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ASYMPTOTICALLY COMPATIBLE SPH-LIKE PARTICLE DISCRETIZATIONS OF ONE DIMENSIONAL LINEAR ADVECTION MODELS*

HWI LEE† AND QIANG DU†

Abstract. Motivated by the smoothed particle hydrodynamics (SPH), we present nonlocal models for linear advection with a variable coefficient in one spatial dimension together with their particle based numerical discretizations. We establish that these numerical methods are robust in the sense that they are convergent as the particle spacing and the smoothing length shrink to zero independently of each other. We demonstrate the important role of nonlocal continuum models to ensure the stability of our numerical methods. The nonlocal models constructed here follow two different strategies: the first model relies on choosing an upwind kernel and the second on introducing a nonlocal viscous term. We study discrete numerical schemes for both models that are in essence particle-like quadrature based finite differences, yet the distinction is clearly drawn in the sense that the scheme for the first model is based on the first moment of the nonlocal kernel while the other is conceived on the basis of renormalized SPH.

Key words. nonlocal models, particle methods, SPH, linear advection, asymptotically compatible scheme, peridynamics

AMS subject classifications. 45A05, 65M12, 65M75, 65R20, 76M28

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1. Introduction.

The smoothed particle hydrodynamics (SPH) has been a popular numerical method for solving problems in fluid dynamics involving complex flows. Originally conceived by Lucy [23] and Gingold and Monaghan [18], the SPH is a meshless Lagrangian numerical technique that is based on smoothing kernel approximations of functions and it has gained popularity due to its simplicity and efficiency. Since its invention the SPH has undergone numerous developments in various aspects, ranging from addressing the shortfalls of the SPH such as tensile instability [26] to the development of the variant of SPH and other related numerical methods such as the finite pointset method [20] and Voronoi-SPH method [2].

Despite the practical developments and theoretical analysis, there has been a limited amount of rigorous mathematical study of the SPH such as its convergence proof under realistic conditions. We point out an earlier result provided by Moussa and Vila [3] that proved the convergence of their SPH schemes to the scalar nonlinear conservation laws. This theoretical result has not only provided an interesting way to stabilize the SPH via introduction of Riemann problems but it has also highlighted one of the shortcomings of the SPH in terms of its consistency. That is, the convergence result in [3] requires the ratio of the particle spacing to the smoothing length of the SPH kernel to vanish to zero. This requirement has been numerically validated in a later work [34] and it is quite typical in the analysis of other particle-like methods. Such a requirement contradicts the often adopted practice of having the smoothing

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†Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027 (hl3001@columbia.edu, qd2125@columbia.edu).

length proportional to particle spacing to allow efficient evaluation of neighboring particle interactions.

Indeed the issues of stability and consistency have long been in the purview of SPH researchers [16, 32] and they are also what our present work is concerned with. In earlier works, we have connected SPH like approximations with nonlocal relaxations of conventional local conservation laws given by partial differential equations. In [13], this connection is mainly provided for a scalar diffusion model, while in [15], the focus is on a steady state linear Stokes system in multiple space dimensions. The latter offers a rigorous analysis of a nonlocal relaxation and asymptotically compatible discretization to the Stokes equation. In fluid mechanics, the advection phenomenon often plays an essential role. We thus are interested in how earlier works are applicable to the study of advection. The particular setting in which we begin our study is the following one dimensional linear advection equation:

$$
\begin{aligned}
u_t(x,t) + \frac{\partial}{\partial x}(c(x)u(x,t)) &= 0, \qquad x \in \mathbb{R}, t > 0, \\
u(x,0) &= u_0(x)\,.
\end{aligned} \tag{1.1}
$$

This is one of the simplest prototypical equations in the study of hyperbolic conservation laws and its simplicity allows us to present our ideas without too many technical complications. Moreover various nonlocal advection models and nonlocal convection and diffusion models have already been studied previously (see [12, 10, 27] and references cited therein) so it serves us particularly well to continue the investigation in this direction. Here, we first propose continuum nonlocal advection models from which we obtain SPH-like particle methods that are asymptotically compatible (AC) [29]; that is, they converge to the nonlocal models with a fixed smoothing length as the spatial discretization gets refined, while they converge to the local PDE as the smoothing length of the kernel and the spatial discretization parameter go to zero simultaneously in an *arbitrary* manner. Having the AC property is crucial for the robustness of the numerical discretization, especially when the number of surrounding interacting particles may not be able to keep increasing in practical applications. While the concept of AC scheme was originally motivated for the robust discretization of peridynamics models and nonlocal diffusions that use a horizon parameter to characterize the range of nonlocal interactions, there are striking similarities with the SPH-like particle discretization where nonlocal smoothing plays essential roles.

Let us note that a nonlocal model of general advection and diffusion processes can take on a form

$$
u_t(t,x) + \int_{\mathbb{R}} (\alpha'_\delta(x,x')u(t,x') - \alpha_\delta(x',x)u(t,x))dx' = 0
$$

for some nonlocal interaction kernels α_δ and α'_δ. The parameter δ, often called the horizon [9], is introduced to characterize the range of interactions. When the kernels α_δ and α'_δ represent jump rates, the nonlocal model can also be used to describe a Markovian jump process [11]. Upon suitable choices of the kernels, we can view the classical advection equation as the local limit of the above nonlocal model. In this work, we present two different constructions that can be localized to the same equation (1.1). The first is the model proposed in [27], which is in fact a nonlocal convection diffusion model. Their model constructs the convection term via the kernel which is not symmetric but biased toward the upwinding direction and this leads to the stability of the model in the sense of satisfying the maximum principle. In our first nonlocal model we will adopt a similar idea by encoding the upwinding in the kernel, but we will take a simpler approach by using kernels that take into account neighboring particles either to the left or the right of the particle depending only on the upwinding direction of that particle. Despite the simplification we show that a stable numerical scheme is obtained when we recast the model as a pairwise interaction model of [10] by introducing the first moment of the kernel. On the other hand, our second continuum nonlocal model is a bona fide SPH kernel approximation with an addition of nonlocal viscosity. We consider the viscosity in the framework of the nonlocal vector calculus [9], which in fact has already been studied in the context of peridynamics [28]. We adopt the idea of *vanishing viscosity* that vanishes in the limit of $\delta \to 0$, where δ is the smoothing length of the SPH kernel. The consistency of a corresponding numerical scheme is ensured by encapsulating the idea of renormalized SPH of Vila [32].

Our nonlocal continuum models are to be discretized on moving particles which may in general exhibit heterogeneous distributions at each instance of time. The AC property hence bears particular importance to our study, which is aimed at making the discretization more robust to the changes in modeling and discretization parameters. It should be enunciated that there exists a large body of literature related to the consistency and convergence property of SPH-like particle methods with different levels of mathematical rigor. Within the SPH community, the dependence of consistency of the SPH on the ratio of the particle spacing h to the smoothing length δ has been relaxed by the introduction of symmetric tensors that renormalize the discrete SPH approximations of the first [25] and second spatial derivatives [16]. The successful applications of the renormalized SPH in various settings [30, 17] motivate the natural implementation of the renormalization based on the numerical quadrature used in [10, 28] that can handle the extreme case of no neighboring particles within the horizon. The resulting outcome is our AC particle method corresponding to our second formulation of nonlocal models that eliminates any condition (including the relaxed condition) on the ratio $\frac{h}{\delta}$.

