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Capacity bounds via operator space methods

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We prove that for generalized dephasing channels, the coherent information and reverse coherent information coincides. It also implies an alternative approach for the strong super-additivity and strong converse of generalized dephasing channels using the operator space technique. Our argument is based on an improved Rényi relative entropy estimate via analyzing the channel's Stinespring space. We also apply this estimate to new examples of quantum channels arising from quantum group co-representation and Kitave's quantum computation model. In particular, we find concrete examples of non-degradable channels that our estimates are tight and give a formula of nontrivial quantum capacity. *Published by AIP Publishing.*
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I. INTRODUCTION

Quantum Shannon theory extends Shannon's information theory to the framework of quantum physics laws. One core topic is to understand the capacity of quantum communication, that is, the maximal rate of transmitting qubits over a quantum channel. Thanks to the Lloyd-Shor-Devetak theorem,^{6,21,27} the quantum capacity $Q(N)$ of a quantum channel N is characterized as follows:

$$Q(N) = \lim_{k \rightarrow \infty} \frac{Q^{(1)}(N^{\otimes k})}{k},$$

$$Q^{(1)}(N) = \sup_{\phi} H(N(\phi)) - H(id \otimes N(|\phi\rangle\langle\phi|)). \quad (1)$$

Here $|\phi\rangle\langle\phi|$ is a purification of the input state ϕ , $H(\phi) = -\text{tr}(\phi \log \phi)$ is the von Neumann entropy, and the supremum runs over all input ϕ . $Q^{(1)}(N)$ is called the coherent information of N , and the quantum capacity $Q(N)$ is an average of $Q^{(1)}$ per use of channel over many uses, so called the regularization of $Q^{(1)}$. Despite this impressive theoretical success, the quantum capacity is computationally intractable in general, due to the increasing dimension in the supremum optimization. Devetak and Shor⁸ proved that for channels in which the environment can be retrieved from receiver's output with the help of another channel, the regularization is trivial and $Q = Q^{(1)}$ gives a "single-letter" formula for quantum capacity. This class of quantum channels is called degradable. For non-degradable channels, there are examples that the regularization is essentially necessary to see the ultimate capacity.^{5,33} Many different approaches have been introduced to give estimates on quantum capacity for particular or general quantum channels.^{15,28,29,31,32} However beyond degradable ones, little is known about the exact value of quantum capacity.

In our previous work,¹¹ we proved an estimate of quantum capacity via analyzing the Stinespring space of a channel. Recall that quantum channels mathematically are completely positive trace preserving maps which send density operators (positive and trace 1) to density operators. Let $\mathbb{B}(H)$ be the

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operators on a Hilbert space H . A quantum channel $N : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_B)$ admits a *Stinespring dilation*

$$N(\rho) = \text{tr}_E(V\rho V^*),$$

where $V : H_A \rightarrow H_B \otimes H_E$ is an Hilbert space isometry and H_E is the Hilbert space of the environment. The range of V , denoted by $X^N := \text{ran}(V) \subset H_B \otimes H_E$, is called the *Stinespring space* of N . A channel and its capacity are determined by its Stinespring space, and more precisely, the operator space structure by viewing $X^N \subset \mathbb{B}(H_E, H_B) \cong H_B \otimes H_E$ as operators from H_E to H_B . This perspective was used (in Ref. 1) to understand Hastings' counterexamples for additivity of minimal output entropy. Indeed, X^N is isomorphic to the input system H_A as Hilbert space, and hence every quantum channel is the restriction of partial trace on $\mathbb{B}(X^N)$. Using the operator structure in $\mathbb{B}(H_E, H_B)$, N can be rewritten as

$$N(|x\rangle\langle y|) = x^*y, x, y \in X^N \subset \mathbb{B}(H_E, H_B). \quad (2)$$

Here and following, we use the bra-ket notation " $|x\rangle$, $\langle y|$ " for Hilbert vectors (or dual vectors) in X^N and write x, y for the corresponding operators in $\mathbb{B}(H_E, H_B)$. Based on this, we consider the modified channel of N as follows:

$$N_f(|x\rangle\langle y|) = x^*fy, x, y \in X^N,$$

where $f \in \mathbb{B}(H_E)$ is a positive operator. N_f is a channel if and only if for all $x, y \in X^N$, $\text{tr}(yx^*f) = \text{tr}(yx^*)$. This modifies the channel via a matrix multiplication on the environment side H_E and, in particular, $N = N_1$ for $f = 1$ the identity. Let R be the C^* -algebra generated by elements $\{yx^* \in \mathbb{B}(H_E) | x, y \in X^N\}$. Our main estimate is as follows:

Theorem I.1. *Suppose $f \in \mathbb{B}(H_E)$ is a positive operator such that $\text{tr}(af) = \text{tr}(a)$ for all $a \in R$. Then*

$$Q(N) \leq Q(N_f) \leq Q(N) + J(f),$$

where the correction term $J(f)$ is given by

$$J(f) := \sup_{\rho \in R} H(\rho) - H(f^{\frac{1}{2}}\rho f^{\frac{1}{2}})$$

and the supremum runs over all density $\rho \in R$.

This generalizes the estimates in Ref. 11 by dropping a stronger assumption that f is independent of R . An interesting application of Theorem I.1 is that it implies that the reverse coherent information (see Sec. II) and coherent information coincide for generalized dephasing channels (also called Schur multiplier channels) and the $J(f)$ expression gives an alternative formula for the quantum capacity. Moreover, this also gives an operator space proof of the strong converse and strong super-additivity of Schur multiplier channels.^{31,36} Theorem I.1 also applies to channels from quantum group representations and Kitaev's quantum computation model. In particular, we also construct a 16-dimensional example of a non-degradable channel that our upper bound is tight.

Our method relies upon the interpolation relations of noncommutative vector-valued L_p space introduced in Refs. 16 and 25. These L_p norms are closely related to the sandwiched Rényi information measure considered in Refs. 22 and 35. The sandwiched Rényi information measure is recently found very useful to obtain the strong converse rate.^{13,31,34,35} Our estimates also give estimates of strong converse rate as well as private classical capacity using an identical argument from Ref. 11. Nevertheless, throughout the paper, our discussion will mainly focus on quantum capacity and briefly mention the parallel results for private capacity and strong converse rate.

We organize this work as follows: Section II gives the proof of the main Theorem I.1. We show in Sec. III that our estimates are tight for Schur multiplier channels and imply that the reverse coherent information and coherent information of Schur multiplier coincide. In Sec. IV, we discuss the quantum group channels constructed in Ref. 17, which includes the group random unitary channel as a special case. Section V provided the explicit 16-dimensional non-degradable channel for which our estimates are tight and give a formula. In Sec. VI, we discuss the examples related to toric codes and Kitaev's model.

II. CAPACITY ESTIMATES VIA CONDITIONAL L_p -SPACES

A. Conditional L_p -spaces

We index Hilbert by capital letters such as $H_A, H_B, \dots$ and write 1_A for the identity operator in $\mathbb{B}(H_A)$, id_A for the identity map, and tr_A for the trace on $\mathbb{B}(H_A)$. We restrict our discussion to finite dimension Hilbert spaces. For $1 \leq p < \infty$, the Schatten p -norm of an operator x in $\mathbb{B}(H)$ is

$$\|x\|_p := tr(|x|^p)^{\frac{1}{p}}, 1 \leq p < \infty.$$

We write $S_p(H)$ for the space equipped with Schatten p -norm and identify $S_\infty(H) = \mathbb{B}(H)$.

