K-theoretic obstructions to bounded t-structures

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Abstract

Schlichting conjectured that the negative K-groups of small abelian categories vanish and proved this for noetherian abelian categories and for all abelian categories in degree -1. The main results of this paper are that $\mathrm{K}_{-1}(E)$ vanishes when E is a small stable ∞-category with a bounded t-structure and that $\mathrm{K}_{-n}(E)$ vanishes for all $n \geqslant 1$ when additionally the heart of E is noetherian. It follows that Barwick's theorem of the heart holds for nonconnective K-theory spectra when the heart is noetherian. We give several applications, to non-existence results for bounded t-structures and stability conditions, to possible K-theoretic obstructions to the existence of the motivic t-structure, and to vanishing results for the negative K-groups of a large class of dg algebras and ring spectra.

Key Words. Negative K-theory, t-structures, abelian categories, K-theory of dg algebras and ring spectra.

Mathematics Subject Classification 2010. Primary: 16E45, 18E30, 19D35. Secondary: 16P40, 18E10, 55P43.

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*Benjamin Antieau was supported by NSF Grants DMS-1461847 and DMS-1552766.
†David Gepner was supported by NSF Grants DMS-1406529 and DMS-1714273.
‡Jeremiah Heller was supported by NSF Grant DMS-1710966.

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1 Introduction

We prove the following theorems about negative and nonconnective K-theory.

Theorem 1.1. *If E is a small stable ∞-category*[1] *with a bounded t-structure, then $\mathrm{K}_{-1}(E) = 0$.*

Theorem 1.2. *If E is a small stable ∞-category equipped with a bounded t-structure such that $E^{\heartsuit}$ is noetherian, then $\mathrm{K}_{-n}(E) = 0$ for $n \geqslant 1$.*

Theorem 1.3 (Nonconnective theorem of the heart)**.** *If E is a small stable ∞-category with a bounded t-structure such that $E^{\heartsuit}$ is noetherian, then the natural map*

$$\mathrm{K}(E^{\heartsuit}) \xrightarrow{\simeq} \mathrm{K}(E)$$

is an equivalence.

The first two theorems generalize results of Schlichting from [Sch06], who proved the theorems in the special case where $E \simeq \mathcal{D}^b(A)$, the bounded derived ∞-category of a small abelian category A. Note that our theorems are much more general than Schlichting's results, as stable ∞-categories with bounded t-structures are typically not bounded derived ∞-categories. The third result follows from the first two and Barwick's theorem of the heart for connective K-theory [Bar15].

The proof of Theorem 1.2 is based on induction, with the base case provided by Theorem 1.1. The proof of Schlichting's result that $\mathrm{K}_{-1}(A) = 0$ for general abelian categories A is not hard, but the proof of Theorem 1.1 is more difficult as it is necessary to find an excisive square playing the same role for E that the square

$$\begin{array}{ccc} \mathcal{D}^b(A) & \longrightarrow & \mathcal{D}^+(A) \\ \downarrow & & \downarrow \\ \mathcal{D}^-(A) & \longrightarrow & \mathcal{D}(A) \end{array}$$

plays for $\mathcal{D}^b(A)$.

In the inductive step, we use stable ∞-categories of endomorphisms and automorphisms of E. We construct an exact sequence

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^{\omega} \to \mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega} \to \mathcal{D}(\mathbb{G}_m, \mathcal{C})^{\omega}$$

of small idempotent complete stable ∞-categories, where $\mathcal{C} = \mathrm{Ind}(E)$ is the ind-completion of E, the superscript ω denotes the subcategory of compact objects, and $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{S}[s]} \otimes \mathcal{C}$, and similarly for $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$. The subscript $\{0\}$ denotes the full subcategory $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ of objects killed by

[1] The conjectures and results of this paper apply equally well to any triangulated category with a bounded t-structure that admits a model, either as a dg category or a stable ∞-category. This includes all examples of triangulated categories with bounded t-structures we have found in the literature. For background on stable ∞-categories, see Section 2.1.

inverting the endomorphism s. Note that the ∞-category $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ differs from the ∞-category used in [BGT16] to define the *K-theory of endomorphisms*. Indeed, the K-theory of endomorphisms of E takes as input the ∞-category of endomorphisms of objects of E. But, these need not be compact in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$. Conversely, the underlying object of a compact object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ need not be compact in $\mathcal{C}$.

The technical input for the inductive step, proven in Corollary 3.17, is that if E is a small stable ∞-category with a bounded t-structure such that $E^\heartsuit$ is noetherian, then the same is true for $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$. This allows an inductive argument because $\mathrm{K}(E)$ is a summand of $\mathrm{K}(\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega)$ and this summand maps trivially to $\mathrm{K}(\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega)$.

Theorem 1.2 can be extended to the case where $E^\heartsuit$ is merely stably coherent; we do so in Section 3.5. We discuss in Sections 3.4 and 3.6 a counterexample to our approach when $E^\heartsuit$ is not noetherian as well as several possible approaches for circumventing this problem. We hope that these more speculative sections will serve to pique the interest of readers thinking about related problems.

Conjectures. Schlichting made the following conjecture in [Sch06].

Conjecture A. *If A is a small abelian category, then $\mathrm{K}_{-n}(A) = 0$ for $n \geqslant 1$.*

Motivated by this, we pose the next two conjectures.

Conjecture B. *If E is a small stable ∞-category with a bounded t-structure, then $\mathrm{K}_{-n}(E) = 0$ for $n \geqslant 1$.*

Conjecture C. *If E is a small stable ∞-category with a bounded t-structure, then the natural map $\mathrm{K}(E^\heartsuit) \to \mathrm{K}(E)$ is an equivalence of nonconnective K-theory spectra.*

Conjecture A is a special case of the second conjecture by setting $E = \mathcal{D}^b(A)$, the bounded derived ∞-category of A, and we will therefore refer to the second as the generalized Schlichting conjecture. The connective part of Conjecture C is Barwick's theorem [Bar15] for connective K-theory: $\mathrm{K}^{\mathrm{cn}}(E^\heartsuit) \simeq \mathrm{K}^{\mathrm{cn}}(E)$, which generalizes the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7] in the case that $E = \mathcal{D}^b(A)$. So, the open part of that conjecture may be rephrased as saying that $\mathrm{K}_{-n}(E^\heartsuit) \to \mathrm{K}_{-n}(E)$ is an isomorphism for all $n \geqslant 1$. Of course, this would follow from Conjecture B together with Conjecture A. In fact, Conjecture B holds if and only if Conjectures A and C hold.

Note that there are examples of stable ∞-categories E with two different bounded t-structures, one having a noetherian heart and the other having a non-noetherian heart. The standard example, due to Thomas and written down in [AP06], is $\mathcal{D}^b(\mathbb{P}^1)$, and was pointed out to us by Calabrese.

Applications. There are three major areas of application of the work in this paper: obstructions to the existence of t-structures (and hence to stability conditions) on $\mathrm{Perf}(X)$ when X is a singular scheme, possible obstructions to the existence of the conjectural motivic t-structure, and vanishing results for the negative K-theory of nonconnective dg algebras and ring spectra. We describe the first two areas briefly below and leave the extensive vanishing results for ring spectra to Section 4.

Stability conditions. Bridgeland introduced in [Bri07] the notion of stability conditions on abelian and triangulated categories. Moreover, he proved in [Bri07, Proposition 5.3] that giving a stability condition on a triangulated category $\mathcal{T}$ is equivalent to giving a bounded t-structure on $\mathcal{T}$ together with a stability condition on $\mathcal{T}^\heartsuit$. A crucial and open problem in the theory of stability conditions is when $\mathrm{Perf}(X)$ admits any stability conditions at all for X a smooth scheme over $\mathbb{C}$. This is open even for general smooth proper threefolds (see for example [BMT14]).

Our methods give K-theoretic obstructions to the existence of bounded t-structures and hence to stability conditions. As far as we are aware these are the first obstructions of any kind to the existence of bounded t-structures.

Corollary 1.4. *Let X be a scheme such that $\mathrm{K}_{-1}(X) \neq 0$. Then, there exists no bounded t-structure (and hence no stability condition) on $\mathrm{Perf}(X)$. If $\mathrm{K}_{-n}(X) \neq 0$ for some $n \geqslant 2$, then there exists no bounded t-structure on $\mathrm{Perf}(X)$ with noetherian heart.*

The corollary applies to a wide variety of singular schemes, even such simple examples as nodal cubic curves, where $\mathrm{K}_{-1}(X) \cong \mathbb{Z}$. Note that when X is noetherian and singular, it is easy to see that the canonical bounded t-structure on $\mathcal{D}^b(X)$ does not restrict to one on $\mathrm{Perf}(X) \subseteq \mathcal{D}^b(X)$. A priori there could be other, exotic t-structures. We propose the following conjecture, which generalizes Corollary 1.4.

Conjecture 1.5. *Let X be a noetherian scheme of finite Krull dimension. If X is not regular, then $\mathrm{Perf}(X)$ admits no bounded t-structure.*

Based on Corollary 1.4, when X is singular, $\mathcal{D}^b(X)$ appears more natural from the point of view of stability conditions.

Motivic t-structures. One of the major open problems in motives (see [Kah05, Section 4.4.3]) is to construct a bounded t-structure on Voevodsky's triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ of rational effective geometric motives over a field k. The heart of this t-structure would be the abelian category of mixed motives. Voevodsky observed in [Voe00] that there can be no integral motivic t-structure when there are smooth projective conic curves over k with no rational points (thus for example when $k = \mathbb{Q}$), although potentially there could be other bounded t-structures that do not satisfy all of the expected properties. Our next corollary implies a possible approach to proving non-existence of any motivic t-structure. Note that the heart of the motivic t-structure is expected to be noetherian.

Corollary 1.6. *If $\mathrm{K}_{-n}(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)_{\mathbb{Q}}) \neq 0$ for some $n \geqslant 1$, then there is no motivic t-structure.*

Using our work, Sosnilo has proved in [Sos17] that in fact a different conjecture of Voevodsky, the nilpotence conjecture of [Voe95], would imply $\mathrm{K}_{-n}(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)_{\mathbb{Q}}) = 0$ for all $n \geqslant 1$. Put another way, if $\mathrm{K}_{-n}(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)_{\mathbb{Q}}) \neq 0$ for some $n \geqslant 1$, then the nilpotence conjecture would also be false.

Outline. Section 2 is dedicated to background on t-structures, proving several new inheritance results about t-structures, K-theoretic excisive squares, and the proof of Theorem 1.1. Section 3 contains the proofs of Theorem 1.2 and 1.3 as well as our thoughts of how one might attempt to prove Conjecture B in general. Section 4 contains our applications to the negative K-theory of ring spectra. In Appendix A, we construct a functorial ∞-categorical model of the stable category of a Frobenius category. This is needed to check that the definition of negative K-theory we use agrees with Schlichting's.

Notation. Throughout, unless otherwise stated, we use homological indexing for chain complexes and objects in stable ∞-categories. The ∞-category of small stable ∞-categories and exact functors is written $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$, while the full subcategory of small *idempotent complete* stable ∞-categories is written $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$. Given a small stable ∞-category E, we denote by $\widetilde{E}$ or $E^{\sim}$ the idempotent completion of E. If $E \subseteq F$ is a fully faithful inclusion such that E is idempotent complete in F, then F/E denotes the Verdier quotient (the cofiber in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$).

If $\mathcal{C}$ is an ∞-category, $\mathrm{Map}_{\mathcal{C}}(M, N)$ is the mapping space of morphisms from M to N in $\mathcal{C}$. Given an idempotent complete stable ∞-category E, $\mathrm{K}(E)$ always denotes the *nonconnective* K-theory spectrum of E, as defined in [BGT13]. We use $\mathrm{K}^{\mathrm{cn}}(E)$ for the connective cover of $\mathrm{K}(E)$, the connective K-theory spectrum of E. Finally, if R is a ring spectrum, Mod_R, Alg_R, and CAlg_R denote the ∞-categories of R-module spectra, $\mathbb{E}_1$-R-algebra spectra (if R is commutative), and $\mathbb{E}_\infty$-R-algebra spectra (if R is commutative), respectively (even if R is discrete). If R is discrete, we let $\mathrm{Mod}^{\heartsuit}_R$, $\mathrm{Alg}^{\heartsuit}_R$, and $\mathrm{CAlg}^{\heartsuit}_R$ denote the ordinary categories of

discrete right R-modules, discrete associative R-algebras (if R is commutative), and discrete commutative R-algebras (if R is commutative), respectively; this notation reflects the fact that the abelian category of discrete right R-modules is equivalent to the heart of the standard t-structure on the stable ∞-category Mod_R.

Acknowledgments. BA and JH thank the Hausdorff Institute for Mathematics in Bonn and DG thanks the Max Planck Institute for Mathematics: these were our hosts during the summer of 2015, when this project was conceived. BA thanks Akhil Mathew for several conversations that summer at HIM, especially about bounded t-structures for compact modules over cochain algebras.

BA thanks John Calabrese, Denis-Charles Cisinski, Michael Gröchenig, Jacob Lurie, Matthew Morrow, Marco Schlichting, Jesse Wolfson, and Matthew Woolf for conversations and emails about material related to this paper. DG thanks Andrew Blumberg and Markus Spitzweck for conversations about material related to this paper. Both BA and DG would especially like to thank Benjamin Hennion for explaining Tate objects and the subtleties behind excisive squares. We all are very grateful for detailed, helpful comments from an anonymous referee.

We also thank the UIC Visitors' Fund, Purdue University, UIUC, and Lars Hesselholt for supporting collaborative visits in 2016.

2 t-structures

We give some background on stable ∞-categories in Section 2.1. After recalling t-structures in Section 2.2, we study induced t-structures on ind-completions and localizations in Section 2.3. In some cases, our results extend results in [BBD82] beyond the setting in which all functors admit left and right adjoints that preserve compact objects (the main assumption in [BBD82]). The ability to construct a t-structure on a localization in certain circumstances will be used later in the paper when we perform the inductive step in our generalization of Schlichting's theorem.

In Section 2.4, we study excisive squares in algebraic K-theory and their connection to adjointability. We prove that $\mathrm{K}_{-1}(E) = 0$ when E is a small stable ∞-category with a bounded t-structure in Section 2.5.

2.1 Stable ∞-categories

For the purposes of studying K-theory, it has been known for some time that triangulated categories are not sufficient. This was the result of work of Schlichting [Sch02], which gave an example of two stable model categories with triangulated equivalent homotopy categories but different K-theories. On the other hand, Toën and Vezzosi [TV04] showed that K-theory is a good invariant of simplicial localizations of Waldhausen categories in the following sense. If C and D are good Waldhausen categories and if the simplicial localizations $L^H C$ and $L^H D$ are equivalent simplicial categories, then $\mathrm{K}(C) \simeq \mathrm{K}(D)$. Thus, the simplicial localization loses some information, like passing to the homotopy category, but not so much that K-theory is inaccessible. These simplicial localizations are a kind of enhancement of the triangulated homotopy categories, and it is now well-understood that K-theory requires some kind of enhancement.

Unfortunately, computations are difficult in the model categories of simplicial categories and dg categories, and it is much easier to work in the setting of ∞-categories. The K-theory of ∞-categories is studied in [BGT13] and [Bar16] and it agrees in all cases with Waldhausen K-theory when both are defined. So, this setting provides a best-of-both-worlds approach to K-theory, where we can not only compute K-theory correctly but we can also compute maps between the inputs. The theory of ∞-categories is not the only way of doing this, but it is by now the most well-developed and it is the most well-suited for the problems we study.

A pointed ∞-category is an ∞-category E with an object 0 that is both initial and final. It is called a zero object of E. A cofiber sequence in a pointed ∞-category is a commutative diagram

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & d \end{array}$$

which is a pushout diagram in the sense of colimits in ∞-categories as developed in [Lur09]. It is standard practice to abbreviate and write $a \to b \to c$ for a cofiber sequence. If $f : a \to b$ is a morphism in E, then a cofiber for f is a cofiber sequence $a \to b \to c$. Cofibers for f are unique up to homotopy. Fiber sequences and fibers are defined similarly.

By definition, a pointed ∞-category is **stable** if it has all cofibers and fibers and if a triangle in E is a fiber sequence if and only if it is a cofiber sequence. It turns out that this definition is equivalent to asking for a pointed ∞-category to have all finite colimits and for the suspension functor $\Sigma : E \to E$ to be an equivalence (see [Lur12, Corollary 1.4.2.27]).

Unlike the case of triangulated categories in which the triangulation is extra *structure* which must be specified, stable ∞-categories are ∞-categories with certain *properties*, and the homotopy category $\mathrm{Ho}(E)$ of a stable ∞-categories is an ordinary category equipped with a *canonical* triangulation. If $\mathcal{C}$ is stable, [Lur12, Theorem 1.1.2.15] says that a sequence $a \to b \to c$ determines a cofiber sequence if and only $a \to b \to c$ is a distinguished triangle in the triangulated homotopy category $\mathrm{Ho}(E)$. For additional details and background about stable ∞-categories, see [Lur12, Chapter 1].

2.2 Definitions and first properties

The notion of a t-structure appears in Beĭlinson-Bernstein-Deligne [BBD82, Definition 1.3.1]. However, as we will work with homological indexing, Lurie's treatment in [Lur12, Definition 1.2.1.1] is more a convenient reference. If E is a stable ∞-category and $x \in E$, we will typically write $x[n]$ for the n-fold suspension $\Sigma^n x$ of x. If $F \subseteq E$ is a full subcategory, we will also write $F[n] \subseteq E$ for the full subcategory spanned by the objects of the form $x[n]$, where x is an object of F.

Definition 2.1. A **t-structure** on a stable ∞-category E consists of a pair of full subcategories $E_{\geqslant 0} \subseteq E$ and $E_{\leqslant 0} \subseteq E$ satisfying the following conditions:

(1) $E_{\geqslant 0}[1] \subseteq E_{\geqslant 0}$ and $E_{\leqslant 0} \subseteq E_{\leqslant 0}[1]$;

(2) if $x \in E_{\geqslant 0}$ and $y \in E_{\leqslant 0}$, then $\mathrm{Hom}_E(x, y[-1]) = 0$;

(3) every $x \in E$ fits into a cofiber sequence $\tau_{\geqslant 0}x \to x \to \tau_{\leqslant -1}x$ where $\tau_{\geqslant 0}x \in E_{\geqslant 0}$ and $\tau_{\leqslant -1}x \in E_{\leqslant 0}[-1]$.

An exact functor $E \to F$ between stable ∞-categories equipped with t-structures is **left t-exact** (resp. **right t-exact**) if it sends $E_{\leqslant 0}$ to $F_{\leqslant 0}$ (resp. $E_{\geqslant 0}$ to $F_{\geqslant 0}$). An exact functor is t-exact if is both left and right t-exact. We set $E_{\geqslant n} = E_{\geqslant 0}[n]$ and $E_{\leqslant n} = E_{\leqslant 0}[n]$.

Example 2.2. (a) If A is a small abelian category, then the bounded derived ∞-category $\mathcal{D}^b(A)$ (see Definition 3.21) admits a canonical t-structure, where $\mathcal{D}^b(A)_{\geqslant n}$ consists of the complexes x such that $\mathrm{H}_i(x) = 0$ for $i < n$, and similarly for $\mathcal{D}^b(A)_{\leqslant n}$.

(b) If A is a Grothendieck abelian category, then the derived ∞-category $\mathcal{D}(A)$ admits a t-structure with the same description as the previous example. This stable ∞-category and its t-structure are studied in [Lur12, Section 1.3.5].

(c) If R is a connective $\mathbb{E}_1$-ring spectrum, then the stable presentable ∞-category Mod_R of right R-module spectra admits a t-structure with $(\mathrm{Mod}_R)_{\geqslant 0} \simeq \mathrm{Mod}_R^{\mathrm{cn}}$, the ∞-category of connective R-module spectra. See for example [Lur12, Proposition 1.4.3.6]. We call this the **Postnikov t-structure**.

Condition (2) implies in fact that the mapping spaces $\mathrm{Map}_E(x, y[-1])$ are contractible for $x \in E_{\geqslant 0}$ and $y \in E_{\leqslant 0}$. This is not generally the case for the mapping spectra. Indeed, if A is a Grothendieck abelian category, then $\pi_0 \mathrm{Map}_{\mathcal{D}(A)}(x[-n], y[-1]) \cong \mathrm{Ext}_A^{n-1}(x, y)$ for $x, y \in A$. (See [Lur12, Proposition 1.3.5.6].)

Lemma 2.3. *The intersection $E_{\geqslant 0} \cap E_{\leqslant 0}$ is the full subcategory of $E_{\geqslant 0}$ consisting of discrete objects. Moreover, the intersection is an abelian category.*

Proof. See [Lur12, Warning 1.2.1.9] for the first statement and [BBD82, Théorème 1.3.6] for the second. □

Definition 2.4. The abelian category $E_{\geqslant 0} \cap E_{\leqslant 0}$ is called the **heart** of the t-structure $(E_{\geqslant 0}, E_{\leqslant 0})$ on E, and is denoted $E^{\heartsuit}$.

Example 2.5. The hearts of the t-structures in Example 2.2 are A in (a), A in (b), and $\mathrm{Mod}_{\pi_0 R}^{\heartsuit}$, the abelian category of right $\pi_0 R$-modules, in (c).

The truncations $\tau_{\geqslant n} x$ and $\tau_{\leqslant n} x$ are functorial in the sense that the inclusions $E_{\geqslant n} \to E$ and $E_{\leqslant n} \to E$ admit right and left adjoints, respectively, by [Lur12, Corollary 1.2.1.6]. Let $\pi_n x = \tau_{\geqslant n} \tau_{\leqslant n} x[-n] \in E^{\heartsuit}$. This functor is homological by [BBD82, Théorème 1.3.6], meaning that there are long exact sequences

$$\cdots \to \pi_{n+1} z \to \pi_n x \to \pi_n y \to \pi_n z \to \pi_{n-1} x \to \cdots$$

in $E^{\heartsuit}$ whenever $x \to y \to z$ is a cofiber sequence in E.

Definition 2.6. A t-structure $(E_{\geqslant 0}, E_{\leqslant 0})$ on a stable ∞-category is **right separated** if

$$\bigcap_{n \in \mathbb{Z}} E_{\leqslant n} = 0.$$

Left separated t-structures are defined similarly. Left and right separated t-structures are called **non-degenerate** in [BBD82].

