recall several facts concerning elliptic measure and Green's functions. For our first results we will only assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is an open set, not necessarily connected with $\partial\Omega$ being AR. Later we will focus on the case where Ω is 1-sided CAD.

Let $Lu = -\text{div}(A \nabla u)$ be a variable coefficient second order divergence form operator with $A(X) = (a_{i,j}(X))_{1 \leq i, j \leq n+1}$ being a real (non-necessarily symmetric) $(n+1) \times (n+1)$ matrix such that $a_{i,j} \in L^\infty(\Omega)$ for $1 \leq i, j \leq n+1$, and A is uniformly elliptic, that is, there exists $1 \leq \Lambda < \infty$ such that

$$\Lambda^{-1} |\xi|^2 \leq A(X) \xi \cdot \xi, \quad |A(X) \xi \cdot \eta| \leq \Lambda |\xi| |\eta|,$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and almost every $X \in \Omega$. We write $L^\top$ to denote the transpose of L , or, in other words, $L^\top u = -\text{div}(A^\top \nabla u)$ with $A^\top$ being the transpose matrix of A .

We say that a function $u \in W_{\text{loc}}^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ in Ω or that $Lu = 0$ in the weak sense in Ω , if

$$\iint_{\Omega} A(X) \nabla u(X) \cdot \nabla \Phi(X) dX = 0, \quad \forall \Phi \in C_0^\infty(\Omega).$$

Associated with L and $L^\top$ one can respectively construct the elliptic measures $\{\omega_L^X\}_{X \in \Omega}$ and $\{\omega_{L^\top}^X\}_{X \in \Omega}$, and the Green functions G_L and $G_{L^\top}$ (see [HMT] for full details). We next present some of the properties that these object satisfy and that will be used throughout this paper.

Lemma 2.23. *Suppose that $\partial\Omega$ is n -dimensional AR. Then there are uniform constants $c \in (0, 1)$ and $C \in (1, \infty)$, depending only on n , AR, and Λ such that for every $x \in \partial\Omega$, and every $r \in (0, \text{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$, then*

$$(2.24) \quad \omega_L^Y(\Delta(x, r)) \geq 1/C > 0.$$

We refer the reader to [Bo, Lemma 1] for the proof in the harmonic case and to [HMT] for general elliptic operators. See also [HKM, Theorem 6.18] and [Z, Section 3].

A proof of the next lemma may be found in [HMT]. We note that, in particular, the AR hypothesis implies that $\partial\Omega$ is Wiener regular at every point (see [HLMN, Lemma 3.27]). In fact, it satisfies the Capacity Density Condition (CDC) (see [Z, Section 3]).

Lemma 2.25. *Let Ω be an open set with n -dimensional AR boundary. Let $Lu = -\operatorname{div}(A \nabla u)$ be as above. There are positive, finite constants C , depending only on dimension, Λ and c_θ , depending on dimension, Λ , and $\theta \in (0, 1)$, such that G_L , the Green function associated with L , satisfies*

$$(2.26) \quad G_L(X, Y) \leq C |X - Y|^{1-n}$$

$$(2.27) \quad c_\theta |X - Y|^{1-n} \leq G_L(X, Y), \quad \text{if } |X - Y| \leq \theta \delta(X), \quad \theta \in (0, 1);$$

$$(2.28) \quad G_L(\cdot, Y) \in C(\overline{\Omega} \setminus \{Y\}) \quad \text{and} \quad G_L(\cdot, Y)|_{\partial\Omega} \equiv 0, \quad \forall Y \in \Omega;$$

$$(2.29) \quad G_L(X, Y) \geq 0, \quad \forall X, Y \in \Omega, \quad X \neq Y;$$

$$(2.30) \quad G_L(X, Y) = G_{L^\top}(Y, X), \quad \forall X, Y \in \Omega, \quad X \neq Y.$$

Moreover, $G_L(\cdot, Y) \in W_{\text{loc}}^{1,2}(\Omega \setminus \{Y\})$ for any $Y \in \Omega$ and satisfies $LG_L(\cdot, Y) = \delta_Y$ in the weak sense in Ω , that is,

$$\iint_{\Omega} A(X) \nabla_Y G_L(X, Y) \cdot \nabla \Phi(X) dX = \Phi(Y), \quad \forall \Phi \in C_0^\infty(\Omega).$$

In particular, $G_L(\cdot, Y)$ is a weak solution to $LG_L(\cdot, Y) = 0$ in the open set $\Omega \setminus \{Y\}$.

Finally, the following Riesz formula holds

$$(2.31) \quad \iint_{\Omega} A^\top(Y) \nabla_Y G_{L^\top}(Y, X) \cdot \nabla \Phi(Y) dY = \Phi(X) - \int_{\partial\Omega} \Phi d\omega^X, \quad \text{for a.e. } X \in \Omega,$$

for every $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$.

Next, we recall a Caffarelli-Fabes-Mortola-Salsa estimate (cf. [CFMS], and [HMT] for the current version).

Lemma 2.32. *Let Ω be a 1-sided CAD domain. Let $B := B(x, r)$, with $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$. Then for $X \in \Omega \setminus 2B$ we have*

$$(2.33) \quad \frac{1}{C} \omega_L^X(\Delta) \leq r^{n-1} G_L(X, X_\Delta) \leq C \omega_L^X(\Delta).$$

The constant in (2.33) depends **only** on Λ , dimension and on the constants in the 1-sided CAD character.

Lemma 2.34. *Suppose that Ω is a 1-sided CAD domain. Let $B := B(x, r)$, $x \in \partial\Omega$, $\Delta := B \cap \partial\Omega$ and $X \in \Omega \setminus 4B$. Then there is a uniform constant C , depending **only** on Λ , dimension and on the constants in the 1-sided CAD character, such that*

$$(2.35) \quad \omega_L^X(2\Delta) \leq C \omega_L^X(\Delta).$$

3. AUXILIARY RESULTS

We have the following Poincaré inequality which is an improvement of [HM1, Lemma 4.8].

Lemma 3.1. *Suppose that Ω is a 1-CAD. Fix $Q_0 \in \mathbb{D}$, a (possibly empty) pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_{Q_0}$, and let $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$. Then for every p , $1 \leq p < \infty$, and for every small $\epsilon > 0$, there is a constant $C_{\epsilon, p}$ such that*

$$(3.2) \quad \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |f(X) - c_{Q, \epsilon}|^p dX \leq C_{\epsilon, p} \ell(Q)^p \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |\nabla f(X)|^p dX,$$

where $c_{Q, \epsilon} := |\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}|^{-1} \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} f dX$. In particular, the previous Poincaré inequality holds for $U_{Q, \epsilon}$ replacing $\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}$.

Proof. Without loss of generality we may assume that $\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q} \neq \emptyset$. We first observe that

$$\begin{aligned}
 (3.3) \quad & \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} \left| f(X) - \frac{1}{|\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}|} \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} f(Y) dY \right|^p dX \\
 & \leq \frac{1}{|\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}|} \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} \iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |f(X) - f(Y)|^p dX dY \\
 & \leq \frac{1}{|\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}|} \sum_{I, J \in \mathcal{W}_{\mathcal{F}(\epsilon \ell(Q)), Q}} \iint_I \iint_{J^*} |f(X) - f(Y)|^p dX dY.
 \end{aligned}$$

Fix now $I, J \in \mathcal{W}_{\mathcal{F}(\epsilon \ell(Q)), Q}$. Note that $\text{dist}(I, J) \lesssim \ell(I) \approx \ell(J) \approx \ell(Q)$ (where the implicit constants depend upon ϵ). By [HM1, Lemma 3.61] there is a chain $\{I_1, I_2, \dots, I_N\} \subset \mathcal{W}_{\mathcal{F}(\epsilon \ell(Q)), Q}$, of bounded cardinality N depending only on dimension, the 1-sided CAD constants of Ω , and ϵ , such that $I_1 = J$, $I_N = I$, $\ell(I_j) \approx \ell(I) \approx \ell(J)$ for each j (again the implicit constants depend upon ϵ), and for which $\cup_{j=1}^N I_j^*$ contains a Harnack Chain which connects the centers of I and J . Moreover, the chain may be constructed so that $I_j^* \cap I_{j+1}^* \neq \emptyset$, $1 \leq j \leq N-1$. Hence, by telescoping and using the standard Poincaré inequality

$$\begin{aligned}
 (3.4) \quad & \left(\iint_{I^*} \iint_{J^*} |f(X) - f(Y)|^p dX dY \right)^{\frac{1}{p}} \\
 & \lesssim \ell(I)^{\frac{n+1}{p}} \left(\left(\iint_{I^*} |f(X) - f_{I^*}|^p dX \right)^{\frac{1}{p}} + \left(\iint_{J^*} |f(Y) - f_{J^*}|^p dY \right)^{\frac{1}{p}} \right) + \ell(I)^{\frac{2(n+1)}{p}} \sum_{j=1}^{N-1} |f_{I_j^*} - f_{I_{j+1}^*}| \\
 & \lesssim \ell(I)^{\frac{n+1}{p}+1} \left(\iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |\nabla f(X)|^p dX \right)^{\frac{1}{p}} + \ell(I)^{\frac{2(n+1)}{p}} \sum_{j=1}^{N-1} |f_{I_j^*} - f_{I_{j+1}^*}|,
 \end{aligned}$$

where we have used the notation $f_{I_j^*} := |I_j^*|^{-1} \iint_{I_j^*} f$, and where the implicit constants depend on p and ϵ . To analyze the last term, take $1 \leq j \leq N-1$. Recall that $I_j^* = (1+\lambda)I_j$ with I_j being a dyadic Whitney cube. The same applies to I_{j+1}^* . Also, by choice of λ and since $I_j^* \cap I_{j+1}^* \neq \emptyset$ it follows that ∂I_j meets ∂I_{j+1} , which in turns implies that $\ell(I_j) \approx \ell(I_{j+1})$. Hence, one can find a cube $\tilde{I} \subset I_j^* \cap I_{j+1}^*$ with $\ell(\tilde{I}) \approx \lambda \ell(I_j) \approx \lambda \ell(I_{j+1})$. Then, by using again the standard Poincaré inequality we conclude

$$\begin{aligned}
 |f_{I_j^*} - f_{I_{j+1}^*}| & \leq |f_{I_j^*} - f_{\tilde{I}}| + |f_{\tilde{I}} - f_{I_{j+1}^*}| \leq \frac{1}{|\tilde{I}|} \iint_{\tilde{I}} |f(X) - f_{I_j^*}| dX + \frac{1}{|\tilde{I}|} \iint_{\tilde{I}} |f(X) - f_{I_{j+1}^*}| dX \\
 & \lesssim \frac{1}{|I_j^*|} \iint_{I_j^*} |f(X) - f_{I_j^*}| dX + \frac{1}{|I_{j+1}^*|} \iint_{I_{j+1}^*} |f(X) - f_{I_{j+1}^*}| dX \\
 & \lesssim \frac{\ell(I_j^*)}{|I_j^*|} \iint_{I_j^*} |\nabla f(X)| dX + \frac{\ell(I_{j+1}^*)}{|I_{j+1}^*|} \iint_{I_{j+1}^*} |\nabla f(X)| dX
 \end{aligned}$$

where the implicit constants depend on n and λ . Now we plug this estimate into (3.4), use that $\ell(I_j^*) \approx \ell(I) \approx \ell(Q)$ (with constants that depend on ϵ), the bounded overlap of the family $\{I_j^*\}_{j=1}^N$ and that N depends upon ϵ (it also depends on I and J , but in a uniformly bounded manner for ϵ fixed):

$$\begin{aligned}
 & \left(\iint_{I^*} \iint_{J^*} |f(X) - f(Y)|^p dX dY \right)^{\frac{1}{p}} \\
 & \lesssim \ell(I)^{\frac{n+1}{p}+1} \left(\iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |\nabla f(X)|^p dX \right)^{\frac{1}{p}} + \ell(I)^{\frac{n+1}{p}+1-\frac{n+1}{p'}} \iint_{\cup_{j=1}^N I_j^*} |\nabla f(Y)| dY
 \end{aligned}$$

$$\begin{aligned}
&\lesssim \ell(I)^{\frac{n+1}{p}+1} \left(\iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |\nabla f(X)|^p dX \right)^{\frac{1}{p}} + \ell(I)^{\frac{n+1}{p}+1-\frac{n+1}{p'}} |\cup_{j=1}^N I_j^*|^{\frac{1}{p'}} \left(\iint_{\cup_{j=1}^N I_j^*} |\nabla f(Y)|^p dY \right)^{\frac{1}{p}} \\
&\lesssim |\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}|^{\frac{1}{p}} \ell(Q) \left(\iint_{\Omega_{\mathcal{F}(\epsilon \ell(Q)), Q}} |\nabla f(Y)|^p dY \right)^{\frac{1}{p}}.
\end{aligned}$$

Both in the last line above as in order to conclude the proof of (3.2) from this inequality and (3.3) we need to observe that

$$\#\mathcal{W}_{\mathcal{F}(\epsilon \ell(Q)), Q} \leq \#\{Q' \in \mathbb{D}_Q : \ell(Q') > \epsilon \ell(Q)\} \leq c_\epsilon.$$

□

The following result is of purely real variable nature and establishes that if a measure satisfies an A_∞ type condition on a cube Q_0 then a stopping time argument allows us to extract a pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_{Q_0}$ such that the averages of the measure for cubes “above” the sawtooth (i.e., in $\mathbb{D}_{\mathcal{F}, Q_0}$) are essentially constant. Additionally, the complement of the union of the cubes in $\mathcal{F}$ are an ample portion of Q_0 , this means that the local sawtooth region $\Omega_{\mathcal{F}, Q_0}$ has an ample contact with Q_0 .

Lemma 3.5. *Let $Q_0 \in \mathbb{D}$ and let μ be a non-negative regular Borel measure on Q_0 . Assume that $\mu \ll \sigma$ on Q_0 and write $k = d\mu/d\sigma$. Assume also that there exist $K_0 \geq 1$, $\theta > 0$ such that*

$$(3.6) \quad 1 \leq \frac{\mu(Q_0)}{\sigma(Q_0)} \leq K_0 \quad \text{and} \quad \frac{\mu(F)}{\sigma(Q_0)} \leq K_0 \left(\frac{\sigma(F)}{\sigma(Q_0)} \right)^\theta, \quad \forall F \subset Q_0.$$

Then, there exists a pairwise disjoint family $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ such that

$$(3.7) \quad \sigma\left(Q_0 \setminus \bigcup_{Q_j \in \mathcal{F}} Q_j\right) \geq K_1^{-1} \sigma(Q_0)$$

and

$$(3.8) \quad \frac{1}{2} \leq \frac{\mu(Q)}{\sigma(Q)} \leq K_0 K_1, \quad \forall Q \in \mathbb{D}_{\mathcal{F}, Q_0},$$

where $K_1 = (4 K_0)^{\frac{1}{\theta}}$

Proof. The proof is based on a stopping time argument similar to those used in the proof of the Kato square root conjecture [HMc, HLMc, AHLMcT], a more refined version appears in [HLMN, HM2].