Let $\mathcal{A} \subset \mathbb{B}(H)$ be a unital C^* -subalgebra. The conditional expectation $E: \mathbb{B}(H) \rightarrow \mathcal{A}$ is the completely positive unital trace preserving map such that

$$tr(ax) = tr(aE(x)), a \in \mathcal{A}, x \in \mathbb{B}(H).$$

Let $2 \leq s, r \leq \infty$, $1 \leq q \leq \infty$ and fix the relation $\frac{1}{q} + \frac{1}{s} + \frac{1}{r} = \frac{1}{p}$. For an operator $x \in \mathbb{B}(H)$, we recall that the conditional $L_{(s,r)}^q(E)$ norm is defined as

$$\|x\|_{L_{(s,r)}^q(E)} := \sup\{\|axb\|_p \mid a, b \in \mathcal{A}, \|a\|_s = \|b\|_r = 1\}. \quad (3)$$

It was proved in Ref. 16 that

- (i) (3) are indeed norms (satisfy the triangle inequality) for all $2 \leq s, r \leq \infty$, $1 \leq p \leq \infty$.
- (ii) $L_{(s,r)}^q(E)$ spaces satisfy the complex interpolation relation that

$$[L_{(s_0,r_0)}^{q_0}(E), L_{(s_1,r_1)}^{q_1}(E)]_\theta = L_{(s,r)}^q(E) \quad (4)$$

isometrically for $0 \leq \theta \leq 1$ and

$$(1-\theta)/s_0 + \theta/s_1 = 1/s, (1-\theta)/r_0 + \theta/r_1 = 1/r, \\ (1-\theta)/q_0 + \theta/q_1 = 1/q.$$

We refer to Ref. 2 for information about complex interpolation. We will mostly use the cases $(s, r, q) = (2p, 2p, \infty)$ and $(s, r, q) = (2p, \infty, \infty)$ that

$$\|x\|_{L_{(2p,2p)}^\infty(E)} = \sup\{\|axb\|_p \mid a, b \in \mathcal{A}, \|a\|_{2p} = \|b\|_{2p} = 1\}, \\ \|x\|_{L_{(2p,\infty)}^\infty(E)} = \sup\{\|ax\|_{2p} \mid a \in \mathcal{A}, \|a\|_{2p} = 1\}.$$

We will use the short notation $L_p^\infty(E) := L_{(2p,2p)}^\infty(E)$ for the symmetric conditional L_p -space. It is clear from the definition that $\|f\|_{L_\infty^\infty(E)} = \|f\|_\infty$ and for positive f , $\|f\|_{L_1^\infty(E)} = \|E(f)\|_\infty$. It is also easy to verify that for any $x \in \mathbb{B}(H)$

$$\|x\|_{L_{(2p,\infty)}^\infty(E)}^2 = \|x^*x\|_{L_{2p}^\infty(E)}.$$

B. TROs and TRO channels

Let H, K be two Hilbert spaces. A closed subspace X of $\mathbb{B}(H, K)$ is a *ternary ring of operators* (TROs) if X is closed under the triple product

$$x, y, z \in X \Rightarrow xy^*z \in X.$$

TROs were first introduced by Hestenes in Ref. 14 and later studied as important classes of operator Hilbert modules.^{18,23} Given a TRO X , its left C^* -algebra L_X and the right C^* -algebra R_X are given by

$$L_X = \text{span}\{xy^* \mid x, y \in X\} \subset \mathbb{B}(H), \\ R_X = \text{span}\{x^*y \mid x, y \in X\} \subset \mathbb{B}(K).$$

X is a natural $L_X - R_X$ bimodule

$$L_X X = X, X R_X = X.$$

In finite dimensions, a TRO is always given by a direct sum of rectangular matrices with multiplicity. Namely, $X \cong \oplus_i (M_{n_i, m_i} \otimes \mathbb{C}1_{l_i})$, where M_{n_i, m_i} is the $n_i \times m_i$ matrix spaces and l_i is the multiplicity of the i th diagonal block $n_i \times m_i$.

Definition II.1. We say a quantum channel $N : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_B)$ is a TRO channel if its Stinespring space $X^N \subset \mathbb{B}(H_E, H_B)$ is a TRO.

Let $N : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_B)$ be a TRO channel with Stinespring space $X^N \subset \mathbb{B}(H_E, H_B)$. N is equivalent to a direct sum of partial traces and the range of N is the left algebra L_X . Indeed, let $X^N \cong \oplus_i (M_{n_i, m_i} \otimes 1_{l_i})$. The diagonal blocks M_{n_i, m_i} are mutually orthogonal subspaces of $H_A \cong (X^N, \|\cdot\|_2)$, and on each subspace, M_{n_i, m_i} as a Stinespring space corresponds to a partial trace $\text{id}_{n_i} \otimes \text{tr}_{m_i}$. Then the total map can be decomposed as an orthogonal sum

$$N(\rho) = \oplus_{i=1} [\text{id}_{n_i} \otimes \text{tr}_{m_i}(P_i \rho P_i)] \otimes \pi_{l_i}, \quad (5)$$

where P_i are projections onto i th block M_{n_i, m_i} and $\pi_{l_i} = \frac{1}{l_i}$ is the completely mixed state in M_{l_i} , the dummy state to the multiplicity [see Ref. 11 (Proposition 2.1) for a detailed argument]. The multiplicity l_i is irrelevant to channel capacity and we will often omit it. From Refs. 11 (Proposition 3.3) and 10 (Theorem 1), the quantum capacity of (5) is $\log \max_i n_i$, the logarithm of the largest size of the invariant system. Moreover, these channels have strong converse and strong additivity. One can see TROs as Stinespring spaces correspond to a special class of quantum channels whose capacity and related quantities are clear.

C. Perturbative capacity estimates

Let $N : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_B)$ be a TRO channel with Stinespring space X . As discussed in the Introduction, we can rewrite N as

$$N(|x\rangle\langle y|) = xy^*, \quad x, y \in X,$$

where $|x\rangle, |y\rangle$ are the vectors in the Hilbert space $(X, \|\cdot\|_2)$. For any $f \in \mathbb{B}(H_E)$, we define the modified map

$$N_f(|x\rangle\langle y|) = xf y^*, \quad x, y \in X.$$

Let $E_R : \mathbb{B}(H_E) \rightarrow R_X$ (respectively, $E_L : \mathbb{B}(H_B) \rightarrow L_X$) be the conditional expectation onto the right algebra R_X (respectively, the left algebra L_X). It is not hard to verify (Proposition 2.3 of Ref. 11) that N_f is a quantum channel if and only if f is positive and $E_R(f) = 1$. In this case,

- (i) the Stinespring dilation of N_f is given by

$$V_f : (X, \|\cdot\|_2) \rightarrow H_B \otimes H_E, \quad V_f(|x\rangle) = |xf^{\frac{1}{2}}\rangle.$$

- (ii) $E_L \circ N_f = N$.

The goal is to compare the capacity of N_f to the TRO channel N , whose capacity is clear. We approach this by analysing the sandwiched Rényi relative entropy introduced in Refs. 22 and 35. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For two densities ρ, σ , the relative entropy D and the sandwiched Rényi relative entropy D_p are defined as follows:

$$D(\rho||\sigma) := \text{tr}(\rho \log \rho - \rho \log \sigma) = \lim_{p \rightarrow 1^+} D_p(\rho||\sigma),$$

$$D_p(\rho||\sigma) := p' \log \|\sigma^{-\frac{1}{2p'}} \rho \sigma^{-\frac{1}{2p'}}\|_p.$$

Here we assume that the support of ρ is dominated by the support of σ ; otherwise, D and D_p are defined as $+\infty$. The following lemma is an improvement of the main technical theorem of Ref. 11:

Lemma II.2. Let $N : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_B)$ be a TRO channel with Stinespring space X , and let N_f be the modified channel defined as above. For any density $\sigma \in L_X \subset \mathbb{B}(H_B)$ and $\rho \in \mathbb{B}(H_A)$,

$$D_p(N(\rho)||\sigma) \leq D_p(N_f(\rho)||\sigma) \leq D_p(N(\rho)||\sigma) + p' \log \|f\|_{L_p^\infty(E_R)}. \quad (6)$$

Proof. By the definition of E_L , $E_L(\sigma) = \sigma$ for $\sigma \in L_X$. Then the lower bound is a direct consequence of the data processing inequality of D_p (see, e.g., Theorem 6 of Ref. 22)

$$D_p(N_f(\rho)||\sigma) \geq D_p(E_L \circ N_f(\rho)||E_L(\sigma)) = D_p(N(\rho)||\sigma).$$