Meanwhile, there have been many developments of other particle methods such as the moving least squares methods [21, 6] and the reproducing kernel particle methods [22, 5] that have been designed to yield more accurate results than the SPH. Connections between SPH and the discrete peridynamics have also been explored in [4]. The underlying views adopted in those methods are nevertheless faithful to an interpretation of the SPH wherein each particle is associated with an interpolant or shape function, and there still is a scarcity of rigorous theoretical analysis of the AC property associated with these methods despite their illustrated effectiveness due to improved consistency. Instead of the particle based interpolant view, we propose a different perspective that utilizes a weighted summation of discrete difference operators, in the same spirit of the well-posed continuum formulation of the nonlocal operators, and with the weights computed analytically. We provide a mathematical justification to confirm the validity of our approach that yields AC particle methods for the first formulation of nonlocal models.

The main contribution of our work is that it is among the first, as far as we are aware, to propose and provide mathematical justifications of particle methods with the full force of the AC property. On one hand, our work can be seen as a extension of the existing AC numerical schemes developed in the setting of nonlocal models discretized on stationary uniform grids. On the other hand, our work suggests *a* possible way to bring in robustness to the SPH, or alike other deterministic particle methods. In doing so, our work highlights the importance of continuum nonlocal models as a bridge between continuum local PDE models and their numerical approximations.

This paper is organized as follows. We propose our first nonlocal advection model and its numerical discretization in section 2. We first present our numerical method on a stationary, general grid and its convergence analysis to the nonlocal model as well as the local PDE as the parameters in our discretization vanish. We then present our numerical schemes with moving particles and their convergence results. Section 3 is centered around our second nonlocal model and it is structured analogously to section 2. Some numerical tests and comparisons are presented in section 4. We then end with some discussions concerning our two approaches in section 5.

2. Model I: Biased interaction kernel. Our first approach is to look for a nonlocal model in which stabilization effect is provided by the kernel that has a built-in bias in deciding which neighboring particles to interact with. Such approaches in the context of nonlocal modeling have been studied previously; see, for example, [10] for one dimensional nonlinear nonlocal conservation laws and [27] for multidimensional linear convection diffusion equations.

2.1. Continuum formulation of Model I. In more specific terms, let us define the kernel

$$w_c^\delta(x,y) = \begin{cases} 1_{y<x}\eta^\delta(y-x) & \text{if } c(x) > 0, \\ 1_{y>x}\eta^\delta(y-x) & \text{otherwise,} \end{cases}$$

where $\eta^\delta(s) = \frac{1}{\delta^2}\eta(\frac{s}{\delta})$ is a scaled kernel for some odd function η. The function η is assumed to satisfy $\eta(z) \geq 0$ on $z \geq 0$ and to be supported on $(-1,1)$. Note the use of the subscript c to elucidate the dependence of the kernel on the velocity field c. We then propose the following nonlocal model (referred to as Model I from here on):

$$u_t(x,t) + \int_{\mathbb{R}} (c(y)u(y,t) - c(x)u(x,t))w_c^\delta(x,y)dy = 0\,, \qquad u(x,0) = u_0(x)\,. \tag{2.1}$$

In order to ensure (at least formal) consistency of Model I with the local PDE, we need to impose a suitable normalization condition on w_c^δ, and to this end fix x and assume without loss of generality $c(x) > 0$. Then we consider the nonlocal operator L^δ defined as

$$L^\delta(u)(x,t) = \int_0^\delta (c(x)u(x,t) - c(x-y)u(x-y,t))\eta^\delta(y)dy \tag{2.2}$$

so that we can rewrite (2.1) as

$$u_t + L^\delta(u) = 0\,.$$

Under the assumption of smooth u and c, direct calculation based on Taylor expansion shows that we need the following condition on the kernel:

$$\int_0^\delta y\eta^\delta(y)dy = 1, \tag{2.3}$$

which we assume from here on.

2.2. Numerical discretization of Model I. We are interested in discretization of (2.2) that is asymptotically compatible, so we follow the approach in [10]. Assuming $c(x) > 0$, we first rewrite the operator $L^\delta(u)(x,t)$,

$$L^\delta(u)(x,t) = \int_0^\delta \frac{c(x)u(x,t) - c(x-y)u(x-y,t)}{y} y\eta^\delta(y)dy,$$

from which we see that the operator L^δ represents a continuum of one-sided finite differences. In comparison with the pairwise interaction nonlocal model of [10], our Model I allows the particle at x_j to interact only with the neighboring paticles to its left. We are motivated to introduce this bias in our model based on the simple fact that the use of the upwinding flux in discretization of the advection PDE with a constant velocity brings about numerical stability *by default* (that is, without explicit consideration of how much numerical viscosity should be added). An extension of that simple idea is what we choose to study in the setting of nonlocal advection with variable speed.

Moving particles at each *fixed* time instant constitutes a grid that is in general irregular, so let us first consider the case of general stationary grid, denoted by $\{x_j\}_{j\in\mathbb{Z}}$. We propose the following discretization:

$$L_h^\delta(u)(x_j) = \sum_{k=1}^{L(j)} (c_j u_j - c_{j-k} u_{j-k}) W_{k,j}^L,$$

where

$$L(j) = \max\{\max\{l : 0 \le l, |x_j - x_{j-l}| \le \delta\}, 1\}$$

and

$$W_{k,j}^L = \frac{1}{x_j - x_{j-k}} \int_{x_j - x_{j-k+1}}^{x_j - x_{j-k}} y\eta^\delta(y)dy + \frac{\mathbf{1}_{k=L(j)}}{x_j - x_{j-k}} \int_{x_j - x_{j-k}}^{\delta} y\eta^\delta(y)dy.$$

For simplicity we adopt the forward Euler time stepping, which then yields the following fully explicit scheme:

$$u_j^{n+1} = u_j^n - \Delta t \sum_{k=1}^{L(j)} (c_j u_j^n - c_{j-k} u_{j-k}^n) W_{k,j}^L. \tag{2.4}$$

Remark 2.1. If the spacing of the grid points is fixed but $\delta \to 0$, then the scheme (2.4) reduces to a local finite difference scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{x_j - x_{j-1}} (c_j u_j^n - c_{j-1} u_{j-1}^n).$$

Remark 2.2. By construction $W_{k,j}^L$ are nonegative with a weighted sum satisfying the identity

$$\sum_{k=1}^{L(j)} (x_j - x_{j-k}) W_{k,j}^L = 1.$$

2.3. Convergence of the numerical scheme to Model I and its local limit. We are interested in the convergence of numerical solutions obtained by the numerical scheme (2.4) in two cases:

$$\begin{cases} \bullet\ \bar{h} \to 0 \text{ with } \delta > 0 \text{ fixed}, \\ \bullet\ \bar{h}, \delta \to 0 \text{ arbitrarily}, \end{cases} \tag{2.5}$$

where $\bar{h} = \sup_i |x_{i+1} - x_i|$. Since the scheme is linear, it suffices to establish consistency and stability. At the expense of assuming smooth solutions and data, we can show the convergence in the L^∞ norm.

Convergence to the nonlocal Model I. Let us first present a consistency estimate.

LEMMA 2.3 (consistency). *Suppose c, u, c', u_x are bounded. If c' and u_x are Lipscthiz, then*

$$\sup_j |L^\delta(u)(x_j,t) - L_h^\delta(u)(x_j,t)| \le C\bar{h}$$

for some constant C independent of δ.

Proof. Let us define

$$G(s) = \frac{c(x)u(x,t) - c(x-s,t)u(x-s,t)}{s}$$

and note that direct calculation shows the uniform boundedness of $G'(s)$. Then applying the argument of Lemma 5.1 in [28] yields the result. ∎

The next lemma is concerned with stability.