Let $\mathcal{F} = \{Q_j\}_j$ be the collection of dyadic cubes contained in Q_0 that are maximal, and therefore pairwise disjoint, with respect to the property that either

$$(3.9) \quad \frac{\mu(Q)}{\sigma(Q)} < \frac{1}{2} \quad \text{and/or} \quad \frac{\mu(Q)}{\sigma(Q)} > K_0 K_1.$$

Note that (3.6) and the fact that $K_1 > 1$ imply that $\mathcal{F} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$. Also, the maximality of the cubes in $\mathcal{F}$ immediately gives (3.8).

On the other hand, we observe that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_1$ corresponds to the family stopping time cubes with respect to the first criterion in (3.9) and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$ is comprised of the maximal cubes for which the first condition in (3.9) fails but the second holds. Set

$$F_0 = Q_0 \setminus \left(\bigcup_{Q_j \in \mathcal{F}} Q_j \right), \quad F_1 = \bigcup_{Q_j \in \mathcal{F}_1} Q_j, \quad F_2 = \bigcup_{Q_j \in \mathcal{F}_2} Q_j,$$

so that $Q_0 = F_0 \cup F_1 \cup F_2$.

We first handle the cubes in $\mathcal{F}_1$ which, by construction, satisfy

$$\mu(F_1) = \sum_{Q_j \in \mathcal{F}_1} \mu(Q_j) \leq \frac{1}{2} \sum_{Q \in \mathcal{F}_1} \sigma(Q) = \frac{1}{2} \sigma(F_1) \leq \frac{1}{2} \sigma(Q_0).$$

On the other hand, the definition of the family $\mathcal{F}_2$ and (3.6) give

$$\sigma(F_2) = \sum_{Q_j \in \mathcal{F}_2} \sigma(Q_j) \leq \frac{1}{K_0 K_1} \sum_{Q_j \in \mathcal{F}_2} \mu(Q_j) = \frac{1}{K_0 K_1} \mu(F_2) \leq \frac{1}{K_0 K_1} \mu(Q_0) \leq \frac{1}{K_1} \sigma(Q_0).$$

This, (3.6), and our choice of K_1 yield

$$\frac{\mu(F_2)}{\sigma(Q_0)} \leq K_0 \left(\frac{\sigma(F_2)}{\sigma(Q_0)} \right)^\theta \leq \frac{K_0}{K_1^\theta} = \frac{1}{4}.$$

Collecting the estimates obtained for F_1 and F_2 , and using again (3.6) we see that

$$\sigma(Q_0) \leq \mu(Q_0) = \mu(F_0) + \mu(F_1) + \mu(F_2) \leq \mu(F_0) + \frac{3}{4} \sigma(Q_0).$$

Hiding the last term on the right hand side and by (3.6) one can conclude that

$$\frac{1}{4} \leq \frac{\mu(F_0)}{\sigma(Q_0)} \leq K_0 \left(\frac{\sigma(F_0)}{\sigma(Q_0)} \right)^\theta,$$

which is (3.7) and the proof is complete. $\square$

With a slight abuse of notation, let Q^0 be either $\partial\Omega$, and in that case $\mathbb{D}_{Q^0} := \mathbb{D}$, or a fixed cube in $\mathbb{D}$, hence $\mathbb{D}_{Q^0}$ is the family of dyadic subcubes of Q^0 . Let $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q^0}}$ be a sequence of non-negative numbers indexed by the dyadic “cubes” in $\mathbb{D}_{Q^0}$, and for any collection $\mathbb{D}' \subset \mathbb{D}_{Q^0}$, we define an associated discrete “measure”

$$(3.10) \quad m_\alpha(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q.$$

We say that m is a “Carleson measure” (with respect to σ) in Q^0 , if

$$(3.11) \quad \|m_\alpha\|_{C(Q^0)} := \sup_{Q \in \mathbb{D}_{Q^0}} \frac{m_\alpha(\mathbb{D}_Q)}{\sigma(Q)} < \infty$$

For simplicity, when $Q^0 = \partial\Omega$ we simply write $\|m_\alpha\|_C$.

Our next result establishes that to show that m_α is a Carleson measure it suffices to check (3.11) only on “sawtooths with an ample contact”:

Lemma 3.12. *Let Q^0 be either $\partial\Omega$ or a fixed cube in $\mathbb{D}$. Let $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q^0}}$ be a sequence of non-negative numbers and consider m_α as defined above. Given $M_1 > 0$ and $K_1 \geq 1$, we assume that for every $Q_0 \in \mathbb{D}_{Q^0}$ there exists a pairwise disjoint family $\mathcal{F}_{Q_0} = \{Q_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ such that*

$$(3.13) \quad \sigma\left(Q_0 \setminus \bigcup_{Q_j \in \mathcal{F}_{Q_0}} Q_j\right) \geq K_1^{-1} \sigma(Q_0)$$

and

$$(3.14) \quad m_\alpha(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) \leq M_1 \sigma(Q_0).$$

Then, m_α is Carleson measure in Q^0 and moreover

$$(3.15) \quad \|m_\alpha\|_{C(Q^0)} = \sup_{Q \in \mathbb{D}_{Q^0}} \frac{m_\alpha(\mathbb{D}_Q)}{\sigma(Q)} \leq K_1 M_1.$$

Proof. We first take a sequence $\mathbb{D}^1 \subset \mathbb{D}^2 \subset \dots \subset \mathbb{D}^N \subset \dots \subset \mathbb{D}_{Q^0}$ such that $\mathbb{D}_{Q^0} = \cup_N \mathbb{D}^N$ and $\#\mathbb{D}^N = N$. For each $N \geq 1$ we let $\alpha_N := \{\alpha_Q^N\}_{Q \in \mathbb{D}_{Q^0}}$ where $\alpha_Q^N := \alpha_Q$ if $Q \in \mathbb{D}^N$ and $\alpha_Q^N := 0$ otherwise. Let $\mathfrak{m}_{\alpha_N}$ be the corresponding discrete measure associated with α_N and set $\ell_N = \min\{\ell(Q) : Q \in \mathbb{D}^N\} > 0$.

We first note that

$$\|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)} = \sup_{Q \in \mathbb{D}_{Q^0}} \frac{\mathfrak{m}_{\alpha_N}(\mathbb{D}_Q)}{\sigma(Q)} = \sup_{Q \in \mathbb{D}_{Q^0}, \ell(Q) \geq \ell_N} \frac{1}{\sigma(Q)} \sum_{Q' \in \mathbb{D}_Q \cap \mathbb{D}^N} \alpha_{Q'} \lesssim \frac{1}{(\ell_N)^n} \sum_{Q' \in \mathbb{D}^N} \alpha_{Q'} < \infty.$$

Fix now $Q_0 \in \mathbb{D}_{Q^0}$ and let $\mathcal{F}_{Q_0}$ be the associated pairwise disjoint family given by our hypotheses. By the definition of $\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$ and by (3.13) we have

$$\begin{aligned} \mathfrak{m}_{\alpha_N}(\mathbb{D}_{Q_0} \setminus \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) &= \sum_{Q_j \in \mathcal{F}_{Q_0}} \mathfrak{m}_{\alpha_N}(\mathbb{D}_{Q_j}) \leq \|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)} \sum_{Q_j \in \mathcal{F}_{Q_0}} \sigma(Q_j) \\ &= \|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)} \sigma\left(\bigcup_{Q_j \in \mathcal{F}_{Q_0}} Q_j\right) \leq (1 - K_1^{-1}) \|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)} \sigma(Q_0). \end{aligned}$$

This and (3.14) yield

$$\frac{\mathfrak{m}_{\alpha_N}(\mathbb{D}_{Q_0})}{\sigma(Q_0)} = \frac{\mathfrak{m}_{\alpha_N}(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0})}{\sigma(Q_0)} + \frac{\mathfrak{m}_{\alpha_N}(\mathbb{D}_{Q_0} \setminus \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0})}{\sigma(Q_0)} \leq M_1 + (1 - K_1^{-1}) \|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)}.$$

Note that this estimate holds for every $Q_0 \in \mathbb{D}_{Q^0}$. Hence, we conclude that

$$\|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)} = \sup_{Q \in \mathbb{D}_{Q^0}} \frac{\mathfrak{m}_{\alpha_N}(\mathbb{D}_Q)}{\sigma(Q)} \leq M_1 + (1 - K_1^{-1}) \|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)}.$$

We can then hide the last term (which is finite as observed above) to obtain $\|\mathfrak{m}_{\alpha_N}\|_{C(Q^0)} \leq K_1 M_1$ and letting $N \rightarrow \infty$ we conclude (3.15). $\square$

4. PROOF OF THE MAIN RESULT

Given $0 < c_0 < 1$, let $\mathcal{B} = \mathcal{B}(c_0)$ denote the collection of $Q \in \mathbb{D}$ for which the c_0 -exterior Corkscrew condition (see Definition 2.16) fails. Set $\alpha := \{\alpha_Q\}_{Q \in \mathbb{D}}$ with

$$(4.1) \quad \alpha_Q := \begin{cases} \sigma(Q), & \text{if } Q \in \mathcal{B}, \\ 0, & \text{otherwise.} \end{cases}$$

Associate to α the discrete measure $\mathfrak{m}_\alpha$ as above, which depends on the parameter c_0 . We are going to prove that under the assumptions in Theorem 1.5 the collection $\mathcal{B}$ satisfies a packing condition, i.e., that $\mathfrak{m}$ is a discrete Carleson measure, provided that c_0 is small enough.

Proposition 4.2. *Under the assumptions of Theorem 1.5, there is c_0 sufficiently small and $M_1 \geq 1$, such that if $\mathcal{B} = \mathcal{B}(c_0)$, its associated measure $\mathfrak{m}_\alpha$ as above satisfies the packing condition*

$$(4.3) \quad \|\mathfrak{m}_\alpha\|_C := \sup_{Q \in \mathbb{D}} \frac{\mathfrak{m}_\alpha(\mathbb{D}_Q)}{\sigma(Q)} = \sup_{Q \in \mathbb{D}} \frac{1}{\sigma(Q)} \sum_{Q' \in \mathcal{B}: Q' \subset Q} \sigma(Q') \leq M_1.$$

The constants c_0 and M_1 depend only upon dimension, Λ , the 1-sided CAD constants, $\|\nabla A\|_\infty$, $\|\nabla A\|_{C(\Omega)}$ and finally q and C in (1.4).

Assuming this result momentarily, we fix a cube $Q \in \mathbb{D}(\partial\Omega)$, and we seek to show that Ω_{ext} has a Corkscrew point relative to Q . Let $\Delta_Q \subset Q$ denote the surface ball defined in (2.13)–(2.14). Take Q_1 , a sub-cube of Q of maximal size contained in Δ_Q , and observe that $\ell(Q_1) \geq c\ell(Q)$. We

claim that there exists $Q' \in \mathbb{D}_{Q_1} \setminus \mathcal{B}$ such that $\ell(Q') \geq 2^{-[M_1]} \ell(Q_1)$ (here $[M_1]$ is the biggest integer smaller than or equal to M_1). Otherwise, by (4.3) (applied to Q_1)

$$([M_1] + 1) \sigma(Q_1) = \sum_{k=0}^{[M_1]} \sum_{\substack{Q \in \mathbb{D}_{Q_1} \\ \ell(Q) = 2^{-k} \ell(Q_1)}} \sigma(Q) \leq \sum_{Q \in \mathcal{B}: Q \subset Q_1} \sigma(Q) \leq M_1 \sigma(Q_1),$$

which readily leads to a contradiction. Hence there is $Q' \in \mathbb{D}_{Q_1} \setminus \mathcal{B}$ such that $\ell(Q') \geq 2^{-[M_1]} \ell(Q_1) \geq c 2^{-[M_1]} \ell(Q)$. Since Q' enjoys the c_0 -exterior Corkscrew condition, so does Q , but with c_0 replaced by $c'_0 = c_0 c 2^{-[M_1]}$. On the other hand, every surface ball contains a cube of comparable diameter, this means that there is an exterior Corkscrew point relative to every surface ball on the boundary, and therefore Ω is NTA, and hence chord-arc. This completes the proof of Theorem 1.5 modulo Proposition 4.2.

To prove Proposition 4.2 we are going to use Lemma 3.12 with $Q^0 = \partial\Omega$. Fix $Q_0 \in \mathbb{D}_{Q^0} = \mathbb{D}$, an arbitrary dyadic cube, and our goal is to obtain (3.14) for some pairwise disjoint family $\mathcal{F}_{Q_0} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ for which (3.13) holds. We note that it suffices to consider the case $\ell(Q_0) < \text{diam}(\partial\Omega)/M_0$ with M_0 large enough (depending only on the allowable parameters). In fact, assuming this, in order to prove the case $\text{diam}(\partial\Omega)/M_0 \leq \ell(Q_0) \leq \text{diam}(\partial\Omega)$ (of course this is meaningful only if $\text{diam}(\partial\Omega) < \infty$), we cover Q_0 by disjoint cubes $\{Q_0^k\}_k$ with $\text{diam}(\partial\Omega)/(2M_0) \leq \ell(Q_0^k) < \text{diam}(\partial\Omega)/M_0$. For each Q_0^k , by the previous case one can find $\mathcal{F}_{Q_0^k}$ so that (3.13) and (3.14) hold with Q_0^k in place of Q_0 . Note that if we set $\mathcal{F}_{Q_0} = \cup_k \mathcal{F}_{Q_0^k}$ we automatically have (3.13) and moreover

$$\begin{aligned} m_\alpha(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) &\leq \sum_k m_\alpha(\mathbb{D}_{\mathcal{F}_{Q_0^k}, Q_0^k}) + \sum_{\substack{Q \in \mathbb{D}_{Q_0} \cap \mathcal{B} \\ \ell(Q) \geq \text{diam}(\partial\Omega)/M_0}} \sigma(Q) \\ &\leq M_1 \sum_k \sigma(Q_0^k) + C_{M_0} \sigma(Q_0) \leq (M_1 + C_{M_0}) \sigma(Q_0). \end{aligned}$$

Thus, we have proved that for every $Q_0 \in \mathbb{D}$, (3.14) holds for some pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ satisfying (3.13). Hence Lemma 3.12 yields (4.3) with some constant M'_1 and hence the proof of Proposition 4.2 would be complete.