The upper bound is equivalent to the following norm inequality:

$$\|\sigma^{-\frac{1}{2p'}} N_f(\rho) \sigma^{-\frac{1}{2p'}}\|_p \leq \|f\|_{L_p^\infty(E_R)} \|\sigma^{-\frac{1}{2p'}} N(\rho) \sigma^{-\frac{1}{2p'}}\|_p.$$

Let $V: H_A \rightarrow H_B \otimes H_E$ be the Stinespring dilation of N . For $\eta \in \mathbb{B}(H_A)$, $V\eta \in \mathbb{B}(H_A, H_E \otimes H_B)$ and we write $\widehat{\eta}$ for the corresponding operator in $\mathbb{B}(H_A \otimes H_E, H_B)$ via the identification $H_B \otimes H_E \cong \mathbb{B}(H_E, H_B)$. Then for $\rho = \eta\eta^*$

$$N(\rho) = \widehat{\eta}\widehat{\eta}^*, N_f(\rho) = \widehat{\eta}(1_A \otimes f)\widehat{\eta}^*.$$

Let $\tilde{X} := \mathbb{B}(H_A, \mathbb{C}) \otimes X$ be the tensor product of X and a column vector space. $\tilde{X}$ is a TRO in $\mathbb{B}(H_A \otimes H_E, H_B)$, and its left algebra is again $L_{(\tilde{X})} = L_X$ and the right algebra is $R_{(\tilde{X})} = \mathbb{B}(H_A) \otimes R_X$. The conditional expectation from $\mathbb{B}(H_A \otimes H_E)$ to $\mathbb{B}(H_A) \otimes R_X$ is $\tilde{E}_R = id_A \otimes E_R$.

It is clear that $\widehat{\eta}$ belongs to $\tilde{X}$ and for $\sigma \in L_X$, $\sigma^{-\frac{1}{2p'}} \widehat{\eta} \in \tilde{X}$ by the left L_X -module property. Therefore for any positive $\rho = \eta\eta^* \in \mathbb{B}(H_A)$,

$$\begin{aligned} & \frac{\|\sigma^{-\frac{1}{2p'}} N_f(\rho) \sigma^{-\frac{1}{2p'}}\|_p}{\|\sigma^{-\frac{1}{2p'}} N(\rho) \sigma^{-\frac{1}{2p'}}\|_p} \\ &= \frac{\|\sigma^{-\frac{1}{2p'}} \widehat{\eta}(1_A \otimes f^{\frac{1}{2}})\|_{2p}^2}{\|\sigma^{-\frac{1}{2p'}} \widehat{\eta}\|_{2p}^2} \\ &\leq \sup_{x \in \tilde{X}, \|x\|_{2p}=1} \|x(1_A \otimes f^{\frac{1}{2}})\|_{2p}^2 \\ &= \sup_{x \in \tilde{X}, \|x\|_{2p}=1} \|(1_A \otimes f^{\frac{1}{2}})x^*x(1_A \otimes f^{\frac{1}{2}})\|_p \\ &= \sup_{x \in \tilde{X}, \|x\|_{2p}=1} \|(x^*x)^{\frac{1}{2}}(1_A \otimes f^{\frac{1}{2}})\|_{2p}^2 \\ &\leq \sup_{a \in R_{\tilde{X}}, \|a\|_{2p}=1} \|a(1_A \otimes f^{\frac{1}{2}})\|_{2p}^2. \end{aligned}$$

The first inequality above is because $\sigma^{-\frac{1}{2p'}} \widehat{\eta} \in \tilde{X}$ and the second inequality is because $(x^*x)^{\frac{1}{2}}$ is an element in the right algebra $R_{\tilde{X}} = \mathbb{B}(H_A) \otimes R_X$. By the definition of the conditional L_p -norms (3),

$$\begin{aligned} & \frac{\|\sigma^{-\frac{1}{2p'}} N_f(\rho) \sigma^{-\frac{1}{2p'}}\|_p}{\|\sigma^{-\frac{1}{2p'}} N(\rho) \sigma^{-\frac{1}{2p'}}\|_p} \leq \|(1_A \otimes f^{\frac{1}{2}})\|_{L_{(2p, \infty)}^\infty(\tilde{E}_R)}^2 \\ &= \|(1_A \otimes f)\|_{L_p^\infty(\tilde{E}_R)}. \end{aligned}$$

Moreover, for any $1 \leq p \leq \infty$, we have

$$\|(1_A \otimes f)\|_{L_p^\infty(id_A \otimes E_R)} = \|f\|_{L_p^\infty(E_R)}.$$

Indeed, for $p = 1$ and $p = \infty$, this is easily verified and the case for general p follows from interpolation. $\square$

The above lemma gives a comparison of Rényi relative entropy for each single input ρ . Passing (6) to the limit $p \rightarrow 1^+$, we obtain the inequality that for relative entropy

$$D(N_f(\rho) || \sigma) \leq D(N(\rho) || \sigma) + \lim_{p \rightarrow 1^+} \frac{p}{p-1} \log \|f\|_{L_p^\infty(E_R)}.$$

Note that for positive f

$$\|f\|_{L_p^\infty(E_R)} = \sup_{\omega} \frac{\|\omega^{\frac{1}{2}} f \omega^{\frac{1}{2}}\|_p}{\|\omega\|_p} = \sup_{\omega} \frac{\|f^{\frac{1}{2}} \omega f^{\frac{1}{2}}\|_p}{\|\omega\|_p},$$

where the supremum runs over all density $\omega \in R_X$. Since $\|f\|_{L_1^\infty(E_R)} = \|E_R(f)\|_\infty = 1$,

$$\begin{aligned} & \lim_{p \rightarrow 1^+} \frac{p \log \|f\|_{L_p^\infty(E_R)}}{p-1} \\ &= \lim_{p \rightarrow 1^+} \frac{\|f\|_{L_p^\infty(E_R)} - 1}{p-1} \\ &= \lim_{p \rightarrow 1^+} \sup_{\omega} \frac{1}{\|\omega\|_p} \frac{\|f^{\frac{1}{2}} \omega f^{\frac{1}{2}}\|_p - 1}{p-1} - \frac{\|\omega\|_p - 1}{p-1} \\ &= \sup_{\omega} H(\omega) - H(f^{\frac{1}{2}} \omega f^{\frac{1}{2}}). \end{aligned}$$

This is the maximum entropy difference of the map

$$M_f : R_X \rightarrow \mathbb{B}(H_E), M_f(\omega) = f^{\frac{1}{2}} \omega f^{\frac{1}{2}}.$$

This map is completely positive trace preserving on R_X by the assumption $E_R(f) = 1$. Define

$$J(f) := \sup_{\omega \in R_X} H(\omega) - H(f^{\frac{1}{2}} \omega f^{\frac{1}{2}}).$$

From the identical argument of Ref. 11 (Corollaries 3.1 and 3.5), we obtain the following capacity estimate:

Theorem II.3. *Let $N : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_B)$ be a TRO channel with Stinespring space X . Then for any $f \in \mathbb{B}(H_E)$ and $E_R(f) = 1$,*

$$Q(N) \leq Q(N_f) \leq Q(N) + J(f).$$

Similar estimates were obtained in Ref. 11 that there is a technical assumption that f is independent of R_X , i.e., there exists a C^* -algebra $\mathcal{A}$ such that $f \in \mathcal{A}$ and for any $a \in \mathcal{A}$, $E_R(a) = \tau(a)1_E$, where $\tau(x) = \frac{1}{|E|} \text{tr}(x)$ is the normalized trace on $\mathbb{B}(H_E)$. Here we only need a minimal assumption that N_f is a channel and the estimates are obtained with the new correction term $J(f)$.