LEMMA 2.4 (L^∞-stability). *Assume c, c' are uniformly bounded. Provided that the following CFL condition holds,*

$$\frac{\Delta t \|c\|_{L^\infty(\mathbb{R})}}{\underline{h}} \le 1,$$

where $\underline{h} = \inf_i |x_{i+1} - x_i|$, then

$$\|u^{n+1}\|_{L^\infty(\mathbb{R})} \le \|u^n\|_{L^\infty(\mathbb{R})}(1 + \Delta t\|c'\|_{L^\infty(\mathbb{R})}) \quad \text{for all } n \ge 0.$$

Proof. We can rewrite the scheme as

$$u_j^{n+1} = (1 - \Delta t c_j \sum_k^{L(j)} W_{k,j}^L)u_j^n + \Delta t \sum_{k=1}^{L(j)} c_j W_{k,j}^L u_{j-k}^n + \Delta t \sum_{k=1}^{L(j)} (c_{j-k} - c_j) W_{k,j}^L u_{j-k}^n.$$

Then since

$$0 \le \Delta t \sum_{k=1}^{L(j)} c_j W_{k,j}^L \le \frac{\Delta t \|c\|_{L^\infty(\mathbb{R})}}{\underline{h}} \sum_{k=1}^{L(j)} (x_j - x_{j-k}) W_{k,j}^L = \frac{\Delta t \|c\|_{L^\infty(\mathbb{R})}}{\underline{h}}$$

and

$$\sum_{k=1}^{L(j)} \left|(c_{j-k} - c_j) W_{k,j}^L\right| \le \|c'\|_{L^\infty(\mathbb{R})} \sum_{k=1}^{L(j)} (x_j - x_{j-k}) W_{k,j}^L = \|c'\|_{L^\infty(\mathbb{R})},$$

the result follows. ∎

We note that one can immediately observe that the CFL condition in Lemma 2.4 is in fact that of *local* PDE. Moreover, if $c(\cdot) = $ constant, we indeed have the maximum principle $\|u^n\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}$.

Asymptotic convergence to the local advection equation (1.1). We note that the CFL condition in Lemma 2.4 is independent of δ. Hence if we choose the time steps Δt to satisfy that CFL condition, the scheme (2.4) is Lax–Richtmyer stable in L^∞ uniformly in δ. It only remains then to show consistency to the local PDE.

LEMMA 2.5 (consistency). *If $u, u_x, u_{xx}, c, c', c''$ are bounded, then*

$$\sup_j \left|L_h^\delta(u)(x_j,t) - (c(x)u(x,t))_x \,|_{x=x_j}\right| = O(\max(\bar{h}, \delta)).$$

Proof. Using Taylor series expansion, we have

$$c(x_j)u(x_j,t) - c(x_{j-k})u(x_{j-k},t) = (c(x)u(x,t))_x\big|_{x=x_j}(x_j - x_{j-k}) + O(|x_j - x_{j-k}|^2).$$

Now for $|x_j - x_{j-1}| > \delta$,

$$\begin{aligned}&\sum_{k=1}^{L(j)}(c(x_j)u(x_j,t) - c(x_{j-k})u(x_{j-k},t))W^L_{k,j}\\&\quad= (c(x)u(x,t))_x\mid_{x=x_j} + O(|x_j - x_{j-1}|)\int_0^\delta y\eta^\delta(y)dy\\&\quad= (c(x)u(x,t))_x\mid_{x=x_j} + O(|x_j - x_{j-1}|),\end{aligned}$$

whereas for $|x_j - x_{j-1}| \leq \delta$,

$$\begin{aligned}&\sum_{k=1}^{L(j)}(c(x_j)u(x_j,t) - c(x_{j-k})u(x_{j-k},t))W^L_{k,j}\\&\quad= (c(x)u(x,t))_x\mid_{x=x_j} + \sum_{k=1}^{L(j)} O(|x_j - x_{j-k}|)(x_j - x_{j+k})W^L_{k,j}\\&\quad= (c(x)u(x,t))_x\mid_{x=x_j} + O(\delta)\sum_{k=1}^{L(j)}(x_j - x_{j+k})W^L_{k,j} = (c(x)u(x))_x\mid_{x=x_j} + O(\delta).\end{aligned}$$

This leads to the conclusion of the lemma. ∎

In summary, we state the following.

THEOREM 2.6. *Let $T > 0$ be a fixed terminal time. Assume c and c' are bounded. Let U^n_j denote the numerical solution given by the scheme* (2.4) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition of Lemma* 2.4. *Let $u(x,t), u^\delta(x,t)$ denote the solutions of* (1.1) *and* (2.1), *respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.*

1. *Suppose c' is Lipschitz in $\mathbb{R}$. If u^δ_x is Lipschitz in $\mathbb{R}$ uniformly in $t \in [0,T]$ and $u^\delta, u^\delta_x, u^\delta_{tt} \in L^\infty(\mathbb{R}\times[0,T])$, then*
$$\sup_j |u^\delta(x_j,T) - U^N_j| \leq C_\delta(\bar{h} + \Delta t)$$
for some $C_\delta > 0$ depending on $\|\cdot\|_{L^\infty(\mathbb{R}\times[0,T])}$ norms of u^δ and u^δ_{tt}, and Lipschitz constant of u^δ_x.
2. *If $c'' \in L^\infty(\mathbb{R})$ and $u, u_x, u_{xx}, u_{tt} \in L^\infty(\mathbb{R}\times[0,T])$, then*
$$\sup_j |u(x_j,T) - U^N_j| = O(\max(\bar{h},\delta)) + O(\Delta t).$$

Proof. The proof follows from the Lax equivalence theorem. ∎

2.4. Moving mesh particle-like approximations. With our convergence results in the case of general nonuniform stationary particles, we can easily establish the corresponding results for moving particles. To this end, we first introduce moving coordinates $x(t)$ subject to some prescribed velocity field $b(x)$, which is not necessarily the underlying velocity field $c(x)$ in a similar spirit as in the arbitrary Euler–Lagrangian methods [19],

$$\dot{x}(t) = b(x(t)), \qquad x(0) = x_0. \tag{2.6}$$

The local PDE can then be rewritten in the Lagrangian frame

$$\frac{d}{dt}u(x(t),t) + b'(x(t))u(x(t),t) + (c'(x(t)) - b'(x(t)))u(x(t),t)) = 0 \tag{2.7}$$

so that our nonlocal integral relaxation of the the flux term yields

$$\begin{aligned}\frac{d}{dt}u(x(t),t) + b'(x(t))u(x(t),t) + \int_{x(t)-\delta}^{x(t)+\delta} ((c(y) - b(y))u(y,t) - \cdots \\ (c(x(t)) - b(x(t)))u(x(t),t))w^{\delta}_{c-b}(x(t),y)dy = 0.\end{aligned} \tag{2.8}$$

Note the dependence of the kernel on the *relative* velocity field $\hat{c} = c - b$. With $\hat{c}_{j,n} = \hat{c}(x_j(n\Delta t)) > 0$, we then obtain the following discretization:

$$u_j^{n+1} = u_j^n - \Delta t(b'_{j,n}u_j^n + \sum_{k=1}^{L(j)}(\hat{c}_{j,n}u_j^n - \hat{c}_{j-k,n}u_{j-k}^n)W^L_{k,j}). \tag{2.9}$$

We assume for simplicity that the initial distribution of particles is uniform with a spacing of size h. We also assume the velocity field b is smooth with a bounded first derivative, which implies that there exist constants C_1 and C_2 depending only on the terminal time T such that

$$C_1(T)h \leq |x_{j+1} - x_j| \leq C_2(T)h$$

for all $j \in \mathbb{Z}$ and $t \in [0,T]$ [24]. Then it is a straightforward extension of the results in subsection 2.3 to establish the following.