In view of the previous observation we fix $Q_0 \in \mathbb{D}$ such that $\ell(Q_0) < \text{diam}(\partial\Omega)/M_0$. We choose M_0 so that if we set $X_0 = X_{\sqrt{M_0}\Delta_{Q_0}}$ one has that $2\kappa_0 r_{Q_0} \leq \delta(X_0) \leq \sqrt{M_0} r_{Q_0}$, where we recall that κ_0 was chosen (depending only on the allowable parameters) so that (2.22) holds. In such a case, $\text{dist}(X_0, T_{Q_0}^{**}) \geq \kappa_0 r_{Q_0}$, hence the pole X_0 will be away from where the argument takes place. By Lemma 2.23 and Harnack's inequality there is $C_0 \geq 1$ depending on the allowable parameters and M_0 such that $\omega_L^{X_0}(Q_0) \geq C_0^{-1}$. We now normalize the elliptic measure and the Green function as follows

$$(4.4) \quad \omega := C_0 \sigma(Q_0) \omega_L^{X_0} \quad \text{and} \quad \mathcal{G}(\cdot) := C_0 \sigma(Q_0) G_L(X_0, \cdot).$$

Note that away from X_0 , $L^\top \mathcal{G}(\cdot) = C_0 \sigma(Q_0) L^\top G_L(X_0, \cdot) = C_0 \sigma(Q_0) L^\top G_{L^\top}(\cdot, X_0) = 0$ (see Lemma 2.25). Moreover by our choice of X_0 , Lemmas 2.32 and 2.34, (2.13), and (2.14) it follows that

$$(4.5) \quad \frac{\mathcal{G}(X_Q)}{\ell(Q)} \approx \frac{\omega(Q)}{\sigma(Q)}, \quad \forall Q \in \mathbb{D}_{Q_0}.$$

On the other hand, since $\omega_L^{X_0}(\partial\Omega) \leq 1$,

$$(4.6) \quad 1 \leq \frac{\omega(Q_0)}{\sigma(Q_0)} \leq C_0.$$

By assumption, $\omega \ll \sigma$, and if $k = d\omega/d\sigma$ denotes the normalized Poisson kernel it follows that, for M_0 is large enough, (1.4), (2.13), and (2.14) yield

$$\left(\int_{Q_0} k(y)^q d\sigma(y) \right)^{\frac{1}{q}} \leq C^{\frac{1}{q}} C_0 =: K_0,$$

where C is the constant in (1.4). As a consequence of that, (4.6) and Hölder's inequality one can derive

$$(4.7) \quad \frac{\omega(F)}{\sigma(Q_0)} = \int_{Q_0} 1_F(y) k(y) d\sigma(y) \leq K_0 \left(\frac{\sigma(F)}{\sigma(Q_0)} \right)^{\frac{1}{q'}}, \quad \forall F \subset Q_0.$$

Hence we can apply Lemma 3.5 to $\mu = \omega$ and obtain a pairwise disjoint family $\mathcal{F}_{Q_0} = \{Q_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ verifying (3.7) and (3.8). Thus, as observed before (see Lemma 3.12) we wish to find M_1 independent of Q_0 such that

$$(4.8) \quad m_\alpha(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) \leq M_1 \sigma(Q_0).$$

Hence, in what follows $Q_0 \in \mathbb{D}$ and $\mathcal{F}_{Q_0}$ is a pairwise disjoint family $\mathcal{F}_{Q_0} = \{Q_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ verifying (3.7) and

$$(4.9) \quad \frac{1}{2} \leq \frac{\omega(Q)}{\sigma(Q)} \leq K_0 K_1, \quad \forall Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0},$$

which is (3.8) with $\mu = \omega$ (see (4.4)).

Let us now fix $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0} \cap \mathcal{B}$ and a point $z_Q \in \Delta_Q \subset Q$. Set $B := B(z_Q, r/4)$, with $r := r_Q \approx \ell(Q)$, and $\Delta := B \cap \partial\Omega$. Take $\Phi \in C_0^\infty(B)$, with $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $\frac{1}{2}B$, and $\|\nabla\Phi\|_\infty \lesssim r^{-1}$. Let $0 < \epsilon \ll 1$ to be chosen and set $\vec{\beta} := |U_{Q,\epsilon}|^{-1} \iint_{U_{Q,\epsilon}} A^\top(Y) \nabla \mathcal{G}(Y) dY$. Recall that (see Section 2.3) $U_{Q,\epsilon} = \Omega_{\mathcal{F}_0(\epsilon\ell(Q)), Q}$ where $\mathcal{F}_0 = \emptyset$ and hence $\mathcal{F}_0(\epsilon\ell(Q))$ is the collection of $Q' \in \mathbb{D}$ such that $\epsilon\ell(Q)/2 < \ell(Q') \leq \epsilon\ell(Q)$. In particular, $Q \in \mathbb{D}_{\mathcal{F}_0(\epsilon\ell(Q)), Q}$ and (2.22) yields $\text{int}(U_Q) \subset U_{Q,\epsilon} \subset T_Q \subset B_Q^*$. Notice that $\mathcal{W}_Q^* \neq \emptyset$ and hence there is $I \in \mathcal{W}_Q^*$ such that $I \subset \text{int}(U_Q)$ with $\ell(Q) \approx \ell(I)$ and consequently $|U_{Q,\epsilon}| \approx \ell(Q)^{n+1}$. Keeping in mind the normalization (4.4), our choice of X_0 and (2.30), we have that $L^\top \mathcal{G} = 0$ in the weak sense in $T_{Q_0}^{**}$. Thus, Caccioppoli's inequality, Harnack's inequality and (4.5) yield that for every $I \in \mathcal{W}_{Q'}^*$, $Q' \in \mathbb{D}_Q \subset \mathbb{D}_{Q_0}$

$$(4.10) \quad \iint_{I^*} |\nabla \mathcal{G}(Y)| dY \lesssim |I| \frac{\mathcal{G}(X(I))}{\delta(X(I))} \approx \ell(Q')^n \mathcal{G}(X_{Q'}) \approx \ell(Q') \omega(Q'),$$

and hence

$$(4.11) \quad |\vec{\beta}| \lesssim \ell(Q)^{-(n+1)} \iint_{T_Q} |\nabla \mathcal{G}(Y)| dY \lesssim \ell(Q)^{-(n+1)} \sum_{Q' \in \mathbb{D}_Q} \sum_{I \in \mathcal{W}_{Q'}^*} \iint_{I^*} |\nabla \mathcal{G}(Y)| dY \\ \lesssim \ell(Q)^{-n} \sum_{k=0}^{\infty} 2^{-k} \sum_{\substack{Q' \in \mathbb{D}_Q \\ \ell(Q') = 2^{-k} \ell(Q)}} \omega(Q') \lesssim \frac{\omega(Q)}{\sigma(Q)} \lesssim 1,$$

where we have used that $\mathcal{W}_{Q'}^*$ has uniformly bounded cardinality and the last estimate follows from (4.9) since $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$.

We next use Lemma 2.34, (2.31), and (2.30) (keeping in mind (4.4), (4.9) and moving slightly the pole X_0 if needed)

$$(4.12) \quad \sigma(Q) \approx \omega(Q) \approx \int_{\partial\Omega} \Phi d\omega = - \iint_{\Omega} A^\top(X) \nabla \mathcal{G}(X) \cdot \nabla \Phi(X) dX \\ = - \iint_{\Omega} (A^\top(X) \nabla \mathcal{G}(X) - \vec{\beta}) \cdot \nabla \Phi(X) dX - \iint_{\mathbb{R}^{n+1}} \vec{\beta} \cdot \nabla \Phi(X) dX + \iint_{\Omega_{\text{ext}}} \vec{\beta} \cdot \nabla \Phi(X) dX$$

$$\begin{aligned}
&= - \iint_{\Omega} (A^{\top}(X) \nabla \mathcal{G}(X) - \vec{\beta}) \cdot \nabla \Phi(X) dX + \iint_{\Omega_{\text{ext}}} \vec{\beta} \cdot \nabla \Phi(X) dX \\
&=: -\mathcal{I} + \mathcal{II}.
\end{aligned}$$

We first estimate $\mathcal{II}$. By [HM1, Lemma 5.7], the failure of the c_0 -exterior Corkscrew property implies that $|\Omega_{\text{ext}} \cap B| \lesssim c_0 r^{n+1}$. This and (4.11) give

$$(4.13) \quad |\mathcal{II}| \lesssim |\vec{\beta}| r^{-1} |\Omega_{\text{ext}} \cap B| \lesssim c_0 r^n \approx c_0 \sigma(Q).$$

To estimate $\mathcal{I}$ we proceed as follows.

$$\begin{aligned}
(4.14) \quad |\mathcal{I}| &\lesssim r^{-1} \iint_{\Omega \cap B} |A^{\top}(X) \nabla \mathcal{G}(X) - \vec{\beta}| dX \\
&\leq r^{-1} \left(\iint_{U_{Q,\epsilon}} |A^{\top}(X) \nabla \mathcal{G}(X) - \vec{\beta}| dX + \iint_{(\Omega \setminus U_{Q,\epsilon}) \cap B} |A^{\top}(X) \nabla \mathcal{G}(X) - \vec{\beta}| dX \right) \\
&=: r^{-1} (\mathcal{I}_1 + \mathcal{I}_2).
\end{aligned}$$

For $\mathcal{I}_1$ we use Hölder's inequality, our choice of $\vec{\beta}$, Lemma 3.1 and the fact that $\delta(X) \approx_{\epsilon} \ell(Q)$ for every $X \in U_{Q,\epsilon}$:

$$\begin{aligned}
(4.15) \quad \mathcal{I}_1 &\lesssim \ell(Q)^{\frac{n+1}{2}} \left(\iint_{U_{Q,\epsilon}} |A^{\top}(X) \nabla \mathcal{G}(X) - \vec{\beta}|^2 dX \right)^{\frac{1}{2}} \leq C_{\epsilon} \ell(Q)^{\frac{n+3}{2}} \left(\iint_{U_{Q,\epsilon}} |\nabla(A^{\top} \nabla \mathcal{G})(X)|^2 dX \right)^{\frac{1}{2}} \\
&\leq C_{\epsilon} r \sigma(Q)^{\frac{1}{2}} \left(\iint_{U_{Q,\epsilon}} |\nabla(A^{\top} \nabla \mathcal{G})(X)|^2 \delta(X) dX \right)^{\frac{1}{2}} =: C_{\epsilon} r \sigma(Q)^{\frac{1}{2}} \Upsilon_{Q,\epsilon}^{\frac{1}{2}}.
\end{aligned}$$

Before estimating $\mathcal{I}_2$ we need to make the following observation. Let $I \in \mathcal{W}$ be such that $I^* \cap B \neq \emptyset$ and pick $Y \in I^* \cap B$. In particular,

$$4 \operatorname{diam}(I) \leq \operatorname{dist}(I, \partial\Omega) \leq |Y - z_Q| < \frac{r}{4} \leq \frac{\ell(Q)}{4} < \frac{\operatorname{diam}(\partial\Omega)}{4M_0}.$$

Recall the construction of Q_I^* , the unique dyadic cube satisfying that $z \in Q_I^*$ and $\ell(Q_I^*) = \ell(I)$, where $z \in \partial\Omega$ is such that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, z)$. We claim that $Q_I^* \subset Q$. To show this let us take $Z \in I$ such that $\operatorname{dist}(I, \partial\Omega) = |Z - z|$. Then

$$|z - x_Q| \leq |z - Z| + |Z - Y| + |Y - z_Q| + |z_Q - x_Q| \leq \operatorname{dist}(I, \partial\Omega) + \operatorname{diam}(I^*) + \frac{r}{4} + r < 2r.$$

This implies that $z \in \Delta(x_Q, 2r) \subset Q$ (cf. (2.13)) and since $\ell(Q_I^*) = \ell(I) < \ell(Q)$ it follows that $Q_I^* \subset Q$ by the dyadic properties.

We are now ready to estimate $\mathcal{I}_2$. We first see that by choice of B we have that $B \cap \Omega \subset T_Q$. Indeed, let $Y \in B \cap \Omega$ and take $I \in \mathcal{W}$ so that $Y \in I$. Note that by the previous observation $Q_I^* \subset Q$. Note also that, as mentioned above, our construction guarantees that $I \in \mathcal{W}_{Q_I^*}^*$. All these yield $Y \in I \subset \operatorname{int}(I^*) \subset \operatorname{int}(U_{Q_I^*}) \subset T_Q$.

Once we have shown that $B \cap \Omega \subset T_Q$ one can easily see that $(\Omega \setminus U_{Q,\epsilon}) \cap B \subset T_Q \setminus U_{Q,\epsilon} \subset \Sigma_{\epsilon} := \{X \in \Omega : \delta(X) \lesssim \epsilon \ell(Q)\}$. Therefore, if ϵ is small enough,

$$(4.16) \quad \mathcal{I}_2 \lesssim |\vec{\beta}| |B \cap \Sigma_{\epsilon}| + \iint_{B \cap \Sigma_{\epsilon}} |\nabla \mathcal{G}| dX \lesssim \epsilon \ell(Q) r^n + \mathcal{I}_3 \approx r \epsilon \sigma(Q) + \mathcal{I}_3,$$

where we have used (4.11) and [HM1, Lemma 5.3]. To estimate $\mathcal{I}_3$, we use again the cubes Q_I^* as above associated with $I \in \mathcal{W}$ with $I^* \cap B \neq \emptyset$. As already mentioned in such an scenario, $Q_I^* \subset Q$ and $I \in \mathcal{W}_{Q_I^*}^*$. We can then invoke (4.10) to obtain

$$I_3 \leq \sum_{\substack{I \in \mathcal{W}: I^* \cap B \neq \emptyset \\ \ell(I) \lesssim \epsilon \ell(Q)}} \iint_{I^*} |\nabla \mathcal{G}(X)| dX \lesssim \sum_{\substack{I \in \mathcal{W}: I^* \cap B \neq \emptyset \\ \ell(I) \lesssim \epsilon \ell(Q)}} \omega(Q_I^*) \ell(Q_I^*) = \sum_{k: 2^{-k} \lesssim \epsilon \ell(Q)} 2^{-k} \sum_{\substack{I \in \mathcal{W}: I^* \cap B \neq \emptyset \\ \ell(I) = 2^{-k}}} \omega(Q_I^*),$$

Notice that for k fixed, the family $\{Q_I^*\}_{I \in \mathcal{W}: \ell(I) = 2^{-k}}$ has bounded overlap, hence

$$(4.17) \quad I_3 \lesssim \omega(Q) \sum_{k: 2^{-k} \lesssim \epsilon \ell(Q)} 2^{-k} \lesssim \epsilon \ell(Q) \omega(Q) \lesssim \epsilon r \sigma(Q),$$

where the last estimate follows again from (4.9) since $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$. Plugging (4.13), (4.14), (4.15), (4.16) and (4.17) into (4.12) we conclude that

$$\sigma(Q) \lesssim c_0 \sigma(Q) + C_\epsilon \sigma(Q)^{\frac{1}{2}} \Upsilon_{Q, \epsilon}^{\frac{1}{2}} + \epsilon \sigma(Q).$$

If ϵ and c_0 are taken small enough (we may assume for later use that $\epsilon = 2^{-N_\epsilon}$ for some $N_\epsilon \in \mathbb{N}$ large enough) the first and third term in the right hand side can be hidden and one easily arrives at

$$(4.18) \quad \sigma(Q) \lesssim_{\epsilon, c_0} \Upsilon_{Q, \epsilon} = \iint_{U_{Q, \epsilon}} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \delta(X) dX \\ \leq \sum_{\substack{Q' \in \mathbb{D}_Q \\ \epsilon \ell(Q) < \ell(Q') \leq \ell(Q)}} \iint_{U_{Q'}} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \delta(X) dX =: \sum_{\substack{Q' \in \mathbb{D}_Q \\ \epsilon \ell(Q) < \ell(Q') \leq \ell(Q)}} \Upsilon_{Q'}.$$

Hence, it has been shown that for a choice of ϵ and c_0 small enough, the previous estimate holds for all $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0} \cap \mathcal{B}(c_0)$. This in turn gives

$$(4.19) \quad m_\alpha(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) = \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0} \cap \mathcal{B}(c_0)} \sigma(Q) \lesssim_{\epsilon, c_0} \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}} \sum_{\substack{Q' \in \mathbb{D}_Q \\ \epsilon \ell(Q) < \ell(Q') \leq \ell(Q)}} \Upsilon_{Q'} \lesssim_\epsilon \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}} \Upsilon_Q,$$

where $\mathcal{F}_{Q_0}^\epsilon$ is the pairwise disjoint family of N_ϵ -descendants (recall that $\epsilon = 2^{-N_\epsilon}$) of the elements of $\mathcal{F}_{Q_0}$:

$$\mathcal{F}_{Q_0}^\epsilon := \bigcup_{Q \in \mathcal{F}_{Q_0}} \{Q' \in \mathbb{D}_Q : \ell(Q') = 2^{-N_\epsilon} \ell(Q) = \epsilon \ell(Q)\}.$$

We next claim that

$$(4.20) \quad \frac{\omega(Q)}{\sigma(Q)} \approx_\epsilon 1, \quad \forall Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0},$$

that is, both estimates in (4.9) can be transmitted from $\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$ to $\mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$, albeit with bounds that may depend on ϵ . To obtain that, fix $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$. By (4.9), we may assume that $Q \notin \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$. This means that there is $Q_1 \in \mathcal{F}_{Q_0} \subset \mathbb{D}_{Q_0}$ such that $Q \subset Q_1$. Since Q_1 splits into its N_ϵ -descendants, we can find $Q'_1 \in \mathcal{F}_{Q_0}^\epsilon$ such that $Q'_1 \cap Q \neq \emptyset$ and $\ell(Q'_1) = \epsilon \ell(Q_1)$. In turn, since $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$, it then follows that $Q'_1 \subsetneq Q \subset Q_1$. We use this, the AR property, the doubling property of ω (Lemma 2.34) and (4.9) (which holds for the dyadic parent of Q_1 , denoted by $\widehat{Q}_1$, since $Q_1 \in \mathcal{F}_{Q_0} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$)

$$\frac{\omega(Q)}{\sigma(Q)} \approx_\epsilon \frac{\omega(\widehat{Q}_1)}{\sigma(\widehat{Q}_1)} \approx 1.$$

This shows our claim.

