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A Chebotarev Variant of the Brun–Titchmarsh Theorem and Bounds for the Lang-Trotter conjectures

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We improve the Chebotarev variant of the Brun–Titchmarsh theorem proven by Lagarias, Montgomery, and Odlyzko using the log-free zero density estimate and zero repulsion phenomenon for Hecke L-functions that were recently proved by the authors. Our result produces an improvement for the best unconditional bounds toward two conjectures of Lang and Trotter regarding the distribution of traces of Frobenius for elliptic curves and holomorphic cuspidal modular forms. We also obtain new results on the distribution of primes represented by positive-definite integral binary quadratic forms.

1 Introduction and Statement of Results

Let $\pi(x;q,a)$ denote the number of primes $p \le x$ such that $p \equiv a \pmod q$. The Siegel–Walfisz theorem states that if $(a,q)=1$ and there exists some constant $A>0$ such that $q \le (\log x)^A$, then

$$\pi(x;q,a) \sim \frac{1}{\varphi(q)}\mathrm{Li}(x), \tag{1.1}$$

where $\mathrm{Li}(x)=\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$. Assuming the generalized Riemann hypothesis, the range of q extends to $q \le x^{1/2-\epsilon}$ for any $\epsilon>0$. A sufficiently strong unconditional improvement

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in the range of q would preclude the existence of a real Landau–Siegel zero for the L-functions of real Dirichlet characters. Since this seems to be beyond the reach of current techniques, it is often useful to trade asymptotic equality in (1.1) for upper and lower bounds of the correct asymptotic order which hold in improved ranges of x.

The first lower bound of this form follows from Fogels' improvements [6] to the ideas of Linnik [15]. These ideas were substantially improved by Heath-Brown [7] and Maynard [16], the latter of whom proved that if q is sufficiently large, then

$$\pi(x;q,a) \gg \frac{\log q}{\varphi(q)\sqrt{q}}\frac{x}{\log x} \qquad \text{for } x \geq q^8. \tag{1.2}$$

(All implied constants in this article are effectively computable. Unless specifically mentioned otherwise, all implied constants in this paper are also absolute.) To describe upper bounds in improved ranges of q, we define $\theta = (\log q)/\log x$. Titchmarsh [25] used Brun's sieve to show that if $\theta < 1$, then

$$\pi(x;q,a) \ll \frac{1}{1-\theta}\frac{x}{\varphi(q)\log x}. \tag{1.3}$$

The implied constant can be made explicit and has been estimated by various authors. The strongest result in this direction for all ranges of q is due to Montgomery and Vaughan [17]; they used the large sieve inequality to prove that if $\theta < 1$, then

$$\pi(x;q,a) \leq \frac{2}{1-\theta}\frac{x}{\varphi(q)\log x}. \tag{1.4}$$

Since the factor of 2 is unlikely to be improved using current techniques, many authors have improved the θ-dependence. To summarize, if q is sufficiently large, then

$$\pi(x;q,a) \leq \frac{(C(\theta)+o(1))}{\varphi(q)}\frac{x}{\log x},$$

where

$$C(\theta) = \begin{cases} (2-(\frac{1-\theta}{4})^6)/(1-\theta) & \text{if } 2/3 \leq \theta < 1, \\ 8/(6-7\theta) & \text{if } 9/20 < \theta < 2/3, \\ 16/(8-3\theta) & \text{if } 1/8 < \theta \leq 9/20, \\ 2 & \text{if } 0 < \theta \leq 1/8, \end{cases} \tag{1.5}$$

the last line being recently proven by Maynard [16]. (See [16] and the sources contained therein for a thorough overview of the problem.) While progress on (1.3) has typically followed from advances in sieve theory and exponential sums, Maynard's proof builds on Heath-Brown's analysis in [7] and uses a log-free zero density estimate for Dirichlet L-functions and careful analysis of Landau–Siegel zeros.

In this article, we consider analogous questions for the distribution of prime ideals in the context of the Chebotarev density theorem. Let L/F be a finite Galois extension of number fields with Galois group G, and let $C \subset G$ be a conjugacy class. Let D_L denote the absolute value of the discriminant of $L/\mathbb{Q}$. To each prime ideal $\mathfrak{p}$ of F that does not ramify in L, there corresponds a certain conjugacy class of automorphisms in G which are attached the prime ideals of L lying above $\mathfrak{p}$. We denote this conjugacy class by the Artin symbol $[\frac{L/F}{\mathfrak{p}}]$. For a fixed conjugacy class $C \subset G$, let

$$\pi_C(x, L/F) := \#\left\{\mathfrak{p} : \mathfrak{p} \text{ unramified in } L, \left[\frac{L/F}{\mathfrak{p}}\right] = C, \mathrm{N}_{F/\mathbb{Q}}\mathfrak{p} \le x\right\}. \tag{1.6}$$

The Chebotarev density theorem, in the effective version proven by Lagarias and Odlyzko [13], states that if $x \ge \exp(10[L:\mathbb{Q}](\log D_L)^2)$, then

$$\pi_C(x, L/F) \sim \frac{|C|}{|G|}\mathrm{Li}(x). \tag{1.7}$$

This subsumes many results in the distribution of primes including the distribution of quadratic nonresidues modulo D for any D, primes in arithmetic progressions, and prime ideals for any number field. As such, we are interested in upper and lower bounds of $\pi_C(x, L/F)$ of the correct order of magnitude with an improved range of x.

A lower bound on $\pi_C(x, L/F)$ with the correct order of magnitude (in the x-aspect) follows from the work of Weiss [27], which was recently made explicit by Thorner and Zaman [24]. Let $H \subset G$ be a largest abelian subgroup such that $H \cap C$ is nonempty, and let K be the fixed field of H. For a character χ in the dual group $\widehat{H}$, let $\mathfrak{f}_\chi$ be the conductor of χ, and define

$$\mathcal{Q}(L/K) = \max\{\mathrm{N}_{K/\mathbb{Q}}\mathfrak{f}_\chi : \chi \in \widehat{H}\}. \tag{1.8}$$

Thorner and Zaman proved that if $x \ge D_K^{694}\mathcal{Q}(L/K)^{521} + D_K^{232}\mathcal{Q}(L/K)^{367}[K:\mathbb{Q}]^{290[K:\mathbb{Q}]}$, then

$$\pi_C(x, L/F) \gg \frac{1}{(D_K\mathcal{Q}(L/K)[K:\mathbb{Q}]^{[K:\mathbb{Q}]})^5}\frac{x}{[L:K]\log x}$$

provided that $D_K\mathcal{Q}(L/K)[K:\mathbb{Q}]^{[K:\mathbb{Q}]}$ is sufficiently large. When this is applied to arithmetic progressions (in which case $L = \mathbb{Q}(e^{2\pi i/q})$ for q sufficiently large and $F = K = \mathbb{Q}$), this yields the bound

$$\pi(x;q,a) \gg \frac{1}{q^5}\frac{x}{\varphi(q)\log x} \quad \text{for } x \geq q^{521}.$$

Up to the quality of the exponents, this is comparable to (1.2).

In analogy with (1.3), Lagarias *et al.* [12] proved that

$$\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x), \qquad \log x \gg (\log D_L)(\log\log D_L)(\log\log\log e^{20}D_L). \tag{1.9}$$

(Serre [22] showed that e^{20} can be replaced with 6.) There are several large sieve inequalities yielding Brun–Titchmarsh type results for counting prime integers in the ring of integers of a number field (e.g., [10, 21]) and for counting prime ideals lying in arithmetic progressions (e.g., [9]), but it appears that (1.9) is the only Brun–Titchmarsh type bound that counts prime ideals with effective field dependence. While the range of x in (1.9) is noticeably less restrictive than the range of x for which (1.7) holds, the range still depends poorly on L; this can be prohibitive for many applications. It does not seem to be the case that sieve methods can produce a range of x that is comparable to (1.4). Using the log-free zero density estimate and zero repulsion results proved by Thorner and Zaman in [24], we improve the range of x in (1.9).

Theorem 1.1. Let L/F be a Galois extension of number fields with Galois group G with $L \neq \mathbb{Q}$. Let C be any conjugacy class of G and let H be an abelian subgroup of G such that $H \cap C$ is nonempty. If K is the subfield of L fixed by H and $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by (1.8), then

$$\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x)$$

provided that

$$x \gg D_K^{246}\mathcal{Q}^{185} + D_K^{82}\mathcal{Q}^{130}[K:\mathbb{Q}]^{246[K:\mathbb{Q}]}. \tag{1.10}$$

□

Remark. For the valid range of x, one can minimize the exponents of D_K and $\mathcal{Q}$ at the expense of a less desirable dependence on $[K:\mathbb{Q}]^{[K:\mathbb{Q}]}$ and vice versa. In particular, the same upper bound for $\pi_C(x, L/F)$ holds when

$$x \gg D_K^{164}\mathcal{Q}^{123} + D_K^{55}\mathcal{Q}^{87}[K:\mathbb{Q}]^{68[K:\mathbb{Q}]} + D_K^{2}\mathcal{Q}^{2}[K:\mathbb{Q}]^{14{,}000[K:\mathbb{Q}]}. \tag{1.11}$$

See the remarks at the end of Section 6 for details. □

Our result always gives an improvement over (1.9). Choosing H to be the cyclic group generated by a fixed element of C, we have that $D_L^{1/|H|} \le D_K\mathcal{Q} \le D_L^{1/\varphi(|H|)}$ (see [27, Section 6]). Moreover, by the classical work of Minkowski, we have that $[K:\mathbb{Q}] \ll \log D_K \le \log D_L$. Therefore, Theorem 1.1 holds when $\log x \gg (\log D_L)(\log\log D_L)$, which is a modest unconditional improvement over (1.9). However, one usually obtains a more significant improvement. For most fields K, the bound $[K:\mathbb{Q}] \ll (\log D_K)/\log\log D_K$ holds. In this case, we may take $\log x \gg \log(D_K\mathcal{Q})$ in Theorem 1.1. Thus Theorem 1.1 holds when $\log x \gg (\log D_L)/\varphi(|H|)$, which noticeably improves (1.9).

Building on [16], we obtain an implied constant that is essentially sharp (short of precluding the existence of Landau–Siegel zeros) when x is sufficiently large in terms of L/F.

Theorem 1.2. Let L/F be a Galois extension of number fields with Galois group G and let C be any conjugacy class of G. Let H be an abelian subgroup of G such that $H \cap C$ is nonempty. If K is the subfield of L fixed by H and $\mathcal{Q} = \mathcal{Q}(L/K)$ is given by (1.8), then

$$\pi_C(x, L/F) < \left\{2 + O\left([K:\mathbb{Q}]x^{-\frac{1}{166[K:\mathbb{Q}]+327}}\right)\right\}\frac{|C|}{|G|}\mathrm{Li}(x)$$

for

$$x \gg D_K^{695}\mathcal{Q}^{522} + D_K^{232}\mathcal{Q}^{367}[K:\mathbb{Q}]^{290[K:\mathbb{Q}]} \tag{1.12}$$

provided that $D_K\mathcal{Q}[K:\mathbb{Q}]^{[K:\mathbb{Q}]}$ is sufficiently large. If any of the following conditions also hold, then the error term can be omitted:

- There exists a sequence of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that K_{j+1}/K_j is a normal extension for all $j = 0, 1, \ldots, n-1$.
- $(2[K:\mathbb{Q}])^{2[K:\mathbb{Q}]^2} \ll D_K\mathcal{Q}^{1/2}$.
- $x \gg [K:\mathbb{Q}]^{334[K:\mathbb{Q}]^2}$. □

In the special case where $L/\mathbb{Q}$ is an abelian Galois extension, we may take $K = \mathbb{Q}$ in Theorem 1.2. Since $\mathbb{Q}/\mathbb{Q}$ is trivially a normal extension, the error term in Theorem 1.2 can be omitted, and we recover Maynard's result in (1.5) for $\theta \leq 1/522$. (See the remark at the end of Section 7 for details.) Another interesting set of primes for which the normal tower condition in Theorem 1.2 applies is the set of primes represented by binary quadratic forms. Suppose $Q(X, Y)$ is a positive-definite primitive binary integral quadratic form with discriminant $-D$. It is well known that such forms, up to SL_2-equivalence, form a group which is isomorphic to the ring class group of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ (see, e.g., [3, Theorem 7.1]). Further, a rational prime p is represented by $Q(X, Y)$ if and only if there exists a prime ideal $\mathfrak{p}$ in $\mathbb{Q}(\sqrt{-D})$ such that its norm equals p and $\mathfrak{p}$ belongs to the corresponding class of $Q(X, Y)$. It follows by the Chebotarev density theorem that

$$\#\{p \leq x : p \text{ is represented by } Q(X, Y)\} \sim \delta_Q \frac{\mathrm{Li}(x)}{h(-D)} \qquad \text{as } x \to \infty, \tag{1.13}$$

where $\delta_Q = 1/2$ if $Q(X, Y)$ is properly equivalent to its opposite and $\delta_Q = 1$ otherwise, and $h(-D)$ is the number of such forms of discriminant $-D$ up to SL_2-equivalence. To obtain an upper bound for the number of such primes, we let $F = \mathbb{Q}(\sqrt{-D})$, and we let L be the ring class field of the order of the discriminant $-D$. Thus $\mathrm{Gal}(L/F)$ is abelian. Applying (1.11) and Theorem 1.2 to L/F, with C equal to the singleton conjugacy class in G corresponding to $Q(X, Y)$, we obtain the following.