THEOREM 2.7. *Let $T > 0$ be a fixed terminal time. Assume c, c', b, b' are uniformly bounded. Let U_j^n denote the numerical solution given by the scheme* (2.9) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition*

$$\frac{\Delta t\|c - b\|_{L^\infty(\mathbb{R})}}{C_1(T)h} \leq 1.$$

Let $u(x,t), u^\delta(x,t)$ denote the solutions of (2.7) *and* (2.8), *respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.*

1. *Suppose c' and b' are Lipschitz in $\mathbb{R}$. If u_x^δ is Lipschitz in $\mathbb{R}$ uniformly in $t \in [0,T]$ and $u^\delta, u_x^\delta, u_{tt}^\delta \in L^\infty(\mathbb{R} \times [0,T])$, then*

$$\sup_j |u^\delta(x_j,T) - U_j^N| \leq C_\delta(\bar{h} + \Delta t)$$

   for some $C_\delta > 0$ depending on $\|\cdot\|_{L^\infty(\mathbb{R}\times[0,T])}$ norms of u^δ and u_{tt}^δ, and Lipschitz constant of u_x^δ.
2. *If $c'', b'' \in L^\infty(\mathbb{R})$ and $u, u_x, u_{xx}, u_{tt} \in L^\infty(\mathbb{R} \times [0,T])$, then*

$$\sup_j |u(x_j,T) - U_j^N| = O(\max(\bar{h},\delta)) + O(\Delta t).$$

Proof. The arguments are analogous to the case of a stationary nonuniform grid in subsection 2.3. ∎

The appearance of the local derivative term in (2.8) suggests that one may consider its nonlocal relaxation to obtain the following fully nonlocal equation:

$$\frac{d}{dt}u(x(t),t) + \left(\int_{x(t)-\delta}^{x(t)+\delta} (b(y) - b(x(t)))w_{\hat{c}}^{\delta}(x(t),y)dy\right) u(x(t),t) + \cdots$$
$$\int_{x(t)-\delta}^{x(t)+\delta} ((c(y) - b(y))u(y,t) - (c(x(t)) - b(x(t)))u(x(t),t))w_{\hat{c}}^{\delta}(x(t),y)dy = 0. \tag{2.10}$$

This formulation fits well with our nonlocal framework when the velocity field $b(x)$ is chosen to be equal to $c(x)$, in which case (2.8) reduces to a local PDE. We can see that the use of the kernel $w_{\hat{c}}^{\delta}$ in our nonlocal relaxation of b' is a convenient choice as it yields, in the case of $\hat{c}_{j,n} > 0$, the following simple numerical discretization:

$$u_j^{n+1} = u_j^n - \Delta t \sum_{k=1}^{L(j)} ((c_{j,n} - b_{j-k,n})u_j^n - \hat{c}_{j-k,n}u_{j-k}^n)W_{k,j}^L. \tag{2.11}$$

The convergence of the above discretization can be shown as follows.

THEOREM 2.8. *Suppose the same assumptions are made as in Theorem* 2.7*. Then with* u *and* u^δ *now denoting the solutions of* (2.7) *and* (2.10)*, respectively, the same conclusions as in Theorem* 2.7 *hold.*

Proof. The proof is exactly analogous to that of Theorem 2.7. ∎

We make some observations concerning the two schemes in (2.9) and (2.11). It is clear that the computational complexity of the latter involves $\mathcal{O}(N)$ more operations than the former, where N denotes the number of particles. In terms of memory storage, however, the former requires additional $\mathcal{O}(N)$ storage for the terms $(b')_j^n$ than the latter, which only needs the values of b_j that have already been computed for the nonlocalized flux. Meanwhile we note that computations of the weights $W_{k,j}^L$ for both schemes can be significantly simplified by choosing a singular kernel such as $\eta^\delta(z) = \frac{1_{(-\delta,\delta)}}{\delta z}$.

3. Model II: Vanishing nonlocal viscosity. It is a well-known practice in the study of conservation laws to seek solutions of an inviscid problem by taking the limit solutions of the corresponding viscous problem as the viscosity effect vanishes. In the particular case of an advection equation, one way to bring in the idea of vanishing viscosity is through advection-diffusion equations where the diffusive terms are introduced through nonlocal operators such as a fractional power of the Laplacian [1, 7, 10]. In this section we present a nonlocal continuum advection-diffusion model which will in turn give rise to another AC particle method for the local advection equation (1.1).

3.1. Continuum formulation of Model II. In more specific terms, we propose the following nonlocal advection-diffusion model (referred to as Model II from here on):

$$u_t(x,t) + \int_{\mathbb{R}} c\left(\frac{x+y}{2}\right)(u(x,t) + u(y,t))w^\delta(y-x)dy - \cdots$$
$$\delta\mu \int_{\mathbb{R}} (u(y,t) - u(x,t))\frac{w^\delta(y-x)}{y-x}dy = 0, \tag{3.1}$$
$$u(x,0) = u_0(x),$$

where $\delta > 0$ is a nonlocal horizon and $\mu > 0$ is a positive coefficient depending only on the velocity field c. We assume that $w^\delta(z) = \frac{1}{\delta^2} w(\frac{z}{\delta})$, where w is an antisymmetric integrable kernel that is supported on $(-1, 1)$ and satisfies $w^\delta(z) \geq 0$ on $z \geq 0$. In the language of nonlocal vector calculus [9], the model can be rewritten as

$$u_t + \mathcal{A}_\delta(u) - \delta\mu\mathcal{D}_\delta(u) = 0,$$

where $\mathcal{A}_\delta$ corresponds to a nonlocal divergence operator with the vector two-point function $\mathbf{v}(x, y) = c(\frac{x+y}{2})u(x, t)$ and the antisymmetric two point function $\alpha = \frac{y-x}{|y-x|}|w^\delta(y - x)|$, whereas $\mathcal{D}_\delta$ corresponds to a nonlocal diffusion operator with the symmetric kernel $\frac{w^\delta(y-x)}{y-x}$. As the focus of this work is on initial Cauchy value problems, discussions on the nonlocal boundary conditions, or constrained values [8], and modifications to the nonlocal diffusion operators near the boundary [13], if any, are not presented here.

Following discussions in [9, 11, 27], it is immediate to see that with a fixed $\delta > 0$ the model is a nonlocal analogue of the modified equation

$$u_t(x, t) + (c(x)u(x, t))_x - \frac{\delta\mu}{2} u_{xx}(x, t) = 0$$

$$\int_{-\delta}^{\delta} h w^\delta(h) dh = 1, \tag{3.2}$$

which we assume from here on. Note that the integrability assumption of w implies that

$$\|w^\delta\|_{L^1(\mathbb{R})} = O\left(\frac{1}{\delta}\right). \tag{3.3}$$

A connection to the SPH can be made by setting $w^\delta(z) = -\partial_z(\frac{1}{\delta}\rho(\frac{z}{\delta}))$, where ρ is a radially symmetric, nonnegative, differentiable function that decreases with increasing radial distances and is compactly supported on $(-1, 1)$ with $\|\rho\|_{L^1(\mathbb{R})} = 1$.

The presence of nonlocal diffusion term in (3.1) suggests that a simple minded discretization of the model may not need *additional* stabilization and that such discretization, with vanishing δ, is expected to converge to the advection equation. This is indeed the case as we establish the convergence results of our discretization of Model II. We assume $\|c\|_{L^\infty(\mathbb{R})} < \infty$ and we choose to take

$$\mu \geq \|c\|_{L^\infty(\mathbb{R})}.$$

This leads to sufficient damping to the nonlocal model to ensure desirable physical features related to stability. For example, for constant velocity c, the above condition implies a maximum principle for the resulting nonlocal convection-diffusion model.