Set $\tilde{\alpha} := \{\tilde{\alpha}_Q\}_{Q \in \mathbb{D}_{Q_0}}$ with

$$(4.21) \quad \tilde{\alpha}_Q := \begin{cases} \Upsilon_Q, & \text{if } Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}, \\ 0, & \text{otherwise.} \end{cases}$$

and associate to $\tilde{\alpha}$ the discrete measure $m_{\tilde{\alpha}}$ as in (3.10). Then, we may immediately see that (4.8) follows from (4.19) and

$$(4.22) \quad m_{\tilde{\alpha}}(\mathbb{D}_{Q_0}) \lesssim_\epsilon \sigma(Q_0).$$

Our goal is then to obtain (4.22) and in order to do that we shall distinguish between two cases depending whether or not A is symmetric. The main idea is that when A is symmetric, in the expression Υ_Q we can replace $\delta(X)$ by $\mathcal{G}(X)$ for every $X \in U_Q$ and for every $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$. Doing this, (4.22) will be obtained by an integration by parts argument. In the non-symmetric case, for the integration by parts to work, we would need to do the same thing but rather than $\mathcal{G}$ (which is essentially $G_L(X_0, \cdot)$, hence a null solution for $L^\top$, cf. (4.4)), we would need to work with essentially $G_{L^\top}(X_0, \cdot)$, hence a null solution for L . The latter would require to perform the stopping time in Lemma 3.5 with $\omega_{L^\top}^{X_0}$. However, it is not clear that one can apply Lemma 3.5 simultaneously to $\omega_L^{X_0}$ and $\omega_{L^\top}^{X_0}$ and obtain a family of cubes whose complement is still ample. We are going to overcome this by another use of Lemma 3.12, hence we will work in $\tilde{Q}_0 \in \mathbb{D}_{Q_0}$ and apply Lemma 3.5 to $\omega_L^{\tilde{X}_0}$ with $\tilde{X}_0$ being effectively a corkscrew point relative to $\tilde{Q}_0$.

4.1. The symmetric case. In this section we assume that A is symmetric.

We start observing that for every $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$ and every $X \in U_Q$ we have that $\delta(X) \approx_\epsilon \mathcal{G}(X)$ for every $X \in U_Q$ in view of Harnack's inequality, (4.5) and (4.20). This and the definitions of the sets $\{U_Q\}_{Q \in \mathbb{D}_{Q_0}}$ and $\Omega_{\mathcal{F}_{Q_0}^\epsilon, Q_0}^*$ (see Section 2.3) yield

$$(4.23) \quad m_{\tilde{\alpha}}(\mathbb{D}_{Q_0}) = \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}} \Upsilon_Q \lesssim_\epsilon \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}} \iint_{U_Q} |\nabla(A \nabla \mathcal{G})(X)|^2 \mathcal{G}(X) dX \\ \lesssim \iint_{\Omega_{\mathcal{F}_{Q_0}^\epsilon, Q_0}^*} |\nabla(A \nabla \mathcal{G})(X)|^2 \mathcal{G}(X) dX.$$

We next take an arbitrary N large enough and define $\mathcal{F}_N := \mathcal{F}_{Q_0}^\epsilon(2^{-N} \ell(Q_0))$ as in Section 2.3. That is, $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$ is the family of maximal cubes of the collection $\mathcal{F}_{Q_0}^\epsilon$ augmented by adding all dyadic cubes of size smaller than or equal than $2^{-N} \ell(Q_0)$. In particular, $Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$ and $\ell(Q) > 2^{-N} \ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N, Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'}, Q_0}$ if $N \leq N'$ and therefore $\Omega_{\mathcal{F}_N, Q_0}^* \subset \Omega_{\mathcal{F}_{N'}, Q_0}^* \subset \Omega_{\mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}}^*$. This and the monotone convergence theorem give that

$$(4.24) \quad \iint_{\Omega_{\mathcal{F}_{Q_0}^\epsilon, Q_0}^*} |\nabla(A \nabla \mathcal{G})(X)|^2 \mathcal{G}(X) dX = \lim_{N \rightarrow \infty} \iint_{\Omega_{\mathcal{F}_N, Q_0}^*} |\nabla(A \nabla \mathcal{G})(X)|^2 \mathcal{G}(X) dX.$$

We now formulate an auxiliary result that will easily lead us to the desired estimate. We note that the following proposition was previously announced, with a sketch of the proof, in [ABHM]. In the sequel, we shall present the full details, and treat also the non-symmetric case (see Proposition 4.36 below).

Proposition 4.25. *Assuming that A is symmetric and given $C_1 \geq 1$, one can find C depending on C_1 and the allowable parameters such that if $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$, $N \geq 1$, is a family of pairwise disjoint dyadic cubes satisfying*

$$(4.26) \quad C_1^{-1} \leq \frac{\omega(Q)}{\sigma(Q)} \leq C_1 \quad \text{and} \quad \ell(Q) > 2^{-N} \ell(Q_0), \quad \forall Q \in \mathbb{D}_{\mathcal{F}_N, Q_0},$$

then

$$(4.27) \quad \iint_{\Omega_{\mathcal{F}_N, Q_0}^*} |\nabla(A \nabla \mathcal{G})(X)|^2 \mathcal{G}(X) dX \leq C \sigma(Q_0).$$

Assuming this result momentarily we see that (4.20) and the construction of $\mathcal{F}_N$ give (4.26). Thus (4.22) follows from (4.23), (4.24), and (4.27). This completes the proof of Proposition 4.2 when A is symmetric, modulo obtaining the just stated proposition.

Proof of Proposition 4.25 for the Laplacian. The case when L is the Laplacian (that is, A is the identity matrix and hence (a), (b), and (c) of Hypothesis 1.1 are trivial) is rather simple and models the general case. To fix ideas, we first present this simple case.

Suppose for now that L is the Laplacian. We first observe that

$$(4.28) \quad \delta(X)|\nabla^2 \mathcal{G}(X)|, |\nabla \mathcal{G}(X)| \lesssim \frac{\mathcal{G}(X)}{\delta(X)} \approx 1, \quad \forall X \in \overline{\Omega_{\mathcal{F}_N, Q_0}^*} =: \overline{\Omega_\star},$$

albeit with bounds that depend on C_1 and the allowable parameters but which are uniform in N . To see the “ $\lesssim$ ” we use the harmonicity of $\mathcal{G}$, $\nabla \mathcal{G}$ and $\nabla^2 \mathcal{G}$; interior estimates, and Caccioppoli’s inequality (we recall that we chose X_0 so that it is away from $T_{Q_0}^*$ and also that $2^{-N} \ell(Q_0) \lesssim \delta(X) \lesssim \ell(Q_0)$ for every $X \in \overline{\Omega_\star}$). The proof “ $\approx$ ” is as follows: given $X \in \overline{\Omega_\star}$, there is $Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ and $I \in \mathcal{W}_Q^*$ such that $X \in I^{**}$. Note that $\delta(X) \approx \delta(X_Q) \approx \ell(I) \approx \ell(Q)$ and $|X - X_Q| \lesssim \ell(Q)$. This, Harnack’s inequality, (4.5), and (4.26) yield as desired

$$\frac{\mathcal{G}(X)}{\delta(X)} \approx \frac{\mathcal{G}(X_Q)}{\ell(Q)} \approx \frac{\omega(Q)}{\sigma(Q)} \approx 1.$$

We now proceed to obtain (4.27) with A being the identity matrix. Write “ ∂ ” to denote a fixed generic derivative. We use that $\mathcal{G}$ and $\partial \mathcal{G}$ are harmonic in $\overline{\Omega_\star}$ to see that in that set the following pointwise equalities hold

$$\operatorname{div} \nabla ((\partial \mathcal{G})^2) = 2 \operatorname{div} [(\partial \mathcal{G}) \nabla (\partial \mathcal{G})] = 2 |\nabla (\partial \mathcal{G})|^2$$

and

$$[\operatorname{div} \nabla ((\partial \mathcal{G})^2)] \mathcal{G} = \operatorname{div} [\nabla ((\partial \mathcal{G})^2) \mathcal{G}] - \nabla ((\partial \mathcal{G})^2) \cdot \nabla \mathcal{G} = \operatorname{div} [\nabla ((\partial \mathcal{G})^2) \mathcal{G} - (\partial \mathcal{G})^2 \nabla \mathcal{G}].$$

Note that $\Omega_\star$ is a finite union of fattened Whitney boxes, thus, its (outward) unit normal ν is well defined a.e. on $\partial \Omega_\star$. Hence the divergence theorem can be applied to obtain

$$(4.29) \quad 2 \iint_{\Omega_\star} |\nabla (\partial \mathcal{G})(X)|^2 \mathcal{G}(X) dX = \int_{\partial \Omega_\star} (\nabla ((\partial \mathcal{G})^2) \mathcal{G} - (\partial \mathcal{G})^2 \nabla \mathcal{G}) \cdot \nu dH^n \\ \lesssim \int_{\partial \Omega_\star} (|\nabla^2 \mathcal{G}| |\nabla \mathcal{G}| \mathcal{G} + |\nabla \mathcal{G}|^3) dH^n \lesssim H^n(\partial \Omega_\star) \lesssim \ell(Q_0)^n \approx \sigma(Q_0),$$

where we have used (4.28), that $\partial \Omega_\star$ is AR (cf. [HM1, Lemma 3.61]) and finally that $\operatorname{diam}(\partial \Omega_\star) \approx \ell(Q_0)$ (note that all bounds are independent of N). From (4.29), we immediately obtain (4.27) in the case of the Laplacian. $\square$

Looking at the previous argument the matrix A being non-constant (for both the symmetric and non-symmetric cases) raises several issues. The first one appears in (4.28): the “ $\approx$ ” is still correct but one does not expect to have the “ $\lesssim$ ” for general matrices A since, as opposed to the constant coefficient case, we no longer have that $\partial \mathcal{G}$ is a null solution of L . As we shall see below in Lemma 4.40, under the assumption that A satisfies (b) of Hypothesis 1.1, one can prove that the estimate for $\nabla \mathcal{G}$ in (4.28) holds pointwise and the estimate for $\nabla^2 \mathcal{G}$ holds in a L^2 -average sense via a Caccioppoli type estimate for second derivatives of solutions. The second issue is that the presence of A in (4.27) makes the algebra significantly more difficult as one has to distribute derivatives and some of them hit A . Finally, because the estimates for $\nabla^2 \mathcal{G}$ hold in an average sense, we cannot integrate by parts as in (4.29). We will solve this by producing some wiggling after incorporating a smooth cut-off of the domain (see Lemma 4.44) which will have the effect of replacing integrals on the boundary by “solid” integrals in a “strip” along the boundary.

Proof of Proposition 4.25. We just need to invoke Proposition 4.36 below with $Q_0 = \tilde{Q}_0$ since (4.37) follows at once from (4.26). Further details are left to the interested reader. $\square$

4.2. The non-symmetric case. As explained above in the non-symmetric case we are going to need to use again Lemma 3.12. Recall that our goal is to show (4.22). Applying Lemma 3.12 with $Q^0 = Q_0$, it suffices to take an arbitrary $\tilde{Q}_0 \in \mathbb{D}_{Q_0}$ and show that there exists a pairwise disjoint family $\tilde{\mathcal{F}}_{\tilde{Q}_0} = \{\tilde{Q}_j\}_j \subset \mathbb{D}_{\tilde{Q}_0} \setminus \{\tilde{Q}_0\}$ such that

$$(4.30) \quad \sigma\left(\tilde{Q}_0 \setminus \bigcup_{\tilde{Q}_j \in \tilde{\mathcal{F}}_{\tilde{Q}_0}} \tilde{Q}_j\right) \geq \tilde{K}_1^{-1} \sigma(\tilde{Q}_0) \quad \text{and} \quad m_{\tilde{\alpha}}(\mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}) \leq \tilde{M}_1 \sigma(\tilde{Q}_0).$$

In order to obtain this we note that $\ell(\tilde{Q}_0) \leq \ell(Q_0) < \text{diam}(\partial\Omega)/M_0$ and set $\tilde{X}_0 = X_{\sqrt{M_0}\Lambda_{\tilde{Q}_0}}$. Recall that by choice of M_0 we have that $2\kappa_0 r_{\tilde{Q}_0} \leq \delta(\tilde{X}_0) \leq \sqrt{M_0} r_{\tilde{Q}_0}$, where κ_0 was chosen (depending only on the allowable parameters) so that (2.22) holds. In such a case, $\text{dist}(\tilde{X}_0, T_{\tilde{Q}_0}^{**}) \geq \kappa_0 r_{\tilde{Q}_0}$, hence the pole $\tilde{X}_0$ will be away from where the argument takes place. By applying Lemma 2.23 and Harnack's inequality we have $\omega_{L^\top}^{\tilde{X}_0}(\tilde{Q}_0) \geq \tilde{C}_0^{-1}$ with $\tilde{C}_0$ depending on the allowable parameters and M_0 . We now take a normalization of the elliptic measure and the Green function for $L^\top$:

$$(4.31) \quad \omega_\top := \tilde{C}_0 \sigma(\tilde{Q}_0) \omega_{L^\top}^{\tilde{X}_0} \quad \text{and} \quad \mathcal{G}_\top(\cdot) := \tilde{C}_0 \sigma(\tilde{Q}_0) G_{L^\top}(\tilde{X}_0, \cdot).$$

As before $L\mathcal{G}_\top = 0$ away from $\tilde{X}_0$, and by our choice of $\tilde{X}_0$, Lemmas 2.32 and 2.34, (2.13), and (2.14) it follows that

$$(4.32) \quad \frac{\mathcal{G}_\top(X_Q)}{\ell(Q)} \approx \frac{\omega_\top(Q)}{\sigma(Q)}, \quad \forall Q \in \mathbb{D}_{\tilde{Q}_0}.$$

Since $\omega_{L^\top}^{\tilde{X}_0}(\partial\Omega) \leq 1$, it follows that $1 \leq \omega_\top(\tilde{Q}_0)/\sigma(\tilde{Q}_0) \leq \tilde{C}_0$. This and the fact that $\omega_{L^\top}$ (and hence $\omega_\top$) is in $A_\infty(\partial\Omega)$ allow us to invoke much as before Lemma 3.5 with $\mu = \omega_\top$ to extract a family of pairwise disjoint cubes $\tilde{\mathcal{F}}_{\tilde{Q}_0} = \{\tilde{Q}_j\}_j \subset \mathbb{D}_{\tilde{Q}_0} \setminus \{\tilde{Q}_0\}$ such that

$$(4.33) \quad \sigma\left(\tilde{Q}_0 \setminus \bigcup_{\tilde{Q}_j \in \tilde{\mathcal{F}}_{\tilde{Q}_0}} \tilde{Q}_j\right) \gtrsim \sigma(\tilde{Q}_0) \quad \text{and} \quad \frac{\omega_\top(Q)}{\sigma(Q)} \approx 1, \quad \forall Q \in \mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0},$$

with implicit constants depending on $\tilde{C}_0$ and the $A_\infty(\partial\Omega)$ character of $\omega_{L^\top}$. Consequently, in view of the previous considerations and Lemma 3.12, it remains to show $m_{\tilde{\alpha}}(\mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}) \leq \tilde{M}_1 \sigma(\tilde{Q}_0)$.