Corollary 1.3. Let $Q(X, Y)$ be a positive-definite primitive binary integral quadratic form with discriminant $-D$, and let $h(-D)$ be the number of such quadratic forms up to SL_2-equivalence. For $x \gg D^{164}$,

$$\#\{p \leq x \colon p \text{ is represented by } Q(X, Y)\} \ll \frac{\mathrm{Li}(x)}{h(-D)} \tag{1.14}$$

with an absolute implied constant. Also, if D is sufficiently large, then for $x \gg D^{695}$,

$$\#\{p \leq x \colon p \text{ is represented by } Q(X, Y)\} < 2\delta_Q \frac{\mathrm{Li}(x)}{h(-D)},$$

where $\delta_Q = 1/2$ if $Q(X, Y)$ is properly equivalent to its opposite and $\delta_Q = 1$ otherwise. □

Remark. Note that (1.9) also implies (1.14) in the much more restricted range $x \gg D^{O(D^{1/2+\epsilon})}$ for any fixed $\epsilon > 0$. On the other hand, Corollary 1.3 gives the range $x \gg D^{O(1)}$, which is comparable (up to the quality of the exponent) to the range $x \gg D^{1+\epsilon}$ predicted by the generalized Riemann hypothesis for Hecke L-functions. □

We use Theorem 1.1 to improve the best unconditional upper bounds for two outstanding conjectures of Lang and Trotter [14]. Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} \tag{1.15}$$

be a holomorphic cusp form of even integral weight $k_f \geq 2$ and level N_f; for simplicity, we assume that $a_f(n) \in \mathbb{Z}$ for all $n \geq 1$. Suppose that f does not have complex multiplication, that the nebentypus of f is trivial, and that f is a newform (i.e., f is a normalized eigenform for the Hecke operators T_p for $p \nmid N_f$ and U_p for $p \mid N_f$). Fix $a \in \mathbb{Z}$, and let

$$\pi_f(x, a) = \#\{p \leq x : a_f(p) = a\}. \tag{1.16}$$

Lang and Trotter conjectured that as $x \to \infty$, we have that

$$\pi_f(x, a) \sim c_{f,a} \begin{cases} \sqrt{x}(\log x)^{-1} & \text{if } k_f = 2, \\ 1 & \text{if } k_f \geq 4, \end{cases}$$

where $c_{f,a} \geq 0$ is a certain constant depending on f and a alone.

In the special case where $k_f = 2$, Elkies [5] proved that $\pi_f(x, 0) \ll_{N_f} x^{3/4}$. In all other cases, Serre proved in 1981 that

$$\pi_f(x, a) \ll_{N_f} \frac{x}{(\log x)^{1+\delta}}$$

for any $\delta < 1/4$; following the ideas of Murty *et al.* [19], Wan [26] improved the range of δ in 1990 to any $\delta < 1$. This was further sharpened by Murty [20] in 1997; he proved that

$$\pi_f(x, a) \ll_{N_f} \frac{x(\log\log x)^3}{(\log x)^2}. \tag{1.17}$$

Using Theorem 1.1, we give a modest improvement.

Theorem 1.4. Let f be a newform of even integral weight $k_f \geq 2$, level N_f, and trivial nebentypus with integral coefficients. If $\pi_f(x, a)$ is given by (1.16), then

$$\pi_f(x, a) \ll_{N_f} \frac{x(\log\log x)^2}{(\log x)^2}. \qquad \square$$

Remark. Theorem 5.1 of [20] actually claims a stronger result than (1.17), but a step in the proof seems not to be justified. The best that the argument appears to give is what we have stated above in (1.17); see Section 9 for details. Note that we recover the claimed result [20, Theorem 5.1]. □

We also consider a different (but closely related) conjecture of Lang and Trotter regarding the Frobenius fields of an elliptic curve. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E without complex multiplication. For a prime $p \nmid N$, let Π_p be the Frobenius endomorphism of $E/\mathbb{F}_p$. Defining $a_E(p) = p+1-\#E(\mathbb{F}_p)$, we have that $\Pi_p^2 - a_E(p)\Pi_p + p = 0$. By Hasse, we know that $|a_E(p)| < 2\sqrt{p}$, so $\mathbb{Q}(\Pi_p)$ in $\mathrm{End}(E/\mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field. For a fixed imaginary quadratic field k with absolute discriminant D_k, let

$$\pi_E(x,k) = \#\{p \le x : \mathbb{Q}(\Pi_p) \cong k\}. \tag{1.18}$$

Lang and Trotter conjectured that as $x \to \infty$,

$$\pi_E(x,k) \sim c_{E,k} \frac{\sqrt{x}}{\log x},$$

where $c_{E,k}$ is a certain constant depending on E and k alone. Using the square sieve, Cojocaru *et al.* [2] proved that

$$\pi_E(x,k) \ll_{N_E,k} \frac{x(\log\log x)^{13/12}}{(\log x)^{25/24}}.$$

Using Murty's version of the Chebotarev density theorem and Serre's method of mixed representations (see [22]), Zywina [30] improved this bound to

$$\pi_E(x,k) \ll_{N_E,k} \frac{x(\log\log x)^2}{(\log x)^2}. \tag{1.19}$$

Using Theorem 1.1, we establish a modest improvement to (1.19).

Theorem 1.5. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E and let k be a fixed imaginary quadratic number field. If $\pi_E(x,k)$ is defined by (1.18) then

$$\pi_E(x,k) \ll_{N_E,k} \frac{x\log\log x}{(\log x)^2}.$$ □

Remark. A similar infinite Galois extension problem is described by Theorem 10 in Section 4.1 of [22], and Theorem 1.1 gives a similar improvement. □

In Sections 2–5, we discuss necessary results on Hecke L-functions and provide the analytic setup for the proofs of Theorems 1.1 and 1.2. These results are then proved in Sections 6–8. Finally, we prove Theorems 1.4 and 1.5 in Section 9.

2 Initial Setup

2.1 Notation

For a number field F, we will use the following notation throughout:

- $\mathcal{O}_F$ is the ring of integers of F.
- $n_F = [F : \mathbb{Q}]$ is the degree of $F/\mathbb{Q}$.
- $D_F = |\mathrm{disc}\,(F/\mathbb{Q})|$ is the absolute value of the discriminant of F.
- $\mathrm{N}_{F/\mathbb{Q}}$ is the absolute field norm of F.
- $\zeta_F(s)$ is the Dedekind zeta function of F.
- $\mathfrak{p}$ is a prime ideal of F.
- $\mathfrak{n}$ is an integral ideal of F.
- $\Lambda_F(\mathfrak{n})$ is the von Mangoldt Λ-function for F given by

$$\Lambda_F(\mathfrak{n}) = \begin{cases} \log \mathrm{N}_{F/\mathbb{Q}}\mathfrak{p} & \text{if } \mathfrak{n} \text{ is a power of a prime ideal } \mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

If it is clear from context, we will write $\mathrm{N} = \mathrm{N}_{F/\mathbb{Q}}$ for convenience.

We also adhere to the convention that all implied constants in all asymptotic inequalities $f \ll g$ or $f = O(g)$ are absolute. If an implied constant depends on a field-independent parameter, such as ϵ, then we use $\ll_\epsilon$ and O_ϵ to denote that the implied constant depends at most on ϵ. All implied constants will be effectively computable.

2.2 Prime ideal counting functions

We briefly recall the definition of an Artin L-function from [18, Chapter 2, Section 2]. Let L/F be a Galois extension of number fields with Galois group G. For each prime ideal $\mathfrak{p}$ of F, and a prime ideal $\mathfrak{P}$ of L lying above $\mathfrak{p}$, we define the decomposition group $D_{\mathfrak{P}}$ to be $\mathrm{Gal}(L_{\mathfrak{P}}/F_{\mathfrak{p}})$, where $L_{\mathfrak{P}}$ (resp. $K_{\mathfrak{p}}$) is the completion of L (resp. K) at $\mathfrak{P}$ (resp. $\mathfrak{p}$). We have a map $D_{\mathfrak{P}}$ to $\mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$ (the Galois group of the residue field extension), which is surjective by Hensel's lemma. The kernel of this map is the inertia group $I_{\mathfrak{P}}$. We thus have the exact sequence

$$1 \to I_{\mathfrak{P}} \to D_{\mathfrak{P}} \to \mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}}) \to 1.$$

The group $\mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$ is cyclic with generator $x \mapsto x^{\mathrm{N}\mathfrak{p}}$, where $\mathrm{N}\mathfrak{p}$ is the cardinality of $k_{\mathfrak{p}}$. We can choose an element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}$ whose image in $\mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$ is this generator. We call $\sigma_{\mathfrak{P}}$ a Frobenius element at $\mathfrak{P}$; it is well-defined modulo $I_{\mathfrak{P}}$. We have that $I_{\mathfrak{P}}$ is trivial for all unramified $\mathfrak{p}$, and for these $\mathfrak{p}$, $\sigma_{\mathfrak{P}}$ is well-defined. For $\mathfrak{p}$ unramified, we denote by $\sigma_{\mathfrak{p}}$ the conjugacy class of Frobenius elements at primes $\mathfrak{P}$ above $\mathfrak{p}$.

Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G, and let ψ denote its character. Let V be the underlying complex vector space on which ρ acts, and let $V^{I_{\mathfrak{P}}}$ be the subspace of V on which $I_{\mathfrak{P}}$ acts trivially. We now define

$$L_{\mathfrak{p}}(s,\psi,L/F) = \begin{cases} \det\,(I_n - \rho(\sigma_{\mathfrak{p}})\mathrm{N}\mathfrak{p}^{-s})^{-1} & \text{if } \mathfrak{p} \text{ is unramified in } L, \\ \det\,(I_n - \rho(\sigma_{\mathfrak{P}})\mid_{V^{I_{\mathfrak{P}}}} \mathrm{N}\mathfrak{p}^{-s})^{-1} & \text{if } \mathfrak{p} \text{ is ramified in } L. \end{cases}$$

This is well defined for all $\mathfrak{p}$, which allows us to define the Artin L-function

$$L(s,\psi,L/F) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s,\psi,L/F)$$

for $\mathrm{Re}\{s\} > 1$. Now, for a conjugacy class $C \subseteq G$, let $g_C \in C$ be arbitrary. Define

$$Z_C(s) := -\frac{|C|}{|G|}\sum_{\psi} \overline{\psi}(g_C)\frac{L'}{L}(s,\psi,L/F), \tag{2.1}$$

where ψ runs over irreducible characters of G and $L(s,\psi,L/F)$ is the associated Artin L-function. Note the definition of $Z_C(s)$ does not depend on the choice of g_C since ψ is the trace of the representation ρ and g_C is conjugate to any other choice. By orthogonality relations for characters (see [8, Section 3] for example),

$$Z_C(s) = \sum_{\mathfrak{n}\subseteq\mathcal{O}_F} \Lambda_F(\mathfrak{n})\Theta_C(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s}, \tag{2.2}$$

where $\Theta_C(\mathfrak{n})$ is supported on integral ideals $\mathfrak{n}$ which are powers of a prime ideal; in particular, for prime ideals $\mathfrak{p}$ unramified in L and $m \geq 1$,

$$\Theta_C(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } [\frac{L/F}{\mathfrak{p}}]^m \subseteq C, \\ 0 & \text{otherwise}, \end{cases} \tag{2.3}$$

and $0 \leq \Theta_C(\mathfrak{p}^m) \leq 1$ if $\mathfrak{p}$ ramifies in L. (This discussion and definition of $\Theta_C(\cdot)$ is also contained in [12, Section 3].) For $x > 1$, define

$$\psi_C(x) := \sum_{\mathrm{N}\mathfrak{n}<x} \Lambda_F(\mathfrak{n})\Theta_C(\mathfrak{n}), \tag{2.4}$$

where the sum is over integral ideals $\mathfrak{n}$ of F. By standard arguments, this prime ideal counting function is related to $\pi_C(x, L/F)$ given by (1.6). Since we are only interested in an upper bound for $\pi_C(x, L/F)$, we give a simpler statement that suffices for our purposes.