Definition 2.7. If E is a stable ∞-category with a t-structure $(E_{\geqslant 0}, E_{\leqslant 0})$, we say that the t-structure is **bounded** if the inclusion

$$E^b = \bigcup_{n \to \infty} E_{\geqslant -n} \cap E_{\leqslant n} \to E$$

is an equivalence. Bounded t-structures are left and right separated.

For example, the t-structure in Example 2.2(1) is bounded.

Lemma 2.8. *If E is a stable ∞-category equipped with a t-structure $(E_{\geqslant 0}, E_{\leqslant 0})$, then the full subcategory $E^b \subseteq E$ is stable and the t-structure on E restricts to a bounded t-structure on E^b.*

Proof. Since $E^b \subseteq E$ is closed under translations (by part (1) of the definition of a t-structure), it is enough to show that it is closed under cofibers in E. Let $x \to y$ be a map in E^b with cofiber z. We must show that z is bounded. We can assume first that x and y are in $E_{\geqslant 0} \cap E_{\leqslant n}$ for some $n > 0$, in which case $z \in E_{\geqslant 0}$ since the inclusion $E_{\geqslant 0} \subseteq E$ preserves and creates colimits. Moreover, $z \to x[1] \to y[1]$ is a fiber sequence in E and $x[1]$ and $y[1]$ are in $E_{\leqslant n+1}$. Since the adjoint $E_{\leqslant n+1} \to E$ preserves limits, it follows that $z \in E_{\leqslant n+1}$. Hence, z is bounded. To conclude, we must show that E^b is closed under truncations in E, which will show that the t-structure on E restricts to a t-structure on E^b. So, suppose that $w \in E^b$, and consider $\tau_{\geqslant 0} w$ in E. We have

only to show that $\tau_{\geqslant 0}w$ is bounded above. Choose $m > 0$ such that $\tau_{\leqslant m}w \simeq 0$. Such an m exists because w is bounded. But, we now have

$$\tau_{\leqslant m}\tau_{\geqslant 0}w \simeq \tau_{\geqslant 0}\tau_{\leqslant m}w \simeq 0,$$

since the truncation functors commute by [Lur12, Proposition 1.2.1.10] or [BBD82, Proposition 1.3.5]. □

Lemma 2.9. *Suppose that $A = E^{\heartsuit}$ is the heart of a t-structure on a stable ∞-category E. If $0 \to x \to y \to z \to 0$ is an exact sequence in A, then $x \to y \to z$ is a cofiber sequence in E.*

Proof. Note that using Lemma 2.8 we can assume that E is bounded. Let w be the cofiber of $x \to y$ in E. Because $E_{\geqslant 0} \subseteq E$ is a left adjoint, we can identify w with the cofiber of $x \to y$ in $E_{\geqslant 0}$. As $E_{\geqslant 0} \xrightarrow{\pi_0} E^{\heartsuit} \simeq A$ is a left adjoint, the sequence $x \to y \to \pi_0 w \to 0$ is exact. But, it is also exact on the left by hypothesis, so that the cofiber c of the natural map $w \to z$ has the property that $\pi_n c = 0$ for all $n \in \mathbb{Z}$. Since bounded t-structures are non-degenerate, this implies $c \simeq 0$ and hence that $w \simeq z$, as desired. □

We leave the proof of the next lemma to the reader.

Lemma 2.10. *Let E and F be stable ∞-categories with t-structures. If $\varphi : E \to F$ is a right (resp. left) t-exact functor, then φ induces a right (resp. left) exact functor $\pi_0\varphi : E^{\heartsuit} \to F^{\heartsuit}$.*

Recall that if A is an abelian category, then $\mathrm{K}_0(A)$ is the Grothendieck group of A, which has generators $[x]$ for $x \in A$ and relations $[y] = [x] + [z]$ whenever x, y, z fit into an exact sequence $0 \to x \to y \to z \to 0$. Similarly, if E is a small stable ∞-category, then $\mathrm{K}_0(E)$ is the free abelian group on symbols $[x]$ for $x \in E$ modulo the relation $[y] = [x] + [z]$ whenever $x \to y \to z$ is a cofiber sequence in E.

It follows from Lemma 2.9 that there is a natural map $\mathrm{K}_0(E^{\heartsuit}) \to \mathrm{K}_0(E)$ when E is equipped with a t-structure.

Lemma 2.11. *If E is a small stable ∞-category equipped with a bounded t-structure, then the natural map $\mathrm{K}_0(E^{\heartsuit}) \to \mathrm{K}_0(E)$ is an isomorphism.*

Proof. Using the boundedness of the t-structure, it is immediate that $\mathrm{K}_0(E^{\heartsuit}) \to \mathrm{K}_0(E)$ is surjective because every object of E is a finite iterated extension of objects in $E^{\heartsuit}$. On the other hand, by assigning to $x \in E$ the sum

$$\sum_{n\in\mathbb{Z}} (-1)^n [\pi_n x],$$

we obtain a map $\mathrm{K}_0(E) \to \mathrm{K}_0(E^{\heartsuit})$, which splits the surjection. □

2.3 Induced t-structures on ind-completions and localizations

We give several results about t-structures on stable ∞-categories. Some of these, especially the equivalence of conditions (i) through (iv) in Proposition 2.20, have not, as far as we are aware, been proved before either for ∞-categories or for triangulated categories, so we treat the subject in greater detail than is strictly needed for the rest of the paper. However, there is some overlap between this section and [Lur, Appendix C] and [HPV16].

A t-structure $(E_{\geqslant 0}, E_{\leqslant 0})$ on a stable ∞-category E is **bounded below** if the natural map

$$E^{-} = \bigcup_{n\in\mathbb{Z}} E_{\geqslant n} \to E$$

is an equivalence and **right complete** if the natural map

$$E \to \lim \left(\cdots \to E_{\geqslant m} \xrightarrow{\tau_{\geqslant m+1}} E_{\geqslant m+1} \to \cdots \right)$$

is an equivalence. **Bounded above** and **left complete** t-structures are defined similarly. A bounded below t-structure is right separated as is a right complete t-structure. Neither converse is true in general.

The following definitions were introduced in [Lur12, Section 1]. A t-structure on a stable presentable ∞-category E is **accessible** if $E_{\geqslant 0}$ is presentable. A t-structure on a stable presentable ∞-category E is **compatible with filtered colimits** if $E_{\leqslant 0}$ is closed under filtered colimits in E.

Example 2.12. Example 2.2(a) is bounded (above and below). It is neither left or right complete, nor is it accessible or compatible with filtered colimits, as these notions are reserved for presentable ∞-categories. Examples 2.2(b) and (c) are right complete, accessible, and compatible with filtered colimits.

The following proposition also appears in [Lur, Lemma C.2.4.3].

Proposition 2.13. *Suppose that E is a small stable ∞-category with a t-structure. Then,* $\mathrm{Ind}(E_{\geqslant 0}) \subseteq \mathrm{Ind}(E)$ *determines the non-negative part of an accessible t-structure on* $\mathrm{Ind}(E)$ *which is is compatible with filtered colimits and such that the inclusion functor $E \to \mathrm{Ind}(E)$ is t-exact. Moreover, if the t-structure on E is bounded below, then* $\mathrm{Ind}(E)$ *is right complete.*

Proof. The functor $\mathrm{Ind}(E_{\geqslant 0}) \to \mathrm{Ind}(E)$ is fully faithful by [Lur09, Proposition 5.3.5.11], and we let $\mathrm{Ind}(E)_{\geqslant 0}$ denote the essential image. Similarly, let $\mathrm{Ind}(E)_{\leqslant -1}$ denote the essential image of the fully faithful functor $\mathrm{Ind}(E_{\leqslant -1}) \to \mathrm{Ind}(E)$. We claim that this pair of subcategories defines a t-structure on $\mathrm{Ind}(E)$. Condition (1) of Definition 2.1 is immediate. Suppose that $x \simeq \mathrm{colim}_{i\in I} x_i$ is in $\mathrm{Ind}(E)_{\geqslant 0}$, where each x_i is in $E_{\geqslant 0}$, and let $y \simeq \mathrm{colim}_{j\in J} y_j$ be in $\mathrm{Ind}(E)_{\leqslant -1}$, with each $y_j \in E_{\leqslant -1}$. Then, by definition of the ind-completion of E,

$$\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq \lim_i \operatorname*{colim}_j \mathrm{Map}_E(x_i, y_j),$$

which is contractible since each $\mathrm{Map}_E(x_i, y_j)$ is contractible. Hence, (2) holds. To verify condition (3), note that if $x \simeq \mathrm{colim}_{i\in I} x_i$ is a filtered colimit of objects $x_i \in E$, then

$$\operatorname*{colim}_i \tau_{\geqslant 0} x_i \to x \to \operatorname*{colim}_i \tau_{\leqslant -1} x_i$$

is a cofiber sequence since cofiber sequences commute with colimits. Hence, (3) holds.

To see that the t-structure is compatible with filtered colimits, note that $y \in \mathrm{Ind}(E)_{\leqslant -1}$ if and only if $\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq 0$ for all $x \in \mathrm{Ind}(E)_{\geqslant 0} \simeq \mathrm{Ind}(E_{\geqslant 0})$. However, this latter condition holds if and only if $\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq 0$ for all $x \in E_{\geqslant 0}$ since $\mathrm{Ind}(E_{\geqslant 0})$ is generated by $E_{\geqslant 0}$ under filtered colimits. Since the objects $x \in E_{\geqslant 0} \subseteq E$ are compact, this condition is closed under filtered colimits in y, as desired.

By construction, the functor $E \to \mathrm{Ind}(E)$ is t-exact, and the t-structure on $\mathrm{Ind}(E)$ is accessible as $\mathrm{Ind}(E)_{\geqslant 0} \simeq \mathrm{Ind}(E_{\geqslant 0})$ is presentable.

To finish the proof, we first show right separatedness. Suppose that y is an object of $\bigcap_{n\in\mathbb{Z}} \mathrm{Ind}(E)_{\leqslant n}$. Since the objects of E are compact generators for $\mathrm{Ind}(E)$, it is enough to show that the mapping spaces $\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq 0$ for all $x \in E$. Fix $x \in E$. We have for all n a natural equivalence

$$\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq \mathrm{Map}_{\mathrm{Ind}(E)}(\tau_{\leqslant n} x, y).$$

However, since the t-structure on E is bounded below, $\tau_{\leqslant n} x \simeq 0$ for n sufficiently small. Therefore, $\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq 0$. Hence, $y \simeq 0$.

Since $\mathrm{Ind}(E)_{\leqslant 0} \to \mathrm{Ind}(E)$ is closed under finite coproducts and filtered colimits it is closed under countable coproducts. Therefore, it follows by the right complete version of [Lur12, Proposition 1.2.1.19] that $\mathrm{Ind}(E)$ is right separated if and only if it is right complete. This completes the proof. □

We will call the t-structure on $\mathrm{Ind}(E)$ constructed in Proposition 2.13 the **induced** t-structure. The proof of the proposition does not extend to show that bounded above t-structures on E induce left complete t-structures on $\mathrm{Ind}(E)$. The obstruction is that the inclusion of $\mathrm{Ind}(E)_{\geqslant n}$ is a left adjoint rather than a right adjoint.

Corollary 2.14. *Let E be a small stable ∞-category with a bounded t-structure. Then, E is idempotent complete.*

Proof. Let F be the idempotent completion of E. Equivalently, $F \simeq \mathrm{Ind}(E)^{\omega}$, the full subcategory of compact objects of $\mathrm{Ind}(E)$. We claim that the t-structure on E extends to a bounded t-structure on F. It is enough to check that the truncation functors $\tau_{\leqslant 0}$ and $\tau_{\geqslant 0}$ on $\mathrm{Ind}(E)$ preserve compact objects. But, if $x \in F$ is a summand of $y \in E$, it follows that $\tau_{\leqslant 0}x$ is a summand of $\tau_{\leqslant 0}y$, and similarly for $\tau_{\geqslant 0}x$. This proves that the t-structure on $\mathrm{Ind}(E)$ restricts to a bounded t-structure on F. The heart $F^{\heartsuit}$ must be the idempotent completion of $E^{\heartsuit}$. But, since abelian categories are idempotent complete, $E^{\heartsuit} \to F^{\heartsuit}$ is an equivalence. Hence, by Lemma 2.11, $\mathrm{K}_0(E) \to \mathrm{K}_0(F)$ is an isomorphism. It follows from Thomason's classification of dense subcategories of triangulated categories that $E \simeq F$. See [Tho97, Theorem 2.1].

We can also avoid appealing to Thomason's result as follows. Given an object $x \in F$ and an integer $n \geqslant 0$, we say that x has amplitude at most n if there is an interval $[a, b]$ with $b - a \leqslant n$ and such that $\pi_i x = 0$ for $i \notin [a, b]$. As the t-structure on F is bounded, every object has amplitude at most n for some integer $n \geqslant 0$. Since $E^{\heartsuit} \simeq F^{\heartsuit}$, if x has amplitude at most 0, then $x \in E$. We proceed by induction on the amplitude. Assume that for every object y of F of amplitude at most $n-1$, where $n \geqslant 1$, we have that y is in the subcategory E. Fix $x \in F$ an object of amplitude at most n and assume, possibly by suspending, that $\pi_i x = 0$ for $i \notin [0, n]$. Consider the fiber sequence $\tau_{\geqslant 1}x \to x \to \pi_0 x$. The objects $\tau_{\geqslant 1}x$ and $\pi_0 x$ have amplitude at most $n-1$ and hence they are in E. But, x is the fiber of $\pi_0 x \to \tau_{\geqslant 1}x[1]$ and $E \subseteq F$ is full. Since E is stable, x is in E, as desired. □

In the rest of this section, we establish an important device for checking when a t-exact fully faithful functor $i : E \to F$ of small stable ∞-categories induces a t-structure on the cofiber $G = \widetilde{F/E}$ in $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$, the ∞-category of small idempotent complete stable ∞-categories and exact functors. Recall that G is equivalent to the idempotent completion of the Verdier localization of F by E (see [BGT13, Proposition 5.13]). We begin with a couple of easy lemmas.

Lemma 2.15. *If $i : E \to F$ is a t-exact (resp. right t-exact, resp. left t-exact) functor of stable ∞-categories equipped with t-structures, then the induced functor $i^* : \mathrm{Ind}(E) \to \mathrm{Ind}(F)$ is t-exact (resp. right t-exact, resp. left t-exact) with respect to the induced t-structures on* $\mathrm{Ind}(E)$ *and* $\mathrm{Ind}(F)$*.*

Proof. The exactness of i^* is immediate as it preserves all small colimits and hence finite limits since $\mathrm{Ind}(E)$ and $\mathrm{Ind}(F)$ are stable. Because $\mathrm{Ind}(E)_{\geqslant 0} \simeq \mathrm{Ind}(E_{\geqslant 0})$ and $\mathrm{Ind}(F)_{\geqslant 0} \simeq \mathrm{Ind}(F_{\geqslant 0})$, it is immediate that $i^* : \mathrm{Ind}(E) \to \mathrm{Ind}(F)$ is right t-exact if i is. The same holds for left t-exactness. □

Lemma 2.16. *Let $i : E \to F$ be a t-exact fully faithful functor of stable ∞-categories equipped with t-structures. Then, the natural map*

$$\mathrm{Ind}(E)^{\heartsuit} \to \mathrm{Ind}(F)^{\heartsuit} \cap \mathrm{Ind}(E)$$

is an exact equivalence of abelian categories.

Proof. Let $x \in \mathrm{Ind}(F)$ be an object of the intersection. Write $x = i^* y$ for some $y \in \mathrm{Ind}(E)$ (which is unique up to equivalence). The fact that i^* is t-exact and fully faithful implies that $\tau_{\geqslant 1}y \simeq 0$ and $\tau_{\leqslant -1}y \simeq 0$. In particular, y is contained in $\mathrm{Ind}(E)^{\heartsuit}$. It follows that the map in the lemma is essentially surjective. That the map is fully faithful follows from the fact that $\mathrm{Ind}(E) \to \mathrm{Ind}(F)$ is fully faithful, while exactness again follows from Lemma 2.15. □

Recall from [Lur12, Proposition 1.4.4.11] that if $\mathcal{C}$ is a stable presentable ∞-category and $\mathcal{C}' \subseteq \mathcal{C}$ is a full presentable subcategory closed under colimits and extensions in $\mathcal{C}$, then $\mathcal{C}' \simeq \mathcal{C}_{\geqslant 0}$ for some accessible t-structure on $\mathcal{C}$. We will say that the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ is the t-structure **generated** by $\mathcal{C}' \subseteq \mathcal{C}$. This

provides a way for defining many t-structures on stable presentable ∞-categories. Note that if $\mathcal{C} \simeq \mathrm{Ind}(E)$, where E is equipped with a t-structure $(E_{\geqslant 0}, E_{\leqslant 0})$, then the induced t-structure on $\mathrm{Ind}(E)$ is a special case of this phenomenon: it is generated by $\mathrm{Ind}(E_{\geqslant 0})$.

Definition 2.17. Let $A \subseteq B$ be an exact fully faithful functor of abelian categories. We identify A with its essential image in B. Say that A is a **weak Serre subcategory** of B if A is closed under extensions in B. We say that A is a **Serre subcategory** of B (or a **localizing subcategory** of B) if A is a weak Serre subcategory and A is additionally closed under taking subobjects and quotient objects in B.

Example 2.18. Let R be a right coherent ring. Then, the category $\mathrm{Mod}_R^{\heartsuit,\omega}$ of finitely presented (discrete) right R-modules is an abelian subcategory of $\mathrm{Mod}_R^{\heartsuit}$. It is always weak Serre, but it is Serre if and only if R is right noetherian.

Lemma 2.19. *Let $E \to F$ be a t-exact fully faithful functor of stable ∞-categories equipped with t-structures. Then, the induced map $E^{\heartsuit} \to F^{\heartsuit}$ exhibits $E^{\heartsuit}$ as a weak Serre subcategory of $F^{\heartsuit}$.*

Proof. The fact that $E^{\heartsuit} \to F^{\heartsuit}$ is exact and fully faithful follows from Lemma 2.10 and the fully faithfulness of $E \to F$. To check that $E^{\heartsuit}$ is closed under extensions in $F^{\heartsuit}$, consider an exact sequence $0 \to x \to y \to z \to 0$ where $x, z \in E^{\heartsuit}$ and $y \in F^{\heartsuit}$. Then, by Lemma 2.9, $x \to y \to z$ is a cofiber sequence in F. Hence, we can rewrite y as the fiber of $z \to x[1]$. Since $E \to F$ is fully faithful and preserves fibers, it follows that y is in the essential image of $E \to F$, as desired. We conclude by using Lemma 2.16. □

The first draft of this paper contained conditions (i) through (iv) of the next proposition. Benjamin Hennion pointed out another condition, (v) below, which is shown to be equivalent to condition (iii) in [HPV16, Proposition A.5].

Proposition 2.20. *Let $i : E \to F$ be a t-exact fully faithful functor of stable ∞-categories equipped with bounded t-structures, and let $j : F \to G$ be the cofiber in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Provide $\mathrm{Ind}(G)$ with the accessible t-structure generated by the smallest extension-closed cocomplete subcategory of $\mathrm{Ind}(G)$ containing the image of $F_{\geqslant 0}$, and equip $\mathrm{Ind}(E)$ and $\mathrm{Ind}(F)$ with the induced t-structures of Proposition 2.13. The following are equivalent:*

(i) *the essential image of the embedding $i^{\heartsuit} : E^{\heartsuit} \to F^{\heartsuit}$ is a Serre subcategory of $F^{\heartsuit}$;*

(ii) *the t-structure on $\mathrm{Ind}(G)$ restricts to a t-structure on G such that $j : F \to G$ is t-exact;*

(iii) *the induced functor $j^* : \mathrm{Ind}(F) \to \mathrm{Ind}(G)$ is t-exact;*

(iv) *the essential image of the embedding $\mathrm{Ind}(E)^{\heartsuit} \to \mathrm{Ind}(F)^{\heartsuit}$ is a Serre subcategory of $\mathrm{Ind}(F)^{\heartsuit}$;*

(v) *the counit map $i^* i_* x \to x$ induces a monomorphism $\pi_0(i^* i_* x) \to \pi_0(x)$ in $\mathrm{Ind}(F)^{\heartsuit}$ for every object x of $\mathrm{Ind}(F)$, where i_* is the right adjoint of $i^* : \mathrm{Ind}(E) \to \mathrm{Ind}(F)$.*

If these conditions hold, then the t-structure on G in (ii) is bounded.

Proof. Assume (i). Write $G' = F/E$ for the Verdier quotient of F by E. In particular, G is the idempotent completion of G'. We will construct a bounded t-structure on G' such that the functors $F \to G'$ and $G' \subseteq G \subseteq \mathrm{Ind}(G)$ are t-exact. By Corollary 2.14, G' will be idempotent complete. This will establish (ii).

Let $L : F \to G'$ denote the quotient functor. We define $\tau_{\geqslant 0} Lx = L\tau_{\geqslant 0} x$, and similarly $\tau_{\leqslant 0} Lx = L\tau_{\leqslant 0}$. It follows that (1) and (3) from Definition 2.1 hold trivially. Now, consider $\mathrm{Hom}_{G'}(Lx, Ly[-1])$, where $x \in F_{\geqslant 0}$

and $y \in F_{\leqslant 0}$. Pick $f \in \mathrm{Hom}_{G'}(Lx, Ly[-1])$. We can represent f by a zig-zag $x \leftarrow z \rightarrow y[-1]$, where the cofiber c of $x \leftarrow z$ is in E. Now, consider the following diagram

$$\begin{array}{ccccc}
\tau_{\geqslant 0}z & \longrightarrow & z & \longrightarrow & \tau_{\leqslant -1}z \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geqslant 0}x & \longrightarrow & x & \longrightarrow & \tau_{\leqslant -1}x \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geqslant 0}c & \longrightarrow & c & \longrightarrow & \tau_{\leqslant -1}c
\end{array}$$

of truncation sequences. (Warning: while the horizontal sequences are always cofiber sequences, only the central vertical sequence is a cofiber sequence in general.) The fact that $y \in F_{\leqslant 0}$ means that the map $z \rightarrow y[-1]$ factors through $\tau_{\leqslant -1}z$. Now, the fact that x is connective means that $\pi_{-n}z \in E$ for all $n \geqslant 1$. This is where we use the fact that $E^{\heartsuit}$ is a Serre subcategory of $F^{\heartsuit}$, to ensure that the quotient $\pi_{-1}z$ of $\pi_0 c$ is also in E. In particular, $\tau_{\geqslant 0}z \rightarrow \tau_{\geqslant 0}x \simeq x$ has cofiber in E (though it is not in general $\tau_{\geqslant 0}c$). The commutative diagram

$$\begin{array}{ccccc}
 & & z & & \\
 & \swarrow & \uparrow & \searrow & \\
x & \longleftarrow & \tau_{\geqslant 0}z & \xrightarrow{0} & y[-1] \\
 & \nwarrow & \downarrow = & \nearrow_{0} & \\
 & & \tau_{\geqslant 0}z & &
\end{array}$$

shows that f is nullhomotopic, which completes the construction of a bounded t-structure on G', which after the fact is idempotent complete, so $G' \simeq G$.