Proposition II.4. *Let $f \in \mathbb{B}(H_E)$ be a positive operator such that $E_R(f) = 1$. Then $J(f) \leq \|E_R(f \log f)\|_\infty$. In particular, $J(f) = \tau(f \log f)$ if f is independent of R_X .*

Proof. We first show that for any positive $a \in \mathbb{B}(H_E)$, $\|a\|_{L_{(2p,\infty)}^\infty(E_R)} \leq \|a^p\|_{L_{(2,\infty)}^\infty(E_R)}$. Assume that $\|a^p\|_{L_{(2,\infty)}^\infty(E_R)} = 1$. Consider the analytic map

$$h : \{0 \leq \text{Re}(z) \leq 1\} \rightarrow \mathbb{B}(H_E), h(z) = a^{pz}.$$

For $z = 1 + it$,

$$\|a^{p(1+it)}\|_{L_{(2p,\infty)}^\infty(E_R)} = \|a^p\|_{L_{(2,\infty)}^\infty(E_R)} = 1.$$

For $z = it$,

$$\|a^{ip t}\|_{L_{(\infty,\infty)}^\infty(E_R)} = \|a^{ip t}\|_\infty = 1.$$

By interpolation (4), we know at $z = \frac{1}{p}$

$$\|a\|_{L_{(2p,\infty)}^\infty(E_R)} \leq 1.$$

Choosing $a = f^{\frac{1}{p}}$, we obtain that

$$\|f\|_{L_{(2p,2p)}^\infty(E_R)} \leq \|f^p\|_{L_{(2,2)}^\infty(E_R)}.$$

Take the limit $p \rightarrow 1^+$,

$$\begin{aligned} \lim_{p \rightarrow 1^+} \frac{\|f\|_{L_p^\infty(E_R)} - 1}{p - 1} &\leq \lim_{p \rightarrow 1^+} \frac{\|f^p\|_{L_2^\infty(E_R)} - 1}{p - 1} \\ &\leq \lim_{p \rightarrow 1^+} \frac{\|f^p\|_{L_2^\infty(E_R)} - \|f\|_{L_2^\infty(E_R)}}{p - 1} \\ &\leq \lim_{p \rightarrow 1^+} \|E_R(\frac{f^p - f}{p - 1})\|_{L_2^\infty(E_R)} \\ &= \|E_R(f \log f)\|_{L_2^\infty(E_R)}. \end{aligned}$$

In addition, if f is independent of R_X , we know

$$\|E_R(f \log f)\|_\infty = \|\tau(f \log f)\|_\infty = \tau(f \log f). \quad \square$$

For discussion of Secs. III–V, we recall here the negative cb-entropy formula in Ref. 11. For a quantum channel N , the negative cb-entropy (also called the reverse coherent information, see Refs. 12 and 37 for its operational meaning) is given by

$$-S_{cb}(N) = \sup_{\phi} H(\phi) - H(\text{id} \otimes N(|\phi\rangle\langle\phi|)),$$

where the supremum runs over all input states ϕ and $|\phi\rangle$ is a purification of ϕ .

Theorem II.5. *Let N be a quantum channel and N_f be a modification of N such that f is independent of the right algebra R_X . Suppose that the complimentary channel $N^E : \mathbb{B}(H_A) \rightarrow \mathbb{B}(H_E)$ is unital up to a scalar, i.e., $N^E(1_A) = \frac{|A|}{|E|} 1_E$. Then*

$$-S_{cb}(N_f) = \log \frac{|A|}{|E|} + \tau(f \log f).$$

III. SCHUR MULTIPLIERS

Schur multipliers, also called generalized dephasing channels, are special cases of Hadamard channels whose input and output systems are of the same dimension. Let M_n be the algebra of $n \times n$ matrices and e_{ij} be the matrix unit at i th-column and j th-row. The Schur multiplier associated with a matrix $(a_{ij})_{i,j=1}^n$ is given by

$$M_a : M_n \rightarrow M_n, M_a(b) = (a_{ij}b_{ij})_{i,j}.$$

For $a = 1$ being the identity operator, the Schur multiplier is the conditional expectation onto the diagonal matrices D

$$E_D(b) = E_D(\sum_{ij} b_{ij} \otimes e_{ij}) = \sum_i b_{ii} \otimes e_{ii}.$$

It is known that M_a is completely positive if and only if a is positive,²³ and moreover, M_a is trace preserving if and only if $a_{ii} = 1$ for all $1 \leq i \leq n$. Thus for any positive a such that $E_D(a) = 1$, the Schur multiplier M_a is a quantum channel. As special cases of Hadamard channels, M_a are known to be degradable hence the regularization is trivial $Q = Q^{(1)}$. Moreover, they are an important class of quantum channels which have strong-additive quantum capacity and strong converse.^{31,34,36}

The conditional expectation $M_1 = E_D$ is a TRO channel, and M_a can be viewed as a modification of M_1 in our setting. Indeed, the Stinespring dilation of E_D is

$$E_D(\rho) = \text{id}_n \otimes \text{tr}_n(V\rho V),$$

where $V : \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n$ is

$$V(|i\rangle) = |i\rangle|i\rangle.$$

The TRO $X = \text{ran}(V) \subset \mathbb{B}(\mathbb{C}^n, \mathbb{C}^n) \cong M_n$ are the diagonal matrices D and a vector $|h\rangle = \sum_i h_i |i\rangle \in \mathbb{C}^n$ corresponds to operator $h = \sum_i h_i e_{ii}$. The channel $M_1 = E_D$ and M_a can be written as

$$E_D(|h\rangle\langle h|) = h^*h, M_a(|h\rangle\langle h|) = h^*ah.$$

Because E_D has a commutative range, E_D is entanglement-breaking and $Q(E_D) = 0$. Then by Theorem II.3,

$$Q(a) \leq J_D(a), J(a) = \sup_{\omega \in D} H(\omega) - H(a^{\frac{1}{2}} \omega a^{\frac{1}{2}}).$$

The main result of this section is to show that this is indeed an equality. It was observed in Ref. 7 that the negative cb -entropy is the derivative of $S_1 \rightarrow S_p$ completely bounded norm at $p = 1$,

$$-S_{cb}(M_a) = \lim_{p \rightarrow 1^+} \frac{1}{p-1} (\|M_a : S_1^n \rightarrow S_p^n\|_{cb} - 1).$$

By Effros-Ruan theorem (see Refs. 9 and 26)

$$\|M_a : S_1^n \rightarrow S_p^n\|_{cb} = \|\chi_a\|_{M_n(S_p^n)},$$

where $\chi_a = \sum_{ij} a_{ij} e_{ij} \otimes e_{ij}$ is the Choi matrix of M_a and for a bipartite matrix $y \in M_n \otimes M_n$, the $M_n(S_p^n)$ norm is

$$\|y\|_{M_n(S_p^n)} = \sup_{b_1, b_2 \in M_n, \|b_1\|_{2p} = \|b_2\|_{2p} = 1} \|(b_1 \otimes 1)y(b_2 \otimes 1)\|_p.$$

In particular, for $p = 1$ and positive y ,

$$\|y\|_{M_n(S_1^n)} = \|tr \otimes id(y)\|_{\infty}.$$

$M_n(S_p^n)$ is the noncommutative vector-valued L_p space introduced by Pisier in Ref. 25 which defines the operator space structure of S_p . It motivates the conditional L_p -norm of subalgebras. $M_n(S_p)$ is a special case of $L_p^{\infty}(E)$ with E being the partial trace $id \otimes tr : M_n \otimes M_n \rightarrow M_n$.

Proposition III.1. Let

$$C := \{\chi_a = \sum_{ij} a_{ij} e_{ij} \otimes e_{ij} | a \in M_n\} \subset M_n \otimes M_n$$

be the space of Choi matrices of all n -dimensional Schur multiplier. Then

- (i) *the map $\chi : M_n \rightarrow C$, $\chi(a) = \chi_a$ is a $*$ -isomorphism.*
- (ii) *C is a 1-complemented subspace of $M_n(S_p^n)$ for all $1 \leq p \leq \infty$, i.e., there exists a projection map P_C from $M_n(M_n)$ onto C such that P_C is contractive on $M_n(S_p^n)$.*
- (iii) *For any $a \in M_n$,*

$$\|a\|_{L_p^{\infty}(E_D)} = \|\chi_a\|_{M_n(S_p^n)}.$$

Namely, $\chi : L_p^{\infty}(E_D) \rightarrow M_n(S_p^n)$ is a Banach space isometry.