Lemma 2.1. If $x > x_0 > 3$, then

$$\pi_C(x, L/F) \leq \frac{\psi_C(x)}{\log x} + \int_{x_0}^{x} \frac{\psi_C(t)}{t \log^2 t} \mathrm{d}t + O(n_F x_0). \qquad \square$$

Proof. Let $t > 1$. We define

$$\tilde{\pi}_C(t) := \sum_{\mathrm{N}\mathfrak{p} < t} \Theta_C(\mathfrak{p}), \qquad \theta_C(t) := \sum_{\mathrm{N}\mathfrak{p} < t} \Theta_C(\mathfrak{p}) \log \mathrm{N}\mathfrak{p},$$

where the sums are over all prime ideals $\mathfrak{p}$ of F. First, observe that, by (2.3), the only difference between $\tilde{\pi}_C(x)$ and $\pi_C(x, L/F)$ is the contribution from the prime ideals $\mathfrak{p}$ of F ramified in L. Since $0 \leq \Theta_C(\mathfrak{p}) \leq 1$ for such prime ideals, we observe that

$$\pi_C(x, L/F) \leq \tilde{\pi}_C(x), \tag{2.5}$$

so it suffices to estimate $\tilde{\pi}_C(x)$. Using partial summation, we see that if $3 < x_0 < x$, then

$$\tilde{\pi}_C(x) = \frac{\theta_C(x)}{\log x} + \int_{x_0}^{x} \frac{\theta_C(t)}{t \log^2 t} \mathrm{d}t + \tilde{\pi}_C(x_0). \tag{2.6}$$

Since there are at most n_F prime ideals above a rational prime p, observe that

$$\tilde{\pi}_C(x_0) \leq \sum_{p < x_0} \sum_{\mathfrak{p} | (p)} 1 \leq n_F \sum_{p < x_0} 1 \ll \frac{n_F x_0}{\log x_0} \ll n_F x_0. \tag{2.7}$$

Moreover, $\theta_C(t) \leq \psi_C(t)$ for all $t > 1$. Combining these observations with (2.5) and (2.6) yields the desired result. ∎

2.3 Choice of weight

Let us define a weight function and describe its properties. This choice of weight can be regarded as a smoothed version of Maynard's weight [16, Equation (5.6)]. It will be used to count prime ideals with norm between $x^{1/2}$ and x.

Lemma 2.2. For any $x \geq 3, \epsilon \in (0, 1/4)$, and positive integer $\ell \geq 1$, select

$$A = \frac{\epsilon}{2\ell \log x}.$$

There exists a real-variable function $f(t) = f(t; x, \ell, \epsilon)$ such that:

(i) $0 \leq f(t) \leq 1$ for all $t \in \mathbb{R}$, and $f(t) \equiv 1$ for $\frac{1}{2} \leq t \leq 1$.

(ii) The support of f is contained in the interval $[\frac{1}{2} - \frac{\epsilon}{\log x}, 1 + \frac{\epsilon}{\log x}]$.

(iii) Its Laplace transform $F(z) = \int_{\mathbb{R}} f(t) e^{-zt} dt$ is entire and is given by

$$F(z) = e^{-(1+2\ell A)z} \cdot \left(\frac{1 - e^{(\frac{1}{2}+2\ell A)z}}{-z} \right) \left(\frac{1 - e^{2Az}}{-2Az} \right)^{\ell}. \tag{2.8}$$

(iv) Let $s = \sigma + it \in \mathbb{C}, \sigma > 0$ and α be any real number satisfying $0 \leq \alpha \leq \ell$. Then

$$|F(-s \log x)| \leq \frac{e^{\sigma\epsilon} x^{\sigma}}{|s| \log x} \cdot (1 + x^{-\sigma/2}) \cdot \left(\frac{2\ell}{\epsilon |s|} \right)^{\alpha}.$$

(v) If $s = \sigma + it \in \mathbb{C}$ and $\sigma > 0$, then

$$|F(-s \log x)| \leq e^{\sigma\epsilon} x^{\sigma}.$$

Moreover,

$$1/2 < F(0) < 3/4, \qquad F(-\sigma \log x) \leq \frac{e^{\epsilon} x^{\sigma}}{\sigma \log x}.$$

(vi) Let $s = -\frac{1}{2} + it \in \mathbb{C}$. Then

$$|F(-s \log x)| \leq \frac{5x^{-1/4}}{\log x} \left(\frac{2\ell}{\epsilon} \right)^{\ell} (1/4 + t^2)^{-\ell/2}.$$

□

Remark. Our choice is motivated by the works of Weiss [27, Lemma 3.2] and the authors [24, Lemma 9.1] on the least prime ideal. Namely, the weight function f depends on a parameter ℓ which will be chosen to be of size $O(n_K)$. This forces f to be $O(n_K)$-times differentiable and hence $F(x + iy)$ will decay like $|y|^{-O(n_K)}$ for fixed $x > 0$ and $|y| \to \infty$. This decay rate will be necessary when applying log-free zero density estimates such as Theorem 4.5 to bound the contribution of zeros which are high in the critical strip. □

Proof.

- For parts (i) and (ii), let $\mathbf{1}_S(\cdot)$ be an indicator function for the set $S \subseteq \mathbb{R}$. For $j \geq 1$, define

$$w(t) := \frac{1}{2A}\mathbf{1}_{[-A,A]}(t), \quad g_0(t) := \mathbf{1}_{[\frac{1}{2}-\ell A, 1+\ell A]}(t), \quad \text{and} \quad g_j(t) := (w * g_{j-1})(t).$$

Since $\int_{\mathbb{R}} w(t)\mathrm{d}t = 1$, one can verify that $f = g_\ell$ satisfies (i) and (ii).
- For part (iii), observe the Laplace transform $W(z)$ of w is given by

$$W(z) = \frac{\mathrm{e}^{Az} - \mathrm{e}^{-Az}}{2Az} = \mathrm{e}^{-Az} \cdot \left(\frac{1 - \mathrm{e}^{2Az}}{-2Az}\right),$$

and the Laplace transform $G_0(z)$ of g_0 is given by

$$G_0(z) = \frac{\mathrm{e}^{-(1/2-\ell A)z} - \mathrm{e}^{-(1+\ell A)z}}{z} = \mathrm{e}^{-(1+\ell A)z} \cdot \left(\frac{1 - \mathrm{e}^{(\frac{1}{2}+2\ell A)z}}{-z}\right).$$

Thus (iii) follows as $F(z) = G_0(z) \cdot W(z)^\ell$.
- For part (iv), we see by (iii) and the definition of A that

$$|F(-s\log x)| \leq \frac{\mathrm{e}^{\sigma\epsilon}x^\sigma}{|s|\log x} \cdot \left(1 + e^{-\sigma\epsilon}x^{-\sigma/2}\right)\left|\frac{1 - \mathrm{e}^{-2As\log x}}{2As\log x}\right|^\ell. \tag{2.9}$$

To bound the above quantity, we observe that

$$\left|\frac{1 - \mathrm{e}^{-w}}{w}\right|^2 \leq \left(\frac{1 - \mathrm{e}^{-a}}{a}\right)^2 \leq 1 \tag{2.10}$$

for $w = a + ib$ with $a > 0$ and $b \in \mathbb{R}$. This observation can be checked in a straightforward manner. Using (2.10), it follows that

$$\begin{aligned}\left|\frac{1 - \mathrm{e}^{-2As\log x}}{2As\log x}\right|^\ell &= \left|\frac{1 - \mathrm{e}^{-2As\log x}}{2As\log x}\right|^\alpha \cdot \left|\frac{1 - \mathrm{e}^{-2As\log x}}{2As\log x}\right|^{\ell-\alpha} \\ &\leq \left(\frac{1 + x^{-2A\sigma}}{2A|s|\log x}\right)^\alpha \cdot 1 \leq \left(\frac{2\ell}{\epsilon|s|}\right)^\alpha.\end{aligned}$$

In the last step, we noted $1+x^{-2A\sigma} \leq 2$ and used the definition of A. Combining this with (2.9) and observing $e^{-\sigma\epsilon} \leq 1$, we deduce the desired bound.

- For part (v), we see by (iii) that

$$|F(-s\log x)| \leq \left(\frac{1}{2}+2\ell A\right) \mathrm{e}^{\sigma\epsilon} x^{\sigma} \cdot \left|\frac{1-\mathrm{e}^{-(\frac{1}{2}+2\ell A)s\log x}}{(\frac{1}{2}+2\ell A)s\log x}\right| \cdot \left|\frac{1-\mathrm{e}^{-2As\log x}}{2As\log x}\right|^{\ell}$$
$$\leq \mathrm{e}^{\sigma\epsilon} x^{\sigma},$$

where the second inequality follows from an application of (2.10) and the observation that $\frac{1}{2}+2\ell A < \frac{1}{2}+\epsilon < 1$. For $s=\sigma>0$, observe that $F(-\sigma\log x)$ is real and positive. Thus, by (iii) and (2.10),

$$F(-\sigma\log x) \leq \mathrm{e}^{\sigma\epsilon} x^{\sigma} \cdot \left(\frac{1-x^{-(\frac{1}{2}+2\ell A)\sigma}}{\sigma\log x}\right) \cdot \left(\frac{1-x^{-2A\sigma}}{2A\sigma\log x}\right)^{\ell}$$
$$\leq \frac{\mathrm{e}^{\sigma\epsilon} x^{\sigma}}{\sigma\log x} \cdot \left(\frac{1-x^{-2A\sigma}}{2A\sigma\log x}\right)^{\ell}$$
$$\leq \frac{\mathrm{e}^{\sigma\epsilon} x^{\sigma}}{\sigma\log x}.$$

This completes the proof of all cases of (iv).

- For part (vi), we shall argue as in (iv). Rearranging (iii), notice that

$$|F(z)| = \left|\mathrm{e}^{(-\frac{1}{2}+2\ell A)z} \cdot \left(\frac{1-\mathrm{e}^{-(\frac{1}{2}+2\ell A)z}}{z}\right)\left(\frac{1-\mathrm{e}^{-2Az}}{2Az}\right)^{\ell}\right|.$$

If $r := \mathrm{Re}\{z\} > 0$, then

$$|F(z)| \leq \mathrm{e}^{(-\frac{1}{2}+2\ell A)r} \cdot \frac{1+\mathrm{e}^{-(\frac{1}{2}+2\ell A)r}}{|z|} \cdot \left(\frac{1+\mathrm{e}^{-2Ar}}{2A|z|}\right)^{\ell}$$
$$\leq \frac{2\mathrm{e}^{(-\frac{1}{2}+2\ell A)r}}{|z|}\left(\frac{1}{A|z|}\right)^{\ell}.$$

If we substitute $z=-s\log x = (\frac{1}{2}-it)\log x$, then it follows by the definition of A that

$$|F(-s\log x)| \leq \frac{2\mathrm{e}^{\epsilon/2}x^{-1/4}}{|\frac{1}{2}+it|\log x}\left(\frac{2\ell}{\epsilon|\frac{1}{2}+it|}\right)^{\ell} \leq \frac{4\mathrm{e}^{\epsilon/2}x^{-1/4}}{\log x}\left(\frac{2\ell}{\epsilon}\right)^{\ell}(1/4+t^2)^{-\ell/2}.$$

This yields (vi) since $4\mathrm{e}^{\epsilon/2} < 5$ for $\epsilon < 1/4$. ■

3 Preliminary Analysis

3.1 A weighted sum of prime ideals

For $x > 3, \epsilon \in (0,1/4)$ and integer $\ell \geq 1$, use the compactly supported weight $f(\cdot) = f(\cdot\,; x, \ell, \epsilon)$ defined in Lemma 2.2 and set

$$S(x) = S_{\ell,\epsilon}(x) := \sum_{\mathfrak{n} \subseteq \mathcal{O}_F} \Lambda_F(\mathfrak{n}) \Theta_C(\mathfrak{n}) f\left(\frac{\log \mathrm{N}\mathfrak{n}}{\log x} \right). \tag{3.1}$$

We reduce our estimation of $\pi_C(x, L/F)$ given by (1.6) to the smoothed version $S(x)$.