The inclusion $G \rightarrow \mathrm{Ind}(G)$ is evidently right t-exact with respect to the t-structure defined above on $G' \simeq G$ and the given t-structure on $\mathrm{Ind}(G)$. Let $x \in F_{\leqslant -1}$. To see left t-exactness, it suffices to check that $\mathrm{Map}_{\mathrm{Ind}(G)}(y, Lx) \simeq 0$ for all $y \in \mathrm{Ind}(G)_{\geqslant 0}$. But, since $\mathrm{Ind}(G)_{\geqslant 0}$ is generated under filtered colimits and extensions by images of the objects $z \in F_{\geqslant 0}$, this result follows from the computation above. Finally, by construction, $F \rightarrow G' \simeq G$ is t-exact. This completes the proof that (i) implies (ii).

Assume (ii). By definition of the t-structure on $\mathrm{Ind}(G)$, the localization functor $L : \mathrm{Ind}(F) \rightarrow \mathrm{Ind}(G)$ is right t-exact. Let $x \in \mathrm{Ind}(F)_{\leqslant -1}$. We must check that $\mathrm{Map}_{\mathrm{Ind}(G)}(y, Lx) \simeq 0$ for all $y \in \mathrm{Ind}(G)_{\geqslant 0}$. To do so, it is enough to check this for y of the form Lz for some $z \in F_{\geqslant 0}$. However, we can write $x \simeq \mathrm{colim}_I x_i$ for a filtered ∞-category I and some $x_i \in F_{\leqslant -1}$ since we use the t-structure on $\mathrm{Ind}(F)$ induced by $\mathrm{Ind}(F_{\geqslant 0})$. Hence,

$$\mathrm{Map}_{\mathrm{Ind}(G)}(Lz, Lx) \simeq \operatorname*{colim}_I \mathrm{Map}_{\mathrm{Ind}(G)}(Lz, Lx_i)$$

since L commutes with colimits and Lz is compact in $\mathrm{Ind}(G)$. As $Lz \in G_{\geqslant 0}$ and $Lx_i \in G_{\leqslant -1}$, (ii) shows that each mapping space in the colimit on the right is contractible, as desired. Hence, (ii) implies (iii).

To see that (iii) implies (iv), note first that the t-structures on $\mathrm{Ind}(E)$ and $\mathrm{Ind}(F)$ are right complete and hence right separated by Proposition 2.13. It follows from Lemma 2.19 that $\mathrm{Ind}(E)^{\heartsuit} \subseteq \mathrm{Ind}(F)^{\heartsuit}$ is weak Serre. Denote by $i^* : \mathrm{Ind}(E) \rightarrow \mathrm{Ind}(F)$ the induced functor, and let $x \subseteq i^*y$ be a subobject, where $x \in \mathrm{Ind}(F)^{\heartsuit}$ and $y \in \mathrm{Ind}(E)^{\heartsuit}$. Then, by t-exactness, $j^*x \subseteq j^*i^*y = 0$, so $j^*x = 0$ in $\mathrm{Ind}(G)^{\heartsuit}$. It follows that j^*x is in $\mathrm{Ind}(F)^{\heartsuit}$ and in $\mathrm{Ind}(E)$. Hence, by Lemma 2.16, $x = i^*z$ for some $z \in \mathrm{Ind}(E)^{\heartsuit}$. Thus, (iv) holds.

Now, suppose that (iv) holds, and let $x \subseteq iy$ for some $y \in E^{\heartsuit}$ and $x \in F^{\heartsuit}$. Then, $x \simeq i^*z$ for some $z \in \mathrm{Ind}(E)^{\heartsuit}$ by hypothesis (iv). However, as an object z of $\mathrm{Ind}(E)$ is compact if and only if i^*z is compact, it follows that in fact $z \in E$. Hence, (iv) implies (i).

The equivalence of (iii) and (v) is [HPV16, Proposition A.5].

Finally, the boundedness of the t-structure on G assuming that (ii) holds follows from the boundedness of the t-structure on F, the essential surjectivity of j up to retracts, and the t-exactness of j. □

2.4 Excisive squares and adjointability

Consider a commutative square

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array} \tag{1}$$

of small idempotent complete stable ∞-categories and fully faithful functors. In this section, we establish general conditions (Lemma 2.29, Proposition 2.30, and Theorem 2.31) which guarantee that the induced map

$$\begin{array}{ccc} \mathrm{K}(E) & \longrightarrow & \mathrm{K}(F) \\ \downarrow & & \downarrow \\ \mathrm{K}(G) & \longrightarrow & \mathrm{K}(H) \end{array} \tag{2}$$

is a pushout square of spectra and hence gives a long exact sequence

$$\cdots \to \mathrm{K}_n(E) \to \mathrm{K}_n(F) \oplus \mathrm{K}_n(G) \to \mathrm{K}_n(H) \to \mathrm{K}_{n-1}(E) \to \cdots$$

of K-groups. We check these conditions in two situations: for Tate objects (as studied in [Hen17]) later in this section and for t-structures in the proof of Theorem 2.35. We include the former for completeness, while the latter is what we need later in the paper. We begin with a standard lemma about pushouts and cofibers.

Lemma 2.21. *Suppose that*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ P & \xrightarrow{g} & Q \end{array}$$

is a commutative diagram in a stable ∞-category. Then, the induced map $\mathrm{cofib}(f) \to \mathrm{cofib}(g)$ *is an equivalence if and only if the square is a pushout square.*

Proof. If the square is a pushout square, then the horizontal cofibers are equivalent (see [Lur09, Lemma 4.4.2.1]). This is true in any ∞-category with pushouts and a terminal object. So, assume that $\mathrm{cofib}(f) \to \mathrm{cofib}(g)$ is an equivalence. Let S be the pushout of P and N over M, and let T be an arbitrary spectrum. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}(\mathrm{cofib}(g), T) & \longrightarrow & \mathrm{Map}(Q, T) & \longrightarrow & \mathrm{Map}(P, T) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(\mathrm{cofib}(f), T) & \longrightarrow & \mathrm{Map}(S, T) & \longrightarrow & \mathrm{Map}(P, T) \end{array}$$

of fiber sequences of mapping spaces. The outer vertical arrows are equivalences by hypothesis. In general, this does not in general let us conclude that the middle vertical arrow is an equivalence. However, because these are fiber sequences of infinite loop spaces, the long exact sequence in homotopy groups shows that $\mathrm{Map}_{\mathrm{Sp}}(Q, T) \to \mathrm{Map}_{\mathrm{Sp}}(S, T)$ is an equivalence for all T. Hence, $S \to Q$ is an equivalence. □

Let $\widetilde{F/E}$ and $\widetilde{H/G}$ denote the cofibers in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ of the horizontal maps in (1). Then, by localization in K-theory, there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}(E) & \longrightarrow & \mathrm{K}(F) & \longrightarrow & \mathrm{K}(\widetilde{F/E}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}(G) & \longrightarrow & \mathrm{K}(H) & \longrightarrow & \mathrm{K}(\widetilde{H/G}) \end{array} \tag{3}$$

in which the horizontal sequences are cofiber sequences. Hence, using Lemma 2.21, in order to check that (2) is a pushout square it suffices (and is necessary) to see that $\mathrm{K}(\widetilde{F/E}) \to \mathrm{K}(\widetilde{H/G})$ is an equivalence. This occurs in particular when $F/E \to H/G$ is an equivalence after idempotent completion.

Definition 2.22. Say that a square as in (1) is an **excisive square** if $\widetilde{F/E} \to \widetilde{H/G}$ is an equivalence.

Remark 2.23. It is easy to check using the full faithfulness of $F/E \to H/G$ that an excisive square is cartesian, so that $E \to F \cap G$ is an equivalence.

Example 2.24. (a) If (1) is a pushout square, then it is an excisive square.

(b) Suppose that $E = 0$ and that $H = \langle F, G\rangle$ is a **semiorthogonal decomposition** of H. Recall that this means that F and G are full stable subcategories of H such that

(i) $F \cap G = 0$,

(ii) every object $x \in H$ can be written in a cofiber sequence $y \to x \to z$ where $y \in G$ and $z \in F$, and

(iii) the mapping spaces $\mathrm{Map}_H(y, z)$ vanish for all $y \in G$ and all $z \in F$.

Under these conditions, it is easy to check by hand that the induced map $F \to H/G$ is an equivalence, which induces a (split) localization sequence $\mathrm{K}(G) \to \mathrm{K}(H) \to \mathrm{K}(F)$. For more details, see [BGT13].

Remark 2.25. Note that despite conditions (i) and (ii), H is not generally the coproduct in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ of F and G. The coproduct is $F \oplus G$, and in that category one has the additional criterion that $\mathrm{Map}_H(z, y) = 0$ for $y \in G$ and $z \in F$. That is, one has an **orthogonal decomposition**. This is a much stronger hypothesis, but it is rarely satisfied in situations of interest. For example, Beĭlinson's decomposition of $\mathcal{D}^b(\mathbb{P}^1_k) \simeq \langle \mathcal{O}, \mathcal{O}(1)\rangle$ gives a semiorthogonal decomposition of $\mathcal{D}^b(\mathbb{P}^1_k)$ which is not orthogonal (see [Huy06, Corollary 8.29]).

In Proposition 2.30 below, we give a criterion for checking that certain squares (1) are excisive squares. Our arguments are based on those of Benjamin Hennion [Hen17, Proposition 4.2], which in turn are based on those of Sho Saito [Sai15]. We need some preliminaries first.

Definition 2.26 (See [Lur12, Definition 4.7.5.13])**.** Consider a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{i^*} & \mathcal{F} \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ \mathcal{G} & \xrightarrow{j^*} & \mathcal{H} \end{array}$$

of ∞-categories such that i^* and j^* admit right adjoints i_* and j_*, respectively. Fix a natural equivalence $\alpha : j^*p^* \simeq q^*i^*$ (necessarily unique up to homotopy). The diagram is **right adjointable** if the natural map $p^*i_* \to j_*j^*p^*i_* \xrightarrow{\alpha} j_*q^*i^*i_* \to j_*q^*$ is an equivalence.

Remark 2.27. In general, the right adjointability of a diagram as in Definition 2.26 is not equivalent to the adjointability of the transpose diagram.

Proposition 2.28. *Consider a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ G & \xrightarrow{j} & H \end{array}$$

of fully faithful exact functors of stable idempotent complete ∞-categories. The following conditions are equivalent:

(1) *the induced commutative diagram*

$$\begin{array}{ccc} \mathrm{Ind}(F) & \xrightarrow{f^*} & \mathrm{Ind}(F/E) \\ {\scriptstyle q^*}\downarrow & & \downarrow{\scriptstyle r^*} \\ \mathrm{Ind}(H) & \xrightarrow{g^*} & \mathrm{Ind}(H/G) \end{array}$$

of stable presentable ∞-categories is right adjointable, where $f : F \to F/E$ and $g : H \to H/G$ are the quotient maps and $r : F/E \to H/G$ is the induced map on the quotients;

(2) *for any $x \in \mathrm{Ind}(F)$, if $i_*x \simeq 0$ in $\mathrm{Ind}(E)$, then $j_*q^*x \simeq 0$ in $\mathrm{Ind}(G)$, where i_* and j_* are right adjoint to i^* and j^*, respectively.*

The functors $f^*, g^*, i^*, \dots$ all preserve colimits and hence admit right adjoints which we will denote by $f_*, g_*, i_*, \dots$ For the proof and the remainder of the section, we will make use of the cofiber sequences $i^*i_*x \to x \to f_*f^*x$ in $\mathrm{Ind}(F)$ when $x \in \mathrm{Ind}(F)$ and $\mathrm{Ind}(E) \xrightarrow{i^*} \mathrm{Ind}(F) \xrightarrow{f^*} \mathrm{Ind}(F/E)$ is a localization sequence.

Proof. Assume (1). Choose $x \in \mathrm{Ind}(F)$ such that $i_*x \simeq 0$. Then, $x \simeq f_*f^*x$. Now, consider the cofiber sequence

$$j^*j_*q^*f_*f^*x \to q^*f_*f^*x \to g_*g^*q^*f_*f^*x \simeq g_*r^*f^*f_*f^*x \simeq g_*r^*f^*x.$$

Adjointability means that the map $q^*f_*f^*x \to g_*r^*f^*x$ is an equivalence so that $j^*j_*q^*f_*f^*x \simeq 0$. Since j^* is fully faithful, this means that $j_*q^*f_*f^*x \simeq j_*q^*x \simeq 0$, as desired.

We prove (2) implies (1). Let $y \in \mathrm{Ind}(F/E)$. Then, the counit map $f^*f_*y \to y$ is an equivalence. Set $x = f_*y$. Consider the commutative diagram

$$\begin{array}{ccccc} q^*i^*i_*x & \longrightarrow & q^*x & \longrightarrow & q^*f_*f^*x \\ \downarrow & & \downarrow & & \downarrow \\ j^*j_*q^*x & \longrightarrow & q^*x & \longrightarrow & g_*g^*q^*x \end{array}$$

of cofiber sequences in $\mathrm{Ind}(G)$. Since $i_*x \simeq i_*f_*y \simeq 0$, we have that $j_*q^*x \simeq 0$ by hypothesis (3). Hence, both terms on the left vanish, so the map $q^*f_*f^*x \to g_*g^*q^*x$ is an equivalence. But, $g_*g^*q^*x \simeq g_*r^*f^*x$. In particular, $g_*r^*y \simeq q^*f_*y$ for all $y \in \mathrm{Ind}(F/E)$. □

Lemma 2.29. *Suppose that a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ G & \xrightarrow{j} & H \end{array}$$

of fully faithful functors of stable idempotent complete ∞-categories satisfies the equivalent conditions of Proposition 2.28. Then, $F/E \to H/G$ is fully faithful.

Proof. We adopt the notation of the proof of the previous proposition. We show that the natural map $\mathrm{Map}_{\mathrm{Ind}(F/E)}(f^*x, f^*y) \to \mathrm{Map}_{\mathrm{Ind}(H/G)}(r^*f^*x, r^*f^*y)$ is an equivalence for all $x, y \in F/E$. There are natural equivalences,

$$\begin{aligned} \mathrm{Map}_{\mathrm{Ind}(H/G)}(r^*f^*x, r^*f^*y) &\simeq \mathrm{Map}_{\mathrm{Ind}(H/G)}(g^*q^*x, r^*f^*y) \\ &\simeq \mathrm{Map}_{\mathrm{Ind}(H)}(q^*x, g_*r^*f^*y) \\ &\simeq \mathrm{Map}_{\mathrm{Ind}(H)}(q^*x, q^*f_*f^*y) \\ &\simeq \mathrm{Map}_{\mathrm{Ind}(F)}(x, f_*f^*y) \\ &\simeq \mathrm{Map}_{\mathrm{Ind}(F/E)}(f^*x, f^*y), \end{aligned}$$

where the third equivalence is via right adjointability and the fourth follows from the fact that q is fully faithful. □

Now, we come to an important test for adjointability. We include it for completeness, as it will not be used in the rest of the paper. Rather, when needed, we will check that the equivalent conditions of Proposition 2.28 are satisfied. However, the proof is similar to one step in the proof of Theorem 2.35.

Proposition 2.30. *Let*

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ G & \xrightarrow{j} & H \end{array}$$

be a commutative square of fully faithful functors in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ such that

(a) *every object y of G is a cofiltered limit $y \simeq \lim_B p(z_\beta)$ such that $jy \simeq \lim_B jpz_\beta$, and*

(b) *the essential image of q consists of j-cocompact objects of H, meaning that the natural map*

$$\operatorname*{colim}_{B^{\mathrm{op}}} \mathrm{Map}_H(jy_\beta, qx) \to \mathrm{Map}_H(\lim_B jy_\beta, qx)$$

is an equivalence for all $x \in F$ whenever the limit $\lim_B y_\beta$ exists in G and j preserves the limit.

Then, the induced map $F/E \to H/G$ is fully faithful.

Proof. By Proposition 2.28 and Lemma 2.29, it suffices to prove that $j_*q^*x \simeq 0$ for all $x \in \mathrm{Ind}(F)$ such that $i_*x \simeq 0$.

So, assume that $i_*x \simeq 0$ for some $x \in \mathrm{Ind}(F)$. Note that $j_*q^*x \simeq 0$ if and only if $\mathrm{Map}_{\mathrm{Ind}(G)}(y, j_*q^*x) \simeq 0$ for all $y \in G$. Note also that q^* preserves filtered colimits. Pick one $y \in G$, and use condition (a) to write

$y \simeq \lim_B pz_\beta$ where j preserves this limit. If we write $\operatorname{colim}_A x_\alpha \simeq x$ for some filtered ∞-category A with $x_\alpha \in F$, then, using the compactness of j^*y, there is a chain of equivalences

$$\begin{aligned}
\operatorname{Map}_{\mathrm{Ind}(G)}(y, j_*q^*x) &\simeq \operatorname{Map}_{\mathrm{Ind}(H)}(j^*y, q^*x) \\
&\simeq \operatorname*{colim}_A \operatorname{Map}_H(jy, qx_\alpha) \\
&\simeq \operatorname*{colim}_A \operatorname{Map}_H(j \lim_B pz_\beta, qx_\alpha) \\
&\simeq \operatorname*{colim}_A \operatorname{Map}_H(\lim_B jpz_\beta, qx_\alpha) \\
&\simeq \operatorname*{colim}_A \operatorname*{colim}_{B^{\mathrm{op}}} \operatorname{Map}_H(jpz_\beta, qx_\alpha) \\
&\simeq \operatorname*{colim}_{B^{\mathrm{op}}} \operatorname*{colim}_A \operatorname{Map}_F(iz_\beta, x_\alpha) \\
&\simeq \operatorname*{colim}_{B^{\mathrm{op}}} \operatorname{Map}_{\mathrm{Ind}(F)}(i^*z_\beta, x) \\
&\simeq \operatorname*{colim}_{B^{\mathrm{op}}} \operatorname{Map}_{\mathrm{Ind}(E)}(z_\beta, i_*x) \\
&\simeq 0,
\end{aligned}$$

where we use condition (b) to justify the fifth equivalence. This completes the proof. □

Theorem 2.31. *Let*

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ G & \xrightarrow{j} & H \end{array}$$

be a commutative square of fully faithful functors in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ *such that* $F/E \to H/G$ *is fully faithful and such that*

(c) *every object x of H is a retract of an object x' such that x' fits in to a cofiber sequence $jy \to x' \to qz$ for some y in G and some z in F.*

Then, the induced square

$$\begin{array}{ccc} \mathrm{K}(E) & \longrightarrow & \mathrm{K}(F) \\ \downarrow & & \downarrow \\ \mathrm{K}(G) & \longrightarrow & \mathrm{K}(H) \end{array}$$

is a pushout square of spectra.

Proof. By Lemma 2.21, it is enough to show that $\mathrm{K}(\widetilde{F/E}) \to \mathrm{K}(\widetilde{H/G})$ is an equivalence, so it is enough to show that $\widetilde{F/E} \to \widetilde{H/G}$ is an equivalence. By hypothesis this functor is fully faithful, so it is enough to check essential surjectivity. Every object of $\widetilde{H/G}$ is a retract of the image of an object x of H, which is in turn a retract of the image of an object x' of H fitting into a cofiber sequence as in (c). Since $H \to H/G$ kills jy, it follows that every object of $\widetilde{H/G}$ is a retract of the image of an object of F. Since $\widetilde{F/E}$ is idempotent complete by definition, $\widetilde{F/E} \to \widetilde{H/G}$ is essentially surjective. □

Remark 2.32. Note that conditions (a) and (b) in Proposition 2.30 can be used to check fully faithfulness of $F/E \to H/G$.

Example 2.33. Conditions (a) through (c) of Proposition 2.30 and Theorem 2.31 are meant to abstract the basic property of **Tate objects.** Given a stable idempotent complete ∞-category E, the ∞-category $\mathrm{Tate}(E)$ of Tate objects in E fits into a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Ind}(E) \\ \downarrow & & \downarrow \\ \mathrm{Pro}(E) & \longrightarrow & \mathrm{Tate}(E), \end{array}$$

and this square satisfies the properties of the theorem (for ease of exposition, we suppress set-theoretic issues and refer the reader to [Hen17] for a careful treatment). To check condition (a), note that *every* object of $\mathrm{Pro}(E)$ can be written as a cofiltered limit, and $\mathrm{Pro}(E) \to \mathrm{Tate}(E)$ preserves cofiltered limits by the universal property of Tate objects (see [Hen17, Theorem 2.6]). Condition (b) follows from the fact that there is a natural embedding $\mathrm{Tate}(E) \to \mathrm{Pro}\,\mathrm{Ind}(E)$ which preserves cofiltered limits by definition of the mapping spaces in a pro-category. Condition (c) follows for example from [Hen17, Corollary 3.4].

The key point about the ∞-category of Tate objects is that, by the theorem, $\mathrm{K}(\mathrm{Tate}(E)) \simeq \Sigma\mathrm{K}(E)$. Indeed, $\mathrm{K}(\mathrm{Ind}(E)) \simeq 0 \simeq \mathrm{K}(\mathrm{Pro}(E))$ because of the existence of countable (co)products, which means that

$$\begin{array}{ccc} \mathrm{K}(E) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{K}(\mathrm{Tate}(E)) \end{array}$$

is a pushout square.