3.2. Numerical discretization of Model II. Similar to our discretization of Model I, we present our numerical discretization of Model II, wherein the kernels of the nonlocal integral operators are integrated analytically. What is distinct in this case, however, is that we adopt the idea presented in the renormalized SPH by Vila [32] instead of considering the first moment of the kernel. Before presenting our discretization on a set of moving particles, we first consider a set of stationary particles $\{x_i\}_{i\in\mathbb{Z}}$ and propose the following discretization:

$$\mathcal{A}_\delta^h(u(x_j,t)) = \frac{1}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} c\left(\frac{x_j + x_{j+k}}{2}\right)(u(x_j,t) + u(x_{j+k},t))W_{j,j+k},$$

$$\mathcal{D}_\delta^h(u(x_j,t)) = \frac{1}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} \frac{(u(x_{j+k},t) - u(x_j,t))}{x_{j+k} - x_j} W_{j,j+k},$$

where $l_j = \max\{\max\{l \in \mathbb{Z}^+ \mid x_j - x_{j-l} \le \delta\}, 1\}$,$r_j = \max\{\max\{r \in \mathbb{Z}^+ \mid x_{j+r} - x_j \le \delta\}, 1\}$,

$$W_{j,j+k} = \int_{x_{j+k-1}}^{x_{j+k}} w^\delta(y - x_j)dy + 1_{k=r_j} \int_{x_{j+k}}^{x_j+\delta} w^\delta(y - x_j)dy$$

for $k > 0$ (analogously for $W_{j,j+k}$ with $k < 0$) and the renormalization factor

$$N_{j,\delta} = \sum_{k=-l_j}^{r_j} (x_{j+k} - x_j)W_{j,j+k}.$$

Let us then summarize the properties of $N_{j,\delta}$ that will be used in our analysis to follow.

LEMMA 3.1. *There exists a constant C such that*

$$0 \le N_{j,\delta} - 1 \le C\frac{\bar{h}}{\delta}.$$

Moreover

$$\frac{1}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} W_{j,j+k} = 0, \qquad \frac{1}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} (x_{j+k} - x_j)W_{j,j+k} = 1 \tag{3.4}$$

and

$$\left| \frac{\sum_{k=-l_j}^{r_j} (x_{j+k} - x_j)^2 W_{j,j+k}}{N_{j,\delta}} \right| = O(\max(\bar{h}, \delta)). \tag{3.5}$$

Proof. Assume without loss of generality $x_j = 0$ and note

$$N_{j,\delta} - 1 = \sum_{k=1}^{r_j} \int_{x_{j+k-1}}^{x_{j+k}+1_{k=r_j}(\delta - x_{j+k})} (y\big|_{y=x_{j+k}} - y)w^\delta(y)dy + \cdots$$
$$\sum_{k=-l_j}^{-1} \int_{x_{j+k}+1_{k=l_j}(-\delta - x_{j+k})}^{x_{j+k+1}} (y\big|_{y=x_{j+k}} - y)w^\delta(y)dy.$$

Then the first claim follows from the nonnegativity of the integrands and (3.3). The identity (3.4) is immediate from the definition of $N_{j,\delta}$. The identity (3.5) is a consequence of the nonnegativity of each $(x_{j+k} - x_j)W_{j,j+k}$ in $N_{j,\delta}$. ∎

We adopt the forward Euler time stepping, thereby obtaining the following explicit scheme:

$$u_j^{n+1} = u_j^n - \Delta t(\mathcal{A}_\delta^h(u_j^n) - \mu \max\{\delta, x_j - x_{j-l_j}, x_{j+r_j} - x_j\}\mathcal{D}_\delta^h(u_j^n)). \tag{3.6}$$

As will be seen in the proof of Lemma 3.3, approximating δ with $\max\{\delta, x_j - x_{j-l_j}, x_{j+r_j} - x_j\}$ in the viscosity term ensures that sufficient numerical viscosity is present when $\delta < x_j - x_{j-l_j}$ or $\delta < x_{j+r_j} - x_j$.

3.3. Convergence of the numerical scheme to Model II and its local limit. Again, we show that the numerical solutions obtained by the scheme (3.6) are convergent in L^∞ norm in the following two regimes outlined in (2.5). Since the scheme (3.6) is linear, we establish consistency and stability.

Convergence to the nonlocal Model II. The consistency is stated first below.

LEMMA 3.2. *If $u, u_x, u_{xx}, c, c', c''$ are uniformly bounded, then*

$$\sup_j |\mathcal{A}_\delta^h(u(x_j,t)) - \mathcal{A}_\delta(u(x_j,t))| = O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right).$$

If u_{xxx} is uniformly bounded, then

$$\sup_j |\mathcal{D}_\delta^h(u(x_j,t)) - \mathcal{D}_\delta(u(x_j,t))| = O(\bar{h}) + O\left(\frac{\bar{h}^3}{\delta^2}\right).$$

Proof. For convenience of notation, let us denote $f(y,t) = c(\frac{x_j+y}{2})(u(x_j,t) + u(y,t))$ and apply Taylor series expansion to write

$$f(y,t) = f(x_j,t) + (y-x_j)f_y(x_j,t) + R_1(y,t),$$

where $R_1(y,t) = \int_{x_j}^y (y-s)f_{yy}(s,t)ds$. Then the antisymmetry of w^δ together with the moment condition subsection 3.1 yields

$$\mathcal{A}_\delta(u(x_j,t)) = f_y(x_j,t) + \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy.$$

On the other hand, we can apply Taylor series expansion and the identities (3.4) to obtain

$$\mathcal{A}_\delta^h(u(x_j,t)) = f_y(x_j,t) + \frac{1}{N_{j,\delta}}\sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k}.$$

We then have

$$\begin{aligned}
&|\mathcal{A}_\delta(u(x_j,t)) - \mathcal{A}_\delta^h(u(x_j,t))| \\
&= \frac{1}{N_{j,\delta}}\left|\sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - N_{j,\delta}\int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy\right| \\
&\le \left|\sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy\right| + \cdots \\
&\quad O\left(\frac{\bar{h}}{\delta}\right)\left|\int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy\right|,
\end{aligned}$$

where the last inequality is due to $N_{j,\delta} \ge 1$ and $N_{j,\delta} = 1 + O(\frac{\bar{h}}{\delta})$ in Lemma 3.1. But $|R_1(y)| \le C\delta^2$ together with (3.3) implies

$$\begin{aligned}
&|\mathcal{A}_\delta(u(x_j,t)) - \mathcal{A}_\delta^h(u(x_j,t))| \\
&\le \left|\sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy\right| + O(\bar{h}).
\end{aligned}$$

Then the first claim follows from the estimate

$$\begin{aligned}
&\left|\sum_{k=1}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy\right| \\
&= \sum_{k=1}^{r_j}\left|\int_{x_{j+k-1}}^{x_{j+k}+1_{k=r_j}(x_j+\delta-x_{j+k})} \left(R_1(y,t)\big|_{y=x_{j+k}} - R_1(y,t)\right)w^\delta(y-x_j)dy\right| \\
&\le \sum_{k=1}^{r_j}\int_{x_{j+k-1}}^{x_{j+k}+1_{k=r_j}(x_j+\delta-x_{j+k})} \left(\underbrace{\left|\int_{x_j}^{y}(x_{j+k}-y)f_{yy}(s,t)ds\right|}_{O(\delta\bar h)} + \cdots \right. \\
&\qquad \left. \underbrace{\left|\int_{y}^{x_{j+k}}(x_{j+k}-s)f_{yy}(s,t)ds\right|}_{O(\bar h^2)}\right)|w^\delta(y-x_j)|dy \le O(\bar h) + O\left(\frac{\bar h^2}{\delta}\right),
\end{aligned}$$

where (3.3) is used in the last inequality, and the analogous estimate

$$\begin{aligned}
&\left|\sum_{k=-l_j}^{-1} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j-\delta}^{x_j} R_1(y,t)w^\delta(y-x_j)dy\right| \\
&\le \sum_{k=-l_j}^{-1}\left|\int_{x_{j+k}+1_{k=-l_j}(x_j-\delta-x_{j+k})}^{x_{j+k+1}} \left(R_1(y,t)\big|_{y=x_{j+k}} - R_1(y,t)\right)w^\delta(y-x_j)dy\right| \\
&\le O(\bar h) + O\left(\frac{\bar h^2}{\delta}\right).
\end{aligned}$$

The second estimate follows analogously from Taylor series expansion of $u(y,t)$ up to third order in y. ∎

So far as the stability of the scheme (3.6) is concerned, our choice of a nonlocal viscosity term ensures sufficient numerical viscosity to show the following L^∞ stability result.