Lemma 3.1. Let $x_0 > \mathrm{e}^4$. Suppose there exist constants $a, b \geq 0$ and $0 \leq c \leq 1/2$, all of which are independent of x, such that $S(x) < \{a + bx^{-c}\} \frac{|C|}{|G|} x$ for all $x \geq x_0$. Then, for all $x \geq x_0$,

$$\pi_C(x, L/F) < \left\{ a + 2bx^{-c} + O\left(\frac{n_L}{x^{1/2}} + \frac{n_L x_0 \log x}{x} \right) \right\} \frac{|C|}{|G|} \mathrm{Li}(x). \qquad \square$$

Proof. If $t > 1$, then

$$\psi_C(t) = \sum_{t^{1/2} \leq \mathrm{N}\mathfrak{n} < t} \Theta_C(\mathfrak{n}) \Lambda_K(\mathfrak{n}) + \psi_C(t^{1/2}). \tag{3.2}$$

The sum in (3.2) is bounded by $S(t)$ in (3.1) because of Lemma 2.2(i), while the secondary term in (3.2) is estimated much like (2.7). Thus, we have that

$$\psi_C(t) \leq S(t) + O(n_F t^{1/2}). \tag{3.3}$$

We substitute (3.3) into Lemma 2.1 and deduce that

$$\pi_C(x, L/F) \leq \frac{S(x)}{\log x} + \int_{x_0}^{x} \frac{S(t)}{t \log^2 t} \mathrm{d}t + O\left(\frac{n_F x^{1/2}}{\log x} + n_F x_0 \right).$$

From our assumption on $S(t)$ for $t \geq x_0$, it follows that

$$\pi_C(x, L/F) < a\frac{|C|}{|G|} \mathrm{Li}(x) + b\frac{|C|}{|G|} \left[\frac{x^{1-c}}{\log x} + \int_{x_0}^{x} \frac{t^{-c}}{\log^2 t} \mathrm{d}t \right] + O\left(\frac{n_F x^{1/2}}{\log x} + n_F x_0 \right). \tag{3.4}$$

Note that if $0 \leq c \leq 1/2$, then $t^{1-c}/\log^2 t$ is an increasing function of t for $t > \mathrm{e}^4$. Since $x_0 > e^4$ and $\mathrm{Li}(x) > \frac{x}{\log x}$ for $x > \mathrm{e}^4$, we conclude that

$$\int_{x_0}^{x} \frac{t^{-c}}{\log^2 t} \mathrm{d}t = \int_{x_0}^{x} \frac{t^{1-c}}{\log^2 t} \frac{\mathrm{d}t}{t} \leq \frac{x^{1-c}}{\log^2 x} \int_{x_0}^{x} \frac{\mathrm{d}t}{t} \leq \frac{x^{1-c}}{\log x} < x^{-c} \, \mathrm{Li}(x). \tag{3.5}$$

The desired result follows from (3.4), (3.5), and the identity $n_L = [L : F] n_F = |G| n_F$. ■

3.2 Reduction to Hecke L-functions

By Mellin inversion, (3.1), and (2.2), it follows that

$$S(x)=\frac{\log x}{2\pi i}\int_{2-i\infty}^{2+i\infty}Z_C(s)F(-s\log x)\mathrm{d}s. \tag{3.6}$$

To shift the contour, we must rewrite $Z_C(s)$, defined by (2.1), in terms of L-functions which exhibit an analytic continuation to the left of $\mathrm{Re}\{s\}=1$.

To this end, let $H\subseteq G$ be an abelian subgroup such that $H\cap C$ is nonempty, and choose g_C in Section 2.2 so that $g_C\in H\cap C$. Let $K=L^H$ be the subfield of L fixed by H. By standard arguments (see [4, Theorem 3.7] and [12, Section 3]), we have that

$$Z_C(s)=-\frac{|C|}{|G|}\sum_{\chi\in\hat{H}}\overline{\chi}(g_C)\frac{L'}{L}(s,\chi,L/K), \tag{3.7}$$

where the sum runs over certain primitive Hecke characters χ of K satisfying

$$\chi(\mathfrak{P})=\chi\left(\left[\frac{L/K}{\mathfrak{P}}\right]\right)$$

for prime ideals $\mathfrak{P}$ of K that are unramified in L. Substituting (3.7) into (3.6), we conclude that

$$S(x)=\frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g_C)\frac{\log x}{2\pi i}\int_{2-i\infty}^{2+i\infty}-\frac{L'}{L}(s,\chi,L/K)F(-s\log x)\mathrm{d}s. \tag{3.8}$$

Henceforth, any sum over χ is over all $\chi\in\hat{H}$. These are equivalently the Hecke characters attached to the abelian extension L/K by class field theory.

3.3 Hecke L-functions

For a more detailed reference on Hecke L-functions, see [13] for example. Suppose L/K is an abelian extension, so all irreducible representations of $\mathrm{Gal}(L/K)$ are 1-dimensional *primitive* Hecke characters χ satisfying

$$\chi(\mathfrak{P})=\chi\left(\left[\frac{L/K}{\mathfrak{P}}\right]\right)$$

for prime ideals $\mathfrak{P}$ of K that are unramified in L. The Hecke L-function of χ is defined by

$$L(s,\chi,L/K)=\sum_{\mathfrak{N}\subseteq\mathcal{O}_K}\chi(\mathfrak{N})\mathrm{N}\mathfrak{N}^{-s}=\prod_{\mathfrak{P}}\left(1-\frac{\chi(\mathfrak{P})}{\mathrm{N}\mathfrak{P}^s}\right)^{-1} \tag{3.9}$$

for $\mathrm{Re}\{s\} > 1$, where the sum is over integral ideals $\mathfrak{N}$ of K and the product is over prime ideals $\mathfrak{P}$ of K. For this subsection only, we write $L(s,\chi) = L(s,\chi,L/K)$ and suppress the implicit dependence of quantities on the extension L/K. Define the completed Hecke L-function $\xi(s,\chi)$ by

$$\xi(s,\chi) = (s(s-1))^{\delta(\chi)} D_\chi^{s/2} \gamma_\chi(s) L(s,\chi), \tag{3.10}$$

where $D_\chi = D_K \mathrm{N}\mathfrak{f}_\chi$, the K-integral ideal $\mathfrak{f}_\chi$ is the conductor of χ, $\delta(\chi)$ is the indicator function for the trivial character, and $\gamma_\chi(s)$ is the *gamma factor of* χ defined by

$$\gamma_\chi(s) = \left[\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right]^{a(\chi)} \cdot \left[\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right]^{b(\chi)}.$$

Here $a(\chi)$ and $b(\chi)$ are certain nonnegative integers satisfying

$$a(\chi) + b(\chi) = n_K. \tag{3.11}$$

It is well known that $\xi(s,\chi)$ is entire of order 1 and satisfies the functional equation

$$\xi(s,\chi) = w(\chi)\xi(1-s,\overline{\chi}),$$

where $w(\chi) \in \mathbb{C}$ is the root number of χ satisfying $|w(\chi)| = 1$. The zeros of $\xi(s,\chi)$ are the nontrivial zeros ρ of $L(s,\chi)$ and are known to satisfy $0 < \mathrm{Re}\{\rho\} < 1$. The trivial zeros ω of $L(s,\chi)$ are given by

$$\operatorname*{ord}_{s=\omega} L(s,\chi) = \begin{cases} a(\chi) - \delta(\chi) & \text{if } \omega = 0, \\ b(\chi) & \text{if } \omega = -1,-3,-5,\ldots, \\ a(\chi) & \text{if } \omega = -2,-4,-6,\ldots, \end{cases} \tag{3.12}$$

and arise as poles of the gamma factor of $L(s,\chi)$.

3.4 Shifting a contour integral

Next we shift the contour (3.8) and bound $S(x)$ in terms of the nontrivial zeros of Hecke L-functions. Henceforth write $S = S(x)$ for simplicity. Recall f depends on the arbitrary quantities $x > 3, \epsilon \in (0,1/4)$ and an integer $\ell \geq 1$.

Lemma 3.2. Assume $\ell \geq 2$. Then

$$\frac{|G|}{|C|}\frac{S}{e^\epsilon x} \leq 1 + \frac{\log x}{e^\epsilon x}\sum_\chi \sum_{\rho_\chi} |F(-\rho_\chi \log x)| + O\left(n_L x^{-1}\log x + x^{-5/4}(2\ell/\epsilon)^\ell \log D_L\right), \tag{3.13}$$

where the outer sum is over all Hecke characters χ of the abelian extension L/K and the inner sum runs over all nontrivial zeros ρ_χ of $L(s,\chi,L/K)$, counted with multiplicity. □

Proof. Shift the contour in (3.8) to the line $\mathrm{Re}\{s\} = -\frac{1}{2}$. This picks up the nontrivial zeros of $L(s,\chi,L/K)$, the simple pole at $s = 1$ when χ is trivial, and the trivial zero at $s = 0$ of $L(s,\chi,L/K)$ of order $r(\chi)$. Overall, we see that

$$\begin{aligned}\frac{|G|}{|C|}S &= \log x\left[F(-\log x) - \sum_{\chi}\overline{\chi}(g_C)\sum_{\rho_\chi}F(-\rho_\chi\log x) + O\left(\sum_{\chi}r(\chi)|F(0)|\right)\right]\\ &\quad+\log x\sum_{\chi}\frac{\overline{\chi}(g_C)}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}-\frac{L'}{L}(s,\chi,L/K)F(-s\log x)\mathrm{d}s,\end{aligned} \tag{3.14}$$

where the sum over $\rho = \rho_\chi$ is over all nontrivial zeros of $L(s,\chi,L/K)$, counted with multiplicity. From (3.11) and (3.12), we see that $r(\chi) \le n_K$. Hence, it follows by Lemma 2.2(v) that

$$F(-\log x) \le \frac{\mathrm{e}^{\epsilon}x}{\log x}, \quad \text{and} \quad \sum_{\chi}r(\chi)|F(0)| \le [L:K]n_K = n_L.$$

For the remaining contour, by [13, Lemma 6.2] and the primitivity of χ, we have that

$$-\frac{L'}{L}(s,\chi,L/K) \ll \log D_\chi + n_K\log(|s|+3),$$

for $\mathrm{Re}\{s\} = -1/2$, where D_χ is defined in (3.10). It follows by Lemma 2.2(vi) that

$$\begin{aligned}&\frac{\log x}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}-\frac{L'}{L}(s,\chi,L/K)F(-s\log x)\mathrm{d}s\\ &\ll x^{-1/4}\left(\frac{2\ell}{\epsilon}\right)^{\ell}\int_{-\infty}^{\infty}\frac{\log D_\chi + n_K\log(|t|+3)}{(1/4+t^2)^{\ell/2}}\mathrm{d}t \ll x^{-1/4}\left(\frac{2\ell}{\epsilon}\right)^{\ell}\log D_\chi,\end{aligned}$$

because $n_K \ll \log D_K \le \log D_\chi$ and $\ell \ge 2$. Summing over χ and using the conductor-discriminant formula yields

$$\log x\sum_{\chi}\frac{\overline{\chi}(g_C)}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}-\frac{L'}{L}(s,\chi,L/K)F(-s\log x)\mathrm{d}s \ll x^{-1/4}\left(\frac{2\ell}{\epsilon}\right)^{\ell}\log D_L.$$

Taking absolute value of both sides in (3.14), multiplying both sides by $(\mathrm{e}^{\epsilon}x)^{-1}$, and combining all of these observations yields the desired result. ■

To analyze the sum over zeros in Lemma 3.2, we require some information about the distribution of zeros of Hecke L-functions.

4 Distribution of Zeros of Hecke L-Functions

In this section, we record various results about L-functions $L(s,\chi,L/K)$ where the extension L/K is abelian and hence χ is a Hecke character of K by class field theory. Associated notation and classical results can be found in Section 2. Henceforth, any sum $\sum_\chi$ or product $\prod_\chi$ is over all characters χ of L/K unless otherwise specified.

4.1 Logarithmic quantity

Let $\delta_0 > 0$ be fixed and sufficiently small. For the remainder of the article, denote

$$\mathscr{L} := \begin{cases} (\frac{1}{3}+\delta_0)\log D_K + (\frac{19}{36}+\delta_0)\log \mathcal{Q} + (\frac{5}{12}+\delta_0) n_K \log n_K & \text{if } n_K^{\frac{5n_K}{6}} \geq D_K^{\frac{4}{3}} \mathcal{Q}^{\frac{4}{9}}, \\ (1+\delta_0)\log D_K + (\frac{3}{4}+\delta_0)\log \mathcal{Q} + \delta_0 n_K \log n_K & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\mathcal{Q} = \mathcal{Q}(L/K) = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \in \widehat{\mathrm{Gal}}(L/K)\}$. Notice that

$$\mathscr{L} \geq (1+\delta_0)\log D_K + (\tfrac{3}{4}+\delta_0)\log \mathcal{Q} + \delta_0 n_K \log n_K \quad \text{and} \quad \mathscr{L} \geq (\tfrac{5}{12}+\delta_0) n_K \log n_K \tag{4.2}$$

unconditionally. We exhibit a bound on the degree of the extension L/K in terms of $\mathscr{L}$.