Remark 2.34. It is possible to build an ∞-category $\mathrm{Tate}^{\kappa}(E)$ of κ-Tate objects out of $\mathrm{Ind}(E)^{\kappa}$ and ${}^{\kappa}\mathrm{Pro}(E)$, the full subcategory of κ-cocompact objects in $\mathrm{Pro}(E)$. This construction has the same properties as $\mathrm{Tate}(E)$ but has the advantage that it is small and hence does not require working in a larger universe. Such an approach is closer to the spirit of this paper and is done for exact categories in [BGW16].

2.5 Vanishing of K_{-1}

We prove our analogue of Schlichting's theorem [Sch06, Theorem 6] in the case of a stable ∞-category admitting a bounded t-structure. The proof differs substantially from that of Schlichting.

Theorem 2.35. *If E is a small stable ∞-category with a bounded t-structure, then $\mathrm{K}_{-1}(E) = 0$.*

Proof. The t-structure on E extends to a t-structure on $\mathrm{Ind}(E)$ with nonnegative objects $\mathrm{Ind}(E_{\geqslant 0}) \simeq \mathrm{Ind}(E)_{\geqslant 0}$ by Proposition 2.13. Let $A = E^{\heartsuit}$ denote the heart of E, and fix κ an uncountable regular cardinal such that E is essentially κ-small. Consider the commutative diagram of fully faithful functors

$$\begin{array}{ccc} E & \xrightarrow{i} & \mathrm{Ind}^{+}_{A}(E) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{Ind}^{-}_{A}(E)^{\kappa} & \xrightarrow{j} & \mathrm{Ind}_{A}(E)^{\kappa}, \end{array} \tag{4}$$

where

- $\mathrm{Ind}^{+}_{A}(E) \subseteq \bigcup_n \mathrm{Ind}(E)_{\leqslant n} \simeq \bigcup_n \mathrm{Ind}(E)_{\leqslant n}$ is the full subcategory of bounded above objects x with $\pi_n x \in A$ for all n,

- $\mathrm{Ind}_A^-(E)^\kappa \subseteq \bigcup_n \mathrm{Ind}(E_{\geqslant n})^\kappa$ is the full subcategory of the κ-compact bounded below objects with $\pi_n(x) \in A$ for all n, and
- $\mathrm{Ind}_A(E)^\kappa$ is the full subcategory of $\mathrm{Ind}(E)$ of objects $x \in \mathrm{Ind}(E)$ such that $\tau_{\leqslant n}x \in \mathrm{Ind}_A^+(E)$ and $\tau_{\geqslant n}x \in \mathrm{Ind}_A^-(E)^\kappa$ for all n.

Note that the inclusion $\mathrm{Ind}_A^+(E) \to \mathrm{Ind}(E)$ factors through $\mathrm{Ind}_A(E)^\kappa$. Indeed, for $x \in \mathrm{Ind}_A^+(E)$, the truncations $\tau_{\leqslant n}x$ are in $\mathrm{Ind}_A^+(E)$ for all n. Moreover, $\tau_{\geqslant n}x$ is bounded and has homotopy objects all in A, so that $\tau_{\geqslant n}x$ is in fact in E; it follows that $\tau_{\geqslant n}x \in \mathrm{Ind}_A^-(E)^\kappa$.

The objects of $\mathrm{Ind}_A^-(E)^\kappa$ are in fact κ-compact in $\mathrm{Ind}(E)$ because $\mathrm{Ind}(E_{\geqslant 0}) \to \mathrm{Ind}(E)$ preserves κ-compact objects as the right adjoint preserves (ω-)filtered colimits, and hence all κ-filtered colimits. For the same reason, $\mathrm{Ind}(E_{\geqslant n}) \to \mathrm{Ind}(E_{\geqslant n-1})$ preserves κ-compact objects. Clearly, if $x \in \mathrm{Ind}_A(E)^\kappa$, then every truncation $\tau_{\geqslant n}x$ is κ-compact, however we do not claim that every κ-compact object x in $\mathrm{Ind}(E)$ with $\pi_n(x) \in A$ is contained in $\mathrm{Ind}_A(E)^\kappa$.

We do claim that

(1) $\mathrm{Ind}_A(E)^\kappa$, $\mathrm{Ind}_A^+(E)$, and $\mathrm{Ind}_A^-(E)^\kappa$ are essentially small idempotent complete stable subcategories of $\mathrm{Ind}(E)$ and

(2) that the t-structure on $\mathrm{Ind}(E)$ restricts to a t-structure on $\mathrm{Ind}_A(E)^\kappa$.

After establishing these facts, we prove that we can apply Theorem 2.31 to the square (4). This gives a pushout square of K-theory spectra which lets us prove in the end that $\mathrm{K}_{-1}(E) = 0$.

In fact $\mathrm{Ind}_A(E)^\kappa$ is contained in $\mathrm{Ind}(E)^\kappa$. Since the bounded below objects of $\mathrm{Ind}_A(E)^\kappa$ are κ-compact, it is enough to check that $\mathrm{Ind}_A^+(E) \subseteq \mathrm{Ind}(E)^\kappa$. Let $x \in \mathrm{Ind}_A^+(E)$, so that in particular $x \in \mathrm{Ind}(E)_{\leqslant n}$ for some n. There are maps

$$\pi_n x[n] \simeq \tau_{\geqslant n}x \to \tau_{\geqslant n-1}x \to \tau_{\geqslant n-2}x \to \cdots \to x. \tag{5}$$

Since the induced t-structure on $\mathrm{Ind}(E)$ is right complete by Proposition 2.13, the colimit of the sequence is equivalent to x. To see this, note that it is enough to prove that $\mathrm{colim}_i \mathrm{Map}(y, \tau_{\geqslant n-i}x) \simeq \mathrm{Map}(y, x)$ for all $y \in E$. However, since the t-structure on E is bounded, any such y is contained in $E_{\geqslant n-i}$ for some i. Using the two cofiber sequences

$$\tau_{\geqslant n-i}x \to \tau_{\geqslant n-j}x \to \tau_{\leqslant n-i-1}\tau_{\geqslant n-j}x$$

and

$$\tau_{\geqslant n-i}x \to x \to \tau_{\leqslant n-i-1}x,$$

for $j \geqslant i$, we see that

$$\mathrm{Map}(y, \tau_{\geqslant n-i}x) \simeq \mathrm{Map}(y, \tau_{\geqslant n-j}x)$$

and

$$\mathrm{Map}(y, \tau_{\geqslant n-i}x) \simeq \mathrm{Map}(y, x)$$

for $j \geqslant i$. This proves that the colimit of (5) is indeed x. However, each object $\tau_{\geqslant m}x$ is actually in E, so this is a κ-small colimit of compact objects and hence of κ-compact objects in $\mathrm{Ind}(E)$. Thus, x is κ-compact by [Lur09, Corollary 5.3.4.15].

That these three ∞-categories are essentially small follows from the fact that $\mathrm{Ind}_A(E)^\kappa \subseteq \mathrm{Ind}(E)^\kappa$ and the fact that $\mathrm{Ind}(E)^\kappa$ is essentially small because every object is the colimit in presheaves on E of a κ-small diagram [Lur09, Proposition 5.3.4.17]. Moreover, $\mathrm{Ind}_A^+(E)$ is idempotent complete because $\mathrm{Ind}(E)_{\leqslant 0}$ and A are idempotent complete, while each $\mathrm{Ind}(E_{\geqslant n})^\kappa$ is idempotent complete since it is closed under κ-small colimits, and in particular it is closed under idempotent completion because κ is uncountable. It follows

that $\mathrm{Ind}_A(E)^\kappa$ is idempotent complete as well since inclusion in $\mathrm{Ind}_A(E)^\kappa$ is given by a condition on the truncations.

These three ∞-categories are closed under suspension and desuspension in $\mathrm{Ind}(E)$, so to see that they are stable, it is enough to show that they are closed under either taking fibers or cofibers in $\mathrm{Ind}(E)$ by [Lur12, Lemma 1.1.3.3] and its opposite version. We first note that if z is the cofiber of a map $f : x \to y$ in $\mathrm{Ind}(E)$ between two objects such that $\pi_n x$ and $\pi_n y$ are in A for all n, then $\pi_n z$ is an extension of objects of A, namely of $\ker(\pi_{n+1}x \to \pi_{n+1}y)$ by $\mathrm{coker}(\pi_n x \to \pi_n y)$. Since A is closed under extension in $\mathrm{Ind}(E)^\heartsuit$, by Lemma 2.19, we see that $\pi_n z \in A$.

Stability of $\mathrm{Ind}^+_A(E)$ follows from the fact the cofiber of a map of bounded above objects is bounded above; stability of $\mathrm{Ind}^-_A(E)^\kappa$ follows from the fact that $\mathrm{Ind}(E_{\geqslant n})^\kappa$ is closed under cofibers in $\mathrm{Ind}(E)$.

We show that $\mathrm{Ind}_A(E)^\kappa$ is stable, by showing that it is closed under taking fibers in $\mathrm{Ind}(E)$. If $z \to x \to y$ is a fiber sequence in $\mathrm{Ind}(E)$ where $x \to y$ is in $\mathrm{Ind}_A(E)^\kappa$, then $\pi_n z \in A$ for all n, so that $\tau_{\leqslant n} z \in \mathrm{Ind}^+_A(E)$ for all n. Hence, it is enough to show that $\tau_{\geqslant n} z \in \mathrm{Ind}(E_{\geqslant n})^\kappa$. However, $\tau_{\geqslant n} : \mathrm{Ind}(E) \to \mathrm{Ind}(E_{\geqslant n})$ preserves limits as it is a right adjoint. Hence, $\tau_{\geqslant n} z$ is the fiber of $\tau_{\geqslant n} x \to \tau_{\geqslant n} y$ in $\mathrm{Ind}(E_{\geqslant 0})$. Now, as we have chosen κ to be uncountable and such that E is essentially κ-small, we see by [Lur09, Proposition 5.4.7.4] that the inclusion $\mathrm{Ind}(E_{\geqslant n})^\kappa \to \mathrm{Ind}(E_{\geqslant n})$ is closed under all finite limits in $\mathrm{Ind}(E_{\geqslant n})$, and in particular under fibers. Hence, $z \in \mathrm{Ind}_A(E)^\kappa$, which completes the proof of claim (1).

Suppose that $x \in \mathrm{Ind}_A(E)^\kappa$. To show that the t-structure on $\mathrm{Ind}(E)$ restricts to $\mathrm{Ind}_A(E)^\kappa$, we show that $\tau_{\geqslant n} x$ and $\tau_{\leqslant n} x$ are in $\mathrm{Ind}_A(E)^\kappa$. In fact, by stability, it is sufficient to check only one of these. Moreover, $\tau_{\geqslant m}\tau_{\geqslant n} x \simeq \tau_{\geqslant m} x$ for $m \geqslant n$, so that if $x \in \mathrm{Ind}^-_A(E)^\kappa$, then so is $\tau_{\geqslant m} x$ for all m. Similarly, $\tau_{\leqslant m}\tau_{\geqslant n} x \in E \subseteq \mathrm{Ind}^+_A(E)$. So, $\tau_{\geqslant m} x \in \mathrm{Ind}^-_A(E)$, which proves that $\mathrm{Ind}_A(E)^\kappa$ inherits the induced t-structure from $\mathrm{Ind}(E)$. This proves (2). Note that by construction the truncation functors on $\mathrm{Ind}_A(E)^\kappa$ preserve $\mathrm{Ind}^+_A(E)$ and $\mathrm{Ind}^-_A(E)^\kappa$, which therefore inherit compatible t-structures.

To complete the proof, we will show that the square (4) satisfies the hypotheses of Theorem 2.31. The validity of condition (c) in the theorem is due to the t-structure, which gives cofiber sequence $\tau_{\geqslant 0} x \to x \to \tau_{\leqslant -1} x$ for every $x \in \mathrm{Ind}_A(E)^\kappa$, where $\tau_{\geqslant 0} x \in \mathrm{Ind}^-_A(E)^\kappa$ and $\tau_{\leqslant -1} x \in \mathrm{Ind}^+_A(E)$. To prove that the induced functor

$$\mathrm{Ind}^+_A(E)/E \to \mathrm{Ind}_A(E)^\kappa/\mathrm{Ind}^-_A(E)^\kappa$$

is fully faithful, we will prove directly that the square satisfies Proposition 2.28(2) and invoke Lemma 2.29.

Let $x \in \mathrm{Ind}(\mathrm{Ind}^+_A(E))$. We need to show that if $i_* x \simeq 0$ then $j_* q^* x \simeq 0$. It is enough to prove that $\mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}^-_A(E)^\kappa)}(y, j_* q^* x) \simeq 0$ for $y \in \mathrm{Ind}^-_A(E)^\kappa$. Choose a filtered ∞-category B and an equivalence $\mathrm{colim}_B x_\beta \simeq x$ where $x_\beta \in \mathrm{Ind}^+_A(E)$ for all β in B. Using adjunctions and compactness, we get a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}^-_A(E)^\kappa)}(y, j_* q^* x) &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}_A(E)^\kappa)}(j^* y, q^* x) \\
&\simeq \operatorname*{colim}_B \mathrm{Map}_{\mathrm{Ind}(E)^\kappa}(jy, q x_\beta) \\
&\simeq \operatorname*{colim}_B \mathrm{Map}_{\mathrm{Ind}(E)}(y, x_\beta) \\
&\simeq \operatorname*{colim}_B \operatorname*{colim}_n \mathrm{Map}_{\mathrm{Ind}(E)}(\tau_{\leqslant n} y, x_\beta) \\
&\simeq \operatorname*{colim}_B \operatorname*{colim}_n \mathrm{Map}_{\mathrm{Ind}^+_A(E)}(\tau_{\leqslant n} y, x_\beta) \\
&\simeq \operatorname*{colim}_n \operatorname*{colim}_B \mathrm{Map}_{\mathrm{Ind}^+_A(E)}(\tau_{\leqslant n} y, x_\beta) \\
&\simeq \operatorname*{colim}_n \mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}^+_A(E))}(i^* \tau_{\leqslant n} y, x) \\
&\simeq \operatorname*{colim}_n \mathrm{Map}_{\mathrm{Ind}(E)}(\tau_{\leqslant n} y, i_* x) \\
&\simeq 0,
\end{aligned}$$

where we use (i) the crucial fact that x_β is bounded above as well as the t-structure to observe that the colimit $\operatorname{colim}_{n\to\infty} \operatorname{Map}_{\operatorname{Ind}(E)}(\tau_{\leqslant n} y, x_\beta)$ stabilizes at $\operatorname{Map}_{\operatorname{Ind}(E)}(y, x_\beta)$ and (ii) that $\tau_{\leqslant n} jy$ is in $E \subseteq \operatorname{Ind}_A^+(E)$ and hence $i^*\tau_{\leqslant n} jy$ is compact in $\operatorname{Ind}(\operatorname{Ind}_A^+(E))$ to prove the eighth equivalence.

It follows that there is a cofiber sequence

$$\mathrm{K}(E) \to \mathrm{K}(\operatorname{Ind}_A^-(E)^\kappa) \oplus \mathrm{K}(\operatorname{Ind}_A^+(E)) \to \mathrm{K}(\operatorname{Ind}_A(E)^\kappa).$$

of K-theory spectra. It is easy to see that the K-theory spectra of the idempotent complete stable ∞-categories $\operatorname{Ind}_A^+(E)$ and $\operatorname{Ind}_A^-(E)^\kappa$ are zero. Indeed, there is an endofunctor $T : \operatorname{Ind}_A^-(E)^\kappa \to \operatorname{Ind}_A^-(E)^\kappa$ given by $T = \bigoplus_{n\geqslant 0} \mathrm{id}[2n]$ since $\operatorname{Ind}(E)^\kappa$ is closed under countable coproducts. We have an equivalence of endofunctors $T \simeq \mathrm{id} \oplus T[2]$. Hence, the identity map on $\mathrm{K}(\operatorname{Ind}_A^-(E)^\kappa)$ is nullhomotopic. The same argument but with desuspensions shows that $\mathrm{K}(\operatorname{Ind}_A^+(E)) \simeq 0$. Hence,

$$\mathrm{K}_0(\operatorname{Ind}_A(E)^\kappa) \cong \mathrm{K}_{-1}(E).$$

Given an object x of $\operatorname{Ind}_A(E)^\kappa$, we have a canonical triangle

$$\tau_{\geqslant 0} x \to x \to \tau_{\leqslant -1} x$$

coming from the t-structure, where $\tau_{\geqslant 0} x$ is in $\operatorname{Ind}_A^-(E)^\kappa$ and $\tau_{\leqslant -1} x$ is in $\operatorname{Ind}_A^+(E)$. But, since K_0 of each of the half-bounded categories is zero, it follows that the class of x in $\mathrm{K}_{-1}(E)$ is also zero. □

3 Induction

This section contains the proofs of the inductive step of our main theorem and the nonconnective theorem of the heart in the noetherian case, their relation to the Farrell-Jones conjecture in negative K-theory for group rings, and a discussion of the major impediments to proving the conjecture in general.

3.1 Dualizability of compactly generated stable ∞-categories

We discuss in this section some technical preliminaries about dualizability we will need later. The material here is basically well-known, but we include it for the sake of completeness.

Recall that an object x in a symmetric monoidal ∞-category $\mathcal{P}$ is **dualizable** if there is another object, $\mathrm{D}x$ together with an evaluation map $\mathrm{ev} : x \otimes \mathrm{D}x \to \mathbb{1}$ and a coevaluation map $\mathrm{coev} : \mathbb{1} \to \mathrm{D}x \otimes x$ such that the composites

$$x \xrightarrow{\mathrm{id}_x \otimes \mathrm{coev}} x \otimes \mathrm{D}x \otimes x \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_x} x$$

and

$$\mathrm{D}x \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_{\mathrm{D}x}} \mathrm{D}x \otimes x \otimes \mathrm{D}x \xrightarrow{\mathrm{id}_{\mathrm{D}x} \otimes \mathrm{ev}} \mathrm{D}x$$

are equivalent to the identities on x and $\mathrm{D}x$, respectively.

In a closed symmetric monoidal ∞-category $\mathcal{P}$, the endofunctor induced by tensoring with a fixed object x has a right adjoint taking y to y^x by definition. Tensoring with x has a left adjoint if and only if x is dualizable, in which case the unit and counit maps of the adjunction are given by tensoring with coev and ev, respectively. Moreover, when x is a dualizable object in the closed symmetric monoidal ∞-category $\mathcal{P}$, there is a natural equivalence $y \otimes x \simeq y^{\mathrm{D}x}$ for $y \in \mathcal{P}$.

Proposition 3.1. *If $\mathcal{C}$ is a compactly generated stable ∞-category, then $\mathcal{C}$ is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ with dual $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp})$.*

Proof. We refer to [Lur12, Section 4.8] for information about the tensor product of stable presentable ∞-categories. Because colimits in $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp})$ are computed pointwise, the evaluation bifunctor $\mathcal{C} \times \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}$ preserves colimits separately in each variable, so we obtain an evaluation map $\mathcal{C} \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}$. We must define a coevaluation map $\mathrm{Sp} \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}) \otimes \mathcal{C}$, which is to say an object of

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}) \otimes \mathcal{C} \simeq \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp})),$$

where $\mathrm{Fun}^{\lim}$ denotes the ∞-category of limit-preserving functors. Using the fact that $\mathcal{C}$ is stable and compactly generated (i.e. $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\omega})$) we have an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\omega}, \mathrm{Sp})$. Moreover, the (restricted) spectral co-Yoneda embedding $h : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\omega}, \mathrm{Sp})$ preserves limits and factors through the full subcategory $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\omega}, \mathrm{Sp}) \subseteq \mathrm{Fun}(\mathcal{C}^{\omega}, \mathrm{Sp})$. This gives the desired limit-preserving functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp})$. It is then routine to verify the triangle identities, so that $\mathcal{C}$ is dualizable with dual $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp})$. For example, consider the composition

$$\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \mathrm{coev}} \mathcal{C} \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}) \otimes \mathcal{C} \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_{\mathcal{C}}} \mathcal{C},$$

which we can write as the composition

$$\mathcal{C} \to \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{C} \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp})) \xrightarrow{\mathrm{ev}} \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}).$$

By definition of the coevaluation map, the composition is the Yoneda embedding $\mathcal{C} \to \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$. Since the natural equivalence $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathcal{C}$ takes the representable functor $h(x)$ to x, we see that the composition is equivalent to the identity. The argument for the dual is similar. □

Lemma 3.2. *If $\mathcal{C}$ and $\mathcal{D}$ are compactly generated stable ∞-categories, then $\mathcal{C} \otimes \mathcal{D}$ is compactly generated by objects the $x \otimes y$, where x and y range over compact generators of $\mathcal{C}$ and $\mathcal{D}$, respectively.*

Proof. We show first that these objects are compact. The mapping spectrum functor

$$(\mathcal{C} \otimes \mathcal{D})(x \otimes y, -) : \mathcal{C} \otimes \mathcal{D} \to \mathrm{Mod}_{\mathbb{S}}$$

preserves filtered colimits if and only if it admits a right adjoint by the adjoint functor theorem, and when this is the case it represents an object in $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathrm{Mod}_{\mathbb{S}})$. By the universal property of $\mathcal{C} \otimes \mathcal{D}$ this occurs if and only if $\mathcal{C}(x, -) \otimes_{\mathbb{S}} \mathcal{D}(y, -) : \mathcal{C} \times \mathcal{D} \to \mathrm{Mod}_{\mathbb{S}}$ preserves filtered colimits in each variable, using the fact that if if $x, x' \in \mathcal{C}^{\omega}$ and $y, y' \in \mathcal{D}^{\omega}$, then

$$(\mathcal{C} \otimes \mathcal{D})(x \otimes y, x' \otimes y') \simeq \mathcal{C}(x, x') \otimes_{\mathbb{S}} \mathcal{D}(y, y').$$

This happens if and only if x and y are both compact.