Proof. (i) is easy to verify. For (ii), the projection map is given by

$$P_C(x) = PxP,$$

where P is the projection from $\mathbb{C}^n \otimes \mathbb{C}^n$ onto the space $\text{span}\{|i\rangle\langle i| | 1 \leq i \leq n\}$. P_C is clearly completely contractive on $M_n(M_n)$. Note that $(tr \otimes id) \circ P_C(x) = E_D \circ (tr \otimes id)(x)$. Then for a positive x ,

$$\begin{aligned} \|P_C(x)\|_{M_n(S_1)} &= \|(tr \otimes id) \circ P_C(x)\|_{\infty} = \|E_D \circ (tr \otimes id)(x)\|_{\infty} \\ &\leq \|(tr \otimes id)(x)\|_{\infty} = \|x\|_{M_n(S_1)}. \end{aligned}$$

For general x , we can write $x = x_1 x_2$ such that

$$\begin{aligned} \|x\|_{M_n(S_1^n)} &= \|x_1 x_1^*\|_{M_n(S_1^n)}^{\frac{1}{2}} \|x_2^* x_2\|_{M_n(S_1^n)}^{\frac{1}{2}} \\ &\geq \|P x_1 x_1^* P\|_{M_n(S_1^n)}^{\frac{1}{2}} \|P x_2^* x_2 P\|_{M_n(S_1^n)}^{\frac{1}{2}} \\ &\geq \|P x_1 x_2 P\|_{M_n(S_1^n)}. \end{aligned} \quad (7)$$

Thus by interpolation, P_C is a contraction on $M_n(S_p)$ for all $1 \leq p \leq \infty$. For (iii), the equality of $p = \infty$ is a consequence of (i). For $p = 1$, because $tr \otimes id(\chi_a) = E_D$, so for positive a

$$\|\chi_a\|_{M_n(S_1^n)} = \|E_D(a)\|_{\infty} = \|a\|_{L_1^{\infty}(E_D)}.$$

The equality in general follows from the similar Cauchy-Schwarz type argument as above (7). Note that (ii) implies that C equipped with $M_n(S_p)$ norms forms an interpolation family. Then the equality in general p follows from interpolation, which completes the proof. $\square$

Remark III.2. The argument above actually shows that χ is a complete isometry. The space C equipped with operator space structure $M_n(S_p)$ can be expressed as Haagerup tensor product $l_{p,c}^\infty \otimes_h l_{p,r}^\infty$ where $l_{p,c}^\infty$ is the subspace of diagonal matrices in $C^n \otimes_h C_p^n$ and $l_{p,r}^\infty$ is the subspace of diagonal matrices in $R_p^n \otimes_h R^n$. Here C^n (respectively, R^n) is the n -dimensional column (row) space and $C_p^n = R_{p'}^n = [C^n, R^n]_1^p$ is the interpolation space. In particular, for $p = 1$, we have shown that at Banach space level $L_2^\infty(E_D)$ is isometric to Grothendieck's factorization norm $l_2^n \otimes_\gamma l_2^n$ (see Ref. 24).

Our main theorem of this section is as follows:

Theorem III.3. *For all Schur multipliers,*

$$-S_{cb}(M_a) = Q^{(1)}(M_a) = Q(M_a) = J_D(a).$$

Proof. Because Schur multipliers M_a are unital, we have $H(M_a(\rho)) \geq H(\rho)$ and hence

$$\begin{aligned} -S_{cb}(M_a) &= \sup_{\rho^{AA'}} H(\rho^A) - H(id \otimes M_a(\rho)) \\ &\leq \sup H(M_a(\rho^A)) - H(id \otimes M_a(\rho)) \\ &\leq Q^{(1)}(M_a). \end{aligned}$$

Moreover, from Theorem II.3,

$$Q^{(1)}(M_a) \leq Q(M_a) \leq J_D(a).$$

Thus it suffices to show that $J_D(a) \leq -S_{cb}(M_a)$. This follows by taking derivatives to the norms inequality from Proposition III.1

$$J_D(a) = \lim_{p \rightarrow 1^+} \frac{\|a\|_{L_2^\infty(E_D)} - 1}{p - 1} \leq \lim_{p \rightarrow 1^+} \frac{\|\chi_a\|_{M_n(S_p^n)} - 1}{p - 1} = -S_{cb}(M_a). \quad \square$$

This above theorem implies that the coherent information and reverse coherent information coincides. It also provides an operator space proof for the quantum strong converse and strong additivity of $Q^{(1)}$ for Schur multiplier channels.

IV. QUANTUM CHANNELS FROM QUANTUM GROUPS

Given a finite group G , two classes of quantum channels constructed from G have been discussed in Refs. 4 and 11. One is the random unitary channel given by

$$N_f : \mathbb{B}(H) \rightarrow \mathbb{B}(H), N_f(\rho) = \sum_{g \in G} f(g) u(g) \rho u(g),$$

where $u : G \rightarrow B(H)$ is a unitary representation of G and f is a probability distribution on G . The other classes are Schur multipliers

$$M_\phi = \mathbb{B}(l_2(G)) \rightarrow \mathbb{B}(l_2(G)), M_\phi(\rho) = (\phi(g^{-1}h) \rho_{gh})_{g,h},$$

where $l_2(G)$ is the Hilbert space spanned by the orthonormal basis $\{|g\rangle\}_{g \in G}$ and $\phi : G \rightarrow \mathbb{C}$ is a positive definite function on G . These two classes of channels are dual to each other in the quantum group sense. In this section, we introduce a general construction of quantum channels from co-representation of a quantum group. Both this work and Ref. 4 are inspired by the unpublished notes (Ref. 17).

We recall the basic definition of quantum groups and refer to Refs. 20 and 30 for more information. A C^* -compact quantum group $(\mathbb{G}, \Delta)$ is a unital C^* -algebra $\mathbb{G}$ equipped with a comultiplication Δ , i.e., a unital $*$ -homomorphism from $\mathbb{G}$ to $\mathbb{G} \otimes \mathbb{G}$ such that

- (i) $(\Delta \otimes id_{\mathbb{G}})\Delta = (id_{\mathbb{G}} \otimes \Delta)\Delta$,
- (ii) each of the sets $\Delta(\mathbb{G})(1_{\mathbb{G}} \otimes \mathbb{G})$ and $(1_{\mathbb{G}} \otimes \mathbb{G})\Delta(\mathbb{G})$ is linearly dense in $\mathbb{G} \otimes \mathbb{G}$.

A compact quantum group $\mathbb{G}$ admits a unique Haar state τ such that

$$(\tau \otimes id)\Delta(x) = \tau(x)1 = (id \otimes \tau)\Delta(x).$$

We denote $L_2(\mathbb{G}, \tau)$ for the Hilbert space equipped with the inner product $\langle x|y \rangle_{\tau} = \tau(x^*y)$. A co-representation of $\mathbb{G}$ is given by a unitary $U \in \mathbb{B}(H) \otimes \mathbb{G}$ satisfying the identity

$$(id \otimes \Delta)U = U_{13}U_{12}, \quad (8)$$

where $U_{12} = U \otimes 1 \in \mathbb{B}(H) \otimes \mathbb{G} \otimes \mathbb{G}$ and $U_{13} = (\Sigma \otimes id)(1 \otimes U)$ and $\Sigma(x \otimes y) = y \otimes x$ is the flip operator. A unitary co-representation is irreducible if the commutant $\{id \otimes \tau((1 \otimes a)U) | a \in \mathbb{G}\}' = \mathbb{C}$ is trivial. Let us denote $\mathcal{A}$ for the algebra generated by $\{id \otimes \tau((1 \otimes a)U) | a \in \mathbb{G}\}$ and $\mathcal{A}'$ for its commutant.

Proposition IV.1. The conditional expectation $E_{\mathcal{A}'} : \mathbb{B}(H) \rightarrow \mathcal{A}'$ is given by

$$N(\rho) = id \otimes \tau(U(\rho \otimes 1)U^*).$$

In particular, N is a TRO channel.