Lemma 4.1. $[L:K] \ll e^{4\mathscr{L}/3}$ and $n_L \ll \mathscr{L} e^{4\mathscr{L}/3}$. □

Proof. Let $\mathfrak{f} = \mathfrak{f}_{L/K}$ be the Artin conductor attached to L/K by class field theory. Let $I(\mathfrak{f})$ be the group of fractional ideals of K relatively prime to $\mathfrak{f}$. By class field theory, there exists a homomorphism $\phi : I(\mathfrak{f}) \to \mathrm{Gal}(L/K)$. Thus $I(\mathfrak{f})/\ker\phi$ is isomorphic to $\mathrm{Gal}(L/K)$. This induces an isomorphism between their respective character groups and therefore,

$$\mathcal{Q}(L/K) = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \in \widehat{\mathrm{Gal}}(L/K)\} = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \in \widehat{I(\mathfrak{f})/\ker\phi}\}.$$

By our previous observations, $|I(\mathfrak{f})/\ker\phi| = |\mathrm{Gal}(L/K)| = [L:K]$. For $\epsilon_0 > 0$ fixed and sufficiently small, we have by [24, Lemma 2.11] that $|I(\mathfrak{f})/\ker\phi| \ll e^{O_{\epsilon_0}(n_K)} D_K^{1/2+\epsilon_0} \mathcal{Q}^{1+\epsilon_0} \ll e^{4\mathscr{L}/3}$ as desired. To bound n_L, observe that $n_L = [L:K]n_K$ and $n_K \ll \mathscr{L}$. ■

4.2 Low-lying zeros

Next, we specify some important zeros of $\prod_\chi L(s,\chi,L/K)$ which will be used in Section 6 to 8. For the remainder of the article, let $\eta > 0$ be sufficiently small and arbitrary.

Consider the multiset of zeros given by

$$\mathcal{Z} := \left\{ \rho \in \mathbb{C} : \prod_{\chi} L(\rho, \chi, L/K) = 0, 0 < \mathrm{Re}\{\rho\} < 1, |\mathrm{Im}(\rho)| \le \eta^{-2} \right\}. \tag{4.3}$$

We select three important zeros of $\mathcal{Z}$ as follows:

- Choose $\rho_1 \in \mathcal{Z}$ such that $\mathrm{Re}\{\rho_1\}$ is maximal. Let χ_1 be its associated Hecke character so $L(\rho_1, \chi_1, L/K) = 0$. Denote

$$\rho_1 = \beta_1 + i\gamma_1 = \left(1 - \frac{\lambda_1}{\mathscr{L}}\right) + i\frac{\mu_1}{\mathscr{L}},$$

  where $\beta_1 = \mathrm{Re}\{\rho_1\}, \gamma_1 = \mathrm{Im}\{\rho_1\}, \lambda_1 > 0$, and $\mu_1 \in \mathbb{R}$.
- Choose $\rho' \in \mathcal{Z} \backslash \{\rho_1, \overline{\rho_1}\}$ satisfying $L(\rho', \chi_1, L/K) = 0$ such that $\mathrm{Re}\{\rho'\}$ is maximal with respect to these conditions. (If ρ_1 is real then $\rho' \in \mathcal{Z} \setminus \{\rho_1\}$ instead, with the other conditions remaining the same.) Similarly denote

$$\rho' = \beta' + i\gamma' = \left(1 - \frac{\lambda'}{\mathscr{L}}\right) + i\frac{\mu'}{\mathscr{L}}.$$

- Choose $\rho_2 \in \mathcal{Z} \backslash \mathcal{Z}_1$ such that $\mathrm{Re}\{\rho_2\}$ is maximal and where $\mathcal{Z}_1$ is the multiset of zeros of $L(s, \chi_1, L/K)$ contained in $\mathcal{Z}$. Let χ_2 be its associated Hecke character so $L(\rho_2, \chi_2, L/K) = 0$. Similarly, denote

$$\rho_2 = \beta_2 + i\gamma_2 = \left(1 - \frac{\lambda_2}{\mathscr{L}}\right) + i\frac{\mu_2}{\mathscr{L}}.$$

If $\lambda_1 < \eta$ then we henceforth refer to ρ_1 as an η-*Siegel zero*. The proof of Theorem 1.2 will be divided according to whether an η-Siegel zero exists or not.

4.3 Zero-free regions

Here we record the current best-known explicit result regarding zero-free regions of Hecke L-functions; see also [1, 11] for earlier results.

Theorem 4.2 (Zaman)**.** For $\mathscr{L}$ sufficiently large depending on η, $\min\{\lambda', \lambda_2\} > 0.2866$. Furthermore, if $\lambda_1 < 0.0875$ then ρ_1 is a simple real zero of $\prod_\chi L(s, \chi, L/K)$ and is associated with a real character χ_1. □

Proof. When L is a narrow ray class field of K to a given modulus and $\eta = 1$ in (4.3), this is implied by [28, Theorems 1.1 and 1.3] since $\mathscr{L}$ satisfies (4.2). For general abelian extensions L/K and any fixed $\eta \in (0, 1)$, one may easily modify [28] to obtain the cited result by following the outline in [24, Section 8]; see [29] for details. ■

4.4 Zero repulsion

Here we record two explicit estimates for zero repulsion when an exceptional zero exists, also known as "Deuring–Heilbronn phenomenon".

Theorem 4.3 (Zaman)**.** Let $\mathscr{L}$ be sufficiently large depending on η. If $\lambda_1 < 0.0875$, then $\min\{\lambda', \lambda_2\} > 0.44$. If $\eta \le \lambda_1 < 0.0875$, then $\min\{\lambda', \lambda_2\} > 0.2103 \log(1/\lambda_1)$. □

Proof. Again, when L is a narrow ray class field of K to a given modulus and $\eta = 1$, this is implied by [28, Theorem 1.4] since $\mathscr{L}$ satisfies (4.2). Similar to the proof of Theorem 4.2, one may modify [28] as outlined in [24, Section 7] to deduce the same theorem for general abelian extensions L/K and $\eta \in (0, 1)$; see [29] for details. ■

Theorem 4.3 is unable to handle exceptional zeros ρ_1 extremely close to 1 due to the requirement $\lambda_1 \ge \eta$. Thus, we include a version of Deuring–Heilbronn phenomenon [24, Theorem 8.3] which repels zeros in the entire critical strip.

Theorem 4.4 (Thorner–Zaman)**.** Let $T \ge 1$ be arbitrary. Suppose χ_1 is a real character and ρ_1 is a real zero. For any character χ of L/K, let $\rho = \beta + i\gamma \ne \rho_1$ be a nontrivial zero of $L(s, \chi, L/K)$ satisfying $1/2 \le \beta < 1$ and $|\gamma| \le T$. For $\mathscr{L}$ sufficiently large, there exists an absolute effectively computable constant $c_1 > 0$ such that

$$\beta < 1 - \frac{\log\left(\dfrac{c_1}{(1-\beta_1)(\mathscr{L} + n_K \log T)}\right)}{81\mathscr{L} + 25 n_K \log T}. \qquad \square$$

4.5 Log-free zero density estimates

Let $\chi \in \widehat{\mathrm{Gal}}(L/K)$ be a Hecke character. Define

$$N(\sigma, T, \chi) := \#\{\rho = \beta + i\gamma : L(\rho, \chi, L/K) = 0, \sigma < \beta < 1, |\gamma| \le T\}$$

for $0 < \sigma < 1$ and $T \ge 1$. Further denote

$$N(\sigma, T) := \sum_{\chi} N(\sigma, T, \chi). \tag{4.4}$$

Amongst all of the results recorded herein on zeros of Hecke L-functions, the proof of Theorem 1.1 only requires the following log-free zero density estimate, which we emphasize does *not* assume $\mathscr{L}$ is sufficiently large. This is a rephrasing of the authors' result [24, Theorem 3.2] using the definition of $\mathscr{L}$ in (4.1).

Theorem 4.5 (Thorner–Zaman). For $0 < \sigma < 1$ and $T \geq 1$, $N(\sigma, T) \ll (e^{162\mathscr{L}} T^{81 n_K + 162})^{1-\sigma}$. □

The proof of Theorem 1.2 also requires a completely explicit zero density estimate for "low-lying" zeros. Define for $0 < \lambda < \mathscr{L}$,

$$\mathcal{N}(\lambda) := \sum_{\chi} N(1 - \tfrac{\lambda}{\mathscr{L}}, \eta^{-2}, \chi). \tag{4.5}$$

Theorem 4.2 states that $\mathcal{N}(0.0875) \leq 1$ and $\mathcal{N}(0.2866) \leq 2$ for $\mathscr{L}$ sufficiently large depending on η. For larger values of λ, we use the following:

Theorem 4.6 (Thorner–Zaman). Assume $\mathscr{L}$ is sufficiently large depending on η. Let $\epsilon_0 > 0$ be fixed and sufficiently small. If $0 < \lambda < \epsilon_0 \mathscr{L}$ then

$$\mathcal{N}(\lambda) \leq e^{162\lambda + 188}.$$

The bounds for $\mathcal{N}(\lambda)$ in [24, Table 1] are superior when $0 < \lambda \leq 1$. □

Proof. See [24, Theorem 8.6] for details. ■

5 Zeros Outside a Low-Lying Rectangle

From Lemma 3.2, it remains to estimate a sum over all nontrivial zeros of all Hecke L-functions $L(s, \chi, L/K)$. In this section, we demonstrate that the contribution of zeros is negligible if the zeros are either high lying or far from the line $\mathrm{Re}\{s\} = 1$. Throughout, we assume $1 \leq B \leq 1000$ is a fixed absolute constant. We begin by considering high-lying zeros.

Lemma 5.1. Let $T_\star \geq 1$ be arbitrary. Let $0 < E < \frac{2}{3}B$ be fixed. Let

$$B > 162 + E, \quad \ell \geq 82 n_K + 162, \quad \tfrac{1}{4} > \epsilon \geq 4\ell x^{-E/(B\ell)}. \tag{5.1}$$

For $x \geq \mathrm{e}^{B\mathscr{L}}$,

$$\frac{\log x}{x}\sum_{\chi}\sum_{\substack{\rho\\|\mathrm{Im}\{\rho\}|>T_\star}}|F(-\rho\log x)| \ll \frac{1}{T_\star}. \tag{5.2}$$

□

Proof. Write $\rho = \beta + i\gamma$ with $\beta = 1 - \frac{\lambda}{\mathscr{L}}$. If $T \geq 1$, then Lemma 2.2(iv) with $\alpha = \ell(1-\beta)$ and our choices of our conditions on ϵ, ℓ, and x imply that

$$\frac{\log x}{x}|F(-\rho\log x)| \leq \frac{2e^{\epsilon}x^{\beta-1}}{T}\left(\frac{2\ell}{\epsilon T}\right)^{\ell(1-\beta)} \leq \frac{4}{T}e^{-(B-E)\lambda}(2T)^{-(82n_K+162)\lambda/\mathscr{L}}. \tag{5.3}$$

Using Theorem 4.5 via partial summation, we see that

$$\begin{aligned}&\frac{T\log x}{x}\sum_{\chi}\sum_{\substack{\rho\\T\leq|\mathrm{Im}\{\rho\}|\leq 2T}}|F(-\rho\log x)|\\ &\ll \frac{\mathrm{e}^{-(B-E-162)\mathscr{L}}}{(2T)^{n_K}} + \left(B - E + \frac{n_K\log(2T)}{\mathscr{L}}\right)\int_0^{\mathscr{L}}\mathrm{e}^{-(B-E-162)\lambda}(2T)^{-n_K\lambda/\mathscr{L}}\mathrm{d}\lambda \ll 1,\end{aligned}$$

since $B > 162 + E$. Overall, this implies that the LHS of (5.2) is

$$\leq \frac{\log x}{x}\sum_{\chi}\sum_{k=0}^{\infty}\sum_{\substack{\rho\\2^kT_\star\leq\mathrm{Im}\{\rho\}<2^{k+1}T_\star}}|F(-\rho\log x)| \ll \frac{1}{T_\star}\sum_{k=0}^{\infty}\frac{1}{2^k} \ll \frac{1}{T_\star},$$

as desired. ■

As we shall see in the next section, an appropriate combination of Lemma 3.2, Theorem 4.5, and Lemma 5.1 suffices to establish Theorem 1.1. For Theorem 1.2, we must also show low-lying zeros far to the left of $\mathrm{Re}\{s\} = 1$ contribute a negligible amount.