By definition, $\mathcal{C} \otimes \mathcal{D}$ is the universal stable presentable ∞-category equipped with a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ preserving small colimits in each variable (see [Lur09, Remark 5.5.3.9]). Suppose that $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a functor preserving colimits in each variable such that $F(x, y) = 0$ for all $x \in \mathcal{C}^{\omega}$ and $y \in \mathcal{C}^{\omega}$. By definition of F and the fact that every object of $\mathcal{C}$ (resp. $\mathcal{D}$) can be written as a small colimit of objects of $\mathcal{C}^{\omega}$ (resp. $\mathcal{D}^{\omega}$), it follows that F vanishes. It follows that the objects

$$\{x \otimes y : x \in \mathcal{C}^{\omega}, y \in \mathcal{D}^{\omega}\}$$

generate $\mathcal{C} \otimes \mathcal{D}$. □

Proposition 3.3. *Let $\mathcal{C}$ be dualizable object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Then for any fully faithful functor $\mathcal{A} \to \mathcal{B}$ in Pr^{L}, the induced functor $\mathcal{A} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{C}$ is fully faithful.*

Proof. Let $\mathcal{D}$ be a dual of $\mathcal{C}$. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{A}\otimes\mathcal{C} & \longrightarrow & \mathrm{Fun}^{\mathrm{L}}(\mathcal{D},\mathcal{A}) & \longrightarrow & \mathrm{Fun}(\mathcal{D},\mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}\otimes\mathcal{C} & \longrightarrow & \mathrm{Fun}^{\mathrm{L}}(\mathcal{D},\mathcal{B}) & \longrightarrow & \mathrm{Fun}(\mathcal{D},\mathcal{B}), \end{array}$$

in which the left hand horizontal maps are equivalences and the right hand horizontal maps are fully faithful. (Technically, $\mathrm{Fun}(\mathcal{D},-)$ lands in a higher universe, but we can restrict to the κ-continuous functors for any κ such that $\mathcal{D}$ is κ-compactly generated.) Moreover, the right hand vertical map is fully faithful since $\mathcal{A}\to\mathcal{B}$ is, by hypothesis, so it follows that each of the other vertical maps is fully faithful. □

Note that we did not actually use the fact that $\mathcal{C}$ was stable in the proof of the above proposition; the same argument works for $\mathcal{C}$ dualizable in Pr^{L}. Unfortunately, there are not so many dualizable objects of Pr^{L}, but as soon as we pass to $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, we obtain a vast supply by Proposition 3.1.

A **localization sequence** in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is a cofiber sequence $\mathcal{B}\to\mathcal{C}\to\mathcal{D}$ such that $\mathcal{B}\to\mathcal{C}$ is fully faithful. The stable presentable ∞-category $\mathcal{D}$ in this case is equivalent to the usual Bousfield localization of $\mathcal{C}$ at the arrows with cofiber in $\mathcal{B}$ by [BGT13, Proposition 5.6].

Example 3.4. The prototypical localization sequence arises from a quasi-compact and quasi-separated scheme X together with a quasi-compact open subscheme $U\subseteq X$ with complement Z. In this case, the functor $\mathcal{D}(X)\to\mathcal{D}(U)$ is a localization with kernel $\mathcal{D}_Z(X)$, the ∞-category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves supported set-theoretically on Z. Hence, $\mathcal{D}_Z(X)\to\mathcal{D}(X)\to\mathcal{D}(U)$ is a localization sequence.

Since localization sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ are cofiber sequences and as the tensor product on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves small colimits in each variable by [Lur12, Remark 4.8.1.23], the previous lemma shows that localization sequences of stable presentable ∞-categories are preserved by tensoring with a given compactly generated stable ∞-category $\mathcal{E}$. Thus, we have proved the following.

Corollary 3.5. *Let $\mathcal{B}\to\mathcal{C}\to\mathcal{D}$ be a localization sequence of stable presentable ∞-categories. Then,*

$$\mathcal{B}\otimes\mathcal{E}\to\mathcal{C}\otimes\mathcal{E}\to\mathcal{D}\otimes\mathcal{E}$$

is a localization sequence for any compactly generated stable ∞-category $\mathcal{E}$.

3.2 Negative K-theory via ∞-categories of automorphisms

In this section, we prove the following theorem, which verifies Conjecture B in many cases.

Theorem 3.6. *If E is a small stable ∞-category equipped with a bounded t-structure such that $E^{\heartsuit}$ is noetherian, then $\mathrm{K}_{-n}(E)=0$ for $n\geqslant 1$.*

Many of our arguments in the proof work in greater generality, and we take care to isolate those parts that are truly special to the situation of a noetherian heart.

Definition 3.7. Throughout this section, $\mathbb{S}[s]=\Sigma^\infty_+\mathbb{N}$ denotes the free commutative $\mathbb{S}$-algebra on the commutative monoid $\mathbb{N}$. Note that $\mathbb{S}[s]$ equivalent to the free $\mathbb{E}_1$-ring spectrum on the sphere spectrum $\mathbb{S}$. Similarly, $\mathbb{S}[s^{\pm1}]=\Sigma^\infty_+\mathbb{Z}$ is the free commutative $\mathbb{S}$-algebra on the commutative monoid $\mathbb{Z}$, or, equivalently, the localization of $\mathbb{S}[s]$ obtained by inverting $s\in\pi_0\mathbb{S}[s]$. These are each flat over $\mathbb{S}$ and have the expected ring of components; that is, $\pi_0(\mathbb{S}[s])\cong\mathbb{Z}[s]$ and $\pi_0(\mathbb{S}[s^{\pm1}])\cong\mathbb{Z}[s^{\pm1}]$, while $\pi_*(\mathbb{S}[s])\cong(\pi_*\mathbb{S})[s]$ and $\pi_*(\mathbb{S}[s^{\pm1}])\cong(\pi_*\mathbb{S})[s^{\pm1}]$. **Warning:** the commutative $\mathbb{S}$-algebra $\mathbb{S}[s]$ is not the free commutative (or $\mathbb{E}_\infty$) algebra on a single element in degree 0.

Notation 3.8. In general, we will use either the notation Mod_R or $\mathcal{D}(R)$ for the stable presentable ∞-category of right R-modules for an $\mathbb{E}_\infty$-ring spectrum R. Moreover, in the special cases of $\mathbb{S}[s]$ and $\mathbb{S}[s^{\pm 1}]$, we will use the suggestive notation $\mathcal{D}(\mathbb{A}^1) = \mathrm{Mod}_{\mathbb{S}[s]}$ and $\mathcal{D}(\mathbb{G}_m) = \mathrm{Mod}_{\mathbb{S}[s^{\pm 1}]}$. Given a stable presentable ∞-category $\mathcal{C}$, we write

$$\mathcal{D}(\mathbb{A}^1, \mathcal{C}) = \mathcal{D}(\mathbb{A}^1) \otimes \mathcal{C} = \mathrm{Mod}_{\mathbb{S}[s]} \otimes \mathcal{C},$$

and similarly for $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$.

To begin, we show that $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ can be identified with the ∞-category of endomorphisms in $\mathcal{C}$, and that $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$ is equivalent to the ∞-category of automorphisms in $\mathcal{C}$.

Definition 3.9. Given an ∞-category $\mathcal{C}$, the functor category

$$\mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C})$$

is the **∞-category of endomorphisms in $\mathcal{C}$** . An object of the ∞-category of endomorphisms consists of a pair (x, e) where x is an object of $\mathcal{C}$ and $e : x \to x$ is an endomorphism. For example, if $\mathcal{C}$ is additive, then $(x, 0)$ (the object x equipped with the zero endomorphism) and (x, id_x) are functorial sections of the forgetful functor $\mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C}) \to \mathcal{C}$. The **$\infty$-category of automorphisms in $\mathcal{C}$** is

$$\mathrm{Fun}(S^1, \mathcal{C}),$$

where $S^1 \simeq B\mathbb{Z}$ is a Kan complex weakly equivalent to $\Delta^1/\partial\Delta^1$. The map of ∞-categories $\Delta^1/\partial\Delta^1 \to S^1$ induces a fully faithful embedding

$$\mathrm{Fun}(S^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C})$$

with essential image those endomorphisms (x, e) such that $e : x \to x$ is an equivalence.

Proposition 3.10. *If $\mathcal{C}$ is a stable presentable ∞-category, then*

(i) $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \simeq \mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C})$*, and*

(ii) $\mathcal{D}(\mathbb{G}_m, \mathcal{C}) \simeq \mathrm{Fun}(S^1, \mathcal{C})$.

Proof. We prove (i), the proof of (ii) being similar. We claim that there is a natural equivalence

$$\mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Mod}_{\mathbb{S}[s]}, \mathcal{C}).$$

It suffices to show that $\mathrm{Mod}_{\mathbb{S}[s]}$ is the free stable presentable ∞-category generated by $\Delta^1/\partial\Delta^1$. This follows from the $(\mathrm{Mod}_*, \mathrm{End})$ adjunction [AG14, Section 3.1] together with the fact that $\mathbb{S}[s] \simeq \mathbb{S}[\mathbb{N}]$ is the free $\mathbb{S}$-algebra on the monoid $\mathbb{N}$, and that the nerve of $\mathbb{N}$ (viewed as a category with one object) is a fibrant replacement for $\Delta^1/\partial\Delta^1$ in the Joyal model structure. For any $\mathbb{S}$-algebra R and any stable presentable ∞-category $\mathcal{C}$, there is a natural equivalence $\mathrm{Mod}_{R^{\mathrm{op}}} \otimes \mathcal{C} \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Mod}_R, \mathcal{C})$. Indeed, Mod_R is compactly generated, and hence dualizable by Proposition 3.1 with dual $\mathrm{Mod}_{R^{\mathrm{op}}}$. In particular, since $\mathbb{S}[s]$ is an $\mathbb{E}_\infty$-ring spectrum, the ∞-category of endomorphisms in a stable presentable ∞-category $\mathcal{C}$ is equivalent to $\mathrm{Mod}_{\mathbb{S}[s]} \otimes \mathcal{C}$. □

We focus now on the case where $\mathcal{C} \simeq \mathrm{Ind}(E)$ is compactly generated by a small stable ∞-category E.

Lemma 3.11. *If $\mathcal{C} \simeq \mathrm{Ind}(E)$ is compactly generated, then $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ is compactly generated by the objects $\mathbb{S}[s] \otimes x =: x[s]$ and $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$ is compactly generated by the objects $\mathbb{S}[s^{\pm 1}] \otimes x =: x[s^{\pm 1}]$ as x ranges over the objects of E.*

Proof. This is a special case of Lemma 3.2. □

Lemma 3.12. *If $\mathcal{C} \simeq \mathrm{Ind}(E)$ is compactly generated, then the natural functor $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \to \mathcal{D}(\mathbb{G}_m, \mathcal{C})$ is a localization with kernel a compactly generated stable presentable ∞-category which we will denote $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$. Moreover, $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ is compactly generated by the compact objects $(x, 0)$ in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ as x ranges over the objects of E.*

The fact that $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ is generated by compact objects that are compact in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ implies that the right adjoint $\mathcal{D}(\mathbb{G}_m, \mathcal{C}) \to \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ to the localization preserves filtered colimits.

Proof. By [Lur12, Proposition 7.2.4.17] or [AG14, Proposition 6.9], we have a localization sequence

$$\mathrm{Mod}_{\mathbb{A}^1,\{0\}} \to \mathrm{Mod}_{\mathbb{A}^1} \to \mathrm{Mod}_{\mathbb{G}_m}$$

of stable presentable ∞-categories, where $\mathrm{Mod}_{\mathbb{A}^1,\{0\}}$ is the full subcategory of $\mathrm{Mod}_{\mathbb{A}^1}$ consisting of $\mathbb{S}[s]$-modules M such that for every $x \in \pi_m(M)$ there exists a positive integer N such that $s^N \cdot x = 0$. Moreover, by [AG14, Proposition 6.9], $\mathrm{Mod}_{\mathbb{A}^1,\{0\}}$ is compactly generated by the object $\mathbb{S}$ when viewed as an $\mathbb{S}[s]$-module; in particular, since $\mathbb{S} \simeq \mathrm{cofib}\left(\mathbb{S}[s] \xrightarrow{t} \mathbb{S}[s]\right)$ is compact as an $\mathbb{S}[s]$-module, $\mathrm{Mod}_{\mathbb{A}^1,\{0\}}$ is compactly generated by compact objects of $\mathrm{Mod}_{\mathbb{A}^1}$. By tensoring with $\mathrm{Ind}(E)$, we obtain the localization sequence we want by Corollary 3.5. The object $\mathbb{S} \otimes x$ is by definition x with the zero endomorphism. □

We turn to the problem of constructing t-structures on ∞-categories of endomorphisms and automorphisms.

Lemma 3.13. *Let $\mathcal{C} = \mathrm{Ind}(E)$ be a compactly generated stable presentable ∞-category with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. The full subcategory $\mathcal{D}(\mathbb{A}^1, \mathcal{C})_{\geqslant 0} \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ of endomorphisms (x, e) where $x \in \mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ defines the non-negative part of a t-structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$, where $(y, f) \in \mathcal{D}(\mathbb{A}^1, \mathcal{C})_{\leqslant 0}$ if and only if $y \in \mathcal{C}_{\leqslant 0}$. Moreover, the truncation functors induced by the t-structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ preserve the full subcategory $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \supseteq \mathcal{D}(\mathbb{G}_m, \mathcal{C})$.*

Proof. Requirement (1) of Definition 2.1 is inherited from $\mathcal{C}$. Since the truncations $\tau_{\geqslant 0}x$ and $\tau_{\leqslant 0}x$ are functorial, there is a cofiber sequence

$$(\tau_{\geqslant 0}x, \tau_{\geqslant 0}(e)) \to (x, e) \to (\tau_{\leqslant -1}x, \tau_{\leqslant -1}(e)).$$

This verifies requirement (3). As for (2), note that the forgetful functor $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \to \mathcal{C}$ detects nullhomotopic maps. This means that if $x \in \mathcal{C}_{\geqslant 0}$ and $y \in \mathcal{C}_{\leqslant -1}$, then

$$\mathrm{Map}_{\mathcal{D}(\mathbb{A}^1,\mathcal{C})}((x, e), (y, f)) \simeq 0$$

for any endomorphisms e of x and f of y. The first claim follows.

If (x, e) is an object of $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$, then e is an automorphism of x, and hence $\tau_{\geqslant 0}(e)$ is an automorphism of $\tau_{\geqslant 0}x$. So, the truncation functors preserve $\mathcal{D}(\mathbb{G}_m, \mathcal{C}) \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$. This proves the second claim. □

Proposition 3.14. *Let $\mathcal{C} = \mathrm{Ind}(E)$ be a compactly generated stable presentable ∞-category with the t-structure induced (in the sense of Proposition 2.13) by a bounded t-structure on E such that $E^\heartsuit$ is noetherian. The t-structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ of the previous lemma restricts to a bounded t-structure with noetherian heart on the full subcategory $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ of compact objects.*

Proof. Let $F \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ be the full subcategory of objects x such that $\tau_{\geqslant n}x$ is compact for all n. It follows immediately that F is idempotent complete. Moreover, F contains all objects of the form $(x[s], s)$ for $x \in E$ since $\tau_{\geqslant n}(x[s], s) \simeq (\tau_{\geqslant n}x, \tau_{\geqslant n}(s))$ and since $\tau_{\geqslant n}x$ is in E if x is in E. Therefore, if F is stable, the inclusion

$F \to \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ is an equivalence. By definition, F is closed under suspension and desuspension. Hence, by [Lur12, Lemma 1.1.3.3], it is enough to show that F is closed under taking cofibers.

Hence, given a cofiber sequence $x \to y \to c$ in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ with $x, y \in F$, we must show that $\tau_{\geqslant 0} c$ is compact. Let d be the cofiber of $\tau_{\geqslant 0} y \to \tau_{\geqslant 0} y$, so that d fits into a second cofiber sequence $d \to \tau_{\geqslant 0} c \to \pi$, where $\pi \in \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$ is the image of $\pi_0 c \to \pi_{-1} x$. As d is compact by the hypothesis on x and y, it is enough to show that π is compact in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$.

Let $A = E^\heartsuit$, and let $A[s]$ be the full subcategory of compact objects in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$. Note that this is well-defined because $\left(\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{S}[s]} \otimes \mathcal{C}_{\geqslant 0}\right)^\heartsuit \simeq \mathrm{Mod}^\heartsuit_{\mathbb{S}[s]} \otimes \mathcal{C}^\heartsuit$ by definition of the tensor product of Grothendieck abelian categories in [Lur]. In general, there is no reason for $A[s]$ to be an abelian category. However, this is implied by the fact that A is noetherian in our case, as this ensures that the kernel of a map between finitely presented objects is again finitely presented. Moreover, it is a consequence that $\pi_n x \in A[s]$ whenever $x \in \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$. Now, we claim that $A[s]$ is noetherian. To see this, note that $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ is compactly generated by the objects $(y[s], s)$ where $y \in E$. It follows that every object of $A[s]$ is obtained in finitely many steps (consisting of taking kernels, cokernels, extensions, sums, and summands) from objects of the form $(x[s], s)$, where $x \in A$. Since A is noetherian, every such x is noetherian and [Swa68, Theorem 3.5] implies that $(x[s], s)$ is noetherian. This is effectively an easy generalization of the Hilbert basis theorem in algebra. Noetherianity implies that $A[s]$ is a Serre subcategory of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$, so that $\pi \subseteq \pi_{-1} x$ is in $A[s]$. We are reduced to proving that if π is an object of $A[s]$, then π is compact as an object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$.

It is convenient for the rest of the proof to write s for the endomorphism of any object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$. Let $F_i \pi = \ker(s^i : \pi \to \pi)$ for $i \geqslant 0$. This is an increasing filtration on π, which stabilizes at some F_N for $N \geqslant 0$ since $A[s]$ is noetherian. Each $F_i\pi / F_{i-1}\pi$ is in fact an object of A as it is a finitely presented object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$ such that s acts as zero. So, inductively, $F_N \pi$ is compact. Let τ be the quotient $\pi / \mathrm{F}_N \pi$. The endomorphism s acts injectively on τ by construction. To see that τ is compact, choose a surjection $a[s] \to \tau$ such that $a \in A$. Let σ be the kernel, and let $\sigma_i \subseteq a \cdot s^i$ be the intersection of the kernel and $a \cdot s^i \subseteq a[s]$ (viewed as an object of $\mathrm{Ind}(A)$). Then, $s : \sigma_i \to \sigma_{i+1}$ and $\sigma \cong \bigoplus_{i \geqslant 0} \sigma_i$. Moreover, $s : \sigma_i \to \sigma_{i+1}$ is an isomorphism for all i. The injectivity follows from the fact that s acts injectively on $a[s]$, while the surjectivity follows (via the snake lemma) from the fact that s acts injectively on τ and $s : a \cdot s^i \to a \cdot s^{i+1}$ is surjective. It follows that $\sigma \cong \sigma_0[s]$. But, since $\sigma_0 \subseteq a$, it follows that σ is compact. Therefore, τ is compact.

Now, to see that the t-structure is bounded, it is enough to see that each object $(x[s], s)$ is bounded for $x \in E$. This is the case by construction. Finally, we have already mentioned that $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega, \heartsuit} \simeq A[s]$ is noetherian. □

Remark 3.15. Noetherianity is used in a couple primary locations in the proof. The first is to check that $\pi_{-1} x$ is finitely presented and that π is therefore itself in $A[s]$. The second is to guarantee that the filtration $F_\bullet \pi$ stabilizes. We return in the next sections to the problem of weakening the noetherian hypothesis.

Lemma 3.16. *Let $\mathcal{C} = \mathrm{Ind}(E)$ be a compactly generated stable presentable ∞-category with a t-structure induced (in the sense of Proposition 2.13) by a bounded t-structure on E such that $E^\heartsuit$ is noetherian. Then, the t-structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ respects $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$.*

Proof. Note that we did not prove in general that the t-structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ restricts to a t-structure on $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$. But, this is true under the noetherianity condition for the compact objects. Indeed, an object x of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ is contained in the subcategory $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$ if and only if s^N acts nullhomotopically on x for some $N \geqslant 0$. If s^N does act nullhomotopically on x, then it does so on $\tau_{\geqslant 0} x$ as well, which shows that if $x \in \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$, then so is $\tau_{\geqslant 0} x$. □

Corollary 3.17. *Let $\mathcal{C} = \mathrm{Ind}(E)$ be a compactly generated stable presentable ∞-category with a t-structure induced (in the sense of Proposition 2.13) by a bounded t-structure on E such that $E^\heartsuit$ is noetherian. Then, there is a bounded t-structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ with noetherian heart.*

Proof. The subcategory

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^{\omega,\heartsuit} \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega,\heartsuit}$$

is in fact a Serre subcategory. Indeed, if $\tau \subseteq \sigma$ is a subobject where s acts nilpotently on σ, then s acts nilpotently on τ as well. Using (i) implies (ii) in Proposition 2.20, we see that there is an induced t-structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ and that the functor $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega \to \mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ is t-exact. Since every object of $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ is a retract of an object in the image of the localization functor and since the t-structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ is bounded, it follows that the t-structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ is bounded too. The abelian category $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^{\omega,\heartsuit}$ is noetherian because it is equivalent to the localization of the noetherian abelian category $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega,\heartsuit}$ by the Serre subcategory $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^{\omega,\heartsuit}$. □

Proof of Theorem 3.6. Let $\mathcal{C} = \mathrm{Ind}(E)$. Applying K-theory to the exact sequence

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega \to \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega \to \mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega,$$

we obtain a cofiber sequence

$$\mathrm{K}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \to \mathrm{K}(\mathbb{A}^1, \mathcal{C}) \to \mathrm{K}(\mathbb{G}_m, \mathcal{C})$$

of nonconnective K-theory spectra.

Consider the exact functor $i : \mathcal{C} \to \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ given by $i(x) \simeq (x, 0)$. Since $(x[s], s) \xrightarrow{s} (x[s], s) \to (x, 0)$ is a cofiber sequence in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$, we see that $(x, 0)$ is compact if x is. Hence, i restricts to a functor $E \to \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$, also denoted i. Moreover, the additivity theorem, applied to this same cofiber sequence, viewed as a cofiber sequence of functors $E \to \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$, induces a nullhomotopic map $\mathrm{K}(E) \to \mathrm{K}(\mathbb{A}^1, \mathcal{C})$. The underlying object functor $u : \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega \to E$ sending (x, e) to x gives $u \circ i \simeq \mathrm{id}_E$. It follows that $\mathrm{K}(E)$ is a summand of $\mathrm{K}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ and that this summand maps trivially to $\mathrm{K}(\mathbb{A}^1, \mathcal{C})$.