Note first that if $\tilde{Q}_0 \notin \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$ then $\tilde{\alpha}_Q = 0$ for every $Q \in \mathbb{D}_{\tilde{Q}_0}$, hence the desired estimate follows trivially since $m_{\tilde{\alpha}}(\mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}) = 0$. Thus we may assume that $\tilde{Q}_0 \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0}$. Write $\mathcal{F}_\star$ to denote the collection of maximal cubes in $(\mathcal{F}_{Q_0}^\epsilon \cap \mathbb{D}_{\tilde{Q}_0}) \cup \tilde{\mathcal{F}}_{\tilde{Q}_0}$ so that $\mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0} \cap \mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0} = \mathbb{D}_{\mathcal{F}_\star, \tilde{Q}_0}$. Hence,

$$(4.34) \quad \begin{aligned} m_{\tilde{\alpha}}(\mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}) &= \sum_{Q \in \mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}} \tilde{\alpha}_Q = \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}^\epsilon, Q_0} \cap \mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}} \Upsilon_Q \\ &= \sum_{Q \in \mathbb{D}_{\mathcal{F}_\star, \tilde{Q}_0}} \iint_{U_Q} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \delta(X) dX \lesssim \iint_{\Omega_{\mathcal{F}_\star, \tilde{Q}_0}^*} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \mathcal{G}_\top(X) dX. \end{aligned}$$

Note that in the last estimate we have used that the sets $\{U_Q\}_{Q \in \mathbb{D}_{\tilde{Q}_0}}$ have bounded overlap and that Harnack's inequality, (4.32) and the second estimate in (4.33) yield

$$\frac{\mathcal{G}_\top(X)}{\delta(X)} \approx \frac{\mathcal{G}_\top(X_Q)}{\ell(Q)} \approx \frac{\omega_\top(Q)}{\sigma(Q)} \approx 1, \quad \forall X \in U_Q, \quad \forall Q \in \mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}.$$

As we did in the symmetric case we take N large enough and define $\mathcal{F}_N := \mathcal{F}_\star(2^{-N} \ell(\tilde{Q}_0))$ as in Section 2.3 so that $Q \in \mathbb{D}_{\mathcal{F}_N, \tilde{Q}_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F}_\star, \tilde{Q}_0}$ and $\ell(Q) > 2^{-N} \ell(\tilde{Q}_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N, \tilde{Q}_0} \subset \mathbb{D}_{\mathcal{F}_{N'}, \tilde{Q}_0}$ if $N \leq N'$ and therefore $\Omega_{\mathcal{F}_N, \tilde{Q}_0}^* \subset \Omega_{\mathcal{F}_{N'}, \tilde{Q}_0}^* \subset \Omega_{\mathbb{D}_{\mathcal{F}_\star, \tilde{Q}_0}}^*$. This and the monotone convergence theorem give that

$$(4.35) \quad \iint_{\Omega_{\mathcal{F}_\star, \tilde{Q}_0}^*} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \mathcal{G}_\top(X) dX = \lim_{N \rightarrow \infty} \iint_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \mathcal{G}_\top(X) dX.$$

We can now state the analog of Proposition 4.25 in this non-symmetric case.

Proposition 4.36. *Given $C_1 \geq 1$, one can find C depending on C_1 and the allowable parameters such that if $\mathcal{F}_N \subset \mathbb{D}_{\tilde{Q}_0}$, $N \geq 1$, is a family of pairwise disjoint dyadic cubes satisfying*

$$(4.37) \quad C_1^{-1} \leq \frac{\omega(Q)}{\sigma(Q)} \leq C_1, \quad C_1^{-1} \leq \frac{\omega_\top(Q)}{\sigma(Q)} \leq C_1, \quad \text{and} \quad \ell(Q) > 2^{-N} \ell(\tilde{Q}_0),$$

for all $Q \in \mathbb{D}_{\mathcal{F}_N, \tilde{Q}_0}$, then

$$(4.38) \quad \iint_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \mathcal{G}_\top(X) dX \leq C \sigma(\tilde{Q}_0).$$

Assuming this result momentarily, we note that (4.37) follows from (4.20), the second item in (4.33) and the construction of $\mathcal{F}_N$. Consequently, (4.38), (4.34), and (4.35) allow us to conclude that $m_{\tilde{\alpha}}(\mathbb{D}_{\tilde{\mathcal{F}}_{\tilde{Q}_0}, \tilde{Q}_0}) \lesssim \sigma(\tilde{Q}_0)$. As observed above, this was the only thing left to obtain (4.22) in the non-symmetric case and the proof of our main result is eventually complete.

Before starting the proof of Proposition 4.36 we need some auxiliary results, whose proofs are postponed until the next section.

Lemma 4.39. *Let Ω be an open set and let A be a uniformly elliptic matrix in Ω . Given $K \geq 0$, there exists C_K depending only on ellipticity and K such that if*

$$(4.40) \quad \sup_{X \in \Omega} |\nabla A(X)| \delta(X) \leq K,$$

then the following hold:

(i) *For every $u \in W_{\text{loc}}^{1,2}(\Omega)$, $u \geq 0$, verifying $Lu = 0$ in Ω in the weak-sense,*

$$(4.41) \quad |\nabla u(X)| \leq C_K \frac{u(X)}{\delta(X)}, \quad \forall X \in \Omega.$$

(ii) *Given any cube $I \subset \mathbb{R}^{n+1}$, if $6I \subset \Omega$ and $u \in W^{1,2}(6I)$ satisfies $Lu = 0$ in $6I$ in the weak-sense,*

$$(4.42) \quad \iint_I |\nabla^2 u(Y)|^2 dY \leq \frac{C_K}{\ell(I)^2} \iint_{2I} |\nabla u(Y)|^2 dY.$$

Proof of Proposition 4.36. Write $\Omega_\star = \Omega_{\mathcal{F}_N, \tilde{Q}_0}^{**}$. Let us first prove that

$$(4.43) \quad |\nabla \mathcal{G}(X)| \lesssim \frac{\mathcal{G}(X)}{\delta(X)} \approx 1, \quad |\nabla \mathcal{G}_\top(X)| \lesssim \frac{\mathcal{G}_\top(X)}{\delta(X)} \approx 1, \quad \forall X \in \overline{\Omega_\star},$$

albeit with bounds that depend on C_1 and the allowable parameters but which are uniform in N . What $|\nabla \mathcal{G}| \lesssim G/\delta$ (respectively $|\nabla \mathcal{G}_\top| \lesssim G_\top/\delta$) follows from Lemma 4.39 applied to $u = \mathcal{G}$ (resp. $u = \mathcal{G}_\top$) where the implicit constant depends on ellipticity and $\|\nabla A\|_\infty$. To justify the use of that lemma we first notice that (4.40) is just our assumption (b). Also, we recall we chose X_0 and $\tilde{X}_0$ so that they are away from $T_{\tilde{Q}_0}^{**}$ (indeed X_0 is away from $T_{\tilde{Q}_0}^{**} \supset T_{\tilde{Q}_0}^{**}$). Hence $L\mathcal{G}_\top = 0$ and $L^\top \mathcal{G} = 0$ in the weak sense in $T_{\tilde{Q}_0}^{**}$ (cf. Lemma 2.25). Finally we observe that $2^{-N} \ell(\tilde{Q}_0) \lesssim \delta(X) \lesssim \ell(\tilde{Q}_0)$ for every $X \in \overline{\Omega_\star}$.

To continue with the proof of (4.43) let $X \in \overline{\Omega_\star}$. Then there is $Q \in \mathbb{D}_{\mathcal{F}_N, \tilde{Q}_0}$ and $I \in \mathcal{W}_Q^*$ such that $X \in I^{***}$. Note that $\delta(X) \approx \delta(X_Q) \approx \ell(I) \approx \ell(Q)$ and $|X - X_Q| \lesssim \ell(Q)$. This, Harnack's inequality, (4.5), (4.32), that $\tilde{Q}_0 \in \mathbb{D}_{Q_0}$ and (4.37) yield as desired

$$\frac{\mathcal{G}(X)}{\delta(X)} \approx \frac{\mathcal{G}(X_Q)}{\ell(Q)} \approx \frac{\omega(Q)}{\sigma(Q)} \approx 1, \quad \frac{\mathcal{G}_\top(X)}{\delta(X)} \approx \frac{\mathcal{G}_\top(X_Q)}{\ell(Q)} \approx \frac{\omega_\top(Q)}{\sigma(Q)} \approx 1.$$

We now proceed to obtain (4.38). We note that by the boundedness of A ,

$$\begin{aligned} \iint_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*} |\nabla(A^\top \nabla \mathcal{G})(X)|^2 \mathcal{G}_\top(X) dX \\ \lesssim_\Lambda \iint_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*} |\nabla A(X)|^2 |\nabla \mathcal{G}(X)|^2 \mathcal{G}_\top(X) dX \\ + \iint_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*} |\nabla^2 \mathcal{G}(X)|^2 \mathcal{G}_\top(X) dX =: I + II. \end{aligned}$$

The estimate for I is easy. Use (4.43) and (2.22) to conclude as desired

$$I \lesssim \iint_{B_{\tilde{Q}_0}^* \cap \Omega} |\nabla A(X)|^2 \delta(X) dX \lesssim \|\nabla A\|_{L^\infty(\Omega)} \|\nabla A\|_{C(\Omega)} \sigma(\Delta_{\tilde{Q}_0}^*) \approx \|\nabla A\|_{L^\infty(\Omega)} \|\nabla A\|_{C(\Omega)} \sigma(\tilde{Q}_0),$$

where the implicit constants are clearly independent of N .

To estimate II we need the following auxiliary lemma whose proof will be postponed until the next the section.

Lemma 4.44. *There exists $\Psi_N \in C_0^\infty(\mathbb{R}^{n+1})$ such that*

$$\begin{aligned} (i) \quad 1_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*} &\lesssim \Psi_N \leq 1_{\Omega_{\mathcal{F}_N, \tilde{Q}_0}^{**}}. \\ (ii) \quad \sup_{X \in \Omega} |\nabla \Psi_N(X)| \delta(X) &\lesssim 1. \\ (iii) \quad \text{Set } \Sigma &= \partial \Omega_{\mathcal{F}_N, \tilde{Q}_0}^*, \\ (4.45) \quad \mathcal{W}_N &:= \bigcup_{Q \in \mathbb{D}_{\mathcal{F}_N, \tilde{Q}_0}} \mathcal{W}_Q^*, \quad \mathcal{W}_N^\Sigma := \{I \in \mathcal{W}_N : \exists J \in \mathcal{W} \setminus \mathcal{W}_N \text{ with } \partial I \cap \partial J \neq \emptyset\}. \end{aligned}$$

Then

$$(4.46) \quad \nabla \Psi_N \equiv 0 \quad \text{in} \quad \bigcup_{I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma} I^{***} \quad \text{and} \quad \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \lesssim \sigma(\tilde{Q}_0),$$

with implicit constants depending on the allowable parameters but uniform in N .

Now we are ready to estimate II . Using the previous lemma we have

$$II \lesssim \iint_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{G}(X)|^2 \mathcal{G}_\top(X) \Psi_N(X) dX \leq \sum_{j=1}^{n+1} \iint_{\mathbb{R}^{n+1}} |\nabla(\partial_j \mathcal{G})(X)|^2 \mathcal{G}_\top(X) \Psi_N(X) dX$$

Write “ ∂ ” to denote a fixed generic derivative. The following observations will be used several times in the proof. Observe that Lemma 4.39 and (4.43) give that $\mathcal{G}, \mathcal{G}_\top, \nabla \mathcal{G}, \nabla \mathcal{G}_\top \in W^{1,2}(\overline{\Omega_\star}) \cap L^\infty(\overline{\Omega_\star})$. Observe also that Ψ_N is supported in $\Omega_\star$, thus $\partial \mathcal{G} \mathcal{G}_\top \Psi_N \in W_0^{1,2}(\Omega_\star)$. Hence we can find $\{\mathcal{G}_k\}_k \subset C_0^\infty(\Omega_\star)$ such that $\mathcal{G}_k \rightarrow \partial \mathcal{G} \mathcal{G}_\top \Psi_N$ in $W^{1,2}(\Omega_\star)$. Note also that $|\nabla A| \in L^\infty(\overline{\Omega_\star}) < \infty$ by our assumption (a). These observations will, in particular, justify that all the integrals below are absolutely convergent.