Proof. For any $x \in \mathcal{A}'$, $(x \otimes 1)U = U(x \otimes 1)$ because for any $y \in \mathbb{B}(H)$, $a \in \mathbb{G}$,

$$\begin{aligned} tr \otimes \tau((x \otimes 1)U(y \otimes a)) &= tr(x(id \otimes \tau(U(1 \otimes a))y)) \\ &= tr((id \otimes \tau((1 \otimes a)U))xy) \\ &= tr \otimes \tau(U(x \otimes 1)(y \otimes a)). \end{aligned}$$

For any $x \in \mathcal{A}'$, $y \in \mathbb{B}(H)$

$$\begin{aligned} tr(N(y)x) &= tr \otimes \tau(U(y \otimes 1)U^*(x \otimes 1)) \\ &= tr \otimes \tau(U(yx \otimes 1)U^*) \\ &= tr \otimes \tau(yx \otimes 1) \\ &= tr(yx), \end{aligned}$$

which implies $N(y) = E_{\mathcal{A}'}(y)$. Hence N is a TRO channel because $N = E_{\mathcal{A}'}$ is a conditional expectation. $\square$

In finite dimensions, the unitary co-representation U can be decomposed as

$$U = \oplus_i U_i \otimes 1_{l_i},$$

where for each i , U_i is an irreducible co-representation on some finite dimensional Hilbert space $\mathbb{C}^{n_i}$ and l_i is the multiplicity. In this situation,

$$\mathcal{A} \cong M_{n_i} \otimes \mathbb{C}1_{l_i}, \mathcal{A}' \cong \mathbb{C}1_{n_i} \otimes M_{l_i},$$

and the quantum capacity of N is $\log(\max_i l_i)$, the logarithm of the largest multiplicity.

For each density $f \in \mathbb{G}$ (positive and $\tau(f) = 1$), the modification of N is

$$N_f : \mathbb{B}(H) \rightarrow \mathbb{B}(H), N_f(\rho) = id \otimes \tau(U(\rho \otimes f)U^*). \quad (9)$$

Assume that $U = \sum_i x_i \otimes a_i$ for some $x_i \in \mathbb{B}(H)$, $a_i \in \mathbb{G}$. N_f is a completely positive trace preserving map with Stinespring dilation

$$V_f : H \rightarrow H \otimes L_2(\mathbb{G}), V_f|h\rangle = \sum_i x_i|h\rangle \otimes |a_i\rangle.$$

From Theorem II.3 and Proposition II.4, we have

Theorem IV.2. Let $U \in \mathbb{B}(H) \otimes \mathbb{G}$ be finite dimensional co-representation of $\mathbb{G}$ and N_f be the quantum group channel defined in (9). Then for any density $f \in \mathbb{G}$

$$Q(N) \leq Q(N_f) \leq Q(N) + \tau(f \log f),$$

and $Q(N)$ is the logarithm of the largest multiplicity of the irreducible decomposition of U .

Example IV.3 (Left regular co-representation). Finite dimensional quantum groups admit a multiplicative unitary $W \in B(L_2(\mathbb{G})) \otimes \mathbb{G}$ which implements the co-multiplication as follows:

$$\Delta(a) = W^*(a \otimes 1)W.$$

Here $\mathbb{G}$ is identified with its GNS representation in $\mathbb{B}(L_2(\mathbb{G}))$. The dual quantum group $\hat{\mathbb{G}}$ is given by

$$\hat{\mathbb{G}} = \{id \otimes \tau(1 \otimes aW) | a \in \mathbb{G}\}.$$

The co-representation given by W is called the left regular co-representation of $\mathbb{G}$. From the Peter-Weyl theorem for compact quantum group, the regular co-representation

$$W = \oplus_i U_i \otimes 1_{n_i}$$

is the direct sum of all finite dimensional irreducible representation $U_i \in M_{n_i}(\mathbb{G})$ (up to unitary equivalent) and the multiplicity n_i equals to the dimension of the co-representation. For a density $f \in \hat{\mathbb{G}}$, the quantum channels constructed from W are

$$N_f(\rho) = id \otimes \tau(W(\rho \otimes f)W^*).$$

For $f = 1$, the range of N is given by $\hat{\mathbb{G}}'$, the commutant of the dual quantum group $\hat{\mathbb{G}}$. Note that for all $f \in \hat{\mathbb{G}}$,

$$tr(N_f(\rho)) = tr \otimes \tau(W(\rho \otimes f)W^*) = tr(\rho)\tau(f).$$

Then any $f \in \mathbb{G}$ is independent to the right algebra of the TRO channel N . Denote $d_{\mathbb{G}} = \max_i n_i$ as the largest dimension of co-representation of $\mathbb{G}$. We obtain from Theorem II.3, Proposition II.4, and Theorem II.5 that

$$\max\{\log d_{\mathbb{G}}, \tau(f \log f)\} \leq Q^{(1)}(N_f) \leq Q(N_f) \leq \tau(f \log f) + \ln d_{\mathbb{G}}.$$

The upper and lower bounds differ up to a factor of 2. This extends the estimates for group left regular representation in Ref. 11 to quantum groups.

V. A NON-DEGRADABLE EXAMPLE

In this section, we present a concrete 16-dimensional channel which is non-degradable and saturates our upper bound. Let $H = \mathbb{C}^4 \otimes \mathbb{C}^4$ be the 16-dimensional Hilbert space and $\{|j\rangle|k\rangle | 1 \leq j \leq 4, 1 \leq k \leq 4\}$ give a tensor product basis. Let P_j be the projection onto $H_j = \mathbb{C}|j\rangle \otimes \mathbb{C}^4 = \text{span}\{|j\rangle \otimes |h\rangle | h \in \mathbb{C}^4\}$. Consider the following TRO space X :

$$X = \text{span}\left\{\sum_{j=1}^4 |j\rangle \otimes \langle h_j | | h_j \rangle \in H_j\right\} \subset B(\mathbb{C}^{16}, \mathbb{C}^4).$$

More explicitly, in standard basis, X are the rectangle matrices as follows:

$$\begin{bmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{bmatrix},$$

where for each i , h_i is a row vector in H_i and 0 's are 4-dimensional row vectors. Then the left algebra $L_X = l_\infty^4 \subset M_4$ as diagonal matrices and $R_X = l_\infty^4 \otimes M_4 \subset M_{16}$ as block diagonal matrices. The corresponding TRO channel N is an orthogonal sum of traces

$$N(\rho) = \begin{bmatrix} \text{tr}(P_1\rho) & 0 & 0 & 0 \\ 0 & \text{tr}(P_2\rho) & 0 & 0 \\ 0 & 0 & \text{tr}(P_3\rho) & 0 \\ 0 & 0 & 0 & \text{tr}(P_4\rho) \end{bmatrix},$$

which has zero quantum capacity. Recall the Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_2 = i\sigma_1\sigma_3.$$

We have the following representation of M_2 into $M_2 \otimes M_2 \otimes M_2 \otimes M_2 \cong M_{16}$:

$$\pi(\sigma_1) = \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \pi(\sigma_3) = \sigma_3 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3, \\ \pi(\sigma_2) = -\sigma_2 \otimes 1 \otimes \sigma_2 \otimes \sigma_2.$$

Its range $\mathcal{A} = \pi(M_2)$ is a subalgebra of M_{16} independent of R_X . Indeed, let $a = \bigoplus_{j=1}^4 a_j$ in M_{16} and

$$f = \sum_{l=0}^3 \alpha_l \sigma_l \in M_2$$

$$\text{tr}(a\pi(f)) = \sum_j \text{tr}(a_j P_j \pi(f) P_j) = \alpha_0 \sum_j \text{tr}(a_j) = \tau(\pi(f)) \text{tr}(a) = \tau(f) \text{tr}(a)$$

because $P_j \pi(\sigma_1) P_j = P_j \pi(\sigma_2) P_j = P_j \pi(\sigma_3) P_j = 0$ for all j . Note that $f = \sum_{l=0}^3 \alpha_l \sigma_l \in M_2$ and $\pi(f) \in$

M_{16} are normalized densities if and only if $\alpha_1, \alpha_2, \alpha_3$ are real and $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 \leq 1$, which is the Bloch sphere. Write $|\alpha| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2}$. The normalized entropy of f and $\pi(f)$ is given by

$$\tau(f \log f) = \tau(\pi(f) \log \pi(f)) = 1 - h\left(\frac{1 + |\alpha|}{2}\right).$$