LEMMA 3.3. *Assume c, c' are bounded. If*

$$\frac{2\Delta t\mu}{\underline{h}} \le 1,$$

then $\|u^{n+1}\|_{L^\infty(\mathbb{R})} \le (1+\Delta t\|c'\|_{L^\infty(\mathbb{R})})\|u^n\|_{L^\infty(\mathbb{R})}$ *for all* $n \ge 0$.

Proof. The scheme can be rewritten as

$$\begin{aligned}
&u_j^{n+1} \\
&= \left(1 - \frac{\Delta t}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}\left(c_{\frac{j,j+k}{2}} + \frac{\max\{\delta, x_j - x_{j-l_j}, x_{j+r_j}-x_j\}\mu}{x_{j+k}-x_j}\right)W_{j,j+k}\right)u_j^n \\
&\quad + \left(-\frac{\Delta t}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}\left(c_{\frac{j,j+k}{2}} - \frac{\max\{\delta, x_j - x_{j-l_j}, x_{j+r_j}-x_j\}\mu}{x_{j+k}-x_j}\right)W_{j,j+k}\right)u_{j+k}^n.
\end{aligned}$$

Then since

$$\frac{\sum_{k=-l_j}^{r_j} |W_{j,j+k}|}{N_{j,j+k}} \le \frac{1}{\underline{h}} \frac{\sum_{k=-l_j}^{r_j} (x_{j+k} - x_j) W_{j,j+k}}{N_{j,j+k}} = \frac{1}{\underline{h}}$$

and

$$\left| \frac{\sum_{k=-l_j}^{r_j} c_{\frac{j,j+k}{2}} W_{j,j+k}}{N_{j,k}} \right| = \left| \frac{\sum_{k=-l_j}^{r_j} (c_{\frac{j,j+k}{2}} - c_j) W_{j,j+k}}{N_{j,k}} \right|$$
$$\le \frac{\frac{1}{2} \|c'\|_{L^\infty(\mathbb{R})} \sum_{k=-l_j}^{r_j} (x_{j+k} - x_j) W_{j,j+k}}{N_{j,k}} = \frac{\|c'\|_{L^\infty(\mathbb{R})}}{2},$$

the claim follows from the positivity of the coefficients of u_j^n and u_{j+k}^n. ∎

Convergence to the local advection equation (1.1). We have the CFL stability condition in Lemma 3.3 that is independent of δ, so it only remains to show consistency.

LEMMA 3.4. *If $u, u_x, u_{xx}, u_{xxx}, c, c', c''$ are uniformly bounded, then*

$$\sup_j |\mathcal{A}_\delta^h(u(x_j,t)) - (c(x)u(x,t^n))_x\big|_{x=x_j}| = O(\max(\bar{h},\delta)),$$
$$\sup_j |\max\{\delta, x_j - x_{j-l_j}, x_{j+r_j} - x_j\} \mathcal{D}_\delta^h(u(x_j,t))| = O(\max(\bar{h},\delta)).$$

Proof. The results follow from Taylor series expansions, the identities (3.4), and the estimate (3.5) in Lemma 3.1. ∎

In summary, we deduce the following.

THEOREM 3.5. *Let $T > 0$ be a fixed terminal time. Assume c, c', c'' are bounded. Let U_j^n denote the numerical solution given by the scheme* (3.6) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition of Lemma 3.3. Let $u(x,t), u^\delta(x,t)$ denote the solutions of* (3.1) *and* (1.1)*, respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.*

1. *If $u^\delta, u_x^\delta, u_{xx}^\delta, u_{xxx}^\delta, u_{tt}^\delta \in L^\infty(\mathbb{R} \times [0,T])$, then*
$$\sup_j |u^\delta(x_j,T) - U_j^N| = O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right) + O\left(\frac{\bar{h}^3}{\delta^2}\right) + O(\Delta t).$$
2. *If $u, u_x, u_{xx}, u_{xxx}, u_{tt} \in L^\infty(\mathbb{R} \times [0,T])$, then*
$$\sup_j |u(x_j,T) - U_j^N| = O(\max(\bar{h},\delta)) + O(\Delta t).$$

Proof. The proof follows from the Lax equivalence theorem. ∎

3.4. A moving particle approximation. It is now straightforward to consider the case of moving particles that advect according to the velocity field $b(x)$. If we recall the PDE in Lagrangian frame (2.7), we see that the corresponding nonlocal Model II is given by

$$\begin{aligned} &\frac{d}{dt}u(x(t),t) + b'(x(t))u(x(t),t) + \cdots \\ &\int_{\mathbb{R}} (c-b)\left(\frac{x(t)+y}{2}\right)(u(x(t),t) + u(y,t)) w^\delta(y - x(t))dy - \cdots \\ &\mu \int_{\mathbb{R}} (u(y,t) - u(x(t),t)) \frac{w^\delta(y-x(t))}{y - x(t)} dy = 0, \end{aligned} \tag{3.7}$$

which in turn leads to the following scheme:

$$
\begin{aligned}
u_j^{n+1} = u_j^{n+1} - \Delta t(b'_{j,n} u_j^n + \mathcal{A}^h_{\delta,b(\cdot)}(u_j^n) - \cdots \\
\mu \max\{\delta, (x_{j,n} - x_{j-l_j,n}), (x_{j+r_j,n} - x_{j,n})\}\mathcal{D}^h_\delta(u_j^n)),
\end{aligned}
\tag{3.8}
$$

where $\mathcal{A}^h_{\delta,b(\cdot)}$ is the modification of $\mathcal{A}^h_\delta$ obtained by replacing c with $c-b$, $N_{j,\delta}$ within $\mathcal{A}^h_{\delta,b(\cdot)}$ and $\mathcal{D}^h_\delta$ is evaluated using $\{x_{i,n}\}_{i\in\mathbb{Z}} = \{x_i(\Delta t n)\}_{i\in\mathbb{Z}}$, and

$$\mu \geq \|c - b\|_{L^\infty(\mathbb{R})}.$$

As in the case of Model I, we take initial distribution of particles to be uniform with a particle spacing of size h, and velocity b to be smooth such that

$$C_1(T)h \leq |x_{j+1} - x_j| \leq C_2(T)h$$

for all $j \in \mathbb{Z}$ and $t \in [0, T]$. Then one can show the following.

THEOREM 3.6. *Let $T > 0$ be a fixed terminal time. Assume c, c', c'', b, b', b'' are bounded. Let U_j^n denote the numerical solution given by the scheme (3.8) at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition*

$$\frac{2\Delta t \mu}{C_1(T)h} \leq 1.$$

Let $u(x,t), u^\delta(x,t)$ denote the solutions of (2.7) and (3.7), respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.