We can now return to our task of estimating II . By ellipticity and using $\langle \cdot, \cdot \rangle$ to denote the inner product on $L^2(\mathbb{R}^{n+1})$ it follows that

$$\begin{aligned}
F_N &:= \iint_{\mathbb{R}^{n+1}} |\nabla(\partial\mathcal{G})(X)|^2 \mathcal{G}_\top(X) \Psi_N(X) dX \lesssim_\Lambda \langle A^\top \nabla(\partial\mathcal{G}), \nabla(\partial\mathcal{G}) \mathcal{G}_\top \Psi_N \rangle \\
&= \langle A^\top \nabla(\partial\mathcal{G}), \nabla(\partial\mathcal{G}) \mathcal{G}_\top \Psi_N \rangle - \frac{1}{2} \langle A^\top \nabla((\partial\mathcal{G})^2), \nabla(\mathcal{G}_\top \Psi_N) \rangle =: \tilde{I} - \frac{1}{2} \tilde{I} \tilde{I}.
\end{aligned}$$

To estimate $\tilde{I}$ we write

$$\tilde{I} = \langle \partial(A^\top \nabla \mathcal{G}), \nabla(\partial\mathcal{G}) \mathcal{G}_\top \Psi_N \rangle - \langle \partial A^\top \nabla \mathcal{G}, \nabla(\partial\mathcal{G}) \mathcal{G}_\top \Psi_N \rangle =: \tilde{I}_1 - \tilde{I}_2.$$

Controlling $\tilde{I}_1$ it is not difficult as the previous observations along with (4.43) give

$$\tilde{I}_1 = \lim_{k \rightarrow \infty} \langle \partial(A^\top \nabla \mathcal{G}), \nabla \mathcal{G}_k \rangle =: \lim_{k \rightarrow \infty} \tilde{I}_{1,k}$$

On the other hand

$$\tilde{I}_{1,k} = \iint_{\mathbb{R}^{n+1}} \partial(A^\top \nabla \mathcal{G} \cdot \nabla \mathcal{G}_k) dX - \langle A^\top \nabla \mathcal{G}, \nabla \partial \mathcal{G}_k \rangle = \tilde{I}_{1,k,1} - \tilde{I}_{1,k,2}.$$

Note that $A^\top \nabla \mathcal{G} \cdot \nabla \mathcal{G}_k \in W^{1,2}(\Omega_\star)$ and is supported in $\overline{\Omega_\star}$. Hence $\tilde{I}_{1,k,1} = 0$ by the divergence theorem. Also, $\tilde{I}_{1,k,2} = 0$ since $L^\top \mathcal{G} = 0$ in the weak-sense in $\Omega_\star$ (cf. (4.4) and Lemma 2.25) and $\partial \mathcal{G}_k \in C_0^\infty(\Omega_\star)$. Therefore $\tilde{I}_{1,k} = 0$ and consequently $\tilde{I}_1 = 0$.

We next estimate $\tilde{I}_2$:

$$\tilde{I}_2 = \langle \partial A^\top \nabla \mathcal{G}, \nabla(\partial\mathcal{G}) \mathcal{G}_\top \Psi_N \rangle + \langle \partial A^\top \nabla \mathcal{G}, \nabla \mathcal{G}_\top \partial \mathcal{G} \Psi_N \rangle + \langle \partial A^\top \nabla \mathcal{G}, \nabla \Psi_N \partial \mathcal{G} \mathcal{G}_\top \rangle =: \tilde{I}_{21} + \tilde{I}_{22} + \tilde{I}_{23}.$$

Note that by (4.43), Lemma 4.44 and (2.22)

$$|\tilde{I}_{22}| \lesssim \iint_{\mathbb{R}^{n+1}} |\nabla A| |\nabla \mathcal{G}|^2 |\nabla \mathcal{G}_\top| \Psi_N dX \lesssim \iint_{T_{\tilde{Q}_0}^{**}} |\nabla A| dX \lesssim \|\nabla A\|_{C(\Omega)} \sigma(\Delta_{\tilde{Q}_0}^*) \lesssim \|\nabla A\|_{C(\Omega)} \sigma(\tilde{Q}_0),$$

where we have used hypothesis (c). Also by (4.43), Lemma 4.44 and (2.22), and Young's inequality we have

$$\begin{aligned}
|\tilde{I}_{21}| &\lesssim \iint_{\mathbb{R}^{n+1}} |\nabla A| |\nabla(\partial\mathcal{G})| \mathcal{G}_\top \Psi_N dX \\
&\lesssim \left(\iint_{\mathbb{R}^{n+1}} |\nabla A|^2 \mathcal{G}_\top \Psi_N dX \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{n+1}} |\nabla(\partial\mathcal{G})|^2 \mathcal{G}_\top \Psi_N dX \right)^{\frac{1}{2}} \\
&\lesssim \left(\iint_{T_{\tilde{Q}_0}^{**}} |\nabla A|^2 \delta(\cdot) dX \right)^{\frac{1}{2}} F_N^{\frac{1}{2}} \\
&\lesssim \| |\nabla A| \delta \|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\nabla A\|_{C(\Omega)}^{\frac{1}{2}} \sigma(\Delta_{\tilde{Q}_0}^*)^{\frac{1}{2}} F_N^{\frac{1}{2}} \\
&\leq C \sigma(\tilde{Q}_0) + \frac{1}{2} F_N.
\end{aligned}$$

Note that C depends on the 1-sided CAD constants, ellipticity, $\| |\nabla A| \delta \|_{L^\infty(\Omega)}$, $\|\nabla A\|_{C(\Omega)}$ and C_1 fixed in the statement of Proposition 4.36. Analogously,

$$|\tilde{I}_{23}| \lesssim \iint_{\mathbb{R}^{n+1}} |\nabla A| |\nabla \Psi_N| \delta(\cdot) dX \lesssim \iint_{T_{\tilde{Q}_0}^{**}} |\nabla A| dX \lesssim \|\nabla A\|_{C(\Omega)} \sigma(\tilde{Q}_0),$$

Collecting the estimates we have obtained we conclude that

$$|\tilde{I}| = |\tilde{I}_2| \leq |\tilde{I}_{21}| + |\tilde{I}_{22}| + |\tilde{I}_{23}| \leq C \sigma(\tilde{Q}_0) + \frac{1}{2} F_N.$$

We next estimate $\tilde{I} \tilde{I}$:

$$\tilde{I} \tilde{I} = \langle A^\top \nabla((\partial\mathcal{G})^2), \nabla \mathcal{G}_\top \Psi_N \rangle + \langle A^\top \nabla((\partial\mathcal{G})^2), \nabla \Psi_N \mathcal{G}_\top \rangle =: \tilde{I} \tilde{I}_1 + \tilde{I} \tilde{I}_2.$$

For $\widetilde{I}\widetilde{I}_2$ we proceed as before, use (4.43), Lemma 4.44, (4.42), Caccioppoli's and Harnack's inequalities to obtain

$$\begin{aligned} |\widetilde{I}\widetilde{I}_2| &\lesssim \iint_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{G}| |\nabla \Psi_N| \delta(\cdot) dX \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \iint_{I^{***}} |\nabla^2 \mathcal{G}| dX \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \frac{|I|^{\frac{1}{2}}}{\ell(I)} \left(\iint_{2I^{***}} |\nabla \mathcal{G}|^2 dX \right)^{\frac{1}{2}} \\ &\lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \frac{|I|^{\frac{1}{2}}}{\ell(I)^2} \left(\iint_{3I^{***}} |\mathcal{G}|^2 dX \right)^{\frac{1}{2}} \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \frac{|I|}{\ell(I)} \frac{\mathcal{G}(X(I))}{\delta(X(I))} \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \lesssim \sigma(\widetilde{Q}_0). \end{aligned}$$

Let us turn our attention to $\widetilde{I}\widetilde{I}_1$:

$$\widetilde{I}\widetilde{I}_1 = \langle A \nabla \mathcal{G}_\top, \nabla((\partial \mathcal{G})^2) \Psi_N \rangle = \langle A \nabla \mathcal{G}_\top, \nabla((\partial \mathcal{G})^2 \Psi_N) \rangle - \langle A \nabla \mathcal{G}_\top, \nabla \Psi_N (\partial \mathcal{G})^2 \rangle =: \widetilde{I}\widetilde{I}_{11} - \widetilde{I}\widetilde{I}_{12}.$$

Notice that $\widetilde{I}\widetilde{I}_1 = 0$ since $L\mathcal{G}_\top$ in the weak sense in $\Omega_\star$ (cf. (4.31) and Lemma 2.25) and $(\partial \mathcal{G})^2 \Psi_N \in W_0^{1,2}(\Omega_\star)$. Hence, another use of (4.43) and Lemma 4.44 produce

$$|\widetilde{I}\widetilde{I}_1| = |\widetilde{I}\widetilde{I}_{12}| \lesssim \iint_{\mathbb{R}^{n+1}} |\nabla \mathcal{G}|^2 |\nabla \mathcal{G}_\top| |\nabla \Psi_N| dX \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \iint_{I^{***}} \frac{1}{\delta(X)} dX \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \lesssim \sigma(\widetilde{Q}_0).$$

Putting things together

$$|\widetilde{I}\widetilde{I}| \leq |\widetilde{I}\widetilde{I}_1| + |\widetilde{I}\widetilde{I}_2| \lesssim \sigma(\widetilde{Q}_0).$$

To conclude the proof we collect the obtained estimates

$$0 \leq F_N = \widetilde{I} - \frac{1}{2} \widetilde{I}\widetilde{I} \leq |\widetilde{I}| + \frac{1}{2} |\widetilde{I}\widetilde{I}| \leq C \sigma(\widetilde{Q}_0) + \frac{1}{2} F_N.$$

Here all the constants are uniform in N . Since F_N is finite by (4.43), Lemma 4.39 and the fact that $\text{supp } \Psi_N \subset \overline{\Omega_\star} \subset \Omega$ we obtain

$$F_N \lesssim \sigma(\widetilde{Q}_0)$$

which readily yields (4.38) with C depending on the 1-sided CAD constants, ellipticity, C_1 fixed in the statement of Proposition 4.36, $\|\nabla A\|_{L^\infty(\Omega)}$, and $\|\nabla A\|_{C(\Omega)}$. $\square$

4.3. Proofs of Lemmas 4.39 and 4.44. In order to get the appropriate scale-invariant estimates in Lemma 4.39 we first present the case of the unit cube and then extend it to Ω by translation and rescaling.

Lemma 4.47. *Let $I_0 := (-\frac{1}{2}, \frac{1}{2})^{n+1} \subset \mathbb{R}^{n+1}$ and let $A \in \text{Lip}(I_0)$ be a uniformly elliptic matrix in I_0 . Given $K \geq 0$ there exists C_K depending only on dimension, ellipticity and K such that if $\|\nabla A\|_{L^\infty(I_0)} \leq K$, then for every $u \in W^{1,2}(I_0) \cap L^\infty(I_0)$, $u \geq 0$, such that $Lu = 0$ in the weak-sense in I_0 we have*

$$(4.48) \quad \sup_{X \in \frac{1}{2}I_0} |\nabla u(X)| \leq C_K \inf_{X \in \frac{1}{2}I_0} u(X).$$

and

$$(4.49) \quad \iint_{\frac{1}{4}I_0} |\nabla^2 u(X)|^2 dX \leq C_K \iint_{\frac{1}{2}I_0} |\nabla u(X)|^2 dX.$$

Proof. To prove (4.48) we invoke [GW, Lemma 3.1] in the open bounded domain $\frac{3}{4}I_0$ and there exist C_K depending on n , ellipticity and K such that

$$\sup_{X \in \frac{3}{4}I_0} |\nabla u(X)| \text{ dist}\left(X, \partial\left(\frac{3}{4}I_0\right)\right) \leq C_K \sup_{X \in \frac{3}{4}I_0} u(X).$$

This and Harnack's inequality give at once (4.48).

We next prove (4.49). Let us first observe that since A is Lipschitz in I_0 , and $u \in W^{1,2}(I_0)$ satisfies $Lu = 0$ in the weak-sense in I_0 , it follows that $u \in W^{2,2}(\frac{3}{4}I_0)$ by [GT, Theorem 8.8]. With this in hand we are going to use a Caccioppoli type argument. Let $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ be a smooth cut-off of $\frac{1}{4}I_0$, that is, $1_{\frac{1}{4}I_0} \leq \varphi \leq 1_{\frac{3}{8}I_0}$ with $\|\nabla\varphi\|_{L^\infty} \leq C_0$. Write “ ∂ ” to denote a fixed generic derivative and observe that since $u \in W^{2,2}(\frac{3}{4}I_0)$ it follows that $\partial u \varphi^2 \in W_0^{1,2}(\frac{1}{2}I_0)$. Hence there exists $\{u_k\}_k \subset C_0^\infty(\frac{1}{2}I_0)$ such that $u_k \rightarrow \partial u \varphi^2$ in $W^{1,2}(\frac{1}{2}I_0)$. If we write Λ for the ellipticity constant of A , we then have

$$\begin{aligned} \mathcal{I} &:= \iint_{\mathbb{R}^{n+1}} |\nabla(\partial u)(X)|^2 \varphi(X)^2 dX \leq \Lambda \iint_{\mathbb{R}^{n+1}} A(X) \nabla(\partial u)(X) \cdot \nabla(\partial u)(X) \varphi(X)^2 dX \\ &= \Lambda \left(\iint_{\mathbb{R}^{n+1}} A(X) \nabla(\partial u)(X) \cdot [\nabla(\partial u \varphi^2)(X) - 2\nabla\varphi(X) (\partial u)(X) \varphi(X)] dX \right) =: \Lambda (\mathcal{I}_1 - 2\mathcal{I}_2). \end{aligned}$$

For $\mathcal{I}_2$ we observe that by the Cauchy-Schwarz inequality

$$|\mathcal{I}_2| = \left| \iint_{\mathbb{R}^{n+1}} A(X) \nabla(\partial u)(X) \cdot \nabla\varphi(X) (\partial u)(X) \varphi(X) dX \right| \leq \Lambda C_0 \mathcal{I}^{\frac{1}{2}} \left(\iint_{\frac{1}{2}I_0} |\nabla u(X)|^2 dX \right)^{\frac{1}{2}}.$$

For $\mathcal{I}_1$ we use the sequence $\{u_k\}_k$ introduced above and note that

$$\begin{aligned} \mathcal{I}_1^k &:= \iint_{\mathbb{R}^{n+1}} A(X) \nabla(\partial u)(X) \cdot \nabla u_k(X) dX \\ &= \iint_{\mathbb{R}^{n+1}} \partial(A \nabla u \cdot \nabla u_k)(X) dX - \iint_{\mathbb{R}^{n+1}} A(X) \nabla u(X) \cdot \nabla(\partial u_k)(X) dX \\ &\quad - \iint_{\mathbb{R}^{n+1}} \partial A(X) \nabla u(X) \cdot \nabla u_k(X) dX \\ &= - \iint_{\mathbb{R}^{n+1}} \partial A(X) \nabla u(X) \cdot \nabla u_k(X) dX. \end{aligned}$$

Here we have used that since $\{u_k\} \subset C_0^\infty(\frac{1}{2}I_0)$ both terms in the second line vanish. In fact the first term is the integral of a derivative of a $W^{1,2}(\mathbb{R}^{n+1})$ compactly supported function, and the second term because $Lu = 0$ in I_0 in the weak sense and $\partial u_k \in C_0^\infty(\frac{1}{2}I_0)$. To continue with our estimate we observe that by Cauchy-Schwarz

$$\begin{aligned} |\mathcal{I}_1| &= \left| \lim_{k \rightarrow \infty} \mathcal{I}_1^k \right| = \left| \iint_{\mathbb{R}^{n+1}} \partial A(X) \nabla u(X) \cdot \nabla(\partial u \varphi^2)(X) dX \right| \\ &\leq \|\nabla A\|_{L^\infty(I_0)} \left(\iint_{\mathbb{R}^{n+1}} |\nabla u(X)| |\nabla(\partial u)(X)| \varphi(X)^2 dX + 2 \iint_{\mathbb{R}^{n+1}} |\nabla u(X)|^2 |\nabla\varphi(X)| \varphi(X) dX \right) \\ &\leq \|\nabla A\|_{L^\infty(I_0)} \left(\mathcal{I}^{\frac{1}{2}} \left(\iint_{\frac{1}{2}I_0} |\nabla u(X)|^2 dX \right)^{\frac{1}{2}} + 2C_0 \iint_{\frac{1}{2}I_0} |\nabla u(X)|^2 dX \right). \end{aligned}$$

Collecting all the obtained estimates we conclude that

$$\mathcal{I} \leq \Lambda (\|\nabla A\|_{L^\infty(I_0)} + 2C_0\Lambda) \mathcal{I}^{\frac{1}{2}} \left(\iint_{\frac{1}{2}I_0} |\nabla u(X)|^2 dX \right)^{\frac{1}{2}} + 2\Lambda C_0 \|\nabla A\|_{L^\infty(I_0)} \iint_{\frac{1}{2}I_0} |\nabla u(X)|^2 dX.$$

From here we can use Young’s inequality with epsilon in the first term on the right hand side, hide $\mathcal{I}$ (which is finite since $u \in W^{2,2}(\frac{3}{4}I_0)$) and the desired estimates follows easily. $\square$

Proof of Lemma 4.39. This result follows easily from Lemma 4.47. For (i), first $u \in L_{\text{loc}}^\infty(\Omega)$ by interior regularity. We take J , any Whitney cube in Ω , and translate and rescale $2J$ so that it becomes $\bar{I}_0$. Note that (4.40) translates into the boundedness of the gradient of the corresponding matrix in Lemma 4.47 (up to some dimensional constants). Hence (4.48) and Harnack’s inequality give as desired (4.41).