Lemma 5.2. Let $0 \leq R \leq \frac{1}{2}\mathscr{L}$ be arbitrary. Assume (5.1) holds. For $x \geq \mathrm{e}^{B\mathscr{L}}$,

$$\frac{\log x}{x}\sum_{\chi}\sideset{}{'}\sum_{\rho}|F(-\rho\log x)| \ll x^{-(B-E-162)R/B\mathscr{L}},$$

where the marked sum $\sum'$ runs over zeros $\rho = \beta + i\gamma$ of $L(s,\chi,L/K)$, counting with multiplicity, satisfying $0 < \beta \leq 1 - R/\mathscr{L}$ and $|\gamma| \leq \epsilon^{-1}$. □

Proof. From our choices of ϵ, ℓ in (5.1) and Theorem 4.5, it follows that

$$N(1 - \tfrac{\lambda}{\mathscr{L}}, \epsilon^{-1}) \ll \mathrm{e}^{162\lambda}(1/\epsilon)^{(81n_K+162)\lambda/\mathscr{L}} \ll \mathrm{e}^{162\lambda}x^{E\lambda/B\mathscr{L}} \ll x^{(162+E)\lambda/B\mathscr{L}}$$

for $0 < \lambda < \mathscr{L}$, where $N(\sigma, T)$ is given by (4.4). Write $\rho = \beta + i\gamma$ with $\beta = 1 - \frac{\lambda}{\mathscr{L}}$ for some nontrivial zero ρ appearing in the marked sum. By Lemma 2.2(iv) with $\alpha = 0$ and Lemma 2.2(v), it follows that

$$\frac{\log x}{x}|F(-\rho \log x)| \ll \begin{cases} x^{-\lambda/\mathscr{L}} & \text{for } |\rho| \geq 1/4, \\ x^{-3/4}\log x & \text{for } |\rho| \leq 1/4. \end{cases} \tag{5.4}$$

To clarify the second inequality, we observe by Lemma 2.2(v) that $|F(-\rho \log x)| \ll x^{\beta} \ll x^{1/4}$ for $|\rho| \leq 1/4$. Thus, by (5.4) and partial summation, we have that

$$\begin{aligned}\frac{\log x}{x}\sum_{\chi}\sum_{|\rho|\geq 1/4}{}' |F(-\rho \log x)| &\ll x^{\frac{-(B-E-162)}{B}} + \frac{\log x}{\mathscr{L}}\int_R^{\mathscr{L}} x^{\frac{-(B-E-162)\lambda}{B\mathscr{L}}}\,\mathrm{d}\lambda \\ &\ll x^{-(B-E-162)R/B\mathscr{L}}.\end{aligned}$$

Moreover, by (5.4), a crude application of [12, Lemma 2.1], and Lemma 4.1, it follows that

$$\frac{\log x}{x}\sum_{\chi}\sum_{\substack{\rho \\ |\rho|\leq 1/4}}{}' |F(-\rho \log x)| \ll [L:K]\mathscr{L}x^{-3/4}\log x \ll x^{-3/4}\mathrm{e}^{2\mathscr{L}}\log x \ll x^{-\frac{3}{4}+\frac{3}{B}}.$$

Combining these estimates yields the desired result since, by our assumptions on B and R, $x^{-(B-E-162)R/B\mathscr{L}} \gg x^{-(B-E-162)/2B} \gg x^{-1/2} \gg x^{-3/4+3/162} \gg x^{-3/4+3/B}$. ■

We package these lemmas into the following convenient proposition.

Proposition 5.3. Let $0 \leq R \leq \frac{1}{2}\mathscr{L}$ be arbitrary. Let $0 < E < \frac{2}{3}B$ be fixed. Assume that

$$B > 162 + E, \qquad \ell \geq 82n_K + 162, \qquad \tfrac{1}{4} > \epsilon \geq 4\ell x^{-E/(B\ell)}. \tag{5.5}$$

If $x \geq \mathrm{e}^{B\mathscr{L}}$ and $S(x)$ is given by (3.1), then

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} \leq 1 + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sum_{\rho}{}^{\star} |F(-\rho \log x)| + O\big(\epsilon + x^{-(B-E-162)R/B\mathscr{L}}\big), \tag{5.6}$$

where the sum $\sum^{\star}$ indicates a restriction to nontrivial zeros ρ of $L(s, \chi, L/K)$, counted with multiplicity, satisfying $1 - R/\mathscr{L} < \mathrm{Re}\{\rho\} < 1$ and $|\mathrm{Im}\{\rho\}| \leq \epsilon^{-1}$. □

Proof. Let $T_\star = 1/\epsilon$. It follows from our hypothesis (5.5) along with Lemma 3.2, Lemma 5.1, and Lemma 5.2 that

$$\frac{|G|}{|C|}\frac{S}{e^{\epsilon}x} \leq 1 + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sideset{}{^\star}\sum_{\rho}|F(-\rho\log x)| \\ + O\left(\epsilon + x^{-(B-E-162)R/B\mathscr{L}} + n_L x^{-1}\log x + x^{-5/4}(2\ell/\epsilon)^{\ell}\log D_L\right). \tag{5.7}$$

It remains to bound the third and fourth expressions in the error term by ϵ. Since $E < B$ and $\ell \geq 244$, we see that

$$\epsilon > x^{-E/B\ell} > x^{-1/\ell} > x^{-1/244}.$$

Moreover, $n_L = n_K[L:K] \ll \mathscr{L}e^{2\mathscr{L}} \ll x^{3/162}$ by Lemma 4.1 and (4.2). Similarly, since $\log D_L = \sum_{\chi}\log D_{\chi} \leq [L:K]\log(D_K\mathcal{Q})$, it follows that

$$(2\ell/\epsilon)^{\ell}\log D_L \ll x^{E/B}\mathscr{L}[L:K] \ll x^{2/3}\mathscr{L}e^{2\mathscr{L}} \ll x^{2/3+3/162}.$$

Applying these estimates in (5.7) yields (5.6). ■

6 Proof of Theorem 1.1

In comparison to Theorem 1.2, the proof of Theorem 1.1 is quite simple, requiring only the log-free zero density estimate of Hecke L-functions given by Theorem 4.5. Recall this result is uniform over all extensions L/F and therefore we do not assume $\mathscr{L}$ is sufficiently large.

Proof of Theorem 1.1. Select

$$B = 244.5, \quad E = 82.1, \quad \ell = 82n_K + 162, \quad \epsilon = 1/8, \quad \text{and } R = 0. \tag{6.1}$$

Let $M_0 > 0$ be a sufficiently large absolute constant. For $x \geq x_0 := e^{244.5\mathscr{L}} + M_0 n_K^{244.5n_K}$, we claim these are valid choices to invoke Proposition 5.3. It suffices to check $\epsilon = \frac{1}{8} \geq 4\ell x^{-E/B\ell}$ for $x \geq x_0$. We need only show $(32\ell)^{B\ell/E} \leq x_0$. This is visible from the fact that

$$(32\ell)^{B\ell/E} \ll n_K^{\frac{244.5}{82.1}(82n_K+162)}e^{O(n_K)} \ll n_K^{244.5n_K} \leq x_0,$$

after enlarging M_0 if necessary. This proves the claim.

Therefore, by Proposition 5.3, we have that $S(x) \ll \frac{|C|}{|G|}x$ for $x \geq x_0$, because the corresponding restricted sum $\sum^\star$ is empty whenever $R = 0$. Let $M \geq 1$ denote the implicit absolute constant in the above estimate for $S(x)$. Thus, by Lemma 3.1 with $x_0 = \mathrm{e}^{244.5\mathscr{L}} + M_0 n_K^{244.5n_K}$, $a = M$ and $b = c = 0$, we have that

$$\pi_C(x, L/F) < \left\{M + O\Big(n_L x^{-1/2} + \frac{n_L \log x}{x}(\mathrm{e}^{244.5\mathscr{L}} + n_K^{244.5n_K})\Big)\right\} \frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \geq x_0$. By Lemma 4.1 and (4.1), notice that $n_L \ll e^{4\mathscr{L}/3} \ll D_K^2 \mathcal{Q}^2 n_K^{n_K}$. Thus, the desired result follows for $x \gg e^{245.9\mathscr{L}} + D_K^2 \mathcal{Q}^2 n_K^{246n_K}$. ■

Remark.

- If one wishes to minimize the value of B and hence minimize the exponents of D_K and $\mathcal{Q}$ in (1.10) then one may alternatively select

$$B = 162.01, \quad E = 0.95, \quad \ell = 82n_K + 162, \quad \epsilon = 1/8, \quad \text{and } R = 0$$

in place of (6.1). Taking $x_0 = \mathrm{e}^{162.01\mathscr{L}} + M_0 n_K^{13{,}999n_K}$, it follows that

$$(32\ell)^{B\ell/E} \ll n_K^{\frac{162.01}{0.95}(82n_K+162)} \mathrm{e}^{O(n_K)} \ll n_K^{13{,}999n_K} \leq x_0.$$

Arguing as above, one deduces $\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x)$ for $x \gg \mathrm{e}^{164\mathscr{L}} + D_K^2 \mathcal{Q}^2 n_K^{14{,}000n_K}$ as claimed in the remark following Theorem 1.1 based on (4.1).
- Similarly, to minimize the exponents of $n_K^{n_K}$ in (1.10), one may alternatively select

$$B = 359.5, \quad E = 197, \quad \ell = 82n_K + 162, \quad \epsilon = 1/8, \quad \text{and } R = 0$$

in place of (6.1). Taking $x_0 = \mathrm{e}^{359.5\mathscr{L}}$, it follows by (4.2) that

$$(32\ell)^{B\ell/E} \ll n_K^{\frac{359.5}{197}(82n_K+162)} \mathrm{e}^{O(n_K)} \ll n_K^{149.65n_K} \leq x_0,$$

since $359.5 \times \frac{5}{12} > 149.7$. Arguing as above, one deduces $\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x)$ for $x \gg \mathrm{e}^{360.9\mathscr{L}} \geq \mathrm{e}^{4\mathscr{L}/3}\mathrm{e}^{359.5\mathscr{L}}$ as claimed in the remark following Theorem 1.1.

□

The following two sections consist of the proof of Theorem 1.2 which is divided into cases depending on how close the zero ρ_1, defined in Section 4.2, is to $\mathrm{Re}\{s\} = 1$. The main steps are similar to the above proof for Theorem 1.1 but need a more refined analysis.

7 Proof of Theorem 1.2: η-Siegel Zero Exists

Let $\eta > 0$ be arbitrary and sufficiently small and let $\mathscr{L}$ be sufficiently large depending only on η. The proof of Theorem 1.2 is divided into Sections 7 and 8 by whether ρ_1 is an η-Siegel zero or not.

For this section, we consider the case when $\lambda_1 < \eta$. By Theorem 4.2, it follows that $\rho_1 = \beta_1 = 1 - \frac{\lambda_1}{\mathscr{L}}$ is a simple real zero and χ_1 is a real Hecke character. Suppose

$$B = 692, \qquad E = 344, \qquad \ell = 82n_K + 162, \qquad 4\ell x^{-344/692\ell} \le \epsilon < 1/4. \tag{7.1}$$

With these choices, we claim for $x \ge e^{692\mathscr{L}}$ that $4\ell x^{-344/692\ell} = o(1)$ as $\mathscr{L} \to \infty$. If n_K is uniformly bounded while $\mathscr{L} \to \infty$ then this is immediate, so we may assume $n_K \to \infty$. By (4.2), notice that $\ell = 82n_K + 162 \le \{196.8 + o(1)\}\frac{\mathscr{L}}{\log n_K} \le 197\frac{\mathscr{L}}{\log n_K}$ for n_K sufficiently large. Thus, for n_K sufficiently large and $x \ge e^{692\mathscr{L}}$, we have that

$$4\ell x^{-344/692\ell} \ll n_K e^{-344\mathscr{L}/\ell} \ll n_K e^{\frac{-344}{197}\log n_K} \ll n_K^{-0.7}.$$

Hence, $4\ell x^{-344/692\ell} = o(1)$ as $n_K \to \infty$. This proves the claim, which implies the condition on ϵ in (7.1) is nonempty for $\mathscr{L}$ sufficiently large.