Now, suppose that $\mathrm{K}_{-m}(F) = 0$ for all $1 \leqslant m \leqslant n$ and all stable ∞-categories F which admit bounded t-structures with noetherian hearts. The remarks above prove that $\mathrm{K}_{-n-1}(E)$ is a subquotient of $\mathrm{K}_{-n}(\mathbb{G}_m, \mathcal{C})$. By Corollary 3.17, there is a bounded t-structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ with noetherian heart. Hence, $\mathrm{K}_{-n}(\mathbb{G}_m, \mathcal{C}) = 0$ by the inductive hypothesis and so $\mathrm{K}_{-n-1}(E) = 0$ as well. □

3.3 The nonconnective theorem of the heart

In this section we prove Conjecture C in the case of a noetherian heart.

Theorem 3.18 (Nonconnective theorem of the heart)**.** *If E is a small stable ∞-category with a bounded t-structure such that $E^\heartsuit$ is noetherian, then the natural map*

$$\mathrm{K}(E^\heartsuit) \xrightarrow{\simeq} \mathrm{K}(E)$$

is an equivalence.

To give the theorem content, we must define $\mathrm{K}(A)$ when A is an abelian category, show that this agrees with other definitions in the literature, and define the map $\mathrm{K}(E^\heartsuit) \to \mathrm{K}(E)$. We will use the terminology and results about prestable ∞-categories of [Lur, Appendix C], which in turn follows work of Krause [Kra15] on homotopy categories of injective complexes.

Lemma 3.19. *If A is a small abelian category, then $\mathrm{Ind}(A)$ is a Grothendieck abelian category, the Yoneda embedding $A \to \mathrm{Ind}(A)$ is exact, and the natural map $A \to \mathrm{Ind}(A)^\omega$ is an equivalence.*

Proof. Since A has finite colimits, $\mathrm{Ind}(A)$ is presentable. Moreover, it is not difficult to see that $\mathrm{Ind}(A)$ is abelian. To see that filtered colimits preserve monomorphisms, use that filtered colimits preserve finite limits. Yoneda is always right exact, and it preserves finite colimits that exist in A. This proves exactness. The last claim follows because A is idempotent complete. □

Definition 3.20. Let $\check{\mathcal{D}}(\mathrm{Ind}(A))$ denote the unseparated derived ∞-category of $\mathrm{Ind}(A)$ as defined in [Lur, Section C.5.8]. It is the dg nerve of the dg category of complexes of injective objects in $\mathrm{Ind}(A)$. In particular, by [Lur12, Remark 1.3.2.3], $\mathrm{Ho}(\check{\mathcal{D}}(\mathrm{Ind}(A)))$ is the homotopy category of injectives as studied by Krause [Kra15]. There is a right complete t-structure on $\check{\mathcal{D}}(\mathrm{Ind}(A))$, and $\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geqslant 0}$ is anticomplete (see [Lur, Section C.5.5]) with an important universal property: it is initial among Grothendieck prestable ∞-categories $\mathcal{C}$ with $\mathcal{C}^{\heartsuit} \simeq \mathrm{Ind}(A)$ (see [Lur, Corollary C.5.8.9]).

Definition 3.21. Let A be a small abelian category. We define the bounded derived ∞-category of A to be $\mathcal{D}^b(A) = \check{\mathcal{D}}(\mathrm{Ind}(A))^{\omega}$. In particular, $\mathcal{D}^b(A)$ is a small idempotent complete stable ∞-category.

Lemma 3.22. *The ∞-category $\check{\mathcal{D}}(\mathrm{Ind}(A))$ is compactly generated by $\mathcal{D}^b(A)$.*

Proof. This is the content of [Kra15, Theorem 4.9]. In the setting of [Lur, Appendix C], we invoke the fact that $\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geqslant 0}$ is coherent (by [Lur, Corollary C.6.5.9]) and anticomplete to conclude that $\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geqslant 0}$ is compactly generated by [Lur, Theorem C.6.7.1]. Since $\check{\mathcal{D}}(\mathrm{Ind}(A))$ is right complete, a compact object of $\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geqslant 0}$ is compact when viewed in $\check{\mathcal{D}}(\mathrm{Ind}(A)) \simeq \mathrm{Sp}(\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geqslant 0})$. Let $y \in \check{\mathcal{D}}(\mathrm{Ind}(A))$. We have to show that if $\mathrm{Map}_{\check{\mathcal{D}}(\mathrm{Ind}(A))}(x, y)$ for all x in $\check{\mathcal{D}}(\mathrm{Ind}(A))^{\omega}_{\geqslant n}$ and all n, then $y \simeq 0$. But, if this condition is satisfied, then $y \in \check{\mathcal{D}}(\mathrm{Ind}(A))_{\leqslant n}$ for all n. Since $\check{\mathcal{D}}(\mathrm{Ind}(A))$ is right separated, $y \simeq 0$. □

Lemma 3.23. *The canonical t-structure on $\check{\mathcal{D}}(\mathrm{Ind}(A))$ restricts to a bounded t-structure on $\mathcal{D}^b(A)$ with heart equivalent to A.*

Proof. This follows from [Lur, Theorem C.6.7.1]. □

Lemma 3.24. *Let $\mathcal{D}' \subseteq \mathcal{D}(\mathrm{Ind}(A))$ denote the full subcategory of objects x such that $\mathrm{H}_n(x) \in A \subseteq \mathrm{Ind}(A)$ for all n and such that $\mathrm{H}_n(x)$ is non-zero for at most finitely many $n \in \mathbb{Z}$. Then, the map $\check{\mathcal{D}}(\mathrm{Ind}(A)) \to \mathcal{D}(\mathrm{Ind}(A))$ restricts to an equivalence $\mathcal{D}^b(A) \to \mathcal{D}'$.*

Proof. The claim can be checked at the level of homotopy categories, which is the other part of [Kra15, Theorem 4.9]. □

Definition 3.25. If A is a small abelian category, then we define $\mathrm{K}(A) = \mathrm{K}(\mathcal{D}^b(A))$, the nonconnective K-theory spectrum of the idempotent complete stable ∞-category $\mathcal{D}^b(A)$ (defined in [BGT13]).

We want to construct "the natural map $\mathrm{K}(E^{\heartsuit}) \to \mathrm{K}(E)$" of the statement of Theorem 3.18.

Proposition 3.26. *Let E be a small stable ∞-category with a bounded t-structure. Then, there is a natural t-exact functor $\mathcal{D}^b(E^{\heartsuit}) \to E$ inducing an equivalence on hearts.*

Proof. To define the natural map in the statement of Theorem 3.18, let E be a small stable ∞-category with a bounded t-structure, and let $A = E^{\heartsuit}$. By [Lur, Corollary C.5.8.9], there is a left exact functor $\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geqslant 0} \to \mathrm{Ind}(E)_{\geqslant 0}$ inducing the equivalence $\mathrm{Ind}(A) \simeq \mathrm{Ind}(E)^{\heartsuit}$.

By [Lur, Proposition C.3.2.1], the induced functor $F : \check{\mathcal{D}}(\mathrm{Ind}(A)) \to \mathrm{Ind}(E)$ is t-exact and induces an equivalence on hearts. It suffices to check that F preserves compact objects. By Lemma 3.24, every compact object of $\check{\mathcal{D}}(\mathrm{Ind}(E))$ is a finite iterated fiber of maps between shifts of objects in $A \subseteq \check{\mathcal{D}}(\mathrm{Ind}(E))^{\heartsuit}$. Thus, it suffices to show that $F(x) \in E$ when $x \in A$. But, this follows from hypothesis. □

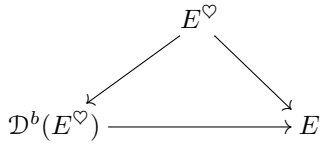
Corollary 3.27. *Let E be a small stable ∞-category with a bounded t-structure. Then, there is a natural map $\mathrm{K}(E^{\heartsuit}) \to \mathrm{K}(E)$ of nonconnective K-theory spectra.*

Proof. Apply K to the exact functor $\mathcal{D}^b(E^{\heartsuit}) \to E$. □

With this in mind, we turn to the proof of the theorem.

Proof of Theorem 3.18. The first step is to use Barwick's theorem of the heart to prove that the induced map $\mathrm{K}^{\mathrm{cn}}(E^{\heartsuit}) \to \mathrm{K}^{\mathrm{cn}}(E)$ of connective K-theory is an equivalence. Philosophically, this *is* Barwick's theorem, but we have defined K-theory in terms of stable ∞-categories instead of using exact ∞-categories.

Consider the commutative triangle



of Waldhausen ∞-categories in the sense of [Bar15]. For the moment, denote by $\mathrm{K}^{\mathrm{Bar}}$ the (connective) K-theory spectrum of a Waldhausen ∞-category, as constructed in [Bar16]. Then, [Bar15, Theorem 6.1] shows that the $\mathrm{K}^{\mathrm{Bar}}(E^{\heartsuit}) \to \mathrm{K}^{\mathrm{Bar}}(\mathcal{D}^b(E^{\heartsuit}))$ and $\mathrm{K}^{\mathrm{Bar}}(E^{\heartsuit}) \to \mathrm{K}^{\mathrm{Bar}}(E)$ are equivalences. Hence, $\mathrm{K}^{\mathrm{Bar}}(E^{\heartsuit}) \to \mathrm{K}^{\mathrm{Bar}}(E)$ is an equivalence.

By [Bar16, Corollary 10.6], $\mathrm{K}^{\mathrm{Bar}}(\mathcal{D}^b(E^{\heartsuit}))$ and $\mathrm{K}^{\mathrm{Bar}}(E)$ are equivalent to the Waldhausen K-theory of suitable Waldhausen categories, and these are in turn equivalent to $\mathrm{K}^{\mathrm{cn}}(\mathcal{D}^b(E^{\heartsuit}))$ and $\mathrm{K}^{\mathrm{cn}}(E)$, respectively, by [BGT13, Theorem 7.8]. This proves the result in connective K-theory.

Now, in the situation of the theorem, both $\mathcal{D}^b(E^{\heartsuit})$ and E are have bounded t-structures with noetherian hearts. It follows from Theorem 3.6 that $\mathrm{K}_{-n}(E^{\heartsuit}) = \mathrm{K}_{-n}(E) = 0$ for $n \geqslant 1$. This completes the proof. □

Remark 3.28. If E is a small stable ∞-category with a bounded t-structure, then $\mathrm{K}^{\mathrm{cn}}(E) \simeq \mathrm{K}^{\mathrm{cn}}(\mathcal{D}^b(E^{\heartsuit}))$ is equivalent to the Quillen K-theory of $E^{\heartsuit}$ viewed as an exact category. This follows from the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7] and a theorem of Waldhausen [TT90, Theorem 1.11.2]. In the end, the theorem of the heart is a generalization of the Gillet-Waldhausen theorem.

Remark 3.29. Note that the negative K-groups of a small abelian category $\mathcal{A}$, defined in this paper as $\mathrm{K}_{-n}(\mathcal{D}^b(\mathcal{A}))$ for $n \geqslant 1$, agree with the negative K-groups of $\mathcal{A}$ as defined by Schlichting [Sch06]. To see this, denote the latter by $\mathrm{K}^{\mathrm{S}}_{-n}(\mathcal{A})$ for the moment.

As it is not necessary for our paper, we only illustrate the argument. Recall that a Frobenius pair is a pair $(\mathcal{E}, \mathcal{E}_0)$ of small Frobenius categories where $\mathcal{E}_0$ is a full subcategory of $\mathcal{E}_0$ such that the embedding $\mathcal{E}_0 \to \mathcal{E}$ preserves projective (or equivalently injective) objects. A morphism of Frobenius pairs $(\mathcal{E}, \mathcal{E}_0) \to (\mathcal{F}, \mathcal{F}_0)$ consists of a functor $\Phi : \mathcal{E} \to \mathcal{F}$ such that $\Phi(M) \in \mathcal{F}_0$ if $M \in \mathcal{E}_0$. (See for example [Sch06].) Let $\mathscr{F}\mathrm{rob}$ denote the ∞-category of Frobenius pairs. Using the functor $\mathcal{D}_{\mathrm{sing}}$ defined in Appendix A, we obtain a functor $\mathcal{D}_{\mathrm{sing}} : \mathscr{F}\mathrm{rob} \to \mathrm{Cat}^{\mathrm{perf}}_{\infty}$ defined by letting

$$\mathcal{D}_{\mathrm{sing}}(\mathcal{E}, \mathcal{E}_0) = (\mathcal{D}_{\mathrm{sing}}(\mathcal{E})/\mathcal{D}_{\mathrm{sing}}(\mathcal{E}_0))^{\sim},$$

the idempotent completion of the Verdier quotient. Note that $\mathcal{D}_{\mathrm{sing}}(\mathcal{E}_0) \to \mathcal{D}_{\mathrm{sing}}(\mathcal{E})$ is fully faithful because any map $\mathcal{E}_0$ that factors through a projective in $\mathcal{E}$ also factors through a projective in $\mathcal{E}_0$.

By the definition of an exact sequence in $\mathscr{F}\mathrm{rob}$ given in [Sch06], $\mathcal{D}_{\mathrm{sing}}$ sends exact sequences to localization sequences. Moreover, if $(\mathcal{E}, \mathcal{E}_0)$ is a flasque Frobenius pair, meaning that there is an endofunctor T of the pair such that $T \simeq T \oplus \mathrm{id}$, then $\mathcal{D}_{\mathrm{sing}}(\mathcal{E}, \mathcal{E}_0)$ is flasque. Using these facts, and the fact that K_0 can be computed either in $\mathscr{F}\mathrm{rob}$ or in $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$, it follows that $\mathrm{K}^{\mathrm{S}}_{-n}(A) \cong \mathrm{K}_{-n}(A)$.

3.4 Counterexamples using non-stably coherent rings

While our proof of Theorem 3.6 uses crucially the hypothesis that the small stable ∞-category E has a bounded t-structure with noetherian heart, much of the proof works for a general E with any bounded t-structure. In particular, the existence of the sequence $\mathrm{K}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \to \mathrm{K}(\mathbb{A}^1, \mathcal{C}) \to \mathrm{K}(\mathbb{G}_m, \mathcal{C})$, where $\mathcal{C} = \mathrm{Ind}(E)$, exists

without any hypothesis on E except that it be small and stable, as does the fact that $\mathrm{K}(E)$ itself is a summand of $\mathrm{K}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$. The fact that $\mathrm{K}_{-1}(E) = 0$ whenever E admits a bounded t-structure provides additional strength to the assertion that $\mathrm{K}_{-n}(E)$ should be zero for all $n \geqslant 1$.

The noetherian hypothesis is used to prove that the t-structure of Lemma 3.13 on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$ restricts to a (bounded) t-structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$. This leads to the inductive step. One may ask if this is true in general, with a different proof. This is not the case.

Let R be an ordinary ring. A finitely presented right R-module M is coherent if every finitely generated submodule $N \subseteq M$ is finitely presented. The ring R is **right coherent** if R is coherent as a right R-module. We say that R is **right regular** if every finitely presented right R-module has finite projective dimension. Finally, we call R **right regular coherent** if it is both right coherent and right regular.

Theorem 3.30. *If R is a right regular coherent ring, then $\mathrm{K}_{-1}(R) = 0$.*

Proof. The right coherence of R means that the category $\mathrm{Mod}_R^{\heartsuit,\omega}$ of finitely presented R-modules is abelian and hence that $\mathcal{D}^b(R) = \mathcal{D}^b(\mathrm{Mod}_R^{\heartsuit,\omega})$ is a well-defined small stable presentable ∞-category. The right regularity of R means that the natural map $\mathrm{Mod}_R^\omega \to \mathcal{D}^b(R)$ is an equivalence. Hence, $\mathrm{K}(R) \simeq \mathrm{K}(\mathcal{D}^b(R))$. We can conclude in either of two ways. We can appeal to Theorem 2.35 using that the equivalence induces a bounded t-structure on Mod_R^ω. Or, we can use that $\mathrm{K}_{-1}(\mathcal{D}^b(R)) = 0$, as proved by Schlichting [Sch06, Theorem 6]. □

Note that if R is an ordinary ring, then $\mathcal{D}(\mathbb{G}_m, \mathrm{Mod}_R) \simeq \mathrm{Mod}_{R[s^{\pm 1}]}$ and the t-structure on $\mathcal{D}(\mathbb{G}_m, \mathrm{Mod}_R)$ induced by that on Mod_R via Lemma 3.13 agrees with the standard t-structure on $\mathrm{Mod}_{R[s^{\pm 1}]}$.

Proposition 3.31. *Suppose that R is right regular coherent but that $R[s^{\pm 1}]$ is not right coherent. Then, the t-structure on $\mathrm{Mod}_{R[s^{\pm 1}]}$ does not restrict to a t-structure on compact objects.*

Proof. Note that $R[s^{\pm 1}]$ is right regular. Let $I \subseteq R[s^{\pm 1}]$ be a finitely generated ideal that is not finitely presented, and let $P : R[s^{\pm 1}]^n \to R[s^{\pm 1}]$ be a chain complex in degrees 1 and 0, with $d(R[s^{\pm 1}]^n) = I$. By our choice of I, $\mathrm{H}_1(P)$ is not a finitely generated $R[s^{\pm 1}]$-module. If the t-structure on $\mathrm{Mod}_{R[s^{\pm 1}]}$ restricted to a t-structure on the compact objects, then $\mathrm{H}_1(P)$ would have to be perfect, and hence finitely presented, a contradiction. □

Example 3.32. Glaz presents an example of Soublin in [Gla89, Example 7.3.13] showing that the ring

$$R = \prod_{m \in \mathbb{Z}} \mathbb{Q}[\![x, y]\!]$$

is right regular coherent, but that $R[s]$ is not coherent. By [Gla89, Theorem 8.2.4(2)], $R[s^{\pm 1}]$ is not coherent either.

This example implies that the strategy used in the previous section to prove Conjecture B in the case of a noetherian heart cannot work for general small stable ∞-categories E equipped with bounded t-structures.

Another strategy is to replace $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ by some stable ∞-category that does have a bounded t-structure. For example, one can consider the abelian closure $\widetilde{\mathrm{coh}}(\mathbb{A}^1, \mathcal{C})$ in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$ of the additive category consisting of $\pi_n x$ as x ranges over all compact objects of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$. Let $\mathcal{D}^b_{\widetilde{\mathrm{coh}}}(\mathbb{A}^1, \mathcal{C}) \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ be the full subcategory of bounded objects x such that $\pi_n x \in \widetilde{\mathrm{coh}}(\mathbb{A}^1, \mathcal{C})$ for all n. This is a small stable ∞-category and the induced t-structure is bounded.

Now, consider the localization sequence

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \cap \mathcal{D}^b_{\widetilde{\mathrm{coh}}}(\mathbb{A}^1, \mathcal{C}) \to \mathcal{D}^b_{\widetilde{\mathrm{coh}}}(\mathbb{A}^1, \mathcal{C}) \to \left(\mathcal{D}^b_{\widetilde{\mathrm{coh}}}(\mathbb{A}^1, \mathcal{C}) / \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \cap \mathcal{D}^b_{\widetilde{\mathrm{coh}}}(\mathbb{A}^1, \mathcal{C})\right)^\sim.$$

Let $\mathcal{D}^b_{\{0\},\widetilde{\mathrm{coh}}}(\mathbb{A}^1,\mathcal{C})$ denote the left-hand side. For this to play the role of the localization sequence $\mathcal{D}_{\{0\}}(\mathbb{A}^1,\mathcal{C})^\omega \to \mathcal{D}(\mathbb{A}^1,\mathcal{C})^\omega \to \mathcal{D}(\mathbb{G}_m,\mathcal{C})^\omega$, we need to guarantee two things:

(1) $\mathrm{K}(E)$ is a summand of $\mathrm{K}(\mathcal{D}^b_{\{0\},\widetilde{\mathrm{coh}}}(\mathbb{A}^1,\mathcal{C}))$;

(2) the abelian subcategory
$$\mathcal{D}^b_{\{0\},\widetilde{\mathrm{coh}}}(\mathbb{A}^1,\mathcal{C})^\heartsuit \subseteq \mathcal{D}^b_{\widetilde{\mathrm{coh}}}(\mathbb{A}^1,\mathcal{C})^\heartsuit$$
is Serre.

Condition (2) is easier and is in fact always true. Condition (1) would follow if x is compact when (x,e) is an object of $\mathcal{D}^b_{\{0\},\widetilde{\mathrm{coh}}}(\mathbb{A}^1,\mathcal{C})$.

3.5 Stable coherence

The next theorem was known to Bass and Gersten.

Theorem 3.33. *Suppose that R is a right regular coherent ring such that $R[s_1,\dots,s_n]$ is right coherent. Then, $\mathrm{K}_{-n-1}(R)=0$.*

Proof. It is enough to note that in this case $R[s_1^{\pm1},\dots,s_n^{\pm1}]$ is right coherent by [Gla89, Theorem 8.2.4(2)]. Hence, $\mathrm{K}_{-1}(R[s_1^{\pm1},\dots,s_n^{\pm1}])=0$. As $\mathrm{K}_{-n-1}(R)$ is a subquotient of $\mathrm{K}_{-1}(R[s_1^{\pm1},\dots,s_n^{\pm1}])$, the result follows from Theorem 3.30. □

The classical proof (due to Bass [Bas73, Section 2]) uses a specific inductive presentation of $\mathrm{K}_{-n-1}(R)$, namely as the cokernel of
$$\mathrm{K}_{-n}(R[s])\oplus\mathrm{K}_{-n}(R[s^{-1}])\to\mathrm{K}_{-n}(R[s^{\pm1}]).$$
(See [TT90, Section 6].) In particular, $\mathrm{K}_{-n-1}(R)$ is a quotient of $\mathrm{K}_{-1}(R[s_1^{\pm1},\dots,s_n^{\pm1}])$, which vanishes if $R[s_1^{\pm1},\dots,s_n^{\pm1}]$ is right regular coherent.