The proof of (4.42) follows easily from (4.49) by rescaling and translation, again interior regularity gives that $u \in L_{\text{loc}}^\infty(6I)$. Details are left to the reader. $\square$

Proof of Lemma 4.44. We recall that given I , any closed dyadic cube in $\mathbb{R}^{n+1}$, we set $I^{**} = (1+2\lambda)I$ and $I^{***} = (1+4\lambda)I$. Let us introduce $\widetilde{I}^{**} = (1+3\lambda)I$ so that

$$(4.50) \quad I^{**} \subsetneq \text{int}(\widetilde{I}^{**}) \subsetneq \widetilde{I}^{**} \subset \text{int}(I^{***}).$$

Given $I_0 := [-\frac{1}{2}, \frac{1}{2}]^{n+1} \subset \mathbb{R}^{n+1}$, fix $\phi_0 \in C_0^\infty(\mathbb{R}^{n+1})$ such that $1_{I_0^{**}} \leq \phi_0 \leq 1_{\widetilde{I}_0^{**}}$ and $|\nabla \phi_0| \lesssim 1$ (the implicit constant will depend on the parameter λ). For every $I \in \mathcal{W} = \mathcal{W}(\Omega)$ we set $\phi_I(\cdot) = \phi_0(\frac{\cdot - X(I)}{\ell(I)})$ so that $\phi_I \in C_0^\infty(\mathbb{R}^{n+1})$, $1_{I^{**}} \leq \phi_I \leq 1_{\widetilde{I}^{**}}$ and $|\nabla \phi_I| \lesssim \ell(I)^{-1}$ (with implicit constant depending only on n and λ).

For every $X \in \Omega$, we let $\Phi(X) := \sum_{I \in \mathcal{W}} \phi_I(X)$. It then follows that $\Phi \in C_{\text{loc}}^\infty(\Omega)$ since for every compact subset of Ω the previous sum has finitely many non-vanishing terms. Also, $1 \leq \Phi(X) \lesssim C_\lambda$ for every $X \in \Omega$ since the family $\{\widetilde{I}^{**}\}_{I \in \mathcal{W}}$ has bounded overlap by our choice of λ . Hence we can set $\Phi_I = \phi_I/\Phi$ and one can easily see that $\Phi_I \in C_0^\infty(\mathbb{R}^{n+1})$, $C_\lambda^{-1} 1_{I^{**}} \leq \Phi_I \leq 1_{\widetilde{I}^{**}}$ and $|\nabla \Phi_I| \lesssim \ell(I)^{-1}$. With this in hand and by recalling the definition of $\mathcal{W}_N$ in (4.45) we set

$$\Psi_N(X) := \sum_{I \in \mathcal{W}_N} \Phi_I(X) = \frac{\sum_{I \in \mathcal{W}_N} \phi_I(X)}{\sum_{I \in \mathcal{W}} \phi_I(X)}, \quad X \in \Omega.$$

We first note that the number of terms in the sum defining Ψ_N is bounded depending on N . Indeed, if $Q \in \mathbb{D}_{\mathcal{F}_N, \widetilde{Q}_0}$ then $Q \in \mathbb{D}_{\widetilde{Q}_0}$ and $2^{-N} \ell(\widetilde{Q}_0) < \ell(Q) \leq \ell(\widetilde{Q}_0)$ which implies that $\mathbb{D}_{\mathcal{F}_N, \widetilde{Q}_0}$ has finite cardinality with bounds depending only on the AR property and N . Also, by construction $\mathcal{W}_Q^*$ has cardinality depending only in the allowable parameters. Hence, $\#\mathcal{W}_N \lesssim C_N < \infty$. This and the fact that each $\Phi_I \in C_0^\infty(\mathbb{R}^{n+1})$ yield that $\Psi_N \in C_0^\infty(\mathbb{R}^{n+1})$. Note also that (4.50) and the definition of $\mathcal{W}_N$ in (4.45) give

$$\text{supp } \Psi_I \subset \bigcup_{I \in \mathcal{W}_N} \widetilde{I}^{**} = \bigcup_{Q \in \mathbb{D}_{\mathcal{F}_N, \widetilde{Q}_0}} \bigcup_{I \in \mathcal{W}_Q^*} \widetilde{I}^{**} \subset \text{int} \left(\bigcup_{Q \in \mathbb{D}_{\mathcal{F}_N, \widetilde{Q}_0}} \bigcup_{I \in \mathcal{W}_Q^*} I^{***} \right) = \text{int} \left(\bigcup_{Q \in \mathbb{D}_{\mathcal{F}_N, \widetilde{Q}_0}} U_Q^{**} \right) = \Omega_{\mathcal{F}_N, \widetilde{Q}_0}^{**}.$$

This, the fact that $\mathcal{W}_N \subset \mathcal{W}$ and the definition of Ψ_N immediately gives that $\Psi_N \leq 1_{\Omega_{\mathcal{F}_N, \widetilde{Q}_0}^{**}}$. On the other hand if $X \in \Omega_{\mathcal{F}_N, \widetilde{Q}_0}^*$ then there exists $I \in \mathcal{W}_N$ such that $X \in I^{**}$ in which case $\Psi_N(X) \geq \Phi_I(X) \geq C_\lambda^{-1}$. This completes the proof of (i).

To obtain (ii) we note that for every $X \in \Omega$

$$|\nabla \Psi_N(X)| \leq \sum_{I \in \mathcal{W}_N} |\nabla \Phi_I(X)| \lesssim \sum_{I \in \mathcal{W}} \ell(I)^{-1} 1_{\widetilde{I}^{**}}(X) \lesssim \delta(X)^{-1}$$

where we have used that if $X \in \widetilde{I}^{**}$ then $\delta(X) \approx \ell(I)$ and also that the family $\{\widetilde{I}^{**}\}_{I \in \mathcal{W}}$ has bounded overlap.

Let us finally address (iii). Fix $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$ and $X \in I^{***}$, and set $\mathcal{W}_X := \{J \in \mathcal{W} : \phi_J(X) \neq 0\}$. We first note that $\mathcal{W}_X \subset \mathcal{W}_N$. Indeed, if $\phi_J(X) \neq 0$ then $X \in \widetilde{J}^{**}$. Hence $X \in I^{***} \cap \widetilde{J}^{**}$ and our choice of λ gives that ∂I meets ∂J , this in turn implies that $J \in \mathcal{W}_N$ since $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. All these imply

$$\Psi_N(X) = \frac{\sum_{J \in \mathcal{W}_N} \phi_J(X)}{\sum_{J \in \mathcal{W}} \phi_J(X)} = \frac{\sum_{J \in \mathcal{W}_N \cap \mathcal{W}_X} \phi_J(X)}{\sum_{J \in \mathcal{W} \cap \mathcal{W}_X} \phi_J(X)} = \frac{\sum_{J \in \mathcal{W}_N \cap \mathcal{W}_X} \phi_J(X)}{\sum_{J \in \mathcal{W}_N \cap \mathcal{W}_X} \phi_J(X)} = 1.$$

Hence $\Psi_N|_{I^{***}} \equiv 1$ for every $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. This and the fact that $\Psi_N \in C_0^\infty(\mathbb{R}^{n+1})$ immediately give that $\nabla \Psi_N \equiv 0$ in $\bigcup_{I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma} I^{***}$.

To complete the proof we need to estimate the sum in (4.46). Recall that $\Sigma = \partial\Omega_{\mathcal{F}_N, \tilde{Q}_0}^* \subset \Omega$ and let $I \in \mathcal{W}_N^\Sigma$. We claim that there exists $Z_I \in \Sigma$ such that $\text{dist}(Z_I, I) \approx \ell(I) \approx \delta(Z_I)$. To prove this we first observe that $\text{int}(I^{**}) \subset \Omega_{\mathcal{F}_N, \tilde{Q}_0}^* \subset \Omega$ since $I \in \mathcal{W}_N$. On the other hand, $I \in \mathcal{W}_N^\Sigma$ implies that there is $J \in \mathcal{W} \setminus \mathcal{W}_N$ such that $\partial I \cap \partial J \neq \emptyset$. In particular, $X(J) \in \mathbb{R}^{n+1} \setminus \Omega_{\mathcal{F}_N, \tilde{Q}_0}^*$ (by our choice of λ) where $X(J)$ is the center of J . Then we can find $Z_I \in \Sigma$ with Z_I in the segment joining $X(J)$ and $X(I)$. Note that $\text{dist}(Z_I, I) \leq |Z_I - X(I)| \leq |X(I) - X(J)| \leq \ell(I)$ since $\partial I \cap \partial J \neq \emptyset$ implies that $\ell(I) \approx \ell(J)$ by the nature of the Whitney cubes. On the other hand since $\text{int}(I^{**}) \subset \Omega_{\mathcal{F}_N, \tilde{Q}_0}^*$ we have that $Z_I \notin \text{int}(I^{**})$, thus $\text{dist}(Z_I, I) \gtrsim \ell(I)$ (with implicit constant depending on λ). Finally, $\ell(I) \approx \ell(J) \approx \delta(Z_I)$.

One we have chosen Z_I we let $\Delta_I^\Sigma = B(Z_I, \delta(Z_I)/2) \cap \Sigma$, which is a surface ball with respect to the domain $\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*$ centered on $Z_I \in \Sigma = \partial\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*$. Since $\partial\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*$ is AR (cf. [HM1, Lemma 3.61]) with bounds that do not depend on N , then it follows that

$$\sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \approx \sum_{I \in \mathcal{W}_N^\Sigma} \delta(Z_I)^n \approx \sum_{I \in \mathcal{W}_N^\Sigma} H^n(\Delta_I^\Sigma).$$

We next see that the family $\{\Delta_I^\Sigma\}_{I \in \mathcal{W}_N^\Sigma}$ has bounded overlap. Indeed, suppose that $\Delta_{I_1}^\Sigma \cap \Delta_{I_2}^\Sigma \neq \emptyset$ and take Y in that intersection. Assume for instance that $\ell(I_1) \leq \ell(I_2)$. then,

$$\delta(Z_{I_2}) \leq |Z_{I_2} - Y| + |Y - Z_{I_1}| + \delta(Z_{I_1}) \leq \frac{1}{2} \delta(Z_{I_2}) + \frac{3}{2} \delta(Z_{I_1})$$

which implies that $\ell(I_2) \approx \delta(Z_{I_2}) \lesssim \delta(Z_{I_1}) \approx \ell(I_1)$. Thus, $\ell(I_1) \approx \ell(I_2)$. Moreover,

$$\text{dist}(I_1, I_2) \leq \text{dist}(I_1, Z_{I_1}) + |Z_{I_1} - Y| + |Y - Z_{I_2}| + \text{dist}(I_2, Z_{I_2}) \lesssim \ell(I_1) + \ell(I_2) \approx \ell(I_1) \approx \ell(I_2).$$

By the properties of the Whitney cubes it then follows that the family $\{\Delta_I^\Sigma\}_{I \in \mathcal{W}_N^\Sigma}$ has bounded overlap. Thus,

$$\sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \approx \sum_{I \in \mathcal{W}_N^\Sigma} H^n(\Delta_I^\Sigma) \lesssim H^n\left(\bigcup_{I \in \mathcal{W}_N^\Sigma} \Delta_I^\Sigma\right) \subset H^n(\Sigma) = H^n(\partial\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*) \lesssim \text{diam}(\partial\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*)^n \lesssim \ell(\tilde{Q}_0)^n,$$

where we have used again that $\partial\Omega_{\mathcal{F}_N, \tilde{Q}_0}^*$ is AR and also that this set is bounded with diameter controlled by $\ell(\tilde{Q}_0)$. This completes the proof of Lemma 4.44. $\square$

APPENDIX A. THE A_∞ PROPERTY IN LIPSCHITZ DOMAINS: THE KENIG-PIPHER ARGUMENT

The result of [KKIPT] allows for a slight condensation of the proof of the results of [KP], albeit with the very same ideas. For the reader's convenience, we supply the shortened proof here following the key part of [KP] essentially unchanged. To be precise, we shall prove the following.

Theorem B. ([KP]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz domain, and suppose that $L = -\text{div} A \nabla$ is an elliptic operator in Ω satisfying Hypothesis 1.1, but with property (c) replaced by the weaker condition (1.8). Then elliptic measure is absolutely continuous with respect to surface measure σ on $\partial\Omega$, and the Poisson kernel satisfies Hypothesis 1.3, with constants depending only on dimension, the Lipschitz character of Ω , and the constants in the modified version of Hypothesis 1.1 that we assume here.*

Sketch of Proof. Since the estimate to be proved, namely (1.4), is local, we may reduce matters to working in a single co-ordinate patch, and thus we may suppose that $\Omega = \{(x, t) \in \mathbb{R}^{n+1} : t > \varphi(x)\}$, where φ is a Lipschitz function. We may then further suppose that $\Omega = \mathbb{R}_+^{n+1}$, the upper half-space, by pulling back under an appropriate mapping (see, e.g., [DKPV]) which preserves the class of

coefficients satisfying the modified Hypothesis 1.1 (i.e., with property (c) replaced by (1.8)). By [KKiPT], we may further reduce matters to proving the Carleson measure estimate

$$(A.1) \quad \sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla u(x, t)|^2 t \, dx dt \leq C \|u\|_\infty^2,$$

for any bounded weak solution of the equation $Lu = 0$ in $\mathbb{R}_+^{n+1}$, and the supremum runs over all cubes $Q \subset \mathbb{R}^n$. At this point we follow the argument of [KP] essentially verbatim.