Now, let $1 \le R \le \frac{1}{2}\mathscr{L}$ be arbitrary. By Proposition 5.3, for $x \ge e^{692\mathscr{L}}$, we have that

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} \le 1 + \frac{x^{-(1-\beta_1)}}{\beta_1} + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sum_{\rho\ne\rho_1}{}^{\star}|F(-\rho\log x)| + O\big(\epsilon + x^{-186R/692\mathscr{L}}\big), \tag{7.2}$$

where $\sum^{\star}$ runs over nontrivial zeros $\rho \ne \rho_1$ of $L(s,\chi,L/K)$, counted with multiplicity, satisfying

$$1 - R/\mathscr{L} < \mathrm{Re}\{\rho\} < 1, \qquad |\mathrm{Im}\{\rho\}| \le \epsilon^{-1}.$$

Note that the β_1 term in (7.2) arises from bounding $F(-\sigma\log x)$ in Lemma 2.2(v) with $\sigma = \beta_1$. We further subdivide our arguments depending on the range of λ_1.

7.1 λ_1 very small ($\frac{2\eta\mathscr{L}}{\log x} \le \lambda_1 < \eta$)

Here select $\epsilon = \eta^2$ and $R = \min\{\frac{1}{82}\log(c_1/\lambda_1), \frac{1}{2}\mathscr{L}\}$ for some fixed sufficiently small $c_1 > 0$. Since $4\ell x^{-344/692\ell} = o(1)$ as $\mathscr{L} \to \infty$, it follows that this choice of ϵ satisfies (7.1) for $\mathscr{L}$ sufficiently large depending only on η.

Hence, by Theorem 4.4, these choices imply that the restricted sum $\sum^\star$ in (7.2) is empty for $\mathscr{L}$ sufficiently large depending only on η. Moreover, we see that

$$x^{-186R/693\mathscr{L}} \le e^{-\frac{186}{82}\log(c_1/\lambda_1)} \ll \lambda_1^2 \ll \eta^2,$$

as $x \ge e^{692\mathscr{L}}$ and $186/82 > 2$. Further, we have that

$$\frac{x^{-(1-\beta_1)}}{\beta_1} = e^{-\lambda_1 \log x/\mathscr{L}}\{1 + O(\lambda_1/\mathscr{L})\} < 1 - \eta + O(\eta^2),$$

since $\frac{2\eta\mathscr{L}}{\log x} \le \lambda_1 < \eta$ and $e^{-t} < 1 - t/2$ for $0 \le t \le 1$. Overall, we conclude that $S(x) < \{2 - \eta + O(\eta^2)\}\frac{|C|}{|G|}x$ for $x \ge e^{692\mathscr{L}}$. By Lemmas 3.1 and 4.1, we conclude that

$$\pi_C(x, L/F) < \{2 - \eta + O\big(\eta^2 + \mathscr{L}e^{1.4\mathscr{L}}(x^{-1/2} + e^{693\mathscr{L}}x^{-1}\log x)\big)\}\frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \ge e^{692\mathscr{L}}$. Hence, in this subcase, Theorem 1.2 (with no error term) follows for $x \ge e^{694.5\mathscr{L}}$ after fixing $\eta > 0$ sufficiently small and recalling $\mathscr{L}$ is sufficiently large.

7.2 λ_1 extremely small ($\lambda_1 < \frac{2\eta\mathscr{L}}{\log x} \le \eta$)

Here select

$$\epsilon = 4\ell x^{-344/692\ell} \quad \text{and} \quad R = \min\left\{\frac{\mathscr{L}}{81\mathscr{L} + 25n_K\log(1/\epsilon)}\log\left(\frac{c_1}{\lambda_1}\cdot\frac{\mathscr{L}}{\mathscr{L} + n_K\log(1/\epsilon)}\right), \frac{1}{2}\mathscr{L}\right\}$$

for some sufficiently small $c_1 > 0$. Again, since $4\ell x^{-344/692\ell} = o(1)$ as $\mathscr{L} \to \infty$, it follows that $\epsilon < 1/4$ for $\mathscr{L}$ sufficiently large so this choice of ϵ satisfies (7.1).

Now, from our choice of R and Theorem 4.4, the restricted sum in (7.2) is empty. For the main term, observe for $\mathscr{L}$ sufficiently large and $\eta > 0$ sufficiently small that

$$\frac{x^{-(1-\beta_1)}}{\beta_1} < \left(1 - \frac{\lambda_1 \log x}{2\mathscr{L}}\right)\left(1 + \frac{\lambda_1}{\mathscr{L}}\right) \le 1 - \frac{\lambda_1 \log x}{3\mathscr{L}},$$

as $\lambda_1 < \frac{2\eta\mathscr{L}}{\log x}$ and $e^{-t} < 1 - t/2$ for $0 \le t \le 1$. To bound the error term in (7.2), notice that

$$81\mathscr{L} + 25n_K\log(1/\epsilon) \le \frac{81}{692}\log x + \frac{344\cdot 25n_K}{692(82n_K + 162)}\log x < \frac{185.9}{692}\log x,$$

by our choice of ϵ and ℓ and since $x \geq e^{693\mathscr{L}}$. Consequently, $R \geq \frac{692\mathscr{L}}{185.9 \log x} \log(\frac{c_1' \mathscr{L}}{\lambda_1 \log x})$ for some sufficiently small $c_1' > 0$, implying

$$x^{-186R/692\mathscr{L}} \ll \left(\frac{\lambda_1 \log x}{\mathscr{L}}\right)^{\frac{186}{185.9}} \ll \eta^{1/2000} \left(\frac{\lambda_1 \log x}{\mathscr{L}}\right),$$

since $\lambda_1 < \frac{2\eta\mathscr{L}}{\log x}$ and $\frac{0.1}{185.9} < \frac{1}{2000}$. Combining these observations into (7.2) implies that

$$\frac{|G|}{|C|} \frac{S(x)}{e^{\epsilon} x} < 2 - \frac{\lambda_1 \log x}{3\mathscr{L}} + O\left(\epsilon + \eta^{1/2000} \cdot \frac{\lambda_1 \log x}{\mathscr{L}}\right) < 2 - 100\lambda_1 + O(\epsilon)$$

as η is sufficiently small. Rearranging and substituting the choice of ϵ and ℓ, we see that

$$S(x) < \left\{2 - 100\lambda_1 + O\big(n_K x^{-\frac{1}{166 n_K + 327}}\big)\right\} \frac{|C|}{|G|} x$$

for $x \geq e^{692\mathscr{L}}$. Now, if $x \geq e^{694.9\mathscr{L}}$ then, by Lemma 4.1, we have that

$$n_L e^{692\mathscr{L}} x^{-1} \log x \ll n_K e^{693.4\mathscr{L}} x^{-1} \log x \ll n_K x^{-1.5/694.9} \log x \ll n_K x^{-1/(166 n_K + 327)}.$$

Similarly, $n_L x^{-1/2} \ll n_K x^{-1/(166 n_K + 327)}$. Thus, by the previous inequality and Lemma 3.1, it follows that

$$\pi_C(x, L/F) < \left\{2 - 100\lambda_1 + O\big(n_K x^{-\frac{1}{166 n_K + 327}}\big)\right\} \frac{|C|}{|G|} \mathrm{Li}(x) \tag{7.3}$$

for $x \geq e^{694.9\mathscr{L}}$. As δ_0 in (4.1) is sufficiently small, this completes the proof of Theorem 1.2 when an η-Siegel zero exists. ∎

Remark.

- In (7.1), we could instead take $B = 502$ and $E = 198$ to establish (7.3) except with an error term of $O(n_K x^{-1/(208 n_K + 411)})$. To improve the error term, we chose the largest values of B and E which did not reduce the valid range of x in Theorem 1.2. This range of x is limited by the case addressed in Section 8.3.
- As stated in Theorem 1.2, we obtain the sharper bound $\pi_C(x, L/F) < 2\frac{|C|}{|G|}\mathrm{Li}(x)$ from (7.3) with good effective lower bounds for λ_1. To see this, notice the error term in (7.3) is $\ll \lambda_1^{1.001}$ provided

$$x \gg \left(\frac{c_1 n_K}{\lambda_1^{1.001}}\right)^{166 n_K + 327} =: x_1,$$

where $c_1 > 0$ is some absolute constant. If the above holds then (7.3) becomes

$$\pi_C(x, L/F) < \{2 - 100\lambda_1 + O(\lambda_1^{1.001})\} \frac{|C|}{|G|} \mathrm{Li}(x).$$

As $\lambda_1 \leq \eta$, this implies $\pi_C(x, L/F) < 2\frac{|C|}{|G|}\mathrm{Li}(x)$ by fixing η sufficiently small. Hence, any effective upper bound on x_1 translates to a range of x where the sharper bound for $\pi_C(x, L/F)$ holds. From the proof of Theorem 1′ in Stark [23], we have that $\lambda_1 \gg \min\{g(n_K)^{-1}, D_K^{-1/n_K}\mathcal{Q}^{-1/2n_K}\}$ where $g(n_K)$ equals 1 if K has a normal tower over $\mathbb{Q}$ and equals $(2n_K)!$ otherwise. If $n_K \leq 10$ and $D_K\mathcal{Q}$ is sufficiently large then we have that

$$x_1 \ll (1/\lambda_1)^{167n_K+328} \ll D_K^{167+328/n_K}\mathcal{Q}^{84+164/n_K} \ll D_K^{495}\mathcal{Q}^{248} \ll x,$$

for x satisfying (1.12), as desired. Thus, we may assume $n_K \geq 10$ in which case we have that

$$\begin{aligned} x_1 &\ll n_K^{167n_K}(1/\lambda_1)^{167n_K+328} \\ &\ll D_K^{167+328/n_K}\mathcal{Q}^{84+164/n_K} n_K^{167n_K} + n_K^{167n_K} g(n_K)^{167n_K+328} \\ &\ll D_K^{200}\mathcal{Q}^{101} n_K^{167n_K} + n_K^{167n_K} g(n_K)^{167n_K+328}. \end{aligned}$$

Therefore, if K has a normal tower over $\mathbb{Q}$ or $(2n_K)! \ll D_K^{1/n_K}\mathcal{Q}^{1/2n_K}$ then

$$x_1 \ll D_K^{200}\mathcal{Q}^{101} n_K^{167n_K} e^{O(n_K)} \ll D_K^{200}\mathcal{Q}^{101} n_K^{168n_K} \ll x,$$

for x satisfying (1.12) and $D_K\mathcal{Q}n_K^{n_K}$ sufficiently large. Otherwise, $g(n_K) \leq (2n_K)! \leq (2n_K)^{2n_K}$ which implies that

$$x_1 \ll D_K^{200}\mathcal{Q}^{101} n_K^{167n_K} + n_K^{333n_K^2}$$

unconditionally. Thus, imposing $x \gg n_K^{334n_K^2}$ in addition to (1.12) also yields the sharper estimate for $\pi_C(x, L/F)$. This completes all cases. □

8 Proof of Theorem 1.2: η-Siegel Zero Does Not Exist

In this section, we assume $\lambda_1 \geq \eta$ for sufficiently small $\eta > 0$ and we will show Theorem 1.2 holds with no error term. Recall $\mathscr{L}$ is sufficiently large depending only on η. Assume $\lambda^\star > 0$ satisfies

$$\lambda^\star < \min\{\lambda', \lambda_2\}, \tag{8.1}$$

where λ' and λ_2 are defined in Section 4.2. Select

$$B > 360, \qquad E = 198, \qquad \ell = 82n_K + 162, \qquad \epsilon = \eta^2, \tag{8.2}$$

and let $R = R(\eta)$ be sufficiently large. We claim these choices satisfy the assumptions of Proposition 5.3. Since $\mathscr{L}$ is sufficiently large depending only on η, it suffices to show, for $x \geq e^{B\mathscr{L}}$, that $4\ell x^{-E/B\ell} = o(1)$ as $\mathscr{L} \to \infty$. We shall argue as in Section 7. If n_K is bounded while $\mathscr{L} \to \infty$ then this is immediate, so we may assume $n_K \to \infty$. By (4.2), notice that $\ell = 82n_K + 162 \leq \{196.8 + o(1)\}\frac{\mathscr{L}}{\log n_K} \leq 197\frac{\mathscr{L}}{\log n_K}$ for n_K sufficiently large. Thus, for n_K sufficiently large and $x \geq e^{B\mathscr{L}}$, we have that

$$4\ell x^{-E/B\ell} \ll n_K e^{-198\mathscr{L}/\ell} \ll n_K e^{-\frac{198}{197}\log n_K} \ll n_K^{-1/197}.$$

Hence, $4\ell x^{-E/B\ell} = o(1)$ for $x \geq e^{B\mathscr{L}}$, as $n_K \to \infty$. This proves the claim.