For $\pi(f) \in \pi(M_2)$ and $\rho \in M_{16}$, we write $\pi(f) = \sum_{1 \leq j, k \leq 4} |j\rangle\langle k| \otimes f_{jk}$ and $\rho = \sum_{1 \leq j, k \leq 4} |j\rangle\langle k| \otimes \rho_{jk}$ according to the tensor decomposition $M_{16} = M_4 \otimes M_4$. The modified channel with symbol $\pi(f)$ is given by

$$N_f(\rho) = \sum_{1 \leq j, k \leq 4} \text{tr}(\rho_{jk} f_{jk}) |j\rangle\langle k|,$$

where f_{ik} are 4×4 matrices as follows:

$$f_{11} = f_{22} = f_{33} = f_{44} = 1, \\ f_{41} = f_{32} = f_{23} = f_{14} = \alpha_1 \sigma_1 \otimes \sigma_1, \\ f_{31} = f_{42} = -f_{13} = -f_{24} = i\alpha_2 \sigma_2 \otimes \sigma_2, \\ f_{12} = f_{21} = -f_{34} = -f_{43} = \alpha_3 \sigma_3 \otimes \sigma_3.$$

The complementary channel of N_f is given by $N_f^E(|h\rangle\langle h|) =$

$$\pi(f^{\frac{1}{2}}) \begin{bmatrix} P_1 |h\rangle\langle h| P_1 & 0 & 0 & 0 \\ 0 & P_2 |h\rangle\langle h| P_2 & 0 & 0 \\ 0 & 0 & P_3 |h\rangle\langle h| P_3 & 0 \\ 0 & 0 & 0 & P_4 |h\rangle\langle h| P_4 \end{bmatrix} \pi(f^{\frac{1}{2}}).$$

Proposition V.1. For any f , $Q(N_f^E) = \log 4$. Hence N_f is not degradable.

Proof. Note that, for $|h\rangle \in P_1 H$

$$\begin{aligned} N_f^E(|h\rangle\langle h|) &= \pi(f)^{\frac{1}{2}} |h\rangle\langle h| \pi(f)^{\frac{1}{2}}, \\ \pi(f)^{\frac{1}{2}} |h\rangle &= \lambda_1 |h\rangle \oplus (\lambda_2 \sigma_1 \otimes \sigma_1 |h\rangle) \\ &\quad \oplus (\lambda_3 \sigma_2 \otimes \sigma_2 |h\rangle) \oplus (\lambda_4 \sigma_3 \otimes \sigma_3 |h\rangle). \end{aligned}$$

The input $|h\rangle$ can be recovered from $\pi(f)^{\frac{1}{2}} |h\rangle$ by a unitary operation. Hence N_f^E can faithfully transmit a 4×4 matrix space, which implies $Q(N_f^E) \geq \log 4$. On the other hand, $Q(N_f^E)$ is at most $\log 4$ because it factors through the conditional expectation onto $M_4 \oplus M_4 \oplus M_4 \oplus M_4$. $\square$

Theorem V.2. For any density $f = 1 + \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3$,

$$Q(N_f) = \tau(f \log f) = 1 - h\left(\frac{1 + |\alpha|}{2}\right).$$

Proof. The upper bound

$$Q(N_f) \leq \tau(f \log f) = 1 - h\left(\frac{1 + |\alpha|}{2}\right)$$

follows from Theorem II.3 and Proposition II.4. Then it suffices to show that this upper bound is achievable. Consider the vector

$$|\xi\rangle = \frac{1}{4}(1 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)|h\rangle \in \mathbb{C}^4,$$

where $|h\rangle$ is an arbitrary unit vector in $\mathbb{C}^4$. $|\xi\rangle$ satisfies that

$$\langle \xi | \sigma_1 \otimes \sigma_1 | \xi \rangle = -\langle \xi | \sigma_2 \otimes \sigma_2 | \xi \rangle = \langle \xi | \sigma_3 \otimes \sigma_3 | \xi \rangle = 1.$$

Then the restriction of N_f on $M_4 \otimes |\xi\rangle\langle \xi|$ is a Schur multiplier

$$\tilde{N}_f(\rho) = N_f(\rho \otimes |\xi\rangle\langle \xi|) = \sum_{j,k} \rho_{jk} \tilde{f}_{jk} |j\rangle\langle k|, \quad \rho = (\rho_{jk}) \in M_4,$$

where $\tilde{f} = 1 + \alpha_1 \sigma_1 \otimes \sigma_1 + \alpha_2 \sigma_2 \otimes \sigma_2 + \alpha_3 \sigma_3 \otimes \sigma_3$. Note that

$$\tilde{\mathcal{A}} = \{\alpha_0 1 + \alpha_1 \sigma_1 \otimes \sigma_1 + \alpha_2 \sigma_2 \otimes \sigma_2 + \alpha_3 \sigma_3 \otimes \sigma_3 \mid \alpha_i \in \mathbb{C}\}$$

is a C^* -algebra independent to the diagonal matrices $D \subset M_4$. Then from Theorem II.5 and the fact $\tilde{N}_f$ is unital, we obtain the lower estimate

$$1 - h\left(\frac{1 + |\alpha|}{2}\right) = \tau(\tilde{f} \log \tilde{f}) = -S_{cb}(\tilde{N}_f) \leq Q^{(1)}(\tilde{N}_f) \leq Q^{(1)}(N_f),$$

which completes the proof. $\square$

The above estimates also imply strong converse and strong-additivity of $Q(N)$. The construction can be easily generalized to higher dimensions by using generalized n -dimensional Pauli matrices

$$X|j\rangle = |j+1\rangle, Z|j\rangle = e^{\frac{2\pi i j}{n}} |j\rangle.$$

They provide new examples of quantum channels which are non-degradable but have computable quantum capacity.

VI. QUANTUM CHANNEL MAPS ARISED FROM ERROR CORRECTION

Many codes in quantum information are designed for quantum computing rather than communication. We should not expect these codes to be optimal in the communication. Rather, these codes protect a particular subspace of *logical qubits* from possible errors, providing the abstraction of ideal quantum circuits within that subspace. Yet the notion of a protected subspace fits our formalism of

conditional expectation exceedingly well. Codes that protect a given subspace are optimal in communication for channels given by the conditional expectation that preserve the protected subspace. Our perturbative estimates then bound the suboptimality of subspace protection for channels that deviate from that conditional expectation.

A. Toric code

In this section, we discuss how the toric code¹⁹ considered by Kitaev fits into the TRO formalism and when it is good for quantum communication and storage. Let us first briefly recall the toric code construction.

Suppose we are given a 2-dimensional crystalline lattice with periodic boundary conditions (a torus). We index the vertex by its coordinate $v = (v_1, v_2)$. Qubits (or spins) live on the edge between adjacent vertices, and we index them via vertex pairs (v, w) . We may also consider the plaquettes, which are the unit squares with a vertex on each corner, indexed by the coordinate $p = (p_1, p_2)$. For each vertex v and plaquette p , the local syndrome observables (also called stabilizer operators) are

$$A_v = X_{v,v+(0,1)}X_{v,v+(0,-1)}X_{v,v+(1,0)}X_{v,v+(-1,0)},$$

$$B_p = Z_{p,p+(0,1)}Z_{p+(0,1),p+(1,1)}Z_{p,p+(1,0)}Z_{p+(1,0),p+(1,1)},$$

where X and Z are the Pauli matrices on the corresponding edge. The protected subspace is the invariant subspace for all the syndrome operators. For a lattice of side length L , there are $L^2 - 1$ vertices and $L^2 - 1$ plaquettes and hence $2(L^2 - 1)$ independent syndrome operators. Then the protected space contains two logical 2 qubits. In the active error correction scheme, a device would constantly measure each of these local observables, applying correction operations that maintain the +1 eigenvalue for all vertices and plaquettes. The logical code space is formed by strings of Pauli operators which create loops of excitations passing through the boundary, e.g.,

$$\bar{Z}_1 = \prod_{i=1}^L Z_{v+(0,i),v+(0,i+1)}, \bar{X}_1 = \prod_{i=1}^L X_{v+(i,0),v+(i+1,0)},$$

$$\bar{Z}_2 = \prod_{i=1}^L Z_{v+(i,0),v+(i+1,0)}, \bar{X}_2 = \prod_{i=1}^L X_{v+(0,i),v+(0,i+1)}$$
(10)

for any fixed vertex v . These operators commute with all A_v and B_p . They are not unique, and in fact, $\bar{Z}_1$ can be any contiguous loops of Pauli Z operators passing through the corresponding boundary. It is the presence or absence of loops that form the logical space, and hence the codes are topologically protected due to their resistance to local errors.