Fix u a bounded weak solution of the equation $Lu = 0$ in $\mathbb{R}_+^{n+1}$. Note that by the preceding reductions, $A = (a_{i,j})_{1 \leq i,j \leq n+1}$ satisfies the modified Hypothesis 1.1 in $\mathbb{R}_+^{n+1}$, with $X := (x, t) \in \mathbb{R}^n \times (0, \infty)$, and $\delta(x, t) = t$. In particular, by property (b), which now becomes $|\nabla A(x, t)| \lesssim 1/t$, we have that $|\nabla u(x, t)| \lesssim t^{-1} \|u\|_\infty$, uniformly in x (see (4.41)).

Observe that if we set $A' := (a_{n+1,n+1})^{-1} A$ (note that $a_{n+1,n+1} \geq \Lambda^{-1} > 0$ by ellipticity), then

$$L'u = -\operatorname{div} A' \nabla u = -\frac{1}{a_{n+1,n+1}} Lu - \nabla \left(\frac{1}{a_{n+1,n+1}} \right) \cdot A \nabla u = -\nabla \left(\frac{1}{a_{n+1,n+1}} \right) \cdot A \nabla u,$$

since $Lu = 0$; i.e., $L'u + \mathbf{B} \cdot \nabla u = 0$, where $\mathbf{B} = (B_1, B_2, \dots, B_{n+1})$, with

$$B_k = \sum_{j=1}^{n+1} \frac{\partial}{\partial x_j} \left(\frac{1}{a_{n+1,n+1}} \right) a_{j,k}, \quad 1 \leq k \leq n+1,$$

and $X_{n+1} = t$. Then by our current assumptions on A , $|\mathbf{B}| \lesssim 1/t$, and $|\mathbf{B}|^2 t \, dx dt$ is a Carleson measure in $\mathbb{R}_+^{n+1}$.

Thus, after relabeling A', L' as A, L , and normalizing so that $\|u\|_\infty \leq 1$, we may suppose that

$$(A.2) \quad a_{n+1,n+1} = 1, \quad Lu + \mathbf{B} \cdot \nabla u = 0, \quad \|u\|_\infty + t \|\nabla u(\cdot, t)\|_\infty \lesssim 1,$$

where $L = -\operatorname{div} A \nabla$, $|\mathbf{B}| \lesssim 1/t$, and $|\mathbf{B}|^2 t \, dx dt$ is a Carleson measure.

Fix a cube $Q \subset \mathbb{R}^n$, we define standard and two-sided Carleson boxes respectively, by

$$R_Q := Q \times (0, \ell(Q)), \quad R_Q^* := Q \times (-\ell(Q), \ell(Q)).$$

Let $\Phi = \Phi_Q \in C_0^\infty(\mathbb{R}^{n+1})$ be a smooth cut-off adapted to R_Q , so that $\operatorname{supp} \Phi \subset R_{2Q}^*$, $\Phi \equiv 1$ in R_Q^* , $0 \leq \Phi \leq 1$, and $\|\nabla \Phi\|_\infty \lesssim 1/\ell(Q)$.

Set

$$d\mu(x, t) := |\nabla A(x, t)|^2 t \, dx dt, \quad d\nu(x, t) := |\mathbf{B}(x, t)|^2 t \, dx dt,$$

and define their respective Carleson norms by

$$(A.3) \quad \|\mu\|_C := \sup_Q \frac{\mu(R_Q)}{|Q|}, \quad \|\nu\|_C := \sup_Q \frac{\nu(R_Q)}{|Q|}.$$

To prove the corresponding estimate in (A.1) for Q , it is routine to see that we can work with u_η, A_η and $\mathbf{B}_\eta$, in place of u, A and $\mathbf{B}$, defined by $u_\eta(x, t) := u(x, t + \eta)$, etc., and then let $\eta \rightarrow 0^+$ provided all our estimates are independent of η . To simplify the presentation we abuse the notation and use u, A and $\mathbf{B}$ to denote respectively u_η, A_η and $\mathbf{B}_\eta$. Notice that (A.2) remains true with bounds uniform in η , and also that $u = u_\eta$ is continuous in $\overline{\mathbb{R}_+^{n+1}}$. We use ellipticity and then the second equation in (A.2) to write

$$(A.4) \quad \begin{aligned} \iint_{R_Q} |\nabla u|^2 t \, dx dt &\leq \iint_{\mathbb{R}_+^{n+1}} |\nabla u|^2 \Phi t \, dx dt \\ &\lesssim \iint_{\mathbb{R}_+^{n+1}} \langle A \nabla u, \nabla u \rangle \Phi t \, dx dt \end{aligned}$$

$$\begin{aligned}
&= \iint_{\mathbb{R}_+^{n+1}} \langle A \nabla u, \nabla (u \Phi t) \rangle dx dt - \iint_{\mathbb{R}_+^{n+1}} \langle A \nabla u, \nabla (\Phi t) \rangle u dx dt \\
&= - \iint_{\mathbb{R}_+^{n+1}} \mathbf{B} \cdot \nabla u u \Phi t dx dt - \iint_{\mathbb{R}_+^{n+1}} \langle A \nabla u, \nabla \Phi \rangle u t dx dt - \iint_{\mathbb{R}_+^{n+1}} \langle A \nabla u, e_{n+1} \rangle u \Phi dx dt \\
&=: -I_1 - I_2 - I_3, ,
\end{aligned}$$

where e_{n+1} denotes the standard unit basis vector in the positive t direction.

We first treat term I_2 . By the last item in (A.2), and the construction of Φ , we find that

$$|I_2| \lesssim \frac{1}{\ell(Q)} \iint_{R_{2Q}} 1 dx dt \lesssim |Q|.$$

Next, we consider term I_3 , which we rewrite as

$$I_3 = \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} a_{n+1,j} (\partial_j u) u \Phi dx dt + \iint_{\mathbb{R}_+^{n+1}} (\partial_t u) u \Phi dx dt =: II + III,$$

since we have reduced to the case that $a_{n+1,n+1} \equiv 1$. Then, since u is continuous in $\overline{\mathbb{R}_+^{n+1}}$,

$$III = \frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} \partial_t (u^2) \Phi dx dt = -\frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} (\partial_t \Phi) u^2 dx dt - \frac{1}{2} \int_{\mathbb{R}^n} u^2 \Phi dx,$$

whence it follows that $|III| \lesssim |Q|$, by the properties of Φ , and the normalization $\|u\|_\infty \leq 1$. We also have

$$\begin{aligned}
II &= \sum_{j=1}^n \frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} a_{n+1,j} \partial_j (u^2) \Phi dx dt \\
&= -\frac{1}{2} \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \partial_t (a_{n+1,j} \partial_j (u^2) \Phi) t dx dt \\
&= -\frac{1}{2} \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \partial_t (a_{n+1,j}) \partial_j (u^2) \Phi t dx dt - \frac{1}{2} \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} a_{n+1,j} \partial_t (\partial_j (u^2)) \Phi t dx dt \\
&\quad - \frac{1}{2} \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} a_{n+1,j} \partial_j (u^2) (\partial_t \Phi) t dx dt \\
&=: -\frac{1}{2} \sum_{j=1}^n (II_{j,1} + II_{j,2} + II_{j,3}),
\end{aligned}$$

where we have integrated by parts in t in the second line. Exactly as for term I_2 , we find that $|II_{j,3}| \lesssim |Q|$, for each j . Integrating by parts horizontally, we find that

$$II_{j,2} = - \iint_{\mathbb{R}_+^{n+1}} \partial_j (a_{n+1,j}) \partial_t (u^2) \Phi t dx dt - \iint_{\mathbb{R}_+^{n+1}} a_{n+1,j} \partial_t (u^2) (\partial_j \Phi) t dx dt =: II'_{j,2} + II''_{j,2}.$$

Note that $|II''_{j,2}| \lesssim |Q|$, for each j , exactly as for term I_2 .

It remains to treat the terms I_1 , $II_{j,1}$, and $II'_{j,2}$, for which we have the cumulative estimate

$$\begin{aligned}
|I_1| + |II_{j,1}| + |II'_{j,2}| &\lesssim \iint_{\mathbb{R}_+^{n+1}} (|\mathbf{B}| + |\nabla A|) |\nabla u| |u| \Phi t dx dt \\
&\lesssim \frac{1}{\varepsilon} (\mu(R_{2Q}) + \nu(R_{2Q})) + \varepsilon \iint_{\mathbb{R}_+^{n+1}} |\nabla u|^2 \Phi t dx dt,
\end{aligned}$$

where ε is at our disposal, and where we have used the definition of Φ and the normalization $\|u\|_\infty \leq 1$. Choosing ε small enough, we may then hide the small term on the left hand side (more precisely in the second term) in (A.4); note that this is finite since we are working with u_η, A_η and $\mathbf{B}_\eta$. Also, by taking $0 < \eta \leq \ell(Q)$, clearly $\mu(R_{2Q}) + \nu(R_{2Q}) \lesssim (\|\mu\|_C + \|\nu\|_C)|Q|$ uniformly on η . Collecting our various estimates, letting $\eta \rightarrow 0^+$ and since Q was arbitrary, we find that (A.1) holds with $C \approx \|\mu\|_C + \|\nu\|_C$. $\square$

REFERENCES

- [ABHM] M. Akman, M. Badger, S. Hofmann and J. M. Martell, Rectifiability and elliptic measures on 1-sided NTA domains with Ahlfors-David regular boundaries, to appear, *Trans. Amer. Math. Soc.* **4**, 19
- [AHLMcT] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, The solution of the Kato Square Root Problem for Second Order Elliptic operators on $\mathbb{R}^n$, *Annals of Math.* **156** (2002), 633–654. 12
- [AGMT] J. Azzam, J. Garnett, M. Mourgoglou, and X. Tolsa, Uniform rectifiability, elliptic measure, square functions and ϵ -approximability, preprint 2016, arXiv:1612.02650. 4
- [AHMNT] J. Azzam, S. Hofmann, J. M. Martell, K. Nyström and T. Toro, A new characterization of chord-arc domains, to appear, *J. European Math. Soc.* 2
- [BiJ] C. Bishop and P. Jones, Harmonic Measure and arclength, *Annals of Math.* **132** (1990), 511–547. 2
- [Bo] J. Bourgain, On the Hausdorff dimension of harmonic measure in higher dimensions, *Invent. Math.* **87** (1987), 477–483. 9
- [CFMS] L. Caffarelli, E. Fabes, S. Mortola and S. Salsa, Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana Univ. Math. J.* **30** (1981), no. 4, 621–640. 10
- [Chr] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.*, **LX/LXI** (1990), 601–628. 5, 6
- [Dah] B. Dahlberg, On estimates for harmonic measure, *Arch. Rat. Mech. Analysis* **65** (1977), 272–288. 2
- [DKPV] B. Dahlberg, C. Kenig, J. Pipher, and G. Verchota, Area integral estimates for higher order elliptic equations and systems, *Annales de l’institut Fourier* **47** (1997), 1425–1461. 28
- [DJ] G. David and D. Jerison, Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals, *Indiana Univ. Math. J.* **39** (1990), no. 3, 831–845. 2, 4
- [DS1] G. David and S. Semmes, Singular integrals and rectifiable sets in $\mathbb{R}^n$: Au-dela des graphes lipschitziens, *Asterisque* **193** (1991). 2, 5, 6
- [DS2] G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Monographs and Surveys **38**, AMS 1993. 2, 3, 6
- [GMT] J. Garnett, M. Mourgoglou, and X. Tolsa, Uniform rectifiability in terms of Carleson measure estimates and ϵ -approximability of bounded harmonic functions, preprint 2016, arXiv:1611.00264. 2, 4
- [GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, Berlin 1983. 26
- [GW] M. Gruter and K-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), 303–342. 25
- [HKM] J. Heinonen, T. Kilpeläinen and O. Martio *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, New York 1993. 9
- [HLMc] S. Hofmann, M. Lacey and A. McIntosh, The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds, *Annals of Math.* **156** (2002), 623–631. 12
- [HLMN] S. Hofmann, P. Le, J.M. Martell and K. Nyström, *The weak- A_∞ property of harmonic and p -harmonic measures*, to appear, *Anal. PDE.* 2, 9, 12
- [HM1] S. Hofmann and J.M. Martell, *Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p* , Ann. Sci. École Norm. Sup. **47** (2014), no. 3, 577–654. 7, 8, 10, 11, 17, 20, 28
- [HM2] S. Hofmann and J.M. Martell, *Uniform Rectifiability and harmonic measure IV: Ahlfors regularity plus Poisson kernels in L^p implies uniform rectifiability*, preprint, arXiv:1505.06499. 2, 12
- [HMM] S. Hofmann, J.M. Martell, and S. Mayboroda, Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions, *Duke Math. J.* **165** (2016), no. 12, 2331–2389. 2
- [HMT] S. Hofmann, J.M. Martell and T. Toro, General divergence form elliptic operators on domains with ADR boundaries, and on 1-sided NTA domains, in progress. 9, 10

- [HMU] S. Hofmann, J.M. Martell and I. Uriarte-Tuero, Uniform Rectifiability and Harmonic Measure II: Poisson kernels in L^p imply uniform rectifiability, *Duke Math. J.* **163** (2014), no. 8, 1601–1654. [2](#)
- [HMc] S. Hofmann and A. McIntosh, The solution of the Kato problem in two dimensions, Proceedings of the Conference on Harmonic Analysis and PDE held in El Escorial, Spain in July 2000, *Publ. Mat.* Vol. extra, 2002, 143–160. [12](#)
- [HMMM] S. Hofmann, D. Mitrea, M. Mitrea, and A.J. Morris. L^p -square function estimates on spaces of homogeneous type and on uniformly rectifiable sets. *Mem. Amer. Math. Soc.* **245** (2017), no. 1159. [6](#)
- [JK] D. Jerison and C. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), no. 1, 80–147. [5](#)
- [Ke] C.E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, **83**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994. [3](#)
- [KKiPT] C. Kenig, B. Kirchheim, J. Pipher and T. Toro, Square functions and the A_∞ property of elliptic measures, *J. Geom. Anal.* **26** (2016), no. 3, 2383–2410. [4](#), [28](#), [29](#)
- [KP] C.E. Kenig and J. Pipher, The Dirichlet problem for elliptic equations with drift terms, *Publ. Mat.* **45**, (2001), 199–217. [1](#), [2](#), [4](#), [28](#), [29](#)
- [MT] M. Mourgoglou and X. Tolsa, Harmonic measure and Riesz transform in uniform and general domains, preprint, *arXiv:1509.08386*. [2](#)
- [Se] S. Semmes, A criterion for the boundedness of singular integrals on hypersurfaces, *Trans. Amer. Math. Soc.* **311** (1989), 501–513. [2](#)
- [St] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. [7](#)
- [Z] Z. Zhao, BMO solvability and the A_∞ condition of the elliptic measure in uniform domains, preprint, *arXiv:1602.00717*. [9](#)

STEVE HOFMANN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: `hofmanns@missouri.edu`

JOSÉ MARÍA MARTELL, INSTITUTO DE CIENCIAS MATEMÁTICAS CSIC-UAM-UC3M-UCM, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS, C/ NICOLÁS CABRERA, 13-15, E-28049 MADRID, SPAIN

E-mail address: `chema.martell@icmat.es`

TATIANA TORO, UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS, SEATTLE, WA 98195-4350, USA

E-mail address: `toro@uw.edu`