1. *If $u^\delta, u^\delta_x, u^\delta_{xx}, u^\delta_{xxx}, u^\delta_{tt} \in L^\infty(\mathbb{R}\times[0,T])$, then*

$$\sup_j |u^\delta(x_j, T) - U_j^N| = O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right) + O\left(\frac{\bar{h}^3}{\delta^2}\right) + O(\Delta t).$$

2. *If $u, u_x, u_{xx}, u_{xxx}, u_{tt} \in L^\infty(\mathbb{R}\times[0,T])$, then*

$$\sup_j |u(x_j, T) - U_j^N| = O(\max(\bar{h}, \delta))) + O(\Delta t).$$

Proof. The arguments are analogous to the case of stationary particles in subsection 3.3. ∎

Following our approach in the case of Model I, we consider the fully nonlocalized equation

$$
\begin{aligned}
\frac{d}{dt}u(x(t),t) + 2\left(\int_\mathbb{R} b\left(\frac{x(t)+y}{2}\right) w^\delta(y - x(t))dy\right) u(x(t),t) + \cdots \\
\int_\mathbb{R} (c-b)\left(\frac{x(t)+y}{2}\right)(u(x(t),t) + u(y,t))w^\delta(y-x(t))dy - \cdots \\
\delta\mu \int_\mathbb{R} (u(y,t) - u(x(t),t))\frac{w^\delta(y-x(t))}{y-x(t)}dy = 0,
\end{aligned}
\tag{3.9}
$$

which in turn leads to the following scheme:

$$
\begin{aligned}
u_j^{n+1} = u_j^{n+1} - \Delta t(\mathcal{B}^h_\delta(u_j^n) + \mathcal{A}^h_{\delta,b(\cdot)}(u_j^n) - \cdots \\
\mu \max\{\delta, (x_{j,n} - x_{j-l_j,n}), (x_{j+r_j,n} - x_{j,n})\}\mathcal{D}^h_\delta(u_j^n)),
\end{aligned}
\tag{3.10}
$$

where

$$\mathcal{B}_\delta^h(u_j^n) = \left(\frac{2}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} b\left(\frac{x_{j,n} + x_{j+k,n}}{2} \right) W_{j,j+k} \right) u_j^n.$$

Then the following can be shown.

THEOREM 3.7. *Suppose the same assumptions are made as in Theorem* 3.6. *Then with u and u^δ now denoting the solutions of* (2.7) *and* (3.9), *respectively, the same conclusions as in Theorem* 3.6 *hold.*

Proof. The proof is exactly analogous to that of Theorem 3.6. ∎

4. Numerical experiments. We present some numerical tests to demonstrate the convergence of our particle schemes. We take the interval $[0, 2\pi]$ as our computational domain and impose periodic boundary conditions. The underlying velocity field is prescribed as $c(x) = \frac{1}{3}\cos(2x)$ and the particle velocity is taken to be $b(x) = \sin(x)$ so that the trajectories of the particles are given analytically by

$$x(t) = 2\cot^{-1}(\exp(c - t)), \qquad \text{where} \qquad c = \log\left(\frac{\cot(x(0))}{2} \right).$$

The terminal time is chosen to be $T = 1$ and the initial data is $u_0(x) = \cos(x)$. By the method of manufactured solutions we take the exact solution to be $u(x,t) = \cos((x - t))$ and solve the corresponding inhomogeneous problems. The nonlocal integrals in the resulting inhomgeneous terms are evaluated using a MATLAB built-in integration routine. In all our numerical experiments, numerical solution errors are measured in L^∞ norm in accordance with our theoretical analysis. The initial distribution of the particles is chosen to be uniform with the discretization parameter $h = \frac{2\pi}{N}$, where N denotes the number of particles. In order to satisfy the CFL conditions, we pick $\Delta t = \frac{h}{4}$ and $\Delta t = \frac{h}{8}$ for all our numerical experiments on Model I and II, respectively, when validating each model individually. On the other hand, we choose the smaller time step size, i.e., $\Delta t = \frac{h}{8}$ for both models when comparing them side by side. Our particular choices of time step sizes are immaterial to demonstrating the theoretically predicted convergence rates of the schemes so far as the step sizes are chosen to meet the CFL conditions. All the convergence results are presented with respect to the spatial parameters δ and h only, which has been the focus of our theoretical analysis thus far presented.

4.1. Tests with Model I. We take the singular kernel $\eta^\delta(z) = \frac{1_{(-\delta,\delta)}}{\delta z}$ and compute the numerical solutions using both schemes (2.9) and (2.11) (referred to as Scheme 1 and Scheme 2 in this subsection).

Example 1: *Fixed horizon.* We fix the horizon length $\delta = 0.2$ and study the convergence of the numerical solutions to the exact solution as $h \to 0$. In particular, we take the mesh refinement path $h = \frac{2\pi}{160}, \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}$. For both Schemes 1 and 2, the convergence rates are expected to be first order in h, which is experimentally validated in Figure 1.

Example 2: *Asymptotic compatibility.* We consider the three parameter refinement paths (i) $(\delta, h) = (4h, h)$, (ii) $(\delta, h) = (\sqrt{h}, h)$, and (iii) $(\delta, h) = (h^2, h)$ to illustrate the convergence in the cases $\delta = \mathcal{O}(h), h = o(\delta)$, and $\delta = o(h)$, respectively. We refine $h \to 0$ along $h = \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}, \frac{2\pi}{2560}$. Our theoretical analysis predicts the convergence rates to be the first order in $\max\{h, \delta\} = \delta, \delta$, and h along the refinement paths (i), (ii) and (iii), respectively, which agrees with our simulation results in Figure 2.

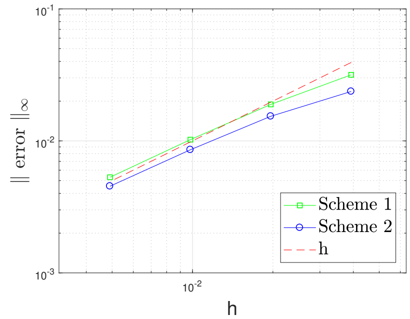
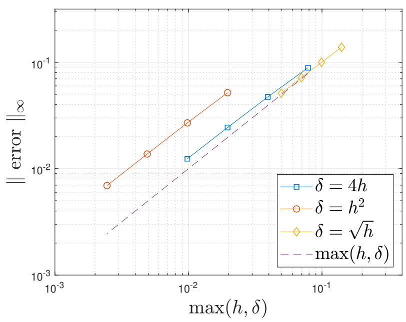
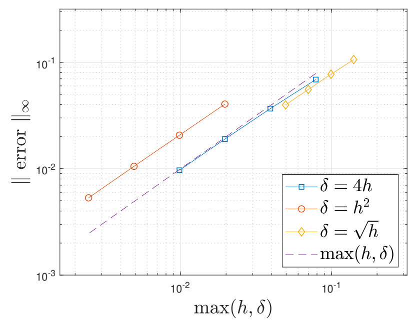


FIG. 1. *Convergence of the numerical solution errors in the* L^∞ *norm.*



(a) Scheme 1



(b) Scheme 2

FIG. 2. *Asymptotic convergence of the numerical solution errors in the* L^∞ *norm.*

4.2. Tests with Model II. We take $w^\delta(z) = -\frac{d}{dz}\left(\frac{1}{\delta}\rho(\frac{z}{\delta})\right)$, where $\rho(z)$ is the classical B-spline kernel [3] defined as

$$\rho(z) = \frac{2}{3}\begin{cases} 1 - \dfrac{3}{2}z^2 + \dfrac{3}{4}z^3 & \text{if } 0 \le |z| \le 1, \\ \dfrac{1}{4}(2-z)^3 & \text{if } 1 \le |z| \le 2, \\ 0 & \text{otherwise} \end{cases}$$

and compute the numerical solutions using both schemes (3.8) and (3.10) (referred to as Scheme 1 and Scheme 2 in this subsection). We take $\mu = \|c - b\|_{L^\infty(\mathbb{R})}$. Note the support of w^δ is in fact $(-2\delta, 2\delta)$, which we will take into account in our numerical experiments in this subsection by letting $\bar{\delta} = 2\delta$.

Example 1: *Fixed horizon.* We fix the horizon $\bar{\delta} = 0.2$ and take the refinement path $h = \frac{2\pi}{160}, \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}$ as in Model I. For both Schemes 1 and 2, the convergence rates are expected to be first order in h, which we can observe in Figure 3.