The conditional expectation onto the protected subspace naturally provides a TRO channel. Let V and P be binary words of length $L^2 - 1$. Each bit $V(v)$ (respectively, $P(p)$) specifies the inclusion or exclusion of the vertex v (respectively, plaquette p). Since $\{A_v, B_p\}$ generate a commutative unitary group, then the conditional expectation onto the protected space is given by

$$N(\rho) = \frac{1}{2^{2L^2-2}} \sum_{V,P} \left(\prod_{v,p} A_v^{V(v)} B_p^{P(p)} \right) \rho \left(\prod_{v,p} A_v^{V(v)} B_p^{P(p)} \right),$$
(11)

where the summation is on the set of all word pairs V, P . $Q(N)$ is the size of the protected subspace, 2 qubits. The modification of N is

$$N_f(\rho) = \sum_{V,P} f_{V,P} \left(\prod_{v,p} A_v^{V(v)} B_p^{P(p)} \right) \rho \left(\prod_{v,p} A_v^{V(v)} B_p^{P(p)} \right),$$
(12)

where $(f_{V,P})$ is a probability measure. Applying Theorem II.3 and Proposition II.4,

$$Q(N) \leq Q(N_f) \leq Q(N) + 2L^2 - 2 + H(f).$$
(13)

Here $H(f) = -\sum f \log f$ is the Shannon entropy of f . These channels characterize randomized local errors in which the protected subspace is immune. If we assume errors are relegated to bit flips X and phase flips Z , respectively (bit flip error and phase flip error, each only happen at one place), the

Kraus operators correspond to only the length one words V and P , which significantly enlarges the protected subspace.

Applying the full set of syndrome operators is equivalent to taking the conditional expectation onto the protected subspace, in essence the channel N . Assume that errors on the protected logical qubits are negligibly rare and that the overall error probability is below the threshold for scalable error correction. Then as L becomes large, the ratio between typical erroneous states and syndrome operator approaches 1 (see Ref. [3](#), p. 43). Consequently, N_f starts to characterize all probable errors. In particular, for a single error (f is concentrated as a pair (v, p)), $Q(N_f) - Q(N)$ approaches the difference between the size of the physical underlying system and the actual capacity of the code. The toric code is a simple model for Sec. [VI B](#) on more general instances of Kitaev's computation model.

B. Drinfel'd double

The toric code generalizes to a theory of non-commutative anyons. For a finite group G , we consider a family of operators $\{A_g, B_h | g, h \in G\}$ satisfying the following commutation relations:

$$A_h A_g = A_{hg}, \quad B_g B_h = \delta_{g,h} B_g, \quad A_g B_h = B_{ghg^{-1}} A_g, \quad \forall g, h \in G. \quad (14)$$

In Kitaev's formalism, these are considered to be local operators. An application of these operators to a ground state creates a pair of anyons, quasiparticles with highly non-trivial exchange relations. Since A_g and B_h are no longer commutative, this system no longer has the simple interpretation of a stabilizer code, but instead corresponds to a more general and powerful model that is capable of non-trivial quantum computation. This model of computation is beyond the scope of this paper, so we refer the readers to Ref. [19](#).

A particular representation of the relation (14) is given by the crossed product. Consider the commutative algebra $l_\infty(G) \subset \mathbb{B}(l_2(G))$ as diagonal matrices. We denote $e_{g,h}$, $g, h \in G$ as the matrix entry in $\mathbb{B}(l_2(G))$ which maps the basis vector $|h\rangle$ to $|g\rangle$. We define the action α of G on $l_\infty(G)$ as the automorphism

$$\alpha_g(e_{h,h}) = W_g e_{h,h} W_g^* = e_{ghg^{-1}, ghg^{-1}}, \quad (15)$$

where $W_g(e_{h,h}) = e_{ghg^{-1}, ghg^{-1}}$ are unitary matrices in $\mathbb{B}(l_2(G))$. The (reduced) crossed product $M = l_\infty(G) \rtimes_\alpha G$ is defined to be the algebra generated by the range of the following two representations on $l_2(G, l_2(G)) \cong l_2(G) \otimes l_2(G)$:

$$\begin{aligned} \pi : l_\infty(G) &\rightarrow B(l_2(G) \otimes l_2(G)), \quad \pi(x) = 1 \otimes x, \\ \tilde{\lambda} : G &\rightarrow B(l_2(G) \otimes l_2(G)), \quad \tilde{\lambda}(g) = \lambda(g) \otimes W_g, \end{aligned}$$

where $\lambda(g)|h\rangle = |gh\rangle$ is the left shift unitary on G . Then we can choose

$$A_g = \lambda(g) \otimes W_g, \quad B_h = 1 \otimes e_{h,h}$$

as explicit forms of Kitaev's operators satisfying the relations (14). Consider the unitary

$$U = \sum_{g,h} (A_g B_h) \otimes (e_{hg,g}) \in \mathbb{B}(l_2(G) \otimes l_2(G) \otimes l_2(G))$$

and the channel $N : \mathbb{B}(l_2(G) \otimes l_2(G)) \rightarrow \mathbb{B}(l_2(G) \otimes l_2(G))$,

$$N(\rho) = id \otimes \text{tr}(U(\rho \otimes \frac{1}{|G|})U^*) = \frac{1}{|G|} \sum_{g,h} A_g B_h \rho (A_g B_h)^*.$$

N is the conditional expectation from $\mathbb{B}(l_2(G) \otimes l_2(G))$ onto the commutant of $l_\infty(G) \rtimes_\alpha G$. Indeed, for any $g_0, h_0 \in G$,

$$\begin{aligned}
A_{g_0} B_{h_0} N(\rho) &= A_{g_0} B_{h_0} \left(\frac{1}{|G|} \sum_{g,h} A_g B_h \rho B_h A_{g^{-1}} \right) \\
&= \frac{1}{|G|} \sum_g A_{g_0 g} B_{g^{-1} h_0 g} \rho B_{g^{-1} h_0 g} A_{g^{-1}} \\
&= \frac{1}{|G|} \sum_g A_{g_0 g} B_{g^{-1} h_0 g} \rho B_{g^{-1} h_0 g} A_{(g_0 g)^{-1}} A_{g_0} \\
&= \frac{1}{|G|} \sum_g A_{g_0 g} B_{g^{-1} h_0 g} \rho B_{g^{-1} h_0 g} A_{(g_0 g)^{-1}} A_{g_0} B_{h_0} \\
&= \frac{1}{|G|} \left(\sum_{g,h} A_{g_0 g} B_{g^{-1} h_0 g} \rho B_{g^{-1} h_0 g} A_{(g_0 g)^{-1}} \right) A_{g_0} B_{h_0} \\
&= N(g) A_{g_0} B_{h_0}.
\end{aligned}$$

For $f \in B(l_2(G))$, the modification of N is given by

$$N_f(\rho) = \sum_{g,g',h,h'} \tau(e_{hg,gg'} e_{g',h'h'}) (A_g B_h) \rho (A_{g'} B_{h'})^*. \quad (16)$$

Then Theorem II.3 and Proposition II.4 imply here

$$Q(N) \leq Q(N_f) \leq Q(N) + \tau(f \log f).$$

As described in the example in Sec. V, such N_f are mixtures of random unitary and generalized dephasing channels, hence not degradable or anti-degradable channels. The practical applications of the Kiteav's model as a communication channel are admittedly unclear, though the connection to anyons does suggest a possible physical realization.

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