Therefore, by Proposition 5.3, it follows that

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} \leq 1 + \frac{\log x}{e^{\epsilon}x}\sum_{\chi}\sideset{}{^\star}\sum_{\rho}|F(-\rho\log x)| + O(\eta^2),$$

for $x \geq e^{B\mathscr{L}}$ and where the sum $\sum^\star$ runs over nontrivial zeros ρ of $L(s,\chi)$, counted with multiplicity, satisfying $\beta > 1 - R/\mathscr{L}$ and $|\gamma| \leq \eta^{-2}$. For a nontrivial zero ρ of a Hecke L-function, write $\rho = \beta + i\gamma = 1 - \frac{\lambda}{\mathscr{L}} + i\frac{\mu}{\mathscr{L}}$. By Lemma 2.2, we see that

$$\frac{\log x}{e^{\epsilon}x}|F(-\rho\log x)| \leq x^{-(1-\beta)} \leq e^{-B\lambda},$$

since $x \geq e^{B\mathscr{L}}$. Extracting ρ_1 and $\overline{\rho_1}$ (or simply ρ_1 if ρ_1 is real) from $\sum^\star$, we deduce by our choice of $\lambda^\star$ in (8.1) that

$$\frac{|G|}{|C|}\frac{S(x)}{e^{\epsilon}x} \leq 1 + m(\rho_1)e^{-B\lambda_1} + \sum_{\chi}\sum_{\substack{\lambda^\star \leq \lambda \leq R \\ |\gamma| \leq \eta^{-2}}} e^{-B\lambda} + O(\eta^2), \tag{8.3}$$

where $m(\rho_1) = 2$ if ρ_1 is complex and $m(\rho_1) = 1$ if ρ_1 is real. To bound the remaining quantities, we must select $\lambda^\star$ for which we further subdivide into cases.

8.1 λ_1 small ($\eta \leq \lambda_1 < 10^{-3}$)

By Theorem 4.2, ρ_1 is a simple real zero attached to a real character χ_1, implying $m(\rho_1) = 1$. Select $B = 361$ and choose $\lambda^\star = 0.2103\log(1/\lambda_1)$, which satisfies (8.1) by Theorem 4.3. Arguing as in [24, Section 10.1.2] and using Theorem 4.6, we may conclude by (8.3) that

$S(x) < \{2 - \eta + O(\eta^2)\}\frac{|C|}{|G|}x$ for $x \geq e^{361\mathscr{L}}$. As in the final arguments of Section 7.1, we use Lemma 3.1 to establish Theorem 1.2 for $x \geq e^{363\mathscr{L}}$.

8.2 λ_1 medium ($10^{-3} < \lambda_1 \leq 0.0875$)

One argues similar to the previous case with some minor changes. Namely, select $B = 593$ and choose $\lambda^\star = 0.44$, and follow [24, Section 10.1.1] to deduce Theorem 1.2 for $x \geq e^{595\mathscr{L}}$.

8.3 λ_1 large ($\lambda_1 \geq 0.0875$)

Select $B = 693$ and $\lambda^\star = 0.2866$ as per Theorem 4.2. Noting $m(\rho_1) \leq 2$ unconditionally, one may argue similarly as per the previous cases and follow [24, Section 11] to deduce Theorem 1.2 for $x \geq e^{694.9\mathscr{L}}$. As δ_0 in (4.1) is sufficiently small, this yields the desired range of x in Theorem 1.2, completing the proof in all cases. ∎

9 Proof of Theorems 1.4 and 1.5

First, we state a slightly weaker (but more convenient) reformulation of Theorem 1.1.

Theorem 9.1. Let L/F be a Galois extension of number fields with Galois group G, and let C be any conjugacy class of G. Let H be an abelian subgroup of G such that $H \cap C$ is nonempty, and let K be the subfield of L fixed by H. Let $\mathcal{P}(L/K)$ be the set of rational primes p such that there is a prime ideal $\mathfrak{p}$ of K with $\mathfrak{p} \mid p$ and $\mathfrak{p}$ ramifies in L, and set

$$M(L/K) = [L:K] D_K^{1/n_K} \prod_{p \in \mathcal{P}(L/K)} p.$$

If $\log x \gg n_K \log(M(L/K) n_K)$, then $\pi_C(x, L/F) \ll \frac{|C|}{|G|}\mathrm{Li}(x)$. □

Proof. If L/K is abelian, then [19, Proposition 2.5] states that

$$\mathcal{Q}(L/K) \leq \left([L:K] \prod_{p \in \mathcal{P}(L/K)} p \right)^{2n_K}.$$

Using the definition of $M(L/K)$, we see that (1.10) is

$$\ll (D_K \mathcal{Q}(L/K) n_K^{n_K})^{246} \ll (n_K M(L/K))^{500 n_K}.$$

The claimed result now follows immediately from Theorem 1.1. ∎

9.1 Proof of Theorem 1.4

Fix a newform f (cf. Section 1) of even integral weight $k_f \geq 2$, level N_f, and trivial nebentypus with integral Fourier coefficients, and fix an integer a. For each prime p, we define $\omega_p = (a_f(p)^2 - 4p^{k_f-1})^{1/2}$. We know from Deligne's proof of the Weil conjectures that $|a_f(p)| \leq 2p^{(k_f-1)/2}$ for all p, so $\mathbb{Q}(\omega_p)$ is an imaginary quadratic extension of $\mathbb{Q}$. Set

$$\pi_f(x, a; \ell) = \#\{p \leq x\colon a_f(p) \equiv a \pmod{\ell} \text{ and } \ell \text{ splits in } \mathbb{Q}(\omega_p)\}.$$

Let $\ell_1 < \ell_2 < \cdots < \ell_t$ be any t odd primes, each less than $\exp(\frac{\log x}{2t})$. By [26, Corollary 4.2], if $t \sim (4/\log 2)\log\log x$, then

$$\pi_f(x, a) \ll \sum_{j=1}^{t} \pi_f(x, a; \ell_j) + \frac{x}{(\log x)^2} \ll (\log\log x) \max_{1\leq j\leq t} \pi_f(x, a; \ell_j) + \frac{x}{(\log x)^2}. \tag{9.1}$$

We proceed to bound $\pi_f(x, a; \ell)$, where $\ell \leq \exp((\log 2)(\log x)/(8\log\log x))$.

Let ℓ be prime, let $\mathbb{F}_\ell$ be the field of ℓ elements, and let Frob_p be the Frobenius automorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at p. For each ℓ, there is a representation

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell) \tag{9.2}$$

which is unramified outside $N_f\ell$ such that for all primes $p \nmid N_f\ell$, we have that $\mathrm{tr}(\rho_{f,\ell}(\mathrm{Frob}_p)) \equiv a_f(p) \pmod{\ell}$ and $\det(\rho_{f,\ell}(\mathrm{Frob}_p)) \equiv p^{k_f-1} \pmod{\ell}$. We have that $\rho_{f,\ell}$ is surjective for all but finitely many ℓ. Let $L = L_\ell$ be the subfield of $\overline{\mathbb{Q}}$ fixed by $\ker \rho_{f,\ell}$. If ℓ is sufficiently large, then $L/\mathbb{Q}$ is a Galois extension, unramified outside of $N_f\ell$, whose Galois group is $G = \{g \in \mathrm{GL}_2(\mathbb{F}_\ell) : \det g \in (\mathbb{F}_\ell^\times)^{k_f-1}\}$.

Define $C = \{A \in G\colon \mathrm{tr}(A) \equiv a \pmod{\ell} \text{ and } \mathrm{tr}(A)^2 - 4\det(A) \in \mathbb{F}_\ell \text{ is a square}\}$. Let B denote the upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_\ell) \cap G$, and let L^B be the subfield of L fixed by B. Let U be the unipotent elements of B, and let L^U be the subfield of L fixed by U. Note that U is a normal subgroup of B and that $B/U \cong \mathrm{Gal}(L^U/L^B)$ is abelian. Let C' be the image of $C \cap B$ in B/U. If x is sufficiently large, then by [30, Lemmas 2.7 and 4.3],

$$\pi_f(x, a; \ell) \ll \pi_{C'}(x, L^U/L^B) + n_{L^B}\left(\frac{\sqrt{x}}{\log x} + \log M(L^U/L^B)\right).$$

Applying Theorem 9.1 to the Chebotarev prime counting functions for each conjugacy class in C', we have that if $\log x \gg n_{L^B}\log(M(L^U/L^B)n_{L^B})$, then

$$\pi_f(x, a; \ell) \ll \frac{|C'|}{|B/U|}\frac{x}{\log x} + n_{L^B}\left(\frac{\sqrt{x}}{\log x} + \log M(L^U/L^B)\right).$$

By [30, Lemma 4.4], we have $|C'|/|B/U| \ll 1/\ell$, $n_{L^B} \ll \ell$, and $\log M(L^U/L^B) \ll_{N_f} \log \ell$. Combining all of our estimates, we find that

$$\pi_f(x,a;\ell) \ll \frac{1}{\ell}\frac{x}{\log x} + \frac{\ell\sqrt{x}}{\log x} + \ell \log N_f \ell, \qquad \log x \gg \ell \log N_f \ell. \tag{9.3}$$

Thus, taking $\ell \sim c' \log x/\log(N_f \log x)$ for some sufficiently small absolute constant $c' > 0$,

$$\pi_f(x,a;\ell) \ll \frac{x\log(N_f \log x)}{(\log x)^2}. \tag{9.4}$$

Now, as before, let $t \in \mathbb{Z}$ satisfy $t \sim 4/(\log 2)\log\log x$, and let $\ell_1 < \ell_2 < \cdots < \ell_t$ be t consecutive primes with $\ell_1 \sim c' \log x/\log(N_f \log x)$. By the prime number theorem, $\ell_j \in [\ell_1, 2\ell_1]$ for all $1 \leq j \leq t$. Therefore, if c' is made sufficiently small, we have that

$$\max_{1\leq j\leq t} \pi_f(x,a;\ell_j) \ll \frac{x\log(N_f \log x)}{(\log x)^2}. \tag{9.5}$$

Theorem 1.4 now follows from inserting the inequality (9.5) into the inequality (9.1).

Remark. Using the Cauchy–Schwarz and Polya–Vinogradov inequalities, Murty [20, Page 304] proved that

$$\pi_f(x,a) \ll \max_{\ell\in[y,2y]} \pi_f(x,a;\ell) + \left(\frac{\pi_f(x,a)x\log y}{y}\right)^{1/2}. \tag{9.6}$$

Using [20, Theorem 4.6], it is subsequently shown that if $\ell \in [y,2y]$ and $y = c'(\log x)/(\log\log x)^2$ for some sufficiently small absolute constant $c' > 0$, then

$$\pi_f(x,a;\ell) \ll \frac{x(\log\log x)^2}{(\log x)^2}. \tag{9.7}$$

It is then claimed in [20] that (9.6) and (9.7) imply $\pi_f(x,a) \ll_{N_f} x(\log\log x)^2/(\log x)^2$. It is not clear to us how to deduce this estimate for $\pi_f(x,a)$ using (9.6) and (9.7). In particular, if $\pi_f(x,a) \gg x/(\log x)^2$, then the aforementioned choice of y forces the secondary term in (9.6) to be $\gg x/(\log x)^{3/2}$. By inserting (9.7) into (9.1) instead of (9.6), one obtains the weaker statement (1.17). The source of our improvement over [20] stems solely from the $\log\log x$ savings over (9.7), which can be seen from (9.4). □

9.2 Proof of Theorem 1.5

The proof of Theorem 1.5 is nearly identical to the proof of [30, Theorem 1.3(ii)] except that we use Theorem 1.1 to bound the ensuing Chebotarev prime counting function instead of using [30, Theorem 2.1(ii)]. The analytic details are very similar to the above proof of Theorem 1.4, but the particular Galois extension to which Theorem 1.1 is applied is different. Following [30, Section 5.2], we apply Theorem 1.1 instead of [30, Theorem 2.1(ii)], which allows us to choose

$$y = \frac{c}{h_k} \frac{\log x}{\log(\frac{D_k}{h_k} \log x)}$$

(where D_k is the absolute discriminant of k and h_k is the class number of k) for some sufficiently small absolute constant $c > 0$. This yields the claimed result.

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