Example 2: *Asymptotic compatibility.* We consider the three parameter refinement paths (i) $(\bar{\delta}, h) = (4h, h)$, (ii) $(\bar{\delta}, h) = (\sqrt{h}, h)$, and (iii) $(\bar{\delta}, h) = (h^2, h)$ along $h = \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}, \frac{2\pi}{2560}$. Figure 4 verifies our theoretical prediction of the first order convergence rate in $\max\{h, \bar{\delta}\} = \bar{\delta}, \bar{\delta}$, and h along the refinement paths (i), (ii), (iii), respectively.

4.3. Comparisons and discussions. Beyond validation of our theoretical analysis, it is of practical interest to conduct additional *empirical* investigation of our numerical schemes. After all it remains to answer important and practical questions such

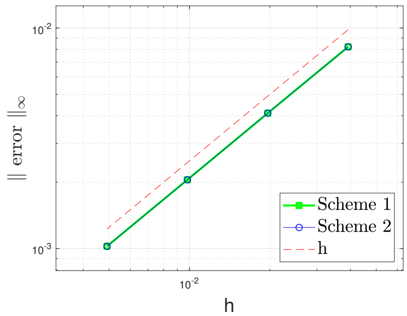
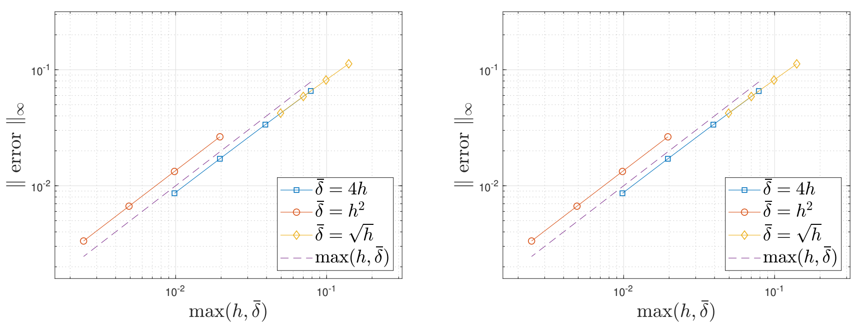


FIG. 3. *Convergence of the numerical solution errors in the L^∞ norm.*



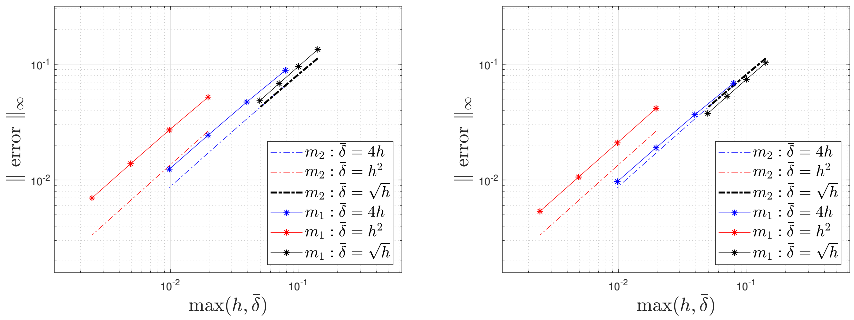
(a) Scheme 1 (b) Scheme 2

FIG. 4. *Asymptotic convergence of the numerical solution errors in the L^∞ norm.*

as which of our particle methods one should use. We focus in particular on comparing the accuracy of the particle methods for Model I and Model II, thereby providing some insights into gauging practical effectiveness of the methods. We choose the kernel $w^\delta(z)$ to be $-2\frac{d}{dz}\left(\frac{1}{\delta}\rho(\frac{z}{\delta})\right)$ for Model I, where ρ is the B-spline kernel. Note that this choice of kernel satisfies the moment condition (2.3). We then perform our numerical experiments for Model I with $h = \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}, \frac{2\pi}{2560}$ along (i) $(\bar{\delta}, h) = (4h, h)$, (ii) $(\bar{\delta}, h) = (\sqrt{h}, h)$, and (iii) $(\bar{\delta}, h) = (h^2, h)$. The numerical results thus obtained are compared with the corresponding results for Model II obtained in the previous subsection.

It can be seen from Figure 5 that the particle methods for Model II produce numerical solutions that are more accurate than those obtained by the methods for Model I except in the regime $\delta = \sqrt{h}$ for the fully nonlocalized schemes. One heuristic explanation of the better accuracy of the former might be that their truncation errors contain approximation of the vanishing second moment of the antisymmetric kernel whereas those for the latter methods involve the *nonvanishing* second moment of the upwinding kernel. On the other hand, the poorer accuracy of the scheme (3.10) relative to the scheme (2.11) when $\delta \gg h$ could be attributed to the pronounced influence of the nonlocal viscosity term in Model II, which is taken to be proportional to δ. This in turn alludes to the importance of nonlocal continuum models in designing effective numerical methods for the corresponding local continuum models.

5. Conclusion. We have presented two nonlocal advection models and their asymptotically compatible particle discretizations. In summary both models are



(a) Schemes (2.9) and (3.8) (b) Schemes (2.11) and (3.10)

FIG. 5. *Comparison of numerical solution errors in* L^∞ *norm:* m_1 *and* m_2 *denote Model* I *and* II*, respectively.*

equipped with stabilizers on the continuum level: one via use of biased kernels and the other via nonlocal diffusion. This is in sharp contrast to the classical SPH in which a simple minded discretization of the continuum model is unstable unless numerical/artificial viscosity is introduced at the discretization level. We reiterate that these built-in stability features are reminiscent of the corresponding ideas well known in the setting of numerical methods for PDE, namely, upwinding and vanishing artificial viscosity.

At the discrete level we can also delineate some differences between Model I and Model II. On one hand, our particle methods for Model I achieve consistency with the local PDE by ensuring that the discrete flux between *each* pair of particles is captured at the correct spatial scale. On the other hand, our particle discretization of Model II assures that the sum of *all* pairwise nonlocal fluxes between a particle and all its neighbors is normalized to the desired local flux value. More on the practical side, we see that the particle methods for Model I admit singular kernels as well as integrable kernels, hence possibly offering a greater flexibility in numerical simulations than the methods for Model II, which rely on integrable kernels. The latter methods, however, are simpler in the sense that they do not require any information of the upwinding directions, which in the case of the former methods is necessary for each moving particle at each instance of time.

Despite the robustness of our particle methods, we point out that they are all limited in terms of their accuracy being only first order accurate. Moreover our particle methods fail to be conservative schemes, hence possibly limited in their applicability to nonlinear problems. On a related note, we remark that the scope of our current study is limited to scalar one dimensional linear problems, though nonlocal formulations of the Stokes systems in multidimensional spaces have also been considered recently [15] and there are natural connections with earlier studies on numerical discretizations of other nonlocal models such as nonlocal diffusion and peridynamics [8, 29]. Indeed the practicality of SPH-like particle methods makes it imperative to extend our current work to the simplest multidimensional setting of two dimensional linear problems, for which some speculations can be offered. As far as the stability is concerned, one may conceive a two dimensional generalization of the one dimensional upwinding kernel so that the domain of nonlocal interactions is limited to a semidisc depending on the direction of the velocity field. However, the generic difficulty of two dimensional interpolation on nonuniformly distributed data [33] poses a significant challenge to determination of suitable quadrature weights to ensure the AC property. The special case of no neighboring particle within the horizon could potentially be treated with a rather simple idea of using the closest particle and seeking appropriate interpolation schemes on just two data points. The more likely situation where there are some neighboring particles present is yet to be further investigated, possibly in similar spirits as in the recent works [31, 14]. In addition, issues related to model nonlinearity and effects of physical boundary are also important topics to be further explored.

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