One reason to prefer our proof is that it extends immediately to small abelian categories A such that $A[s_1,\dots,s_n]$ is abelian, with notation as in the proof of Proposition 3.14. We know of no analogous general result using Bass' methods in the literature. We restate this result separately.

Theorem 3.34. *Let A be a small abelian category such that $A[s_1,\dots,s_n]$ is abelian. Then, $\mathrm{K}_{-n-1}(A)=0$.*

It seems that Schlichting's paper is very close to establishing a result like this. However, the proof given of [Sch06, Lemma 8] relies on the structure of injective modules in a noetherian abelian category to establish the long exact sequence in K-groups allowing one to conclude that that $\mathrm{K}_{-n-1}(A)$ is a subquotient of $\mathrm{K}_{-1}(A[s_1^{\pm1},\dots,s_n^{\pm1}])$. So, that paper allows one to prove the theorem for A noetherian ([Sch06, Remark 7]), but not in this stably coherent setting.

We close this section with a discussion relating these vanishing results and the K-theoretic Farrell-Jones conjecture. None of these results are new (as they all follow from Theorem 3.33), but it serves to illustrate the importance of Conjectures A, B, and C in the non-noetherian case.

The most interesting cases of the conjecture are when $R=\mathbb{Z}$ or when R is an arbitrary regular noetherian commutative ring. Farrell and Jones [FJ95] proved that $\mathrm{K}_{-n}(\mathbb{Z}[V])=0$ for $n\geqslant 2$. If the Farrell-Jones conjecture holds for G, then it follows from the homotopy colimit spectral sequence that $\mathrm{K}_{-n}(\mathbb{Z}[G])=0$ for $n\geqslant 2$ as well. In many cases it is suspected that $K_{-n}(R[G])=0$ for all $n\geqslant 1$. For example, this follows from the Farrell-Jones conjecture when the orders of all finite subgroups of G are invertible in R (see [LR05, Conjecture 79]). Our application to this problem is via a class of groups studied in this setting by Waldhausen [Wal78]. Say that a group G is **regular coherent** (resp. **noetherian**) if $R[G]$ is right regular coherent (resp. **noetherian**) for any regular noetherian commutative ring R.

Theorem 3.35. *Let R be a regular noetherian commutative ring, and let G be a regular coherent group. Then,* $\mathrm{K}_{-n}(R[G]) = 0$ *for* $n \geqslant 1$.

Proof. The key point is that $R[G][s] \cong (R[s])[G]$, so that under the hypotheses, $R[G]$ is stably coherent. The result follows from Theorem 3.33. □

The Farrell-Jones conjecture is known in many cases, and hence the vanishing of negative K-theory is known in many cases over the integers by the result of Farrell and Jones. For a table of known results on the Farrell-Jones conjecture, see [LR05, Section 2.6.3].

Example 3.36. Many groups are regular and coherent. The following list is transcribed from [Wal78]. The group G is regular coherent if it is (1) a free group, (2) a free abelian group, (3) a polycyclic group, (4) a torsion-free one-relator group, (5) a group of the form $\pi_1 M$ where M is a 2-manifold not homeomorphic to $\mathbb{RP}^2$, (6) a sufficiently large 3-manifold group, (7) a group of the form $\pi_1 M$ where M is a submanifold of S^3, (8) a subgroup of a group of one of the above types, or (9) a filtered colimit of inclusions thereof. In particular, for all of these groups, $\mathrm{K}_{-n}(R[G]) = 0$ for $n \geqslant 1$. Example (8) is particularly interesting as regular coherence passes to subgroups by [Wal78, Theorem 19.1] even though this is not known for the Farrell-Jones conjecture.

3.6 Serre cones of abelian categories

We have seen that the straightforward generalization of Schlichting's inductive strategy to prove vanishing of negative K-theory of noetherian abelian categories founders because of the failure of the Serre subcategory condition on the hearts, even though though the weak Serre subcategory condition always holds.

Part of the subtlety of Schlichting's conjecture is that the negative K-theory of an abelian category is defined using derived categories. To date, there is no definition internal to abelian categories. Given a small idempotent complete stable ∞-category E and an uncountable regular cardinal κ, let $\Sigma_\kappa(E)$ be the cofiber in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ fitting into the exact sequence

$$E \to \mathrm{Ind}(E)^\kappa \to \Sigma_\kappa(E).$$

This allows us to define negative K-theory inductively as $\mathrm{K}_{-n}(E) = \mathrm{K}_0(\Sigma_\kappa^{(n)}(E))$ since $\mathrm{K}(\mathrm{Ind}(E)^\kappa) \simeq 0$.

No similar construction is known for the negative K-theory of abelian categories. It would be possible given a positive answer to the next question.

Question 3.37. Suppose that A is a small abelian category. Does there exist an exact fully faithful inclusion of abelian categories $A \subseteq B$ such that $\mathrm{K}(B) \simeq 0$ and such that A is Serre in B?

For example, B might be closed under countable coproducts, which implies the K-acyclicity condition. One natural guess would be to take a category $\mathrm{Ind}(A)^\kappa$ of κ-compact objects for an uncountable cardinal κ. However, $A \subseteq \mathrm{Ind}(A)^\kappa$ is not typically Serre.

Example 3.38. Let R be a non-noetherian coherent ring, and let $\mathrm{coh}_R \subseteq \mathrm{Mod}_R^{\heartsuit,\kappa}$ be the full subcategory of coherent right R-modules inside all κ-compact R-modules (for some regular uncountable cardinal κ). Then, R itself has subobjects (specifically, non-finitely generated ideals) in $\mathrm{Mod}_R^{\heartsuit,\kappa}$ not contained in coh_R.

Remark 3.39. Jacob Lurie informed us that the previous example extends to say that it is not generally true that a small abelian category A admits a fully faithful exact inclusion $A \subseteq B$ where B is closed under countable coproducts and A is Serre inside of B. Indeed, if B has countable coproducts, then for any object M of B, the lattice $\mathrm{Sub}(M)$ of subobjects of M is closed under countable joins. This property will be true for any Serre subcategory. An example where is not true is as follows. Consider the category coh_R of coherent modules for $R = k[x_1, x_2, \ldots]$, the polynomial ring on countably many variables over some field k. This ring

is coherent but not noetherian, so that coh_R is abelian. However, the union of the ideals $(x_1) \subseteq (x_1, x_2) \subseteq \cdots$ is not coherent (or even finitely generated). So, the lattice of coherent subobjects of R is not closed under countable joins. In particular, there is no Serre embedding $\mathrm{coh}_R \subseteq B$ for any B closed under countable coproducts.

Proposition 3.40. *Conjecture C and a positive answer to Question 3.37 imply Conjectures A and B.*

Proof. For $n \geqslant 1$, let $A(n)$ denote the statement that $\mathrm{K}_{-n}(A) = 0$ for all small abelian categories A, and let $B(n)$ denote the statement that $\mathrm{K}_{-n}(E) = 0$ for every small stable ∞-category E with a bounded t-structure. Since we are assuming Conjecture C, $A(n)$ if and only if $B(n)$. So, assume $B(n)$ for some $n \geqslant 1$. It suffices to prove $A(n+1)$.

Let A be a small abelian category, and let $A \subseteq B$ denote the abelian category guaranteed by the hypothesis of the proposition. Conjecture C implies that $\mathrm{K}(A) = \mathrm{K}(\mathcal{D}^b(A)) \simeq \mathrm{K}(\mathcal{D}^b_A(B))$. (Note that even this special case of Conjecture C is open: see [Wei13, Open Problem V.5.3].) Consider the localization sequence

$$\mathcal{D}^b_A(B) \to \mathcal{D}^b(B) \to \left(\mathcal{D}^b(B)/\mathcal{D}^b_A(B)\right)^{\sim}.$$

By our choice of B, $\mathrm{K}(\mathcal{D}^b(B)) \simeq 0$, so

$$\mathrm{K}_{-n}((\mathcal{D}^b(B)/\mathcal{D}^b_A(B))^{\sim}) \to \mathrm{K}_{-n-1}(\mathcal{D}^b_A(B)) \cong \mathrm{K}_{-n-1}(A)$$

is surjective. The quotient $(\mathcal{D}^b(B)/\mathcal{D}^b_A(B))^{\sim}$ has a bounded t-structure by Proposition 2.20. Thus, by $B(n)$, $\mathrm{K}_{-n-1}(A) = 0$. □

Remark 3.41. As a final philosophical remark, note that negative K-theory exists because of the need to idempotent complete when constructing a localization of stable ∞-categories. Since abelian categories are idempotent complete, the sequences $A \to B \to B/A$ are already exact when $A \subseteq B$ is Serre. In particular, the induced map $\mathrm{K}_0(B) \to \mathrm{K}_0(B/A)$ is always surjective for such a localization sequence. It follows that, to the extent it exists along the lines of [BGT13], the universal localizing invariant *of abelian categories* should actually be connective K-theory. This does not imply Schlichting's conjecture by itself, but it would provide some evidence.

4 Some applications

We present here applications of the vanishing results above to the K-theory of dg algebras and of ring spectra.

4.1 Negative K-theory of dg algebras

Our first result has also been proved independently by Denis-Charles Cisinski in unpublished work.

Theorem 4.1. *Let k be a commutative ring, and let A be a cohomological dg k-algebra such that $\mathrm{H}^0(A)$ is semisimple, $\mathrm{H}^i(A)$ is a finitely generated right $\mathrm{H}^0(A)$-module for all i, and $\mathrm{H}^i(A) = 0$ for $i < 0$. Then, $\mathrm{K}_{-n}(A) = 0$ for $n \geqslant 1$.*

Proof. Keller and Nicolás prove in [KN13, Theorem 7.1] that under these hypotheses, Mod^{ω}_A admits a bounded t-structure whose heart is a length category. Recall that a length category is a small abelian category in which every object has finite length. In particular, it is noetherian. The result follows now from Theorem 3.18. □

Example 4.2. Suppose that k is a field and R is a noetherian local commutative k-algebra with maximal ideal $\mathfrak{m}$. Then, the derived endomorphism algebra A of $R/\mathfrak{m}$, which computes $\mathrm{Ext}^*_R(R/\mathfrak{m}, R/\mathfrak{m})$ satisfies the hypotheses of the theorem, and hence the negative K-theory of A vanishes. We can see this in another

way. Let $\mathcal{A} \subseteq \mathrm{Mod}_R^{\heartsuit,\omega}$ be the full subcategory of finitely presented R-modules supported set theoretically on $\operatorname{Spec} R/\mathfrak{m} \subseteq \operatorname{Spec} R$. Then, $\mathcal{D}^b(\mathcal{A})$ is a fully subcategory of $\mathcal{D}^b(R)$ and $R/\mathfrak{m}$ is a compact generator. Hence, $\mathrm{Mod}_A^{\omega} \simeq \mathcal{D}^b(\mathcal{A})$. So, since R is noetherian, $\mathcal{A}$ is noetherian, and the fact that $\mathrm{K}_{-n}(A) = 0$ for $n \geqslant 1$ follows from Schlichting's theorem.

Example 4.3. If k is a field and X is a smooth proper k-scheme, then the algebraic de Rham complex, which computes the algebraic de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X/k)$, satisfies the conditions of the theorem.

4.2 Negative K-theory of periodic and related ring spectra

Let R be a connective ring spectrum. A right R-module M is **π_*-finitely presented** if $\bigoplus_n \pi_n M$ is a finitely presented (right) $\pi_0 R$-module. In particular, this means that M is bounded and that each $\pi_n M$ is a finitely presented $\pi_0 R$-module. A discrete ring R is said to be **right noetherian** if every submodule of a finitely generated R-module is finitely generated. A connective ring spectrum R is **right noetherian** if $\pi_0 R$ is right noetherian and if $\pi_n R$ is finitely generated as a right $\pi_0 R$-module for all $n \in \mathbb{N}$.

Following [MR01], a discrete ring R is said to be **right regular** if every finitely generated discrete (right) R-module has finite projective dimension. A connective ring spectrum R will be said to be **right regular** if $\pi_0 R$ is right regular and if each π_*-finitely presented (right) R-module spectrum M is compact. A connective ring spectrum R will be called **right regular noetherian** if it is right noetherian and right regular.

For the purposes of this section, a map $R \to S$ of ring spectra will be called a **localization** if the induced map $\mathrm{Mod}_R \to \mathrm{Mod}_S$ is a localization with kernel generated by a compact object, or equivalently by a finite set of compact objects.

The next result extends those of Barwick and Lawson in [BL14].

Proposition 4.4. *Let R be a right regular noetherian ring spectrum. Suppose that $R \to S$ is a localization of R such that for a compact R-module M, $S \otimes_R M \simeq 0$ if and only if M is π_*-finitely presented. Then, there is a fiber sequence*

$$\mathrm{K}(\pi_0 R) \to \mathrm{K}(R) \to \mathrm{K}(S)$$

of nonconnective K-theory spectra.

Proof. Let $\mathrm{Mod}_R^{\pi_*\text{-fp}} \subseteq \mathrm{Mod}_R^{\omega}$ be the full subcategory of π_*-finitely presented R-modules. The localization theorem in algebraic K-theory gives a fiber sequence

$$\mathrm{K}(\mathrm{Mod}_R^{\pi_*\text{-fp}}) \to \mathrm{K}(R) \to \mathrm{K}(S).$$

Since R is connective, there is a bounded t-structure on $\mathrm{Mod}_R^{\pi_*\text{-fp}}$ with noetherian heart (the category of finitely presented discrete right R-modules). The result follows from Theorem 3.18. □

Corollary 4.5. *If R is a right regular noetherian ring spectrum and $R \to S$ is a localization of R such that for a compact R-module M, $S \otimes_R M \simeq 0$ if and only if M is π_*-finitely presented, then $\mathrm{K}_{-n}(S) = 0$ for all $n \geqslant 1$.*

Proof. Indeed, $\mathrm{K}_{-n}(\pi_0 R) = 0$ for $n \geqslant 1$ since R is right regular noetherian. Moreover, $\mathrm{K}_{-n}(R) \cong \mathrm{K}_{-n}(\pi_0 R)$ for $n \geqslant 1$ by [BGT13, Theorem 9.53]. □

There are many examples of regular ring spectra admitting localizations satisfying the condition of the theorem. The consequences for negative K-theory are new and require the methods of this paper.

Example 4.6. 1. If R is a ring spectrum with $\pi_* R \cong \pi_0 R[u]$ where $|u| = 2m > 0$ and $\pi_0 R$ is right regular noetherian, then $R \to R[u^{-1}]$ satisfies the conditions of the theorem. In particular, if S is an even periodic ring spectrum with $\pi_0 S$ right regular noetherian, then $\mathrm{K}_{-n}(S) = 0$ for $n \geqslant 1$.

2. In particular, $\mathrm{K}_{-n}(\mathrm{KU}) = 0$ for $n \geqslant 1$. This extends the theorem of Blumberg and Mandell [BM08].

3. Similarly, $\mathrm{K}_{-n}(\mathrm{E}_m) = 0$, $\mathrm{K}_{-n}(\mathrm{K}_m) = 0$, and $\mathrm{K}_{-n}(\mathrm{K}(m)) = 0$ for $n \geqslant 1$ and $m \geqslant 0$, where E_m is the Morava E-theory spectrum, K_m is the 2-periodic Morava K-theory spectrum, and $\mathrm{K}(m)$ is the $2(p^m - 1)$-periodic Morava K-theory spectrum.

4. Barwick and Lawson show in [BL14] that ko is right regular noetherian, and that $\mathrm{ko} \to \mathrm{KO}$ satisfies the hypothesis of the theorem. Hence, $\mathrm{K}_{-n}(\mathrm{KO}) = 0$ for $n \geqslant 1$.

5. They also show that tmf is right regular noetherian, and that $\mathrm{tmf} \to \mathrm{Tmf}$ satisfies the hypothesis of Proposition 4.4. Therefore, $\mathrm{K}_{-n}(\mathrm{Tmf}) = 0$ for $n \geqslant 1$.

Example 4.7. Not all periodic ring spectra concentrated in even degrees satisfy the hypotheses of Example 4.6(1). For example, the Johnson-Wilson theories $\mathrm{E}(m)$ with $m \geqslant 2$ have

$$\pi_*\mathrm{E}(m) = \mathbb{Z}_{(p)}[v_1, \dots, v_{m-1}, v_m^{\pm 1}],$$

where $|v_i| = 2p^i - 2$. Hence, they are periodic with period $2(p^m - 1)$, but they are not concentrated in multiples of this degree. We do not know if $\mathrm{K}_{-n}(\mathrm{E}(m)) = 0$ for $m \geqslant 2$ and $n \geqslant 1$.

4.3 Negative K-theory of cochain algebras

In a different direction, we consider cochain algebras.

Theorem 4.8. *Let X be a compact space and R a regular noetherian discrete commutative ring. There is an equivalence $\oplus_{x \in \pi_0 X} \mathrm{K}(R) \simeq \mathrm{K}(\mathrm{C}^*(X, R))$ of nonconnective K-theory spectra. In particular, $\mathrm{K}_{-n}(\mathrm{C}^*(X, R)) = 0$ for $n \geqslant 1$.*

Proof. It is enough to consider the case when X is connected, so that $X \simeq B\Omega X$. Let

$$\mathrm{Loc}_X(\mathrm{Mod}_R) \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Mod}_R) \simeq \mathrm{Mod}_{\mathrm{C}_*(\Omega X, R)}$$

be the ∞-category of local systems on X with coefficients in the stable ∞-category Mod_R of complexes of R-modules. Since the endomorphism algebra of the constant local system on R is $\mathrm{C}^*(X, R)$, there is a fully faithful functor

$$\mathrm{Mod}_{\mathrm{C}^*(X,R)} \to \mathrm{Mod}_{\mathrm{C}_*(\Omega X, R)}.$$

As R is connective, so is $\mathrm{C}_*(\Omega X, R)$, and hence there is an induced t-structure on $\mathrm{Loc}_X(\mathrm{Mod}_R)$.

If X is compact (in the ∞-category of spaces), then R is compact when viewed as a $\mathrm{C}_*(\Omega X, R)$-module (for example by [DGI06, Proposition 5.3]). But, R corresponds to $\mathrm{C}^*(X, R)$ under the functor above. It follows that $\mathrm{Mod}_{\mathrm{C}^*(X,R)} \to \mathrm{Loc}_X(\mathrm{Mod}_R)$ sends compact objects to bounded objects with respect to the t-structure on $\mathrm{Loc}_X(\mathrm{Mod}_R)$. Moreover, the t-structure restricts to a t-structure on $\mathrm{Mod}_{\mathrm{C}^*(X,R)}$ by Mathew's description [Mat16, Proposition 7.8] of the essential image as the ind-unipotent modules over $\mathrm{C}_*(\Omega X, R)$, a condition which depends only on the action of $\pi_1 X$ on the homotopy groups of the R-module of the underlying local system. Hence, $\mathrm{Mod}^{\omega}_{\mathrm{C}^*(X,R)}$ has a bounded t-structure, with heart easily seen to be the abelian category of finitely presented R-modules.

The theorem now follows immediately from the nonconnective theorem of the heart (Theorem 3.18) and the fact that $\mathrm{K}_{-n}(R) = 0$ for $n \geqslant 1$. □

A Frobenius nerves

We examine an ∞-categorical model of the stable category of a Frobenius category. This material is used in the main body of the paper to verify that Schlichting's definition of the negative K-theory of a small abelian category A agrees with the negative K-theory of the small stable ∞-category $\mathcal{D}^b(A)$, as defined in [BGT13].

Let $\mathcal{E}$ be a small exact category in the sense of Quillen [Qui73]. We will identify $\mathcal{E}$ with a full subcategory of $\mathcal{A} = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Mod}^{\heartsuit}_{\mathbb{Z}})$, the category of left exact additive functors $\mathcal{E}^{\mathrm{op}} \to \mathrm{Mod}^{\heartsuit}_{\mathbb{Z}}$. The (Yoneda) embedding $\mathcal{E} \to \mathcal{A}$ is exact and reflects exactness. Moreover, $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. If, additionally, $\mathcal{E}$ is idempotent complete, then $\mathcal{E}$ is closed under taking kernels of epimorphisms in $\mathcal{A}$. See [TT90, Proposition A.7.16]. In other words, $\mathcal{E}$ satisfies hypothesis [TT90, 1.11.3.1], the key assumption needed for the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7]. We refer to [TT90, Appendix A] in general for details about the Gabriel-Quillen embedding.

Mimicking the definitions in an abelian category, we say that an object P of $\mathcal{E}$ is **projective** if for every admissible epi $M \twoheadrightarrow N$ the induced map $\mathrm{Hom}_{\mathcal{E}}(P, M) \to \mathrm{Hom}_{\mathcal{E}}(P, N)$ is surjective. Dually, an object I of $\mathcal{E}$ is **injective** if $\mathrm{Hom}_{\mathcal{E}}(N, I) \to \mathrm{Hom}_{\mathcal{E}}(M, I)$ is surjective for every admissible mono $M \rightarrowtail N$ in $\mathcal{E}$.

We say that $\mathcal{E}$ has **enough projectives** if for every object M of $\mathcal{E}$ there is an admissible epi $P \twoheadrightarrow M$ where P is projective. Let $\mathcal{E}^{\mathrm{proj}}$ denote the full subcategory of projective objects of $\mathcal{E}$. Similarly, $\mathcal{E}$ has **enough injectives** if for every object M of $\mathcal{E}$ there is an admissible mono $M \rightarrowtail I$ where I is injective.

A **Frobenius category** is an exact category which has enough injectives and projectives and an object of $\mathcal{E}$ is projective if and only if it is injective.

Construction A.1. If $\mathcal{E}$ is a Frobenius category, the **stable category** $\underline{\mathcal{E}}$ of $\mathcal{E}$ has the same objects as $\mathcal{E}$ with morphisms $\mathrm{Hom}_{\underline{\mathcal{E}}}(M, N)$ the quotient of $\mathrm{Hom}_{\mathcal{E}}(M, N)$ by the subgroup of morphisms $f : M \to N$ factoring through a projective (or equivalently injective) object of $\mathcal{E}$.

Remark A.2. The stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$ is triangulated. This was first observed by Happel [Hap87, Theorem 9.4] following ideas of A. Heller [Hel60]. The loopspace of an object M is obtained by taking an admissible exact sequence $\Omega M \rightarrowtail P \twoheadrightarrow M$ with P projective. Then, ΩM is isomorphic to $M[-1]$ in $\underline{\mathcal{E}}$. We will write $\Omega_n M$ for the n-fold iteration $\Omega \cdots \Omega M$. Note that $\Omega_n M$ is not in general a well-defined endofunctor of $\mathcal{E}$, but that it defines an endofunctor of $\underline{\mathcal{E}}$.

Let $\mathcal{E}$ be an idempotent complete exact category. In this section, we will associate to $\mathcal{E}$ a stable ∞-category $\mathcal{D}_{\mathrm{sing}}(\mathcal{E})$, the singularity ∞-category of $\mathcal{E}$, and show that its homotopy category is naturally equivalent to $\underline{\mathcal{E}}$ when $\mathcal{E}$ is Frobenius.

A special case of such a construction can be extracted from Hovey [Hov99, Section 2.2]. A right noetherian ring R is **quasi-Frobenius** if R is injective as a right R-module. See [CR62, Section 58]. In this case, the category $\mathrm{Mod}^{\heartsuit}_R$ of right R-modules is Frobenius, and Hovey constructs a model category structure on $\mathrm{Mod}^{\heartsuit}_R$ whose homotopy category is equivalent to the stable category of $\mathrm{Mod}^{\heartsuit}_R$. Hovey's construction does not seem to generalize because a small Frobenius category need not embed into a Grothendieck abelian category which is also Frobenius. Specifically, there are examples where the Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$ does not preserve injectives. Hence, we take a different approach.

Example A.3. Let k be a field and let G be a locally finite group that is not finite (such as $\mathbb{Q}/\mathbb{Z}$). Then, $k[G]$ is not (right) self-injective by Renault [Ren71], so in particular $\mathrm{Mod}^{\heartsuit}_{k[G]}$ is not Frobenius. On the other hand, G is the filtered colimit of its finite subgroups, and hence $k[G]$ is the filtered colimit of Frobenius sub-algebras along flat transition maps. In particular, $k[G]$ is coherent, for example by [Gla89, Theorem 2.3.3]. It follows that the category of finitely presented (right) $k[G]$-modules is abelian. It is not hard to check that $k[G]$ is injective in $\mathrm{Mod}^{\heartsuit,\omega}_{k[G]}$, which shows that the category of finitely presented $k[G]$-modules is Frobenius.

Let $\mathcal{E}$ be an exact category. Let $\mathrm{Ch}^-(\mathcal{E})$ denote the category of bounded below chain complexes of objects in $\mathcal{E}$. This is a dg category and the dg nerve $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{E}))$ of [Lur12, Construction 1.3.1.6] is a stable ∞-category by [Lur12, Proposition 1.3.2.10]. The homotopy category of $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{E}))$ is the category of bounded chain complexes up to chain homotopy. For simplicity, we will write $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$ for $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{E}))$.

Definition A.4. A complex X in $\mathrm{Ch}^-(\mathcal{E})$ is **acyclic in degree** n if there is a factorization of the differential $X_n \leftarrow X_{n+1} : d_{n+1}$ into

$$X_n \leftarrowtail Z_n \twoheadleftarrow X_{n+1}$$

where $X_n \leftarrowtail Z_n$ is an admissible mono and a kernel for d_n and $X_{n+1} \twoheadrightarrow Z_n$ is an admissible epi and a cokernel for d_{n+2}. The complex X is **acyclic** if it is acyclic in degree n for all $n \in \mathbb{Z}$.

Consider the following full stable subcategories of $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$:

(i) $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$, the dg nerve of the dg category of bounded complexes in $\mathcal{E}$;

(ii) $\mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E})$ and $\mathrm{Ac}^b_{\mathrm{dg}}(\mathcal{E})$, the dg nerve of the dg category of acyclic bounded below and bounded complexes, respectively, in $\mathcal{E}$;

(iii) $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E})$ the dg nerve of the dg category of bounded below complexes in $\mathcal{E}$ which are acyclic in all sufficiently high degrees.

Remark A.5. If $\mathcal{E} \subseteq \mathcal{F}$ is fully faithful, then $\mathrm{Ch}^-(\mathcal{E}) \to \mathrm{Ch}^-(\mathcal{F})$ is fully faithful, which leads to fully faithful maps between all of the subcategories above.

Lemma A.6. *Let $\mathcal{E}$ be an idempotent complete exact category.*

(a) *The stable ∞-categories $\mathrm{Ac}^b_{\mathrm{dg}}(\mathcal{E})$ and $\mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E})$ are idempotent complete in $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$ and $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$, respectively.*

(b) *Any chain complex in $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$ chain homotopy equivalent to an acyclic chain complex is itself acyclic. In other words, $\mathrm{Ac}^b_{\mathrm{dg}}(\mathcal{E})$ and $\mathrm{Ac}^-_{\mathrm{dg}}$ are closed under equivalence in $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$ and $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$, respectively.*

Proof. Let $\mathcal{E} \to \mathcal{A}$ denote the Gabriel-Quillen embedding. We prove first that if $X \in \mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$, then X is acyclic if and only if $\mathrm{H}_*(X) = 0$ when X is viewed as a complex of objects in $\mathcal{A}$. If X is acyclic, then $\mathrm{H}_*(X) = 0$ by definition. We can suppose that X is of the form $0 \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$. Since $\mathrm{H}_*(X) = 0$, it follows that $X_0 \leftarrow X_1$ is surjective. Since $\mathcal{E}$ is idempotent complete, it is closed under kernels of admissible epimorphisms in $\mathcal{A}$. Hence, there is a factorization $X_1 \leftarrowtail Z_1 \twoheadleftarrow X_2$ where Z_1 is the kernel of $X_0 \leftarrow X_1$ and the cokernel of $X_1 \leftarrow X_2$. By induction, the claim follows.

Now, part (a) follows immediately. Indeed, if $X \simeq Y \oplus Z$ in $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$ (resp. $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$) where X is acyclic, then $\mathrm{H}_*(X) = 0$, so $\mathrm{H}_*(Y) = \mathrm{H}_*(Z) = 0$. So, Y and Z are bounded (resp. bounded below) complexes with vanishing homology. By the previous paragraph, they are acyclic.

Part (b) follows as well, since if $X \to Y$ is an equivalence in $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$ with X acyclic, we find that $\mathrm{H}_*(Y) = 0$, so Y is acyclic. □

Definition A.7. Let $\mathcal{E}$ be an idempotent complete exact category. The **bounded derived ∞-category** $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$ of $\mathcal{E}$ is the Verdier quotient

$$\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})/\mathrm{Ac}^b_{\mathrm{dg}}(\mathcal{E}).$$

The homotopy category of $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$ is equivalent to the usual bounded derived category of $\mathcal{E}$. Similarly, the **bounded below derived ∞-category** $\mathcal{D}^-_{\mathrm{dg}}(\mathcal{E})$ is the Verdier quotient

$$\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})/\mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E}),$$

while the **homologically bounded derived ∞-category** $\mathcal{D}^{-,b}_{\mathrm{dg}}(\mathcal{E})$ is the Verdier quotient

$$\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E})/\mathrm{Ac}^{-}_{\mathrm{dg}}(\mathcal{E}).$$

A map in $\mathrm{Ch}^{-}_{\mathrm{dg}}(\mathcal{E})$ which is an equivalence in any of these derived ∞-categories is called a **quasi-isomorphism**. By Lemma A.6(b), these are precisely the maps in $\mathrm{Ch}^{-}_{\mathrm{dg}}(\mathcal{E})$ whose cones are acyclic.

Remark A.8. If $\mathcal{A}$ is a small abelian category viewed as an exact category in the usual way, then $\mathcal{D}^{b}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}^{b}(\mathcal{A})$, where $\mathcal{D}^{b}(\mathcal{A})$ is defined as in Section 3.3 as $\check{\mathcal{D}}(\mathrm{Ind}(\mathcal{A}))^{\omega}$.

Proposition A.9. *The natural functor $\mathcal{D}^{b}_{\mathrm{dg}}(\mathcal{E}) \to \mathcal{D}^{-,b}_{\mathrm{dg}}(\mathcal{E})$ is an equivalence and the natural functor $\mathcal{D}^{-,b}_{\mathrm{dg}}(\mathcal{E}) \to \mathcal{D}^{-}_{\mathrm{dg}}(\mathcal{E})$ is fully faithful.*

Proof. The second functor is trivially fully faithful since $\mathrm{Ac}^{-}_{\mathrm{dg}}(\mathcal{E}) \subseteq \mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E})$. We prove that the composition is fully faithful. For this, it suffices to verify Verdier's criterion [Ver96, Proposition II.2.3.5]. Thus, if $f : M \to X$ is a map in $\mathrm{Ch}^{-}_{\mathrm{dg}}(\mathcal{E})$ with M in $\mathrm{Ac}^{-}_{\mathrm{dg}}(\mathcal{E})$ and X in $\mathrm{Ch}^{b}_{\mathrm{dg}}(\mathcal{E})$, we show that f factors through $M \to M'$ where M' is in $\mathrm{Ac}^{b}_{\mathrm{dg}}(\mathcal{E})$. Choose n such that $X_i = 0$ for $i \geqslant n$. Since M is acyclic, the good truncation $\tau_{\leqslant n}M$ exists in $\mathrm{Ch}^{b}_{\mathrm{dg}}(\mathcal{E})$ and is also acyclic. The map $M \to X$ factors through $M \to \tau_{\leqslant n}M$ since $X_n = 0$.

To see essential surjectivity, let X be in $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E})$ and choose n such that X is acyclic in degrees n and higher. Then, the good truncation $\tau_{\leqslant n}X$ exists in $\mathrm{Ch}^{b}_{\mathrm{dg}}(\mathcal{E})$ and $X \to \tau_{\leqslant n}X$ is a quasi-isomorphism because the cone has zero homology and is hence acyclic by the argument in the proof of Lemma A.6. □

Theorem A.10 (Balmer-Schlichting [BS01, Theorem 2.8])**.** *If $\mathcal{E}$ is idempotent complete, then the derived ∞-category $\mathcal{D}^{b}_{\mathrm{dg}}(\mathcal{E})$ is idempotent complete.*

Proof. This can be checked on the homotopy category, which is done in [BS01]. □

Lemma A.11. *Any complex P in $\mathrm{Ac}^{-}_{\mathrm{dg}}(\mathcal{E}) \cap \mathrm{Ch}^{-}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ is contractible.*

Proof. This follows immediately from the projectivity of the terms of P. □

Remark A.12. It follows that $\mathrm{Ch}^{b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \simeq \mathcal{D}^{b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ and $\mathrm{Ch}^{-}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \simeq \mathcal{D}^{-}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$, since the acyclic complexes are already equivalent to zero in $\mathrm{Ch}^{b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$.

Corollary A.13. *The stable ∞-category $\mathrm{Ch}^{b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ is idempotent complete.*

Proof. This is a special case of Theorem A.10. □

Lemma A.14. *If X is in $\mathrm{Ac}^{-}_{\mathrm{dg}}(\mathcal{E})$ and P is in $\mathrm{Ch}^{-}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$, then any map $f : P \to X$ is chain homotopic to zero.*

Proof. We assume that $P_n = 0$ for $n \leqslant -1$. Let $s_n : P_n \to X_{n+1}$ be the zero map for $n \leqslant -1$. Assume that s_n has been constructed for $n \leqslant N-1$ such that $f_i = d^X_{i+1} \circ s_i + s_{i-1} \circ d^P_i$ for $i \leqslant N-1$. Then,

$$\begin{aligned} d^X_N \circ (f_N - s_{N-1} \circ d^P_N) &= d^X_{N-1} \circ f_N - d^X_n \circ s_{N-1} \circ d^P_N \\ &= d^X_{N-1} \circ f_N - (f_{N-1} - s_{N-2} \circ d^P_{N-1}) \circ d^P_N \\ &= d^X_{N-1} \circ f_N - f_{N-1} \circ d^p_N \\ &= 0 \end{aligned}$$

since f is a map of chain complexes. It follows from the acyclicity of X that $f_N - s_{N-1} \circ d^p_N$ factors through the admissible mono $Z_N \rightarrowtail X_N$. Since $X_{N+1} \to Z_N$ is an admissible epi, there is a lift of $f_N - s_{N-1} \circ d^p_N$ to $s_N : P_N \to X_{N+1}$ such that $d^X_{N+1} \circ s_N = f_N - s_{N-1} \circ d^P_N$. By induction, this proves the existence of a contracting homotopy for f. □

Proposition A.15. *The functors* $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \to \mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$ *and* $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \to \mathcal{D}^-_{\mathrm{dg}}(\mathcal{E})$ *are fully faithful.*

Proof. We use Verdier's criterion [Ver96, Proposition II.2.3.5], which says in our case that if every map $P \to X$ with P in $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ and X in $\mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E})$ factors through a map $X' \to X$ where X' is in $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \cap \mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E})$, then

$$\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})/\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \cap \mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E}) \to \mathcal{D}^-_{\mathrm{dg}}(\mathcal{E})$$

is fully faithful. But, Lemma A.14 says that in fact every such map factors through zero, so the criterion is satisfied. On the other hand, Lemma A.11 says every complex in $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \cap \mathrm{Ac}^-_{\mathrm{dg}}(\mathcal{E})$ is already equivalent to zero, so that the conclusion of Verdier's criterion reduces to the statement of the proposition. The bounded case is similar, or follows from the fully faithfulness of $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \to \mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ and $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E}) \to \mathcal{D}^-_{\mathrm{dg}}(\mathcal{E})$. □

Corollary A.16. *The natural functor* $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \to \mathcal{D}^-_{\mathrm{dg}}(\mathcal{E})$ *is an equivalence.*

Proof. Thanks to the previous proposition it suffices to check essential surjectivity, which follows by taking projective resolutions. □

Definition A.17. Let $\mathcal{E}$ be an idempotent complete exact category. The **singularity ∞-category** $\mathcal{D}_{\mathrm{sing}}(\mathcal{E})$ of $\mathcal{E}$ is the Verdier quotient

$$\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})/\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}).$$

We will write $\mathcal{D}_{\mathrm{sing}}$ for the induced functor from the ∞-category of exact categories and exact functors to $\mathrm{Cat}^{\mathrm{perf}}_\infty$.

Definition A.18. Syzygys play a crucial role in the proof of the next theorem. Let X in $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E})$ be acyclic in degree $n-1$. Then, the nth syzygy $\Omega_n X$ is an object of $\mathcal{E}$, being the kernel of $d_{n-1} : X_{n-1} \to X_{n-2}$. Moreover, in this case, the brutal truncation $\sigma_{\geqslant n} X$ admits a canonical map to $\Omega_n X[n]$. When X is acyclic, $\sigma_{\geqslant n} X \to \Omega_n X[n]$ is a quasi-isomorphism. Finally, if X is a complex of projectives which is acyclic in degree i for $i \geqslant n-1$, then $\Omega_i X \cong \Omega_{i-n}\Omega_n X$ in $\underline{\mathcal{E}}$.

Theorem A.19. *There is a natural equivalence* $\mathrm{Ho}(\mathcal{D}_{\mathrm{sing}}(\mathcal{E})) \simeq \underline{\mathcal{E}}$ *when* $\mathcal{E}$ *is an idempotent complete Frobenius category.*

Proof. The proof of this theorem is due in spirit to Buchweitz [Buc86], though only a special case is given there. For simplicity, we avoid the comparison with the homotopy category of acyclic complexes of projectives, instead giving a direct argument for the equivalence.

There is an evident composition of functors $\mathrm{N}(\mathcal{E}) \to \mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}) \to \mathcal{D}^b_{\mathrm{dg}}(\mathcal{E}) \to \mathcal{D}_{\mathrm{sing}}(E)$, where the first is given by viewing an object of $\mathcal{E}$ as a chain complex concentrated in degree zero. This first functor is evidently fully faithful. The second and third functors are the Verdier quotient functors.

Let $\mathrm{Ch}^{-,b}(\mathcal{E}^{\mathrm{proj}})$ be the full subcategory of $\mathrm{Ch}^-(\mathcal{E}^{\mathrm{proj}})$ consisting of **homologically bounded complexes of projectives**, i.e., those complexes which are acyclic when viewed in $\mathrm{Ch}^-(\mathcal{E})$ in all sufficiently high degrees. It is clear that the natural functor $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \to \mathcal{D}^-(\mathcal{E})$ induces an equivalence $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \simeq \mathcal{D}^{-,b}(\mathcal{E})$. Hence, there are equivalences

$$\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})/\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \simeq \mathcal{D}^{-,b}_{\mathrm{dg}}(\mathcal{E})/\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \simeq \mathcal{D}_{\mathrm{sing}}(\mathcal{E}).$$

We are therefore free in our arguments to replace bounded complexes in $\mathcal{E}$ with homologically bounded complexes of projectives.

We claim first that the functor $\underline{\mathcal{E}} \to \mathcal{D}_{\mathrm{sing}}(\mathcal{E})$ is essentially surjective. Pick a complex X of $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$, and let $P \to X$ be a quasi-isomorphism where P is a bounded below complex of projectives. Choose $n \geqslant 0$ sufficiently large so that P is acyclic in degree i for all $i \geqslant n$. In this case, the brutal truncation $\sigma_{\geqslant i}P$ admits a quasi-isomorphism $\sigma_{\geqslant i}P \to \Omega_i P[i]$ for all $i > n$, where $\Omega_i P$ is some object of $\mathcal{E}$. Fix $i > n$ and extend $\sigma_{\geqslant i}P$ to an unbounded acyclic complex Q of projectives, by taking a projective co-resolution of $\Omega_i P$ (which exists because $\mathcal{E}$ is Frobenius). There is a diagram of morphisms

$$X \leftarrow P \to \sigma_{\geqslant i}P = \sigma_{\geqslant i}Q \leftarrow \sigma_{\geqslant 0}Q \to \Omega_0 Q$$

in $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E})$. The outside arrows are quasi-isomorphisms and hence already equivalences in $\mathcal{D}^{-,b}_{\mathrm{dg}}(\mathcal{E})$. The inside arrows have cones in $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$, and hence they become equivalences in $\mathcal{D}_{\mathrm{sing}}(\mathcal{E})$. But, this shows that $X \simeq \Omega_0 Q$ in $\mathcal{D}^{-,b}_{\mathrm{dg}}(\mathcal{E})/\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \simeq \mathcal{D}_{\mathrm{sing}}(\mathcal{E})$.

To finish the proof, it is enough to prove that $\underline{\mathcal{E}} \to \mathrm{Ho}(\mathcal{D}_{\mathrm{sing}}(\mathcal{E}))$ is fully faithful. We check faithfulness and fullness separately. Let M and N be objects of $\mathcal{E}$. Since $\mathrm{Ch}^-_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}}) \to \mathcal{D}^-_{\mathrm{dg}}(\mathcal{E})$ is fully faithful, by replacing M and N by projective resolutions, we see that $\mathrm{Hom}_{\mathcal{E}}(M,N) \to \pi_0 \mathrm{Map}_{\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})}(M,N)$ is a bijection. Now, consider a diagram

$$M \xleftarrow{s} X \xrightarrow{f} N$$

in $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ with $\mathrm{cone}(s) \in \mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$. Then, $\Omega_n M \leftarrow \Omega_n X$ is an isomorphism up to projective summands for n sufficiently large, so there is an induced map $\Omega_n M \to \Omega_n N$. Since Ω_n is an autoequivalence of $\underline{\mathcal{E}}$, fullness follows.

To prove faithfulness, suppose that $f : M \to N$ maps to zero in $\pi_0 \mathrm{Map}_{\mathcal{D}_{\mathrm{sing}}}(M,N)$. Then, there is $X \xrightarrow{s} M$ such that $\mathrm{cone}(s)$ is quasi-isomorphic to an object of $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$ and $f \circ s$ is zero in $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$. Working with bounded below complexes of projectives, we can assume in fact that $f \circ s$ is nullhomotopic in $\mathrm{Ch}^{-,b}_{\mathrm{dg}}(\mathcal{E}^{\mathrm{proj}})$. In this case, $M \to N$ factors through $X \to \mathrm{cone}(s)$. A sufficiently high syzygy of $\mathrm{cone}(s)$ is projective, so this means that $\Omega_n f$ factors through a projective, and hence is zero in $\underline{\mathcal{E}}$. Again using that Ω_n is an autoequivalence, we find that $f = 0$ in $\underline{\mathcal{E}}$, as desired. □

Example A.20. In general $\mathcal{D}_{\mathrm{sing}}(\mathcal{E})$ is not idempotent complete, and hence neither is $\underline{\mathcal{E}}$. It is enough to find a Gorenstein noetherian commutative ring R with $\mathrm{K}_{-1}(R) \neq 0$, since in this case there is an isomorphism

$$\mathrm{K}_0(\widetilde{\underline{\mathcal{E}}})/\mathrm{K}_0(\underline{\mathcal{E}}) \cong \mathrm{K}_{-1}(R)$$

(as $\mathrm{K}_{-1}(\mathcal{D}^b_{\mathrm{dg}}(R)) = 0$ by Schlichting's theorem). The complete intersection $R = \mathbb{Z}[x_0, x_1]/(x_0 x_1(1 - x_0 - x_1))$ works. In this case, $\mathrm{K}_{-1}(R) \cong \mathbb{Z}$, as can be checked from [Wei84].

Let R be a noetherian commutative ring. The abelian category $\mathrm{Mod}_R^{\heartsuit,\omega}$ of finitely presented discrete R-modules is exact, and its negative K-theory vanishes by Schlichting. Hence, $\mathrm{K}_{-n}(\mathcal{D}_{\mathrm{sing}}(\mathrm{Mod}_R^{\heartsuit,\omega})) \to \mathrm{K}_{-n-1}(R)$ is a surjection for $n \geqslant 0$ and an isomorphism for $n \geqslant 1$. In this way, the singularity category supports the negative K-theory of R and gives one measurement of the singularities of R itself.

When R is not noetherian, the question of whether or not this connection continues is precisely bound up in Schlichting's conjecture. For example, if R is merely coherent, then it is no longer known in general that $\mathrm{K}_{-n}(\mathcal{D}_{\mathrm{sing}}(\mathrm{Mod}_R^{\heartsuit,\omega})) \to \mathrm{K}_{-n-1}(R)$ is an isomorphism for $n \geqslant 1$. This would follow from Conjecture A.

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